$\boxed{2}$ $\langle f, g \rangle = \int_0^1 f(x)\, g(x)\, dx$

$E^* = \langle 1, e^x, e^{-x} \rangle$

a) Base ~~normal~~ ortonormal de E^*

$\varphi_0(x) = 1 \quad \varphi_1(x) = e^x \quad \varphi_2(x) = e^{-x}$

$\lambda_0 \varphi_0 + \lambda_1 \varphi_1 + \lambda_2 \varphi_2 = 0$

$\lambda_0 + \lambda_1 e^x + \lambda_2 e^{-x} = 0 \quad \forall x$

$\cdot e^x$ $\hookrightarrow$ $\lambda_0 e^x + \lambda_1 e^{2x} + \lambda_2 = 0 \quad \forall x$, prenem $z = e^x$ i

$\lambda_0 z + \lambda_1 z^2 + \lambda_2 = 0 \quad \forall z \Rightarrow \lambda_0 = \lambda_1 = \lambda_2 = 0$

Per tant $\varphi_0, \varphi_1, \varphi_2$ és base de E^*. ψ_0, ψ_1, ψ_2 base ortonormal

$\varphi_0(x) = 1 \qquad \int_0^1 dx = 1 \qquad \psi_0 = \varphi_0$

$\tilde{\psi}_1 = \varphi_1 - a\psi_0 \quad , \quad \langle \tilde{\psi}_1, \psi_0 \rangle = \int_0^1 (e^x - a)\,dx = e^x - ax\Big|_0^1 = e-1-a=0$

$a = e-1 \Rightarrow \tilde{\psi}_1 = e^x - (e-1)$; $\|\tilde{\psi}_1\| = \sqrt{\int_0^1 (e^x-(e-1))^2\,dx}$ i

$\psi_1 = \dfrac{\tilde{\psi}_1}{\|\tilde{\psi}_1\|}$

Ara, $\tilde{\psi}_2 = \varphi_2 - b\,\psi_0 - c\,\psi_1$ i $\psi_2 = \dfrac{\tilde{\psi}_2}{\|\psi_2\|}$ (, fer nos)

b) Millor aprox $f^* \in E^*$, a $f(x) = x^2$

$f^* = \langle f, \psi_0\rangle \psi_0 + \langle f, \psi_1 \rangle \psi_1 + \langle f, \psi_2 \rangle \psi_2$

Alternativa: usar equacions normals per la base $\varphi_0, \varphi_1, \varphi_2$

c) $\langle f^*, f^* \rangle = \langle f, \tilde{\psi}_0 \rangle^2 + \langle f, \tilde{\psi}_1\rangle^2 + \langle f, \tilde{\psi}_2\rangle^2$

$\boxed{3}$ $C^1([0,1]) = \{$fg contínues i derivada contínua$\}$ i

$\langle f, g \rangle = f(0)g(0) + \int_0^1 f'(x)g'(x)\,dx$

a) $\langle , \rangle$ és prod. escalar:

i) $\langle f, f \rangle = 0 = f(0)^2 + \int_0^1 f'(x)^2\,dx \Leftrightarrow f(0) = 0$ i $\int_0^1 f'(x)^2 dx = 0$

(Recordem que $\int_a^b g(x)\,dx = 0$ amb g cont i $g \geq 0 \to g = 0$) Per tant

$f'(x) = 0 \quad \forall x \in [0,1] \Rightarrow f(x) = c$ (constant i com $f(0)=0 \Rightarrow$

$c = 0 \Rightarrow f(x) \equiv 0$

ii) bi lineal:

$$\langle \lambda f, g \rangle = \lambda \langle f, g \rangle$$

...

$$\langle f, g \rangle = \langle g, f \rangle$$

$$\langle f_1 f_2, g \rangle = \langle f_1, g \rangle + \langle f_2, g \rangle$$

==== és producte escalar.

