Algorithms for optimisation
Problem Set # 2
Due date: 05/02/2020

1. Compare Newton's method and the secant method on $f(x) = x^2+x^4$ with $x^{(1)} = -3$ and $x^{(0)} = -4$. Run each method for 10 iterations. Make the two following plots:
   I. Plot f vs. The iteration for each method
   II. Plot f' vs. x. Overlay the progression of each method, drawing lines from $(x^{(i)}, f'(x^{(i)}))$ to $(x^{(i+1)}, 0)$ to $(x(i+1), f'(x(i+1)))$ for each transition.

What can you conclude about this comparison?

2. Take the Rosenbrock function : $f = (1-x_1)^2 + 5(x_2 - x_1^2)^2$ **<u>of Problem Set 1.</u>**,with an initial guess of $(x1,x2) = (-1.5,-1.5)$. Find the optimum of this function using
   (a) Newton's method
   (b) Quasi-Newton method (Optional)

Compare the performance to the first order methods of the previous problem set.

Note: Compare the performance of the methods, by reporting

- the number of steps (iteration) taken by your code to reach the optimal point within a certain error (change the error and see how this number changes)
- time to solution
- Plot the steps and compare the paths taken by each method

3. Consider the problem of a (steady) N-dimensional linear system subject to a (steady) forcing : ***Lu+f = 0***, where we look for the f that maximises the energy amplification ***R = u.u /f.f*** .

In this problem, the control variable is the forcing while the state of the system is the response, i.e., ***g = f*** and ***q = u***. The dimension of the control is therefore equal to the dimension of the state. One can transform the maximisation problem into a minimisation problem by minimising the inverse of ***R***, i.e., the ratio of the norm of the forcing to the norm of the response. The problem is formalised in the following way:

Optimise ***J(q,g) = g.g/q.q*** subject to ***F(q,g) = Lq + g = 0***.

The goal is to reach a local minimum of ***J*** acting on ***g***. For the present problem ,

$\partial F/\partial q = L$

$\partial F/\partial g = I$

$\partial J/\partial q = -2q(g.g)/(q.q)^2$

$\partial J/\partial g = 2g/(q.q)$

so that the explicit optimality system is:

$$\begin{aligned}
[\partial F/\partial q]^T a = \partial J/\partial q &\rightarrow L^T a = -2q(g.g)/(q.q)^2 \\
[\partial F/\partial g]^T a = \partial J/\partial g &\rightarrow g = a(q.q)/2 \\
F = 0 &\rightarrow Lq + g = 0
\end{aligned}$$

Solve the optimisation problem sequentially, following:

The method is initialised by giving an initial guess on ***g***, then the loop is:

1. Given the *p-th* guess for the optimal forcing $\boldsymbol{g}^{(p)}$, compute the corresponding response $q^{(p)}$ solving the state equation $\boldsymbol{Lq^{(p)} = -g^{(p)}}$.
2. Compute ***J*** and its relative increment w.r.t the previous iteration. If convergence is reached stop, if not continue.
3. Compute the adjoint state $\boldsymbol{a}^{(p)}$ solving the adjoint equation.
4. Finally update the control vector and get $\boldsymbol{g}^{(p+1)}$ using the optimality condition and go to No. 1.

As an application, we use this iterative method to compute the optimal forcing energy amplification, ***R***, supported by the system defined by the linear operator

$$\begin{bmatrix} -1/Re & 0 \\ 0 & -3/Re \end{bmatrix}$$

for a Reynolds number Re = 40.