Assignment 3 Part 2: Probabilistic Reasoning

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Problem 1

From the given problem description, we can define the transition and observation/sensor matrices right off the bat.

$$\text{Transition matrix, } \mathtt{T} = \begin{pmatrix} 0.2 & 0.8 & 0 & 0 & 0 & 0 \\ 0 & 0.2 & 0.8 & 0 & 0 & 0 \\ 0 & 0 & 0.2 & 0.8 & 0 & 0 \\ 0 & 0 & 0 & 0.2 & 0.8 & 0 \\ 0 & 0 & 0 & 0 & 0.2 & 0.8 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\text{Observation matrix, } \mathtt{O} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}$$

Let, *hot* = H and *cold* = C
Given, sequence of observations for first 3 days: H_1, C_2, C_3

Part 1

P(X_3 | H_1, C_2, C_3)
$= \alpha$ * P(C_3 | X_3) $\sum_{X_2} P(X_3|X_2)P(X_2|H_1, C_2)$
$= \alpha$ * P(C_3 | X_3) $\sum_{X_2} P(X_3|X_2)P(C_2|X_2)\sum_{X_1} P(X_2|X_1)P(X_1|H_1)$

We solve this equation in parts. We start from the rightmost part and use the transition matrix for this. Notice that $P(X_1|H_1 = A)$ = P(X_1 = A) = 1 as we already know that the state on day 1 is A, and all other states will be 0. So we only need to sum over A.

$$\sum_{X_1} P(X_2|X_1)P(X_1|H_1) = \begin{pmatrix} 0.2 \\ 0.8 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} * 1 + 0 = \begin{pmatrix} 0.2 \\ 0.8 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

[The normalizing factor α in this case ends up being 1, so we ignore it here]

$P(X_2|H_1, C_2) = P(C_2|X_2)\sum_{X_1} P(X_2|X_1)P(X_1|H_1)$

$$= \alpha * \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} * \begin{pmatrix} 0.2 \\ 0.8 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0.8 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ [Normalizing factor } \alpha = \tfrac{1}{0.8} = 1.25\text{]}$$

$$= \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

We can deduce that X_2 = B from this solution.

P(X_3 | H_1, C_2, C_3)

$= \alpha * P(C_3|X_3) * \sum_{X_2} P(X_3|X_2)P(X_2|H_1, C_2)$

Here, $P(C_3|X_3) = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$

And X_2 is 1 at only B, 0 at rest, so we only need to sum over B.

$$= \alpha * \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} * (0 + \begin{pmatrix} 0 \\ 0.2 \\ 0.8 \\ 0 \\ 0 \\ 0 \end{pmatrix} + 0)$$

$$= \alpha * \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} * \begin{pmatrix} 0 \\ 0.2 \\ 0.8 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0.2 \\ 0.8 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ [Normalizing factor } \alpha = \tfrac{1}{0.8+0.2} = 1]$$

Part 2

P(X_2 | H_1, C_2, C_3) $= \alpha * P(C_3|X_2) * P(X_2|H_1, C_2)$

From part 1, we already know that $P(X_2|H_1, C_2) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

Which implies that X_2 = B

$P(C_3|X_2)$

$= \sum_{X_3} P(X_3|X_2)P(C_3|X_3)P(E_4, C_3|X_3)$ *[$P(E_4, C_3|X_3)$ = 1 as it is a definite sequence of events]*

$= \sum_{X_3} P(X_3|X_2) * P(C_3|X_3) * \mathbf{1}$ *[this can be directly solved from the transition and observation matrices]*

$$= \begin{pmatrix} 0.2 & 0.8 & 0 & 0 & 0 & 0 \\ 0 & 0.2 & 0.8 & 0 & 0 & 0 \\ 0 & 0 & 0.2 & 0.8 & 0 & 0 \\ 0 & 0 & 0 & 0.2 & 0.8 & 0 \\ 0 & 0 & 0 & 0 & 0.2 & 0.8 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}^T * \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.8 \\ 1 \\ 0.2 \\ 0.8 \\ 1 \\ 1 \end{pmatrix}$$

