Chapter 1: Iterative methods for linear systems
Solutions

Exercise 17.-

a) We consider any basis of $\mathbb{R}^n$: $u_1, \ldots, u_n$ and define the vectors v_i in the following way:

$$v_1 = u_1, v_2 = u_2 - \alpha_{21} v_1, v_3 = u_3 - \alpha_{31} u_1 - \alpha_{32} u_2, \; \ldots, v_n = u_n - \alpha_{n1} u_1 - \cdots - \alpha_{n,n-1} u_{n-1}.$$

If we impose the conditions, we have

$$v_2^T A v_1 = u_2^T A u_1 - \alpha_{21} u_1^T A u_1 = 0,$$

which implies that

$$\alpha_{21} = \frac{u_2^T A u_1}{u_1^T A u_1}.$$

In general,

$$\alpha_{ij} = \frac{u_i^T A v_j}{v_j^T A v_j}, \qquad i > j.$$

Note that $v_j^T A v_j > 0$, since A is positive definite. Of course, there are many ways to obtain vectors with this property.

b) $v_1, \ldots, v_n$ is a basis, since if there exist $\lambda_1, \ldots, \lambda_n$ such that

$$\lambda_1 v_1 + \cdots + \lambda_n v_n = 0,$$

we have

$$0 = v_i^T A(\lambda_1 v_1 + \cdots + \lambda_n v_n) = \lambda_i v_i^T A v_i,$$

and as $v_i^T A v_i > 0$, we deduce that $\lambda_i = 0$, $i = 1, \ldots, n$, which means that the vectors are linearly independent.

c) Recall that $p^{(i)} = v_{i+1}$ and

$$\alpha_k = \frac{(p^{(k)})^T (A x^{(k)} - b)}{(p^{(k)})^T A p^{(k)}}.$$

We write the solution $\bar{x}$ and $x^{(0)}$ in the basis $\{v_1, \ldots, v_n\}$:

$$\bar{x} = \sum_{i=1}^{n} a_i v_i, \qquad x^{(0)} = \sum_{i=1}^{n} b_i v_i.$$

We have that

$$\alpha_0 = \frac{v_1^T (A x^{(0)} - b)}{v_1^T A v_1} = \frac{b_1 v_1^T A v_1 - v_1^T b}{v_1^T A v_1} = b_1 - \frac{v_1^T b}{v_1^T A v_1},$$

using that $v_i^T A v_j = 0$, si $i \neq j$. Therefore,

$$x^{(1)} = x^{(0)} - \left[b_1 - \frac{v_1^T b}{v_1^T A v_1} \right] v_1 = \sum_{i=1}^{n} b_i v_i - \left[b_1 - \frac{v_1^T b}{v_1^T A v_1} \right] v_1,$$

and the first coordinate of $x^{(1)}$ is

$$\frac{v_1^T b}{v_1^T A v_1}.$$

On the other hand, as $A\bar{x} = b$, we have

$$v_1^T A \sum_{i=1}^{n} a_i v_i = a_1 v_1^T A v_1 = v_1^T b.$$

This implies that

$$a_1 = \frac{v_1^T b}{v_1^T A v_1},$$

which is the result we desired.

d) We have seen that taking $p^{(0)} = v_1$ we obtain

$$x^{(1)} = \frac{v_1^T b}{v_1^T A v_1} v_1 + \sum_{i=2}^{n} b_i v_i.$$

Suppose that $p^{(0)} = v_i$. Now, we have

$$\alpha_0 = \frac{v_i^T (A x^{(0)} - b)}{v_1^T A v_1} = b_i - \frac{v_i^T b}{v_i^T A v_i},$$

and

$$x^{(1)} = \frac{v_i^T b}{v_i^T A v_i} v_i + \sum_{j=1, j \neq i}^{n} b_j v_j.$$

In order to compute the i-th coordinate of $\bar{x}$ we do:

$$v_i^T A \sum_{i=1}^{n} a_i v_i = a_i v_i^T A v_i = v_i^T b.$$

Then, $a_i = \frac{v_i^T b}{v_i^T A v_i}$, which means that the i-th components of $\bar{x}$ and $x^{(i)}$ coincide.

e) Note that what we have done does not depends on the initial condition. Therefore, if we apply the method:

$$\begin{aligned}
x^{(1)} &= x^{(0)} - \alpha_0 v_1, \\
x^{(2)} &= x^{(1)} - \alpha_1 v_0, \\
&\cdots \\
x^{(j)} &= x^{(j-1)} - \alpha_{j-1} v_j,
\end{aligned}$$

in each step we only change a component, beginning with first one, after the second, and so on. Thus, afte n steps we obtain the solution, that is $\|x^{(n)} - \bar{x}\| = 0$.