Chapter 3: Nonlinear systems of equations
Fall 2020

1.- Consider $G:\mathbb{R}^n\to\mathbb{R}^n$, $\alpha\in(0,1)$, $r>0$ and let $\|\ \|$ be a norm of $\mathbb{R}^n$, such that $G(0)=0$ and

$$\|G(x)-G(y)\|\le\alpha\|x-y\|\qquad\forall\, x,y\in B_r:=\{z\in\mathbb{R}^n\ :\ \|z\|\le r\}.$$

1. Let $v\in\mathbb{R}^n$ be a vector. Prove that if $\|v\|\le(1-\alpha)r$ then the equation

$$z=G(z)+v$$

has a unique solution in B_r.

2. Take $n=2$, $\|\ \|_\infty$, $v\in\mathbb{R}^2$ and

$$G\begin{pmatrix} z_1\\ z_2\end{pmatrix}=\frac{1}{3}\begin{pmatrix}\sin(z_1+z_2)\\ \cos(z_1-z_2)-1\end{pmatrix}.$$

See that the equation $z=G(z)+v$ has a unique solution in $\mathbb{R}^2$.

3. If we take $z^0=(0,0)^T$ and $z^i=G(z^{i-1})+v$, $i=1,2,\dots$; say, as a function of $\|v\|_\infty$, how many iterates we have to perform in order to have an approximation o the solution with an error less than 10^{-6}, using the $\|\ \|_\infty$ norm.

2.- Consider the equations

$$\begin{aligned} x^3+y^2+x+y-2 &= 0,\\ xy^2+x^2+x-y &= 0,\end{aligned}$$

that have a solution close to $(0.4,0.8)$.

a) Compute this solution with an error smaller than 10^{-8} using the Newton method.

b) Compute this solution with a quasi-Newton method, using the Jacobian matrix only at the initial approximation $(0.4,0.8)$.

3.- Consider the system

$$Ax=b+\epsilon f(x), \tag{1}$$

where A is an $n\times n$ regular matrix such that $a_{ii}\ne0$, $i=1,\dots,n$, $b\in\mathbb{R}^n$, $f:\mathbb{R}^n\longrightarrow\mathbb{R}^n$, and $\epsilon\in\mathbb{R}$.

1. Suppose that there exist a constant $K>0$ such that $\|f(x)-f(y)\|_\infty\le K\|x-y\|_\infty$, for all $x,y\in\mathbb{R}^n$. If we write $A=L+D+U$, where L is strictly lower triangular, D is diagonal and U is strictly upper triangular, we define the following iterative method to find the solution of (1):

$$x^{(k+1)}=D^{-1}\left[b+\epsilon f(x^{(k)})-(L+U)x^{(k)}\right],\qquad k\ge0$$

(a) Prove that if $(x^{(k)})_{k\ge0}\to\alpha$ then $x=\alpha$ is the solution of (1).

(b) If A is strictly diagonally dominant by rows, prove that there exists $\epsilon_0 > 0$ sucb that if $|\epsilon| < \epsilon_0$, for any initial condition $x^{(0)} \in \mathbb{R}^n$, the sequence $(x^{(k)})_k$ has a limit and (1) has a unique solution. Determine ϵ_0 as a function of K, $\beta = \max_{1\le i\le n} \frac{1}{|a_{ii}|}\sum_{j=1,j\neq i}^{n} |a_{ij}|$, i $\gamma = 1/\min_{1\le i\le n} |a_{ii}|$.

2. Now we take

$$A = \begin{pmatrix} 3 & 5 \\ 2 & 3 \end{pmatrix}, \quad b = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad f(x) = \begin{pmatrix} \sin(x_1 + x_2) \\ \cos(x_2) \end{pmatrix},$$

where $x = (x_1, x_2)$.

(a) Prove that if $|\epsilon|$ is small enough, there exists a solution x of (1) close to the solution when $\epsilon = 0$.

(b) Find the Taylor expansion of the solution up to second order in ϵ.

