Group-based beliefs and human capital specialization

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Abstract

Although the overall gender gap in postsecondary degree attainment has reversed over the past forty years, significant heterogeneity persists in terms of which fields men and women choose to study. In this paper, I consider the role of group-based beliefs in explaining these differential convergence rates across fields. I assume a student forms their initial belief about their probability of success in a particular field based on past outcomes for their group type. I then incorporate group-based beliefs into the model of gradual human capital specialization from Alon and Fershtman (2019) to show how these differences in priors can drive human capital specialization decisions amongst otherwise similar agents.

1. Introduction

The gender gap in postsecondary degree attainment has reversed in the US over the past fifty years, as seen in figure 1. The overall gender convergence in human capital has been related to a number of macroeconomic benefits, including a reduction in the gender wage gap (Blau and Kahn 2017) and increased aggregate economic productivity (Hsieh et al. 2019). However, this pattern is not uniformly observed across fields of study. Consider figure 2a, which plots the ratio of women to men completing Bachelor's degrees in several subjects. While the gender ratios of some fields have

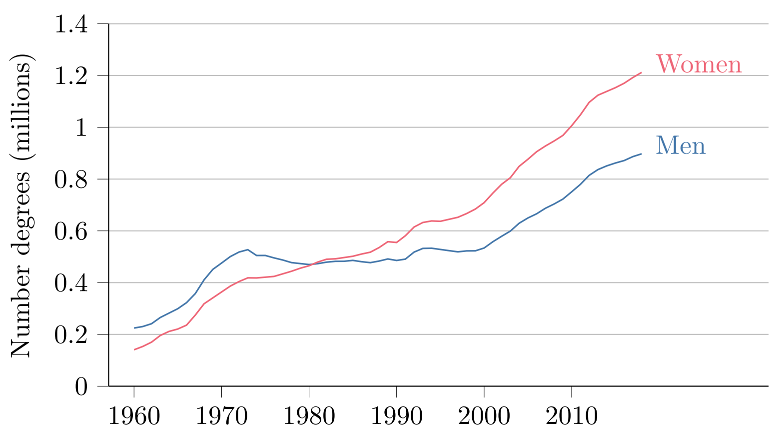

Figure 1. Number of Bachelor's Degrees awarded in US 4-year colleges. Source: IPEDS; Snyder (2013).

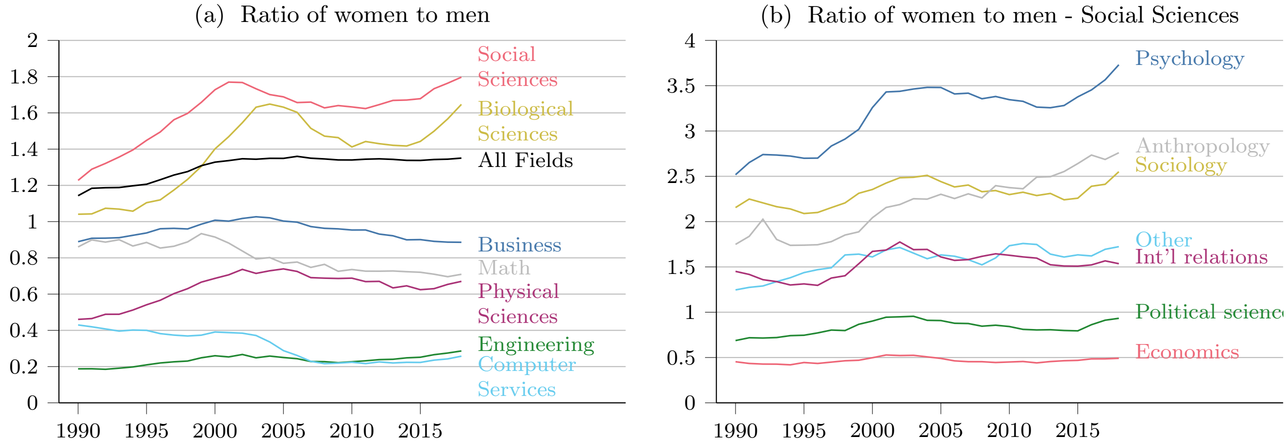

Figure 2. Ratio of women to men completing Bachelor's degrees in U.S. 4-year colleges. Source: IPEDS.

increased since 1990, others have remained flat or worsened. More generally, aggregations of college majors can easily mask heterogeneity in gender convergence across fields (Black et al. 2008). This can be seen when comparing the overall Social Sciences gender ratio from Figure 2a with those of its subfields in 2b. Because differences in major choice across genders appear to matter for labor market outcomes (Sloane, Hurst, and Black 2020), it is worth examining why these differences persist.

This paper addresses this heterogeneity using a model of group-based belief formation and gradual human capital specialization. Building on Alon and Fershtman (2019), I assume individuals belonging to a particular group choose to work or study in heterogeneous fields. Returns to education are stochastic, and underlying abilities are unknown. Agents form beliefs about their unknown abilities based on existing group outcomes, and update these beliefs as they study. I can use this model to analytically characterize the dynamics of student's belief distribution as their education proceeds.

I then use simulations to highlight the different mechanisms of this model. In particular, I highlight the important role that beliefs play in impacting specialization decisions. I use these results to motivate how we can view the model through the lens of statistical discrimination. Future work will identify model parameters and conduct counterfactual exercises. This draft will briefly review possible identification methods and data sources. I will then discuss in the conclusion several possible quantitative exercises this model can be used for.

This paper proceeds as follows. After a brief literature review, I outline the model in section 2. Analytical results are derived in section 3, and implications of the model are explored in section 4. The connection between this model and the theory of statistical discrimination are outlined in section 5. Next steps, including identification and quantitative analysis, are discussed in section 6.

Literature

This paper builds upon the extensive literature on human capital formation (Becker 1962; Ben-Porath 1967; Mincer 1974; Rosen 1983). I expand on Alon and Fershtman's (2019) theoretical model of gradual specialization, a recent contribution to the literature. Their framework closely relates to two classical papers from this field. The first is the seminal Mincer (1974) model of the returns to education. The second is the Roy (1951) model of occupational choice and skill heterogeneity. Alon and Fershtman's (2019) model, and by extension my own, can be viewed as a generalization of the mincerian model of human capital accumulation to include a dynamic Roy model with unknown heterogeneous abilities and sequential learning.

This paper focuses on pre-labor market specialization decisions, specifically college major choice. As such, this project is closely tied to the theoretical and empirical literature on education decisions and college major choice. Altonji's (1993) seminal work on the role of uncertainty in sequential education decisions provides an important theoretical antecedent for this paper. The empirical literature on the determinants of educational choices and returns to education is too extensive to fully outline here; as such, I refer the reader to Patnaik, Wiswall, and Zafar (2020) for an excellent overview of the determinants of college major choice.[1] Although major-specific wages are an important determinant of college major choice in this model, non-pecuniary motivations are central to this analysis. These include sorting on ability (Arcidiacono 2004), preferences (Zafar 2013), peer effects, (Porter and Serra 2020), and beliefs (Arcidiacono et al., forthcoming). Of particular relevance is work on how these determinants differ by gender.

This paper is largely motivated by empirical work documenting the relationship between gender and college major choice. The gender convergence in overall college degree attainment is a well-studied phenomenon, and thoroughly reviewed in Blau and Kahn (2017). However, most of this gender convergence ended by the 1980s, well before parity was achieved (Sloane, Hurst, and Black 2020; England and Li 2006).

Finally, the results of this paper will be closely tied to statistical discrimination literature. The theoretical connection between the model outlined in this paper and the theory of statistical discrimination will be discussed in detail in section 5; relevant literature will be reviewed there.

2. Model of human capital specialization

This section outlines my theory of group-based beliefs and human capital specialization. Please note that the agent's specialization problem and subsequent decision rule follows Alon and Fershtman's (2019) model of gradual human capital specialization; as such, I refer to the reader to their original paper for details.

1. Additional literature reviews are found in Altonji, Blom, and Meghir (2012) and Altonji, Arcidiacono, and Maurel (2016).

2.1. Specialization decision

Assume infinitely lived agents in discrete time choose to work or study among J fields. Specifically, at each time period t, individuals decide to either study a single field j at a postsecondary institution, or to work in a field j, where $j \in \{1, \ldots, J\}$. The variable m_{jt} is equal to one if an agent matriculates and studies field j at time t; otherwise, m_{jt} equals zero. Likewise, ℓ_{jt} indicates whether an agent works in field j at time t. An individual therefore faces the following time constraint in each period t:

$$\sum_{j=1}^{J} (m_{jt} + \ell_{jt}) = 1.$$

Individuals aim to maximize their expected lifetime utility. Let U_j denote the expected per-period utility associated with working in field j, where U_j is a bounded function that is non-decreasing in field-specific wages and in field-specific human capital (w_j and h_{jt}, respectively). The agent's expected lifetime payoff can be written as:

$$\sum_{t=0}^{\infty} \delta^t \sum_{j=1}^{J} U_j(w_j, h_{jt})\ell_{jt}, \tag{1}$$

where $\delta \in (0, 1)$ is the discount rate.

Agents are initially endowed with some level of field-j human capital, h_{j0}, and can stochastically accumulate more field-j human capital by studying. Recall that if $m_{jt} = 1$, an agent matriculates in time period t to take a course in field j. Let s_{jt} indicate whether an agent studying j at time t succeeds and passes that course; if the student succeeds, then $s_{jt} = 1$, otherwise $s_{jt} = 0$. I assume that whether a student passes or fails a particular field-j course is stochastic. Specifically, each student is endowed with some immutable probability of success in field-j courses, denoted θ_j. A student then passes any field-j course with probability θ_j, implying that s_{jt} is a Bernoulli random variable with parameter θ_j.[2] Agents only accumulates human capital when they pass courses. Therefore, an agent's field-specific human capital evolves according to:

$$h_{j,t+1} = h_{jt} + \nu_j s_{jt} m_{jt}, \qquad s_{jt} \sim \text{Bernoulli}(\theta_j), \tag{2}$$

where $\nu_j \geq 0$ is the human capital gain associated with passing the course.[3]

A student's probability of success in a field j course, θ_j, is an ability parameter; students with high values of θ_j are more likely to pass any given class in field j, whereas students with low values of θ_j are more likely to fail. Students do not know their personal value of θ_j, but they have beliefs about what their value of θ_j might be. Their initial beliefs about their own ability is described by the distribution P_{j0}. As students take courses in field-j, they update their belief about what their

2. The variable s_{jt} is assumed to be independent and stationary over time.

3. The per-period expected accumulation of human capital additionally must be non-negative and bounded. Regularity conditions imposed in section 2.3 ensure that is the case.

value of θ_j may be, according to some updating rule Π_j:

$$P_{j,t+1} = \Pi_j(P_{jt}, s_{jt})$$

Overall, students have two incentives for studying a field j: first, they can potentially accumulate j-specific human capital, which directly increases lifetime utility if they specialize in j. Second, studying j reveals information about their ability in that field, which is central to the specialization decision.

It is clear from (1) that agents will only want to work in the field that yields the highest expected lifetime utility. The choice of which field to work in is an individual's *specialization decision*. Agents that plan to specialize in field j will study j to accumulate j-specific human capital, and will eventually endogenously enter the labor market as a field-j specialist. The decision of when to stop studying j and enter the labor market is an agent's *stopping problem*. Specifically, the stopping decision is the time when an agent expects to stop studying j and begin work as a j-specialist, ignoring the existence of other fields.[4]

2.2. Group-based beliefs

Assume each student has a group type, g. To simplify the exposition, consider two groups, men and women ($g \in \{m, f\}$). The distributions of underlying abilities, θ_j, are the same for men and women. However, initial beliefs about underlying abilities, P^g_{j0}, are different for the two groups.

