~~LAW~~ LAX WENDROFF METHODS

MPDATA
multidimensional positive definite advection algorithm

* MPDATA is from LAX WENDROFF FAMILY (LW)
* LW method used central differencing to approximate "correction terms)
* MPDATA uses upwind schemes for correction terms

* MPDATA developed by subtracting spatial approximations of higher order error terms from the first order solution & thus increasing the order of accuracy

* MPDATA is a second order scheme in space & time. Uses explicit time marching

* It is a positive definite scheme. For example if you start with a positive tracer quantity you will end up with a positive tracer at end

* MPDATA made for geophysical applications high Reynolds number, low speed flow

* Focus on sign conserving preserving multidimension advection rather than monotone solutions of hyperbolic conservation laws in one spatial dimension

LAX WENDROFF METHOD

If we have an upwind scheme with backward differencing

Backwards differencing

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$$

$$\frac{\partial u}{\partial t} = -a\frac{\partial u}{\partial x}$$

$$\frac{\partial u}{\partial t} = -a\left(\frac{u_j - u_{j-1}}{\Delta x}\right)$$

This will work if $a > 0$ i.e. positive wave speed

otherwise it will be numerically unstable (Fourier series analysis can prove that)

Similarly Forward difference

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$$

$$\frac{\partial u}{\partial t} = -a\frac{\partial u}{\partial x}$$

$$\frac{\partial u}{\partial t} = -a\left(\frac{u_{j+1} - u_j}{\Delta x}\right)$$

$a < 0$ ie -ve wave speed

So if we want a scheme which introduces numerical dissipation but also does not care about the sign of wave speed we introduce LAX WENDROFF METHODS

$u(x, t+1)$

LAX WENDROFF METHOD

consider Taylor series expansion

$$u(x, t+h) = u + h\frac{\partial u}{\partial t} + \frac{h^2}{2!}\frac{\partial^2 u}{\partial t^2} + O(h^3) \quad \text{--- (1)}$$

i.e. higher order terms

Now consider basic advection equation

$$\frac{\partial u}{\partial t} = -a\frac{\partial u}{\partial x} \quad \text{--- (2)}$$

$$\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2} \quad \text{--- (3)}$$

Now replace time derivatives from (2) & (3) to (1)

$$u(x, t+h) = u + \left(-ah\frac{\partial u}{\partial x}\right) + \frac{a^2 h^2}{2!}\frac{\partial^2 u}{\partial x^2}$$

Replace space derivatives with central differencing

$$u(x, t+h) = u - \frac{1}{2}\left(\frac{ah}{\Delta x}\right)\left(u_{j+1}^n - u_{j-1}^n\right) + \frac{1}{2}\left(\frac{ah}{\Delta x}\right)^2\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right)$$

It can be shown that this would work independent of sign of a i.e. wave speed and stable as long as

$$\left|\frac{ah}{\Delta x}\right| \leq 1$$

$$\mu = -k^2(\nu - \tau k^2)$$

$$s = -k\,(a + \gamma k^2)$$

$$u = e^{\underbrace{-k^2(\nu - \tau k^2)}_{\text{amplitude}}}\; e^{\underbrace{ik[x - (a+\gamma k^2)t]}_{\text{phase}}}$$

amplitude of the solution depends on ν and τ (even derivatives)
Phase depends on a and γ (odd derivatives)

So if take an centered scheme we find out that even derivative get cancelled out

⇒ Means that centered schemes cause no discipation

⇒ So if we want to introduce discipation we have biased stencils (like use more points on one end)

1D MPDATA derivation

Consider $\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$

1^{st} order in time $\frac{u^{n+1} - u^n}{\Delta t} + a\frac{\partial u}{\partial x} = 0$ — (1)

