Convex Optimization homework

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Nonlinear Optimization Systems

Teleinformatics Engineering

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Problem 1 Convexity of some sets. Determine if each set below is convex

(a) $\left\{(x, y) \in \mathbb{R}^2_{++} \mid x/y \leq 1\right\}$

Answer The norm cone is given by

$$C = \left\{(x_1, x_2, \cdots, x_n, t) \in \mathbb{R}^{n+1} \mid \mathbf{x} \in \mathbb{R}^n, \|\mathbf{x}\|_p \leq t\right\} \subseteq \mathbb{R}^{n+1}. \tag{1}$$

When $n = 1$ and $p = 1$ (Manhattan norm), we have that

$$C = \left\{(x, y) \in \mathbb{R}^2 \mid |x|/y \leq 1\right\} \subseteq \mathbb{R}^2, \tag{2}$$

where $t = y$ and $x_1 = x$. From the Equation (2), it is easy to conclude that $y > 0$. Let us further define the function $f : \mathbb{R}^2 \to \mathbb{R}^2$, given by

$$f(x, y) = \begin{bmatrix} |x| & y \end{bmatrix}^\top. \tag{3}$$

This function is clearly convex since the absolute operation is convex. Once it is well-known that the norm cone is convex, and f is a convex function in $C \subseteq \mathrm{dom}\,(f)$, then

$$S = f(C) = \left\{(x, y) \in \mathbb{R}^2_{++} \mid x/y \leq 1\right\} \tag{4}$$

is also convex, which is the set of the question. The Figure 1 shows this set for $0 \leq x \leq 3$ and $0 \leq y \leq 3$.

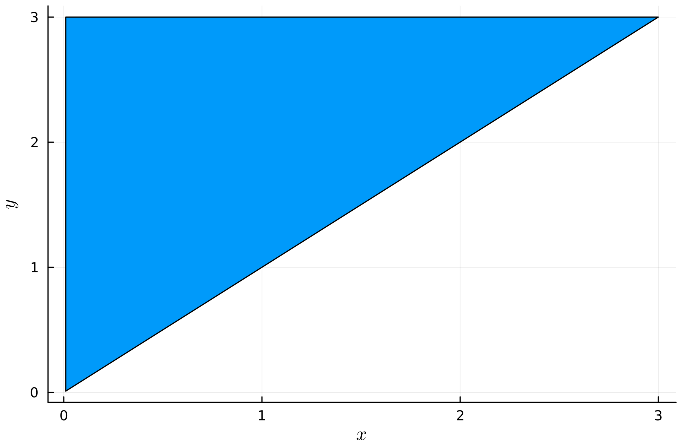

Figure 1: Set of the item a.

(b) $\left\{(x, y) \in \mathbb{R}^2_{++} \mid x/y \geq 1\right\}$

Answer

The set $S = \left\{\mathbf{v} \in \mathbb{R}^2_{++} \mid v_0/v_1 \geq 1\right\}$, where $\mathbf{v} = (v_0, v_1)$, is convex iff the convex combination of a pair of points belonging to S, e.g., $\mathbf{x} = (x_0, x_1)$ and $\mathbf{y} = (y_0, y_1)$, also belong to S. Mathematically,

$$\mathbf{w} = \theta\mathbf{x} + (1-\theta)\mathbf{y} = \begin{bmatrix} \theta x_0 + (1-\theta)y_0 \\ \theta x_1 + (1-\theta)y_1 \end{bmatrix} \tag{5}$$

should belong to S to any $\mathbf{x}, \mathbf{y} \in S$ and $0 \leq \theta \leq 1$. Therefore

$$\frac{\theta x_0 + (1-\theta)y_0}{\theta x_1 + (1-\theta)y_1} \geq 1 \tag{6}$$

$$\theta x_0 + (1-\theta)y_0 \geq \theta x_1 + (1-\theta)y_1 \tag{7}$$

$$\frac{x_0}{x_1}\frac{\theta}{y_1} + \frac{y_0}{y_1}\frac{1-\theta}{x_1} \geq \frac{\theta}{y_1} + \frac{1-\theta}{x_1} \tag{8}$$

This expression always holds for any value of $\mathbf{x}, \mathbf{y} \in S$. Then, $\mathbf{w} \in S$. The Figure 2 shows this set for $[0, 3]$.

