Quasi Normal Modes of Black Holes

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Jan-Apr 2023

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1 Introduction

Every system around us has some natural modes of oscillation, and the frequencies of these modes play a crucial role in understanding the system. Black holes aren't any different as they have a set of characteristic modes known as the Quasi Normal Modes(QNMs) of the black hole[1]. These are generally oscillations with complex-valued frequencies, indicating damping, and thus the name quasi-normal. Unlike normal modes, these quasi-normal modes cannot be superposed to obtain the solution at all points in space-time and, therefore, do not form a complete set. A slight perturbation of the blackhole leads to waves(signal) on an otherwise stationary metric. Although this signal highly depends on initial conditions at early times, the quasi-normal ringing dominates the waveform at intermediate times. The frequencies of these quasi-normal ringing at intermediate times are independent of the initial perturbation and are uniquely characterized by the final black hole's Mass, Charge, and Angular Momentum. At late times the signal shows a power-law tail behaviour.

Quasi Normal Modes of a black hole are of interest for a wide range of things. They carry information about blackhole parameters and can be used to test the no-hair theorem[2]. The energy spectrum and the polarization of QNMs can be used to distinguish between various theories of gravity[3, 4]. Asymptotic forms of the QNM frequencies obtained from a purely classical theory are related to hawking temperature, a quantum phenomenon[5]. Studying QNMs is, therefore, of great importance in understanding the nature of black holes and the fundamental laws of physics that govern them.

Quasi-normal modes are defined as the solutions of the black hole perturbation equation, with boundary conditions being a purely ingoing wave at the event horizon and a purely outgoing wave at infinity [6]. These boundary conditions ensure that the solutions aren't driven by external sources and are mainly characterized by the black hole. The abovementioned equation is generally a linearized perturbation equation obtained by considering first-order perturbations to the metric. Finding quasi-normal frequencies is nothing but an eigenvalue problem for a given perturbation equation and already specified boundary conditions. This is similar to many BVPs, like a string tied on both sides, except that the differential equation is much more complex. There are various approaches to solving this BVP[7], and each one of them has its own merits and demerits.

This report starts by deriving the perturbation equation of the Schwarzschild black hole known as the Regge Wheeler equation in section 2. After a variable change, this equation can be transformed into a time-independent Schrodinger equation(TISE). However, this equation isn't exactly solvable in closed form, and approximations must be made. The potential can be approximated to an inverted parabola which is essentially a WKB method [8] or to Pöschl Teller potential [9, 10] an exactly solvable potential. These approximations have their limitations, as they work well only for the first few modes. Leaver's continued fraction method helps calculate higher overtones(up to 100 overtones) with a high level of accuracy. It involves obtaining a three-term recurrence relation between the coefficients of a series solution, expressing it as a continued fraction equation and solving it to get the QNM frequencies [11]. Nollert's improvement to this method enables it to calculate up to 10,000 overtones [12]. However, calculating such high overtones is computationally intensive due to the large number of terms to be considered in the continued fraction.

In section 3, the perturbation equation for a general Petrov type-D black hole is derived using Newman Penrose formalism. It is then used to obtain the Kerr perturbation equation, known as the Teukolsky equation [13]. The derivation of the Teukolsky equation is much more involved than the Regge-Wheeler equation in the Schwarzschild case. Leaver's continued fraction method for Kerr black holes is discussed, and fundamental QNM frequencies of Kerr black holes are computed using it [11].

2 Schwarzschild Perturbations and QNMs

The Schwarzschild metric is an exact solution to Einstein's field equations, describing a stationary, spherically symmetric, asymptotically flat metric. It is given by,

$$ds^2 = \left(1 - \frac{2M}{r}\right)(dt)^2 - \frac{(dr)^2}{\left(1 - \frac{2M}{r}\right)} - r^2[(d\theta)^2 + \sin^2\theta(d\phi)^2]$$

It denotes the spacetime around a non-rotating uncharged black hole with a mass M.

2.1 Perturbation Equation

Although the Schwarzschild metric has nice symmetries, the perturbed state will, in general, not be stationary and spherically symmetric. Therefore we consider a general non-stationary, axisymmetric metric for the perturbed case which is given by,

$$ds^2 = e^{2\nu}dt^2 - e^{2\psi}(d\phi - q_2dr - q_3d\theta - \Omega dt)^2 - e^{2\mu_2}(dr)^2 - e^{2\mu_3}(d\theta)^2$$

where μ_2, μ_3, ν, ψ, Ω, q_2, q_3 are functions (t, r, θ). Schwarzschild is a special case of this metric where,

$$e^{2\nu} = e^{-2\mu_2} = 1 - \frac{2M}{r}, \quad e^{\mu_3} = r, \quad e^{\psi} = r\sin\theta$$

$$\Omega = q_2 = q_3 = 0$$

Notice that there's a fundamental difference between the two sets of parameters above. When a Schwarzschild metric is perturbed, the perturbations can be classified into two types, namely axial and polar. The coefficients Ω, q_2, q_3 vanish in the Schwarzschild limit as it isn't rotating. If these coefficients are given small values in the perturbed case, it denotes that perturbation provides a small angular momentum to the black hole. These perturbations are called axial perturbations. The perturbations in which the coefficients μ_2, μ_3, ν, ψ are incremented by small values are called polar perturbations. These two perturbations can be completely decoupled and treated independently in the case of Schwarzschild. The perturbation equation is derived by considering metric perturbations, i.e. $g'_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}$ where, $g_{\mu\nu}$ is the background Schwarzschild background and $h_{\mu\nu}$ gives the small perturbation. Let us obtain the perturbation equation for the axial case. As the stress-energy tensor $T_{\mu\nu}$ doesn't include gravitational energy in it, for gravitational perturbations in vacuum $T_{\mu\nu} = \delta T_{\mu\nu} = 0$. This and Einstein's field equations imply that $R_{\mu\nu} = \delta R_{\mu\nu} = 0$.

Considering the equations given by $\delta R_{\phi r}$, $\delta R_{\phi\theta}$,

$$(e^{3\psi+\nu-\mu_2-\mu_3}Q_{23})_{,3} = -e^{3\psi-\nu+\mu_3-\mu_2}Q_{02,0}$$

$$(e^{3\psi+\nu-\mu_2-\mu_3}Q_{23})_{,2} = e^{3\psi-\nu+\mu_2-\mu_3}Q_{03,0}$$

where, $Q_{32} = q_{3,2} - q_{2,3}$, $Q_{0i} = \omega_{,i} - q_{i,0}$ for $i = 1, 2$. Let $Q(t, r, \theta) = \Delta Q_{23} \sin^3\theta$, where $\Delta = r^2 - 2Mr$,

$$\frac{1}{r^4 \sin^3\theta} \frac{\partial Q}{\partial \theta} = -(\Omega_{,2} - q_{2,0})_{,0}$$

$$\frac{\Delta}{r^4 \sin^3\theta} \frac{\partial Q}{\partial r} = +(\Omega_{,3} - q_{3,0})_{,0}$$

Assuming the time dependence to be $e^{-i\omega t}$ and eliminating Ω from above equations, one would obtain,

$$r^4 \frac{\partial}{\partial r}\left(\frac{\Delta}{r^4}\frac{\partial Q}{\partial r}\right) + \sin^3\theta \frac{\partial}{\partial \theta}\left(\frac{1}{\sin^3\theta}\frac{\partial Q}{\partial \theta}\right) + \omega^2 \frac{r^4}{\Delta} Q = 0 \tag{1}$$

Using the separation of variables, $Q(r, \theta) = R(r)\Theta(\theta)$, one finds that the solutions of the angular part $\Theta(\theta)$ are Gegenbauer functions. The radial differential equation can be further simplified into TISE using,

$$r_* = r + 2M \log\left(\frac{r}{2M} - 1\right), R(r) = r\Psi(r)$$

r_* is called the tortoise coordinate and is crucial in further analysis.

$$\left(\frac{d^2}{dr_*^2} + \omega^2\right)\Psi = V\Psi, \quad \text{where,} \quad V = \frac{\Delta}{r^5}[l(l+1)r - 6M]$$

The above equation, called Regge-Wheeler Equation, governs the axial perturbations of the Schwarzschild metric. One can similarly derive the Zerilli equation for the polar perturbations.

$$\left(\frac{d^2}{dr_*^2} + \omega^2\right) Z^{(+)} = V^{(+)}Z^{(+)}, \quad \text{where,} \quad V^{(+)} = \frac{2\Delta}{r^5(nr + 3M)^2}\left[n^2(n+1)r^3 + 3Mn^2r^2 + 9M^2nr + 9M^3\right]$$

It is worth noting that both equations are incredibly similar to the Time Independent Schrodinger Equation(TISE) in Quantum Mechanics. However, the equations aren't exactly solvable, and some approximation must be made to calculate the QNM frequencies. It can be shown that both Regge-Wheeler and Zerilli equations are isospectral[6]. Therefore, for convenience, we use the Regge-Wheeler equation throughout this report.

