

ROTATION

10

10-1 WHAT IS PHYSICS?

As we have discussed, one focus of physics is motion. However, so far we have examined only the motion of **translation**, in which an object moves along a straight or curved line, as in Fig. 10-1a. We now turn to the motion of **rotation**, in which an object turns about an axis, as in Fig. 10-1b.

You see rotation in nearly every machine, you use it every time you open a beverage can with a pull tab, and you pay to experience it every time you go to an amusement park. Rotation is the key to many fun activities, such as hitting a long drive in golf (the ball needs to rotate in order for the air to keep it aloft longer) and throwing a curveball in baseball (the ball needs to rotate in order for the air to push it left or right). Rotation is also the key to more serious matters, such as metal failure in aging airplanes.

We begin our discussion of rotation by defining the variables for the motion, just as we did for translation in Chapter 2. As we shall see, the variables for rotation are analogous to those for one-dimensional motion and, as in Chapter 2, an important special situation is where the acceleration (here the rotational acceleration) is constant. We shall also see that Newton's second law can be written for rotational motion, but we must use a new quantity called *torque* instead of just force. Work and the work–kinetic energy theorem can also be applied to rotational motion, but we must use a new quantity called *rotational inertia* instead of just mass. In short, much of what we have discussed so far can be applied to rotational motion with, perhaps, a few changes.

10-2 The Rotational Variables

We wish to examine the rotation of a rigid body about a fixed axis. A **rigid body** is a body that can rotate with all its parts locked together and without any change in its shape. A **fixed axis** means that the rotation occurs about an axis that does not move. Thus, we shall not examine an object like the Sun, because the parts of the Sun (a ball of gas) are not locked together. We also shall not examine an object like a bowling ball rolling along a lane, because the ball rotates about a moving axis (the ball's motion is a mixture of rotation and translation).

Fig. 10-1 Figure skater Sasha Cohen in motion of (a) pure translation in a fixed direction and (b) pure rotation about a vertical axis. (a: Mike Segar/Reuters/Landov LLC; b: Elsa/Getty Images, Inc.)



(a)



(b)

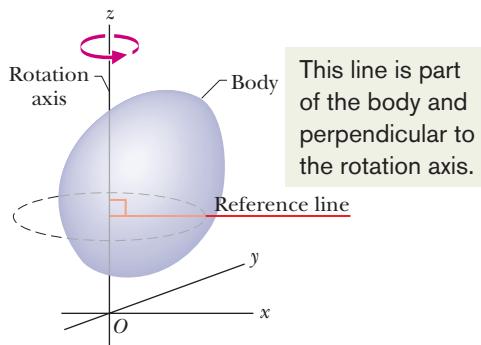


Fig. 10-2 A rigid body of arbitrary shape in pure rotation about the z axis of a coordinate system. The position of the *reference line* with respect to the rigid body is arbitrary, but it is perpendicular to the rotation axis. It is fixed in the body and rotates with the body.

Figure 10-2 shows a rigid body of arbitrary shape in rotation about a fixed axis, called the **axis of rotation** or the **rotation axis**. In pure rotation (*angular motion*), every point of the body moves in a circle whose center lies on the axis of rotation, and every point moves through the same angle during a particular time interval. In pure translation (*linear motion*), every point of the body moves in a straight line, and every point moves through the same *linear distance* during a particular time interval.

We deal now—one at a time—with the angular equivalents of the linear quantities position, displacement, velocity, and acceleration.

Angular Position

Figure 10-2 shows a *reference line*, fixed in the body, perpendicular to the rotation axis and rotating with the body. The **angular position** of this line is the angle of the line relative to a fixed direction, which we take as the **zero angular position**. In Fig. 10-3, the angular position θ is measured relative to the positive direction of the x axis. From geometry, we know that θ is given by

$$\theta = \frac{s}{r} \quad (\text{radian measure}). \quad (10-1)$$

Here s is the length of a circular arc that extends from the x axis (the zero angular position) to the reference line, and r is the radius of the circle.

An angle defined in this way is measured in **radians** (rad) rather than in revolutions (rev) or degrees. The radian, being the ratio of two lengths, is a pure number and thus has no dimension. Because the circumference of a circle of radius r is $2\pi r$, there are 2π radians in a complete circle:

$$1 \text{ rev} = 360^\circ = \frac{2\pi r}{r} = 2\pi \text{ rad}, \quad (10-2)$$

and thus

$$1 \text{ rad} = 57.3^\circ = 0.159 \text{ rev}. \quad (10-3)$$

We do *not* reset θ to zero with each complete rotation of the reference line about the rotation axis. If the reference line completes two revolutions from the zero angular position, then the angular position θ of the line is $\theta = 4\pi$ rad.

For pure translation along an x axis, we can know all there is to know about a moving body if we know $x(t)$, its position as a function of time. Similarly, for pure rotation, we can know all there is to know about a rotating body if we know $\theta(t)$, the angular position of the body's reference line as a function of time.

Angular Displacement

If the body of Fig. 10-3 rotates about the rotation axis as in Fig. 10-4, changing the angular position of the reference line from θ_1 to θ_2 , the body undergoes an **angular displacement** $\Delta\theta$ given by

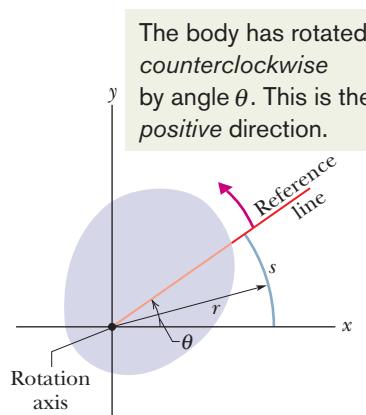
$$\Delta\theta = \theta_2 - \theta_1. \quad (10-4)$$

This definition of angular displacement holds not only for the rigid body as a whole but also for *every particle within that body*.

If a body is in translational motion along an x axis, its displacement Δx is either positive or negative, depending on whether the body is moving in the positive or negative direction of the axis. Similarly, the angular displacement $\Delta\theta$ of a rotating body is either positive or negative, according to the following rule:



An angular displacement in the counterclockwise direction is positive, and one in the clockwise direction is negative.



This dot means that the rotation axis is out toward you.

Fig. 10-3 The rotating rigid body of Fig. 10-2 in cross section, viewed from above. The plane of the cross section is perpendicular to the rotation axis, which now extends out of the page, toward you. In this position of the body, the reference line makes an angle θ with the x axis.

The phrase “*clocks are negative*” can help you remember this rule (they certainly are negative when their alarms sound off early in the morning).



CHECKPOINT 1

A disk can rotate about its central axis like a merry-go-round. Which of the following pairs of values for its initial and final angular positions, respectively, give a negative angular displacement: (a) $-3 \text{ rad}, +5 \text{ rad}$, (b) $-3 \text{ rad}, -7 \text{ rad}$, (c) $7 \text{ rad}, -3 \text{ rad}$?

Angular Velocity

Suppose that our rotating body is at angular position θ_1 at time t_1 and at angular position θ_2 at time t_2 as in Fig. 10-4. We define the **average angular velocity** of the body in the time interval Δt from t_1 to t_2 to be

$$\omega_{\text{avg}} = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{\Delta\theta}{\Delta t}, \quad (10-5)$$

where $\Delta\theta$ is the angular displacement during Δt (ω is the lowercase omega).

The (**instantaneous**) **angular velocity** ω , with which we shall be most concerned, is the limit of the ratio in Eq. 10-5 as Δt approaches zero. Thus,

$$\omega = \lim_{\Delta t \rightarrow 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}. \quad (10-6)$$

If we know $\theta(t)$, we can find the angular velocity ω by differentiation.

Equations 10-5 and 10-6 hold not only for the rotating rigid body as a whole but also for *every particle of that body* because the particles are all locked together. The unit of angular velocity is commonly the radian per second (rad/s) or the revolution per second (rev/s). Another measure of angular velocity was used during at least the first three decades of rock: Music was produced by vinyl (phonograph) records that were played on turntables at “ $33\frac{1}{3}$ rpm” or “45 rpm,” meaning at $33\frac{1}{3}$ rev/min or 45 rev/min.

If a particle moves in translation along an x axis, its linear velocity v is either positive or negative, depending on its direction along the axis. Similarly, the angular velocity ω of a rotating rigid body is either positive or negative, depending on whether the body is rotating counterclockwise (positive) or clockwise (negative). (“Clocks are negative” still works.) The magnitude of an angular velocity is called the **angular speed**, which is also represented with ω .

Angular Acceleration

If the angular velocity of a rotating body is not constant, then the body has an angular acceleration. Let ω_2 and ω_1 be its angular velocities at times t_2 and t_1 , respectively. The **average angular acceleration** of the rotating body in the interval from t_1 to t_2 is defined as

$$\alpha_{\text{avg}} = \frac{\omega_2 - \omega_1}{t_2 - t_1} = \frac{\Delta\omega}{\Delta t}, \quad (10-7)$$

in which $\Delta\omega$ is the change in the angular velocity that occurs during the time interval Δt . The (**instantaneous**) **angular acceleration** α , with which we shall be most concerned, is the limit of this quantity as Δt approaches zero. Thus,

$$\alpha = \lim_{\Delta t \rightarrow 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt}. \quad (10-8)$$

Equations 10-7 and 10-8 also hold for *every particle of that body*. The unit of angular acceleration is commonly the radian per second-squared (rad/s²) or the revolution per second-squared (rev/s²).

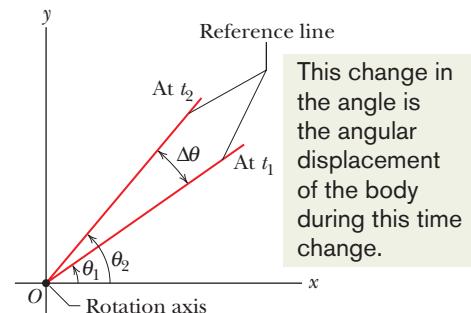


Fig. 10-4 The reference line of the rigid body of Figs. 10-2 and 10-3 is at angular position θ_1 at time t_1 and at angular position θ_2 at a later time t_2 . The quantity $\Delta\theta (= \theta_2 - \theta_1)$ is the angular displacement that occurs during the interval $\Delta t (= t_2 - t_1)$. The body itself is not shown.

Sample Problem

Angular velocity derived from angular position

The disk in Fig. 10-5a is rotating about its central axis like a merry-go-round. The angular position $\theta(t)$ of a reference line on the disk is given by

$$\theta = -1.00 - 0.600t + 0.250t^2, \quad (10-9)$$

with t in seconds, θ in radians, and the zero angular position as indicated in the figure.

- (a) Graph the angular position of the disk versus time from $t = -3.0$ s to $t = 5.4$ s. Sketch the disk and its angular position reference line at $t = -2.0$ s, 0 s, and 4.0 s, and when the curve crosses the t axis.

KEY IDEA

The angular position of the disk is the angular position $\theta(t)$ of its reference line, which is given by Eq. 10-9 as a function of time t . So we graph Eq. 10-9; the result is shown in Fig. 10-5b.

Calculations: To sketch the disk and its reference line at a particular time, we need to determine θ for that time. To do so, we substitute the time into Eq. 10-9. For $t = -2.0$ s, we get

$$\theta = -1.00 - (0.600)(-2.0) + (0.250)(-2.0)^2$$

$$= 1.2 \text{ rad} = 1.2 \text{ rad} \frac{360^\circ}{2\pi \text{ rad}} = 69^\circ.$$

This means that at $t = -2.0$ s the reference line on the disk is rotated counterclockwise from the zero position by $1.2 \text{ rad} = 69^\circ$ (counterclockwise because θ is positive). Sketch 1 in Fig. 10-5b shows this position of the reference line.

