The Stone-Čech compactification βX

Submitted in partial fulfillment of the requirements for the award of the degree of M.Sc in Mathematics

Department of Mathematics

Sikkim University, Gangtok

GUIDE

Dr. Goutam Bhunia

Submitted by:

Tara Prasad Sharma

Roll No.: 21MMT020

Acknowledgement

Undertaking this project has been a truly wonderful experience for me. Throughout the writing of this dissertation I have received a great deal of support and assistance from many people.
First and foremost, I would like to express my profound gratitude to my supervisor Dr. Goutam Bhunia whose expertise, understanding, generous guidance and support made it possible for me to work on a topic that was of great interest to me.

I would also like to thank Dr. Amit Chakraborty, HOD, Department of Mathematics, Sikkim University and all the teaching faculties for organizing weekly seminar to boost our morale and confidence and all the valuable comments I have received from them.

Finally, I would like to express my gratitude to my parents and my friends for their constant support and encouragement which made it possible for me to complete the dissertation.
Name: Tara Prasad Sharma

Place: Gangtok

Date:

CERTIFICATE

This is to certify that this project entitled "The Stone-Čech compactification βX" submitted in partial fulfillment of the degree of MASTER OF SCIENCE (Mathematics) to the Department of Mathematics, Sikkim University, done by Mr. Tara Prasad Sharma, Roll no. 21MMT020 is an authentic review work carried out by him at Sikkim University under my guidance. The matter embodied in this project work has not been submitted earlier for award of any degree or diploma to the best of my knowledge and belief.

Signature of the Student

Signature of the Guide

SELF CERTIFICATE

This is to certify that this project entitled "The Stone-Čech compactification βX" done by me is an authentic review work carried out for the partial fulfillment of the requirements for the award of the degree of M.Sc in Mathematics under the guidance of Dr. Goutam Bhunia. The matter embodied in this project work has not been submitted earlier for award of any degree or diploma to the best of my knowledge and belief.

Signature of the Student

Tara Prasad Sharma

Roll No. 21MMT020

Sikkim University

Abstract

A compactification of a topological space X is a compact Hausdorff space K together with an embedding $e\colon X \to K$ with $e(X)$ dense in K. We will usually identify X with $e(X)$ and consider X as a subspace of K. Our main topic is a very special type of compactification – one in which X is embedded in such a way that every bounded, real-valued continuous function on X will extend continuously to the compactification. Such a compactification of X will be called the Stone-Čech compactification and will be denoted by βX. In this report several constructions of βX will be examined. We will find that βX is a useful device to study relationships between topological characteristics of X and the algebraic structure of the ring $C(X)$ of real-valued continuous functions defined on X and that many topological properties of X can be translated into properties of βX.

Contents

Introduction

The Stone–Čech compactification is a mathematical construction that extends a given topological space to a compact space while preserving certain important properties of the original space. It is named after Marshall Stone and Eduard Čech, who independently developed the concept in the 1930s.

The Stone–Čech compactification is particularly useful in the realm of general topology, where it provides a way to study and analyze non-compact spaces by embedding them into compact spaces. This compactification allows for the exploration of various topological and algebraic properties that may not be readily apparent in the original space.

The Stone–Čech compactification has several notable properties:

1. It is a compact space, meaning that every open cover of βX has a finite subcover. 2. It is a Hausdorff space, ensuring that distinct points can be separated by disjoint open neighborhoods. 3. It is universal among compact Hausdorff spaces in the sense that for any continuous map f from X to a compact Hausdorff space K, there exists a unique continuous extension of f to βX.

The Stone–Čech compactification finds applications in various areas of mathematics, including algebraic topology, functional analysis, and measure theory. It provides a powerful tool for investigating the structure and behavior of non-compact spaces by embedding them into compact spaces that possess desirable properties.

Chapter 1 contains all the prerequisite to study the Stone-Čech compactification theorem, so call preliminaries. Chapter 2, is dedicated to Classical description of βX, chapter 3 is construction of βX by maximal ideal spaces of $C(X)$ and $C^*(X)$, chapter 4 contains, construction of βX via the space of $Z-$ ultrafilters on X. In the 5^{th}chapter there are some know resutlts on $\beta N, \beta Q, \beta R$ has been discussed. Last chapter that is chapter number 6, includes some applications, that is proof of Banach-Stone theorem and Gelfand-Kolmogorav theorem by using Stone-Čech compactification therorm.

Chapter 1

Preliminaries

Note. Open sets in metric spaces provide us with a way of phrasing the definition of continuous functions without mentioning distance. Thus whenever we can carry a reasonable abstract notion of "open set", we can define continuous functions.

Definition 1.0.1. : A topology on a set X is a collection τ of subset of X, called the open sets, satisfying;

1. Any union of elements of τ belongs to τ.

2. Any finite intersection of elements of τ belong to τ.

3. ϕ and X belong to τ.

