

# Convex Optimization homework

Rubem Vasconcelos Pacelli

December 17, 2022

## Nonlinear Optimization Systems

Teleinformatics Engineering

Yuri Carvalho Barbosa Silva, Tarcisio Ferreira Maciel

**Problem 1** Convexity of some sets. Determine if each set below is convex

(a)  $\{(x, y) \in \mathbb{R}_{++}^2 \mid x/y \leq 1\}$

*Answer* The norm cone is given by

$$C = \{(x_1, x_2, \dots, x_n, t) \in \mathbb{R}^{n+1} \mid \mathbf{x} \in \mathbb{R}^n, \|\mathbf{x}\|_p \leq t\} \subseteq \mathbb{R}^{n+1}. \quad (1)$$

When  $n = 1$  and  $p = 1$  (Manhattan norm), we have that

$$C = \{(x, y) \in \mathbb{R}^2 \mid |x|/y \leq 1\} \subseteq \mathbb{R}^2, \quad (2)$$

where  $t = y$  and  $x_1 = x$ . From the Equation (2), it is easy to conclude that  $y > 0$ . Let us further define the function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ , given by

$$f(x, y) = [|x| \quad y]^\top. \quad (3)$$

This function is clearly convex since the absolute operation is convex. Once it is well-known that the norm cone is convex, and  $f$  is a convex function in  $C \subseteq \text{dom}(f)$ , then

$$S = f(C) = \{(x, y) \in \mathbb{R}_{++}^2 \mid x/y \leq 1\} \quad (4)$$

is also convex, which is the set of the question. The Figure 1 shows this set for  $0 \leq x \leq 3$  and  $0 \leq y \leq 3$ .

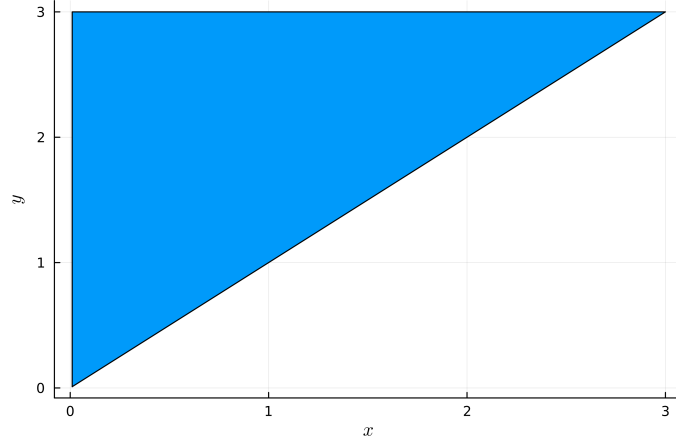


Figure 1: Set of the item a.

$$(b) \quad \{(x, y) \in \mathbb{R}_{++}^2 \mid x/y \geq 1\}$$

*Answer*

The set  $S = \{\mathbf{v} \in \mathbb{R}_{++}^2 \mid v_0/v_1 \geq 1\}$ , where  $\mathbf{v} = (v_0, v_1)$ , is convex iff the convex combination of a pair of points belonging to  $S$ , e.g.,  $\mathbf{x} = (x_0, x_1)$  and  $\mathbf{y} = (y_0, y_1)$ , also belong to  $S$ . Mathematically,

$$\mathbf{w} = \theta \mathbf{x} + (1 - \theta) \mathbf{y} = \begin{bmatrix} \theta x_0 + (1 - \theta) y_0 \\ \theta x_1 + (1 - \theta) y_1 \end{bmatrix} \quad (5)$$

should belong to  $S$  to any  $\mathbf{x}, \mathbf{y} \in S$  and  $0 \leq \theta \leq 1$ . Therefore

$$\frac{\theta x_0 + (1 - \theta) y_0}{\theta x_1 + (1 - \theta) y_1} \geq 1 \quad (6)$$

$$\theta x_0 + (1 - \theta) y_0 \geq \theta x_1 + (1 - \theta) y_1 \quad (7)$$

$$\frac{x_0}{x_1} \frac{\theta}{y_1} + \frac{y_0}{y_1} \frac{1 - \theta}{x_1} \geq \frac{\theta}{y_1} + \frac{1 - \theta}{x_1} \quad (8)$$

This expression always holds for any value of  $\mathbf{x}, \mathbf{y} \in S$ . Then,  $\mathbf{w} \in S$ . The Figure 2 shows this set for  $[0, 3]$ .