To make this explicit, consider the following parameterization of a student's belief distribution. Assume the initial beliefs about θ_j follow a beta distribution with parameters $(\alpha^g_{j0}, \beta^g_{j0})$, implying $P^g_{j0} = \mathcal{B}(\alpha^g_{j0}, \beta^g_{j0})$. To understand why this is a reasonable assumption, recall that our unknown ability parameter, θ_j, is the probability that a student succeeds ($s^g_{jt} = 1$) or fails ($s^g_{jt} = 0$) a field j course at time t. Because the realizations of s_{jt} are independent over time, the sequence of successes and failures for some number of total field j courses taken is a binomial random variable. The beta distribution is a conjugate prior for the binomial distribution, and is thus a natural and tractable choice for modeling beliefs about θ_j (Casella and Berger 2002, pg. 325). Therefore, if we assume that a student updates their beliefs about θ_j using Bayes' rule, their posterior is also a Beta distribution:

$$P^g_{j,t+1} = \mathcal{B}(\alpha^g_{j,t+1}, \beta^g_{j,t+1}), \qquad (\alpha_{j,t+1}, \beta_{j,t+1}) = \begin{cases} (\alpha^g_{jt} + 1, \beta^g_{jt}) & \text{if } s^g_{jt} = 1 \\ (\alpha^g_{jt}, \beta^g_{jt} + 1) & \text{if } s^g_{jt} = 0 \end{cases} \tag{3}$$

To develop intuition about how this assumption influences specialization decisions, it's helpful to proceed with an illustrative, albeit somewhat contrived, parameterization. Let α^g_{j0} and β^g_{j0} denote

4. The point that this decision is made ignoring the existence of other fields is important. This is because I assume that agents do not earn a wage while they are studying. Therefore, an agent's expected field-j stopping time directly impacts their expected field-j payoff.

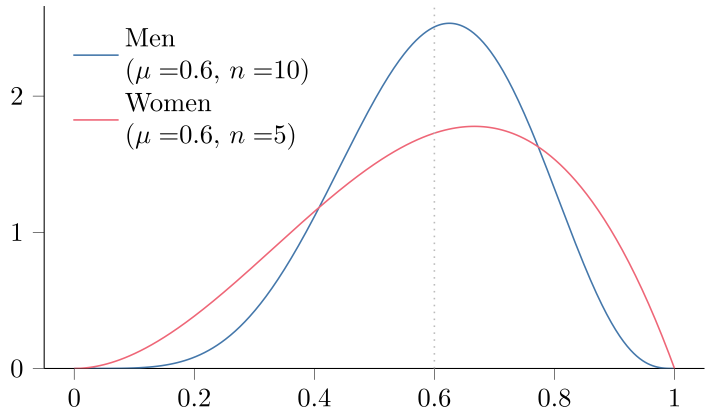

Figure 3. PDF of Beta distribution

the number of type g students who have succeeded and failed in field j at time 0, respectively. As an example, suppose a type g student is forming their initial beliefs about their probability of success in field j, and therefore asks five type g upperclassmen about their experiences in field j. If three of those type g upperclassmen passed the introductory course in field j, while two failed, then the student's initial belief parameters $(\alpha^g_{j0}, \beta^g_{j0})$ would equal $(3, 2)$.

Using this parameterization, the observed group-g success rate, μ^g_{j0}, is given by:

$$\mu^g_{j0} = \frac{\alpha^g_{j0}}{\alpha^g_{j0} + \beta^g_{j0}}.$$

This average is based on a sample of $n^g_{j0} = \alpha^g_{j0} + \beta^g_{j0}$ type-g students. Note that the beta distribution parameters, α^g_{j0} and β^g_{j0}, can be expressed using the average success rate, μ^g_{j0}, and the sample size, n^g_{j0}:

$$\alpha^g_{j0} = \mu^g_{j0} n^g_{j0}, \qquad \beta^g_{j0} = (1 - \mu^g_{j0}) n^g_{j0}.$$

Therefore, an alternative parameterization of prior is given by $\mathcal{B}\left(\alpha^g_{j0}, \beta^g_{j0}\right) = \mathcal{B}\left(\mu^g_{j0} n^g_{j0}, (1 - \mu^g_{j0}) n^g_{j0}\right)$.

Assume the sample size of men is larger than that of women, but that the observed success rate is the same for the two genders:

$$n^m_{j0} > n^f_{j0}, \qquad \mu_{j0} = \mu^m_{j0} = \mu^w_{j0}.$$

Figure 3 provides a numerical example to illustrate how these assumptions affect the priors of men and women. Although women and men have the same probability of success in expectation, women have more initial uncertainty regarding their underlying abilities.

In section 4, I discuss how differential initial beliefs influence specialization decisions. Before moving forward, it is helpful to highlight some shortcomings of the above illustrative example. First, I am being purposefully vague about what "success" means when agents form their initial priors, α^g_{j0} and β^g_{j0}. The example above, in which students solicit feedback from upperclassmen, is helpful for building intuition, and is consistent with the literature highlighting the importance of role-models in specialization decisions; see Porter and Serra (2020) for directly relevant empirical evidence. However, success could mean many things in this model. It could be the number of type g students who graduate into field j, the number of type g professors, the number of students attaining graduate degrees in field j, etc. More generally, while it is illustrative to use the parameters α^g_{j0} and β^g_{j0} to tally the total number of type g successes and failures, it is by no means necessary.

2.3. Optimal policy

To summarize the individual's problem, let h^g_t, P^g_t, m^g_t, ℓ^g_t denote the $J \times 1$ vectors of field-specific human capital, beliefs, study decisions, and labor decisions, respectively. A policy $\pi : (h^g_t, P^g_t) \to (m^g_t, \ell^g_t)$ is optimal if it maximizes lifetime expected utility:

$$\mathbb{E}^{\pi}\left[\sum_{t=0}^{\infty}\delta^t\left(\sum_{j=1}^{J}U_j(h^g_{jt}, w_j)\ell^g_{jt}\right)\middle| h^g_0, P^g_0\right], \tag{4}$$

given the following time constraint:

$$\sum_{j=1}^{J}(m^g_{jt} + \ell^g_{jt}) = 1, \qquad m^g_{jt}, \ell^g_{jt} \in \{0, 1\},$$

subject to the human capital accumulation and belief transition laws:

$$\begin{aligned} h^g_{jt+1} =& h^g_{jt} + \nu_j s^g_{jt} m^g_{jt}, \qquad s^g_{jt} \sim \text{Bernoulli}(\theta_j), \qquad \theta_j \sim P^g_{j0} \equiv \mathcal{B}(\alpha^g_{j0}, \beta^g_{j0}), \\ P^g_{j,t+1} =& \mathcal{B}(\alpha^g_{j,t+1}, \beta^g_{j,t+1}), \qquad (\alpha^g_{j,t+1}, \beta^g_{j,t+1}) = \begin{cases} (\alpha^g_{jt}+1, \beta^g_{jt}) & \text{if } m^g_{jt} = 1 \text{ and } s^g_{jt} = 1 \\ (\alpha^g_{jt}, \beta^g_{jt}+1) & \text{if } m^g_{jt} = 1 \text{ and } s^g_{jt} = 0 \\ (\alpha^g_{jt}, \beta^g_{jt}) & \text{if } m^g_{jt} = 0 \end{cases}. \end{aligned}$$

Alon and Fershtman (2019) characterize the optimal policy to the above problem. To apply their results, first assume the following initial condition:

$$h_{j0} \leq \alpha^g_{j0}\nu_j. \tag{5}$$

This assumption will be discussed in detail in section 3.2. In brief, this condition assures that the stopping problem is monotonic, which in turn implies the optimality of the following policy. Let τ denote the optimal stopping rule defined over $\{s^g_{j1}, s^g_{j2}, \dots\}$. Define the field-$j$ index as the expected lifetime payoff an agent would receive if they commit to studying field-j given their state

(h^g_{jt}, P^g_{jt}):

$$\mathcal{I}_{jt}(h^g_j, P^g_j) = \sup_{\tau \geq 0} \mathbb{E}^\tau \left[\sum_{t=0}^{\infty} \delta^t U_j(h^g_{jt}, w_j) \ell^g_{jt} \middle| (h^g_{j0}, P^g_{j0}) = (h^g_j, P^g_j) \right] \tag{6}$$

Define the graduation region of field j as the states where an agent committed to studying field j would choose to stop studying and enter the labor market:

$$\mathcal{G}_j(h^g_j, P^g_j) = \left\{ (h^g_j, P^g_j) \middle| \arg\max_{\tau \geq 0} \mathbb{E}^\tau \left[\sum_{t=0}^{\infty} \delta^t U_j(h^g_{jt}, w_j) \ell^g_{jt} \middle| (h_j, P^g_j) \right] = 0 \right\} \tag{7}$$

Then the following policy $\pi : (h^g_t, P^g_t) \to (m^g_t, \ell^g_t)$ is optimal:

1. At each $t \geq 0$, choose skill $j^* = \arg\max_{j \in J} \mathcal{I}_j$, breaking ties according to any rule
2. If $(h_{j^*}, P^g_{j^*}) \in \mathcal{G}_j$, then enter the labor market as a j^* specialist. Otherwise, study j^* for an additional period.

3. Discussion of the model

The optimal policy outlined in 2.3 is characterized by two objects: the index (6), which summarizes the expected lifetime payoff associated with studying a field, ignoring all other fields; and the graduation region (7), which defines the states where an agent would choose to stop studying and enter the labor market. This section derives an analytical solution to these objects under the initial condition assumption (5).

Section 3.1 begins by introducing alternative notation to characterize state variables. In section 3.2, I discuss the intuition behind the initial condition assumption (5), and its implications for the optimal stopping problem. This motivates a tractable solution to the graduation region. Section 3.3 uses the results from 3.2 to derive a simplified version of the index. Fully computing the index involves evaluating agent expectations over possible stopping times. The solution to these expectations are derived in section 3.4. A concise summary of how to compute this solution is presented in section 3.5.

Relation to parametric results from Alon and Fershtman (2019)

A key advantage of Alon and Fershtman's (2019) model is its computability under the stronger initial condition assumption $h_{j0} = \alpha_{j0}\nu_j$. This assumption is not out of line with the human capital accumulation function (2), and may be reasonable for simulation exercises when all parameters of the problem are known. However, this assumption presents both theoretical and empirical objections. The goal of this paper is to assess how beliefs impact specialization decisions. In the context of gender, this may involve considering whether a man and woman with similar initial human capital levels but different beliefs make different specialization choices; in the example outlined in section 2.2, this involves assessing whether men and women with similar h_{j0} make different specialization choices when $\alpha_{j0}^m > \alpha_{j0}^w$. However, assuming $h_{j0} = \alpha_{j0}^g \nu_j$ implies that women begin with lower levels of human capital than men in a particular field j, complicating this type of counterfactual analysis. More practically, when bringing the model to the data, I want to be able to control for initial human capital levels when agents make initial specialization choices. Assuming $h_{j0} = \alpha_{j0}^g \nu_j$ effectively eliminates the variable h_{j0}.

For this reason, the sections below focus on the more general monotonic initial condition (5). I present the tractable results outlined in Alon and Fershtman (2019) in the context of a more general solution.

3.1. Comment on state variables

State variables in the general formulation of the model are given by an agent's vector of human capital, h_{jt}, and their beliefs, P_{jt}. An alternative characterization of the agent's state's will be more useful for the remainder of this paper.

Define $\tilde{m}_{jt}$ as the total number of times a student has chosen to matriculate in field j by time t, and define $\tilde{s}_{jt}$ as the total number of times a student has passed their field j courses:

$$\tilde{m}_{jt} = \sum_{n=0}^{t-1} m_{jn}, \qquad \tilde{s}_{jt} = \sum_{n=0}^{t-1} s_{jn}. \tag{8}$$

The individual's state variables at time t are $(h_{jt}, \alpha_{jt}, \beta_{jt})$. Using some simple algebraic transformations,[5] we can now characterize the states at time t using $(\alpha_{j0}, \beta_{j0}, h_{j0}, \tilde{m}_{jt}, \tilde{s}_{jt})$. In words, the agent's state variables at time t are the initial belief parameters α_{j0} and β_{j0}, initial human capital h_{j0}, the endogenous number of field-j courses $\tilde{m}_{jt}$, and the stochastic number of times an agent passed their field-j courses, $\tilde{s}_{jt}$. Given the structure of the problem, there is no need to directly track the evolution of $(\alpha_{jt}, \beta_{jt}, h_{jt})$ over time, because (1) all information about the evolution of beliefs is captured by initial beliefs $(\alpha_{j0}, \beta_{j0})$, course choices $(\tilde{m}_{jt})$, and course outcomes $(\tilde{s}_{jt})$; and (2) all information about human capital evolution is characterized by initial human capital endowments (h_{j0}), course choices $(\tilde{m}_{jt})$, and course outcomes $(\tilde{s}_{jt})$.