Similarly, for $t = 0$, we find $\theta = -1.00 \text{ rad} = -57^\circ$, which means that the reference line is rotated clockwise from the zero angular position by 1.0 rad , or 57° , as shown in sketch 3. For $t = 4.0$ s, we find $\theta = 0.60 \text{ rad} = 34^\circ$ (sketch 5). Drawing sketches for when the curve crosses the t axis is easy, because

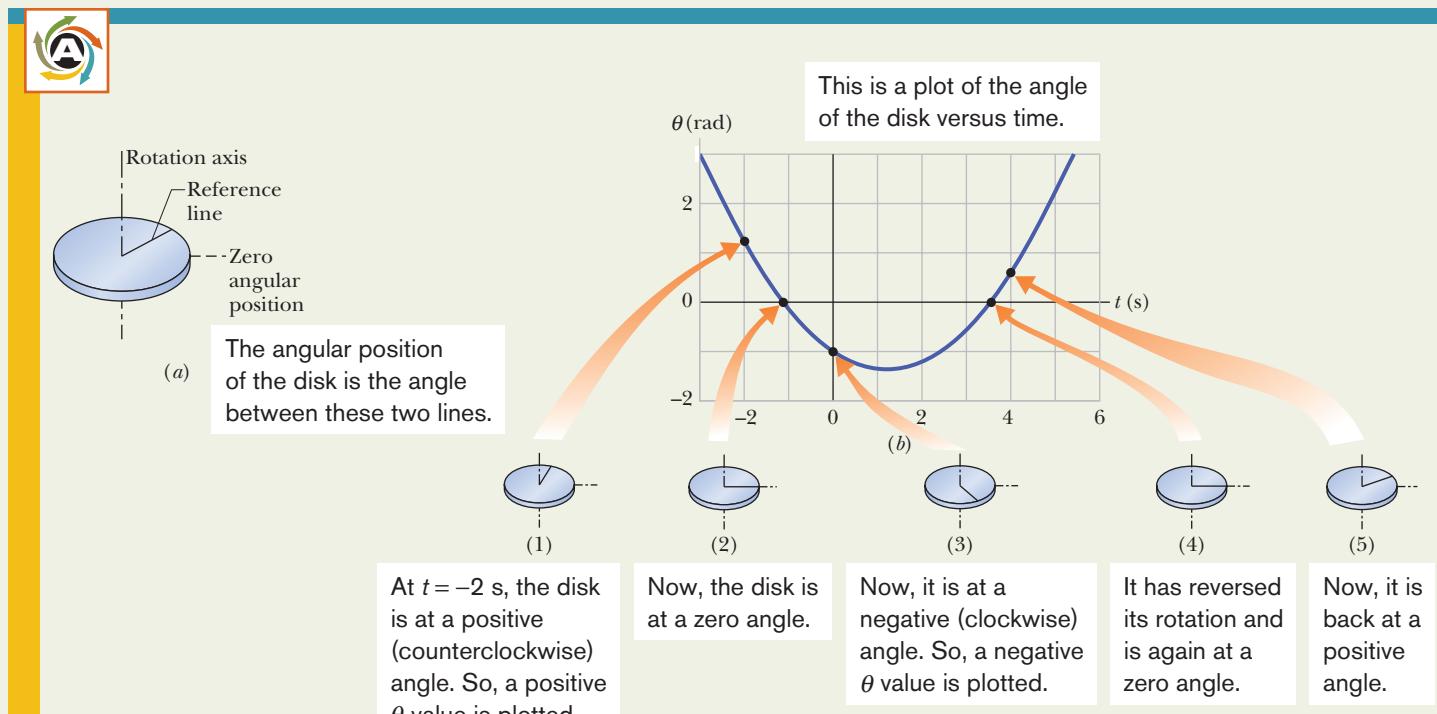


Fig. 10-5 (a) A rotating disk. (b) A plot of the disk's angular position $\theta(t)$. Five sketches indicate the angular position of the reference line on the disk for five points on the curve. (c) A plot of the disk's angular velocity $\omega(t)$. Positive values of ω correspond to counterclockwise rotation, and negative values to clockwise rotation.

then $\theta = 0$ and the reference line is momentarily aligned with the zero angular position (sketches 2 and 4).

- (b) At what time t_{\min} does $\theta(t)$ reach the minimum value shown in Fig. 10-5b? What is that minimum value?

KEY IDEA

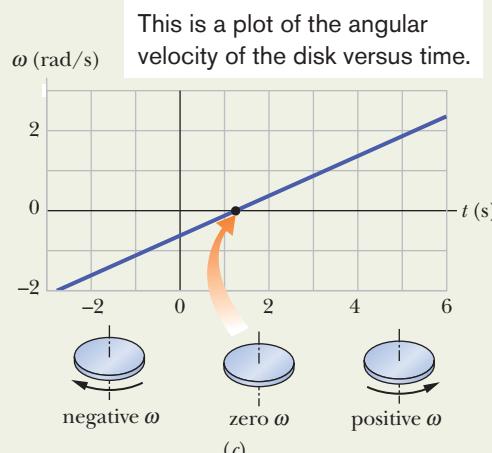
To find the extreme value (here the minimum) of a function, we take the first derivative of the function and set the result to zero.

Calculations: The first derivative of $\theta(t)$ is

$$\frac{d\theta}{dt} = -0.600 + 0.500t. \quad (10-10)$$

Setting this to zero and solving for t give us the time at which $\theta(t)$ is minimum:

$$t_{\min} = 1.20 \text{ s.} \quad (\text{Answer})$$



The angular velocity is initially negative and slowing, then momentarily zero during reversal, and then positive and increasing.

To get the minimum value of θ , we next substitute t_{\min} into Eq. 10-9, finding

$$\theta = -1.36 \text{ rad} \approx -77.9^\circ. \quad (\text{Answer})$$

This *minimum* of $\theta(t)$ (the bottom of the curve in Fig. 10-5b) corresponds to the *maximum clockwise* rotation of the disk from the zero angular position, somewhat more than is shown in sketch 3.

- (c) Graph the angular velocity ω of the disk versus time from $t = -3.0 \text{ s}$ to $t = 6.0 \text{ s}$. Sketch the disk and indicate the direction of turning and the sign of ω at $t = -2.0 \text{ s}, 4.0 \text{ s}$, and t_{\min} .

KEY IDEA

From Eq. 10-6, the angular velocity ω is equal to $d\theta/dt$ as given in Eq. 10-10. So, we have

$$\omega = -0.600 + 0.500t. \quad (10-11)$$

The graph of this function $\omega(t)$ is shown in Fig. 10-5c.

Calculations: To sketch the disk at $t = -2.0 \text{ s}$, we substitute that value into Eq. 10-11, obtaining

$$\omega = -1.6 \text{ rad/s.} \quad (\text{Answer})$$

The minus sign here tells us that at $t = -2.0 \text{ s}$, the disk is turning clockwise (the left-hand sketch in Fig. 10-5c).

Substituting $t = 4.0 \text{ s}$ into Eq. 10-11 gives us

$$\omega = 1.4 \text{ rad/s.} \quad (\text{Answer})$$

The implied plus sign tells us that now the disk is turning counterclockwise (the right-hand sketch in Fig. 10-5c).

For t_{\min} , we already know that $d\theta/dt = 0$. So, we must also have $\omega = 0$. That is, the disk momentarily stops when the reference line reaches the minimum value of θ in Fig. 10-5b, as suggested by the center sketch in Fig. 10-5c. On the graph, this momentary stop is the zero point where the plot changes from the negative clockwise motion to the positive counterclockwise motion.

- (d) Use the results in parts (a) through (c) to describe the motion of the disk from $t = -3.0 \text{ s}$ to $t = 6.0 \text{ s}$.

Description: When we first observe the disk at $t = -3.0 \text{ s}$, it has a positive angular position and is turning clockwise but slowing. It stops at angular position $\theta = -1.36 \text{ rad}$ and then begins to turn counterclockwise, with its angular position eventually becoming positive again.



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Sample Problem**Angular velocity derived from angular acceleration**

A child's top is spun with angular acceleration

$$\alpha = 5t^3 - 4t,$$

with t in seconds and α in radians per second-squared. At $t = 0$, the top has angular velocity 5 rad/s, and a reference line on it is at angular position $\theta = 2$ rad.

(a) Obtain an expression for the angular velocity $\omega(t)$ of the top. That is, find an expression that explicitly indicates how the angular velocity depends on time. (We can tell that there *is* such a dependence because the top is undergoing an angular acceleration, which means that its angular velocity *is* changing.)

KEY IDEA

By definition, $\alpha(t)$ is the derivative of $\omega(t)$ with respect to time. Thus, we can find $\omega(t)$ by integrating $\alpha(t)$ with respect to time.

Calculations: Equation 10-8 tells us

$$d\omega = \alpha dt,$$

so

$$\int d\omega = \int \alpha dt.$$

From this we find

$$\omega = \int (5t^3 - 4t) dt = \frac{5}{4}t^4 - \frac{4}{2}t^2 + C.$$

To evaluate the constant of integration C , we note that $\omega = 5$ rad/s at $t = 0$. Substituting these values in our expression for ω yields

$$5 \text{ rad/s} = 0 - 0 + C,$$

so $C = 5$ rad/s. Then

$$\omega = \frac{5}{4}t^4 - 2t^2 + 5. \quad (\text{Answer})$$

(b) Obtain an expression for the angular position $\theta(t)$ of the top.

KEY IDEA

By definition, $\omega(t)$ is the derivative of $\theta(t)$ with respect to time. Therefore, we can find $\theta(t)$ by integrating $\omega(t)$ with respect to time.

Calculations: Since Eq. 10-6 tells us that

$$d\theta = \omega dt,$$

we can write

$$\begin{aligned} \theta &= \int \omega dt = \int \left(\frac{5}{4}t^4 - 2t^2 + 5\right) dt \\ &= \frac{1}{4}t^5 - \frac{2}{3}t^3 + 5t + C' \\ &= \frac{1}{4}t^5 - \frac{2}{3}t^3 + 5t + 2, \end{aligned} \quad (\text{Answer})$$

where C' has been evaluated by noting that $\theta = 2$ rad at $t = 0$.



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10-3 Are Angular Quantities Vectors?

We can describe the position, velocity, and acceleration of a single particle by means of vectors. If the particle is confined to a straight line, however, we do not really need vector notation. Such a particle has only two directions available to it, and we can indicate these directions with plus and minus signs.

In the same way, a rigid body rotating about a fixed axis can rotate only clockwise or counterclockwise as seen along the axis, and again we can select between the two directions by means of plus and minus signs. The question arises: "Can we treat the angular displacement, velocity, and acceleration of a rotating body as vectors?" The answer is a qualified "yes" (see the caution below, in connection with angular displacements).