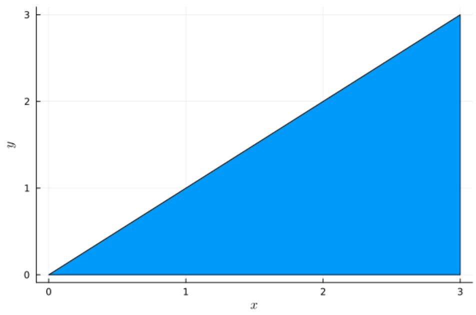

Figure 2: Set of the item b.

(c) $\left\{(x, y) \in \mathbb{R}^2_+ \mid xy \leq 1\right\}$

Answer

If $S = \left\{\mathbf{v} \in \mathbb{R}^2_+ \mid v_0 v_1 \leq 1\right\}$, where $\mathbf{v} = (v_0, v_1)$, then S is convex iff

$$\mathbf{w} = \theta\mathbf{x} + (1-\theta)\mathbf{y} \in S \;\forall\; \mathbf{x}, \mathbf{y} \in S, 0 \leq \theta \leq 1, \tag{9}$$

which is the convex combination of $\mathbf{x}$ and $\mathbf{y}$. Let us prove that this set is nonconvex by contradiction. Once $\mathbf{x}, \mathbf{y} \in S$, we have that

$$x_0 x_1 \leq 1 \tag{10}$$

and

$$y_0 y_1 \leq 1, \tag{11}$$

where $\mathbf{x} = (x_0, x_1), \mathbf{y} = (y_0, y_1)$. The second component of $\mathbf{w} = (w_0, w_1)$ is given by

$$\begin{align} w_1 = \theta x_1 + (1-\theta) y_1 \tag{12} \\ \tag{13} \end{align}$$

If $x_1 \gg 0$ and $y_0 \gg 0$, which leads to $x_0 \longrightarrow 0$ and $y_1 \longrightarrow 0$, respectively, and $\theta = 0.5$, then, from (12), we have that

$$w_1 \approx 0.5 x_1. \tag{14}$$

Since x_1 can be indiscriminality large, we have that $w_1 \gg 0$, which leads to $w_0 \longrightarrow 0$ if $\mathbf{w} \in S$. On the other hand, the first component of w_0 is given by

$$\begin{align} w_0 &= \theta x_0 + (1-\theta) y_0 \tag{15} \\ w_0 &= 0.5 y_0 \tag{16} \end{align}$$

Since y_0 can be indiscriminately large, we have that $w_0 \gg 0$. However, the statement $w_0 \to 0$ should be true in order to $\mathbf{w}$ belong to S. The contradiction leads us to conclude that $\mathbf{w} \notin S$. The Figure 3 shows the set S, and the Figure 4 shows the cone set K in which $\mathbf{x} \preceq_K \mathbf{w} \preceq_K \mathbf{y}$.

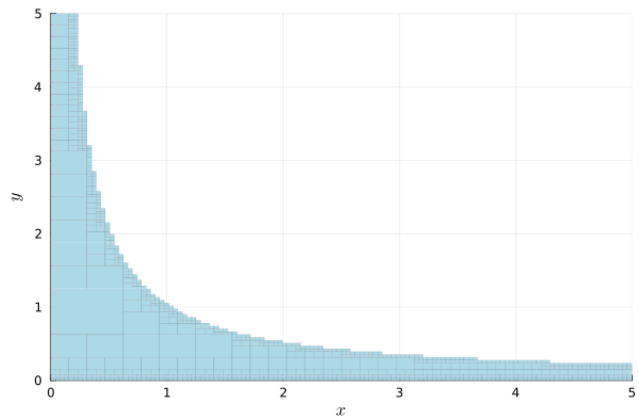

Figure 3: Set of the item c.

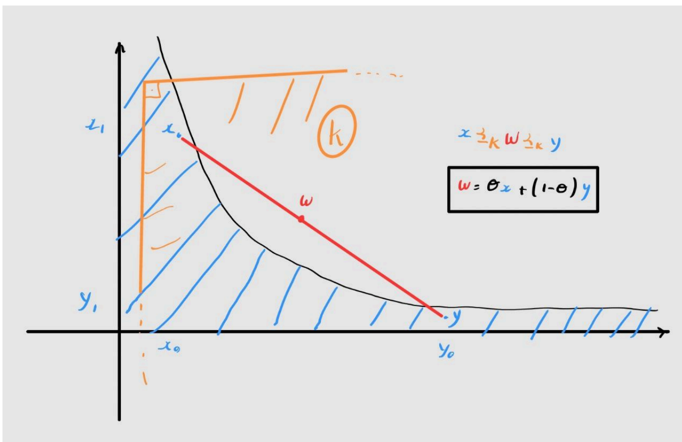

Figure 4: Cone set in which $\mathbf{x} \preceq_K \mathbf{w} \preceq_K \mathbf{y}$.