3.2. Initial condition assumption and optimal stopping time

The optimality policy from section 2.3 is contingent on assuming equation (5) holds, which states that $h_{j0} \leq \nu_j \alpha_{j0}$. Assuming that $h_{j0} \leq \nu_j \alpha_{j0}$ ensures that the stopping problem is monotonic, which in turn implies the optimality the policy outlined in section 2.3. As such, I will occasionally refer to (5) as the *monotonic initial condition* or the *monotonicity assumption*.

It may be helpful to briefly outline why the monotonic initial condition implies optimality of the above policy. Recall that the graduation index, (7), characterizes the states where an individual would stop studying field-j and enter the labor market as a field-j specialist, ignoring all other fields.

5. Specifically, note that (1) $\tilde{m}_{jt} + \alpha_{j0} + \beta_{j0} = \alpha_{jt} + \beta_{jt}$; (2) $\alpha_{jt} = \tilde{s}_{jt} + \alpha_{j0}$; and (3) $h_{jt} = \nu_j \tilde{s}_{jt} + h_{j0}$.

Therefore, this object characterizes the field-specific stopping problem facing an individual. Under the monotonic initial condition (5), the stopping problem for a given field is monotone. Intuitively, monotonicity means that an agent who wants to stop studying j at time t would still want to stop studying j at time $t+1$ if they continued on, independent of stochastic outcomes. Therefore, when an agent at time t chooses whether to stop studying j and enter the labor market as a field-j specialist, they compare their current expected lifetime payoff in j with their expected payoff in the next period. In other words, monotonicity implies the optimality of a one-step-look-ahead comparison for the field-specific stopping problem.

The monotonicity condition (5) ensures that an agent evaluating field j at time t will stop studying j if their expected lifetime payoff in the current t period exceeds their expected lifetime payoff in $t+1$:

$$\frac{1}{1-\delta} w_j h_{jt} \geq \frac{\delta}{1-\delta} w_j \mathbb{E}_t \left[h_{j,t+1} \middle| h_{jt}, \alpha_{jt}, \beta_{jt} \right]$$

This equation can be simplified using the human capital accumulation function (2):

$$h_{jt} \geq \delta(h_{jt} + \nu_j \mathbb{E}_t \left[s_{jt} \middle| h_{jt}, \alpha_{jt}, \beta_{jt} \right]).$$

Recalling that the course outcome s_{jt} is a Bernoulli(θ_j) random variable, this can be written using an agent's beliefs about θ_j at time t:

$$\frac{1-\delta}{\delta} \geq \frac{\nu_j \alpha_{jt}}{h_{jt}(\alpha_{jt} + \beta_{jt})} \tag{9}$$

Using the definitions of $\tilde{m}_{jt}$ and $\tilde{s}_{jt}$ from equation (8), the stopping condition (9) becomes:[6]

$$\tilde{m}_{jt} \geq \frac{\delta}{1-\delta} \left(\frac{\nu_j \alpha_{j0} + \nu_j \tilde{s}_{jt}}{h_{j0} + \nu_j \tilde{s}_{jt}} \right) - \alpha_{j0} - \beta_{j0} \tag{10}$$

Intuitively, this stopping condition says that an agent will stop studying a field j once their total number of completed field-j courses exceeds the right-hand-side inequality. The graduation index (7) can now be written to reflect the stopping condition (10):

$$\mathcal{G}_j = \left\{ \tilde{m}_{jt}, \tilde{s}_{jt}, \alpha_{j0}, \beta_{j0}, h_{j0} \middle| \tilde{m}_{jt} \geq \frac{\delta}{1-\delta} \left(\frac{\nu_j \alpha_{j0} + \nu_j \tilde{s}_{jt}}{h_{j0} + \nu_j \tilde{s}_{jt}} \right) - \alpha_{j0} - \beta_{j0} \right\} \tag{11}$$

6. To replicate this derivation, note $\nu_j \alpha_{jt} = h_{jt} - h_{j0} + \nu_j \alpha_{j0}$. Then (9) implies:

$$h_{jt} \left(\alpha_{jt} + \beta_{jt} - \frac{\delta}{1-\delta} \right) \geq \frac{\delta}{1-\delta} \left(\nu_j \alpha_{j0} - h_{j0} \right)$$

Using the fact that $\tilde{m}_{jt} + \alpha_{j0} + \beta_{j0} = \alpha_{jt} + \beta_{jt}$:

$$h_{jt} \tilde{m_{jt}} + h_{jt} \left(\alpha_{j0} + \beta_{j0} \right) \geq \frac{\delta}{1-\delta} \left(\nu_j \alpha_{j0} - h_{j0} + h_{jt} \right)$$

The simplified stopping condition under monotonicity (10) follows from the fact that $h_{jt} - h_{j0} = \nu_j \tilde{s}_{jt}$

3.3. Simplified index

The goal of this section is to derive a simplified version of the index (6). Recall that the index $\mathcal{I}_j$ from equation (6) characterizes the expected lifetime payoffs associated with specializing in j, ignoring other fields. If the the simplified stopping condition under monotonicity holds at time t (i.e. $(\tilde{m}_{jt}, \tilde{s}_{jt}, \alpha_{j0}, \beta_{j0}, h_{j0}) \in \mathcal{G}_j$), then the agent would expect to enter the labor market (ignoring other fields). Their expected lifetime payoff in j equals their expected lifetime earnings given their current levels of human capital:

$$\frac{1}{1-\delta} w_j h_{jt} = \frac{1}{1-\delta} w_j \left(h_{j0} + \nu_j \tilde{s}_{jt}\right)$$

If $(\tilde{m}_{jt}, \tilde{s}_{jt}, \alpha_{j0}, \beta_{j0}, h_{j0}) \notin \mathcal{G}_j$, then the agent would plan on continuing their studies in j. Their expected lifetime payoff depends on how much human capital they expect to accumulate in j. To make this concrete, let m_j^* denote the total number of periods an agent expects to study field j before entering the labor market. Then the agent expects to be in school for $m_j^* - \tilde{m}_{jt}$ more periods. Because the agent is not earning an income while they are in school, their expected lifetime payoff will be discounted by $\delta^{m_j^* - \tilde{m}_{jt}}$. They then expect to enter the labor market at time $t + m_j^* - \tilde{m}_{jt}$ with some level of human capital, given by $h_{j,t+m_j^*-\tilde{m}_{jt}}$. The index when $(\tilde{m}_{jt}, \tilde{s}_{jt}, \alpha_{j0}, \beta_{j0}, h_{j0}) \notin \mathcal{G}_j$ is then given by:

$$\frac{1}{1-\delta} w_j \mathbb{E}_t \left[\delta^{m_j^* - \tilde{m}_{jt}} h_{j,t+(m_j^*-\tilde{m}_{jt})} \middle| \tilde{m}_{jt}, \tilde{s}_{jt}, \psi_{j0} \right]$$

Therefore, the index (6) is characterized by:

$$\mathcal{I}_j = \begin{cases} \frac{w_j h_{jt}}{1-\delta} & \text{if } (\tilde{m}_{jt}, \tilde{s}_{jt}, \alpha_{j0}, \beta_{j0}, h_{j0}) \in \mathcal{G}_j, \\ \frac{w_j}{1-\delta} \mathbb{E}_t \left[\delta^{m_j^* - \tilde{m}_{jt}} h_{j,t+(m_j^*-\tilde{m}_{jt})} \middle| \tilde{m}_{jt}, \tilde{s}_{jt}, \alpha_{j0}, \beta_{j0}, h_{j0} \right] & \text{otherwise.} \end{cases} \tag{12}$$

The method for evaluation the conditional expectation in equation (12) is described in section 3.4. A summary of these results are in section 3.5.

3.4. Solving the index

This section discusses the analytical solution to the index (12). Specifically, I describe how to evaluate the expected value of discounted human capital accumulation, conditional on initial states:

$$\mathbb{E}_t \left[\delta^{m_j^* - \tilde{m}_{jt}} h_{j,t+(m_j^*-\tilde{m}_{jt})} \middle| \tilde{m}_{jt}, \tilde{s}_{jt}, \alpha_{j0}, \beta_{j0}, h_{j0} \right]. \tag{13}$$

To solve this, I first show how this expectation can be re-written as a function of expected time remaining in school. Computing the index therefore requires finding the conditional probability distribution of stopping times. The remainder of section is devoted to finding this distribution. This is done by first bounding the stopping times, and then recursively defining the probability distribution.

Index in terms of expected time in school

To simplify notation, let ψ_{j0} denote the initial belief parameters and human capital levels:

$$\psi_{j0} = (\alpha_{j0}, \beta_{j0}, h_{j0}) \,.$$

the agent's state when evaluating field j at time t is now determined by $(\tilde{m}_{jt}, \tilde{s}_{jt}, \psi_{j0})$.

Ignoring other fields, an agents expects to study j for $m_j^* - \tilde{m}_{jt}$ additional periods before beginning work as a field-j specialist. Substituting in the human capital accumulation function (2) into (13):

$$\begin{aligned}
&\mathbb{E}_t \left[\delta^{m_j^* - \tilde{m}_{jt}} h_{j,t+(m_j^* - \tilde{m}_{jt})} \middle| \tilde{m}_{jt}, \tilde{s}_{jt}, \psi_{j0} \right] \\
&\qquad = \mathbb{E}_t \left[\delta^{m_j^* - \tilde{m}_{jt}} \left(h_{j0} + \nu_j \tilde{s}_{jt} + \sum_{x=0}^{m_j^* - \tilde{m}_{jt}} s_{j,t+x} \right) \middle| \tilde{m}_{jt}, \tilde{s}_{jt}, \psi_{j0} \right] \\
&\qquad = \mathbb{E}_t \left[\delta^{m_j^* - \tilde{m}_{jt}} \middle| \tilde{m}_{jt}, \tilde{s}_{jt}, \psi_{j0} \right] (h_{j0} + \nu_j \tilde{s}_{jt}) + \mathbb{E}_t \left[\delta^{m_j^* - \tilde{m}_{jt}} \sum_{x=0}^{m_j^* - \tilde{m}_{jt}} s_{j,t+x} \middle| \tilde{m}_{jt}, \tilde{s}_{jt}, \psi_{j0} \right] .
\end{aligned}$$

Two expectations are key. The first is the expected value of discounting the next $m_j^* - \tilde{m}_{jt}$ additional periods. The second is the expected value of the discounted term times the number of of times an agent successfully passes their field-j courses during those $m_j^* - \tilde{m}_{jt}$ periods. To simplify the second probability, use the law of iterated expectations:

$$\begin{aligned}
&\mathbb{E}_t \left[\delta^{m_j^* - \tilde{m}_{jt}} \sum_{x=0}^{m_j^* - \tilde{m}_{jt}} s_{j,t+x} \middle| \tilde{m}_{jt}, \tilde{s}_{jt}, \psi_{j0} \right] \\
&\qquad = \mathbb{E}_t \left[\mathbb{E}_t \left[\delta^{m_j^* - \tilde{m}_{jt}} \sum_{x=0}^{m_j^* - \tilde{m}_{jt}} s_{j,t+x} \middle| m_j^*, \tilde{m}_{jt}, \tilde{s}_{jt}, \psi_{j0} \right] \middle| \tilde{m}_{jt}, \tilde{s}_{jt}, \psi_{j0} \right] \\
&\qquad = \mathbb{E}_t \left[\delta^{m_j^* - \tilde{m}_{jt}} \mathbb{E}_t \left[\sum_{x=0}^{m_j^* - \tilde{m}_{jt}} s_{j,t+x} \middle| m_j^*, \tilde{m}_{jt}, \tilde{s}_{jt}, \psi_{j0} \right] \middle| \tilde{m}_{jt}, \tilde{s}_{jt}, \psi_{j0} \right]
\end{aligned}$$

