Galileo Masterclass Brazil (GMB) 2022

Lecture 2 - Spread Spectrum Ranging

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Outline

Time-Delay Estimation

Signal Properties

Examples for Signal Design

Maximum Likelihood Time-Delay Estimation (1)

For

$$\mathbf{x} = \mathbf{x}[0]$$

let us assume a random variable $\boldsymbol{x}$ has a multivariate Gaussian probability density function (pdf) parameterized by the parameter τ and thus we get

$$\mathrm{p}_{\boldsymbol{x}}(\mathbf{x};\tau) = \frac{1}{(2\pi\sigma_n^2)^{N/2}} \exp\left[-\frac{\|\mathbf{x}-\sqrt{P}\,\mathbf{c}(\tau)\|_2^2}{2\sigma_n^2}\right]$$

The likelihood function with respect to the parameter τ is given as

$$L(\mathbf{x};\tau) = \mathrm{p}_{\boldsymbol{x}}(\mathbf{x};\tau)$$

- $L(\mathbf{x};\tau)$ is a function of the parameter τ, which is to be estimated at a given realization of the random variable $\boldsymbol{x}$
- The pdf $\mathrm{p}_{\boldsymbol{x}}(\mathbf{x};\tau)$ is a function of the realization of the random variable $\boldsymbol{x}$ for a fixed value of the parameter τ

Maximum Likelihood Time-Delay Estimation (2)

Now the maximum likelihood estimator (MLE) can be given as

$$\hat{\tau} = \arg\max_{\tau} \{L(\mathbf{x};\tau)\} = \arg\max_{\tau} \{\log(L(\mathbf{x};\tau))\}.$$

The MLE is asymptotically (large N) unbiased and efficient. When further deriving the estimator we get

$$\begin{aligned} \hat{\tau} &= \arg\max_{\tau} \{\log(L(\mathbf{x};\tau))\} \\ &= \arg\max_{\tau} \left\{\log(1) - \log\left((2\pi\sigma_n^2)^{N/2}\right) - \frac{1}{2\sigma_n^2}\|\mathbf{x} - \sqrt{P}\mathbf{c}(\tau)\|_2^2\right\} \\ &= \arg\max_{\tau} \left\{-\|\mathbf{x}\|_2^2 + 2\sqrt{P}\,\mathbf{x}^{\mathrm{T}}\mathbf{c}(\tau) - P\,\|\mathbf{c}(\tau)\|_2^2\right\} \end{aligned}$$

As the first term does not depend on τ and the third term is constant with $\|\mathbf{c}(\tau)\|_2^2 \approx N, \forall_\tau$ as well as dropping the constant factor $2\sqrt{P}$ we can write

$$\hat{\tau} = \arg\max_{\tau} \{\mathbf{x}^{\mathrm{T}}\mathbf{c}(\tau)\} = \arg\max_{\tau} \{J(\tau)\}$$

Time-Delay Estimation with a Delay Locked Loop (DLL) (1)

In practice, time-delay estimation is performed using a DLL applying a gradient ascent method (step size $\mu > 0$) where the kth iteration can be given as

$$\hat{\tau}[k] = \hat{\tau}[k-1] + \mu \frac{\partial J(\hat{\tau}[k-1])}{\partial \tau}$$

The derivative can be approximated using the central difference quotient of length 2Δ,

$$\begin{aligned}
\hat{\tau}[k] &= \hat{\tau}[k-1] + \frac{\mu}{2\Delta}\left(J(\hat{\tau}[k-1]+\Delta) - J(\hat{\tau}[k-1]-\Delta)\right) \\
&= \hat{\tau}[k-1] + \frac{\mu}{2\Delta}\left(\mathbf{x}^{\mathrm{T}}\mathbf{c}(\hat{\tau}[k-1]+\Delta) - \mathbf{x}^{\mathrm{T}}\mathbf{c}(\hat{\tau}[k-1]-\Delta)\right)
\end{aligned}$$

A stochastic version (considering successive periods k) can be given as

$$\begin{aligned}
\hat{\tau}[k] &= \hat{\tau}[k-1] + \frac{\mu}{2\Delta}\left(\mathbf{x}^{\mathrm{T}}[k-1]\mathbf{c}(\hat{\tau}[k-1]+\Delta)\right. \\
&\qquad \left. - \mathbf{x}^{\mathrm{T}}[k-1]\mathbf{c}(\hat{\tau}[k-1]-\Delta)\right)
\end{aligned}$$

