TIP8419 - Tensor Algebra
Homework 12

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Tensor Kronecker Product Singular Value Decomposition (TKPSVD)

Problem 1 On a previous homework we have implemented the KPSVD (Kronecker Product Singular Value Decomposition) algorithm. Now, we will implement the generalization of that to tensors, namely, the TKPSVD (Tensor Kronecker Product Singular Value Decomposition) algorithm. Consider the N-order tensor $\mathcal{X} \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N}$. Then, a TKPSVD of $\mathcal{X}$ can be written as

$$\mathcal{X} = \sum_{j=1}^{R} \sigma_j \mathcal{A}_j^{(d)} \otimes \mathcal{A}_j^{(d-1)} \otimes \cdots \otimes \mathcal{A}_j^{(1)},$$

where the tensors $\mathcal{A}_j^{(i)} \in \mathbb{R}^{I_1^{(i)} \times I_2^{(i)} \times \cdots \times I_N^{(i)}}$ satisfy

$$||\mathcal{A}_j^{(i)}||_F = 1, \ \prod_{i=1}^{d} I_k^{(i)} = I_k, \ 1 \leq k \leq N.$$

Set $R = 1$ and generate $\mathcal{X} = \sigma \mathcal{A}^{(3)} \otimes \mathcal{A}^{(2)} \otimes \mathcal{A}^{(1)}$, for randomly chosen σ, $\mathcal{A}^{(1)} \in \mathbb{R}^{10\times 4\times 2}$, $\mathcal{A}^{(2)} \in \mathbb{R}^{5\times 2\times 2}$ and $\mathcal{A}^{(3)} \in \mathbb{R}^{2\times 2\times 2}$. To avoid scaling ambiguity issues, normalize the model so that $\|\mathcal{A}^{(i)}\|_F = 1$ for $i = 1, 2, 3$. (In other words, the norm will be absorbed by σ). Then, implement the TKPSVD algorithm that estimate $\mathcal{A}^{(1)}$, $\mathcal{A}^{(2)}$ and $\mathcal{A}^{(3)}$ by solving the following problem

$$(\hat{\mathcal{A}}^{(1)}, \hat{\mathcal{A}}^{(2)}, \hat{\mathcal{A}}^{(3)}) = \underset{\mathcal{A}^{(1)}, \mathcal{A}^{(2)}, \mathcal{A}^{(3)}}{\arg\min} \|\mathcal{X} - \sigma \mathcal{A}^{(3)} \otimes \mathcal{A}^{(2)} \otimes \mathcal{A}^{(1)}\|_F^2.$$

Compare the estimated tensors $\hat{\mathcal{A}}^{(1)}$, $\hat{\mathcal{A}}^{(2)}$ and $\hat{\mathcal{A}}^{(3)}$ with the original (normalized) ones. What can you conclude? Explain the results.

Hint: Use the file "tkpsvd.mat" to validate your result.

Problem 2 Assuming 1000 Monte Carlo experiments, generate $\mathcal{X}_0 = \mathcal{A}^{(3)} \otimes \mathcal{A}^{(2)} \otimes \mathcal{A}^{(1)}$, for randomly chosen $\mathcal{A}^{(1)} \in \mathbb{R}^{10\times 4\times 2}$, $\mathcal{A}^{(2)} \in \mathbb{R}^{5\times 2\times 2}$ and $\mathcal{A}^{(3)} \in \mathbb{R}^{2\times 2\times 2}$ whose elements are drawn from a standard normal distribution (here it is not necessary to normalize them). Let

$\mathcal{X} = \mathcal{X}_0 + \alpha\mathcal{V}$ be a noisy version of $\mathcal{X}_0$, where $\mathcal{V}$ is the additive noise term, whose elements are also drawn from a standard normal distribution. The parameter α controls the power (variance) of the noise term, and is defined as a function of the signal to noise ratio (SNR), in dB, as follows

$$\text{SNR}_{\text{dB}} = 10\log_{10}\left(\frac{||\mathcal{X}_0||_F^2}{||\alpha\mathcal{V}||_F^2}\right). \tag{1}$$

Assuming the SNR range [0, 5, 10, 15, 20, 25, 30] dB, find the estimates $\mathcal{A}^{(1)}$, $\mathcal{A}^{(2)}$ and $\mathcal{A}^{(3)}$ obtained with the TKPSVD algorithm.

Let us define the normalized mean square error (NMSE) measure as follows

$$\text{NMSE}(\mathcal{X}_0) = \frac{1}{1000}\sum_{i=1}^{1000}\frac{\|\hat{\mathcal{X}}_0(i) - \mathcal{X}_0(i)\|_F^2}{\|\mathcal{X}_0(i)\|_F^2}, \tag{2}$$

where $\mathcal{X}_0(i)$ and $\hat{\mathcal{X}}_0(i)$ represent the original data tensor and the reconstructed one at the ith experiment, respectively. For each SNR value, plot the NMSE vs. SNR curve. Discuss the obtained results.

Note: For a given SNR (dB), the parameter α to be used in your experiment is determined from equation (1).