Assignment 3 Part 1: Probabilistic Reasoning

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Problem 1

a

P(A, B, C, D, E) = P(A) * P(B) * P(C) * P(D|A,B) * P(E|B,C)

P(A = true, B = true, C = true, D = true, E = true)
= P(A = true) * P(B = true) * P(C = true) * P(D = true | A = true ∧ B = true) * P(E = true | B = true ∧ C = true)
= 0.2 * 0.5 * 0.8 * 0.1 * 0.3
= **0.0024**

b

P(A, B, C, D, E) = P(A) * P(B) * P(C) * P(D|A,B) * P(E|B,C)

P(A = false, B = false, C = false, D = false, E = false)
= P(A = false) * P(B = false) * P(C = false) * P(D = false | A = false ∧ B = false) * P(E = false | B = false ∧ C = false)
= 0.8 * 0.5 * 0.2 * 0.1 * 0.8
= **0.0064**

c

P(A = false | B = true, C = true, D = true, E = true) = $\frac{P(A=false,B=true,C=true,D=true,E=true)}{\sum_A P(B=true,C=true,D=true,E=true)}$

For brevity we denote P(A = false, B = true, C = true, D = true, E = true) as P(¬A, B, C, D, E) and P(A = true, B = true, C = true, D = true, E = true) as P(A, B, C, D, E)

We can denote the denominator as a constant α. So the expression stands -
P(A = false | B = true, C = true, D = true, E = true) = α * P(¬A, B, C, D, E)
where, $\alpha = \frac{1}{\sum_A P(B,C,D,E)}$

P(¬A, B, C, D, E)
= P(¬A) * P(B) * P(C) P(D | B ∧ ¬A) * P(D | B ∧ C)
= 0.8 * 0.5 * 0.8 * 0.6 * 0.3
= 0.0576

$\sum_A P(B, C, D, E)$
= P(A, B, C, D, E) + P(¬A, B, C, D, E)
= 0.0576 + 0.0024 (from part a)
= $\frac{3}{50}$

So, $\alpha = \frac{50}{3}$

Therefore,
P(A = false | B = true, C = true, D = true, E = true) = $\frac{50}{3}$ * 0.0576 = **0.96**

Problem 2

a

P(B | J = true, M = true) $= \frac{P(B,J,M)}{P(J,M)}$

Let, $\alpha = \frac{1}{P(J,M)}$

P(J, M)

$$= \sum_B \sum_E \sum_A P(B)P(E)P(A|B,E)P(J|A)P(M|A)$$

$$= \sum_A P(J|A)P(M|A) \sum_E P(E) \sum_B P(B)P(A|B,E)$$

$$= \sum_A P(J|A)P(M|A) \sum_E P(E) * [0.001 \begin{pmatrix} 0.95 & 0.05 \\ 0.94 & 0.06 \end{pmatrix} + 0.999 \begin{pmatrix} 0.29 & 0.71 \\ 0.001 & 0.999 \end{pmatrix}]$$

$$= \sum_A P(J|A)P(M|A) \sum_E P(E) \begin{pmatrix} 0.29066 & 0.70934 \\ 0.001939 & 0.998061 \end{pmatrix}$$

$$= \sum_A P(J|A)P(M|A) * 0.002 \begin{pmatrix} 0.29066 \\ 0.70934 \end{pmatrix} + 0.998 \begin{pmatrix} 0.001939 \\ 0.998061 \end{pmatrix}$$

$$= \sum_A P(J|A)P(M|A) * \begin{pmatrix} 0.002516442 \\ 0.997483558 \end{pmatrix}$$

$$= 0.9 * 0.7 * 0.002516442 + 0.05 * 0.01 * 0.997483558$$

$$= 0.002084100239$$

P(B, J, M)

$$= \sum_E \sum_A P(B)P(E)P(A|B,E)P(J|A)P(M|A)$$

$$= P(B) \sum_E P(E) \sum_A P(A|B,E)P(J|A)P(M|A)$$

$$= P(B) \sum_E P(E) * (0.9 * 0.7 * \begin{pmatrix} 0.95 & 0.29 \\ 0.94 & 0.001 \end{pmatrix} + 0.5 * 0.01 * \begin{pmatrix} 0.05 & 0.71 \\ 0.06 & 0.999 \end{pmatrix})$$

$$= P(B) \sum_E P(E) * \begin{pmatrix} 0.598525 & 0.183055 \\ 0.59223 & 0.0011295 \end{pmatrix}$$

$$= P(B) * (0.002 * \begin{pmatrix} 0.598525 \\ 0.183055 \end{pmatrix} + 0.998 * \begin{pmatrix} 0.59223 \\ 0.0011295 \end{pmatrix})$$

$$= \begin{pmatrix} 0.001 * 0.59224259 \\ 0.999 * 0.001493341 \end{pmatrix}$$

$$= \langle 0.00059224259, 0.001491857649 \rangle$$

P(B | J, M)

$$= \frac{P(B,J,M)}{P(J,M)}$$

$= \alpha$ * P(B, J, M)

$$= \frac{1}{0.002084100239} * \langle 0.00059224259, 0.001491857649 \rangle$$

$$= \langle 0.284, 0.716 \rangle$$

Hence, if both John and Mary call, the probability that Burglary happened is **0.284** and not happened is **0.716**.

