Chapter 4: Approximation
Fall 2020

1.- Let E be a real vector space and $\langle \cdot, \cdot \rangle$ a scalar product on E. Let $\{\varphi_0, \varphi_1, \ldots, \varphi_n\}$ be an orthogonal system. Prove that, for any $(c_0, \ldots, c_n) \in \mathbb{R}^{n+1}$,

$$\left\| \sum_{i=0}^{n} c_i \varphi_i \right\|^2 = \sum_{i=0}^{n} c_i^2 \|\varphi_i\|^2.$$

2.- Let E be a real vector space and $\langle \cdot, \cdot \rangle$ a scalar product on E. Prove that two elements f, g of E verify $\langle f, g \rangle = 0$ if and only if $\|\alpha f + g\| \geq \|g\|$ for all $\alpha \in \mathbb{R}$.

3.- Let E be the real linear space $C^0([0,1])$ with the scalar product

$$\langle f, g \rangle = \int_0^1 f(x) g(x)\, dx.$$

Let E^* be the linear subspace of E generated by the functions 1, e^x and e^{-x}.

a) Derive an orthonormal basis of E^*.
b) Compute the best approximation $f^* \in E^*$ to $f(x) = x^2$.
c) Compute $\|f - f^*\|$.

4.- Let E be the space of real functions, defined on $[0,1]$, that are of C^1 class, and define

$$\langle f, g \rangle = f(0)g(0) + \int_0^1 f'(x) g'(x)\, dx. \tag{2}$$

a) Show that $\langle \cdot, \cdot \rangle$ is a scalar product. Is $(f, g) \mapsto \int_0^1 f'(x) g'(x)\, dx$ a scalar product?
b) Using the scalar product (2), derive an orthogonal basis of $\mathbb{R}_2[x]$.
c) Find the best approximation to $f(x) = \cos x$ by an element of $\mathbb{R}_2[x]$.
d) Find the best approximation to $f(x) = \cos x$ by an element of $\{p \in \mathbb{R}_2[x] \ / \ p(0) = 1\}$.

5.- Let us define the set $E = \{f \in C^1_{\mathbb{R}}([-1,1]) \ / \ f(-x) = -f(x) \ \forall x \in [-1,1]\}$, and let us consider the map from $E \times E$ to $\mathbb{R}$ given by

$$(f, g) \in E \times E \mapsto \langle f, g \rangle = \int_{-1}^{1} f'(x) g'(x)\, dx$$

a) Is E a vector space? Is $\langle \cdot, \cdot \rangle$ a scalar product on $C^1_{\mathbb{R}}([-1,1])$? Is $\langle \cdot, \cdot \rangle$ a scalar product on E? Let us define $E_3 = \{p \in E \ / \ p$ is a third degree polynomial $\}$. Is E_3 a vector space? If so, find a basis.
b) Let us define $f_0(x) = \sin x$. Compute the element of E_3 closest to f_0 according to the norm induced by the scalar product $\langle \cdot, \cdot \rangle$.

c) Let su define $F = \{f \in E \ / \ f'(0) = 1\}$, and $F_3 = \{p \in F \ / \ p$ is a third degree polynomial$\}$. Compute the element of F_3 closest to $f_0(x) = \sin x$. Is it possible to use the same scheme to find the element of F_3 closest to $f_0(x) = \cos x$?

6.- Consider the real vector space $E = \mathcal{C}^0([-1,1])$ endowed with the scalar product

$$\langle f, g \rangle = \int_{-1}^{1} w(x) f(x) g(x) dx,$$

where $w(x) > 0$ for all $x \in [-1,1]$ and it is an even and continuous function.

1. Let $f \in E$ be an odd function and $p(x)$ the degree $\leq n$ polynomial which gives the best approximation to the function f with respect to the given scalar product. Prove that $p(x)$ is also an odd function. If f és even prove that $p(x)$, is even. Hint: Any polynomial can be represented as the sum of its even and odd part.
2. Prove that the orthogonal polynomials of odd degree are odd functions and that of even degree are even functions.
3. Suppose that $w(x) = 1 + x^2$. By using the suitable orthogonal polynomials, compute the polynomial of degree ≤ 4 which is the best approximation of the function $f(x) = x^7$. Determine the error of the approximation.