The number of times the agent successfully passes their field-j courses over the next $m_j^* - \tilde{m}_{jt}$ periods is a series of $m_j^* - \tilde{m}_{jt}$ Bernoulli trials with probability θ_j. The agent's expected value of this random variable is given by the sample size multiplied by their ability parameter θ_j. Therefore the previous equation can be written as:

$$
\begin{aligned}
&\mathbb{E}_t\left[\delta^{m_j^*-\tilde{m}_{jt}}\mathbb{E}_t\left[\sum_{x=0}^{m_j^*-\tilde{m}_{jt}} s_{j,t+x}\middle| m_j^*, \tilde{m}_{jt}, \tilde{s}_{jt}, \psi_{j0}\right]\middle| \tilde{m}_{jt}, \tilde{s}_{jt}, \psi_{j0}\right] \\
&\qquad = \mathbb{E}_t\left[\delta^{m_j^*-\tilde{m}_{jt}}\left(m_j^*-\tilde{m}_{jt}\right)\mathbb{E}_t\left[\theta_j \middle| m_j^*, \tilde{m}_{jt}, \tilde{s}_{jt}, \psi_{j0}\right]\middle| \tilde{m}_{jt}, \tilde{s}_{jt}, \psi_{j0}\right] \\
&\qquad = \mathbb{E}_t\left[\delta^{m_j^*-\tilde{m}_{jt}}\left(m_j^*-\tilde{m}_{jt}\right)\frac{\alpha_{j0}+\tilde{s}_{jt}}{\alpha_{j0}+\beta_{j0}+\tilde{m}_{jt}}\middle| \tilde{m}_{jt}, \tilde{s}_{jt}, \psi_{j0}\right] \\
&\qquad = \mathbb{E}_t\left[\delta^{m_j^*-\tilde{m}_{jt}}\left(m_j^*-\tilde{m}_{jt}\right)\middle| \tilde{m}_{jt}, \tilde{s}_{jt}, \psi_{j0}\right]\frac{\alpha_{j0}+\tilde{s}_{jt}}{\alpha_{j0}+\beta_{j0}+\tilde{m}_{jt}}
\end{aligned}
$$

The third line follows from the agent's expected value of θ_j, conditional on their states and their time until completion, m_j^*, according to their belief distribution. We can use this result to further simplify (13) as:

$$
\mathbb{E}_t\left[\delta^{m_j^*-\tilde{m}_{jt}}\middle| \tilde{m}_{jt}, \tilde{s}_{jt}, \psi_{j0}\right]\left(h_{j0}+\nu_j\tilde{s}_{jt}\right)+\mathbb{E}_t\left[\delta^{m_j^*-\tilde{m}_{jt}}\left(m_j^*-\tilde{m}_{jt}\right)\middle| \tilde{m}_{jt}, \tilde{s}_{jt}, \psi_{j0}\right]\frac{\alpha_{j0}+\tilde{s}_{jt}}{\alpha_{j0}+\beta_{j0}+\tilde{m}_{jt}}.
$$

Thus, the key expected value (13) is really the expected value of a function of time remaining in school.

Before proceeding, it's helpful to introduce some simplifying notation, and to re-scale the problem to start at time $t=0$. To simplify notation, let $N=m_j^*-\tilde{m}_{jt}$ denote the time remaining in school after $\tilde{m}_{jt}$. The variable N is capitalized to emphasize the fact that N is a random quantity. Next, note that for any agent evaluating field j at time t with states $(\tilde{m}_{jt}, \tilde{s}_{jt}, \psi_{j0})$, we can always define:

$$
\begin{aligned}
\hat{\alpha}_{j0} &= \alpha_{j0}+\tilde{s}_{jt}, & \hat{\alpha}_{j0}+\hat{\beta}_{j0} &= \alpha_{j0}+\beta_{j0}+\tilde{m}_{jt}, \\
\hat{h}_{j0} &= h_{j0}+\nu_j\tilde{s}_{jt}, & \hat{\psi}_{j0} &= \left(\hat{\alpha}_{j0}, \hat{\beta}_{j0}, \hat{h}_{j0}\right).
\end{aligned}
$$

Therefore, instead of evaluating how many courses an agent has remaining after completing $\tilde{m}_{jt}$ courses, we can re-define the agent's states and evaluate the agent's total expected time in school from $t=0$, before the agent has taken any courses in j. In that vein, I will only condition on the initial states $\psi_{j0}=(\alpha_{j0}, \beta_{j0}, h_{j0})$, and assume $\tilde{s}_{jt}=\tilde{m}_{jt}=0$. Recall that $\tilde{s}_{j,N}$ is the number of times the student successfully passes their courses after matriculating N times. Using the above notation, the monotonic stopping condition (10) is given by:

$$
N \geq \frac{\delta}{1-\delta}\left(\frac{\nu_j\alpha_{j0}+\nu_j\tilde{s}_{j,N}}{h_{j0}+\nu_j\tilde{s}_{j,N}}\right)-\alpha_{j0}-\beta_{j0}, \tag{14}
$$

The goal in subsequent sections is to evaluate the following conditional expectations:

$$
\begin{aligned}
\mathbb{E}_0\left[\delta^N \middle| \alpha_{j0}, \beta_{j0}, h_{j0}\right] &= \sum_{z=0}^{\infty} \delta^z \mathbb{P}\left(N = z \,|\alpha_{j0}, \beta_{j0}, h_{j0}\right) \\
\mathbb{E}_0\left[\delta^N N \middle| \alpha_{j0}, \beta_{j0}, h_{j0}\right] &= \sum_{z=0}^{\infty} \delta^z z \mathbb{P}\left(N = z \,|\alpha_{j0}, \beta_{j0}, h_{j0}\right)
\end{aligned} \tag{15}
$$

The following section discusses the bounds on stopping times, so the above summation is finite.

Bounds on stopping times

This section starts by defining a lower bound larger than zero for all stopping times. I then discuss stopping times with positive probability, and use this discussion to motivate the upper bound for all stopping times.

Lemma 1. Define the positive integer $\underline{n}$ as:

$$
\underline{n} = \min_N \left\{ N \geq \frac{\delta}{1-\delta} - \alpha_{j0} - \beta_{j0} \right\} = \left\lceil \frac{\delta}{1-\delta} \right\rceil - \alpha_{j0} - \beta_{j0}. \tag{16}
$$

Then $\underline{n}$ is a lower bound for stopping times that satisfy the monotonic stopping condition (14).

Proof. For all possible stopping times N and all possible stochastic outcomes $\tilde{s}_{j,N}$, $\frac{\nu_j\alpha_{j0}+\nu_j\tilde{s}_{jN}}{h_{j0}+\nu_j\tilde{s}_{jN}} \geq 1$ under the initial monotonicity condition (5).[7] Therefore, $\underline{n}$ is a lower bound. $\square$

Before defining the upper bound of N, it is helpful to discuss the stopping condition for different potential values of N. Given the stopping condition (14), an agent will decide to stop studying in period $\underline{n}+x$ if:

$$
\begin{aligned}
& \left\lceil \frac{\delta}{1-\delta} \right\rceil - \alpha_{j0} - \beta_{j0} + x \geq \frac{\delta}{1-\delta} \left(\frac{\nu_j\alpha_{j0} + \nu_j\tilde{s}_{j,\underline{n}+x}}{h_{j0} + \nu_j\tilde{s}_{j,\underline{n}+x}} \right) - \alpha_{j0} - \beta_{j0} \\
\implies \quad & \frac{\delta}{1-\delta} + \epsilon_\delta + x \geq \frac{\delta}{1-\delta} \left(\frac{\nu_j\alpha_{j0} + \nu_j\tilde{s}_{j,\underline{n}+x}}{h_{j0} + \nu_j\tilde{s}_{j,\underline{n}+x}} \right) \\
\implies \quad & \epsilon_\delta + x \geq \frac{\delta}{1-\delta} \left(\frac{\nu_j\alpha_{j0} - h_{j0}}{h_{j0} + \nu_j\tilde{s}_{j,\underline{n}+x}} \right) \\
\implies \quad & (\epsilon_\delta + x)\left(h_{j0} + \nu_j\tilde{s}_{j,\underline{n}+x}\right) \geq \frac{\delta}{1-\delta} \left(\alpha_{j0}\nu_j - h_{j0}\right)
\end{aligned} \tag{17}
$$

where the rounding error $\epsilon_\delta \in [0, 1)$ equals the difference between the ceiling of the discount factor $\frac{\delta}{1-\delta}$ and its true value.[8]

7. Define $f(s) = \frac{\nu_j\alpha_{j0}+\nu_j s}{h_{j0}+\nu_j s} = \frac{\nu_j\alpha_{j0}-h_{j0}}{h_{j0}+\nu_j s} + 1$. Because $\nu_j\alpha_{j0} \geq h_{j0}$ and $s \geq 0$, $f(s) \geq 1$.

8. Specifically, if $\frac{\delta}{1-\delta}$ is not an integer, then $\epsilon_\delta = \left\lceil \frac{\delta}{1-\delta} \right\rceil - \left\lfloor \frac{\delta}{1-\delta} \right\rfloor - \left\{ \frac{\delta}{1-\delta} \right\} = 1 - \left\{ \frac{\delta}{1-\delta} \right\}$ is the difference between the ceiling of $\frac{\delta}{1-\delta}$ and $\frac{\delta}{1-\delta}$. If $\frac{\delta}{1-\delta}$ is an integer, then $\epsilon = 0$. Recall that the ceiling of any real number x, denoted $\lceil x \rceil$, is the smallest integer greater than or equal to x. The floor of a x, $\lfloor x \rfloor$, is the largest integer less than or equal

Now let's consider cases where agents would only take $N = \underline{n}$ courses, meaning that $x = 0$. Using the stopping condition (17) with $x = 0$, this only happens if:

$$\begin{aligned} \epsilon_\delta \left(h_{j0} + \nu_j \tilde{s}_{j\underline{n}}\right) &\geq \frac{\delta}{1-\delta}(\alpha_{j0}\nu_j - h_{j0}) \\ \implies \qquad \epsilon_\delta \nu_j \tilde{s}_{j\underline{n}} &\geq \frac{\delta}{1-\delta}\nu_j \alpha_{j0} - \left\lceil \frac{\delta}{1-\delta} \right\rceil h_{j0}. \end{aligned}$$

The agent will only study for $\underline{n}$ periods if this inequality is satisfied for all possible stochastic outcomes. The only stochastic part of this inequality is $\tilde{s}_{j\underline{n}}$, which make take on values between 0 and $\underline{n}$. Therefore, agents will only study for $\underline{n}$ periods if:

$$\begin{aligned} 0 &\geq \frac{\delta}{1-\delta}\nu_j \alpha_{j0} - \left\lceil \frac{\delta}{1-\delta} \right\rceil h_{j0}. \\ \implies \left\lceil \frac{\delta}{1-\delta} \right\rceil h_{j0} &\geq \frac{\delta}{1-\delta}\alpha_{j0}\nu_j. \end{aligned}$$

Combining the above with the initial monotonicity condition (5) implies that an agent will only study for exactly $N = \underline{n}$ periods if:

$$1 \leq \frac{\nu_j \alpha_{j0}}{h_{j0}} \leq \frac{\left\lceil \frac{\delta}{1-\delta} \right\rceil}{\frac{\delta}{1-\delta}}.$$

Because the ratio of the ceiling of the discount factor to its true value will be close to 1, this inequality effectively states that the agent will only study for $N = \underline{n}$ periods if $h_{j0} = \nu_j \alpha_{j0}$ (with some adjustment for rounding error). This is the tractable case evaluated in Alon and Fershtman (2019). Specifically, assuming the slightly stronger initial condition, $h_{j0} = \nu_j \alpha_{j0}$, implies time spent in school N equals $\underline{n}$; an agent who specializes in field j will take exactly N courses in field j. Therefore, the the optimal number of field-j courses is a deterministic function of the agent's initial beliefs. However, for reasons discussed in the overview of section 3, this assumption is not necessarily appropriate for this evaluation. Thus, we now turn to evaluating the upper bound of possible stopping times.

Lemma 2. Define the positive integer $\overline{n}$ as:

$$\overline{n} = \left\lceil \frac{\delta}{1-\delta} \frac{\alpha_{j0}\nu_j}{h_{j0}} \right\rceil - \alpha_{j0} - \beta_{j0} \tag{18}$$

Then $\overline{n}$ is an upper bound for stopping times.

