Exercise list n1
Multlinear transformations and Multlinear rank
Engenharia de Telecomunicações
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Problem 1

Let us define $\mathcal{X} \in \mathbb{C}^{I_1 \times I_2 \times \cdots \times I_N}$ and $\mathbf{U}^{(n)} \in \mathbb{C}^{J_n \times I_n}$, the n-mode product is defined as

$$\mathcal{Y} = \mathcal{X} \times_n \mathbf{U}^{(n)} \tag{1}$$

where $\mathcal{Y} \in \mathbb{C}^{I_1 \times \cdots I_{n-1} \times J_n \times I_{n+1} \times I_N}$ is a tensor obtained multiplying each n-mode fiber of $\mathcal{X}$ by $\mathbf{U}^{(n)}$. The element-wise operation is given by

$$y_{i_1,\cdots,i_{n-1},j_n,i_{n+1},i_N} = \sum_{i_n}^{I_n} x_{i_1,i_2,\cdots,i_N} u_{j_n,i_n} \tag{2}$$

Hence,

$$\begin{aligned} \mathcal{Z} &= T\left\{\mathcal{X}\right\} && (3)\\ &= \mathcal{X} \times_1 \mathbf{U}^{(n)} \times_2 \mathbf{U}^{(2)} \cdots \times_N \mathbf{U}^{(N)} && (4) \end{aligned}$$

gives a multlinear transformation (denoted by $T\left\{\right\}$) , where $\mathcal{Z} \in \mathbb{C}^{J_1 \times J_2 \times \cdots \times J_N}$. The multilinear transformation can be interpreted in two equivalent ways: In the first one, it represents the mathematical operation that leads $\mathcal{X}$ to $\mathcal{Z}$, that is, the values of new tensor stem from changing of the values of $\mathcal{X}$. In the second one, $\mathcal{X}$ is thought as a fixed element in a multidimensional space, and $T\left\{\right\}$ changes the vector basis of that space. In this case, the values of new tensor stem from changing the vector basis used to represent $\mathcal{X}$. In the particular case where $J_n = I_n$, we have that $\mathcal{Z} \in \mathbb{C}^{I_1 \times I_2 \times \cdots \times I_N}$ and each vector basis remains with the same dimensionality.

It is important to notice that there is a *isomorphism* between the mapping from the Eq. (1) and

$$\left[\mathcal{Z}\right]_{(n)} = \mathbf{U}^{(n)} \left[\mathcal{X}\right]_{(n)}, \tag{5}$$

where $[\mathcal{X}]_{(n)}$ is the n-mode unfolding of the tensor $\mathcal{X}$. The n-rank of $\mathcal{Z}$, $R_n = rank\left([\mathcal{Z}]_{(n)}\right)$, is given by

$$
\begin{aligned}
R_n &= rank\left(\left[\mathcal{X} \times_1 \mathbf{U}^{(n)} \times_2 \mathbf{U}^{(2)} \cdots \times_N \mathbf{U}^{(N)}\right]_{(n)}\right) && (6)\\
&= rank\left(\mathbf{U}^{(n)} [\mathcal{X}]_{(n)} \left(\mathbf{U}^{(N)} \otimes \cdots \otimes \mathbf{U}^{(n+1)} \otimes \mathbf{U}^{(n-1)} \otimes \cdots \mathbf{U}^{(1)}\right)^{\mathsf{T}}\right) && (7)\\
&= rank\left(\mathbf{U}^{(n)} [\mathcal{X}]_{(n)} \mathbf{V}\right), && (8)
\end{aligned}
$$

where $\mathbf{V} = \left(\mathbf{U}^{(N)} \otimes \cdots \otimes \mathbf{U}^{(n+1)} \otimes \mathbf{U}^{(n-1)} \otimes \cdots \mathbf{U}^{(1)}\right)^{\mathsf{T}}$, being $\otimes$ the Kronecker product. For the constraint where $\left\{\mathbf{U}^{(n)}\right\}$ is a set of non-singular matrices, we have that

$$
\begin{aligned}
rank\left(\mathbf{U}^{(n)}\right) &= I_n && (9)\\
rank\left([\mathcal{X}]_{(n)}\right) &= r_{x,n} && (10)\\
rank\left(\mathbf{V}\right) &= {I_n}^{N-1} && (11)
\end{aligned}
$$

and

$$R_n = min\left(I_n, r_{x,n}\right) \tag{12}$$

The n-rank of $\mathcal{Z}$ is equal to $\mathcal{X}$ as $r_{x,n}$ does not depends on the multlinear transformation. Hence, the multilinear rank, given by $rank\left(\mathcal{X}\right) - \left(R_1, R2, \cdots, R_N\right)$ is property of the tensor $\mathcal{X}$.