2.2 WKB Method

The potential V in the Regge-Wheeler equation approaches 0, as $r_* \to \pm\infty$. Therefore, a solution of asymptotic form,

$$\Psi = e^{\left(i\int_0^{r_*} \phi dr_*\right)} \quad \text{with the condition} \quad \phi \to \pm\omega, \quad \text{as} \quad r_* \to \pm\infty$$

could be expected. The asymptotic signs of ϕ are chosen to abide by the boundary conditions at infinity and the event horizon. In Quantum Mechanics, WKB approximation is typically used to find the reflected and transmitted amplitudes for a wave incident on the barrier. However, in the case of QNMs, there shouldn't be any incident wave from infinity. Therefore, the transmitted amplitude must be comparable to the reflected amplitude. Using first-order WKB, one always gets an exponential decay ratio between transmitted and reflected given by $e^{-\gamma}$. However, this isn't the case when the two classical turning points r_1 and r_2 are close. This happens when the wave's energy ω^2 is close to the peak of the potential barrier$V_{\max}$. In this case, one should consider second-order WKB and approximate the potential to be quadratic[8, 14].

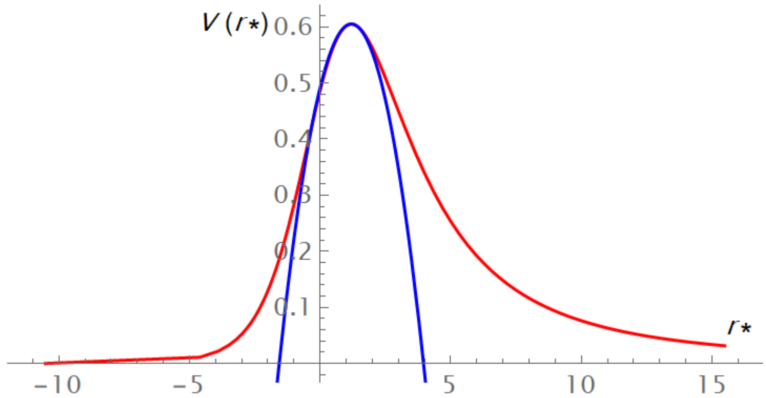

Figure 1: The red curve is Regge-Wheeler Potential, and the blue is the quadratic approximation.

The Regge-Wheeler equation can be written as,

$$\frac{d^2\Psi}{dr_*^2} + p(r_*)\Psi = 0$$

where $p(r_*) = \omega^2 - V(r_*)$. Makig a quadratic approximation to the potential, $p(r_*) = p(r_0) + \frac{1}{2}p_0"(r_* - r_0)^2$ and making the following variable changes,

$$k = \frac{1}{2}p_0", t = (4k)^{\frac{1}{4}} e^{\frac{i\pi}{4}} (r_* - r_0), \rho + \frac{1}{2} = -i\frac{p(r_0)}{(2p_0")^{\frac{1}{2}}}$$

leads to the equation,

$$\frac{d^2\Psi}{dt^2} + \left(\rho + \frac{1}{2} - \frac{t^2}{4}\right)\Psi = 0$$

The solutions of this equation are parabolic cylinder functions and will asymptotically be in the following form, For distances far right from the potential barrier, $r_* >> r_2$,

$$\Psi \approx B\left(4ke^{3i\pi}\right)^{-(\rho+1)/4}(r_*-r_0)^{-(\rho+1)}e^{i\sqrt{k}(r_*-r_0)^2/2}+\left(A+\frac{B\sqrt{2\pi}e^{-i\rho\pi/2}}{\Gamma(\rho+1)}\right)\left(4ke^{i\pi}\right)^{\rho/4}(r_*-r_0)^{\rho}e^{-i\sqrt{k}(r_*-r_0)^2/2}$$

For distances far left from the potential barrier, $r_* << r_1$,

$$\Psi \approx A\left(\frac{4k}{e^{3i\pi}}\right)^{\rho/4}(r_* - r_0)^{\rho}e^{-i\sqrt{k}(r_*-r_0)^2/2} + \left(B - \frac{A\sqrt{2\pi}e^{i\rho\pi/2}}{\Gamma(-\rho)}\right)\left(\frac{4k}{e^{i\pi}}\right)^{-(\rho+1)/4}(r_* - r_0)^{-(\rho+1)}e^{i\sqrt{k}(r_*-r_0)^2/2}$$

The term $e^{i\sqrt{k}(r_*-r_0)^2/2}$ corresponds to waves coming/originating from infinity and the event horizon. The condition that waves are purely ingoing at the event horizon and outgoing at infinity leads to a condition that ρ is a negative integer.

$$\frac{p(r_0)}{(2p_0")^{\frac{1}{2}}} = i\left(-n + \frac{1}{2}\right) \quad n \in \mathbb{Z}^+/\{0\}$$

The minus sign in the R.H.S. term, $-n + 0.5$, differs from the standard literature because of the assumed time dependence $e^{-i\omega t}$. However, the QNM frequencies are the same in both cases. A different way of approaching this method is by finding poles in the scattering matrix.

l	n	WKB	Continued Fraction
2	0	$0.7976 - 0.1765i$	$0.74734 - 0.17792i$
2	1	$0.9067 - 0.4659i$	$0.69342 - 0.54783i$
2	2	$1.0339 - 0.6811i$	$0.60211 - 0.95655i$
3	0	$1.2331 - 0.1846i$	$1.19889 - 0.18541i$
3	1	$1.3239 - 0.5159i$	$1.16529 - 0.56259i$
3	2	$1.4501 - 0.7850i$	$1.10337 - 0.95819i$
4	0	$1.6446 - 0.1876i$	$1.61836 - 0.18833i$
4	1	$1.7204 - 0.5388i$	$1.59326 - 0.56867i$
4	2	$1.8375 - 0.8407i$	$1.54542 - 0.95982i$

Table 1: QNM frequencies obtained from WKB are compared with the QNMs obtained from Leaver's method.

The real part of the QNM frequencies given by the WKB method is decreasing with increasing n. In contrast, the real part of the actual QNM frequencies is increasing. The imaginary part of both frequencies is increasing with increasing n. This method works well for the fundamental $n = 0$ and gives a reasonably good approximate for $n < l$. WKB gets extremely accurate for ($l >> n$), also known as the eikonal limit. Higher order WKB is done up to 3^{rd} and 6^{th} order [15, 16]. It takes enormous effort to calculate the coefficients of higher orders, and still, it fails for higher 'n', especially for $n \gtrsim l$.

2.3 Pöschl Teller Potential

The Regge-Wheeler Potential is not an easy one to solve exactly. However, there are some exactly solvable potentials that behave similarly to Regge-Wheeler's potential in the domain of interest and can provide us with the approximate frequencies of QNMs. One example is the Pöschl Teller Potential. It is given by,

$$V(r_*) = \frac{V_o}{\cosh^2 \alpha(r_* - \bar{r}_*)}$$

$\bar{r}_*$ is the coordinate value r_* of the peak of Regge-wheeler potential, and V_o is the peak value attained. The value of α is chosen such that the second derivatives of both potentials match at the peak.

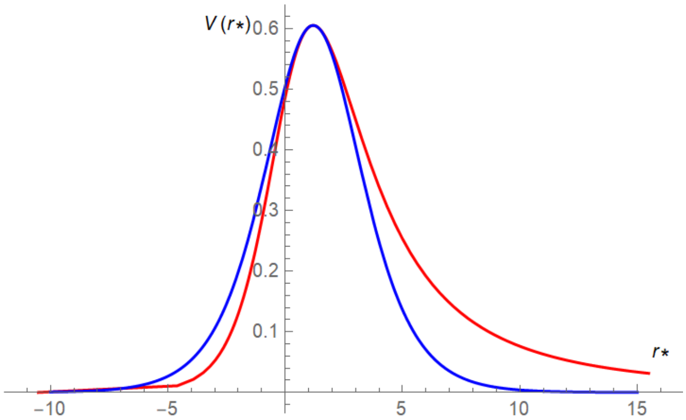

Figure 2: The red curve is Regge-Wheeler Potential, and the blue one is Poschl-Teller.

