Galileo Masterclass Brazil (GMB) 2022

Lecture 6 - Parameter Tracking

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Outline

GNSS Tracking Objectives

GNSS parameter tracking has the objectives:

1. Refine the estimates of Doppler, carrier phase, and time-delay for each satellite based on the initial estimates provided by the acquisition
2. Track Doppler, carrier phase, and time-delay for each satellite over time

Outline

Signal Model

We assume that the Doppler frequency shift was corrected based on the Doppler estimate provided by the acquisition. Thus, the received baseband signal of one satellite can be given as

$$\mathbf{x}[k] = \sqrt{P} g[k] \mathrm{e}^{\mathrm{j}\phi} \mathbf{c}(\tau) + \mathbf{n}[k]$$

where

$$\begin{aligned}
\mathbf{x}[k] &= [x(kN\,T_s), \ldots, x((kN+n)\,T_s), \ldots, x((kN+N-1)\,T_s)]^{\mathrm{T}} \\
\mathbf{n}[k] &= [n(kN\,T_s), \ldots, n((kN+n)\,T_s), \ldots, n((kN+N-1)\,T_s)]^{\mathrm{T}} \\
\mathbf{c}(\tau) &= [c(\tau), \ldots, \ldots, c(nT_s - \tau), \ldots, c((N-1)T_s - \tau)]^{\mathrm{T}}
\end{aligned}$$

as well as $n = 0, 1, \ldots, N-1$, $k = 0, 1, \ldots, K-1$, $N \geq N_d$, and $N/N_d \in \mathbb{Z}$, and $T_s = 1/2B$. For $k = 0$ we can write

$$\mathbf{x} = \sqrt{P} \mathrm{e}^{\mathrm{j}\phi} g[0] \mathbf{c}(\tau) + \mathbf{n}.$$

Maximum Likelihood Estimation (1)

Let us assume a random vector $\boldsymbol{x}$ which has a multivariate Gaussian probability density function (pdf) parameterized by the parameters τ and ϕ, denoted by

$$\mathrm{p}_{\boldsymbol{x}}(\mathbf{x};\tau,\phi) = \frac{1}{(\pi\sigma_n^2)^N}\exp\left[-\frac{\|\mathbf{x}-\sqrt{P}g[0]\mathrm{e}^{\mathrm{j}\phi}\mathbf{c}(\tau)\|_2^2}{\sigma_n^2}\right].$$

The likelihood function with respect to the parameters τ and ϕ is given as

$$L(\mathbf{x};\tau,\phi) = \mathrm{p}_{\boldsymbol{x}}(\mathbf{x};\tau,\phi).$$

Now, the Maximum Likelihood Estimator (MLE) can be given as

$$(\hat{\tau},\hat{\phi}) = \arg\max_{\tau,\phi}\{L(\mathbf{x};\tau,\phi)\} = \arg\max_{\tau,\phi}\{\log\left(L(\mathbf{x};\tau,\phi)\right)\}.$$

Maximum Likelihood Estimation (2)

When further deriving the estimator we get

$$
\begin{aligned}
(\hat{\tau}, \hat{\phi}) &= \arg\max_{\tau,\phi} \left\{ \log(1) - N \log\left(\pi\sigma_n^2\right) \right. \\
&\quad \left. - \frac{1}{\sigma_n^2} \|\mathbf{x} - \sqrt{P} g[0] \mathrm{e}^{\mathrm{j}\phi} \mathbf{c}(\tau)\|_2^2 \right\} \\
&= \arg\max_{\tau,\phi} \left\{ \log(1) - N \log\left(\pi\sigma_n^2\right) \right. \\
&\quad \left. - \|\mathbf{x}\|_2^2 + 2\sqrt{P} g[0] \mathrm{Re}\{\mathbf{x}^{\mathrm{H}} \mathrm{e}^{\mathrm{j}\phi} \mathbf{c}(\tau)\} - P \|\mathbf{c}(\tau)\|_2^2 \right\}.
\end{aligned}
$$

As the first three terms do not depend on τ or ϕ and the last term is constant with $\|\mathbf{c}(\tau)\|_2^2 \approx N, \forall_\tau$ as well as dropping the constant factor $2\sqrt{P}$ we can write

$$
(\hat{\tau}, \hat{\phi}) = \arg\max_{\tau,\phi} \left\{ \mathrm{Re}\{\mathbf{x}^{\mathrm{H}} g[0] \mathrm{e}^{\mathrm{j}\phi} \mathbf{c}(\tau)\} \right\}.
$$

