Convex sets

Set	Comments
Convex hull: • conv $C = \left\{\sum_{i=1}^{k} \theta_i \mathbf{x}_i \mid \mathbf{x}_i \in C, \mathbf{0} \le \boldsymbol{\theta} \le \mathbf{1}, \mathbf{1}^\top \boldsymbol{\theta} = 1\right\}$	• conv C will be the smallest convex set that contains C. • conv C will be a finite set as long as C is also finite.
Affine hull: • aff $C = \left\{\sum_{i=1}^{k} \theta_i \mathbf{x}_i \mid \mathbf{x}_i \in C \text{ for } i = 1, \cdots, k, \mathbf{1}^\top \boldsymbol{\theta} = 1\right\}$	• A will be the smallest affine set that contains C. • Different from the convex set, θ_i is not restricted between 0 and 1 • aff C will always be an infinite set. If aff C contains the origin, it is also a subspace.
Conic hull: • $A = \left\{\sum_{i=1}^{k} \theta_i \mathbf{x}_i \mid \mathbf{x}_i \in C, \theta_i > 0 \text{ for } i = 1, \cdots, k\right\}$	• A will be the smallest convex conic that contains C. • Different from the convex and affine sets, θ_i does not need to sum up 1.
Ray: • $\mathcal{R} = \{\mathbf{x}_0 + \theta \mathbf{v} \mid \theta \ge 0\}$	• The ray is an infinite set that begins in $\mathbf{x}_0$ and extends infinitely in direction of $\mathbf{v}$. In other words, it has a beginning, but it has no end.
Hyperplane: • $\mathcal{H} = \left\{\mathbf{x} \mid \mathbf{a}^\top \mathbf{x} = b\right\}$ • $\mathcal{H} = \left\{\mathbf{x} \mid \mathbf{a}^\top (\mathbf{x} - \mathbf{x}_0) = \mathbf{0}\right\}$ • $\mathcal{H} = \mathbf{x}_0 + a^\perp$	• It is an infinite set $\mathbb{R}^{n-1} \subset \mathbb{R}^n$ that divides the space into two halfspaces. • $a^\perp = \left\{\mathbf{v} \mid \mathbf{a}^\top \mathbf{v} = 0\right\}$ is the set of vectors perpendicular to $\mathbf{a}$. It passes through the origin. • $a^\perp$ is offset from the origin by $\mathbf{x}_0$, which is any vector in $\mathcal{H}$.
Halfspaces: • $\mathcal{H}_- = \left\{\mathbf{x} \mid \mathbf{a}^\top \mathbf{x} \le b\right\}$ • $\mathcal{H}_+ = \left\{\mathbf{x} \mid \mathbf{a}^\top \mathbf{x} \ge b\right\}$	• They are infinite sets of the parts divided by $\mathcal{H}$.
Euclidean ball: • $B(\mathbf{x}_c, r) = \{\mathbf{x} \mid \\|\mathbf{x} - \mathbf{x}_c\\|_2 \le r\}$ • $B(\mathbf{x}_c, r) = \left\{\mathbf{x} \mid (\mathbf{x} - \mathbf{x}_c)^\top (\mathbf{x} - \mathbf{x}_c) \le r\right\}$ • $B(\mathbf{x}_c, r) = \{\mathbf{x}_c + r \\|\mathbf{u}\\| \mid \\|\mathbf{u}\\| \le 1\}$	• $B(\mathbf{x}_c, r)$ is a finite set as long as $r < \infty$. • $\mathbf{x}_c$ is the center of the ball. • r is its radius.
