LLISTA 3 Equacions no lineals

1 $G:\mathbb{R}^n \to \mathbb{R}^n$ $\alpha\in(0,1)$ $r>0$ $G(0)=0$

$$\|G(x)-G(y)\| \leq \alpha\|x-y\| \quad \forall x,y\in B_r=\overline{B(0,r)}$$

a) $v\in\mathbb{R}^n$ Deu que si $\|v\|\leq(1-\alpha)r$ llavors l'equació $z=G(z)+v$ té una única solució a B_r

Definim x_0. Sigui $F(z)=G(z)+v$. (1)

$$\|F(x)-F(y)\| = \|G(x)+v-G(y)-v\| = \|G(x)-G(y)\| \leq$$

per HI

$$\leq \alpha\|x-y\|$$ és una α-contracció $\forall x,y\in B_r$

Definim $x_0=0$ i

$G(0)=0$ xHI

$$x_1=F(x_0)=F(0)=G(0)+v=v, \Rightarrow x_1=v$$

$\|x_1-x_0\|=\|v\|\leq(1-\alpha)\cdot r$ Ara, pel teorema del punt fix o contracció $\exists\bar{x}$ solució única de (1) $\bar{x}\in B_r$. A més,

Teorema 3.3.3 (deu en els apunts)

$$\|x_i-\bar{x}\| \leq \frac{\alpha^i}{1-\alpha}\|x_1-x_0\| = \underbrace{\frac{\alpha^i}{1-\alpha}}_{<1}\|v\|$$

(!)

b) Sigui $n=2$, $\|\cdot\|_\infty$ $v\in\mathbb{R}^2$ i $G\begin{pmatrix}z_1\\z_2\end{pmatrix}=\frac{1}{3}\begin{pmatrix}\sin(z_1+z_2)\\\cos(z_1-z_2)-1\end{pmatrix}$

Veiem que $z=G(z)+v$ te una única sol a $\mathbb{R}^2$

$$G\begin{pmatrix}0\\0\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

$$\|G(x)-G(y)\|_\infty = \frac{1}{3}\cdot\left\|\begin{matrix}\sin(z_1+z_2)-\sin(w_1+w_2)\\\cos(z_1-z_2)-\cos(w_1+w_2)\end{matrix}\right\|$$

Observem que $\|G(x)-G(y)\|_\infty \leq \alpha\cdot\|x-y\|_\infty$ on $\|DG(x)\|_\infty\leq\alpha$

$$DG\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \frac{1}{3} \begin{array}{l} \begin{pmatrix} \frac{1}{3}\cdot\cos(z_1+z_2) & \frac{1}{3}\cos(z_1+z_2) \\ -\frac{1}{3}\sin(z_1-z_2) & \frac{1}{3}\sin(z_1-z_2) \end{pmatrix} \begin{array}{l} \leftarrow \text{primera fila (!)} \\ \leftarrow \text{segona fila} \end{array} \\ \quad\quad \uparrow \text{ deriv respect } z_1 \quad\quad \uparrow \text{ deriv resp. } z_2 \end{array}$$

$\|DG(z)\|_\infty = \frac{1}{3}\max\{|\cos(z_1+z_2)| + |\cos(z_1+z_2)|, |\sin(z_1-z_2)| + |\sin(z_1-z_2)|\}$; $|\sin| \leq 1$, $|\cos| \leq 1$

Per tant $\left\|DG\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right\|_\infty \leq 2/3 \quad \forall (z_1, z_2) \in \mathbb{R}^2 \Rightarrow \boxed{\alpha = 2/3}$

El teorema del valor mig diu que si $F: \mathbb{R}^n \to \mathbb{R}^n$

$$\|F(x) - F(y)\| \leq \max_{z \in \overline{xy}} \|DF(z)\| \cdot \|x-y\|$$

on $\overline{xy}$ segment que uneix x i y.

Si volem que $\|F(x) - F(y)\| \leq K\|x-y\| \quad \forall x, y \in \overline{B(x_0, r)}$

serà cert si $\|DF(x)\| \leq K \quad \forall x \in \overline{B(x_0, r)}$

Observem que cal que $\|v\|_\infty \leq (1-\alpha)r$. Fixat v agafem

$r \geq \dfrac{\|v\|_\infty}{1-\alpha} \Rightarrow \exists$ punt fix únic en $\overline{B(0,r)}$, però

com r pot ser tan gran com vulgui $\Rightarrow \exists!$ punt fix

en $\mathbb{R}^2$ (fent tendir r a ∞)

c) Si prenem $z^0 = (0,0)^T$ i $z^i = G(z^{i-1}) + v \quad i = 1, 2, \dots$

Digueu en funció de $\|v\|_\infty$ el nº d'iterats necessaris

per obtenir un error inferior a 10^{-6} (utilitzeu $\|\,\|_\infty$)

Volem $\|z^{(i)} - \bar{z}\|_\infty \leq 10^{-6}$ on $F(\bar{z}) = \bar{z}$ $z^{(0)} = 0$ i

