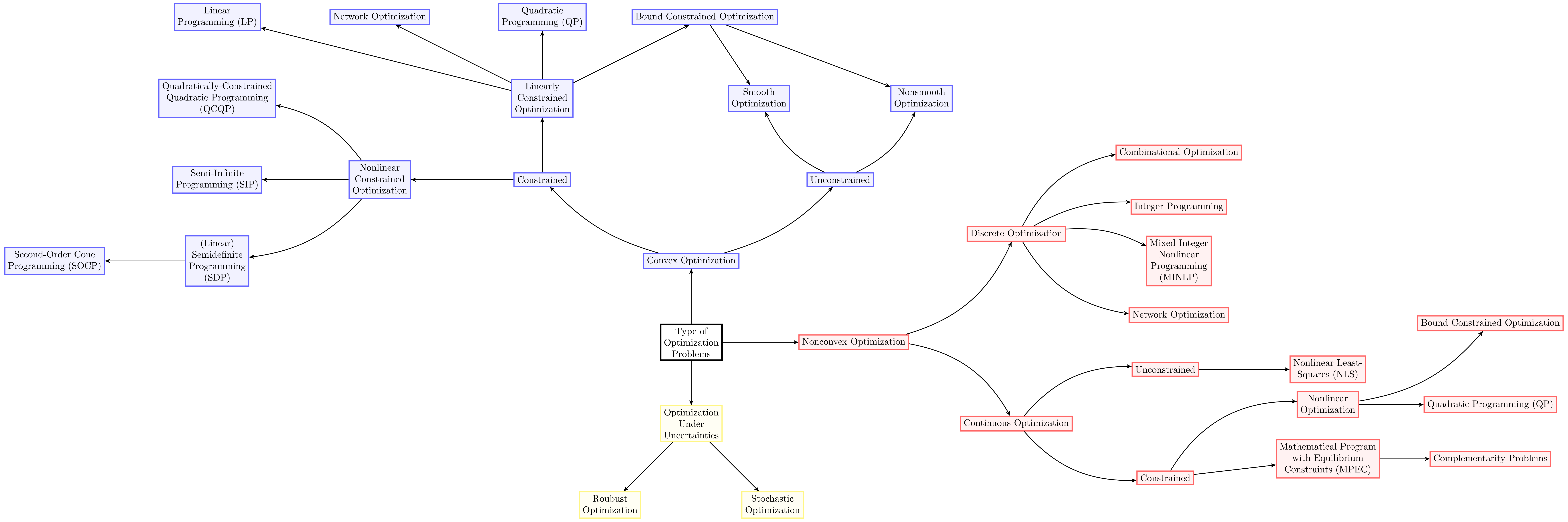
Convex sets

Set	Comments
Convex hull: • conv $C = \left\{\sum_{i=1}^{k} \theta_i \mathbf{x}_i \mid \mathbf{x}_i \in C, \mathbf{0} \preceq \boldsymbol{\theta} \preceq \mathbf{1}, \mathbf{1}^\top \boldsymbol{\theta} = 1\right\}$	• conv C will be the smallest convex set that contains C. • conv C will be a finite set as long as C is also finite.
Affine hull: • aff $C = \left\{\sum_{i=1}^{k} \theta_i \mathbf{x}_i \mid \mathbf{x}_i \in C \text{ for } i = 1, \cdots, k, \mathbf{1}^\top \boldsymbol{\theta} = 1\right\}$	• A will be the smallest affine set that contains C. • Different from the convex set, θ_i is not restricted between 0 and 1 • aff C will always be an infinite set. If aff C contains the origin, it is also a subspace.
Conic hull: • $A = \left\{\sum_{i=1}^{k} \theta_i \mathbf{x}_i \mid \mathbf{x}_i \in C, \theta_i > 0 \text{ for } i = 1, \cdots, k\right\}$	• A will be the smallest convex conic that contains C. • Different from the convex and affine sets, θ_i does not need to sum up 1.
Ray: • $\mathcal{R} = \{\mathbf{x}_0 + \theta \mathbf{v} \mid \theta \geq 0\}$	• The ray is an infinite set that begins in $\mathbf{x}_0$ and extends infinitely in direction of $\mathbf{v}$. In other words, it has a beginning, but it has no end.
Hyperplane: • $\mathcal{H} = \left\{\mathbf{x} \mid \mathbf{a}^\top \mathbf{x} = b\right\}$ • $\mathcal{H} = \left\{\mathbf{x} \mid \mathbf{a}^\top (\mathbf{x} - \mathbf{x}_0) = \mathbf{0}\right\}$ • $\mathcal{H} = \mathbf{x}_0 + a^\perp$	• It is an infinite set $\mathbb{R}^{n-1} \subset \mathbb{R}^n$ that divides the space into two halfspaces. • $a^\perp = \left\{\mathbf{v} \mid \mathbf{a}^\top \mathbf{v} = 0\right\}$ is the set of vectors perpendicular to $\mathbf{a}$. It passes through the origin. • $a^\perp$ is offset from the origin by $\mathbf{x}_0$, which is any vector in $\mathcal{H}$.