We say (X, τ) is a topological space, sometimes abbreviated " X is a topological space".

Examples. Let X be any set and let τ = {a,b}. Then τ is a topology for X, called the trivial (indiscrete) topology for X.

Examples. X = {a,b} and let τ = {ϕ, {a}, X } , then τ is a topology for X.

Examples. Let X be any infinite set, and let the topology consists of the empty set ϕ together with all subsets of X whose complements are finite, this is known as co-finite topology, i.e., complements is finite.

Proof.

$$\tau = \{G \subseteq X \colon X \setminus G \text{is finite}\} \cup \{\phi\}$$

.

1. $\phi \in \tau$, by definition of τ and $X \in \tau$, since $X \setminus X = \phi$ is finite, so $X \in \tau$.

2. Let $G, H \in \tau$. If $G \cap H = \phi$, then $G \cap H \in \tau$. Now we assume that $G \cap H \neq \phi$.

   $\implies X \setminus G$ and $X \setminus H$ are finite.

   $\implies G \neq \phi$ and $H \neq \phi$.

   We have $X\setminus\ (G \cap H) = (X \setminus G) \cap (X \setminus H)$, a finite set. (Since G, H $\in \tau$ and $\neq \phi$).

3. Let $G_i \in \tau\ \forall I$, where I is an index set.

   To show $\bigcup_{i\in I} G_i \in \tau$.

   If $G_i = \phi\ \forall\ i \in I$, then $\bigcup_{i\in I} G_i = \phi \in \tau$.

   So we assume that $G_i \neq \phi$ for some $i_o \in I$.

   $\implies X \setminus Gi_o$ is finite.Then

   $$X \setminus \cup_{i\in I} G_i = \cap_{\in I}(X \setminus G_i) \subseteq X \setminus Gi_o$$

   and so

   $$X \setminus (\cup_{i\in I} G_i)$$

   is finite, hence $\cup_{i\in I} G_i \in \tau$.

□

Definition 1.0.2. A topological space (X , τ) is metrizable if there is metric d on X s.t. the topology induced by d is τ.

Note. A topology on a set can be complicated collection of subsets of a set, and it can be difficult to describe the entire collection. In most cases one describes a subcollection that "generates" the topology.

Definition 1.0.3. Let (X,τ) be a topological space. A basis for τ is a subcollection $\mathscr{B}$ of τ with the property that if U $\in \tau$ then U $= \phi$ or there is a subcollection $\mathscr{B}'$ of $\mathscr{B}$ s.t. U $=$ $\bigcup${B: B $\in \mathscr{B}'$}.

Examples. The collection $\mathscr{B}$ of all open intervals is a basis for the usual topology on $\mathbb{R}$.

Examples. If X is a set, the $\mathscr{B} = \{\{x\} : x \in X\}$ is a basis for the discreate topology on X.

Examples. Let X = {1, 2, 3 } and $\mathscr{B} = \{\{1,2\},\{2,3\},X\}$, then $\mathscr{B}$ is NOT a basis for a topology on X.

Definition 1.0.4. Let (X, τ) be a topological space. A subcollection $\mathscr{C}$ of τ is a "subbasis" for τ, provided the family of all finite intersection of members of τ is a basis for τ.

Examples. If $\mathscr{C}$ is the collection of all intervals of the form (a, ∞), ($-\infty$, b), then $\mathscr{C}$ is a subbasis for the usual topology on $\mathbb{R}$

Definition 1.0.5. Let X and Y be topological space and f a mapping of X onto Y.

f is called a continuous mapping if $f^{-1}(G)$ is open in X whenever G is open in Y.

Definition 1.0.6. An open mapping is continuous if it pull open sets back to open sets, and open if it carries open sets over to open sets.

Definition 1.0.7. f is called embedding if,f is injective, continuous and f: $X \to f(X)$ is homeomorphic.

Definition 1.0.8. f is homeomorphic if f is bijective, continuous and f^{-1} is continuous.

Definition 1.0.9. A space X is compact iff each open cover of X has a finite subcover.

Definition 1.0.10. X is countably compact iff each countable open cover of X has a finite subcover.

Definition 1.0.11. X is compact iff X is countable compact and Lindelöf.

Definition 1.0.12. X is Lindelöf iff every open cover of X has a countable subcover.

Examples. $\mathbb{R}$ is not compact. In fact, the cover of $\mathbb{R}$ by the open set $(-n, n)$ for $n \in \mathbb{N}$, can have no finite subcover.

Examples. All finite sets are compact.