$z^{(i)} = F(z^{(i-1)})$

Com tenim $\boxed{\alpha = 2/3}$ aleshores:

ho imposem

$$\|z^{(i)} - \bar{z}\|_\infty \leq \frac{(2/3)^i}{1/3} \|v\|_\infty < 10^{-6}$$ per tant:

$$\Downarrow$$

$$\left(\frac{2}{3}\right)^i < \frac{1}{3\|v\|_\infty} \cdot 10^{-6} \rightarrow i \log(2/3) \leq \log(3\|v\|_\infty) + \log(10^{-6})$$

$$\Rightarrow \boxed{i \geq \frac{\log(3\|v\|_\infty) + \log 10^{-6}}{\log\left(\frac{2}{3}\right)}} \quad i \geq \frac{\log\left(\frac{10^{-6}}{3\|v\|_\infty}\right)}{\log\left(\frac{2}{3}\right)}$$

$\boxed{2}$ Sistema $\left.\begin{array}{l} x^3+y^2+x+y-2=0 \\ xy^2+x^2+x-y=0 \end{array}\right\}$ té sol. prop de $(0.4, 0.8)$

Fet amb PC

a) Calcular solució amb un error $< 10^{-8}$ utilitzant mètode de Newton (numèric)

$$z^{(k+1)} = z^{(k)} - [Df(z^k)]^{-1} \cdot f(z^k) \qquad z^{(0)} = \begin{pmatrix} 0.4 \\ 0.8 \end{pmatrix}$$

$$\left.\begin{array}{l} f_1(x,y) = x^3+y^2+x+y-2 \\ f_2(x,y) = xy^2+x^2+x-y \end{array}\right\} \quad f = (f_1, f_2)$$

$$Df(x,y) = \begin{pmatrix} 3x+1 & 2y+1 \\ y^2+2x+1 & 2xy-1 \end{pmatrix}$$ Resolem sistema ~~Inversa matriu~~ usant

Gauss ~~en resoldre~~ en el pas k.

Parem per $\|z^{(k)} - z^{(k-1)}\|_\infty < 10^{-8}$

$\|f(z^{(k)})\|_\infty < 10^{-8}$ Convergeix en 4 iterats

Escaneado con CamScanner

b) Resol utilitzant quasi Newton-method utilitzant la matriu Jacobiana en l'aproximació inicial:

$$z^{(k+1)} = z^{(k)} - \left[Df\begin{pmatrix}0.4\\0.8\end{pmatrix}\right]^{-1} f(z^{k}) \quad \text{quasi Newton}$$

5 iterats quasi Newton $z^{(k+1)} = z^{(k)} - [Df(z^{0})]^{-1} \cdot f(z^{0})$

$\boxed{3}$ Sistema $Ax = b + \varepsilon f(x)$ (1) $A_{n\times n}$ regular amb $a_{ii} \neq 0$ $\forall i$ $i=1,\dots,n$ i $b \in \mathbb{R}^n$, $f: \mathbb{R}^n \to \mathbb{R}^n$ i $\varepsilon \in \mathbb{R}$

3a) Suposem $\exists k>0$ t.q. $\|f(x)-f(y)\|_\infty < K\|x-y\|_\infty$ $\forall x,y \in \mathbb{R}^n$. Si escrivim $A = L+D+U$ on L triangular inferior estricte, D diagonal, U triangular sup. estricte. Definim mètode:

$$x^{(k+1)} = D^{-1}\left[b + \varepsilon f(x^{(k)}) - (L+U)x^{(k)}\right] \quad k \geq 0$$

a) Proveu que si $(x^{(k)})_{k\geq 0} \to \alpha$ aleshores $x=\alpha$ és solució de (1)

$\boxed{3}$ $Ax = b + \varepsilon f(x)$

$a_{ii} \neq 0$, $b \in \mathbb{R}^n$, $f: \mathbb{R}^n \to \mathbb{R}^n$, $\varepsilon \in \mathbb{R}$

a) $\| f(x) - f(y) \|_\infty \leq K \| x - y \|_\infty$, $A = L + D + U$

$$x^{(k+1)} = D^{-1} [b + \varepsilon f(x^{(k)}) - (L+U) x^{(k)}] \quad k \geq 0$$

1. Demo si $(x^{(k)})_{k\geq 0} \longrightarrow \alpha$ aleshores $x = \alpha$ és

solució de $Ax = b + \varepsilon f(x)$.

Volem veure $A\alpha = b + \varepsilon f(\alpha)$

Fem el límit:

$$\alpha = D^{-1} (b + \varepsilon f(\alpha) - (L+U)\alpha)$$
$$\Downarrow$$
$$D\alpha = b + \varepsilon f(\alpha) - (L+U)\alpha$$
$$\Downarrow$$
$$\underbrace{(L+U+D)}_{=A} \alpha = b + \varepsilon f(\alpha)$$

2. A estrictament diagonal per files. Demo que

$\exists \varepsilon_0$ t.q si $|\varepsilon| < \varepsilon_0$ $\forall x^{(0)} \in \mathbb{R}$

$(x^{(k)})_n$ té límit i aquest és únic.