Halfspaces: • $\mathcal{H}_- = \left\{\mathbf{x} \mid \mathbf{a}^\top \mathbf{x} \leq b\right\}$ • $\mathcal{H}_+ = \left\{\mathbf{x} \mid \mathbf{a}^\top \mathbf{x} \geq b\right\}$	• They are infinite sets of the parts divided by $\mathcal{H}$.
Euclidean ball: • $B(\mathbf{x}_c, r) = \{\mathbf{x} \mid \Vert\mathbf{x} - \mathbf{x}_c\Vert_2 \leq r\}$ • $B(\mathbf{x}_c, r) = \left\{\mathbf{x} \mid (\mathbf{x} - \mathbf{x}_c)^\top (\mathbf{x} - \mathbf{x}_c) \leq r\right\}$ • $B(\mathbf{x}_c, r) = \{\mathbf{x}_c + r \Vert\mathbf{u}\Vert \mid \Vert\mathbf{u}\Vert \leq 1\}$	• $B(\mathbf{x}_c, r)$ is a finite set as long as $r < \infty$. • $\mathbf{x}_c$ is the center of the ball. • r is its radius.
Ellipsoid: • $\mathcal{E} = \left\{\mathbf{x} \mid (\mathbf{x} - \mathbf{x}_c)^\top \mathbf{P}^{-1} (\mathbf{x} - \mathbf{x}_c) \leq 1\right\}$ • $\mathcal{E} = \{\mathbf{x}_c + \mathbf{A}\mathbf{u} \mid \Vert\mathbf{u}\Vert \leq 1\}$, where $\mathbf{A} = \mathbf{P}^{1/2}$.	• $\mathcal{E}$ is a finite set as long as $\mathbf{P}$ is a finite matrix. • $\mathbf{P}$ is symmetric and positive definite, that is, $\mathbf{P} = \mathbf{P}^\top \succ 0$. • $\mathbf{x}_c$ is the center of the ellipsoid. • The lengths of the semi-axes are given by $\sqrt{\lambda_i}$. • $\mathbf{A}$ is invertible. When it is not, we say that $\mathcal{E}$ is a degenerated ellipsoid (degenerated ellipsoids are also convex).
Norm cone: • $C = \left\{[x_1, x_2, \cdots, x_n, t]^\top \in \mathbb{R}^{n+1} \mid \mathbf{x} \in \mathbb{R}^n, \Vert\mathbf{x}\Vert_p \leq t\right\} \subseteq \mathbb{R}^{n+1}$	• Although it is named "Norm cone", it is a set, not a scalar. • The cone norm increases the dimension of $\mathbf{x}$ in 1. • For $p = 2$, it is called the second-order cone, quadratic cone, Lorentz cone or ice-cream cone.
Proper cone: $K \subset \mathbb{R}^n$ is a proper cone when it has the following properties • K is a convex cone, i.e., $\alpha K \equiv K, \alpha > 0$. • K is closed. • K is solid. • K is pointed, i.e., $-K \cap K = \{\mathbf{0}\}$.	• The proper cone K is used to define the *generalized inequality* (or *partial ordering*) in some set S. For the generalized inequality, one must define both the proper cone K and the set S. • $\mathbf{x} \preceq \mathbf{y} \iff \mathbf{y} - \mathbf{x} \in K$ for $\mathbf{x}, \mathbf{y} \in S$ (generalized inequality) • $\mathbf{x} \prec \mathbf{y} \iff \mathbf{y} - \mathbf{x} \in \text{int } K$ for $\mathbf{x}, \mathbf{y} \in S$ (strict generalized inequality). • There are two cases where K and S are understood from context and the subscript K is dropped out: ▹ When $S = \mathbb{R}^n$ and $K = \mathbb{R}^n_+$ (the nonnegative orthant). In this case, $\mathbf{x} \preceq \mathbf{y}$ means that $x_i \leq y_i$. ▹ When $S = \mathcal{S}^n$ and $K = \mathcal{S}^n_+$ or $K = \mathcal{S}^n_{++}$, where $\mathcal{S}^n$ denotes the set of symmetric $n \times n$ matrices, $\mathcal{S}^n_+$ is the space of the positive semidefinite matrices, and $\mathcal{S}^n_{++}$ is the space of the positive definite matrices. $\mathcal{S}^n_+$ is a proper cone in $\mathcal{S}^n$ (??). In this case, the generalized inequality $\mathbf{Y} \succeq \mathbf{X}$ means that $\mathbf{Y} - \mathbf{X}$ is a positive semidefinite matrix belonging to the positive semidefinite cone $\mathcal{S}^n_+$ in