Ara $(f,g) \longrightarrow \int_0^1 f'(x) \cdot g'(x)\, dx$ <u>No</u> es prod. escalar:

contra exemple: $f(x) = 1 \quad \forall x \in [0,1]$, aleshores

$$(f, f) = \int_0^1 0\, dx = 0$$

b) Base ortogonal de $\mathbb{R}_2[x]$: Prenem:

$\varphi_0(x) = 1 \qquad \varphi_1(x) = x + a \quad$ i $\quad \varphi_2(x) = x^2 + bx + c$

Imposem condicions:

$$\langle \varphi_0, \varphi_1 \rangle = \underbrace{1}_{\varphi_0(0)} \cdot \underbrace{a}_{\varphi_1(0)=a} + \int_0^1 \underbrace{0}_{\varphi_0'=0} \cdot \underbrace{1}_{\varphi_1'=1}\, dx = a = 0$$

són ortogonals φ_0 i φ_1

$$\langle \varphi_2, \varphi_0 \rangle = \underbrace{1}_{\varphi_0(0)=1} \underbrace{c}_{\varphi_2(0)} + \int_0^1 0 \cdot (2x+b)\, dx = c = 0$$

φ_2, φ_1 ortog.

$$\langle \varphi_2, \varphi_1 \rangle = a \cdot 0 + \int_0^1 \underbrace{1}_{\varphi_1'} + \underbrace{(2x+b)}_{\varphi_2'}\, dx = x^2 + bx \Big]_0^1 = 1 + b = 0$$

$$\Downarrow$$

$$b = -1$$

$$\Downarrow$$

$$\boxed{\varphi_0(x) = 1, \quad \varphi_1(x) = x \quad \varphi_2(x) = x^2 - x}$$

Ara

$$\langle fg, h \rangle = f(0) g(0) h(0) + \int_0^1 (f(x) g(x))' h'(x)\, dx$$

$$\langle f, gh \rangle = f(0) \cdot g(0) h(0) + \int_0^1 f'(x) \cdot (g(x) \cdot h(x))'\, dx$$

$\neq$ No podem aplicar teorema

Escaneado con CamScanner

c) Millor aproximació p^* de $f(x)=\cos x$ per un element $\mathbb{R}_2[x]$

Sabem:

$$p^* = \frac{\langle f,\varphi_0\rangle}{\langle\varphi_0,\varphi_0\rangle}\varphi_0 + \frac{\langle f,\varphi_1\rangle}{\langle\varphi_1,\varphi_1\rangle}\varphi_1 + \frac{\langle f,\varphi_2\rangle}{\langle\varphi_2,\varphi_2\rangle}\varphi_2$$

(perquè no sabem si ~~la~~ ~~la~~ és ortonormal, la base de φ_i trobada a b))

casualment: $\left.\begin{array}{l}\langle\varphi_0,\varphi_0\rangle=1\\ \langle\varphi_1,\varphi_1\rangle=1\end{array}\right] \to$ ortonormals

$\langle\varphi_2,\varphi_2\rangle = 1/3\ ] \to$ no ortonormal

$$\Downarrow$$

$$p^* = \langle f,\varphi_0\rangle\varphi_0 + \langle f,\varphi_1\rangle\varphi_1 + \frac{\langle f,\varphi_2\rangle}{\langle\varphi_2,\varphi_2\rangle}\cdot\varphi_2$$

φ $f'(x)=(\cos x)'$

i $\langle f,\varphi_0\rangle = f(0)\cdot\varphi_0(0) + \int_0^1 -\sin x\cdot \underset{\varphi_0'}{0}\,dx = 1\cdot 1+0 = 1$

$\langle f,\varphi_1\rangle = f(0)\cdot\varphi_1(0) + \int_0^1 -\sin x\cdot 1\,dx = 1\cdot 0 + \cos x\Big|_0^1 = \cos 1 - 1$

$\langle f,\varphi_2\rangle = \underset{1}{\underset{\shortparallel}{f(0)}}\cdot\underset{0}{\underset{\shortparallel}{\varphi_2(0)}} + \int_0^1 -\sin x\,(2x-1)\,dx = \cos 1 + 1 - 2\sin 1$

$$\Downarrow$$

$$\boxed{p^* = 1 + \underset{=1\cdot 1}{x} + 3\cdot(\cos 1 + 1 - 2\sin 1)\cdot(x^2-x)}$$

d) Millor aproximació de $f(x)=\cos x$ per un element de $\{p\in\mathbb{R}_2(x)\mid p(0)=1\}$ No és subespai vectorial.