Definition 1.0.13. Let X_α be a set, for each $\alpha \in \Lambda$. The cartesian product of the sets X_α is the

$$\prod_{\alpha\in\Lambda} X_\alpha = \{x \colon \Lambda \to \bigcup_{\alpha\in\Lambda} X_\alpha \colon x(\alpha) \in X_\alpha \text{for each} \alpha \in \Lambda\}$$

Definition 1.0.14. The Tychnonoff topology (or product topology) on $\prod X_\alpha$ is obtained by taking as a base for the open sets, sets of the form $\prod U_\alpha$, where

1. U_α is open in X_α, for each $\alpha \in \Lambda$.

2. For all but finitely many coordinates $U_\alpha = X_\alpha$.

Theorem 1.0.15. *(**Tychonoff**)A non empty product space is compact, iff each factor space is compact.*

Proof. Let a non-empty product space is compact. If the product space is non-empty, then the projection maps are all continuous and onto. So the continuous image of compact space is compact. $\therefore$ each factor space is compact.
Conversely, if each factor space is compact. Let $(x_\lambda)_{\lambda\in\Lambda}$ be an ultranet in $(\prod(x_\lambda))_{\lambda\in\Lambda}$ is an ulternet in X_α and hence converges. (Since X_α is compact.) $\therefore$ (X_α) converges.
Thus the product space is compact. □

Definition 1.0.16. A set Λ is a **directed set** iff there is a relation $\leq$ on Λ satisfying:

1. $\lambda \leq \lambda$, for each $\lambda \in \Lambda$,

2. if $\lambda_1 \leq \lambda_2$ and $\lambda_2 \leq \lambda_3$ then $\lambda_1 \leq \lambda_3$,

3. if λ_1, $\lambda_2 \in \Lambda$ then there is some $\lambda_3 \in \Lambda$ with $\lambda_1 \leq \lambda_3$, $\lambda_2 \leq \lambda_3$.

Definition 1.0.17. A **net** in a a set X is a function $P\colon\ \Lambda \to X$, where Λ is some directed set. The point $P(\lambda)$ is usually denoted x_λ, and we speak of "the net $(x_\lambda)_{\lambda\in\Lambda}$"

Definition 1.0.18. A **subnet** of a net $P\colon\ \Lambda \to X$ is the composition $P \circ \phi$, where $\phi\colon$ $M \to \Lambda$ is an increasing cofinal function from a directed set M to Λ. That is

1. $\phi(\mu_1) \leq \phi(\mu_2)$ whenever $\mu_1 \leq \mu_2$ (ϕ is increasing),

2. for each $\lambda \in \Lambda$, there is some $\mu \in M$ such that $\lambda \leq \phi(\mu)$ (ϕ is cofinal in Λ).

Remark 1.0.19. For $\mu \in M$, the point $P \circ \phi(\mu)$ is often written $(x_{\lambda\mu})$, and we usually speak of "the subnet $(x_{\lambda\mu})$ of (x_λ)".
If (x_λ) is a net in X, a set of the form $\{\ x_\lambda\ :\ \ \lambda \geq \lambda_o\}$, for $\lambda_o \in \Lambda$, is called a tail of (x_λ).

Definition 1.0.20. Let (x_λ) be a net in a space X. Then (x_λ) **converges** to $x \in X$ (written $x_\lambda \to x$) provided for each neighborhood U of x, there is some $\lambda_o \in \Lambda$ such that $\lambda \geq \lambda_o$ implies $x_\lambda \in U$. Thus $x_\lambda \to x$ iff each nhood of x contains a tail of (x_λ). This is sometimes said : (x_λ) converges to x provided it is residually (or eventually) in every ngood of x.

Definition 1.0.21. A sequence (x_n) in a topological space X is said to **converge** to $x \in X$, and we write $x_n \to x$, iff for each ngood U of x, there is some positive integer n_o such that $n \geq n_o$ implies $x_n \in U$. In this case, we say (x_n) is eventually in U.

Definition 1.0.22. A net (x_λ) in a set X is an **ultranet(universal net)** iff for each subset E of X, (x_λ) is either residually in E or residually in $X \setminus E$.

Definition 1.0.23. A topological space X is a T_1 space iff whenever x and y are distinct point in X , there is a neighbourhood of each not containing the other. λ

A normal T_1 space will be called a T_4 space.

Definition 1.0.24. If for each $\lambda \in \Lambda$ $f_\alpha \colon X \to X_\alpha$, then the **evaluation map** $e \colon X \to \prod X_\alpha$ induced by the collection $\{ f_\alpha : \alpha \in \Lambda \}$ is defined as follows : for each $x \in X$, $[\mathrm{e(x)}]_\alpha = f_\alpha(x)$. That is, for $x \in X$, $e(x)$ is the point in $\prod X_\alpha$, whose αth coordinate is $f_\alpha(x)$ for each $\alpha \in \Lambda$.

Definition 1.0.25. Compactification is the process of embedding a given space as a dense subset of some compact Hausdorff space.

Definition 1.0.26. A **compactification** of a space X is an ordered pair (K, h) where K is a compact Hausdorff space and h is an embedding of X as a dense subset of K.

Definition 1.0.27. A compactification of a space X , means a compact space in which X is dense.