Definim

$$\phi(x) = D^{-1} [b + \varepsilon f(x) - (L+U) x]$$

$$\phi(x) - \phi(y) = D^{-1} (\varepsilon f(x) - (L+U)x - \varepsilon f(y) + (L+U) y) -$$

$$- D^{-1} (L+U)(x-y)$$

Escaneado con CamScanner

Tem norma:

$$\|\phi(x)-\phi(y)\|_\infty \leq |\varepsilon| \, \|D^{-1}(f(x)-f(y)) - D^{-1}(L+U)(x-y)\|_\infty \leq$$

$$\leq |\varepsilon| \cdot \|D^{-1}(f(x)-f(y))\|_\infty + \|D^{-1}(L+U)(x-y)\|_\infty \leq$$

$$\leq |\varepsilon| \cdot \|D^{-1}\|_\infty \cdot \|f(x)-f(y)\|_\infty + \|D^{-1}(L+U)\|_\infty \cdot \|x-y\|_\infty \leq$$

$$\leq \left[K \cdot |\varepsilon| \cdot \|D^{-1}\|_\infty + \|D^{-1}(L+U)\|_\infty\right] \cdot \|x-y\|_\infty \leq$$

$$\leq (K \cdot \gamma |\varepsilon| + \beta)$$

Ara, si $K\gamma|\varepsilon| + \beta < 1$ ϕ és una contracció.

$$\boxed{\begin{array}{l} \beta = \max\limits_{1\leq i\leq n} \dfrac{1}{|a_{ii}|} \cdot \sum\limits_{j=1}^{n} |a_{ij}| = \|D^{-1}(L+U)\| < 1 \quad \text{D diag. dom.} \\[2ex] \gamma = \dfrac{1}{\min\limits_{1\leq i\leq n} |a_{ii}|} = \|D^{-1}\|_\infty \end{array}}$$

Aplicant el teorema de contracció o punt (fix)

$$\Downarrow$$

$$|\varepsilon| < \frac{1-\beta}{K\gamma} = \varepsilon_0$$

b)

$$A = \begin{pmatrix} 3 & 5 \\ 2 & 3 \end{pmatrix}, \quad b = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \qquad f(x) = \begin{pmatrix} \sin(x_1+x_2) \\ \cos(x_2) \end{pmatrix}$$

on $x = (x_1, x_2)$

1. Proveu que si ε és prou petita, $\exists$ solució x de $Ax = b + \varepsilon f(x)$, propera a la corresponent $\varepsilon = 0$

$$\left.\begin{array}{l} 3x_1 + 5x_2 = 2 + \varepsilon \sin(x_1+x_2) \\ 2x_1 + 3x_2 = 1 + \varepsilon \cos(x_2) \end{array}\right\} (1)$$

Si $\varepsilon = 0$:

$$3x_1 + 5x_2 = 2$$
$$2x_1 + 3x_2 = 1$$

té solució $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Demanen (1) respecte de ε:

$$(2) \begin{cases} 3x_1' + 5x_2' = \sin(x_1+x_2) + \varepsilon\cos(x_1+x_2)\cdot(x_1'+x_2') \\ 2x_1' + 3x_2' = \cos(x_2) - \varepsilon\sin(x_2)\, x_2' \end{cases}$$

$$\Big\Downarrow \; \varepsilon = 0$$

$$* \begin{cases} 3x_1' + 5x_2' = \sin(-1+1) + \varepsilon\cos(-1+1)\cdot(+1-1) = 0 \\ 2x_1' + 3x_2' = \cos(1) \end{cases}$$

$$F(x_1,x_2) = \begin{pmatrix} 3x_1 + 5x_2 - 2 - \varepsilon\sin(x_1+x_2) \\ 2x_1 + 3x_2 - 1 - \varepsilon\cos(x_2) \end{pmatrix}$$

$$F(-1,1,0) = 0$$

$$D_{(x_1,x_2)}F(-1,1,0) = \begin{pmatrix} 3 & 5 \\ 2 & 3 \end{pmatrix}$$

$$\det D_{(x_1,x_2)}F(-1,1,0) = -1 \neq 0$$

$*$

$$x_1'(0) = 5\cos(1)$$
$$x_2'(0) = -3\cos(1)$$

Ara, Derivem (2)

$$\begin{cases} 3x_1'' + 5x_2'' = \cos(x_1+x_2)\cdot(x_1'+x_2') + \varepsilon \cdot \cos(x_1+x_2) + \cos(x_1+x_2)\cdot(x_1'+x_2') \\ 2x_1'' + 3x_2'' = -\sin(x_2)\, x_2' - \varepsilon\left((\cos(x_2)\cdot(x_2')^2) + \sin(x_2)\, x_2''\right) \end{cases}$$

Fem $\varepsilon = 0$:

$$\begin{cases} 3x_1'' + 5x_2'' = 4\cos(1) \\ 2x_1'' + 3x_2'' = 6\sin(1)\cos(1) \end{cases} \Rightarrow \begin{array}{l} x_1''(0) = 30\sin(1)\cos(1) \\ x_2''(0) = -18\sin(1)\cos(1) + 8\cos(1) \end{array}$$