b

Inference by Enumeration: To apply this method for a chain Bayesian network, we would construct an enumeration tree. As X_i is Boolean variable for all i, the enumeration tree is composed of two complete binary trees, both with depth $n-2$. We then use depth-first search, as in class, to parse the enumeration tree and complete the method. By the properties of depth-first search, parsing the constructed enumeration tree can be done with time complexity $\mathbf{O(2^n)}$. So, the complexity of computing $P(X_1|X_n = true)$ using enumeration is $\mathbf{O(2^n)}$.

Inference by Variable Elimination: We can use factors for this method, where a factor f_i is a matrix indexed by the values of its argument variables (as in class). Note, by construction, the first step of variable elimination is the following (α is the normalization factor):

$$P(X_1|X_n = true) = \alpha P(X_1) ... \sum_{X_{n-2}} P(X_{n-2}|X_{n-3}) \sum_{X_{n-1}} P(X_{n-1}|X_{n-2}) P(X_n = true|X_{n-1})$$

$$P(X_1|X_n = true) = \alpha P(X_1) ... \sum_{X_{n-2}} P(X_{n-2}|X_{n-3}) \sum_{X_{n-1}} f_{X_{n-1}}(X_{n-1}, X_{n-2}) f_{X_n}(X_{n-1})$$

$$P(X_1|X_n = true) = \alpha P(X_1) ... \sum_{X_{n-2}} P(X_{n-2}|X_{n-3}) f_{X_{n-1},X_n}(X_{n-2})$$

Notice how the conclusion of this step produces an expression of the same form as the original problem, but with $n-1$ variables instead of n variables. Critically, the first step (as shown above) can be performed in constant time (**O(1)**), as the factors never involve more than 2 variables as arguments (this is a result of the construction of the problem as a chain). As the first step can be done in constant time and leads to a problem of the same form with $n-1$ variables instead of n variables, it follows by induction on n that the complexity of computing $P(X_1|X_n = true)$ using variable elimination is **O(n)**.

Problem 3

a

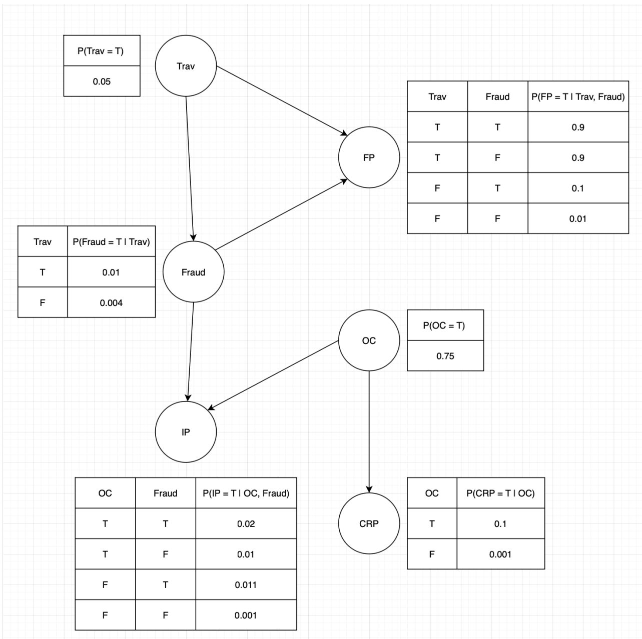

Figure 1: Bayes Network and Conditional Probability Tables

b

$P(Fraud = T) = \sum_{Trav} P(Fraud = T|Trav)P(Trav) = ((0.01)(0.05) + (0.004)(0.95)) = 0.0043$

$P(Fraud = T|FP = T, IP = F, CRP = T) = \alpha P(Fraud = T, FP = T, IP = F, CRP = T)$

$\alpha P(Fraud = T, FP = T, IP = F, CRP = T) = \alpha \sum_{Trav} \sum_{OC} P(Fraud = T, FP = T, IP = F, CRP = T, Trav, OC)$

$\alpha \sum_{Trav} \sum_{OC} P(Fraud = T, FP = T, IP = F, CRP = T, Trav, OC) = \alpha \sum_{Trav} \sum_{OC} P(Trav)P(OC)P(CRP = T|OC)P(Fraud = T|Trav)P(FP = T|Fraud = T, Trav)P(IP = F|Fraud = T, OC)$

