Last time, we found that the vector potential is given by $\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{J}\,dv}{r}$.
In theory this gives $\vec{A}$ for any setup, but in some cases you get an infinite result.
For example, this $\vec{A}$ is infinite for an infinite straight wire (but see Problem 6.5 for a way to get around this problem). However, there are many $\vec{A}$'s that get the $\vec{B} = \vec{\nabla}\times\vec{A}$ job done, for example:

ln(r/R)…but ln(r) - ln(R) and the second term goes out when you take derivative

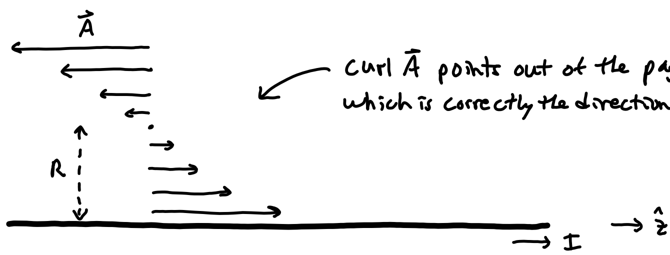
Example: Show that $\vec{A} = -\hat{z}\frac{\mu_0 I}{2\pi}\ln(r)$ is a valid vector potential for a straight wire:
For this $\vec{A}$, only one term in the curl in cylindrical coordinates survives. We obtain:

$$\vec{\nabla}\times\vec{A} = -\frac{\partial A_z}{\partial r}\hat{\theta} = \boxed{\frac{\mu_0 I}{2\pi}\cdot\frac{1}{r}\hat{\theta}} \quad ✓ \quad \leftarrow \text{the correct } \vec{B}$$

We could add on any vector field with zero curl, and we would still have a valid $\vec{A}$.
Actually, the above expression for $\vec{A}$ makes no sense, because it makes no sense to take the log of a dimensionful quantity. So we should write $\ln(r/R)$, where R is any arbitrary length. We then have:

why can't you

$$\vec{A} = -\hat{z}\frac{\mu_0 I}{2\pi}\ln\left(\frac{r}{R}\right)$$



curl $\vec{A}$ points out of the page, which is correctly the direction of $\vec{B}$.

It turns out that $\vec{A}$ isn't that useful for what we'll do in this course. (It's useful in Phys 153 and Phys 232). Instead what we will make frequent use of is the…

Biot-Savart law

We'll derive this law with the help of $\vec{A}$, and then we'll pretty much forget about $\vec{A}$.
We'll restrict the current flow to wires, so $J = \frac{I}{a}$ and $dv = a\,dl \Rightarrow J\,dv = I\,dl$
So we can replace $\vec{J}\,dv$ with $I\,d\vec{l} \Rightarrow$

$$\vec{A} = \frac{\mu_0 I}{4\pi}\int\frac{d\vec{l}}{r} \Rightarrow d\vec{A} = \frac{\mu_0 I}{4\pi}\frac{d\vec{l}}{r}$$

$\Rightarrow$ need $\vec{A}$ as a function of position (that is, need to know $\vec{A}$ at nearby points)

The long way to get $\vec{B}$: Integrate $d\vec{A}$ to get $\vec{A}$, then take the curl to get $\vec{B}$.
Shorter way: Take the curl of $d\vec{A}$ to get $d\vec{B}$, then integrate to get $\vec{B}$. $\leftarrow$ only need to know B at one point.
this result will be the Biot-Savart law

Purcell works out $\vec{\nabla}\times d\vec{A}$ in Cartesian coordinates. Let's do it in a quicker vector-calculus way:

$$d\vec{B} = \vec{\nabla}\times d\vec{A} = \vec{\nabla}\times\left(\frac{\mu_0 I}{4\pi}\frac{d\vec{l}}{r}\right) = \frac{\mu_0 I}{4\pi}\nabla\left(\frac{1}{r}\right)\times d\vec{l}$$

$\leftarrow$ from the identity $\vec{\nabla}\times(f\vec{F}) = f\,\vec{\nabla}\times\vec{F} + (\vec{\nabla}f)\times\vec{F}$
Here, $\vec{F} = d\vec{l}$, which is constant.