7.- Consider the vector space $C^0([0,1])$ endowed with the scalar product $\langle f, g \rangle = \int_0^1 f(x) g(x)\, dx$. Let f be an element of $C^0([0,1])$ such that $\int_0^1 x f(x)\, dx = \frac{1}{2}$ and $\int_0^1 x^2 f(x)\, dx = \frac{1}{3}$.

a) Suppose that $\int_0^1 f(x)\, dx = 0$, and define $M_2 = \{p \in \mathbb{R}_2[x] \ / \ \int_0^1 p(x)\, dx = 0\}$. Find the best approximation to f by an element of M_2.

b) Suppose that $\int_0^1 f(x)\, dx = 1$, and define $D_2 = \{p \in \mathbb{R}_2[x] \ / \ \int_0^1 p(x)\, dx = 1\}$. Find the best approximation to f by an element of D_2.

8.- Compute the monic orthogonal polynomials of degrees 0, 1, 2, 3 and 4, corresponding to the scalar product

$$\langle f, g \rangle = \int_{-1}^{1} (1 + x^2) f(x) g(x)\, dx.$$

9.- Consider $E = C^0([-1,1])$ with the scalar product

$$\langle f, g \rangle = \int_{-1}^{1} \frac{f(x) g(x)}{\sqrt{1 - x^2}}\, dx.$$

Let $f \in E$ defined by $f(x) = \sqrt{1 - x^2}$.

a) Compute the best approximation to f by a 4th degree polynomial p_4.

b) Compute $\|f - p_4\|$, where $\|.\|$ is the norm associated to the previous scalar product.

10.- Let M_n be the set of monic polynomials of degree n, and let us consider the space of real polynomials endowed with the norm

$$\|f\|_2 = \left[\int_{-1}^{1} |f(x)|^2 \, dx\right]^{\frac{1}{2}}.$$

a) If $n = 2$, compute the element of M_n of minimum norm.

b) Compute the element of M_n of minimum norm for an arbitrary value of n (hint: use Legendre polynomials).

Let $A_2 \subset \mathbb{R}_2[x]$ such that $\displaystyle\int_{-1}^{1} p(x)\, dx = 2$.

c) Compute the element of A_2 of minimum norm.

d) Compute the best approximation of x^2 in A_2.

11.- Consider $E = C^0([-1,1])$ with the scalar product $\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\, dx$, and let us denote by $\|.\|$ the associated norm. Let us define M as $\{p \in \mathbb{R}_2[x] \;/\; p(0) = 1\}$.

a) Obtain the element of M of minimum norm. Write the value of this minimum norm.

b) Compute the element of M which is closest to $f(x) = e^x$.

c) If $\bar{M} = \{p \in \mathbb{R}_2[x] \;/\; p(0) = p'(0) = 1\}$, compute the element of $\bar{M}$ which is closest to $f(x) = e^x$.

12.- Let A be a real matrix with n columns and m rows ($m > n$) with full rank, and let us consider the (overdetermined) linear system $Ax = b$ ($x \in \mathbb{R}^n$, $b \in \mathbb{R}^m$). We are interested in computing the vector $x^* \in \mathbb{R}^n$ such that the norm $\|Ax^* - b\|_2$ is small as possible.

a) Write the previous problem as an approximation problem (that is, give two suitable spaces $E^* \subset E$ such that $x^* \in E^*$ is the best approximation to an element of E by an element of E^*).

b) Write the normal equations for the problem of a).

c) Use b) to show that x^* satisfies $A^T A x^* = A^T b$.

d) Explain what happens if A does not have full rank.

13.- Let us define E as $C^0([0, 2\pi])$ with the scalar product

$$\langle f, g\rangle = \int_{0}^{2\pi} f(x)g(x)\, dx.$$

Let us denote by $\|.\|$ the norm associated to this scalar product. Let E_n be the linear subspace of E generated by the functions

$$\{1, \sin(x), \cos(x), \sin(2x), \cos(2x), \ldots, \sin(nx), \cos(nx)\}.$$

a) Compute the best approximation $f_n \in E_n$ to the function $f(x) = x$.

b) Compute $\|f - f_n\|$ and $\lim_{n\to\infty} \|f - f_n\|$. Hint: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

14.- Consider the discrete scalar product defined on $\mathbb{R}_m[x]$,

$$\langle f, g\rangle = \sum_{i=0}^{m} f(x_i)g(x_i), \quad \text{where} \quad x_i = -1 + \frac{2i}{m}.$$

a) Derive, for $m = 4$, the orthogonal polynomials of degrees 0, 1 and 2.

b) Obtain the least-squares approximation polynomials of degrees 1 and 2 to the table

x	-1.0	-0.5	0.0	0.5	1.0
$f(x)$	1.1	0.3	0.1	0.2	0.9

15.- The table

x	0.00	0.20	0.40	0.60	0.80	1.00
$f(x)$	0.76	1.50	0.90	-0.30	-0.38	0.76

corresponds to a periodic signal of period 1. Adjust these values by a function of the type $a + b\sin(2\pi x) + c\cos(2\pi x)$.