Maximum Likelihood Estimation (3)

The derivative of the above log-likelihood with respect to ϕ is

$$\frac{\partial}{\partial\phi}\mathrm{Re}\{\mathbf{x}^{\mathrm{H}}g[0]\mathrm{e}^{\mathrm{j}\phi}\mathbf{c}(\tau)\} = \frac{\partial}{\partial\phi}\mathrm{Re}\{(\cos(\phi)+\mathrm{j}\sin(\phi))g[0]\mathbf{x}^{\mathrm{H}}\mathbf{c}(\tau)\}$$
$$= -\sin(\phi)\mathrm{Re}\{g[0]\mathbf{x}^{\mathrm{H}}\mathbf{c}(\tau)\} - \cos(\phi)\mathrm{Im}\{g[0]\mathbf{x}^{\mathrm{H}}\mathbf{c}(\tau)\}.$$

Equating the above derivative to zero we get

$$\frac{\sin(\phi)}{\cos(\phi)} = \tan(\phi) = -\frac{\mathrm{Im}\{g[0]\mathbf{x}^{\mathrm{H}}\mathbf{c}(\tau)\}}{\mathrm{Re}\{g[0]\mathbf{x}^{\mathrm{H}}\mathbf{c}(\tau)\}} = -\frac{\mathrm{Im}\{\mathbf{x}^{\mathrm{H}}\mathbf{c}(\tau)\}}{\mathrm{Re}\{\mathbf{x}^{\mathrm{H}}\mathbf{c}(\tau)\}}$$

and finally

$$\hat{\phi} = \arctan\left(-\frac{\mathrm{Im}\{\mathbf{x}^{\mathrm{H}}\mathbf{c}(\tau)\}}{\mathrm{Re}\{\mathbf{x}^{\mathrm{H}}\mathbf{c}(\tau)\}}\right).$$

Maximum Likelihood Estimation (4)

After substituting the above in the log-likelihood we get

$$\hat{\tau} = \arg\max_{\tau} \left\{ \mathrm{Re}\left\{ \mathrm{e}^{\mathrm{j}\arctan\left(-\frac{\mathrm{Im}\{\mathbf{x}^{\mathrm{H}}\mathbf{c}(\tau)\}}{\mathrm{Re}\{\mathbf{x}^{\mathrm{H}}\mathbf{c}(\tau)\}}\right)} g[0]\mathbf{x}^{\mathrm{H}}\mathbf{c}(\tau) \right\} \right\}.$$

We can further develop the concentrated criterion and we get[1]

$$\begin{aligned}
\log(L_c(\mathbf{x};\tau)) &= \mathrm{Re}\left\{ \mathrm{e}^{\mathrm{j}\arctan\left(-\frac{\mathrm{Im}\{g[0]\mathbf{x}^{\mathrm{H}}\mathbf{c}(\tau)\}}{\mathrm{Re}\{g[0]\mathbf{x}^{\mathrm{H}}\mathbf{c}(\tau)\}}\right)} g[0]\mathbf{x}^{\mathrm{H}}\mathbf{c}(\tau) \right\} \\
&= \mathrm{Re}\left\{ \mathrm{e}^{\mathrm{j}\arctan(\psi)} g[0]\mathbf{x}^{\mathrm{H}}\mathbf{c}(\tau) \right\} \\
&= \mathrm{Re}\left\{ \left(\cos(\arctan(\psi)) + \mathrm{j}\sin(\arctan(\psi))\right) g[0]\mathbf{x}^{\mathrm{H}}\mathbf{c}(\tau) \right\} \\
&= \mathrm{Re}\left\{ \left(\frac{1}{\sqrt{1+\psi^2}} + \frac{\mathrm{j}\psi}{\sqrt{1+\psi^2}} \right) g[0]\mathbf{x}^{\mathrm{H}}\mathbf{c}(\tau) \right\} \\
&= \frac{\mathrm{Re}\left\{g[0]\mathbf{x}^{\mathrm{H}}\mathbf{c}(\tau)\right\} - \psi\,\mathrm{Im}\left\{g[0]\mathbf{x}^{\mathrm{H}}\mathbf{c}(\tau)\right\}}{\sqrt{1+\psi^2}}
\end{aligned}$$

[1] $\cos(\arctan(\psi)) = \frac{1}{\sqrt{1+\psi^2}}$, $\sin(\arctan(\psi)) = \frac{\psi}{\sqrt{1+\psi^2}}$

Maximum Likelihood Estimation (5)