But $\vec{\nabla}\left(\frac{1}{r}\right) = -\frac{\hat{r}}{r^2}$ $\leftarrow$ because we know that $\frac{q}{4\pi\epsilon_0 r^2}\hat{r} = -\vec{\nabla}\left(\frac{q}{4\pi\epsilon_0 r}\right)$ $\leftarrow$ Coulomb $\vec{E} = -\nabla\phi$

Therefore, $d\vec{B} = \frac{\mu_0 I}{4\pi}\left(\frac{-\hat{r}}{r^2} \times d\vec{l}\right) \Rightarrow \boxed{d\vec{B} = \frac{\mu_0 I}{4\pi} \frac{d\vec{l} \times \hat{r}}{r^2}} = \frac{\mu_0 I}{4\pi} \frac{d\vec{l} \times \vec{r}}{r^3}$

switched $\hat{r}$, $d\vec{l}$ order

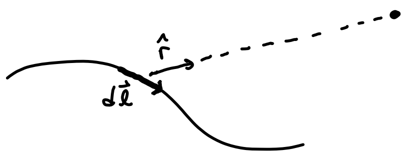
r-hat

r-vector

(Biot – Savart law)

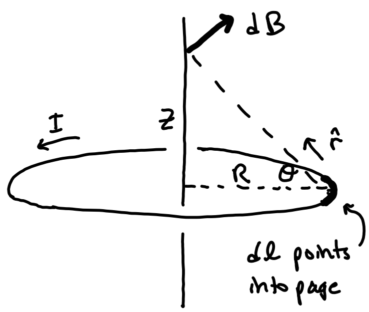
This law is valid for steady currents (but they can be arbitrarily fast).

The $\hat{r}$ vector goes from the element $d\vec{l}$ to the point where you're finding $\vec{B}$.



$d\vec{B} \propto d\vec{l} \times \hat{r}$ points out of the page.
(agrees with right-hand rule for a wire carrying current in the direction of $d\vec{l}$)

Example: Find the magnetic field due to a ring of current, on the axis.



Only the z component of $\vec{B}$ survives, by symmetry.

dl is perp to $\hat{r}$.

from vertical component

$$dB_z = \frac{\mu_0 I}{4\pi} \frac{dl}{r^2} \cos\theta = \frac{\mu_0 I}{4\pi} \frac{dl}{R^2+z^2} \frac{R}{\sqrt{R^2+z^2}}$$

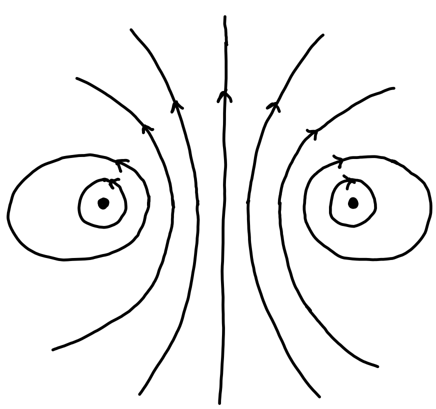
The dl simply integrates to $2\pi R \Rightarrow \boxed{B = \frac{\mu_0 I R^2}{2(R^2+z^2)^{3/2}}}$ on axis

$$= \frac{\mu_0 I R^2}{2 r^3}$$

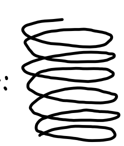
At the center, we have $B = \frac{\mu_0 I}{2R}$.

This field points upward everywhere on the axis, even below the ring.