4.- Consider the equations

$$\begin{aligned} x^3 + y^2 + x + y &= \lambda, \\ xy^2 + x^2 + x - y &= 0. \end{aligned}$$

Note that, for $\lambda = 0$, we have the solution $(x, y) = (0, 0)$.

a) Prove that there exists a value $\delta > 0$ and a function $\lambda \in]-\delta, \delta[\mapsto (\bar{x}(\lambda), \bar{y}(\lambda)) \in \mathbb{R}^2$ such that the values $(\bar{x}(\lambda), \bar{y}(\lambda), \lambda)$ solve the previous equation.

b) Compute $\bar{x}'(\lambda)$ and $\bar{y}'(\lambda)$ for $\lambda = 0$.

c) Use b) to compute an approximation to the solution of the equation for $\lambda = 0.01$.

d) Use the Newton method (you can use the same program as in Exercise 2) to solve the equation (with an error smaller than 10^{-8}) using the result of c) as seed. Compare the number of Newton iterations with those needed if the seed is $x^{(0)} = y^{(0)} = 0$.

5.- Consider the equation

$$f(x, \lambda) = x^2 + \sin(x) + \lambda = 0,$$

with $x = \lambda = 0$ as a solution.

a) Use the Implicit Function Theorem to show that the equation $f(x, \lambda) = 0$ defines a function $\lambda \mapsto \bar{x}(\lambda)$ near $x = \lambda = 0$.

b) Compute the first and second derivatives of function $\bar{x}(\lambda)$ at $\lambda = 0$. Use them to compute an approximation of the solution for $\lambda = 0.01$.

c) Find values (x^*, λ^*) satisfying $f(x^*, \lambda^*) = 0$ and such that the Implicit Function Theorem cannot be applied at (x^*, λ^*). Describe the solutions of $f(x, \lambda) = 0$ near (x^*, λ^*) and discuss if the function $\lambda \mapsto \bar{x}(\lambda)$ defined in a) exists.

d) Give a value $\lambda_0 > 0$ such that, if $|\lambda| < \lambda_0$ then there exists $x(\lambda)$ such that $f(x(\lambda), \lambda) = 0$ and $x(\lambda) \to 0$ when $\lambda \to 0$.

6.- Consider the following system of equations:

$$\begin{aligned} x + y^3 &= \lambda \\ y^2 - x^2 &= \lambda x - \lambda. \end{aligned}$$

It is easy to see that $(x, y, \lambda) = (0, 0, 0)$ and $(x, y, \lambda) = (1, -1, 0)$ are solutions of this system.

a) Show that, for λ close to 0, the system defines a function $\lambda \mapsto (x(\lambda), y(\lambda))$ in a suitable neighbourhood of $\lambda = 0$, such that $x(0) = 1$, $y(0) = -1$. Compute $(x'(0), y'(0))$ and $(x''(0), y''(0))$.

b) For $\lambda = 0$ and $\delta > 0$ small enough, say which is the correct answer, using the Implicit Function Theorem:

  i) There exists a solution close to $(x, y) = (0, 0)$ for all λ such that $|\lambda| < \delta$.

  ii) There exists a solution close to $(x, y) = (0, 0)$ for all λ such that $-\delta < \lambda \leq 0$.

  iii) There exists a solution close to $(x, y) = (0, 0)$ for all λ such that $0 \leq \lambda < \delta$.

7.- Consider the system of equations:

$$\begin{aligned} x + \mu \cos(x + y) &= \mu^2, \\ x + \frac{1}{2}\mu y + \mu x^2 + y^2 &= 0 \end{aligned}$$

1. If $\mu = 1$ ithe system has the solution $(x, y) = 0$. Prove that for μ clos to 1, the equations define a function $(\bar{x}(\mu), \bar{y}(\mu))$ in a suitable neighbourhood of $\mu = 1$. Compute $(\bar{x}'(1), \bar{y}'(1))$, $(\bar{x}''(1), \bar{y}''(1))$.
2. For $\mu = 0$ we also have a solution at $(x, y) = (0, 0)$. If $\delta > 0$ is small enough,use the Implicit Function Theorem to find which option is correct:
   (a) There exists a solution near $x = y = 0$, for all μ such that $|\mu| < \delta$.
   (b) There exists a solution near $x = y = 0$ when $-\delta < \mu \leq 0$.
   (c) There exists a solution near $x = y = 0$ when $0 \leq \mu < \delta$.