Ellipsoid: • $\mathcal{E} = \left\{\mathbf{x} \mid (\mathbf{x} - \mathbf{x}_c)^\top \mathbf{P}^{-1} (\mathbf{x} - \mathbf{x}_c) \le 1\right\}$ • $\mathcal{E} = \{\mathbf{x}_c + \mathbf{A}\mathbf{u} \mid \\|\mathbf{u}\\| \le 1\}$, where $\mathbf{A} = \mathbf{P}^{1/2}$.	• $\mathcal{E}$ is a finite set as long as $\mathbf{P}$ is a finite matrix. • $\mathbf{P}$ is symmetric and positive definite, that is, $\mathbf{P} = \mathbf{P}^\top \succ 0$. • $\mathbf{x}_c$ is the center of the ellipsoid. • The lengths of the semi-axes are given by $\sqrt{\lambda_i}$. • $\mathbf{A}$ is invertible. When it is not, we say that $\mathcal{E}$ is a degenerated ellipsoid (degenerated ellipsoids are also convex).
Norm cone: • $C = \left\{[x_1, x_2, \cdots, x_n, t]^\top \in \mathbb{R}^{n+1} \mid \mathbf{x} \in \mathbb{R}^n, \\|\mathbf{x}\\|_p \le t\right\} \subseteq \mathbb{R}^{n+1}$	• Although it is named “Norm cone”, it is a set, not a scalar. • The cone norm increases the dimension of $\mathbf{x}$ in 1. • For $p = 2$, it is called the second-order cone, quadratic cone, Lorentz cone or ice-cream cone.
Proper cone: $K \subset \mathbb{R}^n$ is a proper cone when it has the following properties • K is a convex cone, i.e., $\alpha K \equiv K, \alpha > 0$. • K is closed. • K is solid. • K is pointed, i.e., $-K \cap K = \{\mathbf{0}\}$.	• The proper cone K is used to define the *generalized inequality* (or *partial ordering*) in some set S. For the generalized inequality, one must define both the proper cone K and the set S. • $\mathbf{x} \preceq \mathbf{y} \iff \mathbf{y} - \mathbf{x} \in K$ for $\mathbf{x}, \mathbf{y} \in S$ (generalized inequality) • $\mathbf{x} \prec \mathbf{y} \iff \mathbf{y} - \mathbf{x} \in \text{int } K$ for $\mathbf{x}, \mathbf{y} \in S$ (strict generalized inequality). • There are two cases where K and S are understood from context and the subscript K is dropped out: ▹ When $S = \mathbb{R}^n$ and $K = \mathbb{R}^n_+$ (the nonnegative orthant). In this case, $\mathbf{x} \preceq \mathbf{y}$ means that $x_i \le y_i$. ▹ When $S = \mathcal{S}^n$ and $K = \mathcal{S}^n_+$ or $K = \mathcal{S}^n_{++}$, where $\mathcal{S}^n$ denotes the set of symmetric $n \times n$ matrices, $\mathcal{S}^n_+$ is the space of the positive semidefinite matrices, and $\mathcal{S}^n_{++}$ is the space of the positive definite matrices. $\mathcal{S}^n_+$ is a proper cone in $\mathcal{S}^n$ (??). In this case, the generalized inequality $\mathbf{Y} \succeq \mathbf{X}$ means that $\mathbf{Y} - \mathbf{X}$ is a positive semidefinite matrix belonging to the positive semidefinite cone $\mathcal{S}^n_+$ in the subspace of symmetric matrices $\mathcal{S}^n$. It is usual to denote $\mathbf{X} \succ \mathbf{0}$ and $\mathbf{X} \succeq \mathbf{0}$ to mean than $\mathbf{X}$ is a positive definite and semidefinite matrix, respectively, where $\mathbf{0} \in \mathbb{R}^{n \times n}$ is a zero matrix. • Another common usage is when $S = \mathbb{R}^n$ and $K = \left\{\mathbf{c} \in \mathbb{R}^n \mid c_1 + c_2 