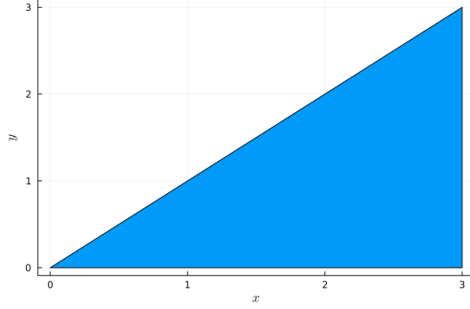


Figure 2: Set of the item b.

$$(c) \quad \{(x, y) \in \mathbb{R}_+^2 \mid xy \leq 1\}$$

*Answer*

If  $S = \{\mathbf{v} \in \mathbb{R}_+^2 \mid v_0 v_1 \leq 1\}$ , where  $\mathbf{v} = (v_0, v_1)$ , then  $S$  is convex iff

$$\mathbf{w} = \theta \mathbf{x} + (1 - \theta) \mathbf{y} \in S \quad \forall \mathbf{x}, \mathbf{y} \in S, 0 \leq \theta \leq 1, \quad (9)$$

which is the convex combination of  $\mathbf{x}$  and  $\mathbf{y}$ . Let us prove that this set is nonconvex by contradiction. Once  $\mathbf{x}, \mathbf{y} \in S$ , we have that

$$x_0 x_1 \leq 1 \quad (10)$$

and

$$y_0 y_1 \leq 1, \quad (11)$$

where  $\mathbf{x} = (x_0, x_1)$ ,  $\mathbf{y} = (y_0, y_1)$ . The second component of  $\mathbf{w} = (w_0, w_1)$  is given by

$$w_1 = \theta x_1 + (1 - \theta) y_1 \quad (12)$$

$$(13)$$

If  $x_1 \gg 0$  and  $y_0 \gg 0$ , which leads to  $x_0 \rightarrow 0$  and  $y_1 \rightarrow 0$ , respectively, and  $\theta = 0.5$ , then, from (12), we have that

$$w_1 \approx 0.5 x_1. \quad (14)$$

Since  $x_1$  can be indiscriminately large, we have that  $w_1 \gg 0$ , which leads to  $w_0 \rightarrow 0$  if  $\mathbf{w} \in S$ . On the other hand, the first component of  $w_0$  is given by

$$w_0 = \theta x_0 + (1 - \theta) y_0 \quad (15)$$

$$w_0 = 0.5 y_0 \quad (16)$$

Since  $y_0$  can be indiscriminately large, we have that  $w_0 \gg 0$ . However, the statement  $w_0 \rightarrow 0$  should be true in order to  $\mathbf{w}$  belong to  $S$ . The contradiction leads us to conclude that  $\mathbf{w} \notin S$ . The Figure 3 shows the set  $S$ , and the Figure 4 shows the cone set  $K$  in which  $\mathbf{x} \leq_K \mathbf{w} \leq_K \mathbf{y}$ .

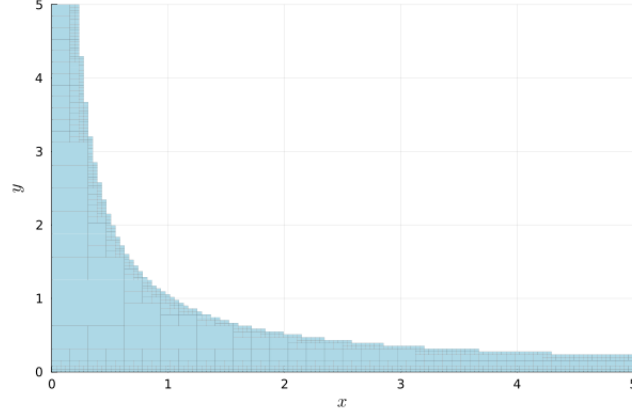


Figure 3: Set of the item c.

