1 Sets

Convex sets	
Set	**Comments**
Convex hull: • $\text{conv } C = \left\{\sum_{i=1}^{k} \theta_i \mathbf{x}_i \mid \mathbf{x}_i \in C \text{ for } i = 1, \cdots, k, \mathbf{0} \preceq \boldsymbol{\theta} \preceq \mathbf{1}, \mathbf{1}^\top \boldsymbol{\theta} = 1\right\}$	• conv C is the smallest convex set that contains C. • conv C is a finite set as long as C is also finite.
Affine hull: • $\text{aff } C = \left\{\sum_{i=1}^{k} \theta_i \mathbf{x}_i \mid \mathbf{x}_i \in C \text{ for } i = 1, \cdots, k, \mathbf{1}^\top \boldsymbol{\theta} = 1\right\}$	• aff C is the smallest affine set that contains C. • aff C is always an infinite set. If aff C contains the origin, it is also a subspace. • Different from the convex set, θ_i is not restricted between 0 and 1
Conic hull: • $A = \left\{\sum_{i=1}^{k} \theta_i \mathbf{x}_i \mid \mathbf{x}_i \in C, \theta_i \geq 0 \text{ for } i = 1, \cdots, k\right\}$	• A is the smallest convex conic that contains C. • Different from the convex and affine sets, θ_i does not need to sum up 1.
Ray: • $\mathcal{R} = \{\mathbf{x}_0 + \theta \mathbf{v} \mid \theta \geq 0\}$	• The ray is an infinite set that begins in $\mathbf{x}_0$ and extends infinitely in direction of $\mathbf{v}$. In other words, it has a beginning, but it has no end. • The ray becomes a convex cone if $\mathbf{x}_0 = \mathbf{0}$.
Hyperplane: • $\mathcal{H} = \left\{\mathbf{x} \mid \mathbf{a}^\top \mathbf{x} = b\right\}$ • $\mathcal{H} = \left\{\mathbf{x} \mid \mathbf{a}^\top (\mathbf{x} - \mathbf{x}_0) = \mathbf{0}\right\}$ • $\mathcal{H} = \mathbf{x}_0 + a^\perp$	• It is an infinite set $\mathbb{R}^{n-1} \subset \mathbb{R}^n$ that divides the space into two halfspaces. • The inner product between $\mathbf{a}$ and any vector in $\mathcal{H}$ yields the constant value b. • $a^\perp = \left\{\mathbf{v} \mid \mathbf{a}^\top \mathbf{v} = 0\right\}$ is the infinite set of vectors perpendicular to $\mathbf{a}$. It passes through the origin. • $a^\perp$ is offset from the origin by $\mathbf{x}_0$, which is any vector in $\mathcal{H}$.
Halfspaces: • $\mathcal{H}_- = \left\{\mathbf{x} \mid \mathbf{a}^\top \mathbf{x} \leq b\right\}$ • $\mathcal{H}_+ = \left\{\mathbf{x} \mid \mathbf{a}^\top \mathbf{x} \geq b\right\}$	• They are infinite sets of the parts divided by $\mathcal{H}$.
Euclidean ball: • $B(\mathbf{x}_c, r) = \{\mathbf{x} \mid \lVert\mathbf{x} - \mathbf{x}_c\rVert \leq r\}$ • $B(\mathbf{x}_c, r) = \left\{\mathbf{x} \mid (\mathbf{x} - \mathbf{x}_c)^\top (\mathbf{x} - \mathbf{x}_c) \leq r^2\right\}$ • $B(\mathbf{x}_c, r) = \{\mathbf{x}_c + r \lVert\mathbf{u}\rVert \mid \lVert\mathbf{u}\rVert \leq 1\}$	• $B(\mathbf{x}_c, r)$ is a finite set as long as $r < \infty$. • $\mathbf{x}_c$ is the center of the ball. • r is its radius.
Ellipsoid: • $\mathcal{E} = \left\{\mathbf{x} \mid (\mathbf{x} - \mathbf{x}_c)^\top \mathbf{P}^{-1} (\mathbf{x} - \mathbf{x}_c) \leq 1\right\}$ • $\mathcal{E} = \left\{\mathbf{x}_c + \mathbf{P}^{1/2} \mathbf{u} \mid \lVert\mathbf{u}\rVert \leq 1\right\}$	• $\mathcal{E}$ is a finite set as long as $\mathbf{P}$ is a finite matrix. • $\mathbf{P}$ is symmetric and positive definite, that is, $\mathbf{P} = \mathbf{P}^\top \succ \mathbf{0}$. It determines how far the ellipsoid extends in every direction from $\mathbf{x}_c$. • $\mathbf{x}_c$ is the center of the ellipsoid. • The lengths of the semi-axes are given by $\sqrt{\lambda_i}$. • When $\mathbf{P}^{1/2} \succeq \mathbf{0}$ but singular, we say that $\mathcal{E}$ is a degenerated ellipsoid (degenerated ellipsoids are also convex).