the subspace of symmetric matrices $\mathcal{S}^n$. It is usual to denote $\mathbf{X} \succ \mathbf{0}$ and $\mathbf{X} \succeq \mathbf{0}$ to mean than $\mathbf{X}$ is a positive definite and semidefinite matrix, respectively, where $\mathbf{0} \in \mathbb{R}^{n \times n}$ is a zero matrix. • Another common usage is when $S = \mathbb{R}^n$ and $K = \left\{\mathbf{c} \in \mathbb{R}^n \mid c_1 + c_2 t + \cdots + c_n t^{n-1} \geq 0, \text{ for } 0 \leq t \leq 1\right\}$. In this case, $\mathbf{x} \preceq_K \mathbf{y}$ means that $x_1 + x_2 t + \cdots + x_n t^{n-1} \leq y_1 + y_2 t + \cdots + y_n t^{n-1}$. • The generalized inequality has the following properties: ▹ If $\mathbf{x} \preceq_K \mathbf{y}$ and $\mathbf{u} \preceq_K \mathbf{v}$, then $\mathbf{x} + \mathbf{u} \preceq_k \mathbf{y} + \mathbf{v}$ (preserve under addition). ▹ If $\mathbf{x} \preceq_K \mathbf{y}$ and $\mathbf{y} \preceq_K \mathbf{z}$, then $\mathbf{x} \preceq_K \mathbf{z}$ (transitivity). ▹ If $\mathbf{x} \preceq_K \mathbf{y}$, then $\alpha\mathbf{x} \preceq_K \mathbf{y}$ for $\alpha \geq 0$ (preserve under nonnegative scaling). ▹ $\mathbf{x} \preceq_K \mathbf{x}$ (reflexivity). ▹ If $\mathbf{x} \preceq_K \mathbf{y}$ and $\mathbf{y} \preceq_K \mathbf{x}$, then $\mathbf{x} = \mathbf{y}$ (antisymmetric). ▹ If $\mathbf{x}_i \preceq_K \mathbf{y}_i$, for $i = 1, 2, \ldots$, and $\mathbf{x}_i \to \mathbf{x}$ and $\mathbf{y}_i \to \mathbf{y}$ as $i \to \infty$, then $\mathbf{x} \preceq_K \mathbf{y}$. • It is called partial ordering because $\mathbf{x} \npreceq_K \mathbf{y}$ and $\mathbf{y} \npreceq_K \mathbf{x}$ for many $\mathbf{x}, \mathbf{y} \in S$. When it happens, we say that $\mathbf{x}$ and $\mathbf{y}$ are not comparable (this case does not happen in ordinary inequality, $<$ and $>$). • $\mathbf{x} \in S$ is the *minimum* element of S with respect to the convex cone K if $\mathbf{x} \preceq_K \mathbf{y}, \forall\, \mathbf{y} \in S$ (for *maximum*, $\mathbf{x} \succeq_K \mathbf{y}, \forall\, \mathbf{y} \in S$). It means that $S \subseteq \mathbf{x} + K$ (for the maximum, $S \subseteq \mathbf{x} - K$), where $\mathbf{x} + K$ denotes the set K shifted from the origin by $\mathbf{x}$. Note that any point in $K + \mathbf{x}$ is greater or equal to $\mathbf{x}$ in the generalized inequality mean. The set S does not necessarily have a minimum (maximum), but the minimum (maximum) is unique if it does. • $\mathbf{x} \in S$ is the *minimal* element of S if $\mathbf{y} \preceq_K \mathbf{x}$ only when $\mathbf{y} = \mathbf{x}$. The same is true for *maximal*. We can have many different minimal (maximal) elements. The mathematical notation for that is $(\mathbf{x} - K) \cap S = \{\mathbf{x}\}$, where $\mathbf{x} - K$ denotes all points that are comparable to $\mathbf{x}$ and less than or equal to $\mathbf{x}$ (for the maximal, we have $(\mathbf{x} + K) \cap S = \{\mathbf{x}\}$). • When $K = \mathbb{R}_+$ and $S = \mathbb{R}$ (ordinary inequality), the minimal is equal to the minimum and the maximal is equal to the maximum. • When we say that a scalar-valued function $f : \mathbb{R}^n \to \mathbb{R}$ is nondecreasing, it means that whenever $\mathbf{u} \preceq \mathbf{v}$, we have $\tilde{h}(\mathbf{u}) \leq \tilde{h}(\mathbf{v})$. Similar results hold for decreasing, increasing, and nonincreasing scalar functions.