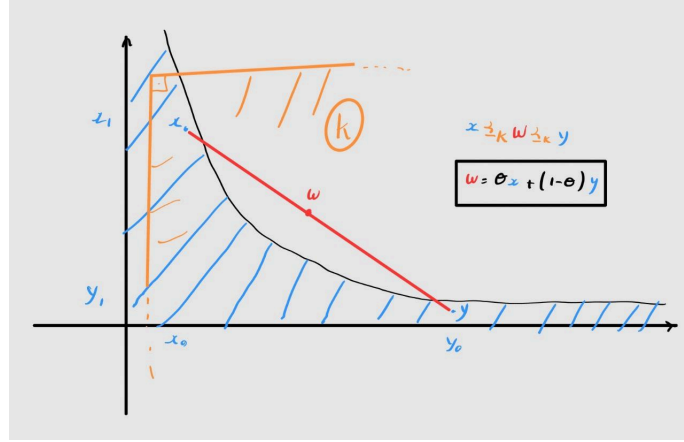


Figure 4: Cone set in which  $\mathbf{x} \leq_K \mathbf{w} \leq_K \mathbf{y}$ .

$$(d) \quad \{(x, y) \in \mathbb{R}_+^2 \mid xy \geq 1\}$$

By using the convex combination, the vector  $\mathbf{w} = (w_0, w_1)$ , which is given by

$$\mathbf{w} = \theta \mathbf{x} + (1 - \theta) \mathbf{y} = \begin{bmatrix} \theta x_0 + (1 - \theta) y_0 \\ \theta x_1 + (1 - \theta) y_1 \end{bmatrix}, \quad (17)$$

shall belong to  $S$ , for any  $\mathbf{x}, \mathbf{y} \in S$ . If it is true, then

$$(\theta x_0 + (1 - \theta)y_0)(\theta x_1 + (1 - \theta)y_1) \geq 1 \quad (18)$$

$$\theta^2 x_0 x_1 + \theta(1 - \theta)y_1 x_0 + (1 - \theta)\theta x_1 y_0 + (1 - \theta)^2 y_0 y_1 \geq 1 \quad (19)$$

$$\theta^2 x_0 x_1 + \theta(1 - \theta)(y_1 x_0 + x_1 y_0) + (1 - \theta)^2 y_0 y_1 \geq 1 \quad (20)$$

$$(x_0 x_1 - y_1 x_0 - x_1 y_0 + y_0 y_1)\theta^2 + (y_1 x_0 + x_1 y_0 - 2y_0 y_1)\theta + y_0 y_1 - 1 \geq 0 \quad (21)$$

$$f(\theta) \geq 0 \quad (22)$$

Note that  $f$  is a second-order function. The previous inequation holds if  $f$  has either no roots or only one root, that is

$$(y_1 x_0 + x_1 y_0 - 2y_0 y_1)^2 - 4(x_0 x_1 - y_1 x_0 - x_1 y_0 + y_0 y_1)(y_0 y_1 - 1) \geq 0 \quad (23)$$

$$(y_1 x_0 + x_1 y_0 - 2y_0 y_1)^2 \geq 4(x_0 x_1 - y_1 x_0 - x_1 y_0 + y_0 y_1)(y_0 y_1 - 1) \quad (24)$$

Let us analyze carefully this inequation

1. If the RHS is greater than the LHS, this inequation is not satisfied. Hence, we must assess it for the worst case, where the LHS is as high as possible, and see whether this inequation holds.
2. The worst case is when  $x_0 \gg 0$  and  $x_1 \gg 0$ , and  $y_0 = y_1 = 1$  because the term  $x_0 x_1$  cannot be quadratically compensated with  $(x_0 x_1)^2$  on the LHS. Otherwise, the higher the values of  $y_0$  and  $y_1$ , the lower the RHS becomes<sup>1</sup>.
3. However, even for this case, the LHS becomes approximately equal to  $(x_0 + x_1)^2 = x_0^2 + 2x_0 x_1 + x_1^2$ , which is greater than  $x_0 x_1$

Therefore, the function  $f$  has no roots and the set  $\{(x, y) \in \mathbb{R}_+^2 \mid xy \geq 1\}$  is convex. The Figure 5 shows this set.