(d) $\left\{(x, y) \in \mathbb{R}^2_+ \mid xy \geq 1\right\}$

By using the convex combination, the vector $\mathbf{w} = (w_0, w_1)$, which is given by

$$\mathbf{w} = \theta\mathbf{x} + (1-\theta)\mathbf{y} = \begin{bmatrix} \theta x_0 + (1-\theta)y_0 \\ \theta x_1 + (1-\theta)y_1 \end{bmatrix}, \tag{17}$$

shall belong to S, for any $\mathbf{x}, \mathbf{y} \in S$. If it is true, then

$$
\begin{aligned}
(\theta x_0 + (1-\theta) y_0)(\theta x_1 + (1-\theta) y_1) &\geq 1 & (18)\\
\theta^2 x_0 x_1 + \theta(1-\theta) y_1 x_0 + (1-\theta)\theta x_1 y_0 + (1-\theta)^2 y_0 y_1 &\geq 1 & (19)\\
\theta^2 x_0 x_1 + \theta(1-\theta)(y_1 x_0 + x_1 y_0) + (1-\theta)^2 y_0 y_1 &\geq 1 & (20)\\
(x_0 x_1 - y_1 x_0 - x_1 y_0 + y_0 y_1)\theta^2 + (y_1 x_0 + x_1 y_0 - 2 y_0 y_1)\theta + y_0 y_1 - 1 &\geq 0 & (21)\\
f(\theta) &\geq 0 & (22)
\end{aligned}
$$

Note that f is a second-order function. The previous inequation holds if f has either no roots or only one root, that is

$$
\begin{aligned}
(y_1 x_0 + x_1 y_0 - 2 y_0 y_1)^2 - 4(x_0 x_1 - y_1 x_0 - x_1 y_0 + y_0 y_1)(y_0 y_1 - 1) \geq 0 \quad (23)\\
(y_1 x_0 + x_1 y_0 - 2 y_0 y_1)^2 \geq 4(x_0 x_1 - y_1 x_0 - x_1 y_0 + y_0 y_1)(y_0 y_1 - 1) \quad (24)
\end{aligned}
$$

Let us analyze carefully this inequation

1. If the RHS is greater than the LHS, this inequation is not satisfied. Hence, we must assess it for the worst case, where the LHS is as high as possible, and see whether this inequation holds.

2. The worst case is when $x_0 \gg 0$ and $x_1 \gg 0$, and $y_0 = y_1 = 1$ because the term $x_0 x_1$ cannot be quadratically compensated with $(x_0 x_1)^2$ on the LHS. Otherwise, the higher the values of y_0 and y_1, the lower the RHS becomes[1].

3. However, even for this case, the LHS becomes approximately equal to $(x_0 + x_1)^2 = x_0^2 + 2x_0 x_1 + x_1^2$, which is greater than $x_0 x_1$

Therefore, the function f has no roots and the set $\left\{(x, y) \in \mathbb{R}_+^2 \mid xy \geq 1\right\}$ is convex. The Figure 5 shows this set.

[1]The quadratic term $4y_0^2 y_1^2$ on the RHS that could increase it is canceled with the very same term $4y_0^2 y_1^2$ on the LHS.

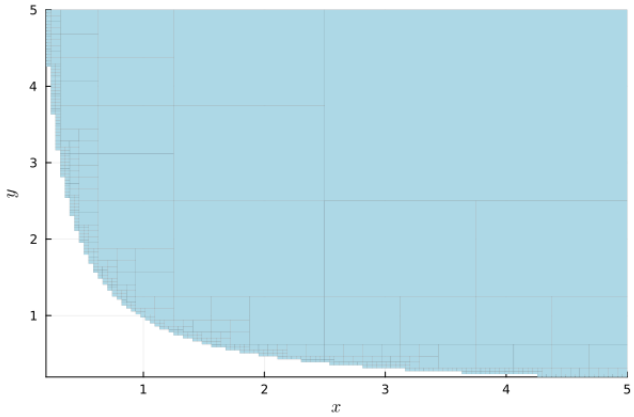

Figure 5: Set of the item d.