Norm cone: • $C = \left\{(x_1, x_2, \cdots, x_n, t) \in \mathbb{R}^{n+1} \mid \mathbf{x} \in \mathbb{R}^n, \lVert\mathbf{x}\rVert_p \leq t\right\} \subseteq \mathbb{R}^{n+1}$	• Although it is named "Norm cone", it is a set, not a scalar. • The cone norm increases the dimension of $\mathbf{x}$ in 1. • For $p = 2$, it is called the second-order cone, quadratic cone, Lorentz cone or ice-cream cone.
Proper cone: $K \subset \mathbb{R}^n$ is a proper cone when it has the following properties • K is a convex cone, i.e., $\alpha K \equiv K, \alpha > 0$. • K is closed. • K is solid. • K is pointed, i.e., $-K \cap K = \{\mathbf{0}\}$.	• The proper cone K is used to define the *generalized inequality* (or *partial ordering*) in some set S. For the generalized inequality, one must define both the proper cone K and the set S. • $\mathbf{x} \preceq \mathbf{y} \iff \mathbf{y} - \mathbf{x} \in K$ for $\mathbf{x}, \mathbf{y} \in S$ (generalized inequality) • $\mathbf{x} \prec \mathbf{y} \iff \mathbf{y} - \mathbf{x} \in \text{int } K$ for $\mathbf{x}, \mathbf{y} \in S$ (strict generalized inequality). • There are two cases where K and S are understood from context and the subscript K is dropped out: ▹ When $S = \mathbb{R}^n$ and $K = \mathbb{R}^n_+$ (the nonnegative orthant). In this case, $\mathbf{x} \preceq \mathbf{y}$ means that $x_i \leq y_i$. ▹ When $S = \mathcal{S}^n$ and $K = \mathcal{S}^n_+$ or $K = \mathcal{S}^n_{++}$, where $\mathcal{S}^n$ denotes the set of symmetric $n \times n$ matrices, $\mathcal{S}^n_+$ is the space of the positive semidefinite matrices, and $\mathcal{S}^n_{++}$ is the space of the positive definite matrices. $\mathcal{S}^n_+$ is a proper cone in $\mathcal{S}^n$ (??). In this case, the generalized inequality $\mathbf{Y} \succeq \mathbf{X}$ means that $\mathbf{Y} - \mathbf{X}$ is a positive semidefinite matrix belonging to the positive semidefinite cone $\mathcal{S}^n_+$ in the subspace of symmetric matrices $\mathcal{S}^n$. It is usual to denote $\mathbf{X} \succ \mathbf{0}$ and $\mathbf{X} \succeq \mathbf{0}$ to mean than $\mathbf{X}$ is a positive definite and semidefinite matrix, respectively, where $\mathbf{0} \in \mathbb{R}^{n \times n}$ is a zero matrix. • Another common usage is when $S = \mathbb{R}^n$ and $K = \left\{\mathbf{c} \in \mathbb{R}^n \mid c_1 + c_2 t + \cdots + c_n t^{n-1} \geq 0, \text{ for } 0 \leq t \leq 1\right\}$. In this case, $\mathbf{x} \preceq_K \mathbf{y}$ means that $x_1 + x_2 t + \cdots + x_n t^{n-1} \leq y_1 + y_2 t + \cdots + y_n t^{n-1}$. • The generalized inequality has the following properties: ▹ If $\mathbf{x} \preceq_K \mathbf{y}$ and $\mathbf{u} \preceq_K \mathbf{v}$, then $\mathbf{x} + \mathbf{u} \preceq_k \mathbf{y} + \mathbf{v}$ (preserve under addition). ▹ If $\mathbf{x} \preceq_K \mathbf{y}$ and $\mathbf{y} \preceq_K \mathbf{z}$, then $\mathbf{x} \preceq_K \mathbf{z}$ (transitivity). ▹ If $\mathbf{x} \preceq_K \mathbf{y}$, then $\alpha \mathbf{x} \preceq_K \mathbf{y}$ for $\alpha \geq 0$ (preserve under nonnegative scaling). ▹ $\mathbf{x} \preceq_K \mathbf{x}$ (reflexivity). ▹ If $\mathbf{x} \preceq_K \mathbf{y}$ and $\mathbf{y} \preceq_K \mathbf{x}$, then $\mathbf{x} = \mathbf{y}$ (antisymmetric). ▹ If $\mathbf{x}_i \preceq_K \mathbf{y}_i$, for $i = 1, 2, \ldots$, and $\mathbf{x}_i \to \mathbf{x}$ and $\mathbf{y}_i \to \mathbf{y}$ as $i \to \infty$, then $\mathbf{x} \preceq_K \mathbf{y}$. • It is called partial ordering because $\mathbf{x} \not\succeq_K \mathbf{y}$ and $\mathbf{y} \not\succeq_K \mathbf{x}$ for many $\mathbf{x}, \mathbf{y} \in S$. When it happens, we say that $\mathbf{x}$ and $\mathbf{y}$ are not comparable (this case does not happen in ordinary inequality, $<$ and $>$). • $\mathbf{x} \in S$ is the *minimum* element of S with respect to the proper cone K if $\mathbf{x} \preceq_K \mathbf{y},\ \forall\, \mathbf{y} \in S$ (for *maximum*, $\mathbf{x} \succeq_K \mathbf{y},\ \forall\, \mathbf{y} \in S$). It means that $S \subseteq \mathbf{x} + K$ (for the maximum, $S \subseteq \mathbf{x} - K$), where $\mathbf{x} + K$ denotes the set K shifted from the origin by $\mathbf{x}$. Note that any point in $K + \mathbf{x}$ is comparable with $\mathbf{x}$ and is greater or equal to $\mathbf{x}$ in the generalized inequality mean. The set S does not necessarily have a minimum (maximum), but the minimum (maximum) is unique if it does. • $\mathbf{x} \in S$ is the *minimal* element of S with respect to the proper cone K if $\mathbf{y} \preceq_K \mathbf{x}$ only when $\mathbf{y} = \mathbf{x}$ (for the *maximal*, $\mathbf{y} \succeq_K \mathbf{x}$ only when $\mathbf{y} = \mathbf{x}$). It means that $(\mathbf{x} - K) \cap S = \{\mathbf{x}\}$ for minimal (for the maximal $(\mathbf{x} + K) \cap S = \{\mathbf{x}\}$), where $\mathbf{x} - K$ denotes the reflected set K shift by $\mathbf{x}$. Note that any point in $\mathbf{x} - K$ is comparable with $\mathbf{x}$ and is less than or equal to $\mathbf{x}$ in the generalized inequality mean. The set S can have many different minimal (maximal) elements. • When $K = \mathbb{R}_+$ and $S = \mathbb{R}$ (ordinary inequality), the minimal is equal to the minimum and the maximal is equal to the maximum. • When we say that a scalar-valued function $f : \mathbb{R}^n \to \mathbb{R}$ is nondecreasing, it means that whenever $\mathbf{u} \preceq \mathbf{v}$, we have $\tilde{h}(\mathbf{u}) \leq \tilde{h}(\mathbf{v})$. Similar results hold for decreasing, increasing, and nonincreasing scalar functions.
Subspace (cone set?) of the symmetric matrices: • $\mathcal{S}^n = \left\{\mathbf{X} \in \mathbb{R}^{n \times n} \mid \mathbf{X} = \mathbf{X}^\top\right\}$	• The positive semidefinite cone is given by $\mathcal{S}^n_+ = \{\mathbf{X} \in \mathbb{R}^{n \times n} \mid \mathbf{X} \succeq \mathbf{0}\} \subset \mathcal{S}^n$. This is the proper cone used to define the generalized inequalities between matrices, e.g., $\mathbf{A} \preceq \mathbf{B}$. • The positive definite cone is given by $\mathcal{S}^n_{++} = \{\mathbf{X} \in \mathbb{R}^{n \times n} \mid \mathbf{X} \succ \mathbf{0}\} \subseteq \mathcal{S}^n_+$. This is the proper cone used to define the generalized inequalities between matrices, e.g., $\mathbf{A} \prec \mathbf{B}$.
Dual cone: • $K^* = \left\{\mathbf{y} \mid \mathbf{x}^\top \mathbf{y} \geq 0,\ \forall\, \mathbf{x} \in K\right\}$	• K^* is a cone, and it is convex even when the original cone K is nonconvex. • K^* has the following properties: ▹ K^* is closed and convex. ▹ $K_1 \subseteq K_2$ implies $K_1^* \subseteq K_2^*$. ▹ If K has a nonempty interior, then K^* is pointed. ▹ If the closure of K is pointed then K^* has a nonempty interior. ▹ K^{**} is the closure of the convex hull of K. Hence, if K is convex and closed, $K^{**} = K$.
Polyhedra: • $\mathcal{P} = \left\{\mathbf{x} \mid \mathbf{a}_j^\top \mathbf{x} \leq b_j, j = 1, \ldots, m, \mathbf{a}_j^\top \mathbf{x} = d_j, j = 1, \cdots, p\right\}$ • $\mathcal{P} = \{\mathbf{x} \mid \mathbf{A}\mathbf{x} \preceq \mathbf{b}, \mathbf{C}\mathbf{x} = \mathbf{d}\}$, where $\mathbf{A} = \begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \ldots & \mathbf{a}_m\end{bmatrix}^\top$ and $\mathbf{C} = \begin{bmatrix}\mathbf{c}_1 & \mathbf{c}_2 & \ldots & \mathbf{c}_m\end{bmatrix}^\top$	• The polyhedron may or may not be an infinite set. • Polyhedron is the result of the intersection of m halfspaces and p hyperplanes. • Subspaces, hyperplanes, lines, rays line segments, and halfspaces are all special cases of polyhedra. • The *nonnegative orthant*, $\mathbb{R}^n_+ = \{\mathbf{x} \in \mathbb{R}^n \mid x_i \leq 0 \text{ for } i = 1, \ldots n\} = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{I}\mathbf{x} \succeq \mathbf{0}\}$, is a special polyhedron.