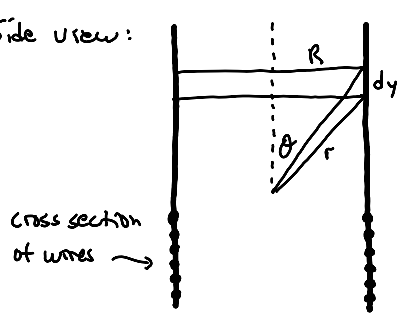
At locations off the axis, a side view of the field looks something like this:



Solenoids: A solenoid is a coil of wire – effectively many rings stacked on top of each other:



Side view:

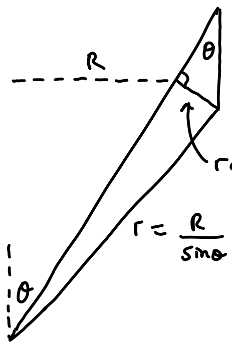


Let there be n windings per unit length (essentially continuous).

The current is $I \Rightarrow$ current per unit length is nI.

Having found above the B from a ring, we must integrate over the rings with height dy.

Expanded view:



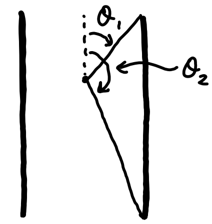
$dy = \frac{r\,d\theta}{\sin\theta} = \frac{R\,d\theta}{\sin^2\theta}$ $\Rightarrow$ current in ring of height dy is $nI\,dy = \frac{nIR\,d\theta}{\sin^2\theta}$

$$\Rightarrow dB = \frac{\mu_0 (dI) R^2}{2r^3} = \frac{\mu_0 R^2}{2r^3}\,\frac{nIR\,d\theta}{\sin^2\theta}$$

Using $R/r = \sin\theta$, this becomes $dB = \frac{\mu_0 n I}{2}\sin\theta\,d\theta$

$$\Rightarrow B = \int_{\theta_1}^{\theta_2} \frac{\mu_0 n I}{2}\sin\theta\,d\theta = \boxed{\frac{\mu_0 n I}{2}(\cos\theta_1 - \cos\theta_2)}$$

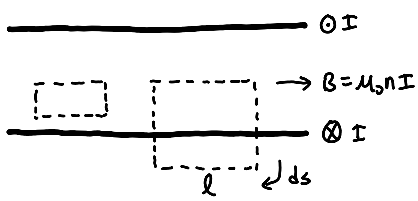
(direction from RH rule)



Infinite solenoid

If we have an infinite solenoid, so that $\theta_1 = 0$ and $\theta_2 = \pi$, then $\boxed{B = \mu_0 n I}$

What about the field outside? Use Ampere's law:



CHECK SOLUTION TO IN CLASS QUIZ.

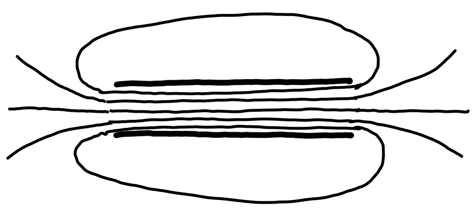
In the left Amperian loop, there is no current enclosed, so $\oint \vec{B}\cdot d\vec{s} = \mu_0 I_{encl}$ tells us that $\vec{B}$ is the same on the two long sides of the rectangle $\Rightarrow$ $\vec{B} = \mu_0 n I$ everywhere inside the solenoid.

In the right Amperian loop, we have $\oint \vec{B}\cdot d\vec{s} = \mu_0 I_{encl} \Rightarrow B_{in}\ell - B_{out}\ell = \mu_0(nI\ell)$ ← current per length

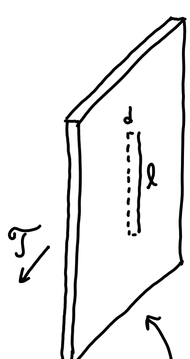
$$\Rightarrow B_{in} - B_{out} = \mu_0 n I \Rightarrow \boxed{B_{out} = 0}$$

So the field is zero everywhere outside the solenoid (or very small for a long but finite solenoid).

This makes sense, because the field lines outside are sparse!