Notem que $p^*(0)=1$. Per tant p^* és la millor aproximació

4

a) $\langle f,g\rangle = \int_{-1}^{1} f'(x)g'(x)\,dx$ en $E = \{f \in C^1[-1,1] : f(-x) = -f(x)\ \forall x \in [-1,1]\}$

i.e funcions senars i contínues $\to C^1$)

E és espai vectorial ✓

És producte escalar ?

.) És bilineal (exercici)

.) simètrica (exercici)

.) si $f=0 \Rightarrow \langle f,f\rangle = 0$ ✓

$\langle f,f\rangle = 0 \Rightarrow f=0$?

contínua positiva (□)

$\langle f,f\rangle = 0 \iff \int_{-1}^{1} (f'(x))^2\,dx = 0 \Rightarrow f'(x) = 0\ \forall x \in [-1,1]$

$\left.\begin{array}{l} \Rightarrow f(x) = c \text{ constants} \quad ; \quad \text{com} \\ f(-x) = -f(x) = -c \\ \quad \overset{\shortparallel}{c} \end{array}\right\} \Rightarrow c = 0 \Rightarrow f \equiv 0$

b) $E_3 = \{p \in E \ , \ p \text{ pol. de grau } 3\}$

E_3 espai vectorial , $p \in E_3 \Rightarrow p(x) = ax^3 + bx^2 + cx + d$

$p(-x) = -p(x)$ (per senar)

$\shortparallel$

$-ax^3 + bx^2 - cx + d = -ax^3 - bx^2 - cx - d \quad \forall x \in [-1,1]$

$\Downarrow$

$b = 0 \ , \ d = 0$

Per tant $p(x) = ax^3 + cx$

Base $P_1(x) = x$ $\quad P_2(x) = x^3$

b) $f_0(x) = \sin x$ $\quad f_0 \in E$

Millor aproximació f^* desde E_3:

$$f^* = c_1 P_1 + c_2 P_2$$

Eq. normals:

$$\begin{pmatrix} \langle P_1, P_1 \rangle & \langle P_1, P_2 \rangle \\ \langle P_1, P_2 \rangle & \langle P_2, P_2 \rangle \end{pmatrix} \cdot \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \langle f_0, P_1 \rangle \\ \langle f_0, P_2 \rangle \end{pmatrix}$$

$$\begin{pmatrix} 2 & 2 \\ 2 & \frac{18}{25} \end{pmatrix} \cdot \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 2 \sin 1 \\ 12 \cos 1 - 6 \sin 1 \end{pmatrix}$$

$\Downarrow$

$c_1 \simeq 0.647611$, $c_2 \simeq 0.193859$

c) $F = \{ f \in E \ , \ f'(0) = 1 \}$ no són espais vectorials

$F_3 = \{ p \in F \mid p \text{ grau} \leq 3 \}$

Millor aproximació de f_0 desde F_3.

Hem de trobar $q_0 \in F_3$ t.q $\| f_0 - q_0 \| \leq \| f_0 - q \| \ \forall q \in F_3$

Escaneado con CamScanner

Si $p \in F_3 \Rightarrow p \in \tilde{F}_3 \Rightarrow p(x) = ax + bx^3$

per def de l'espai $\tilde{F}$

A més, cal $p'(0) = 1 \rightarrow \left. \begin{array}{l} p'(x) = a + 3bx^2 \\ p'(0) = a \end{array} \right\} \rightarrow a = 1$

Per tant $p(x) = x + bx^3$.