---

<sup>1</sup>The quadratic term  $4y_0^2 y_1^2$  on the RHS that could increase it is canceled with the very same term  $4y_0^2 y_1^2$  on the LHS.

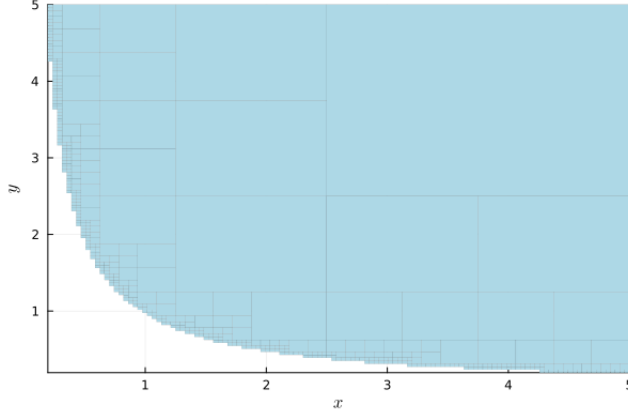


Figure 5: Set of the item d.

### Problem 2

Let  $S = \{\mathbf{\alpha} \in \mathbb{R}^3 | \alpha_1 + \alpha_2 e^{-t} + \alpha_3 e^{-2t} \leq 1.1 \text{ for } t \geq 1\}$ . Is  $S$  affine, a half-space, a convex cone, a convex set, or none of these?

The set  $S$  is affine iff its affine combination is also affine, that is,

$$\mathbf{w} = \theta_1 \mathbf{x} + \theta_2 \mathbf{y} = \begin{bmatrix} \theta_1 x_1 + \theta_2 y_1 \\ \theta_1 x_2 + \theta_2 y_2 \\ \theta_1 x_3 + \theta_2 y_3 \end{bmatrix} \in S \quad \forall \mathbf{x}, \mathbf{y} \in S, \mathbf{1}^\top \boldsymbol{\theta} = 1, \quad (25)$$

where  $\boldsymbol{\theta} = (\theta_1, \theta_2)$ ,  $\mathbf{x} = (x_1, x_2, x_3)$ , and  $\mathbf{y} = (y_1, y_2, y_3)$ . If  $\mathbf{w} \in S$ , then

$$\begin{aligned} \theta_1 x_1 + \theta_2 y_1 + (\theta_1 x_2 + \theta_2 y_2) e^{-t} + (\theta_1 x_3 + \theta_2 y_3) e^{-2t} &\leq 1.1 \\ \theta_1 (x_1 + x_2 e^{-t} + x_3 e^{-2t}) + \theta_2 (y_1 + y_2 e^{-t} + y_3 e^{-2t}) &\leq 1.1 \end{aligned} \quad (26)$$

Once  $\mathbf{x}, \mathbf{y} \in S$ ,  $x_1 + x_2 e^{-t} + x_3 e^{-2t} \leq 1.1$  and  $y_1 + y_2 e^{-t} + y_3 e^{-2t} \leq 1.1$ , the worst case when this inequation might not be satisfied is then  $x_1 + x_2 e^{-t} + x_3 e^{-2t} = y_1 + y_2 e^{-t} + y_3 e^{-2t} = 1.1$ . In this case, we have

$$1.1(\theta_1 + \theta_2) \leq 1.1 \quad (27)$$

$$1.1 \leq 1.1. \quad (28)$$

As the inequation holds,  $S$  is affine, which is consequently convex. The set  $S$  is a cone set iff the result of the conic combination also belongs to  $S$ , that is,

$$\mathbf{w} = \theta_1 \mathbf{x} + \theta_2 \mathbf{y} = \begin{bmatrix} \theta_1 x_1 + \theta_2 y_1 \\ \theta_1 x_2 + \theta_2 y_2 \\ \theta_1 x_3 + \theta_2 y_3 \end{bmatrix} \in S \quad \forall \mathbf{x}, \mathbf{y} \in S, \boldsymbol{\theta} \geq \mathbf{0}. \quad (29)$$