t + \cdots + c_n t^{n-1} \ge 0, \text{ for } 0 \le t \le 1\right\}$. In this case, $\mathbf{x} \preceq_K \mathbf{y}$ means that $x_1 + x_2 t + \cdots + x_n t^{n-1} \le y_1 + y_2 t + \cdots + y_n t^{n-1}$. • The generalized inequality has the following properties: ▹ If $\mathbf{x} \preceq_K \mathbf{y}$ and $\mathbf{u} \preceq_K \mathbf{v}$, then $\mathbf{x} + \mathbf{u} \preceq_k \mathbf{y} + \mathbf{v}$ (preserve under addition). ▹ If $\mathbf{x} \preceq_K \mathbf{y}$ and $\mathbf{y} \preceq_K \mathbf{z}$, then $\mathbf{x} \preceq_K \mathbf{z}$ (transitivity). ▹ If $\mathbf{x} \preceq_K \mathbf{y}$, then $\alpha \mathbf{x} \preceq_K \mathbf{y}$ for $\alpha \ge 0$ (preserve under nonnegative scaling). ▹ $\mathbf{x} \preceq_K \mathbf{x}$ (reflexivity). ▹ If $\mathbf{x} \preceq_K \mathbf{y}$ and $\mathbf{y} \preceq_K \mathbf{x}$, then $\mathbf{x} = \mathbf{y}$ (antisymmetric). ▹ If $\mathbf{x}_i \preceq_K \mathbf{y}_i$, for $i = 1, 2, \ldots$, and $\mathbf{x}_i \to \mathbf{x}$ and $\mathbf{y}_i \to \mathbf{y}$ as $i \to \infty$, then $\mathbf{x} \preceq_K \mathbf{y}$. • It is called partial ordering because $\mathbf{x} \npreceq_K \mathbf{y}$ and $\mathbf{y} \npreceq_K \mathbf{x}$ for many $\mathbf{x}, \mathbf{y} \in S$. When it happens, we say that $\mathbf{x}$ and $\mathbf{y}$ are not comparable (this case does not happen in ordinary inequality, $<$ and $>$). • $\mathbf{x} \in S$ is the *minimum* element of S if $\mathbf{x} \preceq_K \mathbf{y}$ for every $\mathbf{y} \in S$. The set does not necessarily have a minimum, but the minimum is unique if it does. The same is true for *maximum*. The mathematical notation for that is $S \subseteq \mathbf{x} + K$, where $\mathbf{x} + K$ denotes all points that are comparable to $\mathbf{x}$ and greater than or equal to $\mathbf{x}$ (for the maximum, we have $S \subseteq \mathbf{x} - K$). • $\mathbf{x} \in S$ is the *minimal* element of S if $\mathbf{y} \preceq_K \mathbf{x}$ only when $\mathbf{y} = \mathbf{x}$. The same is true for *maximal*. We can have many different minimal (maximal) elements. The mathematical notation for that is $(\mathbf{x} - K) \cap S = \{\mathbf{x}\}$, where $\mathbf{x} - K$ denotes all points that are comparable to $\mathbf{x}$ and less than or equal to $\mathbf{x}$ (for the maximal, we have $(\mathbf{x} + K) \cap S = \{\mathbf{x}\}$). • When $K = \mathbb{R}_+$ and $S = \mathbb{R}$ (ordinary inequality), the minimal is equal to the minimum and the maximal is equal to the maximum.
Dual cone: • $K^* = \left\{\mathbf{y} \mid \mathbf{x}^\top \mathbf{y} \ge 0, \ \forall \ \mathbf{x} \in K\right\}$	• K^* is a cone, and it is convex even when the original cone K is nonconvex. • K^* has the following properties: ▹ K^* is closed and convex. ▹ $K_1 \subseteq K_2$ implies $K_1^* \subseteq K_2^*$. ▹ If K has a nonempty interior, then K^* is pointed. ▹ If the closure of K is pointed then K^* has a nonempty interior. ▹ K^{**} is the closure of the convex hull of K. Hence, if K is convex and closed, $K^{**} = K$.