Si definim: $g_0(x) = f_0(x) - x$ tinc:

$$f^*(x) = x + g^*(x) \quad \text{on} \quad g^*(x) = \tilde{b}x^3$$

$$\|f_0 - g_0\| = \|g_0 - g^*\| \leq \|g_0 - g\| \quad \forall g \text{ t.q. } g(x) = bx^3, b \in \mathbb{R}$$

Notem que $g_0 \in E$ (és senar) i aproximem desde

$$G = \{p \text{ polinomi t.q. } p(x) = ax^3\}.$$

Observem que $G = \langle \varphi \rangle$, $\varphi(x) = x^3$

Apliquem T. de la projecció ortogonal (eq. normals)

$$\text{Si } g^*(x) = \tilde{b}x^3, \quad \tilde{b} = \frac{\langle g_0, p \rangle}{\langle \varphi, \varphi \rangle} = \frac{12\cos 1 - 6\sin 1 - 2}{18/5} = -0.156995$$

→ Si ara considerem $f_0(x) = \cos(x)$ No podem usar el mateix esquema $g_0(x) = \cos x - x \notin E$ (no es senar...)

$\Downarrow$

Per tant $P^* = 0 \Rightarrow p^* = S^* \rightarrow$ ~~p^*~~ p^* és un polinomi ~~~~ senar

b)

P polinomi ortogonal de grau senar és f senar
parell és parell

↳ Podem usar recurrència per demostrar-ho

↳ També es pot demostrar:

$\|P_n\|$ és mínima entre les normes dels pol.

Mònics de grau n

$P_n = S_n + P_n$ si n és senar (P_n parell, S_n senar)

$S_n(x) = x^n + a_{n-2} x^{n-2} + \dots + a_1 x$

$\|P_n\|^2 = \langle S_n + P_n, S_n + P_n \rangle = \langle S_n, S_n \rangle + \langle P_n, P_n \rangle$

com que $S_n \neq 0 \rightarrow P_n = 0 \Rightarrow P_n$ és senar

c) $w(x) = 1 + x^2$ parella

$f(x) = x^7$ senar

per polinomi f^* senar de grau ≤ 4

f^* senar, $f^* = c_1 P_1 + c_3 P_3$ ja que P_1, P_3 són senars i P_2, P_4 parells

$c_1 = \dfrac{\langle f, P_1 \rangle}{\langle P_1, P_1 \rangle}$ $\quad c_3 = \dfrac{\langle f, P_3 \rangle}{\langle P_3, P_3 \rangle}$

usem la recurrència per a obtenir els polinomis

P_1, P_2, P_3 o imposar directament

que $\langle P_1, P_3 \rangle = 0$ $\quad P_1(x) = x$

$P_3(x) = x^3 + ax$

Per l'error: $\|f - f^*\|^2 = \|f\|^2 - \|f^*\|^2$

$\boxed{6}$ P_n, $n = 0, \dots, 4$ pol. ortogonals

$\langle f,g \rangle = \int_{-1}^{1} f(x)g(x)(1+x^2)\,dx$ $\quad (C^0[-1,1], \langle , \rangle)$

espai prehilbertià.

Tenim $\langle fg, h \rangle = \langle f, gh \rangle$ ok ✓

considerem la recurrència:

$P_0 \equiv 1$

$P_{i+1}(x) = (x - \delta_{i+1}) \cdot P_i(x) - \gamma_{i+1}^2 P_{i-1}(x), \quad i \geq 0$

$$\delta_{i+1} = \frac{\langle x P_i, P_i \rangle}{\langle P_i, P_i \rangle}$$

$$\gamma_{i+1}^2 = \begin{cases} 0 & i = 0 \\ \dfrac{\langle P_i, P_i \rangle}{\langle P_{i-1}, P_{i-1} \rangle} & i \geq 1 \end{cases}$$

•) $\underline{\underline{n = 1}}$ $\quad P_1(x) = x - \delta_1$ $\quad \delta_1 = \dfrac{\langle x, 1 \rangle}{\langle 1, 1 \rangle}$