Per tant:

$$x_1(\varepsilon) = -1 + 5\cos(1) + (15\sin(1)\cos(1) - 6\cos(1))\,\varepsilon^2 + O(\varepsilon^2)$$

$$x_2(\varepsilon) = 1 - 3\cos(1)\,\varepsilon + (-9\cdot\sin(1)\cos(1) + 4\cos(1))\,\varepsilon^2 + O(\varepsilon^2)$$

$\boxed{12}$ Entrega:

Diferencial: $\begin{cases} \text{Gradient } \nabla f(x,y) = \begin{pmatrix} \partial f/\partial x \\ \frac{\partial f}{\partial y} \end{pmatrix} \in \mathbb{R}^2 \to \text{vector columna} \\ \text{diferencial}: \; Df(x,y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \in \mathbb{R}^2 \to \text{vector fila} \end{cases}$

vector tangent $\begin{pmatrix} \frac{\partial f}{\partial y} \\ -\frac{\partial f}{\partial x} \end{pmatrix}$

* Hem de guardar vector tangent i comparar amb l'anterior per veure que anem sempre en la mateixa direcció (sentit) xq volem recorre la corba.

× primera predicció $\bar{y}_1 = y_0 + hv$

4 $\left.\begin{array}{l} x^3+y^2+x+y=\lambda \\ xy^2+x^2+x-y=0 \end{array}\right\}$ (1)

per $\lambda=0$ tenim solució $(x,y)=(0,0)$

a) $\underbrace{\lambda \in]-\lambda,\lambda[}$ $\longrightarrow$ $(\bar{x}(\lambda),\bar{y}(\lambda)) \in \mathbb{R}^2$ t.q resol (1)

$\equiv \lambda \in (-\infty,-\lambda] \cup [\lambda,+\infty)$

Tenim $F(x,y) = \begin{pmatrix} x^3+y^2+x+y-\lambda \\ xy^2+x^2+x-y \end{pmatrix}$ $\quad F(0,0)=(0,0)$

$F=(f_1,f_2)$

$$DF(x,y) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial \lambda} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial \lambda} \end{pmatrix} = \begin{pmatrix} 3x^2+1 & 2y+1 & -1 \\ y^2+2x+1 & 2xy-1 & 0 \end{pmatrix}$$

$$DF(0,0) = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix}$$

$$D_{(x,y)}F(0,0) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \Rightarrow \det D_{(x,y)}F(0,0) = -2 \neq 0$$

$\Downarrow$

Pel TFI $\exists \delta>0$, $\exists \bar{x}(\lambda), \bar{y}(\lambda)$ t.q

$F(\bar{x}(\lambda),\bar{y}(\lambda)) = 0 \quad \forall \lambda \in (-\delta,\delta)$; $\bar{x}(0)=0 \quad \bar{y}(0)=0$

b) $\bar{x}'(\lambda)$, $\bar{y}'(\lambda)$?

Derivem (1) respecte de λ:

$$\left.\begin{aligned} 3\bar{x}(\lambda)^2\bar{x}'(\lambda) + 2\bar{y}(\lambda)\bar{y}'(\lambda) + \bar{x}'(\lambda) + \bar{y}'(\lambda) &= 1 \\ \bar{x}'(\lambda)\bar{y}(\lambda)^2 + 2\bar{x}(\lambda)\bar{y}(\lambda)\bar{y}'(\lambda) + 2\bar{x}(\lambda)\bar{x}'(\lambda) + \bar{x}'(\lambda) - \bar{y}'(\lambda) &= 0 \end{aligned}\right\} \Rightarrow$$

$\boxed{\lambda = 0}$

per a) $\bar{x}(0)=0$, $\bar{y}(0)=0$

$$\Rightarrow \left\{\begin{aligned} \bar{x}'(0) + \bar{y}'(0) &= 1 \\ \bar{x}'(0) - \bar{y}'(0) &= 0 \end{aligned}\right. \rightarrow$$

solució:

$$\boxed{\bar{x}'(0) = \frac{1}{2} \qquad \bar{y}'(0) = \frac{1}{2}}$$

c) Aproximació per $\lambda = 0.01$

$$\bar{x}'(\lambda) \overset{\text{Taylor}}{=} \bar{x}(0) + \bar{x}'(0)\lambda + O(\lambda^2) \overset{\bar{x}(0)=0}{=} \frac{1}{2}\lambda + O(\lambda^2)$$

$$\bar{y}'(\lambda) \underset{\text{taylor}}{=} \bar{y}(0) + \bar{y}'(0)\lambda + O(\lambda^2) \underset{\bar{y}(0)=0}{=} \frac{1}{2}\lambda + O(\lambda^2)$$

$\Downarrow$

l'aproximació de $(\bar{x}(0{,}01), \bar{y}(0.01))$ és $(0.005, 0.005)$

d) Newton

Amb 2 iterats arribo a la solució amb condició inicial $(0.005, 0.005)$. Amb condició inicial $(0,0)$ necessitem 3 iterats.