Recalling the Equation (26), we can clearly see that this inequation does not hold for  $\theta_1 \gg 0$ ,  $\theta_2 \gg 0$ , and  $x_1 + x_2 e^{-t} + x_3 e^{-2t} = y_1 + y_2 e^{-t} + y_3 e^{-2t} = 1.1$ . Thus, the set  $S$  is not a cone set. This set might be a close halfspace though, once halfspaces is a set of the form

$$\{\mathbf{x} \mid \mathbf{a}^\top \mathbf{x} \leq b\}. \quad (30)$$

Considering that  $t \geq 1$  is a constant, we can rewrite  $S$  as

$$S = \{\boldsymbol{\alpha} \in \mathbb{R}^3 \mid \mathbf{a}^\top \boldsymbol{\alpha} \leq b\}, \quad (31)$$

where  $\mathbf{a} = (1, e^{-t}, e^{-2t})$  and  $b = 1.1$ . Therefore,  $S$  is a halfspace.

### Problem 3

(a) Explain why  $t - (1/t)\mathbf{u}^\top \mathbf{u}$  is a concave function on  $\text{dom}(f)$ . Hint. Use convexity of the quadratic-over-linear function.

*Answer* The function  $s_1(\mathbf{u}, t) = (1/t)\mathbf{u}^\top \mathbf{u} = \|\mathbf{u}\|^2/t$  is a quadratic-over-linear function, which is known to be convex. The function  $s_2(\mathbf{u}, t) = t - s_1(\mathbf{u}, t)$  is a linear minus a convex function, which yields a concave function.

(b) From this, show that  $-\log(t - (1/t)\mathbf{u}^\top \mathbf{u})$  is a convex function on  $\text{dom} f$ .

*Answer* Let us the define the function  $s_4(\mathbf{u}, t) = s_3 \circ s_2 = \log(s_2(\mathbf{u}, t))$ , where  $s_3(x) = \log(x)$ . Note that, since  $t > \|\mathbf{u}\|$ ,  $\text{dom}(s_2) = \mathbb{R}^n \times \mathbb{R}_{++}$  and  $s_3$  is concave and nondecreasing for this interval. As  $s_2$  is convex, we conclude by composition property that  $s_4$  is concave, thus  $s_5 = -s_4$  is convex.

(c) From this, show that  $f$  is convex.

*Answer*

$$\begin{aligned} f(\mathbf{x}, t) &= -\log(t^2 - \mathbf{x}^\top \mathbf{x}) \\ &= -\log(t(t - \mathbf{x}^\top \mathbf{x}/t)) \\ &= -\log(t) - \log(t - \mathbf{x}^\top \mathbf{x}/t) \\ &= s_6(t) + s_5(\mathbf{u}, t). \end{aligned} \quad (32)$$

We have shown that  $s_5(\mathbf{x}, t) = -\log(t - \mathbf{x}^\top \mathbf{x}/t)$  is convex. Since  $s_6(t) = -\log(t)$  is also convex on  $\text{dom}(f)$ ,  $f = s_6 + s_5$  is convex as well.

**Problem 4** Square and reciprocal of convex and concave functions. For each of the following, determine if the function  $f$  is convex, concave, or neither.

(a)  $f(x) = g^2(x)$

Since  $s(x) = x^2$  is convex and nondecreasing when  $g$  is nonnegative, and  $g$  is convex, then  $f = s \circ g$  is convex as well.

$$(b) \quad f(x) = 1/g(x)$$

Since  $s(x) = 1/x$  is convex and nonincreasing when  $g$  is positive, and  $g$  is concave, then  $f = s \circ g$  is concave.