Dual cone: • $K^* = \left\{\mathbf{y} \mid \mathbf{x}^\top \mathbf{y} \geq 0, \ \forall\, \mathbf{x} \in K\right\}$	• K^* is a cone, and it is convex even when the original cone K is nonconvex. • K^* has the following properties: ▹ K^* is closed and convex. ▹ $K_1 \subseteq K_2$ implies $K_1^* \subseteq K_2^*$. ▹ If K has a nonempty interior, then K^* is pointed. ▹ If the closure of K is pointed then K^* has a nonempty interior. ▹ K^{**} is the closure of the convex hull of K. Hence, if K is convex and closed, $K^{**} = K$.
Polyhedra: • $\mathcal{P} = \left\{\mathbf{x} \mid \mathbf{a}_j^\top \mathbf{x} \leq b_j, j = 1, \ldots, m, \mathbf{a}_j^\top \mathbf{x} = d_j, j = 1, \cdots, p\right\}$ • $\mathcal{P} = \{\mathbf{x} \mid \mathbf{A}\mathbf{x} \preceq \mathbf{b}, \mathbf{C}\mathbf{x} = \mathbf{d}\}$, where $\mathbf{A} = \begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \ldots & \mathbf{a}_m\end{bmatrix}^\top$ and $\mathbf{C} = \begin{bmatrix}\mathbf{c}_1 & \mathbf{c}_2 & \ldots & \mathbf{c}_m\end{bmatrix}^\top$	• The polyhedron may or may not be an infinite set. • Polyhedron is the result of the intersection of m halfspaces and p hyperplanes. • Subspaces, hyperplanes, lines, rays line segments, and halfspaces are all polyhedra. • The *nonnegative orthant*, $\mathbb{R}^n_+ = \{\mathbf{x} \in \mathbb{R}^n \mid x_i \leq 0 \text{ for } i = 1, \ldots n\} = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{I}\mathbf{x} \succeq \mathbf{0}\}$, is a special polyhedron.
Simplex: • $\mathcal{S} = \text{conv } \{\mathbf{v}_m\}_{m=0}^{k} = \left\{\sum_{i=0}^{k} \theta_i \mathbf{v}_i \mid \mathbf{0} \preceq \boldsymbol{\theta} \preceq \mathbf{1}, \mathbf{1}^\top \boldsymbol{\theta} = 1\right\}$ • $\mathcal{S} = \{\mathbf{x} \mid \mathbf{x} = \mathbf{v}_0 + \mathbf{V}\boldsymbol{\theta}\}$, where $\mathbf{V} = \begin{bmatrix}\mathbf{v}_1 - \mathbf{v}_0 & \ldots & \mathbf{v}_n - \mathbf{v}_0\end{bmatrix} \in \mathbb{R}^{n \times k}$ • $\mathcal{S} = \{\mathbf{x} \mid \underbrace{\mathbf{A}_1\mathbf{x} \preceq \mathbf{A}_1\mathbf{v}_0, \mathbf{1}^\top\mathbf{A}_1\mathbf{x} \leq 1 + \mathbf{1}^\top\mathbf{A}_1\mathbf{v}_0}_{\text{Linear inequalities in } x}, \underbrace{\mathbf{A}_2\mathbf{x} = \mathbf{A}_2\mathbf{v}_0}_{\text{Linear equalities in } x}\}$ (Polyhedra form), where $\mathbf{A} = \begin{bmatrix}\mathbf{A}_1 \\ \mathbf{A}_2\end{bmatrix}$ and $\mathbf{A}\mathbf{V} = \begin{bmatrix}\mathbf{I}_{k \times k} \\ \mathbf{0}_{n-k \times n-k}\end{bmatrix}$	• Simplexes are a subfamily of the polyhedra set. • Also called k-dimensional Simplex in $\mathbb{R}^n$. • The set $\{\mathbf{v}_m\}_{m=0}^{k}$ is a affinely independent, which means $\{\mathbf{v}_1 - \mathbf{v}_0, \ldots, \mathbf{v}_k - \mathbf{v}_0\}$ are linearly independent. • $\mathbf{V} \in \mathbb{R}^{n \times k}$ is a full-rank tall matrix, i.e., $\text{rank}(\mathbf{V}) = k$. All its column vectors are independent. The matrix $\mathbf{A}$ is its left pseudoinverse.
α-sublevel set: • $C_\alpha = \{\mathbf{x} \in \text{dom}(f) \mid f(\mathbf{x}) \leq \alpha\}$ (regarding convexity) • $C_\alpha = \{\mathbf{x} \in \text{dom}(f) \mid f(\mathbf{x}) \geq \alpha\}$ (regarding concavity)	• If f is a convex function, then sublevel sets of f are convexes for any $\alpha \in \mathbb{R}$. • The converse is not true: a function can have all its sublevel set convex and not be a convex function.

Functions (or operators) and their implications regarding convexity

Function	Convex?	Comments
Union: $C = A \cup B$	Not in most of the cases.	