Problem 2

Let $S = \left\{\boldsymbol{\alpha} \in \mathbb{R}^3 | \alpha_1 + \alpha_2 e^{-t} + \alpha_3 e^{-2t} \leq 1.1 \text{ for } t \geq 1\right\}$. Is S affine, a halfspace, a convex cone, a convex set, or none of these?

The set S is affine iff its affine combination is also affine, that is,

$$\mathbf{w} = \theta_1 \mathbf{x} + \theta_2 \mathbf{y} = \begin{bmatrix} \theta_1 x_1 + \theta_2 y_1 \\ \theta_1 x_2 + \theta_2 y_2 \\ \theta_1 x_3 + \theta_2 y_3 \end{bmatrix} \in S \; \forall \; \mathbf{x}, \mathbf{y} \in S, \mathbf{1}^\top \boldsymbol{\theta} = 1, \tag{25}$$

where $\boldsymbol{\theta} = (\theta_1, \theta_2)$, $\mathbf{x} = (x_1, x_2, x_3)$, and $\mathbf{y} = (y_1, y_2, y_3)$. If $\mathbf{w} \in S$, then

$$\begin{aligned} \theta_1 x_1 + \theta_2 y_1 + (\theta_1 x_2 + \theta_2 y_2) e^{-t} + (\theta_1 x_3 + \theta_2 y_3) e^{-2t} &\leq 1.1 \\ \theta_1 \left(x_1 + x_2 e^{-t} + x_3 e^{-2t}\right) + \theta_2 \left(y_1 + y_2 e^{-t} + y_3 e^{-2t}\right) &\leq 1.1 \end{aligned} \tag{26}$$

Once $\mathbf{x}, \mathbf{y} \in S$, $x_1 + x_2 e^{-t} + x_3 e^{-2t} \leq 1.1$ and $y_1 + y_2 e^{-t} + y_3 e^{-2t} \leq 1.1$, the worst case when this inequation might not be satisfied is then $x_1 + x_2 e^{-t} + x_3 e^{-2t} = y_1 + y_2 e^{-t} + y_3 e^{-2t} = 1.1$. In this case, we have

$$1.1(\theta_1 + \theta_2) \leq 1.1 \tag{27}$$
$$1.1 \leq 1.1. \tag{28}$$

As the inequation holds, S is affine, which is consequently convex. The set S is a cone set iff the result of the conic combination also belongs to S, that is,

$$\mathbf{w} = \theta_1 \mathbf{x} + \theta_2 \mathbf{y} = \begin{bmatrix} \theta_1 x_1 + \theta_2 y_1 \\ \theta_1 x_2 + \theta_2 y_2 \\ \theta_1 x_3 + \theta_2 y_3 \end{bmatrix} \in S \; \forall \; \mathbf{x}, \mathbf{y} \in S, \boldsymbol{\theta} \geq \mathbf{0}. \tag{29}$$

Recalling the Equation (26), we can clearly see that this inequation does not hold for $\theta_1 \gg 0$, $\theta_2 \gg 0$, and $x_1 + x_2 e^{-t} + x_3 e^{-2t} = y_1 + y_2 e^{-t} + y_3 e^{-2t} = 1.1$. Thus, the set S is not a cone set. This set might be a close halfspace though, once halfspaces is a set of the form

$$\left\{ \mathbf{x} \mid \mathbf{a}^\top \mathbf{x} \leq b \right\}. \tag{30}$$

Considering that $t \geq 1$ is a constant, we can rewrite S as

$$S = \left\{ \boldsymbol{\alpha} \in \mathbb{R}^3 \mid \mathbf{a}^\top \boldsymbol{\alpha} \leq b \right\}, \tag{31}$$

where $\mathbf{a} = (1, e^{-t}, e^{-2t})$ and $b = 1.1$. Therefore, S is a halfspace.

Problem 3

(a) Explain why $t - (1/t)\mathbf{u}^\top \mathbf{u}$ is a concave function on $\mathrm{dom}\,(f)$. Hint. Use convexity of the quadratic-over-linear function.