Poschl-Teller potential approximated the Regge-wheeler potential much better when compared to the parabola in the WKB method. However, there's a significant difference between the actual and approximated potential in their asymptotic behaviour along the positive x-axis. Poschl teller potential falls off exponentially, whereas the Regge wheeler potential falls as r_*^{-2}. The new differential equation to be solved would be,

$$\frac{\partial^2 \Psi}{\partial r_*^2} + \left[\omega^2 - \frac{V_o}{\cosh^2 \alpha(r_* - \bar{r}_*)}\right] = 0 \tag{2}$$

The solutions of the above equation 2 that satisfy the boundary conditions are the QNMs. Performing a variable change $\xi = \left[1 + e^{-2\alpha(r_* - \bar{r}_*)}\right]^{-1}$ leads us to,

$$\xi^2(1-\xi)^2 \frac{\partial^2 \Psi}{\partial \xi^2} - (\xi)(1-\xi)(2\xi - 1)\frac{\partial \Psi}{\partial \xi} + \left[\frac{\omega^2}{4\alpha^2} - \frac{V_o}{\alpha^2}\xi(1-\xi)\right]\Psi = 0$$

The above equation has two regular singular points at $\xi = 0, 1$ and setting $\Psi = (\xi(1-\xi))^{-i\omega/2\alpha} y$ leads to a hypergeometric equation in y.

$$\xi(1-\xi)\partial_\xi^2 + [c - (a+b+1)]\partial_\xi - aby = 0$$

where,

$$a = \frac{\alpha + \sqrt{\alpha^2 - 4V_o} - 2i\omega}{2\alpha}, \quad b = \frac{\alpha - \sqrt{\alpha^2 - 4V_o} - 2i\omega}{2\alpha}, \quad c = 1 - \frac{i\omega}{\alpha}$$

The solutions of hypergeometric equations are well-known,

$$\Psi(\xi) = A\xi^{i\omega/2\alpha}(1-\xi)^{-i\omega/2\alpha}F(a-c+1, b-c+1, 2-c, \xi) + B(\xi(1-\xi))^{-i\omega/2\alpha}F(a, b, c, \xi)$$

The behaviour at the horizon($r_* \to -\infty$) can be analyzing the solution as $\xi \to 0+$. The first term in the solution corresponds to an outgoing wave from the horizon, and the second term is an incoming wave. Applying the boundary conditions will result in $A = 0$. The solutions can also be expressed using a Hyper-geometric function in $1 - \xi$.

$$\Psi(\xi) = A'\xi^{-i\omega/2\alpha}(1-\xi)^{i\omega/2\alpha}F(c-a, c-b, c-a-b+1, 1-\xi) + B'(\xi(1-\xi))^{-i\omega 2\alpha}F(a, b, -c+a+b+1, 1-\xi)$$

The behaviour at infinity ($r_* \to +\infty$) can be inferred by analyzing the solution as $\xi \to 1-$. The first term in the solution corresponds to an incoming wave, and the second term is an outgoing wave at infinity. Applying the boundary conditions will result in $A' = 0$. Writing down both the solutions,

$$\Psi(\xi) = B(\xi(1-\xi))^{-i\omega/2\alpha}F(a, b, c, \xi) \qquad \forall 0 \leq \xi < 1$$

$$\Psi(\xi) = B'(\xi(1-\xi))^{-i\omega/2\alpha}F(a, b, -c+a+b+1, 1-\xi) \qquad \forall 0 < \xi \leq 1$$

The above two solutions should match, and we have a standard identity,

$$F(a, b, c, z) = (1-z)^{c-a-b}\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}F(c-a, c-b, c-a-b+1, 1-z)+$$

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}F(a, b, -c+a+b+1, 1-z)$$

The two solutions match iff the first term in the above identity is zero for all $0 < \xi < 1$, which is only possible if the coefficient is zero. Either $1/\Gamma(a) = 0$ or $1/\Gamma(b) = 0$, which are satisfied when,

$$\omega = \pm\sqrt{V_o - \frac{\alpha^2}{4}} - i\alpha(n + \frac{1}{2}), n \in \mathbb{Z}^+$$

l	n	Pöschl Teller	Continued Fraction
2	0	$0.756547 - 0.181062i$	$0.74734 - 0.17792i$
2	1	$0.756547 - 0.543186i$	$0.69342 - 0.54783i$
2	2	$0.756547 - 0.905311i$	$0.60211 - 0.95655i$
3	0	$1.20484 - 0.186741i$	$1.19889 - 0.18541i$
3	1	$1.20484 - 0.560224i$	$1.16529 - 0.56259i$
3	2	$1.20484 - 0.933707i$	$1.10337 - 0.95819i$
4	0	$1.62286 - 0.189096i$	$1.61836 - 0.18833i$
4	1	$1.62286 - 0.567289i$	$1.59326 - 0.56867i$
4	2	$1.62286 - 0.945482i$	$1.54542 - 0.95982i$

Table 2: QNM frequencies obtained from Poschl Teller, compared with the ones obtained from the Leaver's method.

The real part of the QNM frequencies given by Pöschl Teller potential is constant and is accurate for the fundamental mode. The imaginary part is reasonably accurate for $n \gtrsim l$. Similar to WKB, this method works great in the eikonal limit $l >> n$. However, the QNM frequencies given by this method are more accurate than those given by WKB, and the reason is that Pöschl Teller potential is a better approximation than a parabola.

2.4 Continued Fraction Method

2.4.1 Leaver's Method

This method [11] is originally used to obtain the spectrum of a hydrogen molecule ion. It involves obtaining a three-term recurrence relation among the coefficients of a series expansion and expressing it as a continued fraction. The eigenvalues can be obtained by finding the roots of the continued fraction method. This is thus far the most accurate method to obtain QNM frequencies.

This method starts with the radial equation obtained from 1 without going to the tortoise coordinate.

$$\frac{1}{R(r)}r^4\frac{\partial}{\partial r}\left(\frac{\Delta}{r^4}\frac{\partial R(r)}{\partial r}\right) + \omega^2\frac{r^4}{\Delta} = -\frac{\sin^3\theta}{\Theta(\theta)}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin^3\theta}\frac{\partial\Theta(\theta)}{\partial\theta}\right) = (l+2)(l-1)$$

where, $\Delta = r^2 - 2Mr$. The radial equation after substituting $R(r) = r\Psi(r)$ and taking $2M = 1$ would be,

$$r(r-1)\frac{\partial^2\Psi}{\partial r^2} + \frac{\partial\Psi}{\partial r} + \left[\frac{\omega^2 r^3}{r-1} - l(l-1) + \frac{3}{r}\right]\Psi = 0 \tag{3}$$

This differential equation has regular singular points at $r = 0, r = 1$, and a confluent irregular singular point at $r = \infty$. The singular point at $r = 1$ has indices $\pm i\omega$. The asymptotic solution at infinity will be,

$$\Psi \sim e^{\pm i\omega(r+\ln r)}$$

The boundary conditions at the event horizon and infinity dictate the allowed signs of the asymptotic solutions. At $r = 1$ and $r = \infty$ the solutions will be,

$$\Psi \sim (r-1)^{-i\omega}, \quad \Psi \sim e^{+i\omega(r+ln(r))}$$

respectively. For convenience, let's use $\rho = -i\omega$ in this entire section.

Making a substitution, $\Psi = (r-1)^{\rho} r^{-2\rho} e^{\rho(r-1)} f(r)$ results in,

$$r^2(r-1)\frac{\partial^2 f}{\partial r^2} - r(2\rho r^2 - 4\rho - 1)\frac{\partial f}{\partial r} + f((4\rho^2 + l(l+1))r + 4\rho^2 + 4\rho - 3) = 0$$

In order to express $f(r)$ as a series, one should also be wary of the convergence. If the function f(r) is expressed as a series about the point $r = 1$, the domain of convergence is limited to a unit circle around $r = 1$ due to the presence of the singular point $r = 0$. This issue can be overcome by a variable change, $u = (r-1)/r$. Now, $r = 0, r = 1, r = \infty$ are changed to $u = -\infty, u = 0$ and $u = 1$, respectively, and one can happily assume convergence of the series between $u = 0, 1$ on the real axis.

$$u(1-u)^2\frac{\partial^2 f}{\partial u^2} + (u^2(4\rho+3) - 4u(2\rho+1) + 2\rho + 1)\frac{\partial f}{\partial u} + (u(4\rho^2 + 4\rho - 3) - 8\rho^2 - 4\rho - l(l+1) - 3)f = 0$$

If f is expressed as a power series in u and is plugged into the above differential equation, one would obtain a three-term recurrence relation,

$$\alpha_n a_{n+1} + \beta_n a_n \gamma_n a_{n-1} = 0, \qquad \forall n > 0 \tag{4}$$

where the coefficients are given by,

$$\begin{aligned}
\alpha_n &= n^2 + (2\rho + 2)n + 2\rho + 1, \\
\beta_n &= -(2n^2 + (8\rho + 2)n + 8\rho^2 + 4\rho + l(l+1) - 3), \\
\gamma_n &= n^2 + 4\rho n + 4\rho^2 - 4.
\end{aligned}$$

along with the initial condition,

$$\alpha_0 a_1 + \beta_0 a_1 = 0$$

The final solution for the equation 3 would be,

$$\Psi(r) = (r-1)^{\rho} r^{-2\rho} e^{\rho(r-1)} \sum_{n=0}^{\infty} a_n \left(\frac{r-1}{r}\right)^n \tag{5}$$

The elements a_n and a_{n+1} that lead to physical results have to satisfy an infinite continued fraction relation,

$$\frac{a_{n+1}}{a_n} = \cfrac{-\gamma_{n+1}}{\beta_{n+1} - \cfrac{\alpha_{n+1}\gamma_{n+2}}{\beta_{n+2} - \cfrac{\alpha_{n+2}\gamma_{n+3}}{\beta_{n+3} - \dots}}}$$

