Lista 1 — Ruben Vasconcelos Bcelli

① -

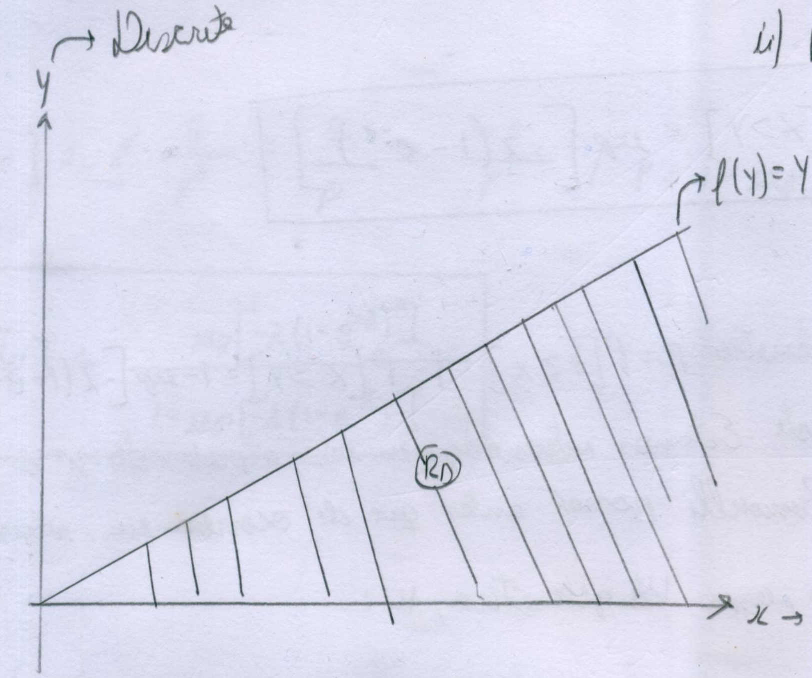
i) $f_X(x) = \lambda e^{-\lambda x}$, $x \geq 0$ (dist. exponencial, contínua), João

ii) $P_Y(y) = \dfrac{\lambda^y}{y!} e^{-\lambda}$, $y \in \mathbb{N}$ (dist de poisson, discreta) Marcos

iii) $f_X(x)$ e $P_Y(y)$ são independentes

a) $P[X > Y]$

i) $g_{X,Y}(x,y) = f_X(x)\, P_Y(y) = \dfrac{\lambda^{y+1}}{y!} e^{-\lambda(x+1)}$ → PDF conjunta, mista



ii) $R_D = \{(x,y) \mid f(y) \leq x < \infty,\ 0 \leq y < \infty\}$

$R_D = \{(x,y) \mid y \leq x < \infty,\ 0 \leq y < \infty\}$

iii) $P[X > Y] = P[(X,Y) \in R_D] = P[\{Y \leq X\} \cap \{0 \leq Y < \infty\}]$

$= \displaystyle\sum_{y=0}^{\infty} \int_y^{\infty} g_{X,Y}(x,y)\, dx = e^{-\lambda} \sum_{y=0}^{\infty} \frac{\lambda^{y+1}}{y!} \int_y^{\infty} e^{-\lambda x}\, dx$; I

(I): $\int_y^{\infty} e^{-\lambda x}\,dx \quad \left.\frac{e^{-\lambda x}}{-\lambda}\right|_y^{\infty} = \frac{-1}{\lambda}\left(\lim_{x\to\infty} e^{-\lambda x} - e^{-\lambda y}\right) = \frac{e^{-\lambda y}}{\lambda}$

cont.:

$$P[X>Y] = \frac{e^{-\lambda}}{\lambda}\sum_{y=0}^{\infty}\frac{\lambda^{y+1}}{y!}e^{-\lambda y} = e^{-\lambda}\sum_{y=0}^{\infty}\frac{(\lambda e^{-\lambda})^y}{y!}$$

iv) $e^3 = \sum_{n=0}^{\infty}\frac{3^n}{n!}$

Subs iv) em iii)

v) $P[X>Y] = e^{-\lambda}\cdot e^{\lambda e^{-\lambda}} \rightarrow \boxed{P[X>Y] = \exp\left[-\lambda(1-e^{-\lambda})\right]}$

b) Seja um teste de Bernoulli c/ parâmetro $p = P[Y \geq X] = 1 - P[X>Y] = 1-\exp\left[-\lambda(1-e^{-\lambda})\right]$, em que p é a prob. de sucesso do teste. Estamos interessados em saber a prob. do nº de falhas que uma sequência de testes de Bernoulli possui antes que se ocorrer um sucesso. Essa dist. de prob. é, por definição, uma VA geométrica, W:

i) $P_W(k) \triangleq p(1-p)^k$

Então $E[W]$ fornece a quantidade média de fracassos $(X>Y)$ antes do primeiro sucesso $(X \leq Y)$.

