Convex Optimization homework

Rubem Vasconcelos Pacelli

December 14, 2022

Nonlinear Optimization Systems
Teleinformatics Engineering
Yuri Carvalho Barbosa Silva, Tarcisio Ferreira Maciel

Problem 1 Convexity of some sets. Determine if each set below is convex
  (a) $\left\{(x, y) \in \mathbb{R}^2_{++} \mid x/y \leq 1\right\}$

Answer The norm cone is given by

$$C = \left\{(x_1, x_2, \cdots, x_n, t) \in \mathbb{R}^{n+1} \mid \mathbf{x} \in \mathbb{R}^n, \|\mathbf{x}\|_p \leq t\right\} \subseteq \mathbb{R}^{n+1}. \qquad (1)$$

When $n = 1$ and $p = 1$ (Manhattan norm), we have that

$$C = \left\{(x, y) \in \mathbb{R}^2 \mid |x|/y \leq 1\right\} \subseteq \mathbb{R}^2, \qquad (2)$$

where $t = y$ and $x_1 = x$. From the Equation (2), it is easy to conclude that $y > 0$. Let us further define the function $f : \mathbb{R}^2 \to \mathbb{R}^2$, given by

$$f(x, y) = \begin{bmatrix} |x| & y \end{bmatrix}^\top. \qquad (3)$$

This function is clearly convex since the absolute operation is convex. Once it is well-known that the norm cone is convex, and f is a convex function in $C \subseteq \mathrm{dom}\,(f)$, then

$$S = f(C) = \left\{(x, y) \in \mathbb{R}^2_{++} \mid x/y \leq 1\right\} \qquad (4)$$

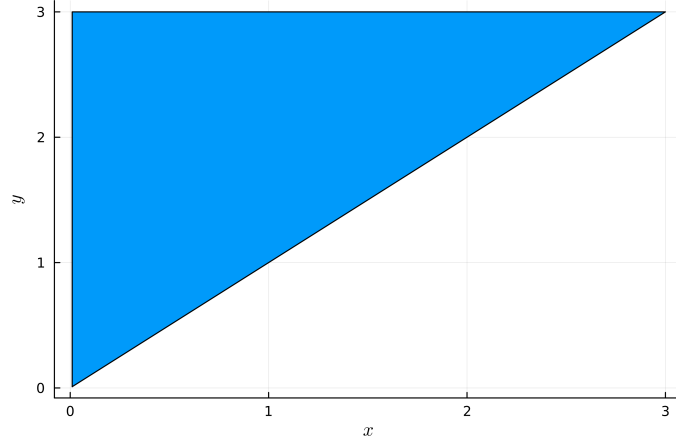is also convex, which is the set of the question. The Figure 1 shows this set for $0 \leq x \leq 3$ and $0 \leq y \leq 3$.

Figure 1: Set of the item a.

(b) $\left\{(x, y) \in \mathbb{R}^2_{++} \mid x/y \geq 1\right\}$

Answer

The set $S = \left\{\mathbf{v} \in \mathbb{R}^2_{++} \mid v_0/v_1 \geq 1\right\}$, where $\mathbf{v} = (v_0, v_1)$, is convex iff the convex combination of a pair of points belonging to S, e.g., $\mathbf{x} = (x_0, x_1)$ and $\mathbf{y} = (y_0, y_1)$, also belong to S. Mathematically,

$$\mathbf{w} = \theta\mathbf{x} + (1 - \theta)\mathbf{y} = \begin{bmatrix} \theta x_0 + (1 - \theta)y_0 \\ \theta x_1 + (1 - \theta)y_1 \end{bmatrix} \tag{5}$$

should belong to S to any $\mathbf{x}, \mathbf{y} \in S$ and $0 \leq \theta \leq 1$. Therefore

$$\frac{\theta x_0 + (1 - \theta)y_0}{\theta x_1 + (1 - \theta)y_1} \geq 1 \tag{6}$$

$$\theta x_0 + (1 - \theta)y_0 \geq \theta x_1 + (1 - \theta)y_1 \tag{7}$$

$$\frac{x_0}{x_1}\frac{\theta}{y_1} + \frac{y_0}{y_1}\frac{1 - \theta}{x_1} \geq \frac{\theta}{y_1} + \frac{1 - \theta}{x_1} \tag{8}$$

This expression always holds for any value of $\mathbf{x}, \mathbf{y} \in S$. Then, $\mathbf{w} \in S$. The Figure 2 shows this set for $[0, 3]$.
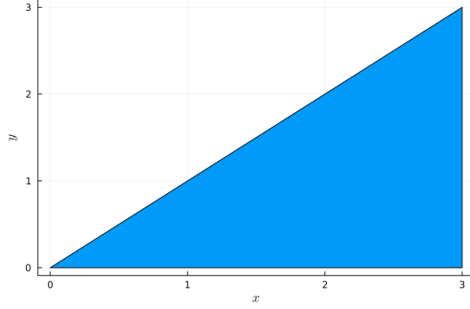
2

Figure 2: Set of the item b.