The usual notation for convenience is as follows,

$$\frac{a_{n+1}}{a_n} = \frac{-\gamma_{n+1}}{\beta_{n+1}-} \frac{\alpha_{n+1}\gamma_{n+2}}{\beta_{n+2}-} \frac{\alpha_{n+2}\gamma_{n+3}}{\beta_{n+3}-} \dots$$

An asymptotic relation for the ratio between a_n and a_{n+1} can be obtained,

$$\frac{a_{n+1}}{a_n} \approx 1 \pm \frac{(2\rho)^{\frac{1}{2}}}{n^{\frac{1}{2}}} + \frac{2\rho - \frac{3}{4}}{n} + \dots$$

The solution set corresponding to the minus sign in the above equation ensures the convergence of solution (5) everywhere between $r = 1$ and ∞. Evaluating the continued fraction expression at $n = 0$ and using the initial condition $\alpha_0 a_1 + \beta_0 a_1 = 0$,

$$0 = \beta_0 - \frac{\alpha_0\gamma_1}{\beta_1-} \frac{\alpha_1\gamma_2}{\beta_2-} \frac{\alpha_2\gamma_3}{\beta_3-} \dots \tag{6}$$

The roots of this equation will be the QNM frequencies of the Schwarzschild black hole. Note that inverting the above equation an arbitrary number of times, n, shouldn't lead to any difference.

$$\left[\beta_n - \frac{\alpha_{n-1}\gamma_n}{\beta_{n-1}-}\frac{\alpha_{n-2}\gamma_{n-1}}{\beta_{n-2}-}\cdots - \frac{\alpha_0\gamma_1}{\beta_0}\right] = \left[\frac{\alpha_n\gamma_{n+1}}{\beta_{n+1}-}\frac{\alpha_{n+1}\gamma_{n+2}}{\beta_{n+2}-}\frac{\alpha_{n+2}\gamma_{n+3}}{\beta_{n+3}-}\cdots\right] \qquad \forall n > 0 \tag{7}$$

For every $n > 0$, (6) and (7) are completely equivalent. Although every solution of (6) is a solution of (7), and vice versa, the function's behaviour on the R.H.S changes with the number of inversions n. The n^{th} quasi-normal mode is usually found to be numerically the most stable root of the n^{th} inversion. The problem of finding the quasi-normal frequencies boils down to numerically finding the roots of the above equation. Numerically the infinite continued fraction on the R.H.S is truncated after a sufficiently large index $n = N$, and the remaining continued fraction(from $n = N + 1$ to ∞) should be approximated to some reasonable value. Root-finding algorithms are used to find the quasi-normal frequencies.

2.4.2 Nollert's Improvement

The truncated continued fraction's convergence worsens for higher overtones, and a large number of terms should be included to attain certain accuracy. This can be overcome by approximating the last term to an asymptotic form. Let $R_N(\omega)$ denote the exact value of the remaining infinite continued fraction,

$$R_N(\omega) := \cfrac{\gamma_{N+1}}{\beta_{N+1} - \cfrac{\alpha_{N+1}\gamma_{N+2}}{\beta_{N+2} - \cfrac{\alpha_{N+2}\gamma_{N+3}}{\beta_{N+3} - \cdots}}} \tag{8}$$

$$= \frac{\gamma_{N+1}}{\beta_{N+1} - \alpha_{N+1}R_{N+1}(\omega)} \tag{9}$$

This $R_N(\omega)$ should be approximated to an asymptotic form so that this asymptotic form can replace the last term in the truncated continued fraction. Approximating R_N to,

$$R_N = \lambda_0 + \frac{\lambda_1}{\sqrt{N}} + \frac{\lambda_2}{N} + ..$$

and substituting this in (9) and letting $N \to \infty$ results,

$$R_N = -1 + \sqrt{\frac{-2i\omega}{N}} + \frac{2i\omega + \frac{3}{4}}{N} + ..$$

The presence of $1/\sqrt{N}$ might be unexpected, but if one considers only integer powers, the asymptotic limit cannot be found. This approximation can be used as the last term of the truncated continued fraction [12].

2.4.3 Results

The continued fraction method is used to find the quasi-normal frequencies. The infinite continued fraction method is truncated at $N = 500$. The following are the QNM frequencies obtained for $l = 2$,

n	QNM frequencies
1	$0.747343 - 0.177924i$
2	$0.693422 - 0.547829i$
3	$0.602107 - 0.956554i$
4	$0.503010 - 1.41030i$
5	$0.415029 - 1.89369i$
6	$0.338599 - 2.39122i$
7	$0.266514 - 2.89584i$
8	$0.185677 - 3.40728i$
9	$0.000000 - 3.9836i$
10	$0.129995 - 4.60435i$
11	$0.155345 - 5.12309i$
12	$0.163028 - 5.63281i$
20	$0.175443 - 9.65386i$
30	$0.146648 - 14.6839i$
40	$0.148211 - 19.6595i$
41	$0.176007 - 20.1908i$

Table 3: QNM frequencies computed for $l = 2$

n	QNM frequencies
1	$1.19889 - 0.185406i$
2	$1.16529 - 0.562596i$
3	$1.10337 - 0.958186i$
4	$1.02392 - 1.38067i$
5	$0.940348 - 1.8313i$
6	$0.862773 - 2.3043i$
7	$0.795319 - 2.79182i$
8	$0.737985 - 3.28769i$
9	$0.689237 - 3.78807i$
10	$0.647366 - 4.2908i$
11	$0.610922 - 4.79471i$
12	$0.578768 - 5.29916i$
20	$0.404208 - 9.33288i$
30	$0.259204 - 14.3691i$
40	$0.0653794 - 19.4288i$
41	$0.0000000 - 20.2896i$

Table 4: QNM frequencies computed for $l = 3$

Even though n^{th} quasi-normal mode is the most stable one for n times inverted equation, the numerically obtained root also significantly depends on the initial value provided to the FindRoot function. Asymptotic expressions of QNM frequencies obtained from WKB or Pöschl Teller should be used as the initial value while searching for the root. For higher overtones, the computation becomes more and more intensive and time-taking, as we need more terms in the continued fraction.

2.5 Hill Determinant Method

Hill determinant method [17] starts from the three-term recurrence relation 4 obtained in the continued fraction method. Non-trivial solutions for that recurrence relation exist only if the following determinant is zero.

$$D = \begin{vmatrix} \beta_0 & \alpha_0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\ \gamma_1 & \beta_1 & \alpha_1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\ 0 & \gamma_2 & \beta_2 & \alpha_2 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & \beta_{n-2} & \alpha_{n-2} & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & \gamma_{n-1} & \beta_{n-1} & \alpha_{n-1} & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \gamma_n & \beta_n & \alpha_n & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{vmatrix} = 0$$

Let D_n be the $(n+1) \times (n+1)$ determinant of D. Then,

$$D_n = \beta_n D_{n-1} - \gamma_n \alpha_n D_{n-2}$$

l	n	Hill Determinant	Continued Fraction
2	0	$0.74734 - 0.17792i$	$0.74734 - 0.17792i$
2	1	$0.69342 - 0.54783i$	$0.69342 - 0.54783i$
2	2	$0.60199 - 0.95643i$	$0.60211 - 0.95655i$
3	0	$1.19889 - 0.18541i$	$1.19889 - 0.18541i$
3	1	$1.16529 - 0.56259i$	$1.16529 - 0.56259i$
3	2	$1.10338 - 0.958146i$	$1.10337 - 0.95819i$
4	0	$1.61836 - 0.188328i$	$1.61836 - 0.18833i$
4	1	$1.59326 - 0.568667i$	$1.59326 - 0.56867i$
4	2	$1.54536 - 0.95982i$	$1.54542 - 0.95982i$

Table 5: QNM frequencies obtained from Hill Determinant are compared with the ones obtained from Leaver's method.

Although the Hill determinant method is easy to understand, it has certain limitations. The stability of the roots cannot be adjusted, and the inaccuracy increases for higher overtones. But the lower overtones are easy to compute to reasonably high accuracy using D_{10} or D_{15}.

3 Kerr Black hole Perturbations and QNMs

The Kerr metric is a stationary, axisymmetric, asymptotically flat metric and is given by,

$$ds^2 = \left(1 - \frac{2Mr}{\Sigma}\right)(dt)^2 + \left(\frac{4Mar\sin^2\theta}{\Sigma}\right)dtd\phi - \left(\frac{\Sigma}{\Delta}\right)(dr)^2 - \Sigma(d\theta)^2 - \sin^\theta\left(\frac{r^2 + a^2 + 2Ma^2r\sin^2\theta}{\Sigma}\right)(d\phi)^2$$

where, M is the mass of the blackhole, aM is the angular momentum, $\Sigma = r^2 + a^2\cos^2\theta$ and $\Delta = r^2 - 2Mr + a^2$. It denotes the spacetime around a rotating uncharged black hole with a given mass M and the rotation parameter a gives angular momentum per unit mass, $0 \leq a \leq M$. Unlike the Schwarzschild case, the perturbation equation of a Kerr black hole is complicated to obtain. Decoupling the variables and getting a single PDE is almost impossible if one proceeds through metric perturbations. However, there's an elegant way to do it through Newman-Penrose(NP) formalism. NP formalism can lead to a more general Petrov type-D perturbation equation, which includes Schwarzschild, Reissner-Nordström, Kerr and Kerr-Newman metrics. Before proceeding to the derivation, we'll look at NP formalism.