Answer The function $s_1(\mathbf{u}, t) = (1/t)\mathbf{u}^\top \mathbf{u} = \|\mathbf{u}\|^2/t$ is a quadratic-over-linear function, which is known to be convex. The function $s_2(\mathbf{u}, t) = t - s_1(\mathbf{u}, t)$ is a linear minus a convex function, which yields a concave function.

(b) From this, show that $-\log(t - (1/t)\mathbf{u}^\top \mathbf{u})$ is a convex function on dom f.

Answer Let us the define the function $s_4(\mathbf{u}, t) = s_3 \circ s_2 = \log(s_2(\mathbf{u}, t))$, where $s_3(x) = \log(x)$. Note that, since $t > \|\mathbf{u}\|$, $\mathrm{dom}\,(s_2) = \mathbb{R}^n \times \mathbb{R}_{++}$ and s_3 is concave and nondecreasing for this interval. As s_2 is convex, we conclude by composition property that s_4 is concave, thus $s_5 = -s_4$ is convex.

(c) From this, show that f is convex.

Answer

$$\begin{aligned} f(\mathbf{x}, t) &= -\log\left(t^2 - \mathbf{x}^\top \mathbf{x}\right) \\ &= -\log\left(t(t - \mathbf{x}^\top \mathbf{x}/t)\right) \\ &= -\log(t) - \log\left(t - \mathbf{x}^\top \mathbf{x}/t\right) \\ &= s_6(t) + s_5(\mathbf{u}, t). \end{aligned} \tag{32}$$

We have shown that $s_5(\mathbf{x}, t) = -\log(t - \mathbf{x}^\top \mathbf{x}/t)$ is convex. Since $s_6(t) = -\log(t)$ is also convex on $\mathrm{dom}\,(f)$, $f = s_6 + s_5$ is convex as well.

Problem 4 Square and reciprocal of convex and concave functions. For each of the following, determine if the function f is convex, concave, or neither.

(a) $f(x) = g^2(x)$

Since $s(x) = x^2$ is convex and nondecreasing when g is nonnegative, and g is convex, then $f = s \circ g$ is convex as well.

(b) $f(x) = 1/g(x)$

Since $s(x) = 1/x$ is convex and nonincreasing when g is positive, and g is concave, then $f = s \circ g$ is concave.

Problem 5

Let us define $\mathbf{x}(k) \in \mathbb{R}^n$ and $u(k) \in \mathbb{R}$, for $k = 0, \ldots, N-1$, as the state of a linear dynamical system and its input signal, respectively, being $n = 3$ and $N = 30$. The system dynamic is governed by the following equation:

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{b}\mathbf{u}(k), \tag{33}$$

where

$$\mathbf{A} = \begin{bmatrix} -1 & 0.4 & 0.8 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \tag{34}$$

and

$$\mathbf{b} = \begin{bmatrix} 7 \\ 2 \\ -6 \end{bmatrix} \tag{35}$$

Assuming that $\mathbf{x}(0) = \mathbf{0}$, solve the following optimization problem:

$$\text{Minimize} \quad \sum_{k=0}^{N-1} f(u(k)) \tag{36}$$

$$\text{Subject to} \quad \mathbf{x}(N) = \mathbf{x}_{des}, \tag{37}$$

where

$$\mathbf{x}_{des} = \begin{bmatrix} 7 \\ 2 \\ -6 \end{bmatrix} \tag{38}$$

and

$$f(a) = \begin{cases} |a| & |a| \leq 1 \\ 2|a| - 1 & |a| > 1 \end{cases}, \tag{39}$$

which is equivalent to $f(a) = \max\{|a|, 2|a| - 1\}$.

Answer

```
using Convex, SCS, Plots, LaTeXStrings

N = 30 # N in the set {0, 1, ..., N}
n = 3 # order of the linear dynamical system
x_des = [7, 2, -6] # constraint -> x(N) == x_des
# model parameters
A = [-1 .4 .8; 1 0 0; 0 1 0]
b = [1, 0, 0.3]

X = Variable(n, N+1) # [x(0) x(1) ... x(N)]
u = Variable(1, N) # [u(0) u(1) ... u(N-1)]
f0 = sum(max(abs(u), 2abs(u)-1)) # objective function
constraints = [
    X[:,2:N+1] == A*X[:,1:N]+b*u, # recursive equation
    X[:,1] == zeros(n), # initial condition
    X[:,N+1] == x_des, # final condition
]
problem = minimize(f0, constraints)
solve!(problem, SCS.Optimizer; silent_solver = true)

fig = plot(vec(u.value), xlabel=L"k", title=L"u(k)", seriestype=:sticks, markershape=:circle,
↪   label="")
savefig(fig, "figs/4.17.png")
```

The optimal solution of the actuator signal is shown in Figure 6.