$\langle x, 1 \rangle = \int_{-1}^{1} x(1+x^2)\,dx = 0$ $\quad P_1(x) = x$

• $\underline{\underline{n=2}}$: $P_2(x) = (x-\delta_2)\, P_1(x) - \gamma_2^2$

$$\delta_2 = \frac{\langle xP_1, P_1\rangle}{\langle P_1, P_1\rangle} \qquad \gamma_2^2 = \frac{\langle P_1, P_1\rangle}{\langle P_0, P_0\rangle}$$

$$\langle xP_1, P_1\rangle = \int_{-1}^{1} x^3 (1+x^2)\, dx = 0$$

$$\gamma_2^2 = \frac{\int_{-1}^{1} x^2 \cdot (1+x^2)dx}{\int_{-1}^{1} (1+x^2)dx} = 2/5$$

$$\Downarrow$$

$$P_2(x) = x^2 - 2/5 = x \cdot x - 2/5 = (x - \underbrace{0}_{=\delta_2}) \cdot \underbrace{x}_{=P_1(x)} - \frac{2}{5}$$

↳ Amb el mateix procediment trobem:

$$P_3(x) = x^3 - \frac{9}{14}x \quad ; \quad P_4(x) = x^4 - \frac{46}{51}x^2 + \frac{37}{357}$$

$\boxed{7}$ $E = C^0([-1,1])$

$$\langle f, g\rangle = \int_{-1}^{1} \frac{f(x)g(x)}{\sqrt{1-x^2}}\, dx \qquad f \in E \qquad f(x) = \sqrt{1-x^2}$$

a) Millor aproximació a f per un pol. de grau ≤ 4

$$P_4 = \sum_{i=0}^{4}{}' c_i T_i(x) \quad , \quad c_i = \frac{\langle T_i, f\rangle}{\langle T_i, T_i\rangle}$$

$$\langle T_i, T_j\rangle = \begin{cases} 0 & \text{si} \quad i \neq j \\ \frac{\pi}{2} & \text{si} \quad i = j \neq 0 \\ \pi & \text{si} \quad i = j = 0 \end{cases}$$

Primers polinomis de Txebitxev

$$T_{n+1}(x) = 2x\,T_n(x) - T_{n-1}(x)$$

$T_0(x) = 1$, $T_1(x) = x$, $T_2(x) = 2x^2 - 1$, $T_3(x) = 4x^3 - 3x$

$T_4(x) = 8x^4 - 8x^2 + 1$

(propietat $T_n(-x) = (-1)^n T_n(x)$)

Tenim:

$<f, T_i> = \int_{-1}^{1} T_i(x)\,dx$ calculem:

$<f, T_0> = 2$, $<f, T_1> = 0$, $<f, T_0> = \frac{-2}{3}$, $<f, T_3> = 0$

$<f, T_4> = \frac{-2}{15}$

$\Downarrow$

$$\boxed{P_4(x) = \frac{2}{\pi} - \frac{4}{3\pi}\cdot(2x^2-1) - \frac{4}{15\pi}(8x^4 - 8x^2 + 1)}$$

b) $\|f - P_4\| = \|f\|^2 - \|P_4\|^2$

$$\|P_4\|^2 = <\sum_{i=0}^{4} c_i T_i, \sum_{i=0}^{4} c_i T_i> = \sum_{i=0}^{4} c_i <T_i, T_i> = \frac{1108}{225\pi}$$

$$\|f\|^2 = \int_{-1}^{1} \underbrace{\sqrt{1-x^2}}_{f} \cdot \underbrace{\cancel{\sqrt{1-x^2}}}_{f} \cdot \underbrace{\frac{1}{\cancel{\sqrt{1-x^2}}}}_{\text{pes}} = \int_{-1}^{1} \sqrt{1-x^2} \overset{x=\cos\theta}{=} \int_{-\pi}^{0} \underbrace{\sqrt{1-\cos^2\theta}}_{=-\sin\theta} \cdot(-\sin\theta)\,d\theta =$$

$$= \int_{-\pi}^{0} \sin^2\theta\,d\theta = \int_{-\pi}^{0} \frac{1}{2}(1 - 2\cos 2\theta)\,d\theta = (\cos^2\theta - \sin^2\theta = \cos 2\theta)\ \frac{\pi}{2}$$

podríem aplicar àrea semicercle radi 1

$\Downarrow$

$$\|f - P_4\| = \sqrt{\frac{\pi}{2} - \frac{1109}{225\pi}}$$

$\boxed{8}$ $\|f\|_2 = \sqrt{\int_{-1}^{1} |f(x)|^2 dx}$ $\|\cdot\|$ associada al producte escalar.