Finally, we get

$$
\begin{aligned}
\log(L_c(\mathbf{x};\tau)) &= \frac{\mathrm{Re}\left\{g[0]\mathbf{x}^{\mathrm{H}}\mathbf{c}(\tau)\right\} + \frac{\mathrm{Im}\{g[0]\mathbf{x}^{\mathrm{H}}\mathbf{c}(\tau)\}}{\mathrm{Re}\{g[0]\mathbf{x}^{\mathrm{H}}\mathbf{c}(\tau)\}}\,\mathrm{Im}\left\{g[0]\mathbf{x}^{\mathrm{H}}\mathbf{c}(\tau)\right\}}{\sqrt{1+\left(\frac{\mathrm{Im}\{g[0]\mathbf{x}^{\mathrm{H}}\mathbf{c}(\tau)\}}{\mathrm{Re}\{g[0]\mathbf{x}^{\mathrm{H}}\mathbf{c}(\tau)\}}\right)^2}} \\
&= \mathrm{Re}\{g[0]\mathbf{x}^{\mathrm{H}}\mathbf{c}(\tau)\}\sqrt{1+\left(\frac{\mathrm{Im}\{g[0]\mathbf{x}^{\mathrm{H}}\mathbf{c}(\tau)\}}{\mathrm{Re}\{g[0]\mathbf{x}^{\mathrm{H}}\mathbf{c}(\tau)\}}\right)^2} \\
&= \sqrt{\mathrm{Re}\{g[0]\mathbf{x}^{\mathrm{H}}\mathbf{c}(\tau)\}^2+\mathrm{Im}\{g[0]\mathbf{x}^{\mathrm{H}}\mathbf{c}(\tau)\}^2} \\
&= |g[0]\mathbf{x}^{\mathrm{H}}\mathbf{c}(\tau)| = |\mathbf{x}^{\mathrm{H}}\mathbf{c}(\tau)|.
\end{aligned}
$$

Thus, the estimator for the time-delay τ can be given as

$$
\hat{\tau} = \arg\max_{\tau}\left\{|\mathbf{x}^{\mathrm{H}}\mathbf{c}(\tau)|\right\} = \arg\max_{\tau}\left\{|\mathbf{x}^{\mathrm{H}}\mathbf{c}(\tau)|^2\right\}.
$$

Feedback Structure (1)

In practice, to maximize $|\mathbf{x}^{\mathrm{H}}\mathbf{c}(\tau)|^2$ a (non-coherent) Delay Locked Loop (DLL) is used, which is the result of applying a gradient ascent method. Hence, a gradient ascent method for the kth iteration can be given as

$$\hat{\tau}[k] = \hat{\tau}[k-1] + \mu_\tau \frac{\partial \left(\log(L_c(\mathbf{x};\hat{\tau}[k-1]))\right)^2}{\partial \tau}.$$

Here, μ_τ is employed to adjust the step size of the gradient method ($\mu_\tau > 0$). The derivative within each iteration is approximated using the central difference quotient of length 2Δ,

$$\begin{aligned} \hat{\tau}[k] &= \hat{\tau}[k-1] + \frac{\mu_\tau}{2\Delta}\Big(\left(\log(L_c(\mathbf{x};\hat{\tau}[k-1]+\Delta))\right)^2 \\ &\quad - \left(\log(L_c(\mathbf{x};\hat{\tau}[k-1]-\Delta))\right)^2 \Big) \\ &= \hat{\tau}[k-1] + \frac{\mu_\tau}{2\Delta}\Big(|\mathbf{x}^{\mathrm{H}}\mathbf{c}(\hat{\tau}[k-1]+\Delta)|^2 \\ &\quad - |\mathbf{x}^{\mathrm{H}}\mathbf{c}(\hat{\tau}[k-1]-\Delta)|^2 \Big). \end{aligned}$$

Feedback Structure (2)

- To derive the central difference quotient, the DLL uses a correlator pair, these two correlators are called early and late correlator, respectively
- To obtain the gradient method using a feedback structure, we consider that the observations are successively collected in intervals with $\mathbf{x}[k]$
- We also take into account that the parameter $\tau[k]$ is not deterministic anymore (constant over k), but changes for different time instances k according to the satellite motion, user motion, and propagation effects

Thus, we can write

$$\begin{aligned}\hat{\tau}[k] \quad &= \quad \hat{\tau}[k-1] + \frac{\mu_\tau}{2\Delta}\Big(|\mathbf{x}^{\mathrm{H}}[k-1]\mathbf{c}(\hat{\tau}[k-1]+\Delta)|^2 \\ &\qquad - |\mathbf{x}^{\mathrm{H}}[k-1]\mathbf{c}(\hat{\tau}[k-1]-\Delta)|^2\Big)\,.\end{aligned}$$