Similar to equation (1) derivation on page (4) for Lax Wendroff method

do Taylor series expansion in time

$$u^{n+1} = u^n + \Delta t \frac{\partial u}{\partial t} + \frac{(\Delta t)^2}{2}\frac{\partial^2 u}{\partial t^2} + \frac{(\Delta t)^3}{3!}\frac{\partial^3 u}{}$$

$$\frac{u^{n+1} - u^n}{\Delta t} = \frac{\partial u}{\partial t} + \frac{(\Delta t)}{2}\frac{\partial^2 u}{\partial t^2} + \frac{(\Delta t)^2}{3!}\frac{\partial^3 u}{\partial x^3}$$

+ (higher order terms)

$$\frac{u^{n+1} - u^n}{\Delta t} = \frac{\partial u}{\partial t} + \frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2} + \frac{(\Delta t)^2}{3!}\frac{\partial^3 u}{\partial x^3} + \ldots$$

or

$$= \frac{\partial u}{\partial t} + \frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2} + O(\Delta t)^2$$

(higher order terms)

so $\frac{u^{n+1} - u^n}{\partial t} = \frac{\partial u}{\partial t} + \frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2} + O(\Delta t)^2$ — (2)

Now take equation (1) & (2)

$$a\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} + \frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2} = O(\Delta t)^2$$ — (3)

Write this as

$$\frac{\partial u}{\partial t} + \frac{\partial a u}{\partial x} = -\frac{\Delta t}{2}\left(a^2 \frac{\partial^2 u}{\partial x^2}\right) + O(\Delta t)^2$$

Now use space stencil

$$\frac{\partial u}{\partial t} + a\left(\frac{q_i - q_{i-1}}{\Delta x}\right) = -\frac{\Delta t}{2}\frac{\partial}{\partial x}\left(a^2 \frac{\partial u}{\partial x}\right) + O(\Delta t)^2$$

if $a > 0$

and $$\frac{\partial u}{\partial t} + a\left(\frac{q_{i+1} - q_i}{\Delta x}\right) = -\frac{\Delta t}{2}\frac{\partial}{\partial x} a^2\left(\frac{\partial u}{\partial x}\right) + O(\Delta t)^2$$

if $a < 0$

Now if the spatial stencils were expanded about the location i and then above two equations are combined such that both -ve & +ve "a" values are taken care off

$$\frac{\partial u}{\partial t} + \frac{\partial a u}{\partial x} = \frac{-\partial}{\partial x}\left(\left(\frac{\Delta t}{2} a^2 - \frac{\Delta x}{2}|a|\right)\frac{\partial u}{\partial x} + O(\Delta t^2, \Delta x^2)\right)$$

Now equation on left side is original convection equation while on the right side is error of first order in space & time

Now this first order error is referred to pseudo velocity in MPDATA

Pseudo velocity is used in approximating the first order error term

The Term $-\frac{\partial}{\partial x}\left(\left(\frac{\Delta t}{2}a^2 - \frac{\Delta x|a|}{2}\right)\frac{\partial u}{\partial x}\right)$

Psuedo velocity term

* Represents first order error
* is a function of the physical velocity (a) and convected quantity 'u'.

* has NO physical significance
* has units of velocity
* In MPDATA we can keep finding these error terms i.e second order third order and then keep finding new psuedo velocity

Now psuedo velocity can be written as

$$a^{(1)} = \frac{1}{2}\left[(\Delta x)|a| - \Delta t\, a^2\right]\frac{1}{u}\frac{\partial u}{\partial x}$$

(This step is not clear)

Non dimensionalize psuedo velocity by $\Delta x / \Delta t$

$$A' = (|A| - A^2)\frac{\Delta x}{2u}\frac{\partial u}{\partial x}$$

Numerical approximation of above equation

$$A^{(1)} = \frac{u_{i+1}^{(1)} - u_i^{(1)}}{u_{i+1}^{(1)} + u_i^{(1)} + \text{epsilon}}\left[|A_{i+\frac{1}{2}}| - \left(A_{i+\frac{1}{2}}\right)^2\right]$$

MPDATA can be expanded to any number of iterations by upating higher order error terms or psuedo velocity

$$u^{k+1} = u^k + \frac{\partial(a^k u^k)}{\partial x} \quad \text{where } a^k = A^{(1)} \text{ for } k=1$$