Definition 1.0.28. A compactification of X is a compact Hausdroff space Y as a subspace such that $\bar{X} = Y$

Examples. $[0, 1]$ is a compactification of $[0, 1)$.
Here $X = [0, 1)$, $\bar{X} = Y = [0, 1]$. Y is compact Hausdorff.

Examples. S^1 is a compactification of $\mathbb{R}$, (under stereographic projection), the ordinal space Ω is a compactification of Ω_0.

Definition 1.0.29. p is said to be prime ideal if p $\neq A$ and whenever $a, b \in A$ and $ab \in A$ then $a \in p$ or $b \in p$.

Definition 1.0.30. Let A be a commutative ring with identity, Spec(A) = {p : p is a prime ideal of A}

Definition 1.0.31. M is said to be maximal ideal, if $M \neq A$ and if there is no ideal J such that $M \subset J \subset A$ and $J \neq A, J \neq M$.

Note. Every maximal ideal is a prime ideal.

Definition 1.0.32. Suppose A be a commutative ring with identity, let $a \in A$, $\mathcal{P}_a = \{ p \in$ Spec(A): $a \notin p \}$

Then $\{ \mathcal{P}_a : a \in A \}$ is a base for some topology on SpecA. This topology is called the **Zariski topology** in Spec (A).

Chapter 2

Classical description of βX

Let $C^*(X) = \{\ f : X \to \mathbb{R}$, where f is bounded and continuous$\}$. The range of each $f \in C^*(X)$ can be taken as a closed bounded interval I_f in $\mathbb{R}$.

Since X is Tychonoff, the collection $C^*(X)$ separates points from closed set in X.

The evaluation map $e\colon\ X \to \prod\{I_f : f \in C^*(x)\}$ defined by $[e(x)]_f = f(x)$ is an embedding of $X \in \prod I_f$.

Note. that under the embedding e, the element f of $c^*(X)$ is transformed into the restriction to $e(x)$ of the fth projection map $\prod_f$, that is

$$\begin{tikzcd} X \arrow[r, "e"] \arrow[dr, "f"'] & e(X) \arrow[d, "\pi_f"] \\ & I_f \end{tikzcd}$$

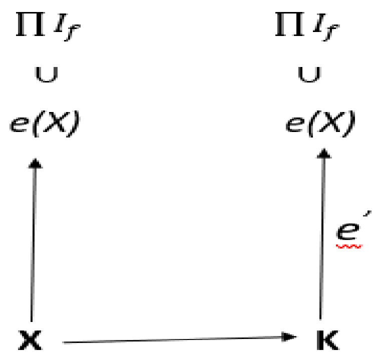
Definition 2.0.1. The Stone - Čech compactification of X is a closure of βX of $e(X)$ in the product $\prod I_f$.

More formally $(\beta X, e)$ is the Stone - Čech compactification of X.

The central useful fact about the Stone- Čech compactification is an extension property, given by the following theorem.

Theorem 2.0.2. *If K is a compact Hausdorff space and $f : X \to K$ is continuous, there is a continuous $F : \beta X \to K$ such that $F \circ e = f$.*

Proof. K is a Tychonoff space and thus can be embedded by and evaluation map e in a cube $\prod\{I_g : g \in C^*(K)\}$. The situation is illustrated in the figure below, $e : X \to e(X)$.



We can define a map $H : \prod I_f \to \prod I_g$ as follows; for each $t \in \prod I_f$, $[H(t)]_g = t_{gof}$. This map is continuous when followed by each projection π_g, in fact $(\pi_g \circ H)(t) = \pi_{gof}(t)$, so H is continuous. Now H takes $e(H)$ into $e'(K)$, for an element of $e(X)$ has the form $e(X)$ for some $x \in X$ and

$$H[e(x)]_g = [e(X)]_{gof} = gof(x) = [e^{'}(f(x))]_g$$

so that $H[e(x)] = e^{'}(f(x))$. But $e(X)$ is dense in βX, so $H[e(X)]$ is dense in $H(\beta X)$ and thus, since $e^{'}(K)$ is closed and contains $H[e(X)]$, $H(\beta X) \subset e^{'}(K)$. Finally, define $F = e^{'-1} \circ (H|\beta X)$. Then $F : \beta X \to K$ is continuous and $Fof = f$ since, for $x \in X$,

$$F \circ e(x) = e^{'-1}[H(e(x)) = e^{'-1}[e^{'}(f(x))] = f(x).$$

$\square$

Remark 2.0.3. Very often it is possible to deal with $e(X)$ directly (as, for example, when dealing with preservation of a topological property in the passage from X to βX).Then X is often written for $e(X)$, so that $X \subset \beta X$, and the above theorem become: **every continuous function from X to a compact space K can be extended to** βX.

Note. Above theorem actually characterizes the Stone - Čech compactification, up to what is called a topological equivalence.