Feedback Structure (3)

To derive a feedback structure for the phase estimate $\hat{\phi}[k]$ with a Phase Locked Loop (PLL) we can use the MLE of ϕ and write

$$\hat{\phi}[k] = \hat{\phi}[k-1] + \mu_\phi \ \text{arctan}\left(-\frac{\mathrm{Im}\{\mathbf{x}^{\mathrm{H}}[k-1]\mathrm{e}^{\mathrm{j}\hat{\phi}[k-1]}\mathbf{c}(\hat{\tau}[k-1])\}}{\mathrm{Re}\{\mathbf{x}^{\mathrm{H}}[k-1]\mathrm{e}^{\mathrm{j}\hat{\phi}[k-1]}\mathbf{c}(\hat{\tau}[k-1])\}}\right)$$

where μ_ϕ is employed to adjust the weighting of the new estimates with respect to the estimates of the previous iteration ($\mu_\phi > 0$).

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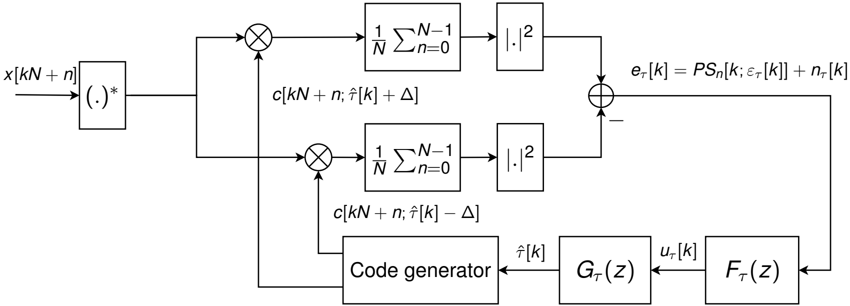
Outline

Non-Coherent DLL (1)



- $e_\tau[k]$, $n_\tau[k]$: Error signal and discriminator noise
- $S_n[k;\varepsilon_\tau[k]]$: S-curve for the non-coherent DLL
- $F_\tau(z)$, $G_\tau(z)$: Loop filter and integrator

Non-Coherent DLL (2)

The S-curve for the non-coherent DLL can be given as

$$S_n[k;\varepsilon_\tau[k]] = |R_c[\varepsilon_\tau[k]-\Delta]|^2 - |R_c[\varepsilon_\tau[k]+\Delta]|^2$$

with

$$\begin{aligned}
R_c[\varepsilon_\tau[k]-\Delta] &= \frac{1}{N}\sum_{n=0}^{N-1} c[kN+n;\tau[k]]c[kN+n;\hat{\tau}[k]+\Delta] \\
R_c[\varepsilon_\tau[k]+\Delta] &= \frac{1}{N}\sum_{n=0}^{N-1} c[kN+n;\tau[k]]c[kN+n;\hat{\tau}[k]-\Delta]
\end{aligned}$$

and the tracking error

$$\varepsilon_\tau[k] = \tau[k] - \hat{\tau}[k].$$

Non-Coherent DLL (3)

The noise component $n_\tau[k]$ of the error signal $e[k]$ can be decomposed into

$$n_\tau[k] = n_e[k] - n_l[k]$$

with

$$\begin{aligned} n_e[k] &= 2\sqrt{P} g[k] R_c[\varepsilon_\tau[k] - \Delta] \left(\cos(\phi[k]) \mathrm{Re}\{\tilde{n}_e[k]\} \right. \\ &\quad \left. + \sin(\phi[k]) \mathrm{Im}\{\tilde{n}_e[k]\}\right) + |\tilde{n}_e[k]|^2 \\ n_l[k] &= 2\sqrt{P} g[k] R_c[\varepsilon_\tau[k] + \Delta] \left(\cos(\phi[k]) \mathrm{Re}\{\tilde{n}_l[k]\} \right. \\ &\quad \left. + \sin(\phi[k]) \mathrm{Im}\{\tilde{n}_l[k]\}\right) + |\tilde{n}_l[k]|^2. \end{aligned}$$

and

$$\begin{aligned} \tilde{n}_e[k] &= \frac{1}{N} \sum_{n=0}^{N-1} n[kN+n]\, c[kN+n; \hat{\tau}[k] + \Delta] \\ \tilde{n}_l[k] &= \frac{1}{N} \sum_{n=0}^{N-1} n[kN+n]\, c[kN+n; \hat{\tau}[k] - \Delta]. \end{aligned}$$

