Sets

Set	Convex?	Comments
Convex hull: • conv $C = \left\{\sum_{i=1}^{k} \theta_i \mathbf{x}_i \mid \mathbf{x}_i \in C, \mathbf{0} \preceq \boldsymbol{\theta} \preceq \mathbf{1}, \mathbf{1}^\mathsf{T}\boldsymbol{\theta} = 1\right\}$	Yes	• conv C will be the smallest convex set that contains C. • conv C will be a finite set as long as C is also finite.
Affine hull: • aff $C = \left\{\sum_{i=1}^{k} \theta_i \mathbf{x}_i \mid \mathbf{x}_i \in C \text{ for } i = 1, \cdots, k, \mathbf{1}^\mathsf{T}\boldsymbol{\theta} = 1\right\}$	Yes.	• A will be the smallest affine set that contains C. • Different from the convex set, θ_i is not restricted between 0 and 1 • aff C will always be an infinite set. If aff C contains the origin, it is also a subspace.
Conic hull: • $A = \left\{\sum_{i=1}^{k} \theta_i \mathbf{x}_i \mid \mathbf{x}_i \in C, \theta_i > 0 \text{ for } i = 1, \cdots, k\right\}$	Yes.	• A will be the smallest convex conic that contains C. • Different from the convex and affine sets, θ_i does not need to sum up 1.
Hyperplane: • $\mathcal{H} = \left\{\mathbf{x} \mid \mathbf{a}^\mathsf{T}\mathbf{x} = b\right\}$ • $\mathcal{H} = \left\{\mathbf{x} \mid \mathbf{a}^\mathsf{T}(\mathbf{x} - \mathbf{x}_0) = \mathbf{0}\right\}$ • $\mathcal{H} = \mathbf{x}_0 + a^\perp$	Yes.	• It is an infinite set $\mathbb{R}^{n-1} \subset \mathbb{R}^n$ that divides the space into two halfspaces. • $a^\perp = \left\{\mathbf{v} \mid \mathbf{a}^\mathsf{T}\mathbf{v} = 0\right\}$ is the set of vectors perpendicular to $\mathbf{a}$. It passes through the origin. • $a^\perp$ is offset from the origin by $\mathbf{x}_0$, which is any vector in $\mathcal{H}$.
Halfspaces: • $\mathcal{H}_- = \left\{\mathbf{x} \mid \mathbf{a}^\mathsf{T}\mathbf{x} \leq b\right\}$ • $\mathcal{H}_+ = \left\{\mathbf{x} \mid \mathbf{a}^\mathsf{T}\mathbf{x} \geq b\right\}$	Yes.	• They are infinite sets of the parts divided by $\mathcal{H}$.
Euclidean ball: • $B(\mathbf{x}_c, r) = \{\mathbf{x} \mid \lVert\mathbf{x} - \mathbf{x}_c\rVert_2 \leq r\}$ • $B(\mathbf{x}_c, r) = \left\{\mathbf{x} \mid (\mathbf{x} - \mathbf{x}_c)^\mathsf{T}(\mathbf{x} - \mathbf{x}_c) \leq r\right\}$ • $B(\mathbf{x}_c, r) = \{\mathbf{x}_c + r\lVert\mathbf{u}\rVert \mid \lVert\mathbf{u}\rVert \leq 1\}$	Yes.	• $B(\mathbf{x}_c, r)$ is a finite set as long as $r < \infty$. • $\mathbf{x}_c$ is the center of the ball. • r is its radius.
Ellipsoid: • $\mathcal{E} = \left\{\mathbf{x} \mid (\mathbf{x} - \mathbf{x}_c)^\mathsf{T}\mathbf{P}^{-1}(\mathbf{x} - \mathbf{x}_c) \leq 1\right\}$ • $\mathcal{E} = \{\mathbf{x}_c + \mathbf{A}\mathbf{u} \mid \lVert\mathbf{u}\rVert \leq 1\}$, where $\mathbf{A} = \mathbf{P}^{1/2}$.	Yes.	• $\mathcal{E}$ is a finite set as long as $\mathbf{P}$ is a finite matrix. • $\mathbf{P}$ is symmetric and positive definite, that is, $\mathbf{P} = \mathbf{P}^\mathsf{T} \succ 0$. • $\mathbf{x}_c$ is the center of the ellipsoid. • The lengths of the semi-axes are given by $\sqrt{\lambda_i}$. • $\mathbf{A}$ is invertible. When it is not, we say that $\mathcal{E}$ is a degenerated ellipsoid (degenerated ellipsoids are also convex).