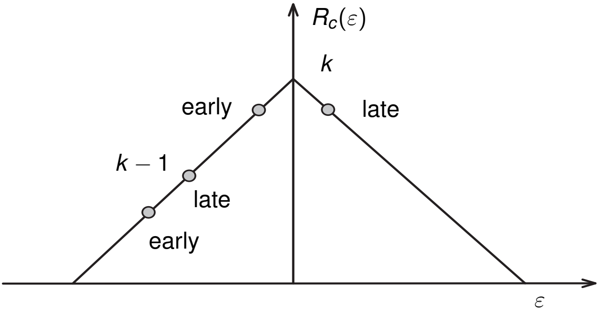
Time-Delay Estimation with a Delay Locked Loop (DLL) (2)

Without noise, assuming that the receiver uses signal matched correlators, and the time-delay tracking error

$$\varepsilon = \tau - \hat{\tau}$$

the discriminator S-curve for a coherent early-late DLL is

$$S(\varepsilon) = R_c(\varepsilon - \Delta) - R_c(\varepsilon + \Delta)$$
$$S[k; \varepsilon[k]] = R_c[\varepsilon[k] - \Delta] - R_c[\varepsilon[k] + \Delta].$$



Cramer-Rao Lower Bound (CRLB) (1)

If

$$\mathrm{E}\left[\frac{\partial \log\left(L(\mathbf{x};\tau)\right)}{\partial \tau}\right] = 0$$

the variance of the time-delay estimation error σ_τ^2 of any unbiased estimator is lower bounded by the Cramer Rao lower bound (CRLB)

$$\mathrm{var}(\hat{\tau}) = \sigma_{\hat{\tau}}^2 \geq \frac{1}{-\mathrm{E}\left[\frac{\partial^2 \log(L(\mathbf{x};\tau))}{\partial \tau^2}\right]} = \frac{\frac{\sigma_n^2}{P}}{\frac{\partial \mathbf{c}^{\mathrm{T}}(\tau)}{\partial \tau}\frac{\partial \mathbf{c}(\tau)}{\partial \tau}}$$

which leads to

$$\sigma_{\hat{\tau}}^2 \geq \frac{B_n}{\frac{P}{N_0}\, 4\pi^2} \frac{1}{\int_{-\infty}^{\infty} f^2\, |P(f)|^2\, df}$$

where B_n is the equivalent noise bandwidth of the generic estimator and

$$\int_{-\infty}^{\infty} |P(f)|^2\, df = 1$$

Cramer-Rao Lower Bound (CRLB) (2)

The term $\int_{-\infty}^{\infty} f^2 \left|P(f)\right|^2 df$ is:

- Second moment of the power spectrum
- Root mean square (RMS) bandwidth
- Gabor bandwidth

It is equal to the curvature of $R_c(\varepsilon)$ at $\varepsilon = 0$

$$\int_{-\infty}^{\infty} f^2 \left|P(f)\right|^2 df = -\frac{1}{4\pi^2} \left.\frac{d^2\ R_c(\varepsilon)}{d\ \varepsilon^2}\right|_{\varepsilon=0}$$

and if $p(t)$ is band-limited to $[-B, B]$ it is upper bounded by

$$\int_{-B}^{B} f^2 \left|P(f)\right|^2 df \leq \int_{-B}^{B} B^2 \left|P(f)\right|^2 df \leq B^2 \ \text{ s.t. } \int_{-B}^{B} \left|P(f)\right|^2 df = 1$$

Thus, $\left|P(f)\right|^2 = \frac{1}{2}\left(\delta(f-B) + \delta(f+B)\right)$ maximizes the Gabor bandwidth and $p(t) = \cos(2\pi Bt)$ or $p(t) = \sin(2\pi Bt)$.