Non-Coherent DLL (4)

The Mth order loop filter $F_\tau(z)$ is a so-called higher order weighted integrator and can be given as

$$F_\tau(z) = \sum_{m=0}^{M-1} a_{\tau m} \frac{z^{-m}}{(1-z^{-1})^m}$$

where

$$\mathcal{Z}\{f_\tau[k]\} = F_\tau(z).$$

An estimate of the time-delay at time instant k can be given by

$$\hat{\tau}[k] = \hat{\tau}[k-1] + K_{G_\tau} u_\tau[k-1]$$

and using the z-transform

$$\mathcal{Z}\{\hat{\tau}[k]\} = \hat{T}(z) = \hat{T}(z)z^{-1} + K_{G_\tau} U_\tau(z) z^{-1}$$

we can derive

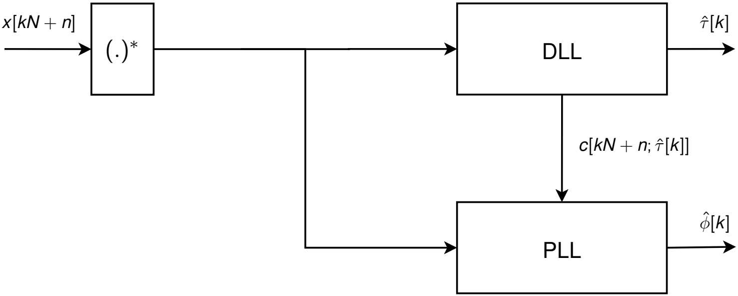
$$G_\tau(z) = \frac{\hat{T}(z)}{U_\tau(z)} = \frac{K_{G_\tau} z^{-1}}{1-z^{-1}}$$

where

$$\mathcal{Z}\{g_\tau[k]\} = G_\tau(z).$$

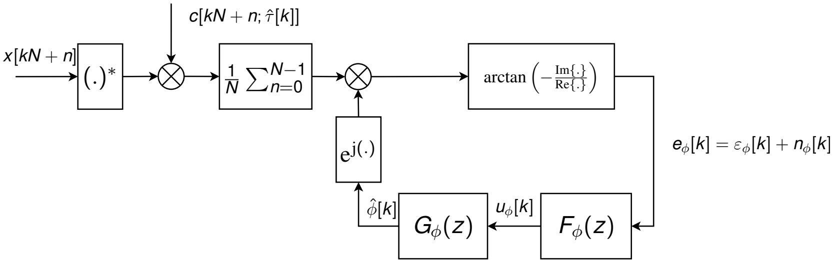
Outline

Costas PLL (1)



- As the DLL is non-coherent, the phase estimate is not fed back to the DLL (MLE)
- The DLL provides a reference signal $c[kN+n;\hat{\tau}[k]]$ to the PLL
- Besides the early and the late correlator included in the DLL the PLL includes the so-called prompt correlator

Costas PLL (2)



- $e_\phi[k]$, $\varepsilon_\phi[k] = \phi[k] - \hat{\phi}[k]$,$n_\phi[k]$: Error signal, phase tracking error, and discriminator noise
- $F_\phi(z)$, $G_\phi(z)$: Loop filter and integrator
- This so-called Costas PLL is insensitive to navigation message $g[k] \in \{-1, 1\}$ bit transitions

Costas PLL (3)

The Mth order loop filter $F_\phi(z)$ can be given as

$$\mathcal{Z}\{f_\phi[k]\} = F_\phi(z) = \sum_{m=0}^{M-1} a_{\phi_m} \frac{z^{-m}}{(1-z^{-1})^m}.$$

An estimate of the phase at time instant k can be given by

$$\hat{\phi}[k] = \hat{\phi}[k-1] + K_{G_\phi} u_\phi[k-1]$$

and using the z-transform

$$\mathcal{Z}\{\hat{\phi}[k]\} = \hat{\Phi}(z) = \hat{\Phi}(z)z^{-1} + K_{G_\phi} U_\phi(z) z^{-1}$$

