Rubem Porelli

Lista 03

03 - a)

$$Z = X_1 + X_2 + \dots + X_K = \sum_{j=1}^{K} X_j \qquad X \sim N(0, \sigma_x^2)$$

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma_X} e^{-\frac{x^2}{2\sigma_x^2}} \xleftrightarrow{\mathcal{F}\{\cdot\}} \phi_X(jw) = e^{-\frac{\sigma_x^2 w^2}{2}}$$ (função característica)

a) i) Como $X_1, \dots, X_n$ são independentes, temos que:

$$f_Z(z) = f_{X_1}(x_1) * f_{X_2}(x_2) * \dots * f_{X_n}(x_n) \xleftrightarrow{\mathcal{F}\{\cdot\}} \phi_Z(jw) = \phi_{X_1}(x_1)\,\phi_{X_2}(x_2) \dots \phi_{X_n}(x_n)$$

ii) $$\phi_Z(w) = \prod_{i=1}^{K} \phi_{X_i}(x_i) = \exp\left[\sum_{i=1}^{K} -\frac{\sigma_x^2 w^2}{2}\right] = \exp\left[-\frac{K\sigma_x^2 w^2}{2}\right]$$

iii) $$f_Z(z) = \mathcal{F}^{-1}\{\phi_Z(w)\} \longrightarrow f_Z(z) = \frac{1}{\sqrt{2\pi K \sigma_x^2}} e^{-\frac{z^2}{2K\sigma_x^2}} \therefore Z \sim N(0, K\sigma_x^2)$$

iv) 1º Passo do projeto do estimador de ML: Definir o parâmetro a ser estimado

$$\theta = K$$

v) 2º: Definir a PDF em função do parâmetro

portanto:

$$f_Z(z|\theta) = \frac{1}{\sqrt{2\pi\theta}\,\sigma_x} e^{-\frac{z^2}{2\theta\sigma_x^2}}$$

vi) 4º " : " o log da função de ML

$$\ln f_Z(z|\theta) = -\frac{1}{2}\ln 2\pi\theta - \ln \sigma_x - \frac{z^2}{2\theta\sigma_x^2}$$

vii) 5º: Derivar o log da função de ML

O parametro θ que maximiza o log da função de ML (denotado por θ^*) é:

$$\frac{\partial}{\partial\theta} f_Z(z|\theta) = 0$$

$$-\frac{1}{2}\,\frac{1}{\cancel{2\pi}\theta^*}\cdot\cancel{2\pi} + \frac{z^2}{2\theta^{*2}\sigma_x^2} = 0 \rightarrow \frac{z^2}{2\theta^{*2}\sigma_x^2} - \frac{1}{2\theta^*} = 0$$

$$\frac{z^2 - \theta^*\sigma_x^2}{2\theta^{*2}\sigma_x^2} = 0 \rightarrow \theta^* = \frac{z^2}{\sigma_x^2}$$

viii) 6º " : Definir o estimador de ML

Portanto o estimador de ML é $\boxed{\hat{\theta}(Z) = \frac{Z^2}{\sigma_x^2}}$

b)

i) $E[\hat{\theta}(Z)] = \frac{1}{\sigma_x^2}E[Z^2] = \frac{1}{\sigma_x^2}E[(Z-E[Z])^2] = \frac{\sigma_Z^2}{\sigma_x^2} = \frac{K\cancel{\sigma_x^2}}{\cancel{\sigma_x^2}} \rightarrow E[\hat{\theta}(Z)] = K$

A polarização desse estimador é:

ii) $B[\hat{\theta}(Z)] = E[\hat{\theta}(Z)] - \theta = K - K = 0$ <u>não polarizado</u>