Outline

Synchronization Accuracy

- The higher the Gabor bandwidth of the signal, the higher the synchronization accuracy that can be achieved in terms of the CRLB
- The second moment of the power spectrum of a signal with bandwidth B is upper bounded by B^2 $\Rightarrow$ The higher the available signal bandwidth , the higher the possible synchronization accuracy
- A high processing gain G is desireable (large bandwidth B) $\Rightarrow$ high synchronization accuracy, high interference robustness, and low MAI-A
- Minimizing the CRLB for τ and maximizing time concentration are contradictive tasks

We can state:

The lower the CRLB for τ (CRLB $\rightarrow \frac{1}{B^2}$) $\Rightarrow$ the lower time concentration of $p(t)$ ($p(t) \rightarrow \sin(2B\pi t)$ or $\cos(2B\pi t)$)

Time Concentration (1)

Time and frequency concentration of $p(t)$ can be given by the quantities:

$$\alpha = \frac{\int_{-T_c/2}^{T_c/2} |p(t)|^2 dt}{\int_{-\infty}^{\infty} |p(t)|^2 dt} \qquad \beta = \frac{\int_{-B}^{B} |P(f)|^2 df}{\int_{-\infty}^{\infty} |P(f)|^2 df}$$

with

$$\int_{-\infty}^{\infty} |P(f)|^2 df = \int_{-\infty}^{\infty} |p(t)|^2 dt = 1$$

and $p(t)$ being strictly band-limited to $[-B, B] \Rightarrow \beta = 1$, and (uncertainty principle of Fourier transform) we get

$$\int_{-T_c/2}^{T_c/2} |p(t)|^2 dt = \alpha < 1$$

Time Concentration (2)

The maximum of the absolute value of the sidelobes of $R_c(\varepsilon)$ is given as

$$\underset{i \in \mathbb{N}}{\forall} |\nu_i| \leq \kappa, \quad \kappa \in [0, 1]$$

ν_i denote the value of $R_c(\varepsilon)$ at the sidelobes, besides the global maximum of $R_c(\varepsilon)$ at $\varepsilon = 0$.

We can state:

1. The higher the sidelobes of $R_c(\varepsilon) \Rightarrow$ the higher the sidelobes of $p(t) \Rightarrow$ the lower time concentration of $p(t)$
2. The higher the sidelobes of $R_c(\varepsilon) \Rightarrow$ the higher $\kappa \Rightarrow$ the less robust the estimation of τ (likelihood has local maxima besides the global maximum)

Multiple Access Interference (1)

- MAI-A and MAI-R can be considered as additional interference components with zero mean
- In general both MAI-A and MAI-R are dependent on the propagation characteristics of the transmitted signal
- U users (*e.g.* visible GNSS satellites) with $u = 1, \ldots, U$ and power P_u causing MAI-A
- V users of another system (*e.g.* visible satellites of a different GNSS) with $v = 1, \ldots, V$ and power P_v are causing MAI-R
- The received signal of another system (*e.g.* different GNSS) in the same frequency band has PSD $\Phi_R(f)$
- The reference PR sequence generator is perfectly synchronized with the received desired signal with power P, so the time-delay τ of the desired signal is known
- The receiver was able to perform down conversion, matched filtering with $P^*(f)$, and sampling at the chip duration

Multiple Access Interference (2)

We can define the statistics of the matched filter output for a WSCS sequence $\{d_m\} \in \{-1, 1\}$ with period $T_d = N_d T_c$ as

$$\mathrm{SINR} = \frac{P}{P_N + P_A + P_R}$$

where the noise power can be given as

$$P_N = \frac{1}{N_d T_c} \frac{N_0}{2} \int_{-\infty}^{\infty} |P(f)|^2 \, df = \frac{N_0}{2 N_d T_c} = \frac{N_0}{2 T_d}$$

the power of the MAI-A can be given as

$$P_A = \frac{1}{2 N_d T_c} \sum_{u=1}^{U} P_u \int_{-\infty}^{\infty} |P(f)|^4 \, df$$

and the power of the MAI-R can be given as

$$P_R = \frac{1}{2 N_d T_c} \sum_{v=1}^{V} P_v \int_{-\infty}^{\infty} |P(f)|^2 \, \Phi_R(f) \, df$$

Multiple Access Interference (3)

- The background noise is assumed as white Gaussian noise with spectral density of $N_0/2$
- The variance due to background noise can be considered as white noise of density $N_0/2$ filtered by the transfer function of the receive filter $P^*(f)$
- All u users (*e.g.* visible GNSS satellites) are assumed to be independent and unsynchronized with the desired signal
- It is assumed that their time-delays are independently uniformly distributed in $[0, T_c]$ and their phases are independently uniformly distributed in $[0, 2\pi]$
- The effect of the u-th user (*e.g.* GNSS satellite) on the matched filter output of the desired signal will be that of white noise passed through the tandem combination of two filters with the transfer functions $|P(f)|^2 \Rightarrow$ MAI-A
- Similar assumptions as for the u users above can be taken for the v users $\Rightarrow$ MAI-R