Simplex: • $S = \text{conv } \{\mathbf{v}_m\}_{m=0}^{k} = \left\{\sum_{i=0}^{k} \theta_i \mathbf{v}_i \mid \mathbf{0} \preceq \boldsymbol{\theta} \preceq \mathbf{1}, \mathbf{1}^\top \boldsymbol{\theta} = 1\right\}$ • $S = \{\mathbf{x} \mid \mathbf{x} = \mathbf{v}_0 + \mathbf{V}\boldsymbol{\theta}\}$, where $\mathbf{V} = \begin{bmatrix}\mathbf{v}_1 - \mathbf{v}_0 & \ldots & \mathbf{v}_n - \mathbf{v}_0\end{bmatrix} \in \mathbb{R}^{n \times k}$ • $S = \{\mathbf{x} \mid \underbrace{\mathbf{A}_1 \mathbf{x} \preceq \mathbf{A}_1 \mathbf{v}_0, \mathbf{1}^\top \mathbf{A}_1 \mathbf{x} \leq 1 + \mathbf{1}^\top \mathbf{A}_1 \mathbf{v}_0}_{\text{Linear inequalities in } x}, \underbrace{\mathbf{A}_2 \mathbf{x} = \mathbf{A}_2 \mathbf{v}_0}_{\text{Linear equalities in } x}\}$ (Polyhedra form), where $\mathbf{A} = \begin{bmatrix}\mathbf{A}_1 \\ \mathbf{A}_2\end{bmatrix}$ and $\mathbf{A}\mathbf{V} = \begin{bmatrix}\mathbf{I}_{k \times k} \\ \mathbf{0}_{n-k \times n-k}\end{bmatrix}$	• Simplexes are a subfamily of the polyhedra set. • Also called k-dimensional Simplex in $\mathbb{R}^n$. • The set $\{\mathbf{v}_m\}_{m=0}^{k}$ is a affinely independent, which means $\{\mathbf{v}_1 - \mathbf{v}_0, \ldots, \mathbf{v}_k - \mathbf{v}_0\}$ are linearly independent. • $\mathbf{V} \in \mathbb{R}^{n \times k}$ is a full-rank tall matrix, i.e., $\text{rank}(\mathbf{V}) = k$. All its column vectors are independent. The matrix $\mathbf{A}$ is its left pseudoinverse.
α-sublevel set: • $C_\alpha = \{\mathbf{x} \in \text{dom}(f) \mid f(\mathbf{x}) \leq \alpha\}$ (regarding convexity), where $f : \mathbb{R}^n \to \mathbb{R}$ • $C_\alpha = \{\mathbf{x} \in \text{dom}(f) \mid f(\mathbf{x}) \geq \alpha\}$ (regarding concavity), where $f : \mathbb{R}^n \to \mathbb{R}$	• If f is a convex (concave) function, then sublevel sets of f are convexes (concaves) for any $\alpha \in \mathbb{R}$. • The converse is not true: a function can have all its sublevel set convex and not be a convex function. • $C_\alpha \subseteq \text{dom}(f)$

2 Fuctions

Functions (or operators) and their implications regarding convexity		
Function	**Convexity**	**Comments**
Union: $C = A \cup B$	Not in most of the cases.	
Intersection: $C = A \cap B$	Yes, if A and B are convex sets.	
Convex function: $f : \mathrm{dom}\,(f) \to \mathbb{R}$ • $f(\theta\mathbf{x} + (1-\theta)\mathbf{y}) \leq \theta f(\mathbf{x}) + (1-\theta) f(\mathbf{y})$, where $0 \leq \theta \leq 1$. • $\mathrm{dom}\,(f)$ shall be a convex set to f be considered a convex function.	Yes.	• Graphically, the line segment between $(\mathbf{x}, f(\mathbf{x}))$ and $(\mathbf{y}, f(\mathbf{y}))$ lies always above the graph f. • In terms of sets, a function is convex iff a line segment within $\mathrm{dom}\,(f)$, which is a convex set, gives an image set that is also convex. • $\mathrm{dom} f$ is convex iff all points for any line segment within $\mathrm{dom}\,(f)$ belong to it. • *First-order condition*: f is convex iff $\mathrm{dom}\,(f)$ is convex and $f(\mathbf{y}) \geq f(\mathbf{x}) + \nabla f(\mathbf{x})^{\mathsf{T}}(\mathbf{y}-\mathbf{x}), \forall\ \mathbf{x}, \mathbf{y} \in \mathrm{dom}\,(f), \mathbf{x} \neq \mathbf{y}$, being $\nabla f(\mathbf{x})$ the gradient vector. This inequation says that the first-order Taylor approximation is a *underestimator* for convex functions. The first-order condition requires that f is differentiable. • If $\nabla f(\mathbf{x}) = \mathbf{0}$, then $f(\mathbf{y}) \geq f(\mathbf{x}), \forall\ \mathbf{y} \in \mathrm{dom}\,(f)$ and $\mathbf{x}$ is a global minimum. • *Second-order condition*: f is convex iff $\mathrm{dom}\,(f)$ is convex and $\mathbf{H} \succeq \mathbf{0}$, that is, the Hessian matrix $\mathbf{H}$ is a positive semidefinite matrix. It means that the graphic of the curvature has a positive (upward) curvature at $\mathbf{x}$. It is important to note that, if $\mathbf{H} \succ \mathbf{0}, \forall\ \mathbf{x} \in \mathrm{dom}\,(f)$, then f is strictly convex. But is f is strictly convex, not necessarily that $\mathbf{H} \succ \mathbf{0}, \forall\ \mathbf{x} \in \mathrm{dom}\,(f)$. Therefore, strict convexity can only be partially characterized.