$\boxed{7}$

$$\left.\begin{array}{l} x+y^3=\lambda \\ y^2-x^2=\lambda x-\lambda \end{array}\right\}$$

$(x,y,\lambda)=(0,0,0) \qquad (x,y,\lambda)=(1,-1,0) \qquad$ solucions

a) $\lambda \simeq 0 \quad x(0)=1 \quad y(0)=-1$

Tenim $F(x,y)=\begin{pmatrix} x+y^3-\lambda \\ y^2-x^2-\lambda x+\lambda \end{pmatrix}$ (1)

$F=(f_1,f_2)$

$$DF(x,y)=\begin{pmatrix} 1 & 3y^2 & -1 \\ -2x & 2y & -x+1 \end{pmatrix}$$

$$DF(1,-1,0)=\begin{pmatrix} 1 & 3 & -1 \\ -2 & -2 & 0 \end{pmatrix} \Rightarrow D_{(x,y)}F(1,-1,0)=\begin{pmatrix} 1 & 3 \\ -2 & -2 \end{pmatrix} \rightarrow \text{regular}$$

Pel TFI, $\exists\, \delta>0$ t.q si $\lambda\in(-\delta,\delta)$ $\exists\, x(\lambda), y(\lambda)$

t.q $F(x(\lambda),y(\lambda),\lambda)=0 \quad \forall \lambda\in(-\delta,\delta)$; $x(0)=1 \quad y(0)=-1$

Derivem (1) respecte de λ:

$$(2)\quad \left.\begin{array}{l} x'+3y^2y'-1=0 \\ 2yy'-2xx'-x-\lambda x'+1=0 \end{array}\right\} \xrightarrow{\boxed{\lambda=0,\ x=1,\ y=-1}} \left\{\begin{array}{l} x'(0)+3y'(0)-1=0 \\ -2y'(0)-2x'(0)-1+1=0 \end{array}\right.$$

$$\Rightarrow \left.\begin{array}{l} x'(0)+3y'(0)=1 \\ -2x'(0)-2y'(0)=0 \end{array}\right\} \Rightarrow$$

solució: $\boxed{x'(0)=\frac{-1}{2} \qquad y'(0)=\frac{1}{2}}$

$\not{b})$ (?)

Derivem (2) respecte de λ:

$$\left.\begin{array}{l} x'' + 6y(y')^2 + 3y^2 y'' = 0 \\ 2(y')^2 + 2yy^2 - 2(x')^2 - 2xx'' - x' - x' - \lambda x'' = 0 \end{array}\right\} \Longrightarrow$$

$$\boxed{\begin{array}{l} \lambda = 0 \\ x = 1 \\ y = -1 \\ x'(0) = -1/2 \\ y'(0) = \frac{1}{2} \end{array}} \Longrightarrow \left\{\begin{array}{l} x''(0) + 6\cdot(-1)\cdot\frac{1}{4} + 3y''(0) = 0 \\ 2\cdot\frac{1}{4} - 2y''(0) - 2\cdot\frac{1}{4} - 2\cdot x''(0) + 1 = 0 \end{array}\right. \Rightarrow \left\{\begin{array}{l} x''(0) + 3y''(0) = 3/2 \\ -2x''(0) - 2y''(0) = -1 \end{array}\right.$$

$\Downarrow$

Solució

$$\boxed{\begin{array}{l} x''(0) = 0 \\ y''(0) = \frac{1}{2} \end{array}}$$

b) $DF(0,0,0) = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix}$ $\quad D_{(x,y)}F(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ matriu

singular (no podem aplicar TFI) i.e no podem posar x, y en funció de λ. Però ho podem arreglar:

$D_{(x,\lambda)}F(0,0) = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \Rightarrow$ regular posem x i λ en funció de y!

$\Downarrow$

Pel TFI $\exists \bar{\delta} > 0$, $\exists\, x(y), \lambda(y)$, $y \in (-\bar{\delta}, \bar{\delta})$ t.q

$F(x(y), y, \lambda(y)) = 0 \quad \forall y \in (-\bar{\delta}, \bar{\delta})$ i $\begin{array}{l} x(0) = 0 \\ \lambda(0) = 0 \end{array}$

Cerquem derivades de $x(y)$, $\lambda(y)$

Derivem (1) respecte de y: $\left(x' = \frac{dx}{dy} \text{ i } \lambda' = \frac{d\lambda}{dy} \right)$

$$\left.\begin{aligned} x' + 3y^2 - \lambda' &= 0 \\ 2y - 2xx' - \lambda' x - \lambda x' + \lambda' &= 0 \end{aligned}\right\} (3) \underset{(\lambda, x, y) = (0,0,0)}{\Longrightarrow} \left.\begin{aligned} x'(0) - \lambda'(0) &= 0 \\ x'(0) &= 0 \end{aligned}\right\}$$

$$\boxed{\begin{aligned} x'(0) &= 0 \\ \lambda'(0) &= 0 \end{aligned}}$$

Derivem (3) respecte de y:

$$\left.\begin{aligned} x'' + 6y - \lambda'' &= 0 \\ 2 - 2(x')^2 - 2xx' - \lambda'' x - \lambda' x' - \lambda' x' - \lambda x'' + \lambda'' &= 0 \end{aligned}\right\} \Rightarrow$$

$$\underset{\boxed{\begin{aligned} &(\lambda, x, y) = (0,0,0) \\ &x'(0) = 0 \\ &y'(0) = 0 \end{aligned}}}{\Longrightarrow} \left.\begin{aligned} x''(0) - \lambda''(0) &= 0 \\ 2 + \lambda''(0) &= 0 \end{aligned}\right\} \Rightarrow \boxed{\begin{aligned} x''(0) &= -2 \\ \lambda''(0) &= -2 \end{aligned}}$$