3.1 Mathematical Preliminaries

The mathematical formulation of GR, Tetrad formalism, and Newman Penrose formalism are prerequisites for deriving the Teukolsky equation. These topics can be read from chapter 1 of S. Chandrashekar's book [6]. However, a few notations that are absolutely necessary for the derivation are given below.

3.1.1 Tetrad Formalism

Local coordinate basis needn't always be the best basis to deal with problems in the General Theory of Relativity. Working in a suitable tetrad basis of four linearly independent vector fields is advantageous, projecting the relevant quantities on the chosen basis. This is the tetrad formalism. At every point in spacetime, we set up a basis of four linearly independent vectors, ${e_{(a)}}^i \quad (a = 1, 2, 3, 4)$. The indices in the parentheses, (a), are tetrad indices, and the other ones, i, are tensor indices. On a metric space, a contravariant vector, by default, has an associated covariant vector. Further, we consider only those sets of vectors to be a tetrad which follows,

$$
{e_{(a)}}^i e_{(b)_i} = \eta_{(a)(b)}, \quad \text{where } \eta_{(a)(b)} \text{ is a constant symmetric matrix}
$$

All quantities defined in GR can be defined/projected on this basis and vice-versa. Raising and lowering indices in tetrad basis is done by $\eta_{(a)(b)}$ similar to the metric tensor in local basis. The definition of Covariant derivative can also be extended to tetrad formalism, and analogues of Christoffel symbols/connection coefficients are the Ricci rotation coefficients.

$$
\gamma_{(c)(a)(b)} := {e_{(b)}}^i e_{(a)k;i} {e_{(c)}}^k
$$

Once you get Christoffel symbols, you can define/project the Riemann tensor, Ricci tensor, and Weyl tensor onto the tetrad basis. Similarly, Ricci and Bianchi's identities can be projected onto the tetrad basis.

3.1.2 Newman-Penrose Formalism

The Newman-Penrose formalism [18] is a tetrad formalism with a particular choice of basis vectors. The choice is a tetrad of null vectors $\mathbf{l}, \mathbf{n}, \mathbf{m}$ and $\bar{\mathbf{m}}$ of which $\mathbf{l}, \mathbf{n}$ are real and $\mathbf{m}, \bar{\mathbf{m}}$ are complex conjugates of another. This formalism is unique in choosing a null basis, contrary to the conventional orthonormal basis. Penrose's belief that light-cone structure is essential in analyzing spacetime was the underlying motivation for selecting a null basis. A tetrad of vectors $\mathbf{l}, \mathbf{n}, \mathbf{m}, \bar{\mathbf{m}}$ is chosen as basis such that,

$$
\mathbf{l} \cdot \mathbf{l} = \mathbf{n} \cdot \mathbf{n} = \mathbf{m} \cdot \mathbf{m} = \bar{\mathbf{m}} \cdot \bar{\mathbf{m}} = 0
$$

$$
\mathbf{l} \cdot \mathbf{m} = \mathbf{l} \cdot \bar{\mathbf{m}} = \mathbf{n} \cdot \mathbf{m} = \mathbf{n} \cdot \bar{\mathbf{m}} = 0
$$

$$
\mathbf{l} \cdot \mathbf{n} = -\mathbf{m} \cdot \bar{\mathbf{m}} = 1
$$

Then, the matrix $\eta_{(a)(b)}$, is a constant symmetric matrix of the form,

$$
[\eta_{(a)(b)}] = [\eta^{(a)(b)}] = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{vmatrix}
$$

The basis vectors, considered directional derivatives, are assigned special symbols,

$$
\mathbf{l} = \mathbf{D}, \quad \mathbf{n} = \boldsymbol{\Delta}, \quad \mathbf{m} = \boldsymbol{\delta}, \quad \mathbf{m}* = \boldsymbol{\delta}^*
$$

The Ricci rotation coefficients, now called the spin coefficients, are also assigned special symbols,

$$
\begin{aligned}
&\kappa = \gamma_{311} \qquad && \rho = \gamma_{314} \qquad && \epsilon = (\gamma_{211} + \gamma_{341})/2 \\
&\sigma = \gamma_{313} && \mu = \gamma_{243} && \gamma = (\gamma_{212} + \gamma_{342})/2 \\
&\lambda = \gamma_{244} && \tau = \gamma_{312} && \alpha = (\gamma_{214} + \gamma_{344})/2 \\
&\nu = \gamma_{242} && \pi = \gamma_{241} && \beta = (\gamma_{213} + \gamma_{343})/2
\end{aligned}
$$

These 12 spin coefficients and their complex conjugates can give all Ricci-rotation coefficients and, thereby, all Christoffel symbols. The Weyl tensor is the traceless part of the Riemann tensor. In the Newman-Penrose formalism, five complex scalars can represent the ten independent components of the Weyl tensor.

$$
\Psi_0 = -C_{1313} = -C_{pqrs} l^p m^q l^r m^s
$$

$$
\Psi_1 = -C_{1213} = -C_{pqrs} l^p n^q l^r m^s
$$

$$
\Psi_2 = -C_{1342} = -C_{pqrs} l^p m^q \bar{m}^r n^s
$$

$$
\Psi_3 = -C_{1242} = -C_{pqrs} l^p n^q \bar{m}^r n^s
$$

$$
\Psi_4 = -C_{2424} = -C_{pqrs} n^p \bar{m}^q n^r \bar{m}^s
$$

where, C_{pqrs} are Weyl tensor components in the local basis and $C_{1313}, C_{1213}, ..$,etc are Weyl tensor components in tetrad basis. Finally, the ten independent components of the Ricci tensor are defined using four real scalars($\Phi_{00}, \Phi_{11}, \Phi_{22}, \Lambda$) and three complex scalars($\Phi_{01}(=\Phi_{10}^*), \Phi_{02}(=\Phi_{20}^*), \Phi_{12}(=\Phi_{21}^*)$),

$$\Phi_{00} = -\frac{R_{11}}{2}; \qquad \Phi_{22} = -\frac{R_{22}}{2}; \qquad \Phi_{02} = -\frac{R_{33}}{2}; \qquad \Phi_{20} = -\frac{R_{44}}{2}$$
$$\Phi_{11} = -\frac{R_{12}+R_{34}}{2}; \qquad \Phi_{01} = -\frac{R_{13}}{2}; \qquad \Phi_{10} = -\frac{R_{14}}{2}$$
$$\Lambda = \frac{R}{24} = \frac{R_{12}-R_{34}}{12}; \qquad \Phi_{12} = -\frac{R_{23}}{2}; \qquad \Phi_{21} = -\frac{R_{24}}{2}$$

Ricci and Weyl tensors together can uniquely determine the Riemann tensor. The Ricci and Bianchi Identities can also be expressed in NP formalism [6].

3.1.3 Tetrad Transformations

An orthonormal frame in tetrad formalism or a null frame in Newman-Penrose formalism isn't unique. The constraints on the basis of vectors are Lorentz invariant. Therefore we have six degrees of freedom in form of six parameters of Lorentz transformations. The Lorentz transformations of the chosen basis vectors $\mathbf{l}, \mathbf{m}, \mathbf{n}, \bar{\mathbf{m}}$ are classified into three types of rotations,

1. Class-I leaves the vector $\mathbf{l}$ unchanged.
$$\mathbf{l} \to \mathbf{l}, \quad \mathbf{m} \to \mathbf{m} + a\mathbf{l}, \quad \bar{\mathbf{m}} \to \bar{\mathbf{m}} + a^*\mathbf{l}, \quad \mathbf{n} \to \mathbf{n} + a^*\mathbf{m} + a\bar{\mathbf{m}} + aa^*\mathbf{l}$$

2. Class-II leaves the vector $\mathbf{n}$ unchanged.
$$\mathbf{n} \to \mathbf{n}, \quad \mathbf{m} \to \mathbf{m} + b\mathbf{n}, \quad \bar{\mathbf{m}} \to \bar{\mathbf{m}} + b^*\mathbf{n}, \quad \mathbf{l} \to \mathbf{l} + b^*\mathbf{m} + b\bar{\mathbf{m}} + bb^*\mathbf{l}$$