Consider the angular velocity. Figure 10-6a shows a vinyl record rotating on a turntable. The record has a constant angular speed ω ($= 33\frac{1}{3}$ rev/min) in the clockwise direction. We can represent its angular velocity as a vector $\vec{\omega}$ pointing along the axis of rotation, as in Fig. 10-6b. Here's how: We choose the length of this vector according to some convenient scale, for example, with 1 cm corresponding to 10 rev/min. Then we establish a direction for the vector $\vec{\omega}$ by using a

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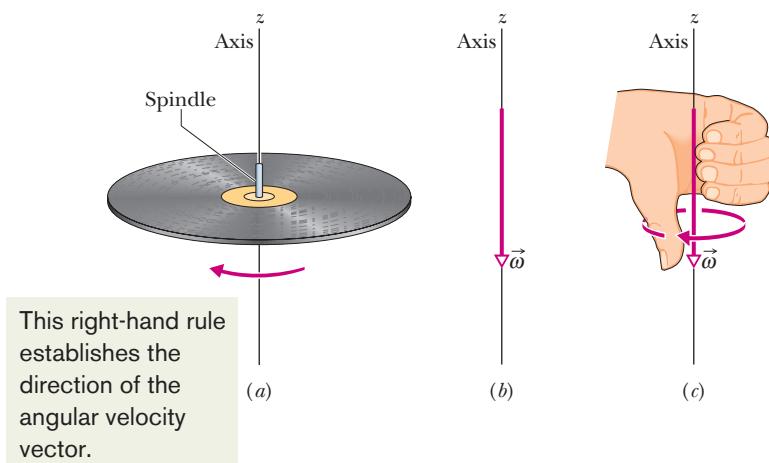


Fig. 10-6 (a) A record rotating about a vertical axis that coincides with the axis of the spindle. (b) The angular velocity of the rotating record can be represented by the vector $\vec{\omega}$, lying along the axis and pointing down, as shown. (c) We establish the direction of the angular velocity vector as downward by using a right-hand rule. When the fingers of the right hand curl around the record and point the way it is moving, the extended thumb points in the direction of $\vec{\omega}$.

right-hand rule, as Fig. 10-6c shows: Curl your right hand about the rotating record, your fingers pointing *in the direction of rotation*. Your extended thumb will then point in the direction of the angular velocity vector. If the record were to rotate in the opposite sense, the right-hand rule would tell you that the angular velocity vector then points in the opposite direction.

It is not easy to get used to representing angular quantities as vectors. We instinctively expect that something should be moving *along* the direction of a vector. That is not the case here. Instead, something (the rigid body) is rotating *around* the direction of the vector. In the world of pure rotation, a vector defines an axis of rotation, not a direction in which something moves. Nonetheless, the vector also defines the motion. Furthermore, it obeys all the rules for vector manipulation discussed in Chapter 3. The angular acceleration $\vec{\alpha}$ is another vector, and it too obeys those rules.

In this chapter we consider only rotations that are about a fixed axis. For such situations, we need not consider vectors—we can represent angular velocity with ω and angular acceleration with α , and we can indicate direction with an implied plus sign for counterclockwise or an explicit minus sign for clockwise.

Now for the caution: Angular *displacements* (unless they are very small) *cannot* be treated as vectors. Why not? We can certainly give them both magnitude and direction, as we did for the angular velocity vector in Fig. 10-6. However, to be represented as a vector, a quantity must *also* obey the rules of vector addition, one of which says that if you add two vectors, the order in which you add them does not matter. Angular displacements fail this test.

Figure 10-7 gives an example. An initially horizontal book is given two 90° angular displacements, first in the order of Fig. 10-7a and then in the order of Fig. 10-7b. Although the two angular displacements are identical, their order is not, and the book ends up with different orientations. Here's another example. Hold your right arm downward, palm toward your thigh. Keeping your wrist rigid, (1) lift the arm forward until it is horizontal, (2) move it horizontally until it points toward the right, and (3) then bring it down to your side. Your palm faces forward. If you start over, but reverse the steps, which way does your palm end up facing? From either example, we must conclude that the addition of two angular displacements depends on their order and they cannot be vectors.

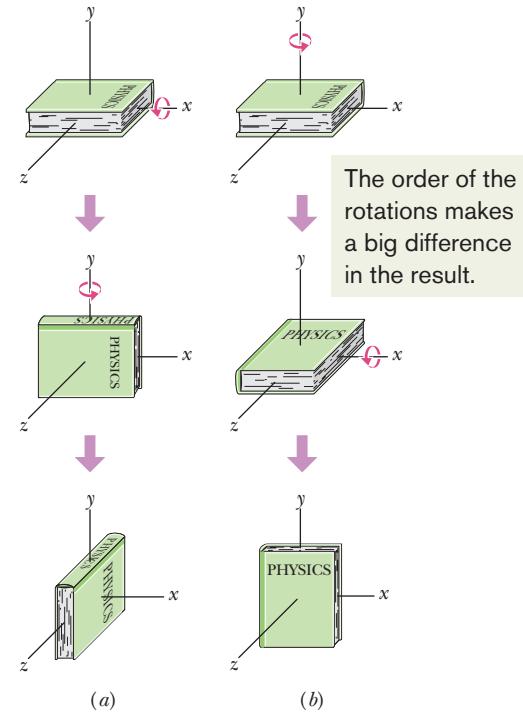


Fig. 10-7 (a) From its initial position, at the top, the book is given two successive 90° rotations, first about the (horizontal) x axis and then about the (vertical) y axis. (b) The book is given the same rotations, but in the reverse order.

10-4 Rotation with Constant Angular Acceleration

In pure translation, motion with a *constant linear acceleration* (for example, that of a falling body) is an important special case. In Table 2-1, we displayed a series of equations that hold for such motion.

In pure rotation, the case of *constant angular acceleration* is also important, and a parallel set of equations holds for this case also. We shall not derive them here, but simply write them from the corresponding linear equations, substituting equivalent angular quantities for the linear ones. This is done in Table 10-1, which lists both sets of equations (Eqs. 2-11 and 2-15 to 2-18; 10-12 to 10-16).

Recall that Eqs. 2-11 and 2-15 are basic equations for constant linear acceleration—the other equations in the Linear list can be derived from them. Similarly, Eqs. 10-12 and 10-13 are the basic equations for constant angular acceleration, and the other equations in the Angular list can be derived from them. To solve a simple problem involving constant angular acceleration, you can usually use an equation from the Angular list (*if you have the list*). Choose an equation for which the only unknown variable will be the variable requested in the problem. A better plan is to remember only Eqs. 10-12 and 10-13, and then solve them as simultaneous equations whenever needed.

CHECKPOINT 2

In four situations, a rotating body has angular position $\theta(t)$ given by (a) $\theta = 3t - 4$, (b) $\theta = -5t^3 + 4t^2 + 6$, (c) $\theta = 2/t^2 - 4/t$, and (d) $\theta = 5t^2 - 3$. To which situations do the angular equations of Table 10-1 apply?

Table 10-1

Equations of Motion for Constant Linear Acceleration and for Constant Angular Acceleration

Equation Number	Linear Equation	Missing Variable	Angular Equation	Equation Number	
(2-11)	$v = v_0 + at$	$x - x_0$	$\theta - \theta_0$	$\omega = \omega_0 + \alpha t$	(10-12)
(2-15)	$x - x_0 = v_0 t + \frac{1}{2}at^2$	v	ω	$\theta - \theta_0 = \omega_0 t + \frac{1}{2}\alpha t^2$	(10-13)
(2-16)	$v^2 = v_0^2 + 2a(x - x_0)$	t	t	$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$	(10-14)
(2-17)	$x - x_0 = \frac{1}{2}(v_0 + v)t$	a	α	$\theta - \theta_0 = \frac{1}{2}(\omega_0 + \omega)t$	(10-15)
(2-18)	$x - x_0 = vt - \frac{1}{2}at^2$	v_0	ω_0	$\theta - \theta_0 = \omega t - \frac{1}{2}\alpha t^2$	(10-16)

Sample Problem

Constant angular acceleration, grindstone

A grindstone (Fig. 10-8) rotates at constant angular acceleration $\alpha = 0.35 \text{ rad/s}^2$. At time $t = 0$, it has an angular velocity of $\omega_0 = -4.6 \text{ rad/s}$ and a reference line on it is horizontal, at the angular position $\theta_0 = 0$.

- (a) At what time after $t = 0$ is the reference line at the angular position $\theta = 5.0 \text{ rev}$?

KEY IDEA

The angular acceleration is constant, so we can use the rota-

tion equations of Table 10-1. We choose Eq. 10-13,

$$\theta - \theta_0 = \omega_0 t + \frac{1}{2}\alpha t^2,$$

because the only unknown variable it contains is the desired time t .

Calculations: Substituting known values and setting $\theta_0 = 0$ and $\theta = 5.0 \text{ rev} = 10\pi \text{ rad}$ give us

$$10\pi \text{ rad} = (-4.6 \text{ rad/s})t + \frac{1}{2}(0.35 \text{ rad/s}^2)t^2.$$

(We converted 5.0 rev to 10π rad to keep the units consis-

tent.) Solving this quadratic equation for t , we find

$$t = 32 \text{ s.} \quad (\text{Answer})$$

Now notice something a bit strange. We first see the wheel when it is rotating in the negative direction and through the $\theta = 0$ orientation. Yet, we just found out that 32 s later it is at the positive orientation of $\theta = 5.0$ rev. What happened in that time interval so that it could be at a positive orientation?

- (b) Describe the grindstone's rotation between $t = 0$ and $t = 32$ s.

Description: The wheel is initially rotating in the negative (clockwise) direction with angular velocity $\omega_0 = -4.6 \text{ rad/s}$, but its angular acceleration α is positive. This initial opposition of the signs of angular velocity and angular acceleration means that the wheel slows in its rotation in the negative direction, stops, and then reverses to rotate in the positive direction. After the reference line comes back through its initial orientation of $\theta = 0$, the wheel turns an additional 5.0 rev by time $t = 32$ s.

- (c) At what time t does the grindstone momentarily stop?

Calculation: We again go to the table of equations for constant angular acceleration, and again we need an equation

We measure rotation by using this reference line.

Clockwise = negative

Counterclockwise = positive

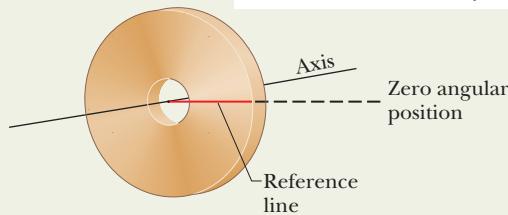


Fig. 10-8 A grindstone. At $t = 0$ the reference line (which we imagine to be marked on the stone) is horizontal.

that contains only the desired unknown variable t . However, now the equation must also contain the variable ω , so that we can set it to 0 and then solve for the corresponding time t . We choose Eq. 10-12, which yields

$$t = \frac{\omega - \omega_0}{\alpha} = \frac{0 - (-4.6 \text{ rad/s})}{0.35 \text{ rad/s}^2} = 13 \text{ s.} \quad (\text{Answer})$$

Sample Problem

Constant angular acceleration, riding a Rotor

While you are operating a Rotor (a large, vertical, rotating cylinder found in amusement parks), you spot a passenger in acute distress and decrease the angular velocity of the cylinder from 3.40 rad/s to 2.00 rad/s in 20.0 rev, at constant angular acceleration. (The passenger is obviously more of a “translation person” than a “rotation person.”)

- (a) What is the constant angular acceleration during this decrease in angular speed?

KEY IDEA

Because the cylinder's angular acceleration is constant, we can relate it to the angular velocity and angular displacement via the basic equations for constant angular acceleration (Eqs. 10-12 and 10-13).

Calculations: The initial angular velocity is $\omega_0 = 3.40 \text{ rad/s}$, the angular displacement is $\theta - \theta_0 = 20.0$ rev, and the angular velocity at the end of that displacement is $\omega = 2.00 \text{ rad/s}$. But we do not know the angular acceleration α and time t , which are in both basic equations.