8.- Consider the system of equations:

$$\begin{aligned} x^4 + ax^2 + bx + c &= 0, \\ 4x^3 + 2ax + b + 2 &= 0, \\ 36x^4 + 12ax^2 + 4x + a^2 &= 0. \end{aligned}$$

1. It is immediate to see that the system has the solution $(x, a, b, c) = (0, 0, -2, 0)$. Prove that for a close to zero, the equation define infinitely differentiable functions $x(a)$, $b(a)$, $c(a)$, such that $x(0) = c(0) = 0$ i $b(0) = -2$.
2. Compute the Taylor expansion of $x(a)$, $b(a)$, $c(a)$ up to second order about $a = 0$.
3. Which of the functions $x(a)$, $b(a)$, and $c(a)$ do have a maximum or a mininum at $a = 0$?

9.- The equations

$$\begin{aligned} x^2+y^2 &= 1, \\ x^2+z^2 &= 1, \end{aligned}$$

have the solution $(x,y,z)=(0,1,1)$.

a) Is there a curve of solutions going through the point $(x,y,z)=(0,1,1)$?

b) If so, study if some of the variables (x, y and z) have a maximum/minimum when going through $(x,y,z)=(0,1,1)$.

10.- The equations

$$\begin{aligned} x^2+y^2 &= 2\lambda, \\ x^2+z^2 &= 2\lambda, \\ y^2+z^2 &= 2, \end{aligned}$$

have, for $\lambda=1$, the solution $(x,y,z)=(1,1,1)$. Discuss the existence of solutions for λ close to 1. If there exist a curve of solutions, compute its Taylor polynomial (with respect to λ) of degree 2.

11.- Assume that $f,g:\mathbb{R}^3\to\mathbb{R}$ are C^1 functions, and consider the equations

$$\begin{aligned} f(x,y,z)=0, \\ g(x,y,z)=0. \end{aligned}$$

a) Assume that there exists a point (x_0,y_0,z_0) such that $f(x_0,y_0,z_0)=g(x_0,y_0,z_0)=0$ and that the vectors $u=\nabla f(x_0,y_0,z_0)$ and $v=\nabla g(x_0,y_0,z_0)$ are different from zero. Moreover, if u and v are not parallel, do the surfaces $f=0$ and $g=0$ intersect on a curve?

b) Consider

$$\begin{aligned} x^4+y^4-2 &= 0, \\ xz+y-2 &= 0, \end{aligned}$$

and note that $(x,y,z)=(1,1,1)$ satisfies both equations. Study the existence of a curve of solutions near $(x,y,z)=(1,1,1)$. Discuss if one of the variables goes through a maximum or minimum at $(x,y,z)=(1,1,1)$ when we move along this curve.

12.- a) The equations

$$\begin{aligned} x^4+y^4 &= 2, \\ (x-1)^2+(y-1)^2 &= \lambda, \end{aligned}$$

have the solution $x=y=1$, $\lambda=0$. Study the existence of a curve of solutions going through this point and, using a Taylor polynomial of suitable degree (larger than 1), give an approximation to the solution (or solutions) for $\lambda=0.01$.

b) Let $f : (x, y) \in \mathbb{R}^2 \mapsto \mathbb{R}$ be a C^1 map, and assume we know a point $(x_0, y_0) \in \mathbb{R}^2$ such that $f(x_0, y_0) = 0$. If $\nabla f(x_0, y_0) \neq 0$ use the Implicit Function Theorem to show, if h is small enough, that the following equations admit two solutions.

$$\begin{aligned} f(x, y) &= 0, \\ (x - x_0)^2 + (y - y_0)^2 &= h^2, \end{aligned}$$