Proof. $\overline{n}$ is an upper bound for stopping times if, for all stochastic outcomes $\tilde{s}_{j\overline{n}}$:

$$\overline{n} \geq \frac{\delta}{1-\delta}\left(\frac{\alpha_{j0}\nu_j + \nu_j \tilde{s}_{j\overline{n}}}{h_{j0} + \nu_j \tilde{s}_{j\overline{n}}}\right) - \alpha_{j0} - \beta_{j0}$$

to x. The fractional part of x, denoted $\{x\}$, is defined by $\{x\} = x - \lfloor x \rfloor$.

Because $\frac{\alpha_{j0}\nu_j+\nu_j\tilde{s}_{j\overline{n}}}{h_{j0}+\nu_j\tilde{s}_{j\overline{n}}}$ is decreasing in $\tilde{s}_{j\overline{n}}$,[9] $\overline{n}$ is an upper bound independent of stochastic outcomes only if the above inequality holds for $\tilde{s}_{j\overline{n}} = 0$ (i.e. when the agent has failed all of their field j courses). Therefore, $\overline{n}$ is an upper bound if:

$$\begin{aligned}\overline{n} \geq & \frac{\delta}{1-\delta}\left(\frac{\alpha_{j0}\nu_j}{h_{j0}}\right) - \alpha_{j0} - \beta_{j0} \\ \Longrightarrow \left\lceil \frac{\delta}{1-\delta}\frac{\alpha_{j0}\nu_j}{h_{j0}} \right\rceil \geq & \frac{\delta}{1-\delta}\left(\frac{\alpha_{j0}\nu_j}{h_{j0}}\right).\end{aligned}$$

Therefore, $\overline{n}$ is an upper bound for stopping times. $\square$

The details above can be used to bound the summations in (15):

$$\begin{aligned}\mathbb{E}\left[\delta^N \middle| \alpha_{j0}, \beta_{j0}, h_{j0}\right] =& \sum_{z=\underline{n}}^{\overline{n}} \delta^z \mathbb{P}\left(N = z \left| \alpha_{j0}, \beta_{j0}, h_{j0}\right.\right) \\ \mathbb{E}\left[\delta^N N \middle| \alpha_{j0}, \beta_{j0}, h_{j0}\right] =& \sum_{z=\underline{n}}^{\overline{n}} \delta^z z \mathbb{P}\left(N = z \left| \alpha_{j0}, \beta_{j0}, h_{j0}\right.\right)\end{aligned}$$

The next section evaluates the above conditional probabilities.

Conditional probabilities

Recall that the lower and upper bounds of N are given by $\underline{n}$ and $\overline{n}$, respectively. The conditional probability that N equals some integer z can be evaluated as:

$$\begin{aligned}\mathbb{P}\left(N = z \left| \psi_{j0}\right.\right) =& \mathbb{P}\left(z \geq \frac{\delta}{1-\delta}\frac{\alpha_{j0}\nu_j + \nu_j\tilde{s}_{jz}}{h_{j0}+\nu_j\tilde{s}_{jz}} - \alpha_{j0} - \beta_{j0} \middle| \psi_{j0}\right) \\ =& \mathbb{P}\left(z - (\underline{n} - \epsilon_\delta) \geq \frac{\delta}{1-\delta}\frac{\alpha_{j0}\nu_j - h_{j0}}{h_{j0}+\nu_j\tilde{s}_{jz}} \middle| \psi_{j0}\right) \\ =& \mathbb{P}\left((z - \underline{n} + \epsilon_\delta)(h_{j0}+\nu_j\tilde{s}_{jz}) \geq \frac{\delta}{1-\delta}(\alpha_{j0}\nu_j - h_{j0}) \middle| \psi_{j0}\right) \qquad (19) \\ =& \mathbb{P}\left((z - \underline{n} + \epsilon_\delta)\tilde{s}_{jz} \geq \frac{\delta}{1-\delta}\alpha_{j0} - \left(\left\lceil\frac{\delta}{1-\delta}\right\rceil + z - \underline{n}\right)\frac{h_{j0}}{\nu_j} \middle| \psi_{j0}\right) \qquad (20)\end{aligned}$$

First consider the case where $N = \underline{n}$. Let $\mathbb{P}_0$ denote this probability. Then, as discussed above:

$$\mathbb{P}_0 = \mathbb{P}\left(N = \underline{n} \left| \psi_{j0}\right.\right) = \begin{cases} 1 & \text{if } 1 \leq \frac{\alpha_{j0}\nu_j}{h_{j0}} \leq \frac{\left\lceil\frac{\delta}{1-\delta}\right\rceil}{\frac{\delta}{1-\delta}} \\ 0 & \text{otherwise.}\end{cases}$$

Before evaluating cases where $N > \underline{n}$, note the probability of stopping at some positive integer

9. Define $f(s) = \frac{\nu_j\alpha_{j0}+\nu_j s}{h_{j0}+\nu_j s} = \frac{\nu_j\alpha_{j0}-h_{j0}}{h_{j0}+\nu_j s} + 1$. Note that $f'(s) = -\frac{\nu_j(\nu_j\alpha_{j0}-h_{j0})}{\left(h_{j0}+\nu_j s\right)^2}$. This is nonpositive when $\nu_j\alpha_{j0} \geq h_{j0}$.

z is always given by:

$$\begin{aligned}\mathbb{P}\left(N=z\middle|\,\psi_{j0}\right) &= \mathbb{P}\left(N=z, N\neq z-1\middle|\,\psi_{j0}\right)+\mathbb{P}\left(N=z, N=z-1\middle|\,\psi_{j0}\right)\\ &= \mathbb{P}\left(N=z\middle|\,N\neq z-1,\psi_{j0}\right)\mathbb{P}\left(N\neq z-1\middle|\,\psi_{j0}\right)\end{aligned}$$

The second line follows from Bayes' Rule, and the fact that an agent would never stop at time z if they already stopped at time $z-1$. In words, this states that the probability of stopping at time z is given by the product of (1) the probability of stopping at time z conditional on having not stopped at time $z-1$; and (2) the probability of having not stopped by $z-1$.

To evaluate the probability that $N=\underline{n}+1$, first evaluate the conditional probability that $N=\underline{n}+1$, conditional on the stopping time not equaling $\underline{n}$:

$$\begin{aligned}\mathbb{P}\left(N=\underline{n}+1\middle|\,N\neq\underline{n},\psi_{j0}\right) =&\mathbb{P}\left((1+\epsilon_\delta)\left(h_{j0}+\nu_j\tilde{s}_{j\underline{n}+1}\right)\geq\frac{\delta}{1-\delta}\left(\alpha_{j0}\nu_j-h_{j0}\right)\middle|\,\psi_{j0}\right)\\ =&\mathbb{P}\left(\tilde{s}_{j\underline{n}+1}\geq\frac{1}{\nu_j}\left(\frac{\delta}{1-\delta}\frac{1}{1+\epsilon_\delta}\left(\alpha_{j0}\nu_j-h_{j0}\right)-h_{j0}\right)\middle|\,\psi_{j0}\right)\\ =&\mathbb{P}\left(\tilde{s}_{j\underline{n}+1}\geq\hat{k}_1\middle|\,\psi_{j0}\right)\end{aligned}$$

The RHS of the above inequality is a function of initial parameters and can be treated as a constant. The variable $\tilde{s}_{j\underline{n}+1}$ is a random variable:

$$\tilde{s}_{j,\underline{n}+1}=\sum_{t=0}^{\underline{n}}s_{jt}\sim\text{Binomial}(\underline{n}+1,\theta_j)$$

Define k_1 as:

$$k_1=\begin{cases}\hat{k}_1-1 & \text{if } \hat{k}_1 \text{ an integer,}\\ \left\lfloor\hat{k}_1\right\rfloor & \text{otherwise.}\end{cases}$$

The conditional probability can be written as:

$$\begin{aligned}\mathbb{P}\left(N=\underline{n}+1\middle|\,N\neq\underline{n},\psi_{j0}\right) =&1-\mathbb{P}\left(\tilde{s}_{j\underline{n}+1}<k_1\middle|\,\psi_{j0}\right)\\ =&1-\sum_{i=0}^{k_1}\binom{\underline{n}+1}{i}\theta^i(1-\theta)^{\underline{n}+1-i}\end{aligned}$$

Now we can fully evaluate the probability that $N = \underline{n} + 1$:

$$\begin{aligned}\mathbb{P}_1 = \mathbb{P}\left(N = \underline{n}+1 \middle|\, \psi_{j0}\right) =& \mathbb{P}\left(N = \underline{n}+1 \middle|\, N \neq \underline{n}, \psi_{j0}\right) \mathbb{P}\left(N \neq \underline{n} \middle|\, \psi_{j0}\right) \\ =& \mathbb{P}\left(N = \underline{n}+1 \middle|\, N \neq \underline{n}, \psi_{j0}\right)\left(1 - \mathbb{P}_0\right) \\ =& \left(1 - \sum_{i=0}^{k_1} \binom{\underline{n}+1}{i} \theta^i (1-\theta)^{\underline{n}+1-i}\right)\left(1 - \mathbb{P}_0\right).\end{aligned}$$

To evaluate the probability that $N = \underline{n} + x$ for some integer x, we have to evaluate:

$$\mathbb{P}_x = \mathbb{P}\left(N = \underline{n}+x \middle|\, \psi_{j0}\right) = \mathbb{P}\left(N = \underline{n}+x \middle| \bigcap_{i=0}^{x-1}\left(N \neq \underline{n}+i\right), \psi_{j0}\right) \mathbb{P}\left(\bigcap_{i=0}^{x-1}\left(N \neq \underline{n}+i\right) \middle| \psi_{j0}\right)$$

First consider the probability that N is not equal any value between the lower bound $\underline{n}$ and $\underline{n}+x-1$. By De Morgan's law and countability additivity:

$$\begin{aligned}\mathbb{P}\left(\bigcap_{i=0}^{x-1}\left(N \neq \underline{n}+i\right) \middle| \psi_{j0}\right) =& 1 - \mathbb{P}\left(\bigcup_{i=0}^{x-1}\left(N = \underline{n}+i\right) \middle| \psi_{j0}\right) \\ =& 1 - \sum_{i=0}^{x-1} \mathbb{P}\left(N = \underline{n}+i \middle|\, \psi_{j0}\right) = 1 - \sum_{i=0}^{x-1} \mathbb{P}_i.\end{aligned}$$

Now we turn to the conditional probability of interest, which can be written as:

$$\begin{aligned}&\mathbb{P}\left(N = \underline{n}+x \middle| \bigcap_{i=0}^{x-1}\left(N \neq \underline{n}+i\right), \psi_{j0}\right) \\ &= \mathbb{P}\Bigg(\left(x + \epsilon_\delta\right)\left(h_{j0} + \nu_j \tilde{s}_{j,\underline{n}+x}\right) \geq \frac{\delta}{1-\delta}\left(\alpha_{j0}\nu_j - h_{j0}\right) \\ &\qquad \Bigg| \bigcap_{k=0}^{x-1} \left(k + \epsilon_\delta\right)\left(h_{j0} + \nu_j \tilde{s}_{j\underline{n}+k-1}\right) < \frac{\delta}{1-\delta}\left(\alpha_{j0}\nu_j - h_{j0}\right)\Bigg).\end{aligned} \tag{21}$$

To simplify this conditional, recall that an agent will decide to keep studying at time $\underline{n} + y$ if:

$$\begin{aligned}\frac{\delta}{1-\delta}\left(\alpha_{j0}\nu_j - h_{j0}\right) >& \left(y + \epsilon_\delta\right)\left(h_{j0} + \nu_j \tilde{s}_{j,\underline{n}+y}\right) \\ =& \left(y - 1 + \epsilon_\delta\right)\left(h_{j0} + \nu_j \tilde{s}_{j,\underline{n}+y-1}\right) + \left(h_{j0} + \nu_j \tilde{s}_{j,\underline{n}+y-1}\right) + (y + \epsilon_\delta)\nu_j s_{j,\underline{n}+y-1} \\ >& \left(y - 1 + \epsilon_\delta\right)\left(h_{j0} + \nu_j \tilde{s}_{j,\underline{n}+y-1}\right)\end{aligned}$$