$P(X_2 \mid H_1, C_2, C_3)$

$$= \alpha * P(C_3|X_2) * P(X_2|H_1, C_2)$$

$$= \alpha * \begin{pmatrix} 0.8 \\ 1 \\ 0.2 \\ 0.8 \\ 1 \\ 1 \end{pmatrix} * \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$ *[The normalizing factor α in this case ends up being 1]*

Part 3

$P(H_4 \mid H_1, C_2, C_3) = \sum_{X_3} P(X_3|H_1, C_2, C_3)P(H_4|X_3)$

From part 1, we already know that $P(X_3 \mid H_1, C_2, C_3) = \begin{pmatrix} 0 \\ 0.2 \\ 0.8 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$P(H_4|X_3) = \sum_{X_4} P(X_4|X_3)P(H_4|X_4)$

$$= \begin{pmatrix} 0.2 & 0.8 & 0 & 0 & 0 & 0 \\ 0 & 0.2 & 0.8 & 0 & 0 & 0 \\ 0 & 0 & 0.2 & 0.8 & 0 & 0 \\ 0 & 0 & 0 & 0.2 & 0.8 & 0 \\ 0 & 0 & 0 & 0 & 0.2 & 0.8 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \text{X} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.2 \\ 0 \\ 0.8 \\ 0.2 \\ 0 \\ 0 \end{pmatrix}$$

$P(H_4 \mid H_1, C_2, C_3) = \sum_{X_3} P(X_3|H_1, C_2, C_3)P(H_4|X_3)$

$$= \begin{pmatrix} 0 \\ 0.2 \\ 0.8 \\ 0 \\ 0 \\ 0 \end{pmatrix}^T \text{X} \begin{pmatrix} 0.2 \\ 0 \\ 0.8 \\ 0.2 \\ 0 \\ 0 \end{pmatrix} = \mathbf{0.64}$$

Part 4

$P(X_4 \mid H_1, C_2, C_3) = \sum_{X_3} P(X_3|H_1, C_2, C_3)P(X_4|X_3)$
Both values have been calculated above

$$= \begin{pmatrix} 0.2 & 0.8 & 0 & 0 & 0 & 0 \\ 0 & 0.2 & 0.8 & 0 & 0 & 0 \\ 0 & 0 & 0.2 & 0.8 & 0 & 0 \\ 0 & 0 & 0 & 0.2 & 0.8 & 0 \\ 0 & 0 & 0 & 0 & 0.2 & 0.8 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}^T \text{X} \begin{pmatrix} 0 \\ 0.2 \\ 0.8 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0.04 \\ 0.32 \\ 0.64 \\ 0 \\ 0 \end{pmatrix}$$

Part 5 (Bonus Question)

$P(H_4, H_5, C_6|H_1, C_2, C_3) = \sum_{X_3} P(X_3|H_1, C_2, C_3)P(H_4, H_5, C_6|X_3)$

The first expression on the right is basically the filtering problem we solved in part 1.

$$P(X_3|H_1, C_2, C_3) = \begin{pmatrix} 0 \\ 0.2 \\ 0.8 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The second expression can be solved sequentially following the `backward` model.

$P(H_4, H_5, C_6|X_3) = \sum_{X_4} P(X_4|X_3)P(H_4|X_4)P(H_5, C_6|X_4)$

$P(H_5, C_6|X_4) = \sum_{X_5} P(X_5|X_4)P(H_5|X_5)P(C_6|X_5)$

$P(C_6|X_5) = \sum_{X_6} P(X_6|X_5)P(C_6|X_6)$

We can start solving from the last equation using the transition and observation matrices.