(2) -

i) Definir V.V.A observados $(\vec{X}, N)$

$$\vec{X} = \left[X[0] \;\; x[1] \;\; \dots \;\; X[N-1] \right]^T$$

ii) Definir o modelo do sinal $(x[n])$

$$x[n] = \alpha_0 + \alpha_1 t[n]$$

iii) " o V.V.A dos parâmetros estimados $(\vec{\theta}, p)$

$$\vec{\theta} = \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix} \qquad p = 2$$

iv) " matriz de observação $(\vec{H} \text{ e } \vec{h}_j)$

$$\vec{X} = \vec{H}\,\vec{\theta} \rightarrow \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix} = \underbrace{\overset{N \times 2}{\begin{bmatrix} 1 & \dots & t[0] \\ 1 & \dots & t[1] \\ \vdots & & \vdots \\ 1 & \dots & t[N-1] \end{bmatrix}}}_{\vec{H} = \left[\vec{h}_1 \;\; \vec{h}_2 \right]} \overset{2 \times 1}{\begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix}}$$

$$\vec{X} = \left[\vec{h}_1 \;\; \vec{h}_2 \right] \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix} \rightarrow \vec{X} = \sum_{j=0}^{p-1=1} \alpha_j \vec{h}_{j+1}$$

v) Definir a função de custo $(J(\vec{\theta}))$

v, i) $J(\theta) = (\vec{X} - \vec{H}\vec{\theta})^T (\vec{X} - \vec{H}\vec{\theta})$

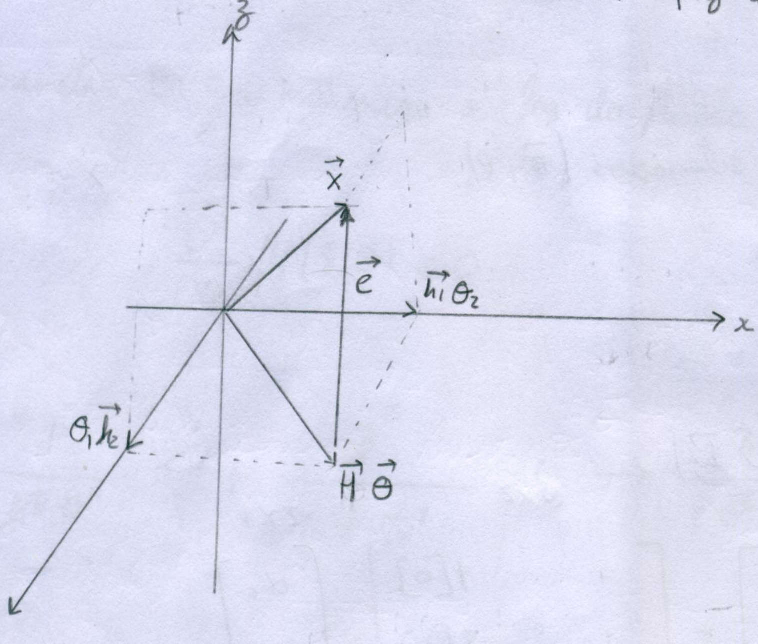
Definindo a distância Euclidiana de um vetor $\vec{Z} = \left[Z_1 \;\; Z_2 \;\; \dots \;\; Z_N \right]^T$ em um espaço R^N como:

$$\|\vec{Z}\| = \sqrt{\sum_i Z_i^2} = \sqrt{\vec{Z}^T \vec{Z}}$$

Então

$$J(\Theta) = \left\|\vec{X} - \vec{H}\vec{\Theta}\right\|^2 = \left\|\vec{X} - \sum_{i=1}^{p} \alpha_i \vec{h_i}\right\|^2 = \|\vec{e}\|^2$$

O objetivo é calcular vetores de observação ($\vec{h_1}$ e $\vec{h_2}$) que minimize a função de custo. Ou seja, o estimador baseado em LS busca minimizar a distância Euclidiana entre o vetor $\vec{X} \in R^N$ e o vetor $\vec{H}\vec{\Theta} \in R^{p=2}$. A fig abaixo mostra um exemplo em que $N=3$ e $p=2$



É possível perceber que $\vec{H}$ é selecionado p/ que $\vec{e}$ seja minimizado, $\vec{e}$ é ortogonal ao subespaço $R^{p=2}$