16.- We define $E = \{f \in \mathcal{C}^2([-1, 1])$ such that $f(0) = f'(0) = 0\}$ and consider the map from $E \times E$ to $\mathbb{R}$ given by

$$(f, g) \in E \times E \mapsto \langle f, g\rangle = \int_{-1}^{1} f''(x)g''(x)dx.$$

a) Is E a vector space? Is $\langle\ ,\ \rangle$ a scalar product on E?

b) Define $E_3 = \{p \in E$ such that p is a polynomial of degree $3\}$. Is E_3 a linear subspace of E? If so, find a basis.

c) Find the best aproximation to $f(x) = \cos x - 1$ by an element of E_3.

17.- We define $E = \mathcal{C}^1([-1, 1])$ endowed with the scalar product

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)dx + \int_{-1}^{1} f'(x)g'(x)dx,$$

1. Prove that $\langle\cdot,\cdot\rangle$ is a scalar product.
2. Prove that the monic orthogonal polynomials of even degree are even functions and those of odd degree are odd functions.
3. Find the monic orthogonal polynomials ofdegrees 0, 1, 2 and 3.
4. Find the best approximation to $f(x) = x^7$ by a forth degree polynomial.

18.- Let $[a, b]$ be and interval. If $m \geq 1$ we define the space $E_{m+1} \subset C([a, b])$ spanned by the system of linearly independent maps $g_0, \ldots, g_m$, where $g_i : [a, b] \to \mathbb{R}$ is continuous for all i. Let $x_0, x_1, \ldots, x_m \in [a, b]$ be $m + 1$ distinct points in $[a, b]$.

1. Show that $\langle \cdot, \cdot \rangle$ defined by

$$\langle f, g \rangle = \sum_{i=0}^{m} f(x_i) g(x_i), \quad \forall f, g \in E_{m+1}, \tag{3}$$

is a scalar product on E_{m+1}, for any choice of distinct points $x_i \in [a, b]$ in E_{m+1} iff for all $f \in E_{m+1} \setminus \{0\}$ f has at most m zeros in $[a, b]$.

2. Suppose that (3) is a scalar product for any choice of $x_0, \ldots, x_m \in [a, b]$, where not two of them are the same. For arbitrary values $y_0, \ldots, y_m \in \mathbb{R}$ show that there exists a unique $g \in E_{m+1}$ such that $y_i = g(x_i)$, $i = 0, \ldots, m$.

3. For arbitrary points $(x_0, y_0), \ldots, (x_m, y_m) \in \mathbb{R}^2$, where $x_0 < x_1 < \cdots < x_m$, we consider the following problem: Find $g_0 \in E_n$, $(n \leq m+1)$ such that

$$\sum_{i=0}^{m} (y_i - g_0(x_i))^2 \leq \sum_{i=0}^{m} (y_i - g(x_i))^2, \quad \forall g \in E_n.$$

If (3) is a scalar product for any choice of distinct values x_i, $i = 0, \ldots, m$, expalin how we can transform this problem to a problem of approximation in a pre-Hilbert space.

4. We have computed the following table of values of a map $f : [0, 3] \to \mathbb{R}$:

x	0	1	2	3
$f(x)$	3	6	16	41

With this table we want to find a map $g_0(x) = a_0 + b_0 e^x$ such that

$$\sum_{i=0}^{3} (f(i) - g_0(i))^2 = \min_{g \in E_2} \sum_{i=0}^{3} (f(i) - g(i))^2,$$

where E_{m+1} is the linear space spanned by $\varphi_j(x) = e^{jx}$, $j = 0, \ldots, m$. Show that this problem can be posed as an approximation problem in pre-Hilbert spaces and solve it using the orthogonal projection theorem.