Change in B at a current sheet



thin sheet, thickness d

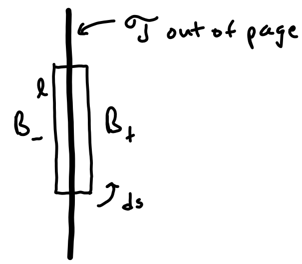
Recall that for the volume current density, J, we have $J \cdot \text{area} = \text{current}$. (units $= \frac{C}{s\,m^2}$)

Define the surface current density, $\mathcal{J}$, as follows:

the current through a height ℓ is $J(\ell d) = (Jd)\ell$

$\Rightarrow$ surface current density is $\boxed{\mathcal{J} \equiv Jd}$. So $\mathcal{J}\cdot \text{length} = \text{current}$ (units $= \frac{C}{s\,m}$)

$\mathcal{J}$ determines the change in B across a sheet, as follows:

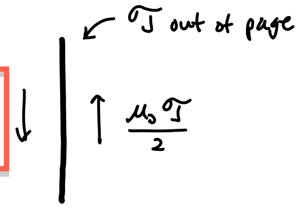


Ampere's law gives:

$$\int \vec{B}\cdot d\vec{s} = \mu_0 I \Rightarrow B_+\ell - B_-\ell = \mu_0(\sigma J\ell)$$
$$\Rightarrow \boxed{B_+ - B_- = \mu_0\sigma J}$$

So the change across the sheet is $\mu_0\sigma J$

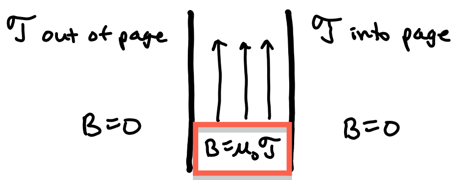
If we have one sheet and nothing else, then the field is $\boxed{\frac{\mu_0\sigma J}{2}}$



(You could also find this field by slicing the sheet up into thin wires and then integrating over these wires.)

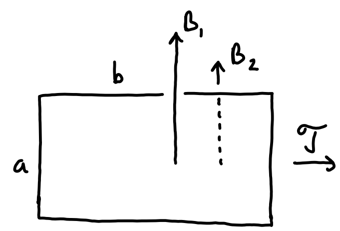
In general, we can superpose on this field any other field.

In the case of two opposite sheets of current, we have:



All statements here have analogous statements in the case of the E from a thin sheet of charge.

Force on a sheet



What is the force on this sheet?

The current is $I = \sigma J a$

$\Rightarrow$ force on length b is $F = IbB = (\sigma J a)\, b\, \frac{B_1+B_2}{2}$ ← avg field

But $B_2 - B_1 = \mu_0\sigma J \Rightarrow \sigma J = (B_2 - B_1)/\mu_0$

So the force per area is $\frac{F}{ab} = \left(\frac{B_2-B_1}{\mu_0}\right)\left(\frac{B_2+B_1}{2}\right) = \frac{1}{2\mu_0}(B_2^2 - B_1^2)$ (use right-hand rule to get the direction)

Or we can just write this as $\boxed{\sigma J\,\frac{B_1+B_2}{2}}$, just like the electric formula $\sigma\,\frac{E_1+E_2}{2}$.

The force on each of the above sheets with opposite currents is outward.

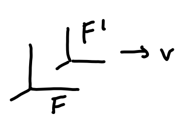
↖ agrees with two-wire demo.

Transformations of $\vec{E}$, $\vec{B}$

If frame F′ moves with velocity $\vec{v}$ w.r.t. frame F, then the $\vec{E}$ and $\vec{B}$ components in the two frames are related by:

$$\boxed{\begin{array}{ll} \vec{E}'_{\parallel} = \vec{E}_{\parallel} & \vec{E}'_{\perp} = \gamma(\vec{E}_{\perp} + \vec{v}\times\vec{B}_{\perp}) \\ \vec{B}'_{\parallel} = \vec{B}_{\parallel} & \vec{B}'_{\perp} = \gamma(\vec{B}_{\perp} - \frac{\vec{v}}{c^2}\times\vec{E}_{\perp}) \end{array}}$$

← $\vec{v}$ is the velocity of F′ w.r.t. F.



(In Gaussian units, both $\vec{v}$ and $\vec{v}/c^2$ are replaced by $\vec{\beta}$.