Convex function: $f : \mathrm{dom}\,(f) \to \mathbb{R}$ • $f(\theta\mathbf{x} + (1-\theta)\mathbf{y}) \geq \theta f(\mathbf{x}) + (1-\theta) f(\mathbf{y})$, where $0 \leq \theta \leq 1$. • $\mathrm{dom}\,(f)$ shall be a concave set to f be considered a concave function.	Concave	yes
Affine function $f : \mathbb{R}^n \to \mathbb{R}^m$ • $f(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}$, where $\mathbf{A} \in \mathbb{R}^{m \times n}, \mathbf{b} \in \mathbb{R}^m, \mathbf{x} \in \mathbb{R}^n$	Yes, if the domain $S \subseteq \mathbb{R}^n$ is a convex set, then its image $f(S) = \{f(\mathbf{x}) \mid \mathbf{x} \in S\} \subseteq \mathbb{R}^m$ is also convex.	• f is an affine function iff $f(\theta\mathbf{x} + (1-\theta)\mathbf{y}) = \theta f(\mathbf{x}) + (1-\theta) f(\mathbf{y})$, where $\theta \in \mathbb{R}$. • The affine function, $f(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}$, is a broader category that encompasses the linear function, $f(\mathbf{x}) = \mathbf{A}\mathbf{x}$. The linear function has its origin fixed at $\mathbf{0}$ after the transformation, whereas the affine function does not necessarily have it (when not, this makes the affine function nonlinear). Graphically, we can think of an affine function as a linear transformation plus a shift from the origin of $\mathbf{b}$. • A special case of the linear function is when $\mathbf{A} = \mathbf{c}^{\mathsf{T}}$. In this case, we have $f(\mathbf{x}) = \mathbf{c}^{\mathsf{T}}\mathbf{x}$, which is the inner product between the vector $\mathbf{c}$ and $\mathbf{x}$. • The inverse image of C, $f^{-1}(C) = \{\mathbf{x} \mid f(\mathbf{x}) \in C\}$, is also convex. • The *linear matrix inequality* (LMI), $\mathbf{A}(\mathbf{x}) = x_1\mathbf{A}_1 + \cdots + x_n\mathbf{A}_n \preceq \mathbf{B}$, is a special case of affine function. In other words, $f(S) = \{\mathbf{x} \mid \mathbf{A}(\mathbf{x}) \preceq \mathbf{B}\}$ is a convex set if S is convex. Many optimization problems can be formulated as LMI problems and solved optimally.
Constant function $f : \mathbb{R} \to \mathbb{R}$ • $f(\theta\mathbf{x} + (1-\theta)\mathbf{y}) = f(\mathbf{x})$, where $\theta \in \mathbb{R}$.	Convex and concave.	
Exponential function $f : \mathbb{R} \to \mathbb{R}$ • $f(x) = e^{ax} \in \mathbb{R}$, where $a \in \mathbb{R}$	Convex.	
Quadratic function $f : \mathbb{R}^n \to \mathbb{R}$ • $f(\mathbf{x}) = a\mathbf{x}^{\mathsf{T}}\mathbf{P}\mathbf{x} + \mathbf{p}^{\mathsf{T}}\mathbf{x} + r \in \mathbb{R}$, where $\mathbf{x}, \mathbf{p} \in \mathbb{R}^n, \mathbf{P} \in \mathbb{R}^{n \times n}$, and $a, b \in \mathbb{R}$	It depends on the matrix $\mathbf{P}$: • f is convex iff $\mathbf{P} \succeq \mathbf{0}$. • f is strictly convex iff $\mathbf{P} \succ \mathbf{0}$. • f is concave iff $\mathbf{P} \preceq \mathbf{0}$. • f is strictly concave iff $\mathbf{P} \prec \mathbf{0}$.	
Power function $f : \mathbb{R}_{++} \to \mathbb{R}$ • $f(x) = x^a$	It depends on a • f is convex iff $a \geq 1$ or $a \leq 0$. • f is concave iff $0 \leq a \leq 1$.	
Power of absolute value: $f : \mathbb{R} \to \mathbb{R}$ • $f(x) = \lvert x\rvert^p$, where $p \leq 1$.	Convex.	