Intersection: $C = A \cap B$	Yes, if A and B are convex sets.	
Convex function: $f : \text{dom}(f) \to \mathbb{R}$ • $f(\theta\mathbf{x} + (1-\theta)\mathbf{y}) \leq \theta f(\mathbf{x}) + (1-\theta) f(\mathbf{y})$, where $0 \leq \theta \leq 1$. • $\text{dom}(f)$ shall be a convex set to f be considered a convex function.	Yes.	• Graphically, the line segment between $(\mathbf{x}, f(\mathbf{x}))$ and $(\mathbf{y}, f(\mathbf{y}))$ lies always above the graph f. • In terms of sets, a function is convex iff a line segment within $\text{dom}(f)$, which is a convex set, gives an image set that is also convex. • $\text{dom} f$ is convex iff all points for any line segment within $\text{dom}(f)$ belong to it. • *First-order condition*: f is convex iff $\text{dom}(f)$ is convex and $f(\mathbf{y}) \geq f(\mathbf{x}) + \nabla f(\mathbf{x})^\top (\mathbf{y} - \mathbf{x}), \forall\, \mathbf{x}, \mathbf{y} \in \text{dom}(f), \mathbf{x} \neq \mathbf{y}$, being $\nabla f(\mathbf{x})$ the gradient vector. This inequation says that the first-order Taylor approximation is a *underestimator* for convex functions. The first-order condition requires that f is differentiable. • If $\nabla f(\mathbf{x}) = \mathbf{0}$, then $f(\mathbf{y}) \geq f(\mathbf{x}), \forall\, \mathbf{y} \in \text{dom}(f)$ and $\mathbf{x}$ is a global minimum. • *Second-order condition*: f is convex iff $\text{dom}(f)$ is convex and $\mathbf{H} \succeq \mathbf{0}$, that is, the Hessian matrix $\mathbf{H}$ is a positive semidefinite matrix. It means that the graphic of the curvature has a positive (upward) curvature at $\mathbf{x}$. It is important to note that, if $\mathbf{H} \succ \mathbf{0}, \forall\, \mathbf{x} \in \text{dom}(f)$, then f is strictly convex. But is f is strictly convex, not necessarily that $\mathbf{H} \succ \mathbf{0}, \forall\, \mathbf{x} \in \text{dom}(f)$. Therefore, strict convexity can only be partially characterized.
Affine function $f : \mathbb{R}^n \to \mathbb{R}^m$ • $f(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}$, where $\mathbf{A} \in \mathbb{R}^{m \times n}, \mathbf{b} \in \mathbb{R}^m, \mathbf{x} \in \mathbb{R}^n$	Yes, if the domain $S \subseteq \mathbb{R}^n$ is a convex set, then its image $f(S) = \{f(\mathbf{x}) \mid \mathbf{x} \in S\} \subseteq \mathbb{R}^m$ is also convex.	• The affine function, $f(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}$, is a broader category that encompasses the linear function, $f(\mathbf{x}) = \mathbf{A}\mathbf{x}$. The linear function has its origin fixed at $\mathbf{0}$ after the transformation, whereas the affine function does not necessarily have it (when not, this makes the affine function nonlinear). Graphically, we can think of an affine function as a linear transformation plus a shift from the origin of $\mathbf{b}$. • A special case of the linear function is when $\mathbf{A} = \mathbf{c}^\top$. In this case, we have $f(\mathbf{x}) = \mathbf{c}^\top \mathbf{x}$, which is the inner product between the vector $\mathbf{c}$ and $\mathbf{x}$. • The inverse image of C, $f^{-1}(C) = \{\mathbf{x} \mid f(\mathbf{x}) \in C\}$, is also convex. • The *linear matrix inequality* (LMI), $\mathbf{A}(\mathbf{x}) = x_1\mathbf{A}_1 + \cdots + x_n\mathbf{A}_n \preceq \mathbf{B}$, is a special case of affine function. In other words, $f(S) = \{\mathbf{x} \mid \mathbf{A}(\mathbf{x}) \preceq \mathbf{B}\}$ is a convex set if S is convex. Many optimization problems can be formulated as LMI problems and solved optimally.
Exponential function $f : \mathbb{R} \to \mathbb{R}$ • $f(x) = e^{ax} \in \mathbb{R}$, where $a \in \mathbb{R}$	Yes.	
Quadratic function $f : \mathbb{R}^n \to \mathbb{R}$ • $f(\mathbf{x}) = a\mathbf{x}^\top \mathbf{P}\mathbf{x} + \mathbf{p}^\top \mathbf{x} + r \in \mathbb{R}$, where $\mathbf{x}, \mathbf{p} \in \mathbb{R}^n, \mathbf{P} \in \mathbb{R}^{n \times n}$, and $a, b \in \mathbb{R}$	It depends on the matrix $\mathbf{P}$: • f is convex iff $\mathbf{P} \succeq \mathbf{0}$. • f is strictly convex iff $\mathbf{P} \succ \mathbf{0}$. • f is concave iff $\mathbf{P} \preceq \mathbf{0}$. • f is strictly concave iff $\mathbf{P} \prec \mathbf{0}$.	