$P(C_6|X_5)$

$$= \sum_{X_6} P(X_6|X_5)P(C_6|X_6)$$

$$= \begin{pmatrix} \sum_{X_6} P(X_6|X_5 = A) * P(C_6|X_6) \\ \sum_{X_6} P(X_6|X_5 = B) * P(C_6|X_6) \\ \sum_{X_6} P(X_6|X_5 = C) * P(C_6|X_6) \\ \sum_{X_6} P(X_6|X_5 = D) * P(C_6|X_6) \\ \sum_{X_6} P(X_6|X_5 = E) * P(C_6|X_6) \\ \sum_{X_6} P(X_6|X_5 = F) * P(C_6|X_6) \end{pmatrix}$$

$$= \begin{pmatrix} 0 + 0.8 + 0 + 0 + 0 + 0 \\ 0 + 0.2 + 0.8 + 0 + 0 + 0 \\ 0 + 0 + 0.2 + 0 + 0 + 0 \\ 0 + 0 + 0 + 0 + 0.8 + 0 \\ 0 + 0 + 0 + 0 + 0.2 + 0.8 \\ 0 + 0 + 0 + 0 + 0 + 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0.8 \\ 1 \\ 0.2 \\ 0.8 \\ 1 \\ 1 \end{pmatrix}$$

$P(H_5, C_6|X_4)$

$$= \sum_{X_5} P(X_5|X_4)P(H_5|X_5)P(C_6|X_5)$$

$$= \begin{pmatrix} \sum_{X_5} P(X_5|X_4 = A) * P(H_5|X_5) * P(C_6|X_5) \\ \sum_{X_5} P(X_5|X_4 = B) * P(H_5|X_5) * P(C_6|X_5) \\ \sum_{X_5} P(X_5|X_4 = C) * P(H_5|X_5) * P(C_6|X_5) \\ \sum_{X_5} P(X_5|X_4 = D) * P(H_5|X_5) * P(C_6|X_5) \\ \sum_{X_5} P(X_5|X_4 = E) * P(H_5|X_5) * P(C_6|X_5) \\ \sum_{X_5} P(X_5|X_4 = F) * P(H_5|X_5) * P(C_6|X_5) \end{pmatrix} = \begin{pmatrix} 0.8 * 0.2 * 1 \\ 0 \\ 0.8 * 0.8 * 1 \\ 0.8 * 0.2 * 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.16 \\ 0 \\ 0.64 \\ 0.16 \\ 0 \\ 0 \end{pmatrix}$$

$P(H_4, H_5, C_6|X_3)$

$$= \sum_{X_4} P(X_4|X_3)P(H_4|X_4)P(H_5, C_6|X_4)$$

$$= \begin{pmatrix} \sum_{X_4} P(X_4|X_3 = A) * P(H_4|X_4) * P(H_5, C_6|X_4) \\ \sum_{X_4} P(X_4|X_3 = B) * P(H_4|X_4) * P(H_5, C_6|X_4) \\ \sum_{X_4} P(X_4|X_3 = C) * P(H_4|X_4) * P(H_5, C_6|X_4) \\ \sum_{X_4} P(X_4|X_3 = D) * P(H_4|X_4) * P(H_5, C_6|X_4) \\ \sum_{X_4} P(X_4|X_3 = E) * P(H_4|X_4) * P(H_5, C_6|X_4) \\ \sum_{X_4} P(X_4|X_3 = F) * P(H_4|X_4) * P(H_5, C_6|X_4) \end{pmatrix} = \begin{pmatrix} 0.16 * 0.2 * 1 \\ 0 \\ 0.8 * 0.16 * 1 \\ 0.16 * 0.2 * 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.032 \\ 0 \\ 0.128 \\ 0.032 \\ 0 \\ 0 \end{pmatrix}$$

$$P(H_4, H_5, C_6|X_3) = \begin{pmatrix} 0 \\ 0.2 \\ 0.8 \\ 0 \\ 0 \\ 0 \end{pmatrix} * \begin{pmatrix} 0.032 \\ 0 \\ 0.128 \\ 0.032 \\ 0 \\ 0 \end{pmatrix} = 0.8 * 0.128 = \mathbf{0.1024}$$

Problem 2

Part 1

The Bellman Equation, as defined in class, is: $V^\pi(s) = R(s, \pi(s)) + \gamma \sum_{s' \in States} T(s, \pi(s), s')V^\pi(s')$.