3. Class-III leaves the directions of $\mathbf{l}, \mathbf{n}$ unchanged and rotate $\mathbf{m}, \bar{\mathbf{m}}$ by an angle θ in $(\mathbf{m}, \bar{\mathbf{m}})$ plane.
$$\mathbf{l} \to A\mathbf{l}, \quad \mathbf{n} \to A^{-1}\mathbf{n}, \quad \mathbf{m} \to e^{i\theta}\mathbf{m}, \quad \bar{\mathbf{m}}e^{-i\theta} \to \bar{\mathbf{m}}$$

where a, b are two complex functions and A, θ are two real functions. The effect of Class-I rotation on Weyl scalars in Newman Penrose formalism is,

$$\begin{aligned} -\Psi_0 = C_{1313} &\to C_{pqrs}l^p(m^q + al^q)l^r(m^s + al^s) \\ &= C_{pqrs}l^p m^q l^r m^s = -\Psi_0 \end{aligned}$$

Similarly, one can do this for all Weyl scalars,

$$\Psi_0 \to \Psi_0, \quad \Psi_1 \to \Psi_1 + a^*\Psi_0, \quad \Psi_2 \to \Psi_2 + 2a^*\Psi_1 + (a^*)^2\Psi_0$$
$$\Psi_3 \to \Psi_3 + 3a^*\Psi_2 + 3(a^*)^2\Psi_1 + (a^*)^3\Psi_0, \quad \Psi_4 \to \Psi_4 + 4a^*\Psi_3 + 6(a^*)^2\Psi_2 + 4(a^*)^3\Psi_1 + (a^*)^4\Psi_0 \tag{10}$$

One can see how spin coefficients transform as well. The effect of Class-II rotation is extremely similar to Class-I rotation. Class-III rotations affect Weyl scalars as follows,

$$\Psi_0 \to A^{-2}e^{2i\theta}\Psi_0, \quad \Psi_1 \to A^{-1}e^{i\theta}\Psi_1, \quad \Psi_2 \to \Psi_2$$
$$\Psi_3 \to A^2 e^{-2i\theta}\Psi_3, \quad \Psi_4 \to Ae^{-i\theta}\Psi_4$$

3.1.4 Petrov Classification

The degrees of freedom in choosing a null tetrad can be used to vanish some of the Weyl scalars. How many scalars can be made to disappear? This question leads us to Petrov Classification. It is clear from eq 10 that $\Psi_4^{(1)}$ can be made to vanish by choosing an appropriate value for a so that the polynomial in a vanishes. If one or more of the roots coincide, the tensor is said to be algebraically special. And various ways in which roots coincide lead to Petrov classification. There are 6 Petrov types, but we'll look at Petrov type-D in this report. All physically relevant blackhole metrics come under Petrov type-D.

1. Petrov type-D - Two distinct twice repeating roots of the polynomial, $b_1 = b_2 \neq b_3 = b_4$). In this case, the polynomial and its derivative vanish at $b = b_1 = b_2$. The derivative of the polynomial is precisely the equation for Ψ_1. Therefore, a suitable rotation of class-II followed by a class-I rotation can make Ψ_0, Ψ_1, Ψ_3, Ψ_4 zero. In that case, the only non-vanishing Weyl scalar is Ψ_2.

It can be proved that if the field is algebraically special and is of Petrov type-D, then the congruences formed by the principal null directions, l and n, must be geodesic and shear-free (i.e., $\kappa = \sigma = \nu = \lambda = 0$ when $\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0$). The converse is also true. This is a corollary of the Goldberg-Sachs theorem [19].

3.2 Petrov type-D perturbation Equation

This section derives a general Petrov type-D perturbation equation using NP formalism [13]. In Schwarzschild, the perturbation equation is derived from metric perturbation and the perturbed Ricci tensor components. However, this method wouldn't work in more complicated metrics like Kerr. It wouldn't lead to a set of separable PDEs, and one would be stuck with a set of coupled differential equations. However, using NP formalism, one can obtain a perturbation equation not just for Kerr but for a general Petrov type-D metric. Here, instead of perturbing the metric, one would deal with a perturbed null tetrad.

Choosing a suitable metric for a Petrov type-D metric makes it possible to get vanishing $\Psi_0, \Psi_1, \Psi_3, \Psi_4$. In a vacuum, the Bianchi Identities give,

$$\begin{aligned} R_{13[13;4]} &= 0 \quad \Rightarrow \quad \kappa = 0, \qquad R_{42[21;4]} = 0 \quad \Rightarrow \quad \lambda = 0 \\ R_{13[13;2]} &= 0 \quad \Rightarrow \quad \sigma = 0, \qquad R_{42[43;2]} = 0 \quad \Rightarrow \quad \nu = 0 \end{aligned}$$

Therefore, for a Petrov type-D metric,

$$\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0 \quad , \quad \kappa = \lambda = \sigma = \nu = 0 \tag{11}$$

The beauty of the Newman Penrose formalism is it reduces the analysis to a minimum and makes it more elegant. It's quite remarkable that the 4-D black hole solutions of general relativity are all of Petrov-type D. To do perturbation theory in the formalism of Newman-Penrose, one uses the perturbed geometry given by $\mathbf{l} = \mathbf{l}^A + \mathbf{l}^B$ and similarly for the other null vectors. Here $\mathbf{l}^A$ is the unperturbed null vector, and $\mathbf{l}^B$ is the perturbation in the null vector. Similarly, all quantities can be written in the form $\Psi_2 = \Psi_2^A + \Psi_2^B$, where Ψ_2^B is the first order perturbation term. We know that for an unperturbed Petrov type-D metric,

$$\Psi_0^A = \Psi_1^A = \Psi_3^A = \Psi_4^A = 0 \quad , \quad \kappa^A = \lambda^A = \sigma^A = \nu^A = 0$$

Considering Bianchi Identities $R_{13[13|4]} = 0, \quad R_{13[13|2]} = 0$ and Ricci Identity for $\quad R_{1313}$,

$$\begin{aligned} (4\alpha - \pi - \boldsymbol{\delta}^*)\Psi_0 + (\mathbf{D} - 4\rho - 2\epsilon)\Psi_1 + 3\kappa\Psi_2 = {} & (2\alpha^* + 2\beta - \boldsymbol{\delta} - \pi^*)\Phi_{00} + (\mathbf{D} - 2\epsilon - 2\rho^*)\Phi_{01} \\ & - 2\sigma\Phi_{10} + 2\kappa\Phi_{11} + \kappa^*\Phi_{02} \end{aligned}$$

$$\begin{aligned} (4\gamma - \mu - \boldsymbol{\Delta})\Psi_0 + (\delta - 4\tau - 2\beta)\Psi_1 + 3\sigma\Psi_2 = {} & (2\beta - \boldsymbol{\delta} - 2\pi^*)\Phi_{01} + (\mathbf{D} - 2\epsilon + 2\epsilon^* - \rho^*)\Phi_{02} \\ & + \lambda^*\Phi_{00} - 2\sigma\Phi_{11} - 2\kappa\Phi_{12} \end{aligned}$$

$$(\mathbf{D} - 3\epsilon + \epsilon^* - \rho - \rho^*)\sigma - (\boldsymbol{\delta} + \pi^* - \tau - 3\beta - \alpha^*)\kappa - \Psi_0 = 0$$

The Ricci tensor terms in the above equation are given by Einstein equations,

$$\Phi_{00} = -\frac{R_{11}}{2} = -\frac{R_{pq}l^p l^q}{2} = 4\pi(T_{pq}l^p l^q - \frac{1}{2}g_{pq}Tl^p l^q)$$

Using,

$$g_{pq} = e_{(a)p}e^{(a)}{}_q = l_p n_q + l_q n_p - m_p\bar{m}_q - \bar{m}_p m_q \qquad \Rightarrow \qquad \Phi_{00} = 4\pi T_{11}$$

and so on. Since the values $\Psi_0^A, \Psi_1^A, \kappa^A, \sigma^A, \lambda^A$ and the tensor $\boldsymbol{\Phi}^A$ vanish, the above equations up to first order perturbation can be written as,

$$(4\alpha - \pi - \boldsymbol{\delta}^*)^A\Psi_0^B + (\mathbf{D} - 4\rho - 2\epsilon)^A\Psi_1^B + 3\kappa^B\Psi_2^A = 4\pi[(2\alpha^* + 2\beta - \boldsymbol{\delta} - \pi^*)^A T_{11}^B + (\mathbf{D} - 2\epsilon - 2\rho^*)^A T_{13}^B] \tag{12}$$

$$(4\gamma - \mu - \boldsymbol{\Delta})^A\Psi_0^B + (\delta - 4\tau - 2\beta)^A\Psi_1^B + 3\sigma^B\Psi_2^A = 4\pi[(2\beta - \boldsymbol{\delta} - 2\pi^*)^A T_{13}^B + (\mathbf{D} - 2\epsilon + 2\epsilon^* - \rho^*)^A T_{33}^B] \tag{13}$$

$$(\mathbf{D} - 3\epsilon + \epsilon^* - \rho - \rho^*)^A\sigma^B - (\boldsymbol{\delta} + \pi^* - \tau - 3\beta - \alpha^*)^A\kappa^B - \Psi_0^B = 0 \tag{14}$$