Polyhedra: • $\mathcal{P} = \left\{\mathbf{x} \mid \mathbf{a}_j^\top \mathbf{x} \le b_j, j = 1, \ldots, m, \mathbf{a}_j^\top \mathbf{x} = d_j, j = 1, \cdots, p\right\}$ • $\mathcal{P} = \{\mathbf{x} \mid \mathbf{A}\mathbf{x} \preceq \mathbf{b}, \mathbf{C}\mathbf{x} = \mathbf{d}\}$, where $\mathbf{A} = \begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \ldots & \mathbf{a}_m\end{bmatrix}^\top$ and $\mathbf{C} = \begin{bmatrix}\mathbf{c}_1 & \mathbf{c}_2 & \ldots & \mathbf{c}_m\end{bmatrix}^\top$	• The polyhedron may or may not be an infinite set. • Polyhedron is the result of the intersection of m halfspaces and p hyperplanes. • Subspaces, hyperplanes, lines, rays line segments, and halfspaces are all polyhedra. • The *nonnegative orthant*, $\mathbb{R}^n_+ = \{\mathbf{x} \in \mathbb{R}^n \mid x_i \le 0 \text{ for } i = 1, \ldots n\} = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{I}\mathbf{x} \succeq \mathbf{0}\}$, is a special polyhedron.
Simplex: • $\mathcal{S} = \text{conv } \{\mathbf{v}_m\}_{m=0}^{k} = \left\{\sum_{i=0}^{k} \theta_i \mathbf{v}_i \mid \mathbf{0} \preceq \boldsymbol{\theta} \preceq \mathbf{1}, \mathbf{1}^\top \boldsymbol{\theta} = 1\right\}$ • $\mathcal{S} = \{\mathbf{x} \mid \mathbf{x} = \mathbf{v}_0 + \mathbf{V}\boldsymbol{\theta}\}$, where $\mathbf{V} = \begin{bmatrix}\mathbf{v}_1 - \mathbf{v}_0 & \ldots & \mathbf{v}_n - \mathbf{v}_0\end{bmatrix} \in \mathbb{R}^{n \times k}$ • $\mathcal{S} = \{\mathbf{x} \mid \underbrace{\mathbf{A}_1 \mathbf{x} \preceq \mathbf{A}_1 \mathbf{v}_0, \mathbf{1}^\top \mathbf{A}_1 \mathbf{x} \le 1 + \mathbf{1}^\top \mathbf{A}_1 \mathbf{v}_0}_{\text{Linear inequalities in } x}, \underbrace{\mathbf{A}_2 \mathbf{x} = \mathbf{A}_2 \mathbf{v}_0}_{\text{Linear equalities in } x} \}$ (Polyhedra form), where $\mathbf{A} = \begin{bmatrix}\mathbf{A}_1 \\ \mathbf{A}_2\end{bmatrix}$ and $\mathbf{A}\mathbf{V} = \begin{bmatrix}\mathbf{I}_{k \times k} \\ \mathbf{0}_{n-k \times n-k}\end{bmatrix}$	• Simplexes are a subfamily of the polyhedra set. • Also called k-dimensional Simplex in $\mathbb{R}^n$. • The set $\{\mathbf{v}_m\}_{m=0}^{k}$ is a affinely independent, which means $\{\mathbf{v}_1 - \mathbf{v}_0, \ldots, \mathbf{v}_k - \mathbf{v}_0\}$ are linearly independent. • $\mathbf{V} \in \mathbb{R}^{n \times k}$ is a full-rank tall matrix, i.e., $\text{rank}(\mathbf{V}) = k$. All its column vectors are independent. The matrix $\mathbf{A}$ is its left pseudoinverse.

Functions (or operators) and their implications regarding convexity

Function	Convex?	Comments
Union: $C = A \cup B$	Not in most of the cases.	
Intersection: $C = A \cap B$	Yes, if A and B are convex sets.	