Chapter 3

βX via the maximal ideal spaces of $C(X)$ and $C^*(X)$

Theorem 3.0.1. The maximal ideals in $C^*(X)$ are precisely the sets

$$M^{*p} = \{f \in C^*(X) : f^{\beta}(p) = 0\}$$

. *($p \in \beta X$,) and they are distinct for distinct p.*

Theorem 3.0.2. *(GELFAND-KOLMOGOROFF).*For the maximal ideals in C(X), we have

$$M^p = \{f \in C(X) : p \in cl_{\beta X} Z_x(f)\}$$

($p \in \beta X$).

As we have seen, the mapping $p \to M^{*p}$ is one-one from βX onto the set $\mathcal{M}^* = \mathcal{M}^*(X)$ of all maximal ideal in $C^*(X)$. Accordingly, it may be used to define a topology on $\mathcal{M}^*$ - simply by transferring that of βX. The image of X under this transfer is, of course, the subspace of all fixed maximal ideals. The topology on $\mathcal{M}^*$ can be described intrinsically as follows. Define

$$\mathcal{I}^*(f) = \{M \in \mathcal{M}^* : f \in M\}$$

$(f \in C^*(X))$. Since we have $f \in M^{*p}$ if and only if $f^{\beta}(p) = 0$ - hence that $M^{*p} \in \mathcal{I}^*(f)$ if and only if $p \in Z_{\beta X}(f^{\beta})$. Consequently, the mapping $p \to M^{*p}$ carries the basic family of closed sets $Z_{\beta X}(f^{\beta})$ in βX onto the family of sets $\mathcal{I}^*(\{)$ in $\mathcal{M}^*$; therefore this latter family constitutes a base in $\mathcal{M}^*$.

Thus the topology defined is called the *Stone topology* on $\mathcal{M}^*$, endowed with the Stone topology, is called the *structure space* of C^*.

In the Stone topology on the set $\mathcal{M} = \mathcal{M}(X)$ of all maximal ideals in C(X), the sets

$$\mathcal{I}(f) = \{M \in \mathcal{M} : f \in M\}$$

$(f \in C(X))$ form a base for the closed sets. By definition, the sets

$$\{p \in \beta X : Z(f) \in A^p\}$$

form a base for the closed set in βX. Therefore the mapping $p \to M^p$ is a homeomorphism of βX onto $\mathcal{M}$. Again, the image of X under the homeomorphism is the subspace of fixed maximal ideals.

It follows that the mapping $M^p \to M{*}p$ is homeomorphism between the structure spaces $\mathcal{M}$ and $\mathcal{M}^*$. It should be noted, however, that in spite of the formal similarity between the basic closed sets in the definitions of the respective Stone topologies, the homeomorphism does not lead, in general, to a correspondence between the bases. In fact, we have

$$M^{*p} \in \mathcal{I}^*(f) \text{ if and only if } p \in Z_{\beta X}(f^{\beta}),$$

as we have seen in previous theorem, by the Gelfand-Kolmogoroff theorem, on the other hand,

$$M^{*p} \in \mathcal{I}^*(f) \text{ if and only if } p \in cl_{\beta X} Z_X(f),$$

while $Z_{\beta X}(f^{\beta}) \supset cl_{\beta X} Z_X(f)$ for every $f \in C^*(X)$, it is easy to construct examples where the two sets are not the same. What is more to the point, the family of all sets $Z_{\beta X} Z_X(f)$ (for $f \in C^*(X)$). Observe that to obtain the family of all sets $\mathcal{I}(f)$, it is enough to let f range over the bounded functions.

Chapter 4

βX via the space of Z-ultrafilters on X

CONSTRUCTION OF βX

Theorem 4.0.1. *COMPACTIFICATION THEOREM. Every (completely regular) space X has a compactification βX, with the equivalent properties.*

I. *(Stone) Every continuous mapping τ from X into any compact space Y has a continuous extension $\bar{\tau}$ from βX into Y.*

II. *(STONE -ČECH) Every function f in $C^*(X)$ has an extension to a function f^β in $C(\beta X)$.*

III. *(ČECH) Any two disjoint zero-sets in X have disjoint closure in βX.*

IV. *For any two zero-sets Z_1 and Z_2 in X, $cl_{\beta X}(Z_1 \cap Z_2) = cl_{\beta X} Z_1 \cap cl_{\beta X} Z_2$.*

V. *Distinct z-ultrafilters on X have distinct limits in βX.*

Furthermore, βX is unique, in the following sense: if a compactification T of X satisfies any one of the listed conditions, then there exists a homeomorphism of βX onto T that leaves X pointwise fixed.

The assertion that βX satisfies (I) will be referred to a Stone's theorem , *and the mapping $\bar{\tau}$ will be called the* Stone extension of τ *into* Y.

To prove the theorem use use the following results.

Theorem 4.0.2. *Let X be dense in T. The following statements are equivalent.*

1. Every continuous mapping τ from X into any compact space Y has an extension to a continuous mapping from T into Y.
2. X is C^*-embedded in T.
3. Any two disjoint zero-sets in X have disjoint closure in T.
4. For any two zero-sets Z_1 and Z_2 in X,$cl_T(Z_1 \cap Z_2) = cl_T Z_1 \cap cl_T Z_2$.

