Solució al problema 13 I

a) Sabem que 1, $\cos(x)$, $\sin(x)$, $\cos(2x)$, $\sin(2x)$, ..., $\cos(nx)$, $\sin(nx)$ són orthogonals, i $\left\langle \frac{1}{2}, \frac{1}{2} \right\rangle = \frac{\pi}{2}$, $\langle \sin(kx), \sin(kx) \rangle = \langle \cos(kx), \cos(kx) \rangle = \pi$. Per tant, formen una base de E_n. Aleshores,

$$f_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos(kx) + b_k \sin(kx),$$

on

$$\begin{aligned} a_0 &= \frac{\left\langle \frac{1}{2}, x \right\rangle}{\left\langle \frac{1}{2}, \frac{1}{2} \right\rangle} = \frac{2}{\pi} \int_0^{2\pi} \frac{1}{2} x \, dx = 2\pi, \\ a_k &= \frac{\langle \cos(kx), x \rangle}{\langle \cos(kx), \cos(kx) \rangle} = \frac{1}{\pi} \int_0^{2\pi} x \cos(kx) \, dx = 0, \\ b_k &= \frac{\langle \sin(kx), x \rangle}{\langle \sin(kx), \sin(kx) \rangle} = \frac{1}{\pi} \int_0^{2\pi} x \sin(kx) \, dx = -\frac{2}{k}. \end{aligned}$$

Solució al problema 13 II

Per tant,

$$f_n(x) = \pi - \sum_{k=1}^{n} \frac{2}{k} \sin(kx).$$

b) Recordem que $\|f - f_n\|^2 = \|f\|^2 - \|f_n\|^2$.

$$\begin{aligned}
\|f\|^2 &= \int_0^{2\pi} x^2\,dx = \frac{8}{3}\pi^3, \\
\|f_n\|^2 &= 4\pi^2 \|\tfrac{1}{2}\|^2 + \sum_{k=1}^{n} \frac{4}{k^2} \|\sin(kx)\|^2 = \\
&= 4\pi^2 \frac{\pi}{2} + \sum_{k=1}^{n} \frac{4}{k^2}\pi = \\
&= 2\pi^3 + 4\pi \sum_{k=1}^{n} \frac{1}{k^2}.
\end{aligned}$$

Solució al problema 13 III

Així,

$$\|f - f_n\| = \left[\frac{2}{3}\pi^3 - 4\pi \sum_{k=1}^{n} \frac{1}{k^2}\right]^{\frac{1}{2}}.$$

Finalment, $\lim_{n\to\infty} \|f - f_n\| = 0$.