This is simply the monotonicity of the stopping problem in reverse; if an agent would decide to continue on at time $\underline{n} + y$, then they also would have wanted to continue on at time $\underline{n} + y - 1$. This simplifies the conditional expression in equation (21):

$$
\begin{aligned}
&\mathbb{P}\left(N=\underline{n}+x \middle| \bigcap_{i=0}^{x-1}\left(N \neq \underline{n}+i\right), \psi_{j0}\right) \\
&\quad=\mathbb{P}\left(\left(x+\epsilon_\delta\right)\left(h_{j0}+\nu_j \tilde{s}_{j,\underline{n}+x}\right) \geq \frac{\delta}{1-\delta}\left(\alpha_{j0}\nu_j-h_{j0}\right)\right. \\
&\qquad\qquad \left.\middle| \left(x-1+\epsilon_\delta\right)\left(h_{j0}+\nu_j \tilde{s}_{j\underline{n}+x-1}\right)<\frac{\delta}{1-\delta}\left(\alpha_{j0}\nu_j-h_{j0}\right)\right)
\end{aligned}
$$

This probability can be re-written to reflect the fact that $\tilde{s}_{j,\underline{n}+x}=\tilde{s}_{j,\underline{n}+x-1}+s_{j,\underline{n}+x-1}$. In words, the total number of successes seen by time $\underline{n}+x$ equals the total number of successes seen by time $\underline{n}+x-1$ plus the course outcome during period $\underline{n}+x-1$:

$$
\begin{aligned}
&\mathbb{P}\left(\left(x+\epsilon_\delta\right)\left(h_{j0}+\nu_j \tilde{s}_{j,\underline{n}+x-1}\right)+\left(x+\epsilon_\delta\right)\nu_j s_{j,\underline{n}+x-1} \geq \frac{\delta}{1-\delta}\left(\alpha_{j0}\nu_j-h_{j0}\right)\right. \\
&\qquad\qquad\qquad\qquad \left.\middle| \left(x-1+\epsilon_\delta\right)\left(h_{j0}+\nu_j \tilde{s}_{j\underline{n}+x-1}\right)<\frac{\delta}{1-\delta}\left(\alpha_{j0}\nu_j-h_{j0}\right)\right).
\end{aligned}
$$

Define the random variables Y and Z and the constant c as:

$$
\begin{aligned}
&Y=g(\tilde{s}_{j,\underline{n}+x-1})=\left(x+\epsilon_\delta\right)\left(h_{j0}+\nu_j \tilde{s}_{j,\underline{n}+x-1}\right), \qquad c=\frac{\delta}{1-\delta}\left(\alpha_{j0}\nu_j-h_{j0}\right), \\
&Z=h(s_{j,\underline{n}+x-1})=\left(x+\epsilon_\delta\right)\nu_j s_{j,\underline{n}+x-1}
\end{aligned}
$$

The conditional probability that $N=\underline{n}+x$ for $x>1$ can now be written as:

$$
\mathbb{P}\left(Y+Z \geq c \middle| Y<c\frac{x+\epsilon_\delta}{x-1+\epsilon_\delta}\right),
$$

where Y and Z are independent random variables whose distributions are one-to-one functions of binomial distributions:

$$
\begin{aligned}
\mathbb{P}(Y=y)=&\mathbb{P}\left(g\left(\tilde{s}_{j,\underline{n}+x-1}\right)=y\right)=\mathbb{P}(\tilde{s}_{j,\underline{n}+x-1}=g^{-1}(y))) \\
=&\binom{\underline{n}+x-2}{g^{-1}(y)}\theta_j^{g^{-1}(y)}(1-\theta_j)^{\underline{n}+x-2-g^{-1}(y)}, \\
\mathbb{P}(Z=z)=&\mathbb{P}\left(h\left(s_{j,\underline{n}+x-1}\right)=z\right)=\mathbb{P}\left(s_{j,\underline{n}+x-1}=h^{-1}(z)\right) \\
=&\theta_j^{h^{-1}(z)}\left(1-\theta_j\right)^{1-h^{-1}(z)}.
\end{aligned}
$$

The joint conditional distribution can be solved using Theorem 20.3 in Billingsley (2012, pg. 280).

3.5. Computing agent behavior

This section summarizes how to compute an agent's behavior given states $(\tilde{m}_{jt}, \tilde{s}_{jt}, \alpha_{j0}, \beta_{j0}, h_{j0}) = (\tilde{m}_{jt}, \tilde{s}_{jt}, \psi_{j0})$ and assuming the initial monotonicity assumption. Recall that the graduation region under the initial monotonicity assumption is given by (11):

$$\mathcal{G}_j = \left\{ \tilde{m}_{jt}, \tilde{s}_{jt}, \alpha_{j0}, \beta_{j0}, h_{j0} \middle| \tilde{m}_{jt} \geq \frac{\delta}{1-\delta} \left(\frac{\nu_j \alpha_{j0} + \nu_j \tilde{s}_{jt}}{h_{j0} + \nu_j \tilde{s}_{jt}} \right) - \alpha_{j0} - \beta_{j0} \right\}$$

The index (12) under the initial monotonicity assumption (5) is given by:

$$\mathcal{I}_j(\tilde{m}_{jt}, \tilde{s}_{jt}, \psi_{j0}) = \begin{cases} \frac{w_j}{1-\delta} h_{jt} & (\tilde{m}_{jt}, \tilde{s}_{jt}, \psi_{j0}) \in \mathcal{G}_j, \\ \frac{w_j}{1-\delta} \mathbb{E}_t \left[\delta^{m_j^* - \tilde{m}_{jt}} h_{j,t+(m_j^* - \tilde{m}_{jt})} \middle| \tilde{m}_{jt}, \tilde{s}_{jt}, \psi_{j0} \right] & \text{otherwise,} \end{cases}$$

1. To find the index for all fields j, first determine whether an agent is in their graduation region (11).

   a. If the agent is in their graduation region for field j, set the j-index equal to $\frac{w_{jt} h_{jt}}{1-\delta}$. **Skip to step 6.**

   b. If the agent is not in their graduation region, the index must be calculated using their expected accumulation of human capital, which is a function of expected time remaining in j:

$$\begin{aligned} \mathbb{E}_t \left[\delta^{m_j^* - \tilde{m}_{jt}} h_{j,t+(m_j^* - \tilde{m}_{jt})} \middle| \tilde{m}_{jt}, \tilde{s}_{jt}, \psi_{j0} \right] &= \mathbb{E}_t \left[g\left(m_j^* - \tilde{m}_{jt}\right) \middle| \tilde{m}_{jt}, \tilde{s}_{jt}, \psi_{j0} \right] \\ &= \mathbb{E}_t \left[\delta^{m_j^* - \tilde{m}_{jt}} \left(h_{j0} + \nu_j \tilde{s}_{jt} + \left(m_j^* - \tilde{m}_{jt}\right) \frac{\alpha_{j0} + \tilde{s}_{jt}}{\alpha_{j0} + \beta_{j0} + \tilde{m}_{jt}} \right) \middle| \tilde{m}_{jt}, \tilde{s}_{jt}, \psi_{j0} \right] \end{aligned}$$

   Proceed to the next step.

2. Re-index the problem to simplify analysis:

$$\begin{aligned} \hat{\alpha}_{j0} &= \alpha_{j0} + \tilde{s}_{jt}, & \hat{\alpha}_{j0} + \hat{\beta}_{j0} &= \alpha_{j0} + \beta_{j0} + \tilde{m}_{jt}, \\ \hat{h}_{j0} &= h_{j0} + \nu_j \tilde{s}_{jt}, & \hat{\psi}_{j0} &= \left(\hat{\alpha}_{j0}, \hat{\beta}_{j0}, \hat{h}_{j0} \right), \end{aligned}$$

   This way, we are considering how many courses an agent is expecting to study from time $t = 0$, instead of how many remaining courses an agent expects to study at an arbitrary t.

3. Let $N = m_{jt}^* - \tilde{m}_{jt}$ denote the number of periods an agent expects to study, and let $\underline{n}$ and $\overline{n}$ denote the lower and upper bound of N, respectively. These bounds are given by:

$$
\begin{aligned}
\underline{n} &= \underline{m_{jt}^{*}-\tilde{m}_{jt}} = \left\lceil \frac{\delta}{1-\delta} \right\rceil - \hat{\alpha}_{j0} - \hat{\beta}_{j0} \\
\overline{n} &= \overline{m_{jt}^{*}-\tilde{m}_{jt}} = \begin{cases} \underline{n} & \text{if } 1 \leq \frac{\hat{\alpha}_{j0}\nu_j}{\hat{h}_{j0}} \leq \frac{\left\lceil \frac{\delta}{1-\delta} \right\rceil}{\frac{\delta}{1-\delta}} \\ \left\lceil \frac{\delta}{1-\delta} \frac{\hat{\alpha}_{j0}\nu_j}{\hat{h}_{j0}} \right\rceil - \hat{\alpha}_{j0} - \hat{\beta}_{j0}, & \text{otherwise.} \end{cases}
\end{aligned}
$$

4. Find the probability distribution of stopping between $\underline{n}$ and $\overline{n}$ conditional on $\left(\hat{\alpha}_{j0}, \hat{\beta}_{j0}, \hat{h}_{j0}\right)$. First evaluate the probability that $N = \underline{n}$: .

$$
\mathbb{P}_0 = \mathbb{P}\left(N = \underline{n} \middle| \hat{\alpha}_{j0}, \hat{\beta}_{j0}, \hat{h}_{j0}\right) = \begin{cases} 1 & \text{if } 1 \leq \frac{\alpha_{j0}\nu_j}{h_{j0}} \leq \frac{\left\lceil \frac{\delta}{1-\delta} \right\rceil}{\frac{\delta}{1-\delta}} \\ 0 & \text{otherwise.} \end{cases}
$$

Next evaluate the probability of stopping at $N = \underline{n} + 1$:

$$
\begin{aligned}
\mathbb{P}_1 = \mathbb{P}\left(N = \underline{n} + 1 | \hat{\psi}_{j0}\right) &= \mathbb{P}\left(N = \underline{n} + 1 | N \neq \underline{n}, \psi_{j0}\right)\left(1 - \mathbb{P}_0\right) \\
&=
\end{aligned}
$$

For any integer $x > 1$ such that $\underline{n} + x < \overline{n}$, the probability of stopping at $\underline{n} + x$ is given by:

$$
\mathbb{P}_x = \mathbb{P}\left(N = \underline{n} + x | \hat{\psi}_{j0}\right) =
$$

Finally, the probability of stopping at the upper bound $\overline{n}$ is given by:

$$
\mathbb{P}_{\overline{n}-\underline{n}} = \mathbb{P}\left(N = \overline{n} \middle| \hat{\alpha}_{j0}, \hat{\beta}_{j0}, \hat{h}_{j0}\right) = 1 - \sum_{n=0}^{\overline{n}-\underline{n}-1} \mathbb{P}_n
$$

5. Compute expected discounted accumulation of human capital

$$
\mathbb{E}_t\left[g(N) | \hat{\psi}_{j0}\right] = \sum_{n=\underline{n}}^{\overline{n}} g(N)\, \mathbb{P}\left(N = n \middle| \hat{\psi}_{j0}\right)
$$

And set the index equal to $\frac{w_j}{1-\delta}\mathbb{E}_t\left[g(N) | \hat{\psi}_{j0}\right]$.

6. Follow the optimal policy outlined in section 2.3. If the agent chooses to study this period, return to step one in the next period.

4. Implications of model

An agent's specialization decision is affected by field, individual, and group characteristics. This section illustrates how these factors motivate an individual's behavior using a simplified version of the model. Section 4.1 develops a version of the model where an agent chooses between two completely symmetric fields. Simulations are explored in section 4.2 to illustrate how different factors influence decision making. In particular, I emphasize the role that beliefs play in an agent's specialization decision.