Logarithm function: $f : \mathbb{R}_{++} \to \mathbb{R}$ • $f(x) = \log x$	Concave.	
Negative entropy function: $f : \mathbb{R}_{+} \to \mathbb{R}$ • $f(x) = x \log x$	Convex.	• When it is defined $f(x)\big\rvert_{x=0} = 0$, $\mathrm{dom}\,(f) = \mathbb{R}$.
Minkowski distance, p-norm function, or l_p norm function: $f : \mathbb{R}^n \to \mathbb{R}$ • $f(\mathbf{x}) = \lVert\mathbf{x}\rVert_p$, where $p \in \mathbb{N}_{++}$.	Convex.	• It can be proved by triangular inequality.
Maximum element: $f : \mathbb{R}^n \to \mathbb{R}$ • $f(\mathbf{x}) = \max\{x_1, \ldots, x_n\}$.	Convex.	
Pointwise maximum (maximum function): $f : \mathbb{R}^n \to \mathbb{R}$ • $f(\mathbf{x}) = \max\{f_1(\mathbf{x}), \ldots, f_n(\mathbf{x})\}$.	f is convex if $f_1, \ldots, f_n$ are convex functions.	• Its domain $\mathrm{dom}\,(f) = \bigcap_{i=1}^{n} \mathrm{dom}\,(f_i)$ is also convex.
Pointwise infimum: • $f(\mathbf{x}) = \inf_{\mathbf{y} \in \mathcal{A}} g(\mathbf{x}, \mathbf{y})$.	f is concave if g is concave for each $\mathbf{y} \in \mathcal{A}$.	• For each value of x, we have an infinite set of points $g(x, y)\rvert_{y \in \mathcal{A}}$. The value $f(x)$ will be the greatest value in the codomain of f that is less than or equal this set. • $\mathrm{dom}\,(f) = \left\{x \mid (x, y) \in \mathrm{dom}\,(g)\ \forall\ y \in \mathcal{A},\ \inf_{y \in \mathcal{A}} g(x, y) > -\infty\right\}$.
Pointwise supremum: • $f(\mathbf{x}) = \sup_{\mathbf{y} \in \mathcal{A}} g(\mathbf{x}, \mathbf{y})$.	f is convex if g is convex for each $\mathbf{y} \in \mathcal{A}$.	• For each value of x, we have an infinite set of points $g(x, y)\rvert_{y \in \mathcal{A}}$. The value $f(x)$ will be the least value in the codomain of f that is greater than or equal this set. • $\mathrm{dom}\,(f) = \left\{x \mid (x, y) \in \mathrm{dom}\,(g)\ \forall\ y \in \mathcal{A},\ \sup_{y \in \mathcal{A}} g(x, y) < \infty\right\}$. • In terms of epigraphs, the pointwise supremum of the infinite set of functions $g(x, y)\rvert_{y \in \mathcal{A}}$ corresponds to the intersection of the following epigraphs: $\mathrm{epi}\ f = \bigcap_{y \in \mathcal{A}} \mathrm{epi}\ g(\cdot, y)$
Minimum function: $f : \mathbb{R}^n \to \mathbb{R}$ • $f(\mathbf{x}) = \min\{f_1(\mathbf{x}), \ldots, f_n(\mathbf{x})\}$.	Nonconvex and nonconcave in most of the cases.	
Log-sum-exp function: $f : \mathbb{R}^n \to \mathbb{R}$ • $f(\mathbf{x}) = \log\left(e^{x_1} + \cdots + e^{x_n}\right)$	Convex.	• This function is interpreted as the approximation of the maximum element function, since $\max\{x_1, \ldots, x_n\} \leq f(\mathbf{x}) \leq \max\{x_1, \ldots, x_n\} + \log n$
Geometric mean function $f : \mathbb{R}^n \to \mathbb{R}$ • $f(\mathbf{x}) = \left(\Pi_{i=1}^{n} x_i\right)^{1/n}$	Convex.	
Log-determinant function $f : \mathcal{S}_{++}^{n} \to \mathbb{R}$ • $f(\mathbf{X}) = \log\lvert\mathbf{X}\rvert$	Convex.	• $\mathbf{X}$ is positive semidefinite, i.e., $\mathbf{X} \succ \mathbf{0} \therefore \mathbf{X} \in \mathcal{S}_{++}^{n}$.