Power function $f : \mathbb{R}_{++} \to \mathbb{R}$ • $f(x) = x^a$	It depends on a • f is convex iff $a \geq 1$ or $a \leq 0$. • f is concave iff $0 \leq a \leq 1$.	
Power of absolute value: $f : \mathbb{R} \to \mathbb{R}$ • $f(x) = \lvert x\rvert^p$, where $p \leq 1$.	Yes.	
Logarithm function: $f : \mathbb{R}_{++} \to \mathbb{R}$ • $f(x) = \log x$	Yes.	
Negative entropy function: $f : \mathbb{R}_+ \to \mathbb{R}$ • $f(x) = x \log x$	Yes	• When it is defined $f(x)\vert_{x=0} = 0$, $\text{dom}(f) = \mathbb{R}$.
Minkwoski distance, p-norm function, or l_p norm function: $f : \mathbb{R}^n \to \mathbb{R}$ • $f(\mathbf{x}) = \Vert\mathbf{x}\Vert_p$, where $p \in \mathbb{N}_{++}$.	Yes.	• It can be proved by triangular inequality.
Maximum element: $f : \mathbb{R}^n \to \mathbb{R}$ • $f(\mathbf{x}) = \max\{x_1, \ldots, x_n\}$.	Yes.	
Maximum function: $f : \mathbb{R}^n \to \mathbb{R}$ • $f(\mathbf{x}) = \max\{f_1(\mathbf{x}), \ldots, f_n(\mathbf{x})\}$.	Yes, if $f_1, \ldots, f_n$ are convex function.	
Minimum function: $f : \mathbb{R}^n \to \mathbb{R}$ • $f(\mathbf{x}) = \min\{f_1(\mathbf{x}), \ldots, f_n(\mathbf{x})\}$.	Not in most of the cases.	
Log-sum-exp function: $f : \mathbb{R}^n \to \mathbb{R}$ • $f(\mathbf{x}) = \log(e^{x_1} + \cdots + e^{x_n})$	Yes.	• This function is interpreted as the approximation of the maximum element function, since $\max\{x_1, \ldots, x_n\} \leq f(\mathbf{x}) \leq \max\{x_1, \ldots, x_n\} + \log n$
Geometric mean function $f : \mathbb{R}^n \to \mathbb{R}$ • $f(\mathbf{x}) = \left(\Pi_{i=1}^{n} x_i\right)^{1/n}$	Yes	
Log-determinant function $f : \mathcal{S}^n_{++} \to \mathbb{R}$ • $f(\mathbf{X}) = \log\lvert\mathbf{X}\rvert$	Yes	• $\mathbf{X}$ is positive semidefinite, i.e., $\mathbf{X} \succ \mathbf{0} \therefore \mathbf{X} \in \mathcal{S}^n_{++}$.
Composite function $f = h \circ g : \mathbb{R}^n \to \mathbb{R}$ • $f = g \circ h$, i.e., $f(\mathbf{x}) = (h \circ g)(\mathbf{x}) = h(g(\mathbf{x}))$, where: ▹ $g : \mathbb{R}^n \to \mathbb{R}^k$. ▹ $h : \mathbb{R}^k \to \mathbb{R}$. ▹ $\text{dom}(f) = \{\mathbf{x} \in \text{dom}(g) \mid g(\mathbf{x}) \in \text{dom}(h)\}$.	• Scalar composition: the following statements hold for $k = 1$ and $n \geq 1$, i.e., $h : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$: ▹ f is convex if h is convex, $\tilde{h}$ is nondecreasing, and g is convex. In this case, $\text{dom}(h)$ is either $(-\infty, a]$ or $(-\infty, a)$. ▹ f is convex if h is convex, $\tilde{h}$ is nonincreasing, and g is concave. In this case, $\text{dom}(h)$ is either $[a, \infty)$ or (a, ∞). ▹ f is concave if h is concave, $\tilde{h}$ is nondecreasing, and g is concave. ▹ f is concave if h is concave, $\tilde{h}$ is nonincreasing, and g is convex. • Vector composition: the following statements hold for $k \geq 1$ and $n \geq 1$, i.e., $h : \mathbb{R}^k \to \mathbb{R}$ and $\boldsymbol{g} : \mathbb{R}^n \to \mathbb{R}^k$. Hence, $g(\mathbf{x}) = (g_1(\mathbf{x}), g_2(\mathbf{x}), \ldots, g_k(\mathbf{x}))$ is a vector-valued function (or simply, vector function), where $g_i : \mathbb{R}^k \to \mathbb{R}$ for $1 \leq i \leq k$. ▹ f is convex if h is is convex, $\tilde{h}$ is nondecreasing