4.1. Choice between symmetric fields

Assume a student can choose to work or study in one of two fields, field X or field Y. Utility in field $j \in \{X, Y\}$ at time t is equal to income:

$$U_j(w_j, h^g_{jt})\ell^g_{jt} = w_j h^g_{jt} \ell^g_{jt} \tag{22}$$

Wages in fields X and Y are equal and will be normalized to 1 ($w_X = w_Y = 1$), as are returns to successfully studying human capital ($\nu_X = \nu_Y = 1$). The student's underlying abilities in the two fields, θ_X and θ_Y, are both equal to 0.5. Therefore, the student has a 50% chance of passing any given field X or field Y course. Finally, I assume the student's beliefs about their own abilities in field X and Y are equal to the uniform prior:[10]

$$P_{X,0} = \mathcal{B}(\alpha_{X,0}, \beta_{X,0}) = \mathcal{B}(1,1), \qquad P_{Y,0} = \mathcal{B}(\alpha_{Y,0}, \beta_{Y,0}) = \mathcal{B}(1,1),$$

For tractability, I modify the assumption (5) as follows:

$$h^g_{j0} = \nu_j \alpha^g_{j0}. \tag{23}$$

Equation (23) is consistent with the human capital accumulation function in equation (2). Further, this assumption ensures that the number of periods an agent studies in school is a deterministic function of initial beliefs, as discussed in section 3.4.[11] The role of beliefs can be more clearly seen in these simulations because all agents who specialize in field j with the same initial beliefs will take the same number of courses in j.[12]

The point at which an agent has "specialized" in a particular field j is not clearly defined in the model. I intuitively describe what specialization looks like in the simulations below, but it

10. Note that if $(\alpha, \beta) = (1, 1)$, the beta distribution $\mathcal{B}(\alpha, \beta)$ equals the uniform distribution over $[0, 1]$. This distribution can be seen graphically in figure 5a.

11. This is specifically addressed while finding lower and upper bounds for stopping times in section 3.4. Assuming $h_{j0} = \nu_j \alpha_{j0}$ implies that all agents specializing in j with initial beliefs $(\alpha_{j0}, \beta_{j0})$ will take exactly $\left\lceil \frac{\delta}{1-\delta} \right\rceil - \alpha_{j0} - \beta_{j0}$ courses in j before entering the labor force.

12. It is worth emphasizing that this assumption ties together initial beliefs and initial human capital in a way that may not be desirable for counterfactual exercises. See the discussion at the start of section 3 for details.

is helpful to provide some concrete definition of specialization. In the figures below, an agent has "specialized" in a field j once they could fail all remaining courses in j, and would switch fields. Mathematically, this is represented by the following condition, letting m_j^* denote the number of courses an agent with beliefs $(\alpha_{j0}, \beta_{j0})$ would take in j before specializing:

$$\frac{1}{1-\delta}\delta^{m_j^* - \tilde{m}_{jt}} w_j h_{jt} > \mathcal{I}_k(\tilde{m}_{kt}, \tilde{s}_{kt}, \alpha_{k0}, \beta_{k0}, h_{k0}), \quad \forall k \neq j.$$

The left-hand side of this inequality is the agent's lifetime payoff of specializing in field j if they fail all of their remaining courses in that field. This would imply they do not accumulate any more human capital in field j, so their lifetime payoff is based on their current levels of human capital, h_{jt}. The right-hand side of this inequality is the expected lifetime payoff associated will all other fields.

4.2. Simulations

Each subplot in figure 4 plots the fraction of simulated agents choosing to study field X or field Y at each time period t. Recall that agents studying field j at time t will either pass and successfully accumulate human capital ($s_{jt}^g = 1$) or they will fail ($s_{jt}^g = 0$), where $s_{jt}^g \sim \text{Bernoulli}(\theta_j)$. The student then updates their beliefs about their own underlying ability, θ_j. Line movements in figure 4 are caused by agents switching fields in response to updated beliefs. Eventually, students will specialize in one field and enter the labor market as a field-X or field-Y specialist. The line for any field j ends once any agent specializing in j stops studying and enters the labor market. Therefore, the length of the lines in figure 4 denote the minimum amount of time an agents spends studying before becoming a field-j specialist. Specialization in figure 4 is generally represented by a flattening of the curve; once a student has made their specialization decision, they no longer switch fields. To make this explicit, I use the definition of specialization from section 4.1, and define specialization as the point when an agent could fail all of their remaining field j courses, and would still choose to specialize in that field. The median point by which simulated agents have made their specialization decision is represented by the dashed vertical line in each plot.

The baseline scenarios in figures 4a and 4b illustrate these dynamics. Figure 4a plots the baseline scenario for 10,000 simulations; figure 4b plots the first 50 of these simulations. Our first takeaway is that the agent's specialization decision in the baseline is effectively a coin flip. This is most clearly seen in figure 4a; at all time periods, approximately 50% of the agents are studying field X and and 50% are studying field Y. This should be expected, as fields X and Y are completely symmetric.

Some of the more subtle decision dynamics can only be seen with fewer observations. Therefore, 4b zooms in on the first 50 of these simulations. Note that the fraction of students studying field X or field Y moves in early periods, but flattens out in later periods. This is because students at the beginning of their education will update their beliefs in response to course outcomes. These

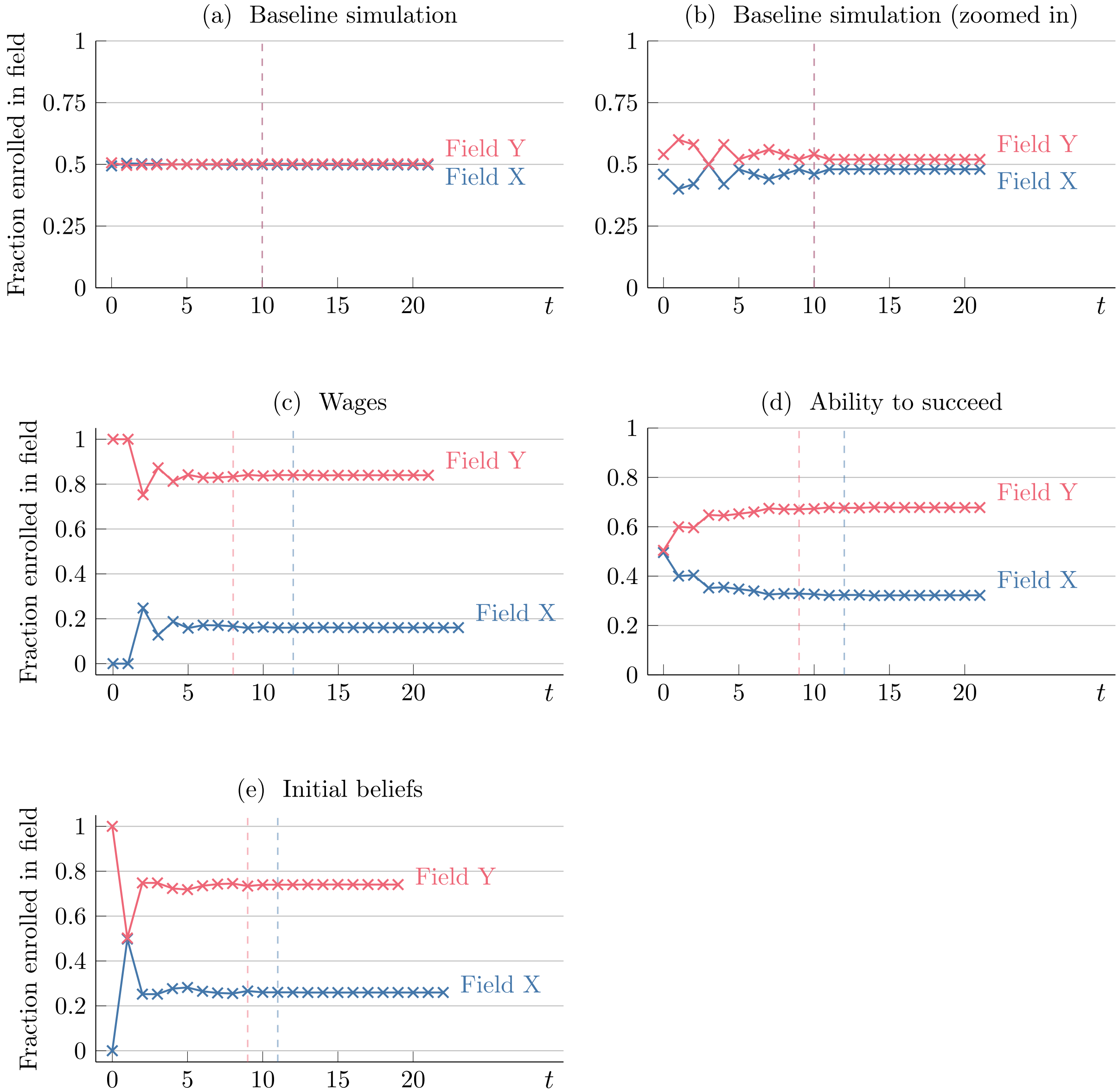

Figure 4. Simulations of simple version of model. Figure (a) presents the baseline for $N = 10,000$ simulations; figure (b) does the same for the first 50 simulations. The remaining figures have $N = 10,000$ simulations. Figure (c) repeats the simulations for $w_X = 1$ and $w_Y = 1.5$. Figure (d) repeats the simulations when $\theta_X = 0.4$ and $\theta_Y = 0.6$. Figure (e) repeats the simulations when $(\alpha_{X0}, \beta_{X0}) = (1, 1)$ and $(\alpha_{Y0}, \beta_{Y0}) = (2, 2)$.

updated beliefs may cause students to switch fields, shifting the composition of simulated agents studying X or Y. In later periods, simulated agents have made their specialization decision and no longer switch fields. This specialization is represented by the flattening of the lines in figure 4b. As in 4a, approximately 50% of agents specialize in field X, and 50% specialize in field Y.

The remainder of figure 4 plots variations of the baseline for $N = 10,000$ simulations. In

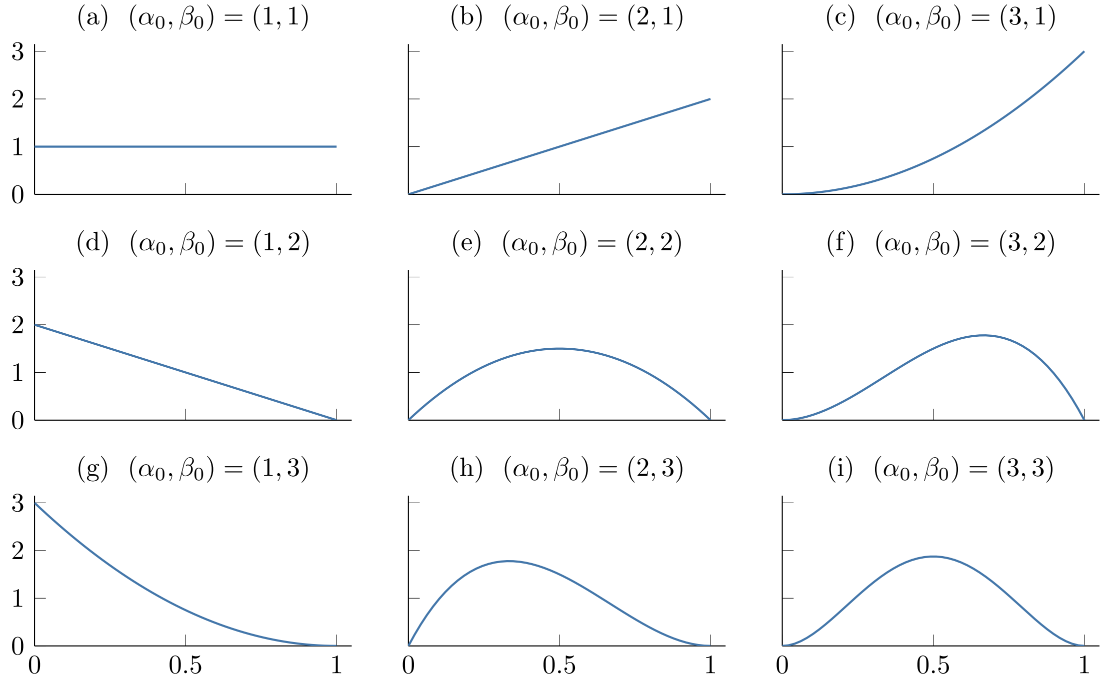

Figure 5. Evolution of the Beta distribution $\mathcal{B}(\alpha_0, \beta_0)$ for different values of (α_0, β_0).

figure 4c, wages in field Y are 50% higher than wages in field X. All other variables are identical to the baseline scenario. Unsurprisingly, higher wages in Y drive specialization into that field. Because the expected lifetime payoff is so much higher, approximately 80% of agents choose to specialize in Y.