Norm cone: • $C = \left\{[x_1, x_2, \cdots, x_n, t]^\mathsf{T} \in \mathbb{R}^{n+1} \mid \mathbf{x} \in \mathbb{R}^n, \lVert\mathbf{x}\rVert_p \leq t\right\} \subseteq \mathbb{R}^{n+1}$	Yes.	• Although it is named "Norm cone", it is a set, not a scalar. • The cone norm increases the dimension of $\mathbf{x}$ in 1. • For $p = 2$, it is called the second-order cone, quadratic cone, Lorentz cone or ice-cream cone.
Polyhedra: • $\mathcal{P} = \left\{\mathbf{x} \mid \mathbf{a}_j^\mathsf{T}\mathbf{x} \leq b_j, j = 1, \ldots, m, \mathbf{a}_j^\mathsf{T}\mathbf{x} = d_j, j = 1, \cdots, p\right\}$ • $\mathcal{P} = \{\mathbf{x} \mid \mathbf{A}\mathbf{x} \preceq \mathbf{b}, \mathbf{C}\mathbf{x} = \mathbf{d}\}$, where $\mathbf{A} = \begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \ldots & \mathbf{a}_m\end{bmatrix}^\mathsf{T}$ and $\mathbf{C} = \begin{bmatrix}\mathbf{c}_1 & \mathbf{c}_2 & \ldots & \mathbf{c}_m\end{bmatrix}^\mathsf{T}$	Yes.	• Polyhedron is the intersection of m halfspaces. • The polyhedron may or may not be an infinite set. • Subspaces, hyperplanes, lines, rays line segments, and halfspaces are all polyhedra. • The *nonnegative orthant*, $\mathbb{R}_+^n = \{\mathbf{x} \in \mathbb{R}^n \mid x_i \leq 0 \text{ for } i = 1, \ldots n\} = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{I}\mathbf{x} \succeq \mathbf{0}\}$, is a special polyhedron.
Simplex: • $\mathcal{S} = \text{conv } \{\mathbf{v}_m\}_{m=0}^{k} = \left\{\sum_{i=0}^{k} \theta_i \mathbf{v}_i \mid \mathbf{0} \preceq \boldsymbol{\theta} \preceq \mathbf{1}, \mathbf{1}^\mathsf{T}\boldsymbol{\theta} = 1\right\}$ • $\mathcal{S} = \left\{\mathbf{x} \mid \mathbf{A}_1\mathbf{x} \preceq \mathbf{A}_1\mathbf{v}_0, \mathbf{A}_2\mathbf{x} = \mathbf{A}_2\mathbf{v}_0, \mathbf{1}^\mathsf{T}\mathbf{A}_1\mathbf{x} \leq 1 + \mathbf{1}^\mathsf{T}\mathbf{A}_1\mathbf{v}_0\right\}$ • $\mathcal{S} = \{\mathbf{x} \mid \mathbf{A}_1\mathbf{x} \prec \mathbf{b}, \mathbf{C}\mathbf{x} = \mathbf{d}\}$ (Polyhedra form), where $\mathbf{b} = \mathbf{A}_1\mathbf{v}_0, \mathbf{C} = \mathbf{A}_2, \mathbf{d} = \mathbf{A}_2\mathbf{v}_0$	Yes.	• Simplexes are another family of polyhedra. • Also called k-dimensional Simplex in $\mathbb{R}^n$. • The set $\{\mathbf{v}_m\}_{m=0}^{k}$ is a affinely independent set.
$C = A \cup B$	Not always.	
$C = A \cap B$	Yes, if A and B are convex sets.	

- All convex set is quasiconvex, but not all quasiconvex is convex.
- It is possible to solve quasiconvex functions, even if it is not convex (see Algorithm 4.1). But not all quasiconvex functions that are nonconvex can be solved(?).
- Superlevel set (a set) 3.3.6, all convex functions have all convex α sub-level set, but not all functions that have convex α sub-level set are convex (see slide 3.11).