Proof. We commence with the proof of uniqueness. By above result, if T satisfies one of (I) - (V), it satisfies *all* of them. By (I), the identity mapping on X, which is a continuous mapping into the compact space T, has a Stone extension from all of βX into T; similarly, it has a Stone extension from T into βX. Since, *Let X be dense in each of the Hausdorff spaces S and T. If the identity mapping on X has continuous extensions σ from S into T, and τ from T into S, the σ is a homeomorphism onto, and $\sigma^{-1} = \tau$.*
We turn now to the construction of βX. There is to be a one-one correspondence between the z-filter covering to its corresponding point. Now, we have such a correspondence between the fixed z-ultrafilters and the points of X, hence X constitutes a ready-made index set for the fixed z-ultrafilters. We increase it in any convenient way to an index set for the family of all z-ultrafilters.

a. *The points of βX are defined to be the elements of this enlarged indes set.*

   The family of all z-ultrafilters on X is written

$$(A^p)_{p \in \beta X}$$

with the understanding that for $p \in X$, A^p represents the (fixed) z-ultrafilter with limit p (i.e., the family of all zero-sets containing p). When emphasis is desirable, we shall denote A^p by A_p, for $p \in X$; thus,$A_p = Z[M_p]$. The topology on βX will be defined in such a way that p is the limit of the z-ultrafilter A^p for every $p \in \beta X$, not only for $p \in X$.

In what follows, Z will always stand for a zero-sets in the given topological space X. Let us write

$$\bar{Z} = \{p \in \beta X : Z \in A^p\}$$

,

that is, $p \in \bar{Z}$ if and only if $Z \in A^p$. In particular, since X itself belongs to every z-ultrafilter, we have $\bar{X} = \beta X$.

We know that $Z_1 \cup Z_2 \in A^p$ if and only if $Z_1 \in A^p$ or $Z_2 \in A^p$; therefore

$$\bar{Z}_1 \cup \bar{Z}_2 = \overline{Z_1 \cup Z_2}.$$

And since θ belongs to no z- ultrafilter, $\bar{\theta} = \theta$. Thus, the family of set $\bar{Z}$ is closed under finite union and contains the empty set.

b. *βX is made into a topological space by taking the family of all sets $\bar{Z}$ as a base for the closed sets.*

Let us verify that the X is subspace βX. Evidently, $p \in \bar{Z} \cap X$ if and only if $Z \in A_p$, which is to say that $p \in Z$. So $\bar{Z} \cap X = Z$. Thus, the identity mapping on X carries the family of basic closed sets in the relative topology onto a family; therefore it is a homeomorphism.

Next, we show that X is dense in βX. In fact, we shall prove, more generally, that

$$cl_{\beta X} Z = \bar{Z}$$

, from which the conclusion $cl_{\beta X} = \bar{X} = \beta X$ follows. We know that $Z \subset \bar{Z}$, whence $clZ \subset \bar{Z}$. On the other hand, for every basic closed set $\bar{z'}$ containing Z, we have

$$Z^{'} = \bar{Z}' \cap X \supset Z,$$

, so that $\bar{Z}' \supset \bar{Z}$. Therefore, $clZ \supset \bar{Z}$.

We now have:

c. $p \in cl_{\beta X} Z$ if and only if $Z \in A^p$.

Since $Z_1 \cap Z_2\ inA^p$ if and only if $Z_1 \in A^p$ and $Z_2 \in A^p$, this immediately yields (IV). According, the proof will be complete as soon as we know that βX is compact - for then, as already mentioned, all five of the listed conditions will hold.

To see, first of all, that βX is a Hausdorff space, consider any two distinct points p and $p^{'}$. Choose disjoint zero-sets $A \in A^p$ and $A^{'} \in A^{p'}$ there exist a zero-set Z disjoint from A, and a zero-set $Z^{'}$ disjoint from $A^{'}$, such that $Z \cup Z^{'} = X$. Evidently, $Z \notin A^p$ and $Z^{'} \notin A^{p'}$; that is to say, $p \notin clZ$ and $p^{'} \notin clZ^{'}$. Since

$$clZ \cup clZ^{'} = \beta X$$

, the neighborhoods $\beta X - clZ$ of p, and $\beta X - clZ^{'}$ of $p^{'}$, are disjoint.

Finally, consider any collection of basic closed sets clZ with the finite intersection property, Z ranging over some family $\mathscr{B}$. By (IV), already established, $\mathscr{B}$ itself also has the finite intersection property.

Consequently, $\mathscr{B}$ is embeddable in a z-ultrafilter A^p, and we have

$$p \in \cap_{z \in A^p} clZ \subset \cap_{z \in \mathscr{B}} clZ$$

, so that the latter intersection is nonempty. Therefore βX is compact.