we can derive

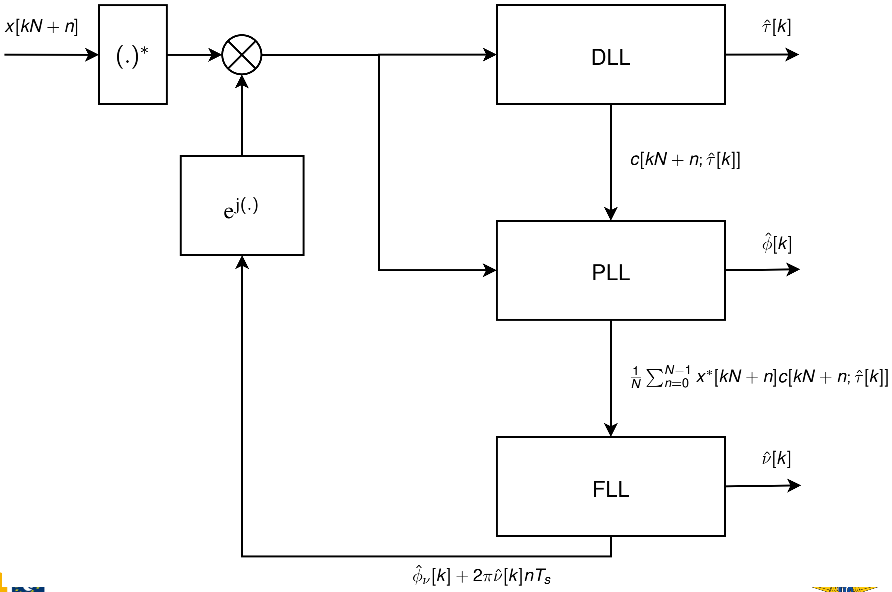
$$\mathcal{Z}\{g_\phi[k]\} = G_\phi(z) = \frac{\hat{\Phi}(z)}{U_\phi(z)} = \frac{K_{G_\phi} z^{-1}}{1-z^{-1}}.$$

Outline

Estimation of Doppler Shift (1)

- The acquisition provides an initial estimate of the Doppler frequency shift
- This initial estimate in general is not exact enough or, as the satellite or the receiver is moving the Doppler shift is changing over time
- The estimate of the Doppler frequency shift needs to be refined and adapted over time (tracking)
- The Doppler estimate can be refined using a feedback structure and the Doppler frequency shift including the Doppler phase are corrected before feeding the signal to the DLL and the PLL
- Initially the estimate of the Doppler provided by acquisition is used and then updated over time
- The Doppler phase $\phi_\nu[k]$ is updated for each period k with duration T_d

Estimation of Doppler Shift (2)



Estimation of Doppler Shift (3)

After initializing the Doppler correction based on the estimate provided by the acquisition we can further refine the estimate based on the time derivative of the phase. Thus, an estimate of the Doppler frequency can be given as

$$\hat{\nu}[k] = \frac{\hat{\phi}[k] - \hat{\phi}[k-1]}{2\pi T_d}.$$

Introducing the MLE of the phase we can write

$$\hat{\nu}[k] = \frac{\arctan\left(-\frac{\mathrm{Im}\{\mathbf{x}^{\mathrm{H}}[k]\mathbf{c}(\hat{\tau}[k])\}}{\mathrm{Re}\{\mathbf{x}^{\mathrm{H}}[k]\mathbf{c}(\hat{\tau}[k])\}}\right) - \arctan\left(-\frac{\mathrm{Im}\{\mathbf{x}^{\mathrm{H}}[k-1]\mathbf{c}(\hat{\tau}[k-1])\}}{\mathrm{Re}\{\mathbf{x}^{\mathrm{H}}[k-1]\mathbf{c}(\hat{\tau}[k-1])\}}\right)}{2\pi T_d}.$$

Alternatively, we can write[2]

$$\tan(\hat{\phi}[k] - \hat{\phi}[k-1]) = \frac{\tan(\hat{\phi}[k]) - \tan(\hat{\phi}[k-1])}{1 + \tan(\hat{\phi}[k])\tan(\hat{\phi}[k-1])}$$

[2] $\tan(x \pm y) = \frac{\tan(x) \pm \tan(y)}{1 \mp \tan(x)\tan(y)}$

Estimation of Doppler Shift (4)