in each argument of $\mathbf{x}$, and $\{g_i\}_{i=1}^{k}$ is a set of convex functions. ▹ f is convex if h is is convex, $\tilde{h}$ is nonincreasing in each argument of $\mathbf{x}$, and $\{g_i\}_{i=1}^{k}$ is a set of concave functions. ▹ f is concave if h is is concave, $\tilde{h}$ is nondecreasing in each argument of $\mathbf{x}$, and $\{g_i\}_{i=1}^{k}$ is a set of concave functions. Where $\tilde{h}$ is the extended-value extension of the function h, which assigns the value ∞ $(-\infty)$ to the point not in $\text{dom}(h)$ for h convex (concave).	• The composition function allows us to see a large class of functions as convex (or concave). • For scale composition, the remarkable ones are: ▹ If g is convex then $f(x) = h(g(\mathbf{x})) = \exp g(\mathbf{x})$ is convex. ▹ If g is concave and $\text{dom}(g) \subseteq \mathbb{R}_{++}$, then $f(\mathbf{x}) = h(g(\mathbf{x})) = \log g(\mathbf{x})$ is concave. ▹ If g is concave and $\text{dom}(g) \subseteq \mathbb{R}_{++}$, then $f(\mathbf{x}) = h(g(\mathbf{x})) = 1/g(\mathbf{x})$ is convex. ▹ If g is convex and $\text{dom}(g) \subseteq \mathbb{R}_+$, then $f(\mathbf{x}) = h(g(\mathbf{x})) = g^p(\mathbf{x})$ is convex, where $p \geq 1$. ▹ If g is convex then $f(\mathbf{x}) = h(g(\mathbf{x})) = -\log(-g(x))$ is convex, where $\text{dom}(f) = \{\mathbf{x} \mid g(\mathbf{x}) < 0\}$. • For vector composition, we have the following examples: ▹ If $g(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}$ is an affine function, then $f = h \circ g$ is convex (concave) if h is convex (concave). ▹ Let $h(\mathbf{x}) = x_{[1]} + \cdots + x_{[r]}$ be the sum of the r largest components of $\mathbf{x} \in \mathbb{R}^k$. If $g_1, g_2, \ldots, g_k$ are convex, where $\text{dom}(g_i) = \mathbb{R}^n$, then $f = h \circ g$, which is the pointwise sum of the largest g_i's, is convex. ▹ $f = h \circ g$ is a convex function when $h(\mathbf{x}) = \log\left(\sum_{i=1}^{k} e^{x_i}\right)$ and $g_1, g_2, \ldots, g_k$ are convex functions. ▹ For $0 < p \leq 1$, the function $h(\mathbf{x}) = \left(\sum_{i=1}^{k} x_i^p\right)^{1/p}$, where $\text{dom}(h) = \mathbb{R}^n_+$, is concave. If $g_1, g_2, \ldots, g_k$ are concaves (convexes) and nonnegatives, then $f = h \circ g$ is concave (convex).
Nonnegative weighted sum: $f : \text{dom}(f) \to \mathbb{R}$ • $f(\mathbf{x}) = \sum_{i=1}^{m} w_i f_i(\mathbf{x})$, where $w \geq 0$.	• If $f_1, f_2, \ldots, f_m$ are convex or concave functions, then f is a convex or concave function, respectively. • If $f_1, f_2, \ldots, f_m$ are strictly convex or concave functions, then f is a strictly convex or concave function, respectively.	• Special cases is when $f = wf$ (a nonnegative scalar factor) and $f = f_1 + f_2$ (sum).
Integral function $f : \mathbb{R}^n \to \mathbb{R}$: • $f(\mathbf{x}) = \int_{\mathcal{A}} w(\mathbf{y}) g(\mathbf{x}, \mathbf{y})\, \mathrm{d}\mathbf{y}$, where $\mathbf{y} \in \mathcal{A} \subseteq \mathbb{R}^m$, and $w : \mathbb{R}^m \to \mathbb{R}$.	• If g is convex in $\mathbf{x}$ for each $\mathbf{y} \in \mathcal{A}$ and if $w(\mathbf{y}) \geq 0, \ \forall\, \mathbf{y} \in \mathcal{A}$, then f is convex (provided the integral exists).	