Então:

$$\vec{e} \perp \vec{H} \longrightarrow (\vec{X}-\vec{H}\vec{\Theta})^T\vec{H} = \vec{0}^T \rightarrow \left((\vec{X}-\vec{H}\vec{\Theta})^T(\vec{H}^T)^T\right) = \vec{0}^T$$

$$\left(\vec{H}^T(\vec{X}-\vec{H}\vec{\Theta})\right)^T = \vec{0}^T \rightarrow \vec{H}^T\vec{X} - \vec{H}^T\vec{H}\vec{\Theta} = \vec{0} \rightarrow \vec{\Theta} = (\vec{H}^T\vec{H})^{-1}\vec{H}^T\vec{X}$$

VI) Calcular $\overrightarrow{J_{min}}$

$$J(\vec{\Theta}^*) = J_{min} = (\vec{X}-\vec{H\Theta})^T(\vec{X}-\vec{H\Theta}) = \vec{X}^T(\vec{X}-\vec{H\Theta}) - \overbrace{(\vec{H\Theta})^T\vec{\epsilon}}^{\text{ortogonais } (=0)}$$

$$J(\vec{\Theta}^*) = X^TX - \vec{X}^T\vec{H\Theta^*} \therefore \Theta^* = (H^TH)^{-1}\vec{H}^T\vec{X}$$

matriz de projeção ortogonal

$$J_{min} = J(\Theta^*) = X^TX - X^TH(H^TH)^{-1}H^TX \rightarrow \boxed{J_{min} = X^T\left(I - H(H^TH)^{-1}H^T\right)X} = X^T(I-P)X$$

vi) Calculos θ^*

$$\vec{\theta}^* = \underbrace{\left(H^T H\right)^{-1}}_{\text{I}} H^T x$$

I:

$$\left(\overset{H^T}{\begin{bmatrix} 1 & 1 & \cdots & 1 \\ t[0] & t[1] & \cdots & t[N-1] \end{bmatrix}} \overset{H}{\begin{bmatrix} 1 & t[0] \\ \vdots & \vdots \\ 1 & t[N-1] \end{bmatrix}} \right)^{-1} = \begin{bmatrix} N & \sum t[i] \\ \sum t[i] & \sum t^2[i] \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} \sum t^2[i] & -\sum t[i] \\ -\sum t[i] & N \end{bmatrix}$$

Cont:

$$\vec{\theta}^* = \overset{2\times 2}{\begin{bmatrix} \sum t^2[i] & -\sum t[i] \\ -\sum t[i] & N \end{bmatrix}} \overset{2\times N}{\begin{bmatrix} 1 & 1 & \cdots & 1 \\ t[0] & t[1] & \cdots & t[N-1] \end{bmatrix}} \vec{X}$$

$$= \overset{2\times N}{\begin{bmatrix} \sum t^2[i] - t[0]\sum t[i] & \sum t^2[i] - t[1]\sum t[i] & \cdots \\ Nt[0] - \sum t[i] & Nt[1] - \sum t[i] & \cdots \end{bmatrix}} \overset{N\times 1}{\begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix}}$$

$$\theta^* = \begin{bmatrix} \sum_{k}^{N-1} x[k]\left(\sum_i t^2[i] - t[k]\sum_i t[i]\right) \\ \sum_{k}^{N-1} x[k]\left(N t[k] - \sum_i t[i]\right) \end{bmatrix} = \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix}$$

vii) Equações do sistema

$$\boxed{x = \sum_{k=0}^{N-1} x[k]\left(\sum_{i=0}^{N-1} t^2[i] - t[k]\sum_{i=0}^{N-1} t[i]\right) + t\left(\sum_{k=0}^{N-1} x[k]\left(N\,t[k] - \sum_{i=0}^{N-1} t[i]\right)\right)}$$

(6) -

$$x[n] = Ar^n + \omega[n] \qquad p/ \quad 0 < n \leq N-1$$

i) $\omega \sim N(0,\sigma^2)$ iid $\rightarrow$ $x \sim N(Ar^n, \sigma^2)$ OBS: $\theta = A$