Per tant:

$$\lambda(y) = \frac{1}{2} \lambda''(0) y^2 + O(y^3) = -y^2 + O(y^3) \rightsquigarrow \underline{\underline{\lambda \text{ negativa}}}$$

si y és prou petita.

El correcte és situació 2 enunciat:

$\exists$ solució per $-\delta < \lambda \leq 0$

$\boxed{7}$

$$\left.\begin{array}{l} \overbrace{x^4+ax^2+bx+c}^{g_1}=0 \\ \underbrace{4x^3+2ax+b+2}_{g_2}=0 \\ \underbrace{36x^4+12ax^2+4x+a^2}_{g_3}=0 \end{array}\right\} (1)$$

a) $(x,a,b,c)=(0,0,-2,0)$

$x(a),\ b(a),\ c(a) \qquad x(0)=c(0)=0 \qquad b(0)=-2$

$$g(x,a,b,c)=\begin{cases} g_1(x,a,b,c) \\ g_2(x,a,b,c) \\ g_3(x,a,b,c) \end{cases}$$

$$Dg(0,0,-2,0)=\begin{pmatrix} -2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 4 & 0 & 0 & 0 \end{pmatrix}=\begin{pmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial a} & \frac{\partial g_1}{\partial b} & \frac{\partial g_1}{\partial c} \\ & \cdots & & \end{pmatrix}$$

tenim $\begin{vmatrix} -2 & 0 & 1 \\ 0 & 1 & 0 \\ 4 & 0 & 0 \end{vmatrix}=-4\neq 0$ Aplicant el TFI

Derivem (1) respecte de a tenint en compte que

$x=x(a)\ ,\ b=b(a)\ ,\ c=c(a)\ ,\ x(0)=0,\ b(0)=-2,\ C(0)=0$

$$\left.\begin{array}{l} 4x^3x'+x^2+2axx'+b'x+bx'+c'=0 \\ 12x^2x'+2x+2ax'+b'=0 \\ 144x^3x'+12x^2+24axx'+4x'+2a=0 \end{array}\right\} (2) =$$

$$=\left.\begin{array}{l} -2x'(0)+c'(0)=0 \\ b'(0)=0 \\ 4x'(0)=0 \end{array}\right\} \quad x'(0)=b'(0)=c'(0)=0 \quad (!) \rightarrow$$

Tornem a deriva ((2)) i avaluem a $a=0$

$$\begin{cases} 12x^2(x')^2 + 4x^3x'' + 2xx' + 2xx' + 2a(x')^2 + 2axx'' + b''x + b'x' + b'x' + bx'' + c'' = 0 \\ g_2'' \\ g_3'' \end{cases}$$

Ara avaluant en $a=0$ i usem també que $x'(0) = b'(0) = c'(0) = 0$ ← (!)

i queda:

$$\left.\begin{array}{r} -2x''(0) + c''(0) = 0 \\ b''(0) = 0 \\ 4x''(0) + 2 = 0 \end{array}\right\} \Rightarrow x''(0) = -1/2 \ , \ b''(0) = 0 \ \text{ i } \ c''(0) = -1$$

c) x i c tenen un màxim en $a=0$

Calculem $x'''(0)$, $b'''(0)$, $c'''(0)$ (com abans) i obtenim

$b'''(0) = 3$, per tant no té màxim ni mínim

$\boxed{9}$

$$\left.\begin{array}{l} x^2+y^2 = 2\lambda \\ x^2+z^2 = 2\lambda \\ y^2+z^2 = 2 \end{array}\right\}$$ Si $\lambda = 1$ tenim solució $(x,y,z) = (1,1,1)$

Existeixen sol. per λ proper a 1? Calcular polinomi de Taylor de grau 2.

~~Tenim~~ Calculem $DF(x,y,z,\lambda)$ i avaluem en $(1,1,1,1)$:

Tenim $F(x,y,z,\lambda) = 0$ ~~⇔~~ $F(x,y,z,\lambda) = \begin{pmatrix} x^2+y^2-2\lambda \\ x^2+z^2-2\lambda \\ y^2+z^2-2 \end{pmatrix}$

$$DF(x,y,z,\lambda) = \begin{pmatrix} 2x & 2y & 0 & -2 \\ 2x & 0 & 2z & -2 \\ 0 & 2y & 2z & 0 \end{pmatrix}$$

$$DF(1,1,1,1) = \begin{pmatrix} 2 & 2 & 0 & -2 \\ 2 & 0 & 2 & -2 \\ 0 & 2 & 2 & 0 \end{pmatrix}, \quad \det \underset{\substack{(x,y,z) \\ \shortparallel \\ (1,1,1)}}{DF} = -16 \neq 0 \Rightarrow$$

$\Rightarrow$ Usant TFI: Sí existeixen solucions per λ prop de 1.