$\square$

Remark 4.0.3. The space βX is know as the *Stone -Čech compactification* of X. Incidentally, the equivalance of (II) with (III) is an immediate consequence of Urysohn's extension theorem (because disjoint closed set in βX are completely separated).

Chapter 5

Some known results on $\beta\mathbb{N}$, $\beta\mathbb{Q}$ and $\beta\mathbb{R}$

Theorem 5.0.1. If S is open-and closed in X, then *$cl_{\beta X}S$ and $cl_{\beta X}(X - S)$ are complementary open sets in βX.*

Theorem 5.0.2. *An isolated point of X is in βX; and X is open in βX if and only if X is locally compact.*

Theorem 5.0.3. *S is C^*- embedded in X if and only if $cl_{\beta X}S = \beta S$. Where S is subspace-not necessarily dense of X*

By theorem 5.0.2, $\mathbb{N}$ is open in $\beta\mathbb{N}$, and $\mathbb{R}$ is open in $\beta\mathbb{R}$, but $\mathbb{Q}$ is not open in $\beta\mathbb{Q}$. The space $\beta\mathbb{N}$. More specifically, every point of $\mathbb{N}$ is an isolated point of $\beta\mathbb{N}$. These are the only isolated points, of course, since $\mathbb{N}$ is dense in $\beta\mathbb{N}$.
By theorem 5.0.1, the closure of $\beta\mathbb{N}$ of every subset of $\mathbb{N}$ is open in $\beta\mathbb{N}$. The points p of $\beta\mathbb{N}-\mathbb{N}$ are in one-one correspondence with the free ultrafilters A^p on $\mathbb{N}$, with A^p converging to p. Hence every neighborhood of p meets $\mathbb{N}$ in a member of A^p. On the other hand, if $\mathbb{Z} \in A^p$, then cl($\mathbb{Z}$) is an open neighborhood of p. Next, $\beta\mathbb{N}$ is totally disconnected: given distinct points p and q, choose $\mathbb{Z} \in A^p - A^q$; then cl$\mathbb{Z}$ is an open-and-closed set containing p but not q.

By theorem 5.0.3, The subset N_1 of odd integers is C^*-embedded in $\mathbb{N}$. Therefore $clN_1 = \beta N_1$, and hence clN_1 is homeomorphic with $\beta\mathbb{N}$. Similarly for the subset N_2 of even integers. Thus, $\beta\mathbb{N}$ is expressible as the union of two disjoint copies of itself.

We can also decompose $\mathbb{N}$ into infinitely many disjoint infinite sets A_n $(n \in \mathbb{N})$. The sets clA_n are then disjoint open-and-closed subsets of $\beta\mathbb{N}$, and each is homeomorphic with $\beta\mathbb{N}$. Now, though, their union

$$T = \cap_n clA_n$$

is not all of $\beta\mathbb{N}$, as a compact space cannot be a union of infinitely many disjoint open sets. However, T is dense in $\beta\mathbb{N}$; in fact, $\mathbb{N} \subset T \subset \beta\mathbb{N}$, so that $\beta T = \beta\mathbb{N}$.

Let us choose a point $p_n \in clA_n - \mathbb{N}$, and define

$$D = \{p_1, p_2, ..\}$$

. Trivially, D is a discrete subspace of $\beta\mathbb{N}$; so D is homeomorphic with $\mathbb{N}$. Moreover, D is C^*-embedded (in fact, C-embedded) in T: to extend a function f on D to a continuous function on T, simply assing the constant value $f(P_n)$ to each point of clA_n. It follows that D is C^* -embedded in βT-which is $\beta\mathbb{N}$. By theorem 5.0.3, $cl_{\beta\mathbb{N}} = \beta D$. Now, since D is contained in the closed set $\beta\mathbb{N} - \mathbb{N}$, so is cl D; thus, $\beta\mathbb{N} \supset \mathbb{N} \supset \beta D$.

1. $\beta\mathbb{N} - \mathbb{N}$ contains a copy of $\beta\mathbb{N}$.

   Let f be a function in $C^*(\mathbb{N})$ that assumes all rational values in [0,1]. Then every real number in [0,1] belongs to the closure of $f[\mathbb{N}]$ is all of [0,1]. In particular, the cardinal of $\beta\mathbb{N}$ must be at least the cardinal c of the continuum. (As a matter of fact, its cardinal is 2^c).

   The space $\beta\mathbb{Q}$. This space, too, is totally disconnected. For, distinct points p and q are contained in disjoint closed neighborhoods U and V, respectively. Then $U \cap \mathbb{Q}$ and $V \cap \mathbb{Q}$ are disjoint closed sets in $\mathbb{Q}$. Consequently, there is an open-and-closed set E in $\mathbb{Q}$ containing $U \cap \mathbb{Q}$ and disjoint from $V \cap \mathbb{Q}$. The open-and-closed set $cl_{\beta\mathbb{Q}}E \in \beta\mathbb{Q}$ then contains p but not q.