ii) $E[W] = \sum_{k=0}^{\infty} k\,P_W(k) = p\sum_{k=0}^{\infty} k(1-p)^k = p\sum_{k=0}^{\infty} k q^k$ (II)

(II):

- $\sum_{k=0}^{\infty} q^k = \dfrac{1}{1-q}$, suponde $q<1$

- $\dfrac{d}{dk}\left(\sum_{k=0}^{\infty} q^k\right) = \sum_{k=0}^{\infty} k\, q^{k-1} = \dfrac{1}{(1-q)^2} = \dfrac{1}{p^2}$

- $\sum_{k=0}^{\infty} k\, q^k = \dfrac{q}{p^2}$

cont:

$$E[W] = \cancel{p}\,\frac{q}{p^2} = \frac{q}{p} = \frac{1-p}{p} \;/\; p = 1-\exp\left[-\lambda\left(1-e^{-\lambda}\right)\right]$$

$$\boxed{E[W] = \frac{\exp\left[-\lambda(1-e^{-\lambda})\right]}{1-\exp\left[-\lambda(1-e^{-\lambda})\right]}}$$

(2) -

i) $f_{X,Y}(x,y) = c\,\exp\left(-x^2 - y^2 + xy\right)$

a) $\iint_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = 1 \longrightarrow c\iint_{-\infty}^{\infty} \exp\left(-x^2-y^2+xy\right)dx\,dy = 1$

$$c\iint_{-\infty}^{\infty} \exp\left(-\left(x^2 - xy - 0{,}25y^2\right)\right)\exp\left(-0{,}75y^2\right)dx\,dy = 1$$

$$c\int_{-\infty}^{\infty} \exp\left(\frac{3}{4}y^2\right) \underbrace{\int_{-\infty}^{\infty} \exp\left[-\left(x-0{,}5y\right)^2\right] dx}_{=\sqrt{\pi}}\, dy = 1 \rightarrow \sqrt{\pi}c \underbrace{\int_{-\infty}^{\infty} \exp\left(\frac{3}{4}y^2\right) dy}_{\sqrt{\pi \frac{4}{3}}} = 1$$

$$\frac{2\pi c}{\sqrt{3}} = 1 \rightarrow \boxed{c = \frac{\sqrt{3}}{2\pi}}$$

OBS: $\int_{-\infty}^{\infty} e^{-(ax\pm b)^2} = \sqrt{\frac{\pi}{a}}$

ii) $f_{X,Y}(x,y) = \frac{\sqrt{3}}{2\pi} \exp\left(-x^2 - y^2 + xy\right)$

b)

i) $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy = c \int_{-\infty}^{\infty} \exp\left(-\left(y^2 - xy + 0{,}25x^2\right)\right) \exp\left(-0{,}75x^2\right) dy$.

$$= c\, e^{-\frac{3}{4}x^2} \int_{-\infty}^{\infty} \exp\left(-\left(y^2 - 0{,}5x\right)\right) dy = \boxed{\frac{\sqrt{3\pi}}{2\pi} e^{-\frac{3}{4}x^2} = f_X(x)}$$

ii) $f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx = c \int_{-\infty}^{\infty} \exp\left(-(x^2 - xy + 0{,}25y^2\right) \exp\left(-0{,}75y^2\right) dx$

$$= c \int_{-\infty}^{\infty} \exp\left(-\left(x - 0{,}5y\right)^2\right) dx \; \exp\left(-\frac{3}{4}y^2\right) \longrightarrow \boxed{f_Y(y) = \frac{\sqrt{3\pi}}{2\pi} e^{-\frac{3}{4}y^2}}$$

iii) $f_{X,Y}(x,y) = \frac{\sqrt{3}}{2\pi} \exp(-x^2 - y^2 + xy) \neq f_X(x) f_Y(x) = \frac{3}{4\pi} \exp(-0{,}75x^2 - 0{,}75y^2)$

como $f_{X,Y}(x,y) \neq f_X(x) f_Y(y)$, X e Y não são independentes

iv) como $f_X(x) = f_Y(x)$, X e Y são identicamente distribuídos

c)