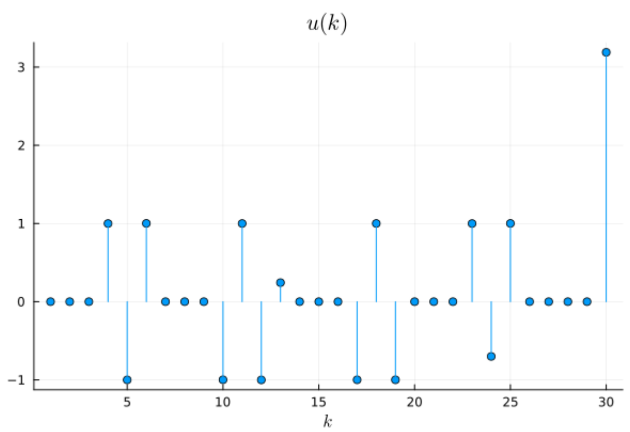

Figure 6: Optimum solution of the actuator signal.

Problem 6 Consider the optimization problem

$$\begin{aligned} \text{minimize} \quad & x_1^2 && (40)\\ \text{subject to} \quad & x_1 \le 1, && (41)\\ & x_1^2 + x_2^2 \le 2 && (42) \end{aligned}$$

with $\mathbf{x} = (x_1, x_2)$. Determine whether each of the following statements is true or false.

(a) The point $(-1, 1)$ is a solution.

Answer True. Since $(-1, 1) \in S$, where S is the feasible set, $(-1, 1)$ is a possible solution for the problem. It is not the optimal solution, though. For this problem, $\mathbf{x}^\star \approx (-1, 0)$.

(b) The optimal value is 1.

Answer True.

```
using Convex, SCS

x = Variable(2)

f0 = square(x[1]) # objective function
constraints = [
  x[1] ≤ -1,
  square(x[1]) + square(x[2]) ≤ 2
]
problem = minimize(f0, constraints)
solve!(problem, SCS.Optimizer; silent_solver = true)

println(round(problem.optval, digits = 2)) # it prints 1
```

(c) The problem is convex.

Answer True. The objective function and its domain are convex. The constraints are also convex. Then this optimization problem is convex.

(d) The problem has multiple solutions.

Answer False. Since this problem is convex, one can state that

$$\exists!\, \mathbf{x}^\star \in \mathbb{R}^n \mid f_0(\mathbf{x}^\star) \le f_0(\mathbf{y})\ \forall\, \mathbf{y} \in S \subset \mathrm{dom}(f), \tag{43}$$

where S is the feasible set.

Problem 7 Consider the quadratic program

$$
\begin{array}{ll}
\text{minimize} & x_1^2 + x_2^2 - x_1 x_2 - x_1 \\
\text{subject to} & x_1 + 2x_2 \leq u_1 \\
& x_1 - 4x_2 \leq u_2 \\
& 5x_1 - 76x_2 \leq 1
\end{array}
$$

Solve this QP, for parameter values $u_1 = -2, u_2 = -3$, to find optimal primal variable values $x_1^\star, x_2^\star$, and optimal dual variable values $\lambda_1^\star, \lambda_2^\star$, and $\lambda_3^\star$. Let $p^\star$ denote the optimal objective value. Verify that the KKT conditions hold for the optimal primal and dual variables you found (within reasonable numerical accuracy).

Answer

```
using Convex, SCS
# data parameters
Q = [1 -1/2; -1/2 2];
f = [-1, 0];
A = [1 2; 1 -4; 5 76];
b = [-2, -3, 1];


# item 1 (a)
x = Variable(2);
constraints = A*x <= b;
p = minimize(quadform(x,Q)+f'*x, constraints);
solve!(p, SCS.Optimizer);
lambda = constraints.dual;
p_star = p.optval;
println(lambda);
println(p_star);
println(x.value);
println(A*x.value - b);
println(2*Q*x.value + f + A'*lambda);
```