(c) $\{(x, y) \in \mathbb{R}_+^2 \mid xy \leq 1\}$

Answer

If $S = \{\mathbf{v} \in \mathbb{R}_+^2 \mid v_0 v_1 \leq 1\}$, where $\mathbf{v} = (v_0, v_1)$, then S is convex iff

$$\mathbf{w} = \theta \mathbf{x} + (1 - \theta)\mathbf{y} \in S \; \forall \; \mathbf{x}, \mathbf{y} \in S, 0 \leq \theta \leq 1, \tag{9}$$

which is the convex combination of $\mathbf{x}$ and $\mathbf{y}$. Let us prove that this set is nonconvex by contradiction. Once $\mathbf{x}, \mathbf{y} \in S$, we have that

$$x_0 x_1 \leq 1 \tag{10}$$

and

$$y_0 y_1 \leq 1, \tag{11}$$

where $\mathbf{x} = (x_0, x_1), \mathbf{y} = (y_0, y_1)$. The second component of $\mathbf{w} = (w_0, w_1)$ is given by

$$w_1 = \theta x_1 + (1 - \theta)y_1 \tag{12}$$

$$\tag{13}$$

If $x_1 \gg 0$ and $y_0 \gg 0$, which leads to $x_0 \to 0$ and $y_1 \to 0$, respectively, and $\theta = 0.5$, then, from (12), we have that

$$w_1 \approx 0.5 x_1. \tag{14}$$

Since x_1 can be indiscriminality large, we have that $w_1 \gg 0$, which leads to $w_0 \to 0$ if $\mathbf{w} \in S$. On the other hand, the first component of w_0 is given by

$$w_0 = \theta x_0 + (1 - \theta)y_0 \tag{15}$$

$$w_0 = 0.5 y_0 \tag{16}$$

3

Since y_0 can be indiscriminately large, we have that $w_0 \gg 0$. However, the statement $w_0 \to 0$ should be true in order to $\mathbf{w}$ belong to S. The contradiction leads us to conclude that $\mathbf{w} \notin S$. The Figure 3 shows the set S, and the Figure 4 shows the cone set K in which $\mathbf{x} \preceq_K \mathbf{w} \preceq_K \mathbf{y}$.
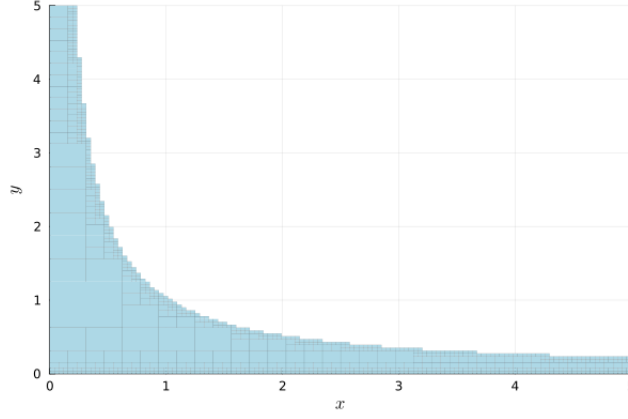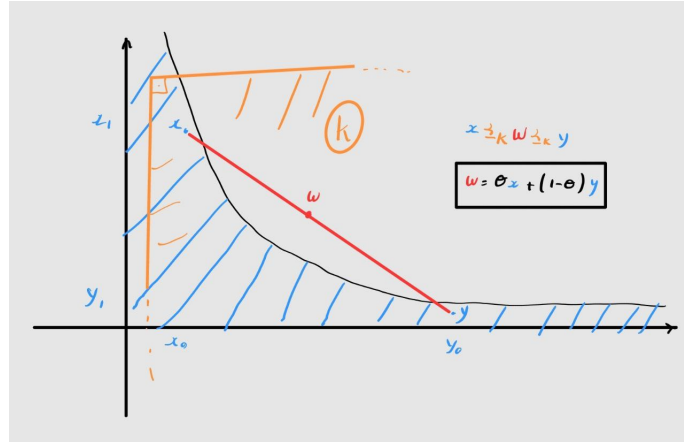

Figure 3: Set of the item c.