$\alpha \sum_{Trav} \sum_{OC} P(Trav)P(OC)P(CRP = T|OC)P(Fraud = T|Trav)P(FP = T|Fraud = T, Trav)P(IP = F|Fraud = T, OC) = \alpha \sum_{Trav} P(Trav)P(Fraud = T|Trav)P(FP = T|Fraud = T, Trav) \sum_{OC} P(OC)P(CRP = T|OC)P(IP = F|Fraud = T, OC)$

$\sum_{OC} P(OC)P(CRP = T|OC)P(IP = F|Fraud = T, OC) = ((0.75)(0.1)(0.98) + (0.25)(0.001)(0.989)) = 0.07374725$

$\alpha \sum_{Trav} P(Trav)P(Fraud = T|Trav)P(FP = T|Fraud = T, Trav) \sum_{OC} P(OC)P(CRP = T|OC)P(IP = F|Fraud = T, OC) = \alpha(0.07374725) \sum_{Trav} P(Trav)P(Fraud = T|Trav)P(FP = T|Fraud = T, Trav)$

$\sum_{Trav} P(Trav)P(Fraud = T|Trav)P(FP = T|Fraud = T, Trav) = ((0.05)(0.01)(0.9)+(0.95)(0.004)(0.1)) = 0.00083$

$\alpha(0.07374725) \sum_{Trav} P(Trav)P(Fraud = T|Trav)P(FP = T|Fraud = T, Trav) = \alpha(0.07374725)(0.00083) = \alpha(0.0000612102175)$

$\alpha = 1/(P(Fraud = T, FP = T, IP = F, CRP = T) + P(Fraud = F, FP = T, IP = F, CRP = T))$

$P(Fraud = T, FP = T, IP = F, CRP = T) = 0.0000612102175$

$P(Fraud = F, FP = T, IP = F, CRP = T) = \sum_{Trav} \sum_{OC} P(Fraud = F, FP = T, IP = F, CRP = T, Trav, OC)$

$\sum_{Trav} \sum_{OC} P(Fraud = F, FP = T, IP = F, CRP = T, Trav, OC) = \sum_{Trav} \sum_{OC} P(Trav)P(OC)P(CRP = T|OC)P(Fraud = F|Trav)P(FP = T|Fraud = F, Trav)P(IP = F|Fraud = F, OC)$

$\sum_{Trav} \sum_{OC} P(Trav)P(OC)P(CRP = T|OC)P(Fraud = F|Trav)P(FP = T|Fraud = F, Trav)P(IP = F|Fraud = F, OC) = \sum_{Trav} P(Trav)P(Fraud = F|Trav)P(FP = T|Fraud = F, Trav) \sum_{OC} P(OC)P(CRP = T|OC)P(IP = F|Fraud = F, OC)$

$\sum_{OC} P(OC)P(CRP = T|OC)P(IP = F|Fraud = F, OC) = ((0.75)(0.1)(0.99) + (0.25)(0.001)(0.999)) = 0.07449975$

$\sum_{Trav} P(Trav)P(Fraud = F|Trav)P(FP = T|Fraud = F, Trav) \sum_{OC} P(OC)P(CRP = T|OC)P(IP = F|Fraud = F, OC) = (0.07449975) \sum_{Trav} P(Trav)P(Fraud = F|Trav)P(FP = T|Fraud = F, Trav)$

$\sum_{Trav} P(Trav)P(Fraud = F|Trav)P(FP = T|Fraud = F, Trav) = ((0.05)(0.99)(0.9)+(0.95)(0.996)(0.01)) = 0.054012$

$(0.07449975) \sum_{Trav} P(Trav)P(Fraud = F|Trav)P(FP = T|Fraud = F, Trav) = (0.07449975)(0.054012) = 0.004023880497$

$P(Fraud = F, FP = T, IP = F, CRP = T) = 0.004023880497$

$1/(P(Fraud = T, FP = T, IP = F, CRP = T)+P(Fraud = F, FP = T, IP = F, CRP = T)) = 1/((0.0000612102175)+(0.004023880497)) = 244.792605572$

$\alpha = 244.792605572$

$\alpha(0.0000612102175) = (244.792605572)(0.0000612102175) = 0.0149838086294$

$P(Fraud = T|FP = T, IP = F, CRP = T) = 0.0149838086294$

The prior probability that the current transaction is a fraud ($P(Fraud = T)$) is **0.0043**.
The probability that the current transaction is a fraud once we have verified that it is a foreign transaction, but not an internet purchase and that the card holder purchased computer related accessories in the past week ($P(Fraud = T|FP = T, IP = F, CRP = T)$) is **0.0149838086294**.