For convenience, labels A will be dropped from all unperturbed quantities. However, label B is still used to represent perturbed quantities. Bianchi Identities $R_{13[21|4]} = 0$ and $R_{13[43|2]} = 0$ give,

$$D\Psi_2 = 3\rho\Psi_2, \qquad \delta\Psi_2 = 3\tau\Psi_2$$

Using these equations one can write (14) as follows,

$$(\mathbf{D} - 3\epsilon + \epsilon^* - 4\rho - \rho^*)\Psi_2\sigma^B - (\boldsymbol{\delta} + \pi^* - 4\tau - 3\beta - \alpha^*)\Psi_2\kappa^B - \Psi_2\Psi_0^B = 0 \tag{15}$$

The commutation relations of NP tetrad vectors along with Ricci identities corresponding to R_{1312}, R_{3143} and $(R_{1213} - R_{3413})/2$ along with the properties (11) of Petrov type-D together give an Identity,

$$[\mathbf{D} - (a+1)\epsilon + \epsilon^* + b\rho - \rho^*)(\boldsymbol{\delta} - a\beta + b\tau) - [\boldsymbol{\delta} - (a+1)\beta - \alpha^* + \pi^* + b\tau)](\mathbf{D} - a\epsilon + b\rho) = 0 \tag{16}$$

Operate $(\boldsymbol{\delta} + \pi^* - 4\tau - 3\beta - \alpha^*)$ on equation (12) and $(\mathbf{D} - 3\epsilon + \epsilon^* - 4\rho - \rho^*)$ on equation (13) and subtract one equation from the other. Use Identity (16) with $a = 2, b = -4$ to eliminate Ψ_1^B terms along with equation (15) to get,

$$[(\mathbf{D} - 3\epsilon + \epsilon^* - 4\rho - \rho^*)(\boldsymbol{\Delta} - 4\gamma + \mu) - (\boldsymbol{\delta} + \pi^* - 4\tau - 3\beta - \alpha^*)(\boldsymbol{\delta}^* - 4\alpha + \pi) - 3\Psi_2]\Psi_0^B = 4\pi T_0 \tag{17}$$

where T_0 is given by,

$$\begin{aligned} T_0 = (\boldsymbol{\delta} + \pi^* - 4\tau - 3\beta - \alpha^*)[(\mathbf{D} - 2\epsilon - 2\rho^*)T_{13}^B - (\boldsymbol{\delta} + \pi^* - 2\alpha^* - 2\beta)T_{11}^B] \\ - (\mathbf{D} - 3\epsilon + \epsilon^* - 4\rho - \rho^*)[(\mathbf{D} - 2\epsilon + 2\epsilon^* - \rho^*)T_{33}^B - (\boldsymbol{\delta} - 2\beta + 2\pi^*)T_{13}^B] \end{aligned} \tag{18}$$

This is the decoupled equation for Ψ_0^B. The NP formalism is invariant under the interchange $\mathbf{l} \leftrightarrow \mathbf{n}$,$\mathbf{m} \leftrightarrow \bar{\mathbf{m}}$. One can therefore derive an equation for Ψ_4^B by applying the interchange on (17) and (18),

$$[(\boldsymbol{\Delta} + 3\gamma - \gamma^* + 4\mu + \mu^*)(\mathbf{D} + 4\epsilon - \rho) - (\boldsymbol{\delta}^* - \tau^* + 4\pi + 3\alpha + \beta^*)(\boldsymbol{\delta} + 4\beta - \tau) - 3\Psi_2]\Psi_4^B = 4\pi T_4 \tag{19}$$

where T_4 is given by,

$$\begin{aligned} T_4 = (\boldsymbol{\Delta} + 3\gamma - \gamma^* + 4\mu + \mu^*)[(\boldsymbol{\delta}^* - 2\tau^* + 2\alpha)T_{24}^B - (\boldsymbol{\Delta} + \mu^* - 2\gamma^* + 2\gamma)T_{44}^B] \\ - (\boldsymbol{\delta}^* - \tau^* + 4\pi + 3\alpha + \beta^*)[(\boldsymbol{\delta}^* + 2\alpha + 2\beta^* - \tau^*)T_{22}^B - (\boldsymbol{\Delta} + 2\gamma + 2\mu^*)T_{24}^B] \end{aligned} \tag{20}$$

These are the perturbation equations for a general Petrov type-D metric. The physical interpretation of Ψ_0 could be incoming transverse radiation, and of Ψ_1 be outgoing transverse radiation. Ψ_1, Ψ_3 can be considered longitudinal radiation or gauge quantities. Ψ_2 is called the Coulomb term, representing the effects due to gravitational monopole source.

3.2.1 Teukolsky Equation

The perturbation equation of the Kerr Black hole can be obtained from the general Petrov type-D equation. A set of null vectors can be obtained from null geodesics,

$$l^i = \left(\frac{r^2 + a^2}{\Delta}, 1, 0, \frac{a}{\Delta}\right), \quad n^i = \frac{(r^2 + a^2, -\Delta, 0, a)}{2\Sigma}$$

There's some freedom in choosing $m, \bar{m}$ following the constraints laid by NP formalism,

$$m^i = \frac{(ia\sin\theta, 0, 1, i/\sin\theta)}{\sqrt{2}(r + ia\cos\theta)}$$

The non-vanishing spin coefficients are,

$$\rho = -\frac{1}{r - ia\cos\theta}, \quad \beta = -\rho^*\frac{\cot\theta}{2\sqrt{2}}, \quad \mu = \frac{\rho 62\rho^*\Delta}{2}, \quad \gamma = \mu + \frac{\rho\rho^*(r - M)}{2}$$

$$\pi = \frac{ia\rho^2\sin\theta}{\sqrt{2}}, \quad \tau = -\frac{ia\rho\rho^*\sin\theta}{\sqrt{2}}, \quad \alpha = \pi - \beta^*$$

The fact that the spin coefficients $\kappa, \sigma, \lambda, \nu$ are vanishing shows that the chosen null geodesics are shear-free. Using Goldberg-Sachs theorem, It can be concluded that Kerr space-time is Petrov type-D. It can also be shown that $\Psi_0, \Psi_1, \Psi_3, \Psi_4$ vanish on this basis. Ψ_2 is given by,

$$\Psi_2 = R_{ijkl}l^i m^j n^k \bar{m}^l = -\frac{M}{r^3}$$

Using all this, one can derive Teukolsky's master equation for gravitational perturbations. Teukolsky's master perturbation equation is given as,

$$\begin{aligned} \left[\frac{(r^2 + a^2)^2}{\Delta} - a^2\sin^2\theta\right]\frac{\partial^2\psi}{\partial t^2} - \frac{1}{\Delta^s}\frac{\partial}{\partial r}\left(\Delta^{s+1}\frac{\partial\psi}{\partial r}\right) - 2s\left[\frac{M(r^2 - a^2)}{\Delta} - r - ia\cos\theta\right]\frac{\partial\psi}{\partial t} + \frac{4Mar}{\Delta}\frac{\partial^2\psi}{\partial t\partial\phi} \\ - \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) - 2s\left[\frac{a(r - M)}{\Delta} + \frac{i\cos\theta}{\sin^2\theta}\right]\frac{\partial\psi}{\partial\phi} - \left[\frac{a^2}{\Delta} - \frac{1}{\sin^2\theta}\right]\frac{\partial^2\psi}{\partial\phi^2} + s^2\cot^2\theta\psi - s\psi = 4\pi T \end{aligned} \tag{21}$$

This master equation validates the Kerr black hole's massless scalar, neutrino, electromagnetic and gravitational perturbations [13].

- Massless scalar field perturbations

$$\psi = \Phi, \quad s = 0, \quad \Box\Phi = 4\pi T$$

- Electromagnetic field perturbations

$$\psi = \phi_0, \quad s = 1, T = J_0 = (\boldsymbol{\delta} - \beta - \alpha^* - 2\tau + \pi^*)J_l - (\mathbf{D} - \epsilon + \epsilon^* - 2\rho - \rho^*)J_m$$

$$\psi = \rho^{-2}\phi_2, \quad s = -1, T = \rho^{-2}J_2 = \rho^{-2}(\boldsymbol{\Delta} + \gamma - \gamma^* + 2\mu + \mu^*)J_{m^*} - (\boldsymbol{\delta^*} + \alpha + \beta^* + 2\pi - \tau^*)J_n$$

- Gravitational Perturbations

$$\psi = \psi_0^B, \quad s = 2, \quad T = 2T_0$$

  where, T_0 is given in equation (18)

$$\psi = \rho^{-4}\psi_4^B, \quad s = -2, \quad T = 2\rho^{-4}T_4$$

  where, T_4 is given in equation (20)