•) Càlcul Polinomi Taylor

Ara $x = x(\lambda)$ $\quad y = y(\lambda)$ $\quad z = z(\lambda)$

$$\left.\begin{array}{l} 2xx' + 2yy' - 2 = 0 \\ 2xx' + 2zz' - 2 = 0 \\ 2yy' + 2zz' = 0 \end{array}\right\} \xRightarrow[(x,y,z)=(1,1,1)]{} \left.\begin{array}{l} 2x'(1) + 2y'(1) - 2 = 0 \\ 2x'(1) + 2z'(1) - 2 = 0 \\ 2y'(1) + 2z'(1) = 0 \end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{l} x'(1) = 1 \\ y'(1) = 0 \\ z'(1) = 0 \end{array}\right\} (!)$$

$\downarrow$ volem grau 2: (tornem a derivar)

$$\left.\begin{array}{l} 2(x')^2 + 2xx'' + 2(y')^2 + 2yy'' = 0 \\ 2(x')^2 + 2xx'' + 2(z')^2 + 2zz'' = 0 \\ 2(y')^2 + 2yy'' + 2(z')^2 + 2zz'' = 0 \end{array}\right\} \xRightarrow{(!) + (\lambda = 1)}$$

$$\left.\begin{array}{l} x = x(1) = 1 \\ x' = x'(1) = 1 \\ y' = y'(1) = 0 \\ y = y(1) = 1 \end{array}\right\} \rightarrow \left.\begin{array}{l} 2 + 2x''(1) + 2y''(1) = 0 \\ 2 + 2x''(1) + 2z''(1) = 0 \\ 2y''(1) + 2z''(1) = 0 \end{array}\right\} \Rightarrow \begin{array}{l} x''(1) = -1 \\ y''(1) = 0 \\ z''(1) = 0 \end{array}$$

Per tant, pol. de Taylor de grau 2:

$$\begin{pmatrix} x(\lambda) \\ y(\lambda) \\ z(\lambda) \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + (\lambda-1)\cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \frac{1}{2!}(\lambda-1)^2 \cdot \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}}_{\text{Pol. Taylor Grau 2}} + \overset{o(|\lambda-1|^3)}{\cancel{o(\lambda^3)}}$$

$\boxed{10}$ $f, g : \mathbb{R}^3 \to \mathbb{R}$, C^1

$f(x,y,z) = 0$

$g(x,y,z) = 0$

a) Suposem que $\exists (x_0, y_0, z_0)$ t.q. $f(x_0,y_0,z_0) = g(x_0,y_0,z_0) = 0$

i, $u = \nabla f(x_0,y_0,z_0) \in \mathbb{R}^3$ (gradient), $v = \nabla g(x_0,y_0,z_0) \in \mathbb{R}^3$

i $u, v \neq 0$ i <u>no</u> són para l·lels, aleshores

$f=0$ i $g=0$ superfícies intersequen en una corba?

Sí, en un entorn de (x_0, y_0, z_0) ja que:

1º) $\nabla f(x,y,z) := \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{pmatrix}$ i per tant

2º) $DF(x_0,y_0,z_0) = \begin{pmatrix} \frac{\partial f}{\partial x}(x_0,y_0,z_0) & \frac{\partial f}{\partial y}(x_0,y_0,z_0) & \frac{\partial f}{\partial z}(x_0,y_0,z_0) \\ \frac{\partial g}{\partial x}(x_0,y_0,z_0) & \frac{\partial g}{\partial y}(x_0,y_0,z_0) & \frac{\partial g}{\partial z}(x_0,y_0,z_0) \end{pmatrix}$

$= \begin{pmatrix} \nabla f(x,y,z)^T \\ \nabla g(x,y,z)^T \end{pmatrix}$ No paral·lels vol dir:

rang $DF(x_0,y_0,z_0) = 2$

- Si $D_{(x,y)}F(x_0,y_0,z_0)$ és regular, pel TFI $x=x(z)$ $y=y(z)$

$\downarrow$

Tenim una corba associada $(x(z), y(z), z)$

- Si $D_{(x,z)}F(x_0,y_0,z_0)$ és regular, anàlogament tenim corba $(x(y), y, z(y))$

- Si $D_{(y,z)}F(x_0,y_0,z_0)$ és regular... $(x, y(x), z(x))$

b) $x^4+y^4-2=0 \;\; /\!\!/ \; f(x,y,z)$

$g(x,y,z) = xz+y-2=0$ $\Big\}$ aleshores $(x,y,z)=(1,1,1)$

$$\nabla f(x,y,z) = \begin{pmatrix} 4x^3 \\ 4y^3 \\ 0 \end{pmatrix} \to \nabla f(1,1,1) = \begin{pmatrix} 4 \\ 4 \\ 0 \end{pmatrix}$$

$$\nabla g(x,y,z) = \begin{pmatrix} z \\ 1 \\ x \end{pmatrix} \Rightarrow \nabla g(1,1,1) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