To eliminate the unknown t , we use Eq. 10-12 to write

$$t = \frac{\omega - \omega_0}{\alpha},$$

which we then substitute into Eq. 10-13 to write

$$\theta - \theta_0 = \omega_0 \left(\frac{\omega - \omega_0}{\alpha} \right) + \frac{1}{2} \alpha \left(\frac{\omega - \omega_0}{\alpha} \right)^2.$$

Solving for α , substituting known data, and converting 20 rev to 125.7 rad, we find

$$\begin{aligned} \alpha &= \frac{\omega^2 - \omega_0^2}{2(\theta - \theta_0)} = \frac{(2.00 \text{ rad/s})^2 - (3.40 \text{ rad/s})^2}{2(125.7 \text{ rad})} \\ &= -0.0301 \text{ rad/s}^2. \end{aligned} \quad (\text{Answer})$$

- (b) How much time did the speed decrease take?

Calculation: Now that we know α , we can use Eq. 10-12 to solve for t :

$$\begin{aligned} t &= \frac{\omega - \omega_0}{\alpha} = \frac{2.00 \text{ rad/s} - 3.40 \text{ rad/s}}{-0.0301 \text{ rad/s}^2} \\ &= 46.5 \text{ s.} \end{aligned} \quad (\text{Answer})$$



Additional examples, video, and practice available at WileyPLUS

10-5 Relating the Linear and Angular Variables

In Section 4-7, we discussed uniform circular motion, in which a particle travels at constant linear speed v along a circle and around an axis of rotation. When a rigid body, such as a merry-go-round, rotates around an axis, each particle in the body moves in its own circle around that axis. Since the body is rigid, all the particles make one revolution in the same amount of time; that is, they all have the same angular speed ω .

However, the farther a particle is from the axis, the greater the circumference of its circle is, and so the faster its linear speed v must be. You can notice this on a merry-go-round. You turn with the same angular speed ω regardless of your distance from the center, but your linear speed v increases noticeably if you move to the outside edge of the merry-go-round.

We often need to relate the linear variables s , v , and a for a particular point in a rotating body to the angular variables θ , ω , and α for that body. The two sets of variables are related by r , the *perpendicular distance* of the point from the rotation axis. This perpendicular distance is the distance between the point and the rotation axis, measured along a perpendicular to the axis. It is also the radius r of the circle traveled by the point around the axis of rotation.

The Position

If a reference line on a rigid body rotates through an angle θ , a point within the body at a position r from the rotation axis moves a distance s along a circular arc, where s is given by Eq. 10-1:

$$s = \theta r \quad (\text{radian measure}). \quad (10-17)$$

This is the first of our linear–angular relations. *Caution:* The angle θ here must be measured in radians because Eq. 10-17 is itself the definition of angular measure in radians.

The Speed

Differentiating Eq. 10-17 with respect to time—with r held constant—leads to

$$\frac{ds}{dt} = \frac{d\theta}{dt} r.$$

However, ds/dt is the linear speed (the magnitude of the linear velocity) of the point in question, and $d\theta/dt$ is the angular speed ω of the rotating body. So

$$v = \omega r \quad (\text{radian measure}). \quad (10-18)$$

Caution: The angular speed ω must be expressed in radian measure.

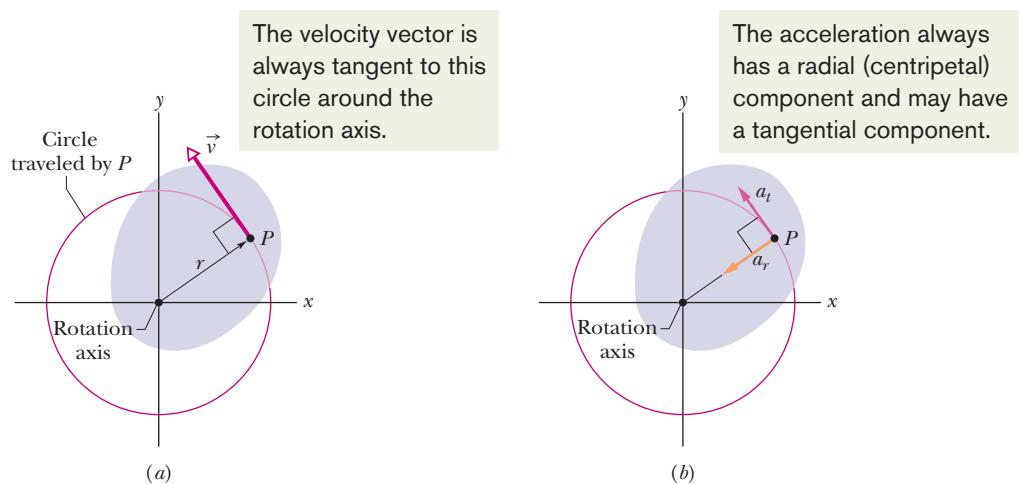
Equation 10-18 tells us that since all points within the rigid body have the same angular speed ω , points with greater radius r have greater linear speed v . Figure 10-9a reminds us that the linear velocity is always tangent to the circular path of the point in question.

If the angular speed ω of the rigid body is constant, then Eq. 10-18 tells us that the linear speed v of any point within it is also constant. Thus, each point within the body undergoes uniform circular motion. The period of revolution T for the motion of each point and for the rigid body itself is given by Eq. 4-35:

$$T = \frac{2\pi r}{v}. \quad (10-19)$$

This equation tells us that the time for one revolution is the distance $2\pi r$ traveled in one revolution divided by the speed at which that distance is traveled.

Fig. 10-9 The rotating rigid body of Fig. 10-2, shown in cross section viewed from above. Every point of the body (such as P) moves in a circle around the rotation axis. (a) The linear velocity \vec{v} of every point is tangent to the circle in which the point moves. (b) The linear acceleration \vec{a} of the point has (in general) two components: tangential a_t and radial a_r .



Substituting for v from Eq. 10-18 and canceling r , we find also that

$$T = \frac{2\pi}{\omega} \quad (\text{radian measure}). \quad (10-20)$$

This equivalent equation says that the time for one revolution is the angular distance 2π rad traveled in one revolution divided by the angular speed (or rate) at which that angle is traveled.

The Acceleration

Differentiating Eq. 10-18 with respect to time—again with r held constant—leads to

$$\frac{dv}{dt} = \frac{d\omega}{dt} r. \quad (10-21)$$

Here we run up against a complication. In Eq. 10-21, dv/dt represents only the part of the linear acceleration that is responsible for changes in the *magnitude* v of the linear velocity \vec{v} . Like \vec{v} , that part of the linear acceleration is tangent to the path of the point in question. We call it the *tangential component* a_t of the linear acceleration of the point, and we write

$$a_t = \alpha r \quad (\text{radian measure}), \quad (10-22)$$

where $\alpha = d\omega/dt$. *Caution:* The angular acceleration α in Eq. 10-22 must be expressed in radian measure.

In addition, as Eq. 4-34 tells us, a particle (or point) moving in a circular path has a *radial component* of linear acceleration, $a_r = v^2/r$ (directed radially inward), that is responsible for changes in the *direction* of the linear velocity \vec{v} . By substituting for v from Eq. 10-18, we can write this component as

$$a_r = \frac{v^2}{r} = \omega^2 r \quad (\text{radian measure}). \quad (10-23)$$

Thus, as Fig. 10-9b shows, the linear acceleration of a point on a rotating rigid body has, in general, two components. The radially inward component a_r (given by Eq. 10-23) is present whenever the angular velocity of the body is not zero. The tangential component a_t (given by Eq. 10-22) is present whenever the angular acceleration is not zero.

**CHECKPOINT 3**

A cockroach rides the rim of a rotating merry-go-round. If the angular speed of this system (*merry-go-round + cockroach*) is constant, does the cockroach have (a) radial acceleration and (b) tangential acceleration? If ω is decreasing, does the cockroach have (c) radial acceleration and (d) tangential acceleration?

Sample Problem**Linear and angular variables, roller coaster speedup**

In spite of the extreme care taken in engineering a roller coaster, an unlucky few of the millions of people who ride roller coasters each year end up with a medical condition called *roller-coaster headache*. Symptoms, which might not appear for several days, include vertigo and headache, both severe enough to require medical treatment.

Let's investigate the probable cause by designing the track for our own *induction roller coaster* (which can be accelerated by magnetic forces even on a horizontal track). To create an initial thrill, we want each passenger to leave the loading point with acceleration g along the horizontal track. To increase the thrill, we also want that first section of track to form a circular arc (Fig. 10-10), so that the passenger also experiences a centripetal acceleration. As the passenger accelerates along the arc, the magnitude of this centripetal acceleration increases alarmingly. When the magnitude a of the net acceleration reaches $4g$ at some point P and angle θ_P along the arc, we want the passenger then to move in a straight line, along a tangent to the arc.

- (a) What angle θ_P should the arc subtend so that a is $4g$ at point P ?

KEY IDEAS

- At any given time, the passenger's net acceleration \vec{a} is the vector sum of the tangential acceleration \vec{a}_t along the track and the radial acceleration \vec{a}_r toward the arc's center of curvature (as in Fig. 10-9b).
- The value of a_r at any given time depends on the angular speed ω according to Eq. 10-23 ($a_r = \omega^2 r$, where r is the radius of the circular arc).
- An angular acceleration α around the arc is associated with the tangential acceleration a_t along the track according to Eq. 10-22 ($a_t = \alpha r$).
- Because a_t and r are constant, so is α and thus we can use the constant angular-acceleration equations.

Calculations: Because we are trying to determine a value for angular position θ , let's choose Eq. 10-14 from among the constant angular-acceleration equations:

$$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0). \quad (10-24)$$

For the angular acceleration α , we substitute from Eq. 10-22:

$$\alpha = \frac{a_t}{r}. \quad (10-25)$$

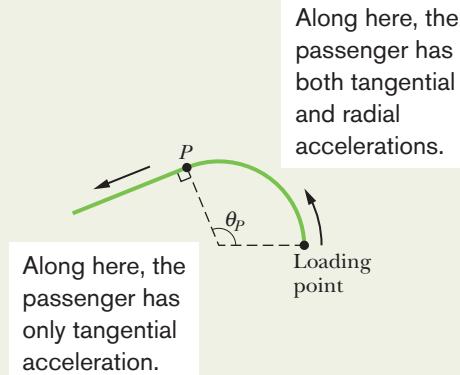


Fig. 10-10 An overhead view of a horizontal track for a roller coaster. The track begins as a circular arc at the loading point and then, at point P , continues along a tangent to the arc.

We also substitute $\omega_0 = 0$ and $\theta_0 = 0$, and we find

$$\omega^2 = \frac{2a_t\theta}{r}. \quad (10-26)$$

Substituting this result for ω^2 into

$$a_r = \omega^2 r \quad (10-27)$$

gives a relation between the radial acceleration, the tangential acceleration, and the angular position θ :

$$a_r = 2a_t\theta. \quad (10-28)$$

Because \vec{a}_t and \vec{a}_r are perpendicular vectors, their sum has the magnitude

$$a = \sqrt{a_t^2 + a_r^2}. \quad (10-29)$$

Substituting for a_r from Eq. 10-28 and solving for θ lead to

$$\theta = \frac{1}{2} \sqrt{\frac{a^2}{a_t^2} - 1}. \quad (10-30)$$

When a reaches the design value of $4g$, angle θ is the angle θ_P we want. Substituting $a = 4g$, $\theta = \theta_P$, and $a_t = g$ into Eq. 10-30, we find

$$\theta_P = \frac{1}{2} \sqrt{\frac{(4g)^2}{g^2} - 1} = 1.94 \text{ rad} = 111^\circ. \quad (\text{Answer})$$

- (b) What is the magnitude a of the passenger's net acceleration at point P and after point P ?