$$\langle f, g \rangle = \int_{-1}^{1} f(x) g(x) \, dx$$

a) Si $n=2$ calcular l'element de norma mínima de M_2 $\Rightarrow$ és el polinomi ortogonal mònic de grau 2.

$\rightarrow$ $P(x) = x^2 - \frac{1}{5}$ (pol. de Legendre normalitzat $\equiv$ mònic)

$\rightarrow$ També podrien aplicar criteri general, però moltes operacions

$\rightarrow$ Per qualsevol n ... en general cert.

•) $A_2 \subset \mathbb{R}[x]$ t.q. $\int_{-1}^{1} p(x) dx = 2$

$\uparrow$
(conjunt de polinomis de grau ≤ 2)

a) Element de A_2 de norma mínima:

$\rightarrow$ Observem que A_2 no és subespai vectorial, no podem aplicar teorema de projecció ortonal o equacions normals...

Si $p \in A_2$, aleshores què passa amb $1-p$?

$$\int_{-1}^{1} (1-p(x))\,dx = 2 - \int_{-1}^{1} p(x)\,dx = 0$$

Per tant si $p \in A_2 \Rightarrow \|p\|_2 = \|1-\underbrace{(1-p)}_{=q}\|_2$

$$\Downarrow$$

$$\int_{-2}^{2} q(x)\,dx = 0$$

$$\Downarrow$$

Si resolem el problema de trobar $q^* \in A_2'$ t.q

$\|1-q^*\| \leq \|1-q\| \quad \forall q \in A_2'$ on $A_2' = \{q \in \mathbb{R}_2[x] : \int_{-1}^{1} q(x)\,dx = 0\}$

Aleshores $1-p^* = q^* \Rightarrow \boxed{p^* = 1-q^*}$ és la solució del problema original.

Tenim A_2' és subespai vectorial:

Si $q(x) = ax^2+bx+c$ i $q \in A_2'$ aleshores

$$\int_{-1}^{1} ax^2+bx+c\,dx = a\frac{x^3}{3} + b\frac{x^2}{2} + cx \Big]_{-1}^{1} = \frac{2}{3}a + 2c = 0$$

Per tant $0 = \frac{1}{3}a + c \Rightarrow a+3c = 0 \Rightarrow a = -3c$

$q(x) = -3cx^2 + bx + c$, Així exemple de base:

$P_1(x) = x$

$P_2(x) = 3x^2 - 1$

Ara $q^* = c_1 P_1 + c_2 P_2$. i les equacions normals:
(no sabem si la base és ortogonal)

$$\begin{pmatrix} \langle P_1, P_1 \rangle & \langle P_1, P_2 \rangle \\ \langle P_2, P_1 \rangle & \langle P_2, P_2 \rangle \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \langle \cancel{q} 1, P_1 \rangle \\ \langle 1, P_2 \rangle \end{pmatrix}$$

En el nostre cas : $\left.\begin{array}{l} \langle P_1, P_1 \rangle = 2/3 \\ \langle P_1, P_2 \rangle = 0 \\ \langle P_2, P_2 \rangle = 8/5 \end{array}\right\}$ casualment és ortogonal

$\Downarrow$

Calculem i: $\langle 1, P_1 \rangle = 0$ i $\langle 1, P_2 \rangle = 0$

$\Downarrow$

$q^* = 0 \Rightarrow p^* = 1$ //

$\boxed{9}$ $E = \mathcal{C}^0([-1,1])$

$\langle f, g \rangle = \int_{-1}^{1} f(x)\, g(x)\, dx \qquad M = \{p \in \mathbb{R}_2[x] \,,\, p(0) = 1\}$

a) Element de norma mínima?