I = Informação de Fisher

ii)

$$I_n(\theta) = E\left[\left(\frac{\partial \ln f_{\vec{x}}(\vec{x}|\theta)}{\partial\theta}\right)^2\right] = -E\left[\frac{\partial^2 \ln f_{\vec{x}}(\vec{x}|\theta)}{\partial^2\theta}\right]$$

iii) $$\ln f_{\vec{x}}(\vec{x}|\theta) = \ln \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(\frac{-1}{2\sigma^2}\sum_{i=0}^{N-1}(x_i - \theta r^i)^2\right)$$

$$\ln f_{\vec{x}}(\vec{x}|\theta) = -\ln\sqrt{2\pi}\,\sigma - \frac{1}{2\sigma^2}\sum_{i=0}^{N-1}(x_i - \theta r^i)^2$$

iv) $$\frac{\partial^2}{\partial^2\theta}\ln f_{\vec{x}}(\vec{x}|\theta) = -\frac{1}{2\sigma^2}\sum_{i=0}^{N-1}\frac{\partial^2}{\partial^2\theta}(x_i - \theta r^i)^2$$

v) $$\frac{\partial}{\partial\theta} = -2(x_i - \theta r^i)r^i$$ vi) $$\frac{\partial^2}{\partial^2\theta}(x_i - \theta r^i)^2 = 2r^{2i}$$

Subs. vi) em iv):

vii) $$\frac{\partial^2 \ln f_x(\vec{x}|\theta)}{\partial^2\theta} = -\frac{1}{\sigma^2}\sum_{i=0}^{N-1}(r^2)^i = -\frac{1}{\sigma^2}\,\frac{(r^2)^0 - r^{2N}}{1-r^2} = -\frac{1-r^{2N}}{\sigma^2(1-r^2)}$$

viii) $I_N(\theta) = -E\left[\frac{\partial^2}{\partial^2\theta} \ln f_{\vec{X}}(\vec{x}|\theta)\right] \rightarrow I_N(\theta) = \frac{1-r^{2N}}{\sigma^2(1-r^2)}$

O limite de Cramer-Rao é

ix) $\boxed{VAR\left[\hat{\theta}(\vec{X}_N)\right] \geq \frac{\sigma^2(1-r^2)}{1-r^{2N}} = \frac{1}{I_N(\theta)}}$

Um estimador eficiente é aquele cuja variância alcança o CRB, e o CRLB só pode ser alcançado se, e somente se:

x) $\frac{\partial}{\partial\theta} f_{\vec{X}}(\vec{x}|\theta) = I(A)\left(\hat{\theta}(\vec{X}_N) - \theta\right) \therefore \frac{\partial}{\partial\theta} f_{\vec{X}_N}(\vec{x}_N|\theta) = \frac{1}{\sigma^2}\sum_{i=0}^{N-1}(x_i - \theta r^i)r^i$

$= \frac{1}{\sigma^2}\sum_{i=0}^{N-1} x_i r^i - \frac{\theta}{\sigma^2}\sum_{i=0}^{N-1}(r^2)^i = \text{''} - \frac{\theta}{\sigma^2}\frac{1-r^{2N}}{1-r^2}$

$= \frac{1-r^{2N}}{\sigma^2(1-r^2)}\left(\frac{1-r^2}{1-r^{2N}}\sum_{i=0}^{N-1} x_i r^i - \theta\right) = I_N(\theta)\left(\hat{\theta}(\vec{X}_N) - \theta\right)$

em que $\hat{\theta}(\vec{X}_N) = \frac{1-r^2}{1-r^{2N}}\sum_{i=0}^{N-1} x_i r^i$ Portanto, o estimador eficiente existe e $VAR\left[\hat{\theta}(\vec{X}_N)\right] = \frac{1}{I_N(\theta)}$

P/ quais valores de r que $\hat{\theta}(\vec{X}_N)$ é consistente:

xii) $VAR\left[\hat{\theta}(\vec{X}_N)\right] \geq \frac{\sigma^2(r^2-1)}{r^{2N}-1}$

Se $r > 1$, então:

$\lim_{N\to\infty} \frac{\sigma^2(r^2-1)}{r^{2N}-1} = \sigma^2(r^2-1)\lim_{N\to\infty}\frac{1}{r^{2N}-1} = 0$

portanto, se $r > 1$, $\hat{\theta}(\vec{X}_N)$ é consistente

② - b) Se

$$f[i] = i$$

do item a), temos que

i) $$\alpha_0 = \sum_{k=0}^{N-1} x[k]\left(\sum_{i=0}^{N-1} f^2[i] = i^2 - k\sum_{i=0}^{N-1} f[i] = i\right) = \sum_{k=0}^{N-1} x[k]\left(\sum_{i=1} i^2 - k\sum_{i=1} i\right)$$

$$= \sum_{k=0}^{N-1} x[k]\left(\frac{(N-1)N(2N-1)}{6} - k\,\frac{N(N-1)}{2}\right) \rightarrow$$ Ref. de Summation of Series. Jolley, 1961.

$$\boxed{\alpha_0 = \frac{(N-1)N(2N-1)}{6}\sum_{k=0}^{N-1} x[k] - \frac{N(N-1)}{2}\sum_{k=0}^{N-1} k\,x[k]}$$

i) $$\alpha_1 = \sum_{k=0}^{N-1} x[k]\left(N k - \sum_{i=1}^{N-1} i\right)$$

$$\boxed{\alpha_1 = N\sum_{k=0}^{N-1} k\,x[k] - \frac{N(N-1)}{2}\sum_{k=0}^{N-1} x[k]}$$

(4) -

i) $\vec{X}_N = [x[0] \; x[1] \dots x[N-1]]^H$; $x[i]$, p/ $0 \le i \le N-1$, são VA iid

ii) $x[i] \sim N(\mu_x, \sigma_x^2)$ p/ $0 \le i \le N-1$, $\mu_x \sim N(0, \sigma_\mu^2)$

iii) σ_x^2 e σ_μ^2 são valores conhecidos

A PDF a posterior é:

iv) $$f_{\mu_x}(\mu_x | \vec{x}_N) = \frac{1}{\sqrt{2\pi}\,\sigma_\mu} \exp\left(-\frac{\mu_x^2}{2\sigma_\mu^2}\right) = \frac{f_{\vec{X}_N}(\vec{x}_N | \mu_x) f_{\mu_x}(\mu_x)}{f_{\vec{X}_N}(\vec{x}_N)}$$

A PDF do V.V.A $\vec{X}_N$ é:

v) $$f_{\vec{X}_N}(\vec{x}_N) = \prod_{i=0}^{N-1} f_{x[i]}(x[i]) = \frac{1}{\sqrt{2\pi}\,\sigma_x} \exp\left(-\frac{1}{2\sigma_x^2} \sum_{i=0}^{N-1} (x[i] - \mu_x)^2\right)$$

O log da PDF a posteriori é:

vi) $$\ln f_{\mu_x}(\mu_x | \vec{X}_N) = \ln f_{\vec{X}_N}(\vec{x}_N | \mu_x) + \ln f_{\mu_x}(\mu_x) - \ln f_{\vec{X}_N}(\vec{x}_N)$$

O valor de μ_x que maximiza a PDF a posterior é:

vii) $$\frac{\partial}{\partial \mu_x} f_{\mu_x}(\mu_x | \vec{x}_N) = \frac{\partial}{\partial \mu_x} \ln f_{\vec{X}_N}(\vec{x}_N | \mu_x) + \frac{\partial}{\partial \mu_x} f_{\mu_x}(\mu_x) = 0$$

viii) $$\frac{\partial}{\partial \mu_x} \ln f_{\vec{X}_N}(\vec{x}_N | \mu_x) = \underbrace{\cancel{\frac{\partial}{\partial \mu_x} \ln \frac{1}{\sqrt{2\pi}\,\sigma_x}}}_{=0} - \frac{1}{2\sigma_x^2} \sum_{i=0}^{N-1} \frac{\partial}{\partial \mu_x} (x[i] - \mu_x)^2$$

$$\frac{\partial}{\partial \mu} \ln f_{\vec{X}_N}(\vec{x}_N | \mu_x) = \frac{1}{\sigma_x^2} \sum_{i=0}^{N-1} (x[i] - \mu_x) = \frac{1}{\sigma_x^2} \sum_{i=0}^{N-1} x[i] - \frac{N\mu_x}{\sigma_x^2}$$