Further, we can write[3]

$$\tan(\hat{\phi}[k]-\hat{\phi}[k-1]) = \frac{-\frac{\mathrm{Im}\{\mathbf{x}^{\mathrm{H}}[k]\mathbf{c}(\hat{\tau}[k])\}}{\mathrm{Re}\{\mathbf{x}^{\mathrm{H}}[k]\mathbf{c}(\hat{\tau}[k])\}}+\frac{\mathrm{Im}\{\mathbf{x}^{\mathrm{H}}[k-1]\mathbf{c}(\hat{\tau}[k-1])\}}{\mathrm{Re}\{\mathbf{x}^{\mathrm{H}}[k-1]\mathbf{c}(\hat{\tau}[k-1])\}}}{1+\frac{\mathrm{Im}\{\mathbf{x}^{\mathrm{H}}[k]\mathbf{c}(\hat{\tau}[k])\}\mathrm{Im}\{\mathbf{x}^{\mathrm{H}}[k-1]\mathbf{c}(\hat{\tau}[k-1])\}}{\mathrm{Re}\{\mathbf{x}^{\mathrm{H}}[k]\mathbf{c}(\hat{\tau}[k])\}\mathrm{Re}\{\mathbf{x}^{\mathrm{H}}[k-1]\mathbf{c}(\hat{\tau}[k-1])\}}}$$

$$=-\frac{\mathrm{Im}\{\mathbf{x}^{\mathrm{H}}[k]\mathbf{c}(\hat{\tau}[k])\}\mathrm{Re}\{\mathbf{x}^{\mathrm{H}}[k-1]\mathbf{c}(\hat{\tau}[k-1])\}-\mathrm{Im}\{\mathbf{x}^{\mathrm{H}}[k-1]\mathbf{c}(\hat{\tau}[k-1])\}\mathrm{Re}\{\mathbf{x}^{\mathrm{H}}[k]\mathbf{c}(\hat{\tau}[k])\}}{\mathrm{Re}\{\mathbf{x}^{\mathrm{H}}[k]\mathbf{c}(\hat{\tau}[k])\}\mathrm{Re}\{\mathbf{x}^{\mathrm{H}}[k-1]\mathbf{c}(\hat{\tau}[k-1])\}+\mathrm{Im}\{\mathbf{x}^{\mathrm{H}}[k]\mathbf{c}(\hat{\tau}[k])\}\mathrm{Im}\{\mathbf{x}^{\mathrm{H}}[k-1]\mathbf{c}(\hat{\tau}[k-1])\}}.$$

Finally, we can write

$$\hat{\nu}[k] = \arctan\left(-\frac{Q[k]I[k-1]-Q[k-1]I[k]}{I[k]I[k-1]+Q[k]Q[k-1]}\right)/(2\pi T_d)$$

where

$$\begin{aligned} Q[k] &= \frac{1}{N}\mathrm{Im}\left\{\sum_{n=0}^{N-1} x^*[kN+n]c[kN+n;\hat{\tau}[k]]\right\} \\ I[k] &= \frac{1}{N}\mathrm{Re}\left\{\sum_{n=0}^{N-1} x^*[kN+n]c[kN+n;\hat{\tau}[k]]\right\}. \end{aligned}$$

[3] $\tan(-x) = -\tan(x)$

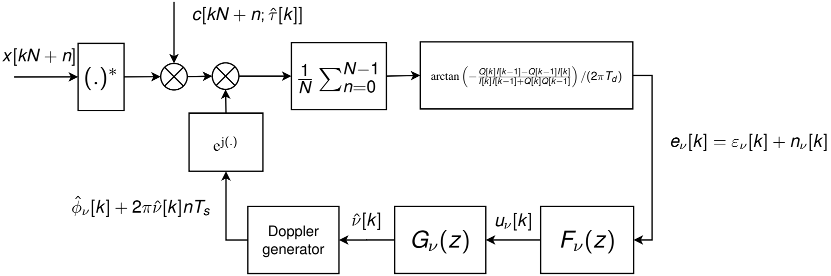
Estimation of Doppler Phase

An estimate of the Doppler phase for each period k can be given as

$$\hat{\phi}_\nu[k] = \hat{\phi}_\nu[k-1] + 2\pi\hat{\nu}[k]T_d.$$

- The Doppler phase can be initialized with $\hat{\phi}_\nu[0] = 0$
- The Doppler phase estimate also includes changes of the Doppler frequency over time

FLL (1)



- $e_\nu[k]$, $\varepsilon_\nu[k]=\nu[k]-\hat{\nu}[k]$, $n_\nu[k]$: Error signal, Doppler tracking error, and discriminator noise
- $F_\nu(z)$, $G_\nu(z)$: Loop filter and integrator
- This FLL is insensitive to navigation message $g[k]\in\{-1,1\}$ bit transitions

FLL (2)

The Mth order loop filter $F_\nu(z)$ can be given as

$$\mathcal{Z}\{f_\nu[k]\} = F_\nu(z) = \sum_{m=0}^{M-1} a_{\nu_m} \frac{z^{-m}}{(1 - z^{-1})^m}.$$