Definim $M' = \{p \in \mathbb{R}_2[x] : p(0) = 0\}$ sub. vectorial

Problema, trobar $p^* \in M$ t.q $\|p^*\| \leq \|p\| \quad \forall p \in M$

Si $p \in M$, $p = 1 - q$, $q \in M'$. $p^* = 1 - q^*$ on

$\|1 - q^*\| \leq \|1 - q\| \quad \forall q \in M'$. Base M': $\varphi_1(x) = x$, $\varphi_2(x) = x^2$.

$q^* = c_1 \varphi_1 + c_2 \varphi_2$.

$$\begin{pmatrix} \langle \varphi_1, \varphi_1 \rangle & \langle \varphi_1, \varphi_2 \rangle \\ \langle \varphi_2, \varphi_1 \rangle & \langle \varphi_2, \varphi_2 \rangle \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$$

$$\langle 1, \varphi_1 \rangle = 0$$

$$\langle 1, \varphi_2 \rangle = \frac{2}{3} = \int_{-1}^{1} x^2\, dx$$

$$\langle \varphi_1, \varphi_1 \rangle = \frac{2}{3}, \quad \langle \varphi_1, \varphi_2 \rangle = 0, \quad \langle \varphi_2, \varphi_2 \rangle = 2/5$$

Per tant: $C_1 = 0$ i $C_2 = 5/3 \Rightarrow q^* = \frac{5}{3}x^2$

$$\Downarrow$$

$$p^*(x) = 1 - \frac{5}{3}x^2$$

$\|1-q^*\|_2^2$

$$\|p^*\|_2^2 = \|1\|_2^2 - \|q^*\|_2^2 \overset{(!)}{=} \cancel{4\pi(a^2+c^2) = 4 - \frac{25}{9} = \frac{36-25}{9} = 11/9}$$

$$\overset{(!)}{\underset{\varphi_1=0}{=}} \int_{-1}^{1} dx - \int_{-1}^{1} c_2^2\varphi_2^2\, dx \underset{\varphi_2=x^2}{=} 2 - \int_{-1}^{1} \frac{25}{9}x^4\, dx = 2 - \frac{25}{9}\cdot\frac{2}{5} = \frac{90-50}{45} = 40/45 = 8/9$$

b) $\bar{M} = \{p \in \mathbb{R}_2[x] \,/\, p(0) = p'(0) = 1\}$ $p^* \in M^*$ més propera de $f(x) = e^x$.

Si $p \in \bar{M} \Rightarrow$
$$\left.\begin{array}{l} p(x) = ax^2 + bx + c \rightarrow p(0) = 1 = c \\ p'(x) = 2ax + b \rightarrow p'(0) = 1 = b \end{array}\right\} \Rightarrow p(x) = ax^2 + x + 1$$

Considerem $\bar{M}' = \{q \in \mathbb{R}_2[x] : q(0) = q'(0) = 0\}$

~~Aleshores si $q \in \bar{M}' \Rightarrow q(x) =$~~

base $\bar{M}'$ és $\varphi(x) = x^2$

$$\|f - p^*\| \leq \|f - p\| \quad \forall p \in \bar{M}$$

$$p(x) = 1 + x + q(x), \quad q(x) \in \bar{M}', \quad p(x) \in \bar{M}$$

Si $p^*(x) = 1 + x + q^*(x) = 1 + \varphi_1(x) + q_1^*(x)$

$$\|f - p^*\| = \|f - 1 - x - q^*(x)\| \leq \|f - 1 - \varphi_1 - q\| = \|f - p\|, \; p \in \bar{M}$$

$$\boxed{\begin{array}{l} \forall q \in \bar{M}' \\ p = 1 + \varphi_1 + q \end{array}}$$