ix) $\frac{\partial}{\partial \mu_x} l_{\mu_x}(\mu_x) = \cancel{\frac{\partial}{\partial \mu_x} \ln \frac{1}{\sqrt{2\pi}\,\sigma_\mu}}^{=0} - \frac{\partial}{\partial \mu_x} \frac{\mu_x^2}{2\sigma_\mu^2}$

$$\frac{\partial}{\partial \mu_x} l_{\mu_x}(\mu_x) = -\frac{\mu_x}{\sigma_\mu^2}.$$

Sub. viii) e ix) em vii)

$$\frac{\partial}{\partial \mu_x} l_{\mu_x}(\mu_x \mid \vec{x}_N) = \frac{1}{\sigma_x^2}\sum_{i=0}^{N-1} x[i] - \frac{N\mu_x}{\sigma_x^2} - \frac{\mu_x}{\sigma_\mu^2} = 0$$

$$\mu_x\left(\frac{N\sigma_\mu^2 + \sigma_x^2}{\cancel{\sigma_x^2}\,\sigma_\mu^2}\right) = \frac{1}{\cancel{\sigma_x^2}}\sum_{i=0}^{N-1} x[i] \rightarrow \boxed{\mu_x = \frac{\sigma_\mu^2}{\sigma_x^2 + N\sigma_\mu^2}\sum_{i=0}^{N-1} x[i]}$$

(7) -

Seja

$$Y = \sum_{i=0}^{N-1} x^2[i] \qquad \cdot E[Y] = K\sigma_x^2 \quad \cdot VAR[Y] = 2K\sigma_x^4 \quad \cdot X \sim N(0, \sigma_x^2)$$

i) $\vec{Y}_N = [Y[0]\ Y[1]\ \dots\ Y[N-1]]^H \longrightarrow$ V.V.A

Os parâmetros a serem estimados são:

ii) $\theta_1 = K$ e $\theta_2 = \sigma_x^2$

De acordo com o método dos momentos, os momentos probabilísticos (teóricos) devem ser iguais aos momentos estatísticos (amostrais) de Y, ou seja:

1º momento

iii) $E[Y] = \overline{Y_N} \longrightarrow K\sigma_x^2 = \theta_1\theta_2 = \frac{1}{N}\sum_{i=0}^{N-1} Y[i] = \overline{Y_N} \longrightarrow \theta_1 = \frac{\overline{Y_N}}{\theta_2}$

Sabendo que

iv) $VAR[Y] = E\left[(Y - E[Y])^2\right] = E[Y^2] - E[Y]^2 \longrightarrow E[Y^2] = VAR[Y] + E[Y]^2$

Então:

v) $E\left[Y^2\right] = VAR[Y] + E[Y]^2 = 2\theta_1\theta_2^2 + \theta_1^2\theta_2^2 = \frac{1}{N}\sum_{i=0}^{N-1} Y^2[i] = \overline{Y_N^2}$

Subst iii) em v), temos:

vi) $2\frac{\overline{Y_N}}{\cancel{\theta_2}}\theta_2^{\cancel{2}} + \frac{\overline{Y_N}^2}{\cancel{\theta_2^2}}\cancel{\theta_2^2} = \overline{Y_N^2} \rightarrow \boxed{\theta_2 = \frac{\overline{Y_N^2} - \overline{Y_N}^2}{2\overline{Y_N}}}$

Subst. vi) em iii), temos

$$\boxed{\theta_1 = \frac{2\overline{Y_N}^2}{\overline{Y_N^2} - \overline{Y_N}^2}}$$

7 -

$$Y[i] = X + W[i] \quad p/ \ i = 1, 2$$

OBS: $W[1]$ e $W[2]$ são iid

i) $W[1] \sim N(0, 1)$

ii) X é cte e desconhecida $(\theta = X)$

iii)

$$E\left[W[2]\right] = 0 \qquad VAR\left[W[2]\right] = \begin{cases} 1, & x \geq 0 \\ 2, & x < 0 \end{cases}$$

iv) $\vec{Y_2} = \left[Y[1] \ \ Y[2]\right]^H$

v) $\theta = X$

vi) $Y[1] \sim N(\theta, 1)$ e $Y[2] \begin{cases} \sim N(\theta, 1) \ p/ \ \theta \geq 0 \\ \sim N(\theta, 2) \ p/ \ \theta \leq 0 \end{cases}$