Seja Z_a a V.A auxiliar, então

i) $Z = g_1(X,Y) = X - 2Y$ ii) $Z_a = g_2(X,Y) = X$

cujas transformações inversas são dadas por

iii) $X = h_1(Z, Z_a) = Z_a$ iv) $Y = h_2(Z, Z_a) = \frac{Z_a - Z}{2}$

Definindo os V.VA

v) $\vec{X} = (X,Y)$ vi) $\vec{Z} = (Z, Z_a)$

Então:

vii) $f_{\vec{Z}}(z, z_a) = \frac{f_{\vec{X}}(h_1(z,z_a), h_2(z,z_a))}{J(x,y)}$

viii) $J(x,y) = \begin{vmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \\ \frac{\partial z_a}{\partial x} & \frac{\partial z_a}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & -2 \\ 1 & 0 \end{vmatrix} = 2$

Subs viii) em vii)

ix) $$f_{Z,Z_a}(z,z_a) = \frac{f_{X,Y}\left(z_a, \frac{z_a - z}{2}\right)}{2} = \frac{\sqrt{3}}{4\pi} \exp\left(-z_a^2 + \frac{z_a^2 - z z_a}{2} - \frac{(z_a - z)^2}{4}\right)$$

$$f_{Z,Z_a}(z,z_a) = \frac{\sqrt{3}}{4\pi} \exp\left(\frac{-4z_a^2 + 2z_a^2 \cancel{-2zz_a} - z_a^2 \cancel{+2zz_a} - z^2}{4}\right)$$

$$f_{Z,Z_a}(z,z_a) = \frac{\sqrt{3}}{4\pi} \exp\left(\frac{-3z_a^2 - z^2}{4}\right)$$

x) mas $Z_a = X$, então $z_a = x$

Sub x) em ix):

xi) $$f_{Z,X}(z,x) = \frac{\sqrt{3}}{4\pi} \exp\left(\frac{-(3x^2+z)}{4}\right)$$

3 -

Independente

i) $$P[\{X<25\} \cap \{Y<8\}] = P[X<25]\,P[Y>8] = \int_0^{25} f_X(x)\,dx \int_8^{9} f_Y(y)\,dy$$

$$= \frac{3}{12500}\,\frac{1}{64} \int_0^{25} x\,dx \int_8^{9} (y-9)^2 = 0{,}00390625$$

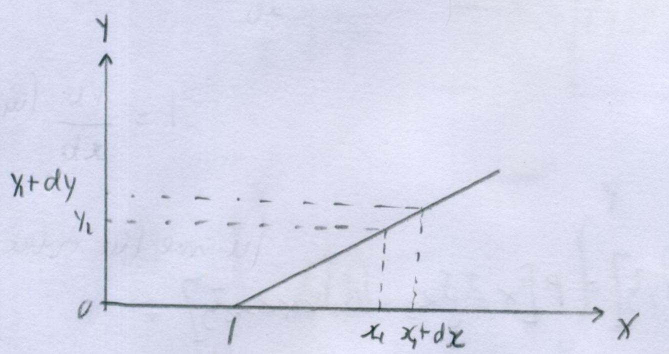
④ -

i) $F_X(x) = 1 - e^{-2x}$ p/ $x > 0$

a) $Y = X - 1$ p/ $X > 1$

i) $f_X(x) = \frac{d}{dx} F_X(x) = 2e^{-2x}$ p/ $x > 0$, $f_X(x) = 0$ p/ $x < 0$

ii) Assume-se que $Y = 0$ p/ $X \leq 1$



iii) Os eventos $C_Y = \{ y_1 \leq Y \leq y_1 + dy \}$ e $B_X = \{ x_1 \leq X \leq x_1 + dx \}$ são eventos equivalentes

iv) $P[Y \in C_Y] = P[X \in B_X] = f_Y(y_1)|dy| = f_X(x_1)|dx| \rightarrow f_Y(y_1) = \dfrac{f_X(x_1)}{\left|\dfrac{dy}{dx}\right|}$