**Problem 5**

Let us define  $\mathbf{x}(k) \in \mathbb{R}^n$  and  $u(k) \in \mathbb{R}$ , for  $k = 0, \dots, N-1$ , as the state of a linear dynamical system and its input signal, respectively, being  $n = 3$  and  $N = 30$ . The system dynamic is governed by the following equation:

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{b}u(k), \quad (33)$$

where

$$\mathbf{A} = \begin{bmatrix} -1 & 0.4 & 0.8 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad (34)$$

and

$$\mathbf{b} = \begin{bmatrix} 7 \\ 2 \\ -6 \end{bmatrix} \quad (35)$$

Assuming that  $\mathbf{x}(0) = 0$ , solve the following optimization problem:

$$\text{Minimize} \quad \sum_{k=0}^{N-1} f(u(k)) \quad (36)$$

$$\text{Subject to} \quad \mathbf{x}(N) = \mathbf{x}_{des}, \quad (37)$$

where

$$\mathbf{x}_{des} = \begin{bmatrix} 7 \\ 2 \\ -6 \end{bmatrix} \quad (38)$$

and

$$f(a) = \begin{cases} |a| & |a| \leq 1 \\ 2|a| - 1 & |a| > 1 \end{cases}, \quad (39)$$

which is equivalent to  $f(a) = \max \{|a|, 2|a| - 1\}$ .

*Answer*



---

```

1  using Convex, SCS, Plots, LaTeXStrings
2
3  N = 30 # N in the set {0, 1, ..., N}
4  n = 3 # order of the linear dynamical system
5  x_des = [7, 2, -6] # constraint -> x(N) == x_des
6  # model parameters
7  A = [-1 .4 .8; 1 0 0; 0 1 0]
8  b = [1, 0, 0.3]
9
10 X = Variable(n, N+1) # [x(0) x(1) ... x(N)]
11 u = Variable(1, N) # [u(0) u(1) ... u(N-1)]
12 f0 = sum(max(abs(u), 2abs(u)-1)) # objective function
13 constraints = [
14     X[:,2:N+1] == A*X[:,1:N]+b*u, # recursive equation
15     X[:,1] == zeros(n), # initial condition
16     X[:,N+1] == x_des, # final condition
17 ]
18 problem = minimize(f0, constraints)
19 solve!(problem, SCS.Optimizer; silent_solver = true)
20
21 fig = plot(vec(u.value), xlabel=L"k", title=L"u(k)", seriestype=:sticks,
22 ↪ markershape=:circle, label="")
23 savefig(fig, "figs/4.17.png")

```

---

The optimal solution of the actuator signal is shown in Figure 6.

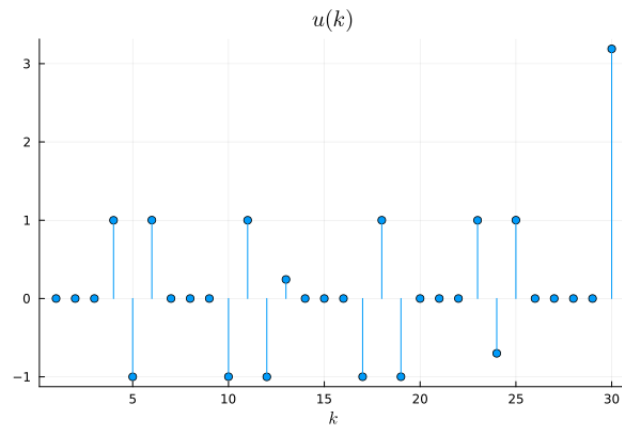


Figure 6: Optimum solution of the actuator signal.

**Problem 6** Consider the optimization problem

$$\text{minimize } x_1^2 \tag{40}$$

$$\text{subject to } x_1 \leq 1, \tag{41}$$

$$x_1^2 + x_2^2 \leq 2 \tag{42}$$

with  $\mathbf{x} = (x_1, x_2)$ . Determine whether each of the following statements is true or false.

(a) The point  $(-1, 1)$  is a solution.

Since  $(-1, 1) \in \mathcal{S}$ , where  $\mathcal{S}$  is the feasible set,  $(-1, 1)$  is a possible solution for the problem.

(b) The optimal value is 1.

(c) The problem is convex.

(d) The problem has multiple solutions.