The proofs of these are in the textbook. Let's just look at some special cases to help convince you that they're true. ⤳

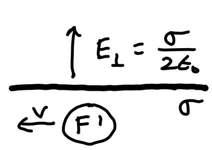
Case 1

true in any case.

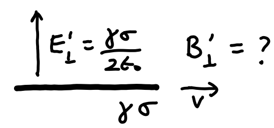
If the sources are at rest in F, then $\vec{B}_{\parallel} = \vec{B}_{\perp} = 0 \Rightarrow \vec{E}'_{\parallel} = \vec{E}_{\parallel}$ and $\vec{E}'_{\perp} = \gamma E_{\perp}$ ✓

Additionally, we have $\vec{B}'_{\perp} = -\gamma \frac{\vec{v}}{c^2} \times \vec{E}_{\perp}$, and this is indeed correct because consider the case of a single sheet:

Frame F:



Frame F':



What is $B'_{\perp}$? First note that a general expression for $\mathcal{J}$ is $\mathcal{J} = \sigma v$, because

$$I = \frac{dq}{dt} = \frac{\sigma(v\,dt)\ell}{dt} = (\sigma v)\ell \equiv \mathcal{J}\ell$$

(arbitrary length in transverse direction)

$$\text{So}\quad B'_{\perp} = \frac{\mu_0 \mathcal{J}}{2} = \frac{\mu_0(\gamma\sigma)v}{2} = \frac{\gamma\sigma v}{2\epsilon_0 c^2} = \frac{\gamma v}{c^2}\frac{\sigma}{2\epsilon_0} = \frac{\gamma v}{c^2} E_{\perp} \quad ✓$$

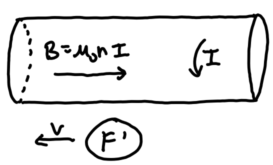
And $B'_{\perp}$ points out of the page, which is correctly the direction of $-\vec{v} \times \vec{E}_{\perp}$ ✓

of F' w.r.t. F

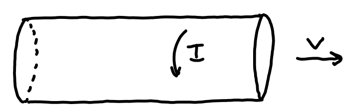
Case 2

To see why $\vec{B}'_{\parallel} = \vec{B}_{\parallel}$, consider a solenoid:

Frame F:



Frame F'



In F', n increases by γ (length contraction)

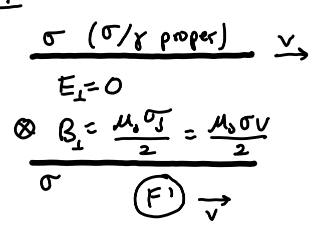
but I decreases by γ (time dilation)

$\Rightarrow B' = \mu_0 n' I'$ equals $B = \mu_0 n I$ ✓

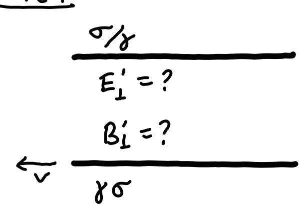
Case 3:

If $\vec{E}_{\perp} = 0$, then $\vec{B}'_{\perp} = \gamma \vec{B}_{\perp}$ and $E'_{\perp} = \gamma \vec{v} \times \vec{B}_{\perp}$. Consider this setup:

Frame F



Frame F'



$$B'_{\perp} = \frac{\mu_0 \mathcal{J}'}{2} = \frac{\mu_0(\gamma\sigma)v}{2} = \gamma B_{\perp} \quad ✓$$

$$E'_{\perp} = \frac{1}{2\epsilon_0}\left(\gamma\sigma - \frac{\sigma}{\gamma}\right) = \frac{1}{2\epsilon_0}\gamma\sigma\left(1 - \frac{1}{\gamma^2}\right) = \frac{1}{2\epsilon_0}\gamma\sigma \cdot \beta^2 = \frac{\gamma\sigma v^2}{2\epsilon_0 c^2} = \frac{\mu_0 \gamma\sigma v^2}{2}$$

$$= \gamma v\left(\frac{\mu_0 \sigma v}{2}\right) = \gamma v B_{\perp} \quad \text{(upward)}$$

the direction of $\vec{v} \times \vec{B}$ ✓