Composite function $f = h \circ g : \mathbb{R}^n \to \mathbb{R}$ • $f = g \circ h$, i.e., $f(\mathbf{x}) = (h \circ g)(\mathbf{x}) = h(g(\mathbf{x}))$, where: ▹ $g : \mathbb{R}^n \to \mathbb{R}^k$. ▹ $h : \mathbb{R}^k \to \mathbb{R}$. ▹ $\mathrm{dom}\,(f) = \{\mathbf{x} \in \mathrm{dom}\,(g) \mid g(\mathbf{x}) \in \mathrm{dom}\,(h)\}$.	• Scalar composition: the following statements hold for $k = 1$ and $n \geq 1$, i.e., $h : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$: ▹ f is convex if h is convex, $\tilde{h}$ is nondecreasing, and g is convex. In this case, $\mathrm{dom}\,(h)$ is either $(-\infty, a]$ or $(-\infty, a)$. ▹ f is convex if h is convex, $\tilde{h}$ is nonincreasing, and g is concave. In this case, $\mathrm{dom}\,(h)$ is either $[a, \infty)$ or (a, ∞). ▹ f is concave if h is concave, $\tilde{h}$ is nondecreasing, and g is concave. ▹ f is concave if h is concave, $\tilde{h}$ is nonincreasing, and g is convex. • Vector composition: the following statements hold for $k \geq 1$ and $n \geq 1$, i.e., $h : \mathbb{R}^k \to \mathbb{R}$ and $\boldsymbol{g} : \mathbb{R}^n \to \mathbb{R}^k$. Hence, $g(\mathbf{x}) = (g_1(\mathbf{x}), g_2(\mathbf{x}), \ldots, g_k(\mathbf{x}))$ is a vector-valued function (or simply, vector function), where $g_i : \mathbb{R}^k \to \mathbb{R}$ for $1 \leq i \leq k$. ▹ f is convex if h is is convex, $\tilde{h}$ is nondecreasing in each argument of $\mathbf{x}$, and $\{g_i\}_{i=1}^{k}$ is a set of convex functions. ▹ f is convex if h is is convex, $\tilde{h}$ is nonincreasing in each argument of $\mathbf{x}$, and $\{g_i\}_{i=1}^{k}$ is a set of concave functions. ▹ f is concave if h is is concave, $\tilde{h}$ is nondecreasing in each argument of $\mathbf{x}$, and $\{g_i\}_{i=1}^{k}$ is a set of concave functions. Where $\tilde{h}$ is the extended-value extension of the function h, which assigns the value ∞ $(-\infty)$ to the point not in $\mathrm{dom}\,(h)$ for h convex (concave).	• The composition function allows us to see a large class of functions as convex (or concave). • For scale composition, the remarkable ones are: ▹ If g is convex then $f(x) = h(g(\mathbf{x})) = \exp g(\mathbf{x})$ is convex. ▹ If g is concave and $\mathrm{dom}\,(g) \subseteq \mathbb{R}_{++}$, then $f(\mathbf{x}) = h(g(\mathbf{x})) = \log g(\mathbf{x})$ is concave. ▹ If g is concave and $\mathrm{dom}\,(g) \subseteq \mathbb{R}_{++}$, then $f(\mathbf{x}) = h(g(\mathbf{x})) = 1/g(\mathbf{x})$ is convex. ▹ If g is convex and $\mathrm{dom}\,(g) \subseteq \mathbb{R}_{+}$, then $f(\mathbf{x}) = h(g(\mathbf{x})) = g^p(\mathbf{x})$ is convex, where $p \geq 1$. ▹ If g is convex then $f(\mathbf{x}) = h(g(\mathbf{x})) = -\log(-g(x))$ is convex, where $\mathrm{dom}\,(f) = \{\mathbf{x} \mid g(\mathbf{x}) < 0\}$. • For vector composition, we have the following examples: ▹ If $g(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}$ is an affine function, then $f = h \circ g$ is convex (concave) if h is convex (concave). ▹ Let $h(\mathbf{x}) = x_{[1]} + \cdots + x_{[r]}$ be the sum of the r largest components of $\mathbf{x} \in \mathbb{R}^k$. If $g_1, g_2, \ldots, g_k$ are convex, where $\mathrm{dom}\,(g_i) = \mathbb{R}^n$, then $f = h \circ g$, which is the pointwise sum of the largest g_i's, is convex. ▹ $f = h \circ g$ is a convex function when $h(\mathbf{x}) = \log\left(\sum_{i=1}^{k} e^{x_i}\right)$ and $g_1, g_2, \ldots, g_k$ are convex functions. ▹ For $0 < p \leq 1$, the function $h(\mathbf{x}) = \left(\sum_{i=1}^{k} x_i^p\right)^{1/p}$, where $\mathrm{dom}\,(h) = \mathbb{R}_{+}^{n}$, is concave. If $g_1, g_2, \ldots, g_k$ are concaves (convexes) and nonnegatives, then $f = h \circ g$ is concave (convex).
Nonnegative weighted sum: $f : \mathrm{dom}\,(f) \to \mathbb{R}$ • $f(\mathbf{x}) = \sum_{i=1}^{m} w_i f_i(\mathbf{x})$, where $w \geq 0$.	• If $f_1, f_2, \ldots, f_m$ are convex or concave functions, then f is a convex or concave function, respectively. • If $f_1, f_2, \ldots, f_m$ are strictly convex or concave functions, then f is a strictly convex or concave function, respectively.	• Special cases is when $f = wf$ (a nonnegative scaling) and $f = f_1 + f_2$ (sum).