No són paral·lels $\exists$ solució

$$DF(x,y,z) = \begin{pmatrix} 4 & 4 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

Podem posar $\left.\begin{array}{l} x=x(y) \\ z=z(y) \end{array}\right\}$ $\sim$ la derivada 1ª de x serà diferent de zero i calculem

z té mínim

$\boxed{11}$

a) $$\left.\begin{array}{l} x^4+y^4=2 \\ (x-1)^2+(y-1)^2=\lambda \end{array}\right\}(1) \qquad x=y=1 \qquad \lambda=0$$

$$F(x,y,\lambda)=\begin{pmatrix} x^4+y^4-2 \\ (x-1)^2+(y-1)^2-\lambda \end{pmatrix}$$

$$DF(x,y,\lambda)=\begin{pmatrix} 4x^3 & 4y^3 & 0 \\ 2(x-1) & 2(y-1) & -1 \end{pmatrix} \Rightarrow DF(1,1,0)=\begin{pmatrix} 4 & 4 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

com $\begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} \neq 0$, $D_{(y,\lambda)}F(1,1,0)$ és una matriu regular

Aplicant TFI $\Rightarrow \exists\, y(x), \lambda(x)$, $|x|$ prou petit.

$(x, y(x), \lambda(x))$ corba. Calculem derivades:

Derivem x respecte de (1)

$$\left.\begin{array}{l} 4x^3+4y^3y'(x)=0 \\ 2(x-1)+2(y-1)y'(x)=\lambda'(x) \end{array}\right\} \text{ Avaluem en } x=y=1,\ \lambda=0:$$

$$\Downarrow$$

$$\left\{\begin{array}{ll} 4+4y'(1)=0 & \Rightarrow y'(1)=-1 \\ 0=\lambda'(x) & \Rightarrow \lambda'(1)=0 \end{array}\right.$$

(com $\lambda'(1)=0$... Tornem a derivar:

$$\left.\begin{array}{l} 12x^2+12y^2\cdot(y'(x))^2+4y^3y''(x)=0 \\ 2+2(y)\cdot(y'(x))^2+2(y-1)y''(x)=\lambda''(x) \end{array}\right\} \begin{array}{l} \text{Avaluem en} \\ x=1 \quad y(1)=1 \quad \lambda(1)=0 \\ y'(1)=-1 \quad \lambda'(1)=0 \end{array}$$

$$\Downarrow$$

$$\left.\begin{array}{l} 12+12+4y''(1)=0 \\ 2+2=\lambda''(1) \end{array}\right\} \Rightarrow \boxed{\begin{array}{l} y''(1)=6 \\ \lambda''(1)=4 \end{array}}$$

Desenv. de Taylor segon ordre:

$$y(x) = y(1) + y'(1)\cdot(x-1) + \frac{1}{2!}y''(1)\cdot(x-1)^2 + O((x-1)^3)$$

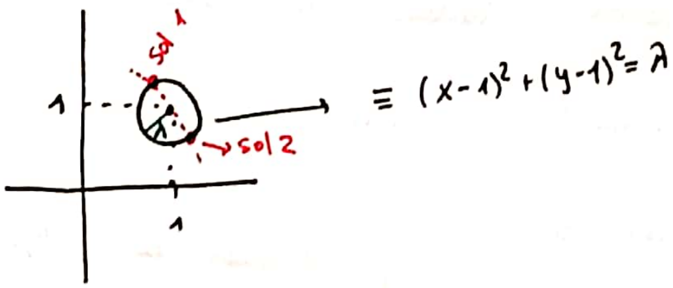
$$y(x) = 1 - 1\cdot(x-1) - 3(x-1)^2 + O((x-1)^3) \quad (!!)$$

(!) $\lambda(x) = 2(x-1)^2 + O((x-1)^3)$

Si x és prou a prop de 1 $\underset{x\neq 1}{\lambda(x) > 0}$. Per tant,

Si $\lambda > 0$, petita, tenim dues solucions (x_0, y_0), (x_1, y_1)

t.q $\left.\begin{array}{l}(x_i-1)^2 + (y_i-1)^2 = \lambda \\ x_i^4 + y_i^4 = 1\end{array}\right\} \quad i=0,1$



Aproximació de x per $\lambda = 0.01$ tenim: (usem (!))

$0.01 = 2\cdot(x-1)^2 \rightarrow 0.1 \simeq \pm\sqrt{2}\cdot(x-1) \rightarrow x \simeq 1 \pm \sqrt{0.005}$ per tant

$$\boxed{\begin{array}{l} x_0 \simeq 1.0707\ldots \\ x_1 \simeq 0.9292\ldots \end{array}}$$

Per trobar y: (usem (!!))

$y_i \simeq 1 - (x_i - 1) - 3(x_i - 1)^2 \quad i=0,1$

$$\boxed{\begin{array}{l} y_0 \simeq 0.9142 \\ y_1 \simeq 1.0537 \end{array}}$$