Any mapping τ of $\beta\mathbb{N}$ onto $\mathbb{Q}$ is continuous mapping into the compact space $\beta\mathbb{Q}$; as such, it has a Stone extension $\overline{\tau}$ from all of $\beta\mathbb{N}$ into $\beta\mathbb{Q}$. Since the range of $\overline{\tau}$ is a compact set in $\beta\mathbb{Q}$, and contains the dense set $\mathbb{Q}$, it must be all of $\beta\mathbb{Q}$. Thus, $\beta\mathbb{Q}$ is continuous image of $\beta\mathbb{N}$. On the other hand, by theorem 5.0.3, implies that

$$cl_{\beta\mathbb{Q}}\mathbb{N} = \beta\mathbb{N}$$

. Therefore $\beta\mathbb{N}$ is equipotent with $\beta\mathbb{N}$.

Since $\beta\mathbb{Q}$ is compact, every neighborhood of a point contains a compact neighborhood. But, clearly, no compact neighborhood can be contained entirely in $\mathbb{Q}$. It follows that $\beta\mathbb{Q} - \mathbb{Q}$ is dense in $\beta\mathbb{Q}$. (This argument is general and shows that if T is any locally compact space containing $\mathbb{Q}$, then T - $\mathbb{Q}$ is dense in T.)

The space $\beta\mathbb{R}$. As above, $\beta\mathbb{R}$ is continuous image of $\beta\mathbb{N}$, and

2.

$$cl_{\beta\mathbb{R}}N = \beta\mathbb{N}$$

Hence $\beta\mathbb{R}$ is equipotent with $\beta\mathbb{N}$.

Let $\mathbb{R}^+$ denote the subspace of all nonnegative reals, and $\mathbb{R}^-$ the subspace of non-positive reals. Since the closure of a connected set is always connected, $cl_{\beta\mathbb{R}}R^+$ and $cl_{\beta\mathbb{R}}R^-$, as well as $\beta\mathbb{R} = cl\mathbb{R}$, are connected.

Trivally, R^+ is C^*-embedded in $\mathbb{R}$, so that $clR^+ = \beta R^+$. Since R^+ is homeomorphic with R^-, clR^+ is homeomorphic with clR^-, and $clR^+ - R^+$ with $clR^- - R^-$. Every neighborhood of a point of $clR^+ - R^+$ meets R^+ is locally compact, it is open in clR^+R, whence $clR^+ - R^+$ is compact.

Obviously, $clR^+ \cup clR^- = \beta\mathbb{R}$. The arctangent function in $C(\mathbb{R})$ has a continuous extension to all of $\beta\mathbb{R}$; clearly, this extension assumes the sole value $\frac{\pi}{2}$ on $clR^+ - R^+$, and the sole value $-\frac{\pi}{2}$ on $clR^- - R^-$. Therefore $\beta\mathbb{R} - \mathbb{R}$.

Finally, we show that $clR^+ - R^-$ is connected. If not, it admits a continuous function that assumes precisely the values 0 and 1. This has an extension to a function $f \in C(clR^+)$ (since $clR^+ - R^+$) is compact), and f must assume the value $\frac{1}{2}$ on an unbounded set in R^+, and hence also at some point of $clR^+ - R^-$. This contradiction shows that $clR^+ - R^-$ is connected. Thus, $\beta\mathbb{R} - \mathbb{R}$ is the union of two disjoint, homeomorphic connected sets.

Chapter 6

Applications of the Stone-Čech compactification βX

Theorem 6.0.1. *Banach-Stone theorem. For any two compact spaces X and $Y \cong y$ if and only if $C(X) \cong C(Y)$.*

Proof. Suppose that two compact space X and Y are homeomorphism then $C(X) \cong C(Y)$. Conversely, suppose that $C(X) \cong C(Y)$ for any two compact space X and Y. Then $\mathcal{M}(C(X)) \cong \mathcal{M}(C(Y))$, that is $\beta X \cong \beta Y$, that is $X \cong Y$.

□

Theorem 6.0.2. *Gelfand-Kolmogorav theorem: Let $M^p = \{f \in C(X) : p \in cl_{\beta X} Z(f)\}$. The maximal ideals of the ring $C(X)(p \in \beta X)$ are precisely M^p's $(p \in \beta X)$.*

Proof. □

Bibliography

[1] Munkres, J.R. (2000) *Topology: a First course*,Prentice - Hall of India Ltd, New Delhi.

[2] Willard, S. (1970) *General Topology,* Addision Wesley Publishing Company, Massachusetts.

[3] Gillman Leonard, Jerson Meyer (1960), *Rings of Continous functions*, Dover Publication Inc, New York.

[4] Simmons, G.F. (2003). *Introduction of Topology and Modern Analysis* (4th edition), McGraw Hill.