Convex function: $f : \mathbb{R}^n \to \mathbb{R}$ • $f(\theta \mathbf{x} + (1 - \theta)\mathbf{y}) \le \theta f(\mathbf{x}) + (1 - \theta) f(\mathbf{y})$, where $0 \le \theta \le 1$. • dom $f \subseteq \mathbb{R}^n$ shall be a convex set to f be a convex function.	Yes.	• Affine (and therefore also linear) functions are examples of convex functions. • Graphically, the line segment between $(\mathbf{x}, f(\mathbf{x}))$ and $(\mathbf{y}, f(\mathbf{y}))$ lies always above the graph f. • In terms of sets, a function is convex iff a line segment within dom f, which is a convex set, gives an image set that is also convex. • dom$f$ is convex iff all points for any line segment within dom f belong to it. • *First-order condition*: f is convex iff dom f is convex and $f(\mathbf{y}) \ge f(\mathbf{x}) + \nabla f(\mathbf{x})^\top (\mathbf{y} - \mathbf{x}), \forall \ \mathbf{x}, \mathbf{y} \in \text{dom } f, \mathbf{x} \neq \mathbf{y}$, being $\nabla f(\mathbf{x})$ the gradient vector. This inequation says that the first-order Taylor approximation is a *underestimator* for convex functions. The first-order condition requires that f is differentiable. • If $\nabla f(\mathbf{x}) = \mathbf{0}$, then $f(\mathbf{y}) \ge f(\mathbf{x}), \forall \ \mathbf{y} \in \text{dom } f$ and $\mathbf{x}$ is a global minimum. • *Second-order condition*: f is convex iff dom f is convex and $\mathbf{H} \succeq \mathbf{0}$, that is, the Hessian matrix $\mathbf{H}$ is a positive semidefinite matrix. It means that the graphic of the curvature has a positive (upward) curvature at $\mathbf{x}$. It is important to note that, if $\mathbf{H} \succ \mathbf{0}, \forall \ \mathbf{x} \in \text{dom } f$, then f is strictly convex. But is f is strictly convex, not necessarily that $\mathbf{H} \succ \mathbf{0}, \forall \ \mathbf{x} \in \text{dom } f$. Therefore, the strict convexity can only be partially characterized.
Affine function $f : \mathbb{R}^n \to \mathbb{R}^m$ • $f(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}$, where $\mathbf{A} \in \mathbb{R}^{m \times n}, \mathbf{b} \in \mathbb{R}^m, \mathbf{x} \in \mathbb{R}^n$	Yes, if the domain $S \subseteq \mathbb{R}^n$ is a convex set, then its image $f(S) = \{f(\mathbf{x}) \mid \mathbf{x} \in S\} \subseteq \mathbb{R}^m$ is also convex.	• The affine function, $f(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}$, is a broader category that encompasses the linear function, $f(\mathbf{x}) = \mathbf{A}\mathbf{x}$. The linear function has its origin fixed at $\mathbf{0}$ after the transformation, whereas the affine function does not necessarily have it (when not, this makes the affine function nonlinear). Graphically, we can think of an affine function as a linear transformation plus a shift from the origin of $\mathbf{b}$. • Similarly, the inverse image of C, $f^{-1}(C) = \{\mathbf{x} \mid f(\mathbf{x}) \in C\}$, is also convex. • The *linear matrix inequality* (LMI), $\mathbf{A}(\mathbf{x}) = x_1 \mathbf{A}_1 + \cdots + x_n \mathbf{A}_n \preceq \mathbf{B}$, is a special case of affine function. In other words, $f(S) = \{\mathbf{x} \mid \mathbf{A}(\mathbf{x}) \preceq \mathbf{B}\}$ is a convex set if S is convex. Many optimization problems can be formulated as LMI problems and solved optimally.
Perspective function $f : \mathbb{R}^{n+1} \to \mathbb{R}^n$ • $f(\mathbf{x}, t) = \mathbf{x}/t$, where $\mathbf{x} \in \mathbb{R}^n, t \in \mathbb{R}$.	Yes, if the domain $S \subseteq \text{dom } f$ is a convex set, then its image $f(S) = \{f(\mathbf{x}) \mid \mathbf{x} \in S\} \subseteq \mathbb{R}^n$ is also convex.	• dom $f = \mathbb{R}^n \times \mathbb{R}_{++}$ • The perspective function decreases the dimension of the domain. Its effect is similar to the camera zoom. • The inverse image is also convex, that is, if $C \subseteq \mathbb{R}^n$ is convex, then $f^{-1}(C) = \left\{(\mathbf{x}, t) \in \mathbb{R}^{n+1} \mid \mathbf{x}/t \in C, t > 0\right\}$ is also convex.