The field X line in figure 4c is longer than the field Y line, implying that agents who specialize in X spend more time in school. To understand why this happens, first note that all agents begin their education studying Y because of the higher relative wages. However, after two periods, a large fraction of agents switch from studying Y to studying X. This is due to agents (randomly) failing their first two courses in field Y, and switching into field X. The reason agents switch fields can be seen by the evolution of their belief distributions, shown in figure 5. The student's initial belief distribution is plotted in figure 5a. A student that fails their first course in field Y updates their beliefs about their underlying ability in Y to the distribution plotted in figure 5d; if they fail their second course in Y, they update their beliefs to 5g. As we can see from figure 5g, a student that fails their first two classes in Y will believe they likely have a lower ability in that field. As such, if they choose to specialize in Y, they would not expect to successfully accumulate much human capital over the course of their studies, implying a lower expected lifetime payoff. As a result, these agents switch to studying field X, in spite of the lower wages. This switching leads to more overall time in school; as mentioned above, equation (23) implies that the number of periods an agents spends studying field X or Y before becoming a specialist is a deterministic function of initial beliefs. Agents' initial beliefs about their abilities in X and Y are the same, and as such, agents specializing in either X or Y will study their chosen discipline for the same number of periods. However, because all agents spend their first two periods studying Field Y, those who specialize in Field X will study for a minimum of two more periods.

Figure 4d augments the baseline scenario so agents have a higher ability in field Y. Specifically, probability of success in any given field X course, θ_X, equals 0.4, whereas the probability of success in field Y is given by $\theta_Y = 0.6$. Unsurprisingly, a higher ability in field Y drives specialization into that field.

We now turn to the impact of differential priors on specialization dynamics, plotted in Figure 4e. I assume simulated agents are initially more certain about their abilities in field Y relative to field X. Specifically, I assume a student's initial prior about their ability in Y is given by $P_{Y0} = \mathcal{B}(2, 2)$; this corresponds to the distribution plotted in figure 5e. Their initial prior about their ability in X continues to equal the uniform distribution, $P_{X0} = \mathcal{B}(1, 1)$. Note that agents have the same belief about their probability of success in X and Y in expectation. However, the variances of the initial distributions suggest that agents have more certainty about their underlying ability in field Y than in field X.

The first consequence of this assumption is that agents specializing in field X study for more periods than those specializing in field Y, as shown in figure 4e. As mentioned above, equation (23) implies that the number of periods an agent spends studying j before specializing in that field is a deterministic function of the agent's initial beliefs. Agents have more initial uncertainty about their abilities in X than in Y, and therefore they will study X for more periods before specializing in that field. The second takeaway from 4e is that all agents begin their education studying field Y. Agents know that if they become field Y specialists, they will finish their education earlier, and begin earning an income sooner. The prospect of ending their education earlier drives agents to initially study field Y.

The key takeaway from figure 4e is that increased initial certainty about field Y abilities causes more agents to specialize in field Y. [13] Although agents are equally likely to succeed in fields X and Y, and although the payoffs for specializing in these fields are the same, differential initial beliefs about underlying abilities drives the majority of simulated agents to specialize in field Y. Thus, initial beliefs play a key role in specialization decisions. I now consider how those beliefs are formed, and the consequences of forming those beliefs based on existing group outcomes.

5. Connection to statistical discrimination

This model can be viewed through the lens of statistical discrimination. To see this, I first briefly review the definition of statistical discrimination, and discuss some relevant literature.

13. It's worth emphasizing that this is not driven by risk aversion across fields; assuming linear utility in (22) ensures that agents are risk neutral across fields. Rather, concavity due to discounting ensures that agents are risk averse across time.

5.1. Statistical discrimination literature

Fundamentally, statistical discrimination is a theory whereby inequality results from rational agents forming expectations based on existing group characteristics. To better understand this definition, it helps to briefly review models of taste-based discrimination that preceded it. The canonical model of taste-based discrimination in labor markets, as formulated by Becker (1957), assumes prejudiced employers receive disutility from hiring employees belonging to a particular group.[14] A key implication of the Becker model is that discriminatory firms will be less profitable than firms that do not discriminate. Thus, long-run neoclassical analysis suggests that discrimination will be driven out of the marketplace, an implication that appears incongruous with the persistence of unexplained wage differentials between groups of workers.[15]

The theory of statistical discrimination, as first formulated by Arrow (1972) and Phelps (1972), grew out of this critique. The classical analysis formulated in Aigner and Cain (1977) assumes that a job applicant's group type is one variable an employer uses for inference about their unknown true productivity.[16] If groups have different aggregate characteristics, and these characteristics are controlled for by the employer-as-statistician, then individuals with the same ability from different group may have different expected productivities. Statistical discrimination therefore presents a different view of inequality than the Becker model; unequal outcomes may not be the result of prejudice or distaste for certain group types, but rather the result of rational decision making.

Whether or not an unequal outcome arises from taste-based or statistical discrimination matters from a policy perspective. It is worth emphasizing that whether a discriminatory outcome arises from explicit prejudice or statistical inference is often irrelevant in a legal sense. As emphasized in Lundberg and Startz (1983), statistical discrimination is still discrimination, and is often illegal.

5.2. Connection to model

As discussed above, statistical discrimination is often discussed through the lens of labor market discrimination. In this case, employers do not know the true productivity of their potential employee. Unequal outcomes arise from employers-as-statisticians using group-based information for inference.

Fundamentally, the model outlined in sections 2 and 3 is a model of statistical discrimination. Rational beliefs may be formed based existing aggregate group outcomes. Section 4 demonstrates

14. For a review of the Becker model and details on testable implications, see Charles and Guryan (2008), who find empirical evidence for the existence of taste-based discrimination in the U.S. labor market.

15. A number of authors have incorporated search frictions into the Becker model to explain the long-run persistence of racial wage gaps. See Lang and Lehmann (2012) for a review of the literature. It is worth noting that racial wage gaps cannot persist in a taste-based model with firm entry; discriminatory firms will always be less profitable.

16. Note that much of the canonical discrimination literature primarily focuses on labor market discrimination, whereby employers discriminate offer lower wages to employees of a particular group. These theories of discrimination can easily be extended to alternative contexts.

Table 1. Model parameters

Parameter	Description
J	Number of fields
δ	Discount factor
h_{j0}	Initial field-j human capital
$(\alpha_{j0}, \beta_{j0})$	Initial field-j ability beliefs

how differences in beliefs can lead to unequal results. In this model, individual's do not know their true productivity. Students-as-statisticians use group-based information for inference. The resulting inequality of outcomes can be classified as statistical discrimination.

To convince the reader this inequality of outcomes should indeed be classified as statistical discrimination, consider the definition set forth in Lundberg and Startz (1983):

> Economic discrimination exists when groups with equal average initial endowments of productive ability do not receive equal average compensation in equilibrium.

This definition is employed to explicitly account for the fact that the existence of unequal outcomes may impact pre-labor market human capital investment decisions.

6. Future work: Identification and quantitative exercises

Working identifying model parameters and utilizing the model in counterfactual exercises is ongoing. Here, I briefly describe that work.

6.1. Identification overview

The parameters driving agent behavior in the model are summarized in table 1. The first part of table 1 lists aggregate parameters that impact all agents. The first key parameter for this analysis are the number of fields of study, J. I utilize both IPEDS and ACS data for this analysis, and therefore wanted a classification system that applies across both fields. The discount rate, δ, is currently assumed to equal 0.96.

Parameters that govern individual agent heterogeneity are in the second half of table 1. The primary identification problem for this analysis is uncovering the group-based belief parameters $(\alpha^g_{j0}, \beta^g_{j0})$. Assuming we have data on the agent's choice of field at time t, we can utilize some type of conditional logit method and find the parameters that maximize the following likelihood:

$$\log \mathcal{L} = \log \sum_{i=1}^{n} \sum_{t=1}^{\infty} \sum_{j=1}^{J} m_{ijt} \log P(m_{ijt} = 1 | \overline{m}_{ijt}, \overline{s}_{ijt}, \alpha_{j0}, \beta_{j0}, h_{j0}, \theta_{ji})$$

Thus, identifying the model entails specifying the appropriate choice probability.

One immediate concern associated with this process is time endogeneity; an agent's choice of a field at time t is related to their choice at time $t+1$. A simple way to avoid this endogeneity is to identify model parameters from time $t = 0$ (i.e. the beginning of the agent's education). More sophisticated methods should take advantage of the problem's recursive structure to utilize more of an agent's panel.

Let $G_j(x)$ denote the CDF of the index j. Recall from section 3.4 that the agent's expected lifetime payoff associated with field j is a function of their expected time in school, denoted N:

$$\begin{aligned}G_j(x) =& \mathbb{P}\left(\mathcal{I}_{jt} < x \middle| \tilde{m}_{jt}, \tilde{s}_{jt}, \psi_{j0}\right)\\ =& \mathbb{P}\Bigg(\frac{1}{1-\delta} w_j \left(\mathbb{E}\left[\delta^N \middle| \cdot\right](h_{j0} + \nu_j \tilde{s}_{jt}) + \mathbb{E}\left[\delta^N N \middle| \cdot\right](h_{j0} + \nu_j \tilde{s}_{jt}) \frac{\alpha_{j0} + \tilde{s}_{jt}}{\alpha_{j0} + \beta_{j0} + \tilde{m}_{j0}}\right)\\ & \quad < x \Bigg| \tilde{m}_{jt}, \tilde{s}_{jt}, \psi_{j0}\Bigg)\end{aligned}$$

The agent's probability distribution over possible stopping times is solved in section 3.4. The solution to $G_j(x)$ depends only on the distribution of h_{j0} and the distribution of θ_j. The choice probability can then be written as:

$$\begin{aligned}\mathbb{P}\left(m_{jt} = 1 \middle| \tilde{m}_t, \tilde{s}_t, \psi_0\right) =& \mathbb{P}\left(\mathcal{I}_{jt} > \mathcal{I}_{kj} \middle| \tilde{m}_t, \tilde{s}_t, \psi_0, \forall k \neq j\right)\\ =& \int \prod_{k \neq j} G_k\left(x \middle| \tilde{m}_{kt}, \tilde{s}_{kt}, \psi_{k0}\right) dG_j\left(x \middle| \tilde{m}_{jt}, \tilde{s}_{jt}, \psi_{j0}\right)\end{aligned}$$

My goal is to analytically characterize the CDF $G_j(x)$, and find helpful constraints on the distributions of h_{j0} and θ_j such that I can identify $(\alpha^g_{j0}, \beta^g_{j0})$ from the data.

Data sources

Characteristics of all U.S. postsecondary institutions are reported in the Integrated Postsecondary Education Data System (IPEDS) data. These data are collected annually by the National Center for Educational Statistics and describe the universe of institutions that participate in federal student financial aid programs. The empirical motivation for this analysis relies on the IPEDS Completion Surveys, which describe all degrees and certificates awarded at postsecondary institutions by field of study, gender, and race.[17] University-level graduation rates are estimated using the IPEDS graduation surveys. Details on IPEDS graduation surveys are in the Appendix.

The 2012/17 Beginning Postsecondary Students Longitudinal Study, conducted by the Na-

17. For more details on the IPEDS series, please visit https://nces.ed.gov/ipeds/. IPEDS data are available from 1986 until the present, though I begin empirical analysis in 1990 due to changes in how fields of study are classified. For earlier data, such as those used in Figure 1, I supplement the IPEDS series with data from Snyder (1993). However, it is worth noting that the predecessor to IPEDS series is the Higher Education General Information Survey (HEGIS), available through the International Archive of Education Data at University of Michigan (https://www.icpsr.umich.edu/web/ICPSR/series/00030). As such, I refer the reader to the HEGIS series for a more detailed portrait of postsecondary education statistics than available in Snyder (1993).

tional Center for Education Statistics, can be used to identify initial levels of human capital and group-level beliefs. This longitudinal study collects education and employment data from a nationally representative sample of first-time beginning postsecondary students. Respondents are initially surveyed in 2011-2012, the beginning of their postsecondary studies. Follow-up surveys were conducted three and six years after they began their studies.[18] Key for this analysis are the BPS transcript studies, which contain information on first-year major choice, high school performance, and outcomes. An application for restricted-use BPS data is pending.

6.2. Quantitative exercises

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