A informação de Fisher é dada por:

vii) $I_2(\theta) = E\left[\left(\frac{\partial}{\partial\theta} l_{\vec{Y_2}}(\vec{Y_2})\right)^2\right] = -E\left[\frac{\partial^2}{\partial^2\theta} l_{\vec{Y_2}}(\vec{Y_2})\right]$

$f_{\vec{Y_2}}(\vec{Y_2})$ é dado por:

vii)

$$f_{\vec{Y_2}}(\vec{Y_2}|\theta) = \frac{1}{\sqrt{(2\pi)^n |K|}} \exp\left(-\frac{1}{2}(\vec{Y_2} - \vec{m_Y})^T K^{-1} (\vec{Y_2} - \vec{m_Y})\right) \quad \therefore n=2$$

em que $m_Y = [\theta \ \theta]^T$ e

$$K = \begin{bmatrix} VAR[Y[1]] & COV[Y[1], Y[2]] \\ COV(Y[2], Y[1]) & VAR[Y[2]] \end{bmatrix} = \begin{bmatrix} \sigma_{Y_1}^2 & 0 \\ 0 & \sigma_{Y_2}^2 \end{bmatrix}, \text{ em que } \sigma_{Y_2}^2 \begin{cases} = 1 \text{ p/ } x > 0 \\ = 2 \text{ p/ } x < 0 \end{cases}$$

$$|K| = \sigma_{Y_2}^2 \sigma_{Y_1}^2 \text{ e } K^{-1} = \frac{1}{\sigma_{Y_1}^2 \ \sigma_{Y_2}^2} \underbrace{(cof(A))^T}_{= Adj(A)} = \frac{1}{\sigma_{Y_1}^2 \sigma_2^2} \begin{bmatrix} \sigma_{Y_2}^2 & 0 \\ 0 & \sigma_{Y_1}^2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sigma_{Y_1}^2} & 0 \\ 0 & \frac{1}{\sigma_{Y_2}^2} \end{bmatrix}$$

Então

$$f_{\vec{Y_2}}(\vec{Y_2}) = \frac{1}{2\pi \sigma_{Y_2}} \exp\left(-\frac{1}{2} \underbrace{\begin{bmatrix} (Y[1]-\theta) & (Y[2]-\theta) \end{bmatrix}}_{1\times 2} \underbrace{\begin{bmatrix} \frac{1}{\sigma_{Y_1}^2} & 0 \\ 0 & \frac{1}{\sigma_{Y_2}^2} \end{bmatrix}}_{2\times 2} \begin{bmatrix} Y[1]-\theta \\ Y[2]-\theta \end{bmatrix}\right)$$

$$f_{\vec{Y_2}}(\vec{Y_2}) = \frac{1}{2\pi \sigma_{Y_2}} \exp\left(-\frac{1}{2} \underbrace{\begin{bmatrix} \frac{Y[1]-\theta}{\sigma_{Y_1}^2} & \frac{Y[2]-\theta}{\sigma_{Y_2}^2} \end{bmatrix}}_{1\times 2} \underbrace{\begin{bmatrix} Y[1]-\theta \\ Y[2]-\theta \end{bmatrix}}_{2\times 1}\right)$$

$$f_{\vec{Y_2}}(\vec{Y_2}) = \frac{1}{2\pi \sigma_{Y_2} \sigma_{Y_1}} \exp\left(-\frac{1}{2}\left(\frac{(Y[1]-\theta)^2}{\sigma_{Y_1}^2} + \frac{(Y[2]-\theta)^2}{\sigma_Y^2}\right)\right)$$