Reasoning: At P , a has the design value of $4g$. Just after P is reached, the passenger moves in a straight line and no longer has centripetal acceleration. Thus, the passenger has only the acceleration magnitude g along the track. Hence,

$$a = 4g \text{ at } P \quad \text{and} \quad a = g \text{ after } P. \quad (\text{Answer})$$

Roller-coaster headache can occur when a passenger's head undergoes an abrupt change in acceleration, with the

acceleration magnitude large before or after the change. The reason is that the change can cause the brain to move relative to the skull, tearing the veins that bridge the brain and skull. Our design to increase the acceleration from g to $4g$ along the path to P might harm the passenger, but the abrupt change in acceleration as the passenger passes through point P is more likely to cause roller-coaster headache.



Additional examples, video, and practice available at WileyPLUS

10-6 Kinetic Energy of Rotation

The rapidly rotating blade of a table saw certainly has kinetic energy due to that rotation. How can we express the energy? We cannot apply the familiar formula $K = \frac{1}{2}mv^2$ to the saw as a whole because that would give us the kinetic energy only of the saw's center of mass, which is zero.

Instead, we shall treat the table saw (and any other rotating rigid body) as a collection of particles with different speeds. We can then add up the kinetic energies of all the particles to find the kinetic energy of the body as a whole. In this way we obtain, for the kinetic energy of a rotating body,

$$\begin{aligned} K &= \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + \frac{1}{2}m_3v_3^2 + \dots \\ &= \sum \frac{1}{2}m_i v_i^2, \end{aligned} \quad (10-31)$$

in which m_i is the mass of the i th particle and v_i is its speed. The sum is taken over all the particles in the body.

The problem with Eq. 10-31 is that v_i is not the same for all particles. We solve this problem by substituting for v from Eq. 10-18 ($v = \omega r$), so that we have

$$K = \sum \frac{1}{2}m_i(\omega r_i)^2 = \frac{1}{2}\left(\sum m_i r_i^2\right)\omega^2, \quad (10-32)$$

in which ω is the same for all particles.

The quantity in parentheses on the right side of Eq. 10-32 tells us how the mass of the rotating body is distributed about its axis of rotation. We call that quantity the **rotational inertia** (or **moment of inertia**) I of the body with respect to the axis of rotation. It is a constant for a particular rigid body and a particular rotation axis. (That axis must always be specified if the value of I is to be meaningful.)

We may now write

$$I = \sum m_i r_i^2 \quad (\text{rotational inertia}) \quad (10-33)$$

and substitute into Eq. 10-32, obtaining

$$K = \frac{1}{2}I\omega^2 \quad (\text{radian measure}) \quad (10-34)$$

as the expression we seek. Because we have used the relation $v = \omega r$ in deriving Eq. 10-34, ω must be expressed in radian measure. The SI unit for I is the kilogram-square meter ($\text{kg} \cdot \text{m}^2$).

Equation 10-34, which gives the kinetic energy of a rigid body in pure rotation, is the angular equivalent of the formula $K = \frac{1}{2}Mv_{\text{com}}^2$, which gives the kinetic energy of a rigid body in pure translation. In both formulas there is a factor of $\frac{1}{2}$. Where mass M appears in one equation, I (which involves both mass and its distribution)

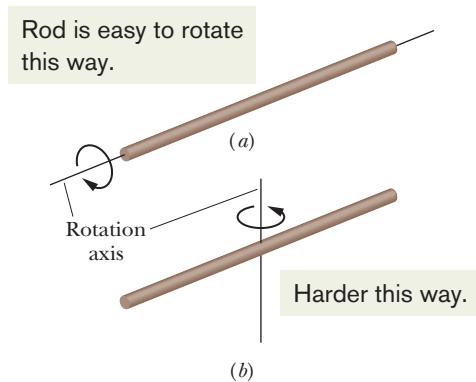


Fig. 10-11 A long rod is much easier to rotate about (a) its central (longitudinal) axis than about (b) an axis through its center and perpendicular to its length. The reason for the difference is that the mass is distributed closer to the rotation axis in (a) than in (b).

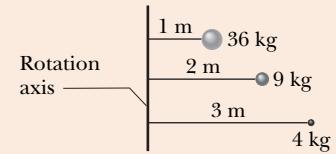
appears in the other. Finally, each equation contains as a factor the square of a speed—translational or rotational as appropriate. The kinetic energies of translation and of rotation are not different kinds of energy. They are both kinetic energy, expressed in ways that are appropriate to the motion at hand.

We noted previously that the rotational inertia of a rotating body involves not only its mass but also how that mass is distributed. Here is an example that you can literally feel. Rotate a long, fairly heavy rod (a pole, a length of lumber, or something similar), first around its central (longitudinal) axis (Fig. 10-11a) and then around an axis perpendicular to the rod and through the center (Fig. 10-11b). Both rotations involve the very same mass, but the first rotation is much easier than the second. The reason is that the mass is distributed much closer to the rotation axis in the first rotation. As a result, the rotational inertia of the rod is much smaller in Fig. 10-11a than in Fig. 10-11b. In general, smaller rotational inertia means easier rotation.



CHECKPOINT 4

The figure shows three small spheres that rotate about a vertical axis. The perpendicular distance between the axis and the center of each sphere is given. Rank the three spheres according to their rotational inertia about that axis, greatest first.



10-7 Calculating the Rotational Inertia

If a rigid body consists of a few particles, we can calculate its rotational inertia about a given rotation axis with Eq. 10-33 ($I = \sum m_i r_i^2$); that is, we can find the product mr^2 for each particle and then sum the products. (Recall that r is the perpendicular distance a particle is from the given rotation axis.)

If a rigid body consists of a great many adjacent particles (it is *continuous*, like a Frisbee), using Eq. 10-33 would require a computer. Thus, instead, we replace the sum in Eq. 10-33 with an integral and define the rotational inertia of the body as

$$I = \int r^2 dm \quad (\text{rotational inertia, continuous body}). \quad (10-35)$$

Table 10-2 gives the results of such integration for nine common body shapes and the indicated axes of rotation.

Parallel-Axis Theorem

Suppose we want to find the rotational inertia I of a body of mass M about a given axis. In principle, we can always find I with the integration of Eq. 10-35. However, there is a shortcut if we happen to already know the rotational inertia I_{com} of the body about a *parallel* axis that extends through the body's center of mass. Let h be the perpendicular distance between the given axis and the axis through the center of mass (remember these two axes must be parallel). Then the rotational inertia I about the given axis is

$$I = I_{\text{com}} + Mh^2 \quad (\text{parallel-axis theorem}). \quad (10-36)$$

This equation is known as the **parallel-axis theorem**. We shall now prove it.

Proof of the Parallel-Axis Theorem

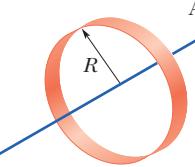
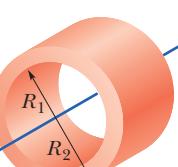
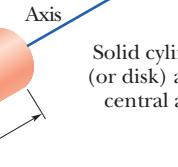
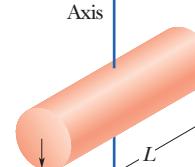
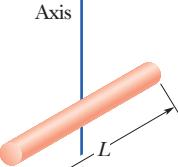
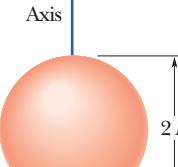
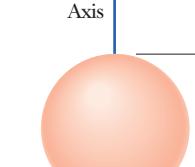
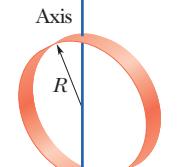
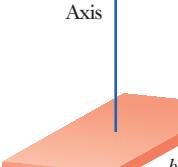
Let O be the center of mass of the arbitrarily shaped body shown in cross section in Fig. 10-12. Place the origin of the coordinates at O . Consider an axis through O

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Table 10-2

Some Rotational Inertias

 $I = MR^2$	 $I = \frac{1}{2}M(R_1^2 + R_2^2)$	 $I = \frac{1}{2}MR^2$
 $I = \frac{1}{4}MR^2 + \frac{1}{12}ML^2$	 $I = \frac{1}{12}ML^2$	 $I = \frac{2}{5}MR^2$
 $I = \frac{2}{3}MR^2$	 $I = \frac{1}{2}MR^2$	 $I = \frac{1}{12}M(a^2 + b^2)$

perpendicular to the plane of the figure, and another axis through point P parallel to the first axis. Let the x and y coordinates of P be a and b .

Let dm be a mass element with the general coordinates x and y . The rotational inertia of the body about the axis through P is then, from Eq. 10-35,

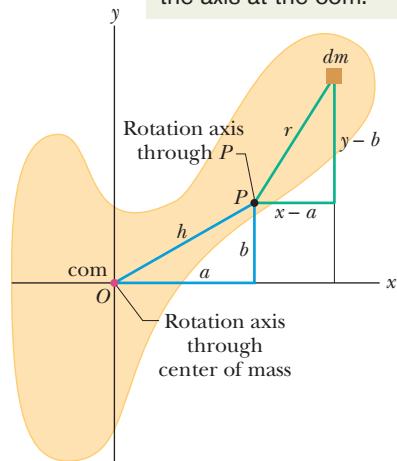
$$I = \int r^2 dm = \int [(x - a)^2 + (y - b)^2] dm,$$

which we can rearrange as

$$I = \int (x^2 + y^2) dm - 2a \int x dm - 2b \int y dm + \int (a^2 + b^2) dm. \quad (10-37)$$

From the definition of the center of mass (Eq. 9-9), the middle two integrals of Eq. 10-37 give the coordinates of the center of mass (multiplied by a constant) and thus must each be zero. Because $x^2 + y^2$ is equal to R^2 , where R is the distance from O to dm , the first integral is simply I_{com} , the rotational inertia of the body about an axis through its center of mass. Inspection of Fig. 10-12 shows that the last term in Eq. 10-37 is Mh^2 , where M is the body's total mass. Thus, Eq. 10-37 reduces to Eq. 10-36, which is the relation that we set out to prove.

We need to relate the rotational inertia around the axis at P to that around the axis at the com.



CHECKPOINT 5

The figure shows a book-like object (one side is longer than the other) and four choices of rotation axes, all perpendicular to the face of the object. Rank the choices according to the rotational inertia of the object about the axis, greatest first.

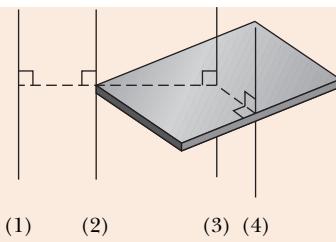


Fig. 10-12 A rigid body in cross section, with its center of mass at O . The parallel-axis theorem (Eq. 10-36) relates the rotational inertia of the body about an axis through O to that about a parallel axis through a point such as P , a distance h from the body's center of mass. Both axes are perpendicular to the plane of the figure.

Sample Problem

Rotational inertia of a two-particle system

Figure 10-13a shows a rigid body consisting of two particles of mass m connected by a rod of length L and negligible mass.

- (a) What is the rotational inertia I_{com} about an axis through the center of mass, perpendicular to the rod as shown?

KEY IDEA

Because we have only two particles with mass, we can find the body's rotational inertia I_{com} by using Eq. 10-33 rather than by integration.