Define-se:

v) $x = h(y) = y + 1$ vi) $\left|\dfrac{dy}{dx}\right| = 1$

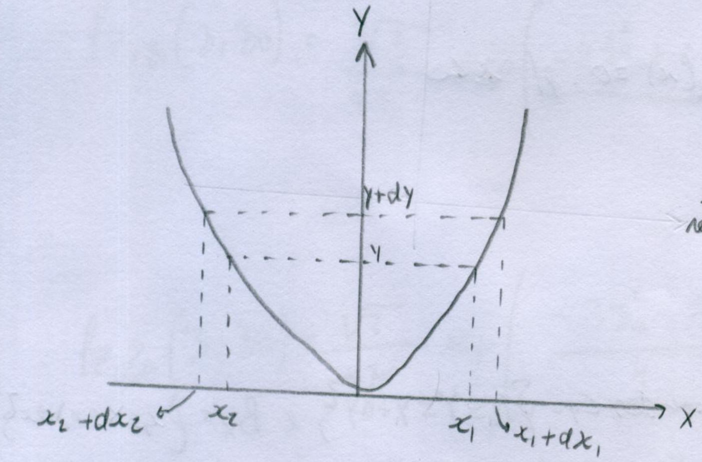
Sub. v) em iv)

vii) $f_Y(y) = \dfrac{f_X(h(y))}{\left|\dfrac{dy}{dx}\right|} = \dfrac{f_X(y+1)}{1} \rightarrow \boxed{f_Y(y) = 2e^{-(y+1)}}$

b) $y = x^2 = g(x)$

i) $f_X(x) = 2e^{-2x}$

ii) $X = h(Y) = \sqrt{Y}$



iii) O evento $C_Y = \{ y \le Y \le y + dy \}$ e

$B_X = \{ (x_2 + dx_2 \le X \le x_2) \cup (x_1 \le X \le x_1 + dx_1) \} = B_{X_1} \cup B_{X_2}$

são eventos equivalentes. Note ainda que $x_2 = -x_1$ e $x_1 > 0$.

iv) $P[Y \in C_Y] \overset{\text{Equivalência}}{=} P[X \in B_Y] \rightarrow P[Y \in \{ y \le Y \le y + dy \}] = P[X \in \{ x_2 + dx_2 < X < x_2 \}] +$

$P[X \in \{ x_1 < X \le x_1 + dx_1 \}]$

$$f_Y(y)\,|dy| = f_X(x_1)\,|dx_1| + f_X(x_2)\,|dx_2| \rightarrow f_Y(y) = \sum_{k=1}^{2} \frac{f_X(x_k)}{\left| \frac{dy}{dx_k} \right|}$$

como $y > 0$ e $x_1 > 0$, então

v) $x_1 = \sqrt{y} \rightarrow \dfrac{dx_1}{dy} = \dfrac{1}{2} \dfrac{1}{\sqrt{y}} = \dfrac{1}{2\sqrt{y}}$ $\therefore$ $x_1 = h_1(y) = \sqrt{y}$

vi) $x_2 = -x_1 = -\sqrt{y} \rightarrow \dfrac{dx_2}{dy} = \dfrac{-1}{2\sqrt{y}}$ $\therefore$ $x_2 = h_2(y) = -\sqrt{y}$

Sendo V) e Vi) em iV)

Vii) $f_Y(y) = \dfrac{f_X(x_1)}{2\sqrt{y}} + \dfrac{f_X(x_2)}{2\sqrt{y}} = \dfrac{f_X(h_1(y))}{2\sqrt{y}} + \dfrac{f_X(h_2(y))}{2\sqrt{y}}$

Como

$h_2(y) = -\sqrt{y} < 0 \longrightarrow f_X(h_2(y)) = f_X(-\sqrt{y}) = 0$

• $f_X(x) = 0$ p/ $x < 0$

vii)

$$f_Y(y) = \frac{f_X(\sqrt{y})}{2\sqrt{y}} \longrightarrow \boxed{f_Y(y) = \frac{e^{-2\sqrt{y}}}{\sqrt{y}} \text{ , p/ } y > 0}$$

viii) $F_Y(y) = \displaystyle\int_0^y f_Y(t)\, dt = \int_0^y \frac{e^{-2\sqrt{t}}}{\sqrt{t}}\, dt$

• $u = -2\sqrt{t} \rightarrow \dfrac{du}{dt} = -2 \, \dfrac{1}{2} \, \dfrac{1}{\sqrt{t}} = \dfrac{-1}{\sqrt{t}} \rightarrow \underline{dt = -\sqrt{t}\, du}$

• $t = 0 \rightarrow u = 0$ • $t = y \rightarrow u = -2\sqrt{y}$

$$F_Y(y) = \int_0^{2\sqrt{y}} \frac{-e^u}{\cancel{\sqrt{t}}} \cancel{\sqrt{t}}\, du = -\int_0^{2\sqrt{y}} e^u\, du = -e^u \Big|_0^{-2\sqrt{y}} = -\left(e^{-2\sqrt{y}} - 1\right)$$

$$\boxed{F_Y(y) = 1 - e^{-2\sqrt{y}}}$$