Integral function $f : \mathbb{R}^n \to \mathbb{R}$: • $f(\mathbf{x}) = \int_{\mathcal{A}} w(\mathbf{y}) g(\mathbf{x}, \mathbf{y})\, \mathrm{d}\mathbf{y}$, where $\mathbf{y} \in \mathcal{A} \subseteq \mathbb{R}^m$, and $w : \mathbb{R}^m \to \mathbb{R}$.	If g is convex in $\mathbf{x}$ for each $\mathbf{y} \in \mathcal{A}$ and if $w(\mathbf{y}) \geq 0$, $\forall\ \mathbf{y} \in \mathcal{A}$, then f is convex (provided the integral exists).	
Perspective function $f : \mathbb{R}^n \times \mathbb{R}_{++} \to \mathbb{R}^n$ • $f(\mathbf{x}, t) = \mathbf{x}/t$, where $\mathbf{x} \in \mathbb{R}^n, t \in \mathbb{R}$.	Yes, if $S \subseteq \mathrm{dom}\,(f)$ is a convex set, then its image, $f(S) = \{f(\mathbf{x}) \mid \mathbf{x} \in S\} \subseteq \mathbb{R}^n$, is also convex.	• The perspective function decreases the dimension of the function domain since $\dim(\mathrm{dom}\,(f)) = n + 1$. • Its effect is similar to the camera zoom. • The inverse image is also convex, that is, if $C \subseteq \mathbb{R}^n$ is convex, then $f^{-1}(C) = \left\{(\mathbf{x}, t) \in \mathbb{R}^{n+1} \mid \mathbf{x}/t \in C, t > 0\right\}$ is also convex. • A special case is when $n = 1$, which is called *quadratic-over-linear function*.
Projective (or linear-fractional) function, $f : \mathbb{R}^n \to \mathbb{R}^m$ • $f = p \circ g$, i.e., $f(\mathbf{x}) = (p \circ g)(\mathbf{x}) = p(g(\mathbf{x}))$, where ▹ $g : \mathbb{R}^n \to \mathbb{R}^{m+1}$ is an affine function given by $g(\mathbf{x}) = \begin{bmatrix}\mathbf{A} \\ \mathbf{c}^{\mathsf{T}}\end{bmatrix}\mathbf{x} + \begin{bmatrix}\mathbf{b} \\ d\end{bmatrix}$, being $\mathbf{A} \in \mathbb{R}^{m \times n}, \mathbf{b} \in \mathbb{R}^m, \mathbf{c} \in \mathbb{R}^n$, and $d \in \mathbb{R}$. ▹ $p : \mathbb{R}^{m+1} \to \mathbb{R}^m$ is the perspective function. • $f(\mathbf{x}) = \mathcal{P}^{-1}(\mathbf{Q}\mathcal{P}(\mathbf{x}))$ ▹ $\mathcal{P}(\mathbf{x}) = \{(t\mathbf{x}, t) \mid t \geq 0\} \subset \mathbb{R}^{n+1}$ ▹ $\mathbf{Q} = \begin{bmatrix}\mathbf{A} & \mathbf{b} \\ \mathbf{c}^{\mathsf{T}} & d\end{bmatrix} \in \mathbb{R}^{(m+1)\times(n+1)}$	Yes, if $S \subseteq \mathrm{dom}\,(f)$ is a convex set, then its image, $f(S) = \{f(\mathbf{x}) \mid \mathbf{x} \in S\} \subseteq \mathbb{R}^n$, is also convex.	• The linear and affine functions are special cases of the linear-fractional function. • $\mathrm{dom}\,(f) = \left\{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{c}^{\mathsf{T}}\mathbf{x} + d > 0\right\}$ • $\mathcal{P}(\mathbf{x}) \subset \mathbb{R}^{n+1}$ is a ray set that begins at the origin and its last component takes only positive values. For each $\mathbf{x} \in \mathrm{dom}\,(f)$, it is associated a ray set in $\mathbb{R}^{n+1}$ in this form. This (projective) correspondence between all points in $\mathrm{dom}\,(f)$ and their respective sets $\mathcal{P}$ is a biunivocal mapping. • The linear transformation $\mathbf{Q}$ acts on these rays, forming another set of rays. • Finally we take the inverse projective transformation to recover $f(\mathbf{x})$.
Epigraph: • $\mathrm{epi}\ f = \{(\mathbf{x}, t) \mid \mathbf{x} \in \mathrm{dom}\,(f), t \geq f(\mathbf{x})\}$	• The function f is convex iff its epigraph is convex.	• Visually, it is the graph above the $(\mathbf{x}, f(\mathbf{x}))$ curve.
Hypograph: • $\mathrm{hypo}\ f = \{(\mathbf{x}, t) \mid \mathbf{x} \in \mathrm{dom}\,(f), t \geq f(\mathbf{x})\}$	• The function f is concave iff its hypograph is convex.	• Visually, it is the graph below the $(\mathbf{x}, f(\mathbf{x}))$ curve.