3.3 Continued Fraction Method

Teukolsky equation 21 can be Fourier transformed in t, ϕ . Or, it can also be assumed that the time dependence in ψ is $e^{-i\omega t}$ and angular dependence is $e^{im\phi}$. Taking $\psi = R_{lm}(r)S_{lm}(\cos\theta)$, one can obtain two separate ODEs. The differential equation in θ is given by,

$$[(1-u^2)S_{lm,u}]_{,u} + \left[a^2\omega^2u^2 - 2a\omega su + s + A_{lm} - \frac{(m+su)^2}{1-u^2}\right]S_{lm} = 0 \tag{22}$$

where $u = \cos\theta$. The radial differential equation is given by,

$$\Delta R_{lm,rr} + (s+1)(2r-1)R_{lm,r} + V(r)R_{lm} = 0 \tag{23}$$

where,

$$V(r) = \{[(r^2+a^2)^2\omega^2 - 2am\omega r + a^2m^2 + is(am(2r-1) -]\omega(r^2-a^2))]\Delta^{-1} + [2is\omega r - a^2\omega^2 - A_{lm}]\}$$

The boundary conditions for equation 22 are that it should be finite at both the singular point. At singular point $u = +1$ the index is $\pm(m+s)/2$ and at singular point $u = -1$ the index is $\pm(m-s)/2$. Therefore the solution for the equation 22 will be of the form,

$$S_{lm}(u) = e^{a\omega u}(1+u)^{|m-s|/2}(1-u)^{|m+s|/2}\sum_{n=0}^{\infty} a_n(1+u)^n \tag{24}$$

The coefficients a_n in the above solution are related by a three-term recurrence relation given by,

$$\begin{aligned} \alpha_0^\theta a_1 + \beta_0^\theta a_0 = 0, \\ \alpha_n^\theta a_{n+1} + \beta_n^\theta a_{n+1} + \gamma_n^\theta a_{n-1} = 0, \quad \forall n \in \mathbb{Z}^+ \end{aligned} \tag{25}$$

where the recurrence coefficients are given by,

$$\begin{aligned} \alpha_n^\theta &= -2(n+1)(n+2k_1+1) \\ \beta_n^\theta &= n(n-1) + 2n(k_1+k_2+1-2a\omega) - [2a\omega(2k_1+s+1) - (k_1+k_2)(k_1+k_2+1)] - [a^2\omega^2 + s(s+1) + A_{lm}] \\ \gamma_n^\theta &= 2a\omega(n+k_1+k_2+s) \end{aligned} \tag{26}$$

The eigenvalue A_{lm} can be found for a given ω by solving the continued fraction equation,

$$0 = \beta_0^\theta - \frac{\alpha_0^\theta\gamma_1^\theta}{\beta_1^\theta -}\frac{\alpha_1^\theta\gamma_2^\theta}{\beta_2^\theta -}\frac{\alpha_2^\theta\gamma_3^\theta}{\beta_3^\theta -}\dots \tag{27}$$

The radial solution $R_{lm}(r)$ can be found similarly. The two regular singular points, r_+, r_-, are the roots of $\Delta = r^2 - r + a = 0$. For convenience let's define a parameter $b = (1-4a^2)^{1/2}$, then $r_\pm = (1\pm b)/2$. The indices at event horizon $r = r_+$ are $i\sigma_+$ and $-s - i\sigma_+$, where $\sigma_+ = (\omega r_+ - am)/b$. The first index corresponds to outcoming waves from the event horizon, and the other index corresponds to ingoing waves. The solutions to equation 23 at $r \to \infty$ are given by,

$$R_{lm}(r) \sim r^{-1-i\omega} e^{-i\omega r} \quad , \quad R_{lm}(r) \sim r^{-1-2s+i\omega} e^{+i\omega r}$$

Radial boundary conditions for QNMs are,

$$\lim_{r \to r_+} R_{lm}(r) \sim (r - r_+)^{-s-i\sigma_+} \quad , \quad \lim_{r \to \infty} R_{lm}(r) \sim (r)^{-1-2s+i\sigma_+} e^{i\omega r}$$

These correspond to outgoing waves at infinity and ingoing waves at the horizon. The solution can be expressed as,

$$R_{lm} = e^{i\omega r}(r - r_-)^{-1-s+i\omega+i\sigma_+}(r - r_+)^{-s-i\sigma_+} \sum_{n=0}^{\infty} d_n \left(\frac{r - r_+}{r - r_-}\right)^n \tag{28}$$

where a three-term recurrence relation defines the coefficients d_n,

$$\begin{aligned} \alpha_0^r d_1 + \beta_0^r d_0 &= 0, \\ \alpha_n^r d_{n+1} + \beta_n^r d_{n+1} + \gamma_n^r a_{n-1} &= 0, \quad \forall n \in \mathbb{Z}^+ \end{aligned} \tag{29}$$

where the recurrence coefficients are given by,

$$\begin{aligned} \alpha_n^r &= n^2 + (c_0 + 1)n + c_0 \\ \beta_n^r &= -2n^2 + (c_1 + 2)n + c_3 \\ \gamma_n^r &= n^2 + (c_2 - 3)n + c_4 - c_2 + 2 \end{aligned} \tag{30}$$

where the constants c_n are defined by,

$$\begin{aligned} c_0 &= 1 - s - i\omega - \frac{2i}{b}\left(\frac{\omega}{2} - am\right) \\ c_1 &= -4 + 2i\omega(2 + b) + \frac{4i}{b}\left(\frac{\omega}{2} - am\right) \\ c_2 &= s + 3 - 3i\omega - \frac{2i}{b}\left(\frac{\omega}{2} - am\right) \\ c_3 &= \omega^2(4 + 2b - a^2) - 2am\omega - s - 1 + (2 + b)i\omega - A_{lm} + \frac{4\omega + 2i}{b}\left(\frac{\omega}{2} - am\right) \\ c_4 &= s + 1 - 2\omega^2 - (2s + 3)i\omega - \frac{4\omega + 2i}{b}\left(\frac{\omega}{2} - am\right) \end{aligned} \tag{31}$$

The QNM frequencies ω can be found by solving the continued fraction equation,

$$0 = \beta_0^r - \frac{\alpha_0^r \gamma_1^r}{\beta_1^r -} \frac{\alpha_1^r \gamma_2^r}{\beta_2^r -} \frac{\alpha_2^r \gamma_3^r}{\beta_3^r -} \cdots \tag{32}$$

In the limit $a = 0$, this method is the same as the continued fraction method in the Schwarzschild case, as it should be. The n^{th} quasi-normal mode is found to be numerically the most stable root of the n^{th} inversion. Numerically the infinite continued fraction on the RHS is truncated after a sufficiently large index $n = N$, and the remaining continued fraction(from $n = N + 1$ to ∞) should be approximated. Root-finding algorithms and fixed point iteration methods are used to solve numerically for A_{lm}, ω_{lm} simultaneously.

l	**m**	**a**	**QNM frequencies**
2	0	0	$0.747343 - 0.177925i$
2	0	0.1	$0.750248 - 0.177401i$
2	0	0.2	$0.759363 - 0.175653i$
2	0	0.3	$0.776108 - 0.171989i$
2	0	0.4	$0.803835 - 0.164313i$
2	0	0.45	$0.824009 - 0.154946i$
2	0	0.49	$0.844508 - 0.147065$
2	1	0	$0.747343 - 0.177925i$
2	1	0.1	$0.776495 - 0.176977i$
2	1	0.2	$0.815958 - 0.174514i$
2	1	0.3	$0.871937 - 0.169128i$
2	1	0.4	$0.960461 - 0.155910i$
2	1	0.45	$1.032583 - 0.139609i$
2	1	0.49	$1.128310 - 0.103285i$

Table 6: Fundamental QNM frequency for $l = 2, m = 0, 1$. The QNM frequencies are accurate up to 5^{th} decimal.

4 Conclusion

Metric perturbation equations for Schwarzschild, namely Regge-Wheeler and Zerilli, are derived. QNM frequencies of Schwarzschild Blackhole are obtained using WKB, Poschl Teller, Continued fraction and Hill Determinant method. WKB and Pöschl Teller methods are found to be accurate under eikonal limits, $l >> n$, with Poschl-Teller being better than WKB. However, both approaches fail to work for higher overtones, $n \gtrsim l$. The continued fraction method is thus far the most precise method that can obtain hundreds of QNM frequencies, but it is computationally intensive. Although Hill determinant method is computationally less intensive, it is unstable for higher overtones. The perturbation equation for a general Petrov type-D metric is derived using Newman-Penrose formalism. The Kerr perturbation equation, namely the Teukolsky equation, is also derived. QNM modes of the Kerr black hole are computed using the continued fraction method.

Better approximation methods and the perturbation techniques of Quantum Mechanics can be employed to obtain the QNMs. Investigating the QNM frequencies of black holes in asymptotically Ads and dS spacetimes would also be interesting, as it will give us an insight into the stability of black holes in the respective spacetimes. The asymptotic behaviour of the QNM frequencies obtained using the Monodromy method would be interesting to analyse. Getting the higher-order perturbation equation and computing the QNM frequencies for the corresponding equation would give an insight into the validity of linear approximation.

All codes used to calculate QNM frequencies are available here.

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