An estimate of the phase at time instant k can be given by

$$\hat{\nu}[k] = \hat{\nu}[k-1] + K_{G_\nu} u_\nu[k-1]$$

and using the z-transform

$$\mathcal{Z}\{\hat{\nu}[k]\} = \hat{N}(z) = \hat{N}(z)z^{-1} + K_{G_\nu} U_\nu(z)z^{-1}$$

we can derive

$$\mathcal{Z}\{g_\nu[k]\} = G_\nu(z) = \frac{\hat{N}(z)}{U_\nu(z)} = \frac{K_{G_\nu} z^{-1}}{1 - z^{-1}}.$$

Outline

Non-Coherent DLL

The estimate $\hat{\tau}[k]$ of a non-coherent DLL based on the error signal without discriminator noise can be given as

$$\hat{\tau}[k] = (g_\tau[k] * f_\tau[k]) * PS_n[k;\varepsilon_\tau[k]]$$

and we can linearize the S-curve as

$$\hat{\tau}[k] = (g_\tau[k] * f_\tau[k]) * PS_n'[k;0](\tau[k] - \hat{\tau}[k])$$

where

$$S_n'[k,0] = \left.\frac{\partial S_n[k;\varepsilon_\tau[k]]}{\partial \varepsilon_\tau[k]}\right|_{\varepsilon_\tau[k]=0}.$$

Using the z-transform we get

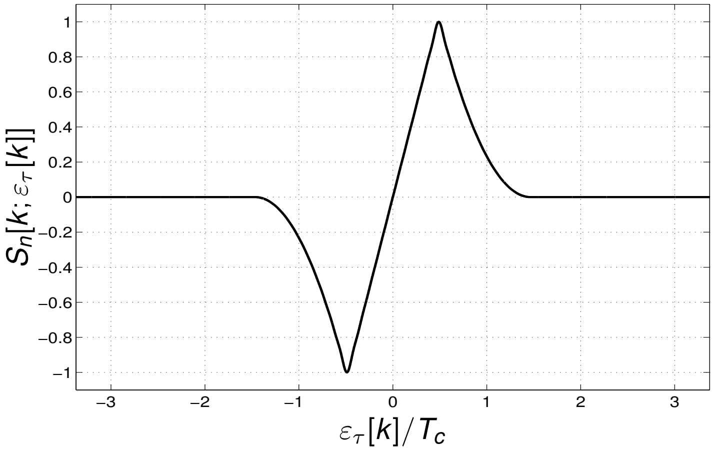
$$\hat{T}(z) = G_\tau(z)F_\tau(z)PS_n'[k;0](T(z) - \hat{T}(z)).$$

The closed loop transfer function can be given as

$$H_\tau(z) = \frac{\hat{T}(z)}{T(z)} = \frac{PS_n'[k;0]G_\tau(z)F_\tau(z)}{1 + PS_n'[k;0]G_\tau(z)F_\tau(z)}.$$

S-Curve of a Non-Coherent DLL

Discriminator $S_n[k; \varepsilon_\tau[k]]$ for a rectangular chip pulse (e.g. GPS C/A, Galileo E5b) with $\Delta = 0.5T_c$



Costas PLL

The estimate $\hat{\phi}[k]$ of a Costas PLL with an $\arctan$-discriminator considering no discriminator noise can be given as

$$\hat{\phi}[k] = (g_\phi[k] * f_\phi[k]) * (\phi[k] - \hat{\phi}[k]).$$

Using the z-transform we get

$$\hat{\Phi}(z) = G_\phi(z)F_\phi(z)(\Phi(z) - \hat{\Phi}(z)).$$

The closed loop transfer function can be given as

$$H_\phi(z) = \frac{\hat{\Phi}(z)}{\Phi(z)} = \frac{G_\phi(z)F_\phi(z)}{1 + G_\phi(z)F_\phi(z)}.$$

FLL

The estimate $\hat{\nu}[k]$ of a FLL with an $\arctan$-discriminator considering no discriminator noise can be given

$$\hat{\nu}[k] = (g_\nu[k] * f_\nu[k]) * (\nu[k] - \hat{\nu}[k]).$$

Using the z-transform we get

$$\hat{N}(z) = G_\nu(z)F_\nu(z)(N(z) - \hat{N}(z)).$$

The closed loop transfer function can be given as

$$H_\nu(z) = \frac{\hat{N}(z)}{N(z)} = \frac{G_\nu(z)F_\nu(z)}{1 + G_\nu(z)F_\nu(z)}.$$