Calculations: For the two particles, each at perpendicular distance $\frac{1}{2}L$ from the rotation axis, we have

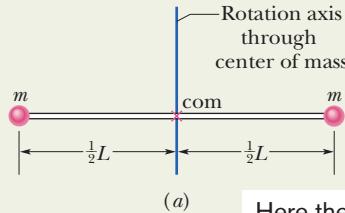
$$\begin{aligned} I &= \sum m_i r_i^2 = (m)(\frac{1}{2}L)^2 + (m)(\frac{1}{2}L)^2 \\ &= \frac{1}{2}mL^2. \end{aligned} \quad (\text{Answer})$$

- (b) What is the rotational inertia I of the body about an axis through the left end of the rod and parallel to the first axis (Fig. 10-13b)?

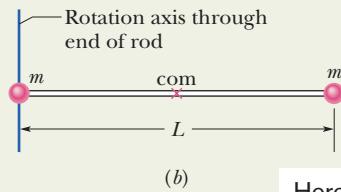
KEY IDEAS

This situation is simple enough that we can find I using either of two techniques. The first is similar to the one used in part (a). The other, more powerful one is to apply the parallel-axis theorem.

First technique: We calculate I as in part (a), except here the perpendicular distance r_i is zero for the particle on the left and



Here the rotation axis is through the com.



Here it has been shifted from the com without changing the orientation. We can use the parallel-axis theorem.

Fig. 10-13 A rigid body consisting of two particles of mass m joined by a rod of negligible mass.

L for the particle on the right. Now Eq. 10-33 gives us

$$I = m(0)^2 + mL^2 = mL^2. \quad (\text{Answer})$$

Second technique: Because we already know I_{com} about an axis through the center of mass and because the axis here is parallel to that "com axis," we can apply the parallel-axis theorem (Eq. 10-36). We find

$$\begin{aligned} I &= I_{\text{com}} + Mh^2 = \frac{1}{2}mL^2 + (2m)(\frac{1}{2}L)^2 \\ &= mL^2. \end{aligned} \quad (\text{Answer})$$

Sample Problem

Rotational inertia of a uniform rod, integration

Figure 10-14 shows a thin, uniform rod of mass M and length L , on an x axis with the origin at the rod's center.

- (a) What is the rotational inertia of the rod about the perpendicular rotation axis through the center?

KEY IDEAS

(1) Because the rod is uniform, its center of mass is at its center. Therefore, we are looking for I_{com} . (2) Because the rod is a continuous object, we must use the integral of Eq. 10-35,

$$I = \int r^2 dm, \quad (10-38)$$

to find the rotational inertia.

Calculations: We want to integrate with respect to coordi-

nate x (not mass m as indicated in the integral), so we must relate the mass dm of an element of the rod to its length dx along the rod. (Such an element is shown in Fig. 10-14.) Because the rod is uniform, the ratio of mass to length is the same for all the elements and for the rod as a whole. Thus, we can write

$$\frac{\text{element's mass } dm}{\text{element's length } dx} = \frac{\text{rod's mass } M}{\text{rod's length } L}$$

$$\text{or } dm = \frac{M}{L} dx.$$

We can now substitute this result for dm and x for r in Eq. 10-38. Then we integrate from end to end of the rod (from $x = -L/2$ to $x = L/2$) to include all the elements. We find

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$$\begin{aligned}
 I &= \int_{x=-L/2}^{x=+L/2} x^2 \left(\frac{M}{L} \right) dx \\
 &= \frac{M}{3L} \left[x^3 \right]_{-L/2}^{+L/2} = \frac{M}{3L} \left[\left(\frac{L}{2} \right)^3 - \left(-\frac{L}{2} \right)^3 \right] \\
 &= \frac{1}{12} ML^2. \quad (\text{Answer})
 \end{aligned}$$

This agrees with the result given in Table 10-2e.

- (b) What is the rod's rotational inertia I about a new rotation axis that is perpendicular to the rod and through the left end?

KEY IDEAS

We can find I by shifting the origin of the x axis to the left end of the rod and then integrating from $x = 0$ to $x = L$. However,

here we shall use a more powerful (and easier) technique by applying the parallel-axis theorem (Eq. 10-36), in which we shift the rotation axis without changing its orientation.

Calculations: If we place the axis at the rod's end so that it is parallel to the axis through the center of mass, then we can use the parallel-axis theorem (Eq. 10-36). We know from part (a) that I_{com} is $\frac{1}{12}ML^2$. From Fig. 10-14, the perpendicular distance h between the new rotation axis and the center of mass is $\frac{1}{2}L$. Equation 10-36 then gives us

$$\begin{aligned}
 I &= I_{\text{com}} + Mh^2 = \frac{1}{12}ML^2 + (M)\left(\frac{1}{2}L\right)^2 \\
 &= \frac{1}{3}ML^2. \quad (\text{Answer})
 \end{aligned}$$

Actually, this result holds for any axis through the left or right end that is perpendicular to the rod, whether it is parallel to the axis shown in Fig. 10-14 or not.

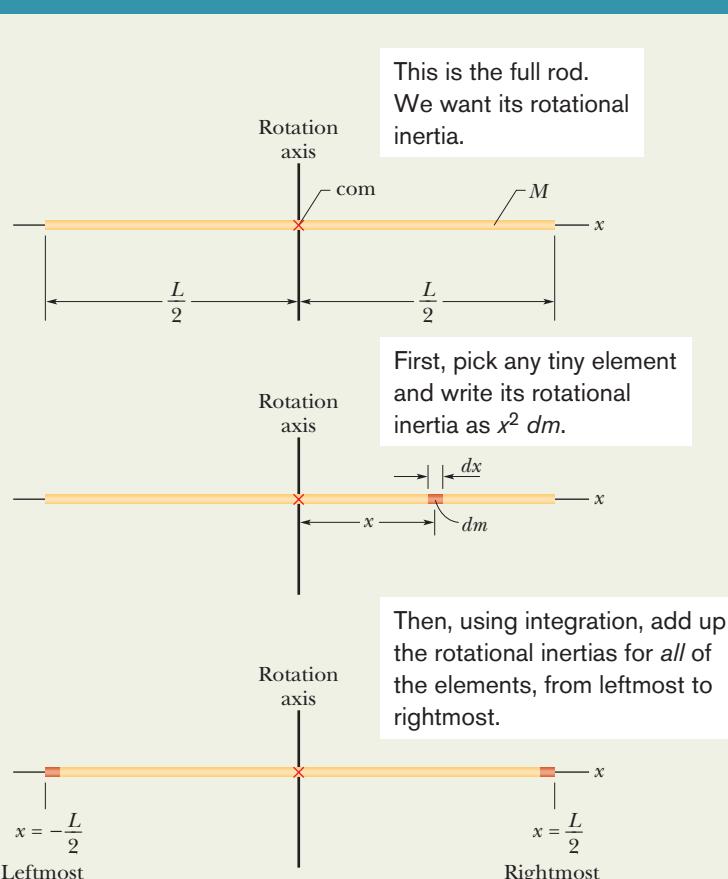


Fig. 10-14 A uniform rod of length L and mass M . An element of mass dm and length dx is represented.



Additional examples, video, and practice available at WileyPLUS

Sample Problem

Rotational kinetic energy, spin test explosion

Large machine components that undergo prolonged, high-speed rotation are first examined for the possibility of failure in a *spin test system*. In this system, a component is *spun up* (brought up to high speed) while inside a cylindrical arrangement of lead bricks and containment liner, all within a steel shell that is closed by a lid clamped into place. If the rotation causes the component to shatter, the soft lead bricks are supposed to catch the pieces for later analysis.

In 1985, Test Devices, Inc. (www.testdevices.com) was spin testing a sample of a solid steel rotor (a disk) of mass $M = 272$ kg and radius $R = 38.0$ cm. When the sample reached an angular speed ω of 14 000 rev/min, the test engineers heard a dull thump from the test system, which was located one floor down and one room over from them. Investigating, they found that lead bricks had been thrown out in the hallway leading to the test room, a door to the room had been hurled into the adjacent parking lot, one lead brick had shot from the test site through the wall of a neighbor's kitchen, the structural beams of the test building had been damaged, the concrete floor beneath the spin chamber had been shoved downward by about 0.5 cm, and the 900 kg lid had been blown upward through the ceiling and had then crashed back onto the test equipment (Fig. 10-15). The exploding pieces had not penetrated the room of the test engineers only by luck.

How much energy was released in the explosion of the rotor?

KEY IDEA

The released energy was equal to the rotational kinetic energy K of the rotor just as it reached the angular speed of 14 000 rev/min.



Fig. 10-15 Some of the destruction caused by the explosion of a rapidly rotating steel disk. (Courtesy Test Devices, Inc.)

Calculations: We can find K with Eq. 10-34 ($K = \frac{1}{2}I\omega^2$), but first we need an expression for the rotational inertia I . Because the rotor was a disk that rotated like a merry-go-round, I is given by the expression in Table 10-2c ($I = \frac{1}{2}MR^2$). Thus, we have

$$I = \frac{1}{2}MR^2 = \frac{1}{2}(272 \text{ kg})(0.38 \text{ m})^2 = 19.64 \text{ kg} \cdot \text{m}^2.$$

The angular speed of the rotor was

$$\begin{aligned}\omega &= (14\,000 \text{ rev/min})(2\pi \text{ rad/rev})\left(\frac{1 \text{ min}}{60 \text{ s}}\right) \\ &= 1.466 \times 10^3 \text{ rad/s.}\end{aligned}$$

Now we can use Eq. 10-34 to write

$$\begin{aligned}K &= \frac{1}{2}I\omega^2 = \frac{1}{2}(19.64 \text{ kg} \cdot \text{m}^2)(1.466 \times 10^3 \text{ rad/s})^2 \\ &= 2.1 \times 10^7 \text{ J.}\end{aligned}\quad (\text{Answer})$$

Being near this explosion was quite dangerous.



Additional examples, video, and practice available at WileyPLUS

10-8 Torque

A doorknob is located as far as possible from the door's hinge line for a good reason. If you want to open a heavy door, you must certainly apply a force; that alone, however, is not enough. Where you apply that force and in what direction you push are also important. If you apply your force nearer to the hinge line than the knob, or at any angle other than 90° to the plane of the door, you must use a greater force to move the door than if you apply the force at the knob and perpendicular to the door's plane.

Figure 10-16a shows a cross section of a body that is free to rotate about an axis passing through O and perpendicular to the cross section. A force \vec{F} is applied at point P , whose position relative to O is defined by a position vector \vec{r} . The directions of vectors \vec{F} and \vec{r} make an angle ϕ with each other. (For simplicity, we consider only forces that have no component parallel to the rotation axis; thus, \vec{F} is in the plane of the page.)

To determine how \vec{F} results in a rotation of the body around the rotation axis, we resolve \vec{F} into two components (Fig. 10-16b). One component, called the *radial component* F_r , points along \vec{r} . This component does not cause rotation, because it acts along a line that extends through O . (If you pull on a door parallel to the plane of the door, you do not rotate the door.) The other component of \vec{F} , called the *tangential component* F_t , is perpendicular to \vec{r} and has magnitude $F_t = F \sin \phi$. This component *does* cause rotation. (If you pull on a door perpendicular to its plane, you can rotate the door.)