Figure 4: Cone set in which $\mathbf{x} \preceq_K \mathbf{w} \preceq_K \mathbf{y}$.

Since $x_1 \gg 0$, $w_0 \to 0$

(d) $\left\{(x, y) \in \mathbb{R}_+^2 \mid xy \geq 1\right\}$

By using the convex combination, the vector $\mathbf{w} = (w_0, w_1)$, which is given by

$$\mathbf{w} = \theta\mathbf{x} + (1 - \theta)\mathbf{y} = \begin{bmatrix} \theta x_0 + (1 - \theta)y_0 \\ \theta x_1 + (1 - \theta)y_1 \end{bmatrix},$$ \hfill (17)

4

shall belong to S, for any $\mathbf{x}, \mathbf{y} \in S$. If it is true, then

$$(\theta x_0 + (1-\theta)y_0)(\theta x_1 + (1-\theta)y_1) \geq 1 \quad (18)$$

$$\theta^2 x_0 x_1 + \theta(1-\theta)y_1 x_0 + (1-\theta)\theta x_1 y_0 + (1-\theta)^2 y_0 y_1 \geq 1 \quad (19)$$

$$\theta^2 x_0 x_1 + \theta(1-\theta)(y_1 x_0 + x_1 y_0) + (1-\theta)^2 y_0 y_1 \geq 1 \quad (20)$$

$$(x_0 x_1 - y_1 x_0 - x_1 y_0 + y_0 y_1)\theta^2 + (y_1 x_0 + x_1 y_0 - 2y_0 y_1)\theta + y_0 y_1 - 1 \geq 0 \quad (21)$$

$$f(\theta) \geq 0 \quad (22)$$

Note that f is a second-order function. The previous inequation holds if f has either no roots or only one root, that is

$$(y_1 x_0 + x_1 y_0 - 2y_0 y_1)^2 - 4(x_0 x_1 - y_1 x_0 - x_1 y_0 + y_0 y_1)(y_0 y_1 - 1) \geq 0 \quad (23)$$

$$(y_1 x_0 + x_1 y_0 - 2y_0 y_1)^2 \geq 4(x_0 x_1 - y_1 x_0 - x_1 y_0 + y_0 y_1)(y_0 y_1 - 1) \quad (24)$$

Let us analyze carefully this inequation

1. If the RHS is greater than the LHS, this inequation is not satisfied. Hence, we must assess it for the worst case, where the LHS is as high as possible, and see whether this inequation holds.

2. The worst case is when $x_0 \gg 0$ and $x_1 \gg 0$, and $y_0 = y_1 = 1$ because the term $x_0 x_1$ cannot be quadratically compensated with $(x_0 x_1)^2$ on the LHS. Otherwise, the higher the values of y_0 and y_1, the lower the RHS becomes[1].

3. However, even for this case, the LHS becomes approximately equal to $(x_0 + x_1)^2 = x_0^2 + 2x_0 x_1 + x_1^2$, which is greater to $x_0 x_1$

Therefore, the function f has no roots and the set $\{(x, y) \in \mathbb{R}_+^2 \mid xy \geq 1\}$ is convex. The Figure shows this set.

---
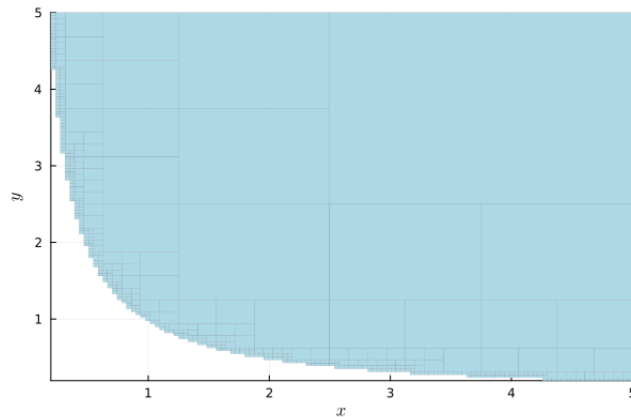
[1] The quadratic term $4y_0^2 y_1^2$ on the RHS that could increase it is canceled with the very same term $4y_0^2 y_1^2$ on the LHS.

Problem 2

Let $S = \left\{ \boldsymbol{\alpha} \in \mathbb{R}^3 | \alpha_1 + \alpha_2 e^{-t} + \alpha_3 e^{-2t} \leq 1.1 \text{ for } t \geq 1 \right\}$. Is S affine, a halfspace, a convex cone, a convex set, or none of these?