Multiple Access Interference (4)

The ratio of the SNR to the SINR can be given as

$$\begin{aligned}\Delta\mathrm{SNR} &= \frac{\mathrm{SNR}}{\mathrm{SINR}} = \frac{P/P_N}{P/(P_N+P_A+P_R)} = 1+\frac{P_A+P_R}{P_N} \\ &= 1+\sum_{u=1}^{U}\frac{P_u}{N_0}\int_{-B}^{B}|P(f)|^4 df+\sum_{v=1}^{V}\frac{P_v}{N_0}\int_{-\infty}^{\infty}|P(f)|^2\,\Phi_R(f)df\end{aligned}$$

The variance of the MAI-A component can be lower bounded by

$$\int_{-B}^{B}|P(f)|^4\,df \geq \frac{1}{2B}$$

with equality iff

$$|P(f)|^2 = \frac{1}{2B}\,,\ -B \leq f \leq B, \quad p(t) = \sqrt{2B}\,\frac{\sin(2\pi Bt)}{2\pi Bt}$$

Multiple Access Interference (5)

A proof can be given by using the Schwarz inequality:

$$\int_{-B}^{B} |X_1(f)|^2 \, df \int_{-B}^{B} |X_2(f)|^2 \, df \geq \left[\int_{-B}^{B} X_1(f) X_2(f) \, df \right]^2$$

Suppose that $X_1(f) = |P(f)|^2$, $X_2(f) = 1$ and $\int_{-B}^{B} |P(f)|^2 \, df = 1$, it follows that

$$\int_{-B}^{B} |P(f)|^4 \, df \cdot 2\,B \geq \left[\int_{-B}^{B} |P(f)|^2 \, df \right]^2 = 1$$

Thus we get,

$$\int_{-B}^{B} |P(f)|^4 \, df \geq \frac{1}{2B}$$

Multiple Access Interference (6)

- Assumptions for MAI-A and MAI-R are derived following IS-95 and CDMA2000 standards
- In GNSS MAI-A and MAI-R are called spectral separation
- The term $\int_{-\infty}^{\infty} |P(f)|^2 \; \Phi_R(f) df$ is called spectral separation coefficient (SSC)
- The term $\int_{-\infty}^{\infty} |P(f)|^4 df$ is called self SSC
- MAI-A and MAI-R has to be considered in signal design (chip pulse shape design) or can be treated in the receiver (multi-user detection and mitigation)

We define the CRLB-I as the CRLB which considers noise plus interference (MAI-A, MAI-R):

$$\tilde{\sigma}_{\hat{\tau}}^2 \geq \sigma_{\hat{\tau}}^2 \cdot \Delta\mathrm{SNR}$$

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Time-Delay Estimation

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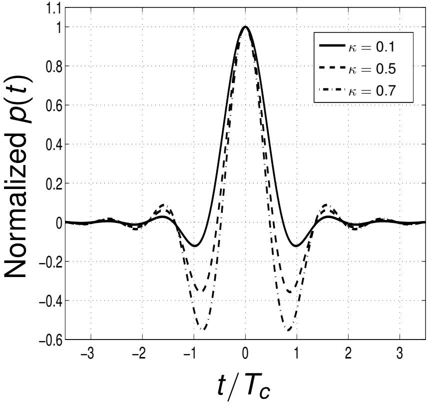
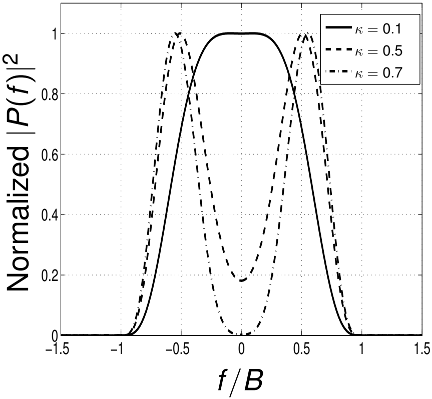
Example Pulse Shapes

In order to illustrate the previously discussed signal or pulse shape properties we consider:

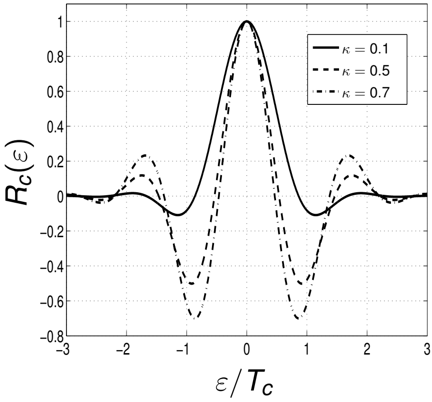
- Three example pulse shapes $p(t)$ with $\kappa = 0.1$, $\kappa = 0.5$, $\kappa = 0.7$, $B = 1.023$ MHz, and $BT_c = 1$
- $P_u = -154\,\mathrm{dBW}$ with $U = 11$
- $N_0 = -204\,\mathrm{dBW/Hz}$
- c denotes the speed of light
- For this example we assume that no MAI-R is present
- The multipath error envelope gives the maximum bias of a DLL in case that in addition to the line-of-sight signal a single reflective multipath signal with signal-to-multipath ration of 6 dB is present
- The envelope is defined by the cases if the multipath signal has a relative phase of 0 or of π with respect to the line-of-sight signal

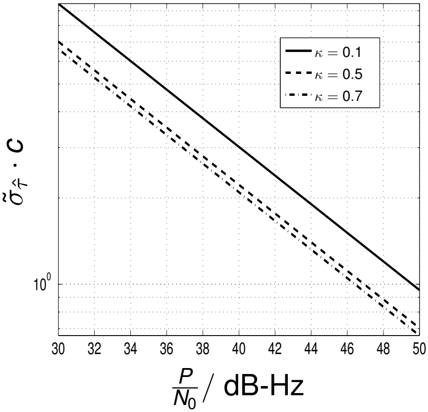
Time and Frequency Domain





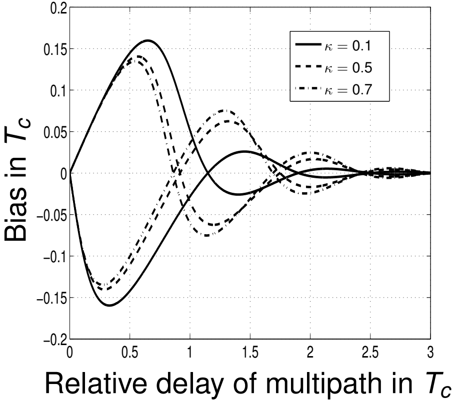
Autocorrelation and CRLB-I

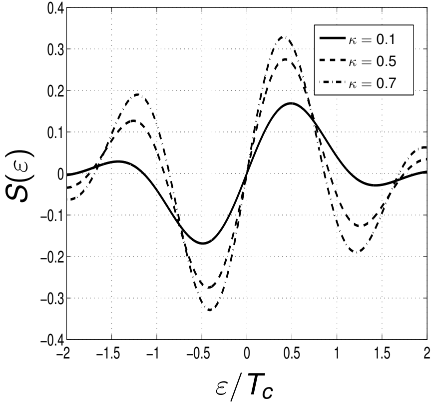




$p(t)$	$\kappa = 0.1$	$\kappa = 0.5$	$\kappa = 0.7$	Lower bound MAI-A
MAI-A	0.79	0.80	0.99	0.54
ΔSNR	1.79 (2.53 dB)	1.80 (2.55 dB)	1.99 (3.0 dB)	1.54 (1.87 dB)

Multipath Error Envelope and S-Curve





Summary

Signal properties for time-delay estimation and GNSS

1. The higher the Gabor bandwidth of the signal $\Rightarrow$ the higher the synchronization accuracy that can be achieved in terms of the CRLB
2. Minimizing the CRLB for τ and maximizing time concentration are contradictive tasks
3. The higher the processing gain G (large bandwidth B) $\Rightarrow$ the higher synchronization accuracy and the higher interference robustness
4. The lower the CRLB for τ (CRLB $\to \frac{1}{B^2}$) $\Rightarrow$ the lower time concentration of $p(t)$ ($p(t) \to \sin(2B\pi t)$ or $\cos(2B\pi t)$)
5. The higher the sidelobes of $R_c(\varepsilon) \Rightarrow$ the higher the sidelobes of $p(t) \Rightarrow$ the lower time concentration of $p(t)$
6. The higher the sidelobes of $R_c(\varepsilon) \Rightarrow$ the higher $\kappa \Rightarrow$ the less robust estimation of τ (likelihood has local maxima besides the global maximum)
7. MAI-A and MAI-R have to be considered in the signal design (chip pulse shape design) or can be treated in the receiver (multi-user detection and mitigation)