The ability of \vec{F} to rotate the body depends not only on the magnitude of its tangential component F_t , but also on just how far from O the force is applied. To include both these factors, we define a quantity called **torque** τ as the product of the two factors and write it as

$$\tau = (r)(F \sin \phi). \quad (10-39)$$

Two equivalent ways of computing the torque are

$$\tau = (r)(F \sin \phi) = rF_t \quad (10-40)$$

and

$$\tau = (r \sin \phi)(F) = r_{\perp}F, \quad (10-41)$$

where r_{\perp} is the perpendicular distance between the rotation axis at O and an extended line running through the vector \vec{F} (Fig. 10-16c). This extended line is called the **line of action** of \vec{F} , and r_{\perp} is called the **moment arm** of \vec{F} . Figure 10-16b shows that we can describe r , the magnitude of \vec{r} , as being the moment arm of the force component F_t .

Torque, which comes from the Latin word meaning “to twist,” may be loosely identified as the turning or twisting action of the force \vec{F} . When you apply a force to an object—such as a screwdriver or torque wrench—with the purpose of turning that object, you are applying a torque. The SI unit of torque is the newton-meter ($N \cdot m$). *Caution:* The newton-meter is also the unit of work. Torque and work, however, are quite different quantities and must not be confused. Work is often expressed in joules ($1 J = 1 N \cdot m$), but torque never is.

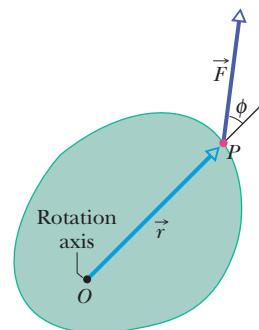
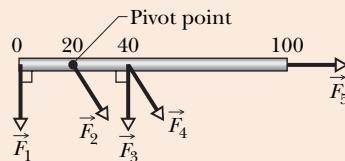
In the next chapter we shall discuss torque in a general way as being a vector quantity. Here, however, because we consider only rotation around a single axis, we do not need vector notation. Instead, a torque has either a positive or negative value depending on the direction of rotation it would give a body initially at rest: If the body would rotate counterclockwise, the torque is positive. If the object would rotate clockwise, the torque is negative. (The phrase “clocks are negative” from Section 10-2 still works.)

Torques obey the superposition principle that we discussed in Chapter 5 for forces: When several torques act on a body, the **net torque** (or **resultant torque**) is the sum of the individual torques. The symbol for net torque is τ_{net} .

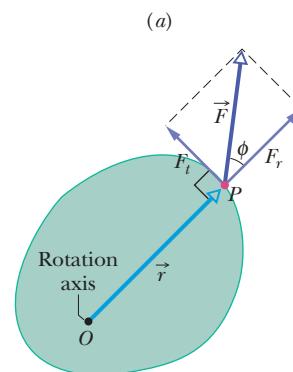


CHECKPOINT 6

The figure shows an overhead view of a meter stick that can pivot about the dot at the position marked 20 (for 20 cm). All five forces on the stick are horizontal and have the same magnitude. Rank the forces according to the magnitude of the torque they produce, greatest first.

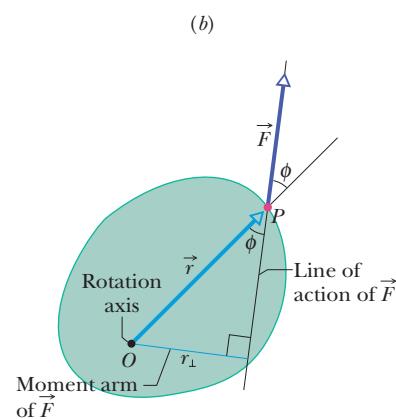


The torque due to this force causes rotation around this axis (which extends out toward you).



(a)

But actually only the *tangential* component of the force causes the rotation.



(b)

You calculate the same torque by using this moment arm distance and the full force magnitude.

(c)

Fig. 10-16 (a) A force \vec{F} acts on a rigid body, with a rotation axis perpendicular to the page. The torque can be found with (a) angle ϕ , (b) tangential force component F_t , or (c) moment arm r_{\perp} .

The torque due to the tangential component of the force causes an angular acceleration around the rotation axis.

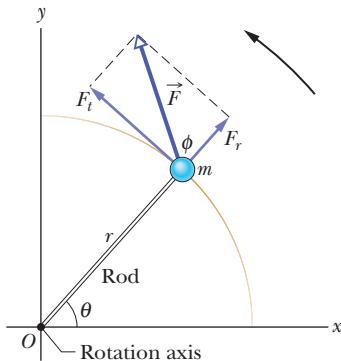


Fig. 10-17 A simple rigid body, free to rotate about an axis through O , consists of a particle of mass m fastened to the end of a rod of length r and negligible mass. An applied force \vec{F} causes the body to rotate.

10-9 Newton's Second Law for Rotation

A torque can cause rotation of a rigid body, as when you use a torque to rotate a door. Here we want to relate the net torque τ_{net} on a rigid body to the angular acceleration α that torque causes about a rotation axis. We do so by analogy with Newton's second law ($F_{\text{net}} = ma$) for the acceleration a of a body of mass m due to a net force F_{net} along a coordinate axis. We replace F_{net} with τ_{net} , m with I , and a with α in radian measure, writing

$$\tau_{\text{net}} = I\alpha \quad (\text{Newton's second law for rotation}). \quad (10-42)$$

Proof of Equation 10-42

We prove Eq. 10-42 by first considering the simple situation shown in Fig. 10-17. The rigid body there consists of a particle of mass m on one end of a massless rod of length r . The rod can move only by rotating about its other end, around a rotation axis (an axle) that is perpendicular to the plane of the page. Thus, the particle can move only in a circular path that has the rotation axis at its center.

A force \vec{F} acts on the particle. However, because the particle can move only along the circular path, only the tangential component F_t of the force (the component that is tangent to the circular path) can accelerate the particle along the path. We can relate F_t to the particle's tangential acceleration a_t along the path with Newton's second law, writing

$$F_t = ma_t.$$

The torque acting on the particle is, from Eq. 10-40,

$$\tau = F_t r = ma_t r.$$

From Eq. 10-22 ($a_t = \alpha r$) we can write this as

$$\tau = m(\alpha r)r = (mr^2)\alpha. \quad (10-43)$$

The quantity in parentheses on the right is the rotational inertia of the particle about the rotation axis (see Eq. 10-33, but here we have only a single particle). Thus, using I for the rotational inertia, Eq. 10-43 reduces to

$$\tau = I\alpha \quad (\text{radian measure}). \quad (10-44)$$

For the situation in which more than one force is applied to the particle, we can generalize Eq. 10-44 as

$$\tau_{\text{net}} = I\alpha \quad (\text{radian measure}), \quad (10-45)$$

which we set out to prove. We can extend this equation to any rigid body rotating about a fixed axis, because any such body can always be analyzed as an assembly of single particles.

CHECKPOINT 7

The figure shows an overhead view of a meter stick that can pivot about the point indicated, which is to the left of the stick's midpoint. Two horizontal forces, \vec{F}_1 and \vec{F}_2 , are applied to the stick. Only \vec{F}_1 is shown. Force \vec{F}_2 is perpendicular to the stick and is applied at the right end. If the stick is not to turn, (a) what should be the direction of \vec{F}_2 , and (b) should F_2 be greater than, less than, or equal to F_1 ?



Sample Problem

Newton's 2nd law, rotation, torque, disk

Figure 10-18a shows a uniform disk, with mass $M = 2.5 \text{ kg}$ and radius $R = 20 \text{ cm}$, mounted on a fixed horizontal axle. A block with mass $m = 1.2 \text{ kg}$ hangs from a massless cord that is wrapped around the rim of the disk. Find the acceleration of the falling block, the angular acceleration of the disk, and the tension in the cord. The cord does not slip, and there is no friction at the axle.

KEY IDEAS

- (1) Taking the block as a system, we can relate its acceleration a to the forces acting on it with Newton's second law ($\vec{F}_{\text{net}} = m\vec{a}$).
- (2) Taking the disk as a system, we can relate its angular acceleration α to the torque acting on it with Newton's second law for rotation ($\tau_{\text{net}} = I\alpha$).
- (3) To combine the motions of block and disk, we use the fact that the linear acceleration a of the block and the (tangential) linear acceleration a_t of the disk rim are equal.

Forces on block: The forces are shown in the block's free-body diagram in Fig. 10-18b: The force from the cord is \vec{T} , and the gravitational force is \vec{F}_g , of magnitude mg . We can now write Newton's second law for components along a vertical y axis ($F_{\text{net},y} = ma_y$) as

$$T - mg = ma. \quad (10-46)$$

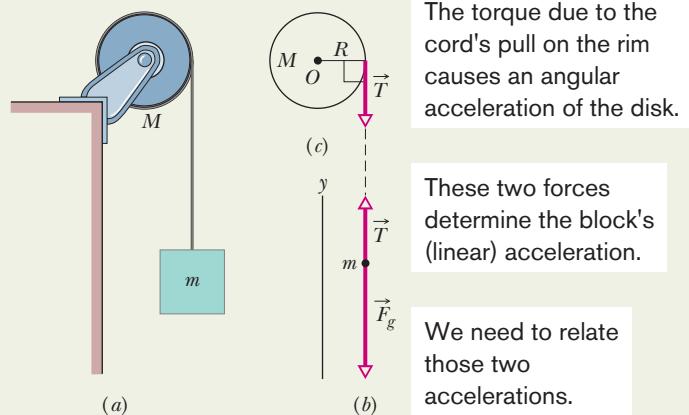
However, we cannot solve this equation for a because it also contains the unknown T .

Torque on disk: Previously, when we got stuck on the y axis, we switched to the x axis. Here, we switch to the rotation of the disk. To calculate the torques and the rotational inertia I , we take the rotation axis to be perpendicular to the disk and through its center, at point O in Fig. 10-18c.

The torques are then given by Eq. 10-40 ($\tau = rF_t$). The gravitational force on the disk and the force on the disk from the axle both act at the center of the disk and thus at distance $r = 0$, so their torques are zero. The force \vec{T} on the disk due to the cord acts at distance $r = R$ and is tangent to the rim of the disk. Therefore, its torque is $-RT$, negative because the torque rotates the disk clockwise from rest. From Table 10-2c, the rotational inertia I of the disk is $\frac{1}{2}MR^2$. Thus we can write $\tau_{\text{net}} = I\alpha$ as

$$-RT = \frac{1}{2}MR^2\alpha. \quad (10-47)$$

This equation seems useless because it has two unknowns, α and T , neither of which is the desired a . However, mustering physics courage, we can make it useful



The torque due to the cord's pull on the rim causes an angular acceleration of the disk.

These two forces determine the block's (linear) acceleration.

We need to relate those two accelerations.