$$f_{\vec{Y_2}}(\vec{Y}) = \frac{1}{\sqrt{2\pi}\,\sigma_{y_1}} \exp\left(-\frac{(Y[1]-\theta)^2}{2\sigma_{y_1}^2}\right) \frac{1}{\sqrt{2\pi}\,\sigma_{y_2}} \exp\left(-\frac{(Y[2]-\theta)^2}{2\sigma_{y_2}^2}\right)$$

o log de $f_{\vec{Y_2}}(\vec{Y})$ é:

$$\ln f_{\vec{Y_2}}(\vec{Y}) = \ln \frac{1}{\sqrt{2\pi}\,\sigma_{y_1}} - \frac{(Y[1]-\theta)^2}{2\sigma_{y_1}^2} + \ln \frac{1}{\sqrt{2\pi}\,\sigma_{y_2}} - \frac{(Y[2]-\theta)^2}{2\sigma_{y_2}^2}$$

Sua derivada é (e $\sigma_{y_1}^2 = VAR[Y[1]] = 1$)

ix)

$$\frac{\partial}{\partial\theta} f_{\vec{Y_2}}(\vec{Y_2}) = Y[1]-\theta + \frac{Y[2]-\theta}{\sigma_{y_2}^2} = \frac{Y[1]\sigma_{y_2} - \theta\sigma_{y_2} + Y[2]-\theta}{\sigma_{y_2}^2}$$

$$= -\frac{\theta(\sigma_{y_2}+1)}{\sigma_{y_2}^2} + \frac{\sigma_{y_2}Y[1]+Y[2]}{\sigma_{y_2}^2}$$

x)

$$\frac{\partial^2}{\partial^2\theta} f_{\vec{Y_2}}(\vec{Y_2}) = -\frac{\sigma_{y_2}^2+1}{\sigma_{y_2}^2}$$

Subst. x) em Vii):

x)

$$I_N(\theta) = -E\left[\frac{\partial^2 f_{\vec{Y_2}}(\vec{Y_2}|\theta)}{\partial^2\theta}\right] = \frac{\sigma_{y_2}^2+1}{\sigma_y^2} = \begin{cases} 2 & \text{se } x \geq 0 \\ \frac{3}{2} & \text{se } x \leq 0 \end{cases}$$

o limite de Cramer Rao é

$$VAR\left[\hat{\theta}(\vec{Y_2})\right] \geq \frac{1}{I_N(\theta)} \geq \begin{cases} \frac{1}{2} & \text{se } x > 0 \\ \frac{2}{3} & \text{se } x \leq 0 \end{cases}$$

⑥ -

$$x[n] = \Pi^n + v[n] \quad n = 0, 1, \dots, N-1$$

$$v[n] \sim N(0, \sigma^2) \quad \therefore x[n] \sim N(\Pi^n, \sigma^2) \rightarrow x[n] \sim (\theta^n, \sigma^2)$$

i) $\theta = \Pi$ ii) $\vec{x}_n = [x[0]\ x[1]\ \dots\ x[N-1]]^T$

Assumindo que $x[i]$ são V.A iid p/ $i = 0, 1, \dots, N-1$, o log da função de ML é:

iii) $$\ln f_{\vec{x}_n}(\vec{x}_n|\theta) = \sum_{i=0}^{N-1} \ln f_{x[i]}(x[i]|\theta)$$

$$= \sum_{i=0}^{N-1} \ln \frac{1}{\sqrt{2\pi}\sigma} - \sum_{i=0}^{N-1} \frac{(x[i] - \theta^i)^2}{2\sigma^2}$$

iv) $$\frac{\partial}{\partial\theta} \ln f_{\vec{x}_n}(\vec{x}_n|\theta) = + \sum_{i=0}^{N-1} \frac{x[i] - \theta^i}{\sigma^2} \cdot i\, \theta^{i-1}$$

O método de ML deve ser escolhido tal que

v) $$\frac{\partial}{\partial\theta} \ln f_{\vec{x}_n}(\vec{x}_n|\theta) = 0 \longrightarrow \sum_{i=0}^{N-1} i\,\theta^{i-1}(x[i] - \theta^i) = 0$$

A equação acima é não linear e deve ser calculada através de métodos numéricos computacionais