Problem 2 O sinal de tempo contínuo $x_c(t)$ com a transformada de Fourier $X_c(j\Omega)$ mostrada na Figura 1 passa pelo sistema mostrado na Figura 2. Determine o intervalo de valores de T para o qual $x_r(t) = x_c(t)$

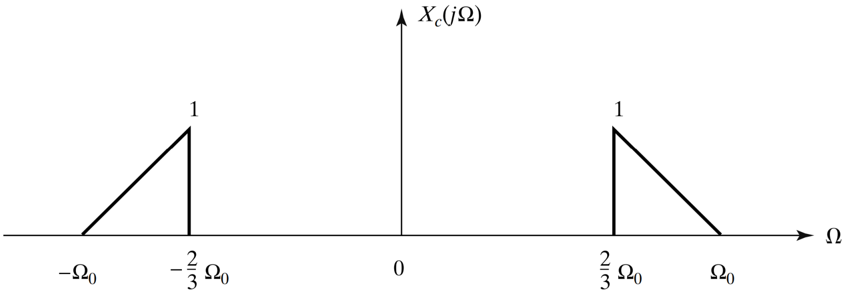

Figura 1: Figura para solução do Problem 1.

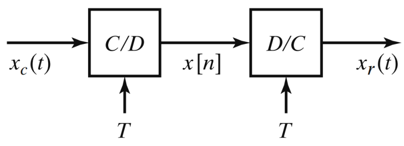

Figura 2: Figura para solução do Problem 1.

Resposta

Problem 3 Considere a representação do processo de amostragem seguido pela reconstrução mostrada na Figura 3.

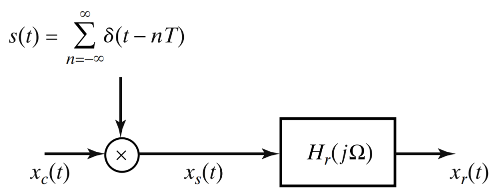

Figura 3: Figura para solução do Problem 2.

Assuma que o sinal de entrada seja

$$x_c(t) = 2\cos(100\pi t - \pi/4) + \cos(300\pi t + \pi/3) \quad -\infty < t < \infty$$

A resposta em frequência do filtro de reconstrução é

$$H_r(j\Omega) = \begin{cases} T, & |\Omega| \leq \pi/T \\ 0, & |\Omega| > \pi/T \end{cases}$$

(a) Determine a transformada de Fourier de tempo contínuo $X_c(j\Omega)$ e a esboce como uma função de Ω.

(b) Suponha que $f_s = 1/T = 500$ amostras/s e esboce a transformada de Fourier $X_s(j\Omega)$ em função de Ω para $-2\pi/T \leq \Omega \leq 2\pi/T$. Qual é a saída $x_r(t)$ nesse caso? (Você deverá ser capaz de dar uma expressão exata para $x_r(t)$.)

Resposta

Problem 4 Desenhe o diagrama de fluxo de sinais para a implementação na forma direta I do sistema LIT com função de sistema

$$H(z) = \frac{1 - \frac{1}{2}z^{-2}}{1 - \frac{1}{4}z^{-1} - \frac{1}{8}z^{-2}} \tag{13}$$

Resposta

Problem 5 Determine a resposta ao impulso $h[n]$ para um filtro FIR de fase linear com comprimento $M = 4$, cuja a resposta em frequência satisfaça

$$H_r(0)\,, \quad H_r\left(\tfrac{\pi}{2}\right) = \tfrac{1}{2}.$$

Resposta

Problem 6 Use a transformação bilinear para converter o filtro analógico

$$H(s) = \frac{s+1}{(s+0,1)}^{2} + 9$$

para um filtro digital IIR. Selecione $T = 0,1$.

Problem 7 Repita a questão anterior, mas desta vez use o método da invariância ao impulso. Por fim, compare a localização dos zeros com a localização dos zeros obtidas na questão anterior.