Fig. 10-18 (a) The falling block causes the disk to rotate. (b) A free-body diagram for the block. (c) An incomplete free-body diagram for the disk.

with this fact: Because the cord does not slip, the linear acceleration a of the block and the (tangential) linear acceleration a_t of the rim of the disk are equal. Then, by Eq. 10-22 ($a_t = \alpha r$) we see that here $\alpha = a/R$. Substituting this in Eq. 10-47 yields

$$T = -\frac{1}{2}Ma. \quad (10-48)$$

Combining results: Combining Eqs. 10-46 and 10-48 leads to

$$\begin{aligned} a &= -g \frac{2m}{M + 2m} = -(9.8 \text{ m/s}^2) \frac{(2)(1.2 \text{ kg})}{2.5 \text{ kg} + (2)(1.2 \text{ kg})} \\ &= -4.8 \text{ m/s}^2. \end{aligned} \quad (\text{Answer})$$

We then use Eq. 10-48 to find T :

$$\begin{aligned} T &= -\frac{1}{2}Ma = -\frac{1}{2}(2.5 \text{ kg})(-4.8 \text{ m/s}^2) \\ &= 6.0 \text{ N.} \end{aligned} \quad (\text{Answer})$$

As we should expect, acceleration a of the falling block is less than g , and tension T in the cord ($= 6.0 \text{ N}$) is less than the gravitational force on the hanging block ($= mg = 11.8 \text{ N}$). We see also that a and T depend on the mass of the disk but not on its radius. As a check, we note that the formulas derived above predict $a = -g$ and $T = 0$ for the case of a massless disk ($M = 0$). This is what we would expect; the block simply falls as a free body. From Eq. 10-22, the angular acceleration of the disk is

$$\alpha = \frac{a}{R} = \frac{-4.8 \text{ m/s}^2}{0.20 \text{ m}} = -24 \text{ rad/s}^2. \quad (\text{Answer})$$

10-10 Work and Rotational Kinetic Energy

As we discussed in Chapter 7, when a force F causes a rigid body of mass m to accelerate along a coordinate axis, the force does work W on the body. Thus, the body's kinetic energy ($K = \frac{1}{2}mv^2$) can change. Suppose it is the only energy of the body that changes. Then we relate the change ΔK in kinetic energy to the work W with the work–kinetic energy theorem (Eq. 7-10), writing

$$\Delta K = K_f - K_i = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 = W \quad (\text{work-kinetic energy theorem}). \quad (10-49)$$

For motion confined to an x axis, we can calculate the work with Eq. 7-32,

$$W = \int_{x_i}^{x_f} F dx \quad (\text{work, one-dimensional motion}). \quad (10-50)$$

This reduces to $W = Fd$ when F is constant and the body's displacement is d . The rate at which the work is done is the power, which we can find with Eqs. 7-43 and 7-48,

$$P = \frac{dW}{dt} = Fv \quad (\text{power, one-dimensional motion}). \quad (10-51)$$

Now let us consider a rotational situation that is similar. When a torque accelerates a rigid body in rotation about a fixed axis, the torque does work W on the body. Therefore, the body's rotational kinetic energy ($K = \frac{1}{2}I\omega^2$) can change. Suppose that it is the only energy of the body that changes. Then we can still relate the change ΔK in kinetic energy to the work W with the work–kinetic energy theorem, except now the kinetic energy is a rotational kinetic energy:

$$\Delta K = K_f - K_i = \frac{1}{2}I\omega_f^2 - \frac{1}{2}I\omega_i^2 = W \quad (\text{work-kinetic energy theorem}). \quad (10-52)$$

Here, I is the rotational inertia of the body about the fixed axis and ω_i and ω_f are the angular speeds of the body before and after the work is done, respectively.

Also, we can calculate the work with a rotational equivalent of Eq. 10-50,

$$W = \int_{\theta_i}^{\theta_f} \tau d\theta \quad (\text{work, rotation about fixed axis}), \quad (10-53)$$

where τ is the torque doing the work W , and θ_i and θ_f are the body's angular positions before and after the work is done, respectively. When τ is constant, Eq. 10-53 reduces to

$$W = \tau(\theta_f - \theta_i) \quad (\text{work, constant torque}). \quad (10-54)$$

The rate at which the work is done is the power, which we can find with the rotational equivalent of Eq. 10-51,

$$P = \frac{dW}{dt} = \tau\omega \quad (\text{power, rotation about fixed axis}). \quad (10-55)$$

Table 10-3 summarizes the equations that apply to the rotation of a rigid body about a fixed axis and the corresponding equations for translational motion.

Proof of Eqs. 10-52 through 10-55

Let us again consider the situation of Fig. 10-17, in which force \vec{F} rotates a rigid body consisting of a single particle of mass m fastened to the end of a massless rod. During the rotation, force \vec{F} does work on the body. Let us assume that the

Table 10-3

Some Corresponding Relations for Translational and Rotational Motion

Pure Translation (Fixed Direction)		Pure Rotation (Fixed Axis)	
Position	x	Angular position	θ
Velocity	$v = dx/dt$	Angular velocity	$\omega = d\theta/dt$
Acceleration	$a = dv/dt$	Angular acceleration	$\alpha = d\omega/dt$
Mass	m	Rotational inertia	I
Newton's second law	$F_{\text{net}} = ma$	Newton's second law	$\tau_{\text{net}} = I\alpha$
Work	$W = \int F dx$	Work	$W = \int \tau d\theta$
Kinetic energy	$K = \frac{1}{2}mv^2$	Kinetic energy	$K = \frac{1}{2}I\omega^2$
Power (constant force)	$P = Fv$	Power (constant torque)	$P = \tau\omega$
Work–kinetic energy theorem	$W = \Delta K$	Work–kinetic energy theorem	$W = \Delta K$

only energy of the body that is changed by \vec{F} is the kinetic energy. Then we can apply the work–kinetic energy theorem of Eq. 10-49:

$$\Delta K = K_f - K_i = W \quad (10-56)$$

Using $K = \frac{1}{2}mv^2$ and Eq. 10-18 ($v = \omega r$), we can rewrite Eq. 10-56 as

$$\Delta K = \frac{1}{2}mr^2\omega_f^2 - \frac{1}{2}mr^2\omega_i^2 = W. \quad (10-57)$$

From Eq. 10-33, the rotational inertia for this one-particle body is $I = mr^2$. Substituting this into Eq. 10-57 yields

$$\Delta K = \frac{1}{2}I\omega_f^2 - \frac{1}{2}I\omega_i^2 = W,$$

which is Eq. 10-52. We derived it for a rigid body with one particle, but it holds for any rigid body rotated about a fixed axis.

We next relate the work W done on the body in Fig. 10-17 to the torque τ on the body due to force \vec{F} . When the particle moves a distance ds along its circular path, only the tangential component F_t of the force accelerates the particle along the path. Therefore, only F_t does work on the particle. We write that work dW as $F_t ds$. However, we can replace ds with $r d\theta$, where $d\theta$ is the angle through which the particle moves. Thus we have

$$dW = F_t r d\theta. \quad (10-58)$$

From Eq. 10-40, we see that the product $F_r r$ is equal to the torque τ , so we can rewrite Eq. 10-58 as

$$dW = \tau d\theta. \quad (10-59)$$

The work done during a finite angular displacement from θ_i to θ_f is then

$$W = \int_{\theta_i}^{\theta_f} \tau d\theta,$$

which is Eq. 10-53. It holds for any rigid body rotating about a fixed axis. Equation 10-54 comes directly from Eq. 10-53.

We can find the power P for rotational motion from Eq. 10-59:

$$P = \frac{dW}{dt} = \tau \frac{d\theta}{dt} = \tau \omega,$$

which is Eq. 10-55.

Sample Problem

Work, rotational kinetic energy, torque, disk

Let the disk in Fig. 10-18 start from rest at time $t = 0$ and also let the tension in the massless cord be 6.0 N and the angular acceleration of the disk be -24 rad/s^2 . What is its rotational kinetic energy K at $t = 2.5 \text{ s}$?

KEY IDEA

We can find K with Eq. 10-34 ($K = \frac{1}{2}I\omega^2$). We already know that $I = \frac{1}{2}MR^2$, but we do not yet know ω at $t = 2.5 \text{ s}$. However, because the angular acceleration α has the constant value of -24 rad/s^2 , we can apply the equations for constant angular acceleration in Table 10-1.

Calculations: Because we want ω and know α and $\omega_0 (= 0)$, we use Eq. 10-12:

$$\omega = \omega_0 + \alpha t = 0 + \alpha t = \alpha t.$$

Substituting $\omega = \alpha t$ and $I = \frac{1}{2}MR^2$ into Eq. 10-34, we find

$$\begin{aligned} K &= \frac{1}{2}I\omega^2 = \frac{1}{2}\left(\frac{1}{2}MR^2\right)(\alpha t)^2 = \frac{1}{4}M(R\alpha t)^2 \\ &= \frac{1}{4}(2.5 \text{ kg})[(0.20 \text{ m})(-24 \text{ rad/s}^2)(2.5 \text{ s})]^2 \\ &= 90 \text{ J.} \end{aligned} \quad (\text{Answer})$$

KEY IDEA

We can also get this answer by finding the disk's kinetic energy from the work done on the disk.



Additional examples, video, and practice available at WileyPLUS

Calculations: First, we relate the *change* in the kinetic energy of the disk to the net work W done on the disk, using the work–kinetic energy theorem of Eq. 10-52 ($K_f - K_i = W$). With K substituted for K_f and 0 for K_i , we get

$$K = K_i + W = 0 + W = W. \quad (10-60)$$

Next we want to find the work W . We can relate W to the torques acting on the disk with Eq. 10-53 or 10-54. The only torque causing angular acceleration and doing work is the torque due to force \vec{T} on the disk from the cord, which is equal to $-TR$. Because α is constant, this torque also must be constant. Thus, we can use Eq. 10-54 to write

$$W = \tau(\theta_f - \theta_i) = -TR(\theta_f - \theta_i). \quad (10-61)$$

Because α is constant, we can use Eq. 10-13 to find $\theta_f - \theta_i$. With $\omega_i = 0$, we have

$$\theta_f - \theta_i = \omega_i t + \frac{1}{2}\alpha t^2 = 0 + \frac{1}{2}\alpha t^2 = \frac{1}{2}\alpha t^2.$$

Now we substitute this into Eq. 10-61 and then substitute the result into Eq. 10-60. Inserting the given values $T = 6.0 \text{ N}$ and $\alpha = -24 \text{ rad/s}^2$, we have

$$\begin{aligned} K &= W = -TR(\theta_f - \theta_i) = -TR\left(\frac{1}{2}\alpha t^2\right) = -\frac{1}{2}TR\alpha t^2 \\ &= -\frac{1}{2}(6.0 \text{ N})(0.20 \text{ m})(-24 \text{ rad/s}^2)(2.5 \text{ s})^2 \\ &= 90 \text{ J.} \end{aligned} \quad (\text{Answer})$$

REVIEW & SUMMARY

Angular Position To describe the rotation of a rigid body about a fixed axis, called the **rotation axis**, we assume a **reference line** is fixed in the body, perpendicular to that axis and rotating with the body. We measure the **angular position** θ of this line relative to a fixed direction. When θ is measured in **radians**,

$$\theta = \frac{s}{r} \quad (\text{radian measure}), \quad (10-1)$$

where s is the arc length of a circular path of radius r and angle θ . Radian measure is related to angle measure in revolutions and degrees by

$$1 \text{ rev} = 360^\circ = 2\pi \text{ rad.} \quad (10-2)$$

Angular Displacement A body that rotates about a rotation axis, changing its angular position from θ_1 to θ_2 , undergoes an **angular displacement**

$$\Delta\theta = \theta_2 - \theta_1, \quad (10-4)$$

where $\Delta\theta$ is positive for counterclockwise rotation and negative for clockwise rotation.

Angular Velocity and Speed If a body rotates through an angular displacement $\Delta\theta$ in a time interval Δt , its **average angular velocity** ω_{avg} is

$$\omega_{\text{avg}} = \frac{\Delta\theta}{\Delta t}. \quad (10-5)$$

The **(instantaneous) angular velocity** ω of the body is

$$\omega = \frac{d\theta}{dt}. \quad (10-6)$$

Both ω_{avg} and ω are vectors, with directions given by the **right-hand rule** of Fig. 10-6. They are positive for counterclockwise rotation and negative for clockwise rotation. The magnitude of the body's angular velocity is the **angular speed**.