Perspective function $f : \mathbb{R}^n \times \mathbb{R}_{++} \to \mathbb{R}^n$ • $f(\mathbf{x}, t) = \mathbf{x}/t$, where $\mathbf{x} \in \mathbb{R}^n, t \in \mathbb{R}$.	Yes, if $S \subseteq \text{dom}(f)$ is a convex set, then its image, $f(S) = \{f(\mathbf{x}) \mid \mathbf{x} \in S\} \subseteq \mathbb{R}^n$, is also convex.	• The perspective function decreases the dimension of the function domain since $\dim(\text{dom}(f)) = n + 1$. • Its effect is similar to the camera zoom. • The inverse image is also convex, that is, if $C \subseteq \mathbb{R}^n$ is convex, then $f^{-1}(C) = \left\{(\mathbf{x}, t) \in \mathbb{R}^{n+1} \mid \mathbf{x}/t \in C, t > 0\right\}$ is also convex. • A special case is when $n = 1$, which is called *quadratic-over-linear function*.
Projective (or linear-fractional) function, $f : \mathbb{R}^n \to \mathbb{R}^m$ • $f = p \circ g$, i.e., $f(\mathbf{x}) = (p \circ g)(\mathbf{x}) = p(g(\mathbf{x}))$, where ▹ $g : \mathbb{R}^n \to \mathbb{R}^{m+1}$ is an affine function given by $g(\mathbf{x}) = \begin{bmatrix}\mathbf{A} \\ \mathbf{c}^\top\end{bmatrix}\mathbf{x} + \begin{bmatrix}\mathbf{b} \\ d\end{bmatrix}$, being $\mathbf{A} \in \mathbb{R}^{m \times n}, \mathbf{b} \in \mathbb{R}^m, \mathbf{c} \in \mathbb{R}^n$, and $d \in \mathbb{R}$. ▹ $p : \mathbb{R}^{m+1} \to \mathbb{R}^m$ is the perspective function. • $f(\mathbf{x}) = \mathcal{P}^{-1}(\mathbf{Q}\mathcal{P}(\mathbf{x}))$ ▹ $\mathcal{P}(\mathbf{x}) = \{(t\mathbf{x}, t) \mid t \geq 0\} \subset \mathbb{R}^{n+1}$ ▹ $\mathbf{Q} = \begin{bmatrix}\mathbf{A} & \mathbf{b} \\ \mathbf{c}^\top & d\end{bmatrix} \in \mathbb{R}^{(m+1) \times (n+1)}$	Yes, if $S \subseteq \text{dom}(f)$ is a convex set, then its image, $f(S) = \{f(\mathbf{x}) \mid \mathbf{x} \in S\} \subseteq \mathbb{R}^n$, is also convex.	• The linear and affine functions are special cases of the linear-fractional function. • $\text{dom}(f) = \left\{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{c}^\top \mathbf{x} + d > 0\right\}$ • $\mathcal{P}(\mathbf{x}) \subset \mathbb{R}^{n+1}$ is a ray set that begins at the origin and its last component takes only positive values. For each $\mathbf{x} \in \text{dom}(f)$, it is associated a ray set in $\mathbb{R}^{n+1}$ in this form. This (projective) correspondence between all points in $\text{dom}(f)$ and their respective sets $\mathcal{P}$ is a biunivocal mapping. • The linear transformation $\mathbf{Q}$ acts on these rays, forming another set of rays. • Finally we take the inverse projective transformation to recover $f(\mathbf{x})$.
Epigraph: • epi $f = \{(\mathbf{x}, t) \mid \mathbf{x} \in \text{dom}(f), t \geq f(\mathbf{x})\}$	• Visually, it is the graph above the $\mathbf{x}, f(\mathbf{x})$ curve. • The function f is convex iff its epigraph is convex.	
Hypograph: • hypo $f = \{(\mathbf{x}, t) \mid \mathbf{x} \in \text{dom}(f), t \geq f(\mathbf{x})\}$	• Visually, it is the graph below the $\mathbf{x}, f(\mathbf{x})$ curve. • The function f is concave iff its hypograph is convex.	


Linear Programming (LP)
Network Optimization
Quadratic Programming (QP)
Bound Constrained Optimization
Quadratically-Constrained Quadratic Programming (QCQP)
Linearly Constrained Optimization
Smooth Optimization
Nonsmooth Optimization
Semi-Infinite Programming (SIP)
Nonlinear Constrained Optimization
Constrained
Unconstrained
Combinational Optimization
Integer Programming
Discrete Optimization
Mixed-Integer Nonlinear Programming (MINLP)
Second-Order Cone Programming (SOCP)
(Linear) Semidefinite Programming (SDP)
Convex Optimization
Network Optimization
Bound Constrained Optimization
Type of Optimization Problems
Nonconvex Optimization
Unconstrained
Nonlinear Least-Squares (NLS)
Nonlinear Optimization
Quadratic Programming (QP)
Optimization Under Uncertainties
Continuous Optimization
Mathematical Program with Equilibrium Constraints (MPEC)
Complementarity Problems
Constrained
Roubust Optimization
Stochastic Optimization