Part 2

$\forall s \in States : V^\pi_{k+1}(s) = R(s, \pi(s)) + \gamma \sum_{s' \in States} T(s, \pi(s), s')V^\pi_k(s')$

State	V^π_0
A	0
B	0

$V^\pi_1(A) = R(A, \pi(A)) + \gamma(T(A, \pi(A), A)V^\pi_0(A) + T(A, \pi(A), B)V^\pi_0(B))$

$V^\pi_1(A) = R(A, 1) + \gamma(T(A, 1, A)V^\pi_0(A) + T(A, 1, B)V^\pi_0(B))$

$V^\pi_1(A) = 0 + (1)((1)(0) + (0)(0)) = 0$

$V^\pi_1(B) = R(B, \pi(B)) + \gamma(T(B, \pi(B), A)V^\pi_0(A) + T(B, \pi(B), B)V^\pi_0(B))$

$V^\pi_1(B) = R(B, 1) + \gamma(T(B, 1, A)V^\pi_0(A) + T(B, 1, B)V^\pi_0(B))$

$V^\pi_1(B) = 5 + (1)((0)(0) + (1)(0)) = 5$

State	V^π_0	V^π_1
A	0	0
B	0	5

$V^\pi_2(A) = R(A, \pi(A)) + \gamma(T(A, \pi(A), A)V^\pi_1(A) + T(A, \pi(A), B)V^\pi_1(B))$

$V^\pi_2(A) = R(A, 1) + \gamma(T(A, 1, A)V^\pi_1(A) + T(A, 1, B)V^\pi_1(B))$

$V^\pi_2(A) = 0 + (1)((1)(0) + (0)(5)) = 0$

$V^\pi_2(B) = R(B, \pi(B)) + \gamma(T(B, \pi(B), A)V^\pi_1(A) + T(B, \pi(B), B)V^\pi_1(B))$

$V^\pi_2(B) = R(B, 1) + \gamma(T(B, 1, A)V^\pi_1(A) + T(B, 1, B)V^\pi_1(B))$

$V^\pi_2(B) = 5 + (1)((0)(0) + (1)(5)) = 10$

State	V_0^π	V_1^π	V_2^π
A	0	0	0
B	0	5	10

Part 3

$\forall s \in States : \pi_{new}(s) = \arg\max_{a \in A}[R(s,a) + \gamma \sum_{s' \in States} T(s,a,s')V_2^\pi(s')]$

$R(A,1) + \gamma(T(A,1,A)V_2^\pi(A) + T(A,1,B)V_2^\pi(B)) = 0 + (1)((1)(0) + (0)(10)) = 0$

$R(A,2) + \gamma(T(A,2,A)V_2^\pi(A) + T(A,2,B)V_2^\pi(B)) = (-1) + (1)((0.5)(0) + (0.5)(10)) = 4$

$4 > 0 \implies \pi_{new}(A) = 2$

$R(B,1) + \gamma(T(B,1,A)V_2^\pi(A) + T(B,1,B)V_2^\pi(B)) = 5 + (1)((0)(0) + (1)(10)) = 15$

$R(B,2) + \gamma(T(B,2,A)V_2^\pi(A) + T(B,2,B)V_2^\pi(B)) = 0 + (1)((0)(0) + (1)(10)) = 10$

$15 > 10 \implies \pi_{new}(B) = 1$

$\pi_{new} = (\pi_{new}(A), \pi_{new}(B)) = (2,1)$