Escaneado con CamScanner

Si $q^* \in \bar{M}'$ és la millor aprox a $g(x) = f(x) - 1 - x$,

llavors $p^* = 1 + \varphi_1 + q^*$ és la millor aprox a $f(x) = e^x$

Clarament $q^*_{M} = c \cdot x^2$ (doncs base de $\bar{M}'$ és $\varphi(x) = x^2$)

$$c = \frac{\langle g, \varphi \rangle}{\langle \varphi, \varphi \rangle} = \frac{5}{2} e - \frac{25}{2} e^{-1} - \frac{5}{3}$$

$g = f(x) - 1 - x = e^x - 1 - x$

i $p^*(x) = 1 + x + cx^2$

$\boxed{10}$ $A_{m \times n}$ $(m > n)$ rang maxim

$Ax = b$ sistema sobre determinat

$x^* \in \mathbb{R}^4$ t.q $\|Ax^* - b\|_2$ mínima

a) Sigui $E^* = \operatorname{Im}(A) \subset \mathbb{R}^m$, espai generat per $a_1, \dots, a_n \in \mathbb{R}^m$

$A = (a_1, \dots, a_n)$ linealment indep.

Donat $b \in \mathbb{R}^m$ trobar $y^* \in E^*$ t.q $\|y^* - b\|_2 \leq \|y - b\|_2$

on $y \in E^*$. Tenim $y^* = \underbrace{c_1 a_1 + \dots + c_n a_n}_{= A \cdot \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}}$. Utilitzem

b) equacions normals:

$$\begin{pmatrix} \langle a_1, a_1 \rangle & \dots & \langle a_1, a_n \rangle \\ \vdots & & \\ \langle a_n, a_1 \rangle & \dots & \langle a_n, a_n \rangle \end{pmatrix} \cdot \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} \langle a_1, b \rangle \\ \vdots \\ \langle a_n, b \rangle \end{pmatrix}$$

i $\langle x, y \rangle = \sum_{i=1}^{m} x_i y_i \quad x, y \in \mathbb{R}^m$, $x^* = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$

Escaneado con CamScanner

c) Si $x^* = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$ // observem: $A^TA = \begin{pmatrix} a_1^T \\ \vdots \\ a_n^T \end{pmatrix} \cdot (a_1, \dots, a_n) = \begin{pmatrix} \langle a_1, a_1 \rangle & \dots & \langle a_1, a_n \rangle \\ \langle a_n, a_1 \rangle & \dots & \langle a_n, a_n \rangle \end{pmatrix}$

També $A^Tb = \begin{pmatrix} \langle a_1, b \rangle \\ \vdots \\ \langle a_n, b \rangle \end{pmatrix}$

i per tant tenim que resoldre $A^TAx = A^Tb$

d) A no té rang màx.

Siguin $a_{i_1}, \dots a_{i_k}$ l.i. (vectors columna d'A)

obtenim $\underset{\text{únic}}{b^*} \in \operatorname{Im} A$ t.q $\|b^* - b\|_2$ és mínima.

En aquest cas $b^* = Ax^*$, x^* no és únic

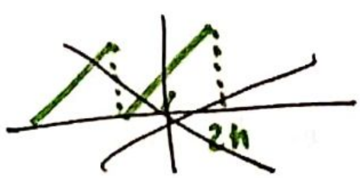
$\boxed{11}$ $E = C^0([0, 2\pi])$

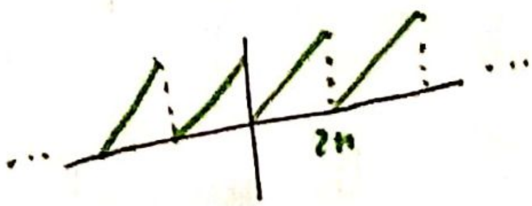
$\langle f, g \rangle = \int_0^{2\pi} f(x) g(x)\, dx$

En general per: $1, \sin x\ \cos x, \sin 2x, \cos 2x, \dots, \sin(nx), \cos(nx)$

a) $f(x) = x$ millor aprox

Podem extendre com a f periòdica, però no serà contínua





$\langle \frac{1}{2}, \frac{1}{2} \rangle = \frac{\pi}{2}$

$\langle \sin(kx), \sin(kx) \rangle = \pi$

$\langle \cos(kx), \cos(kx) \rangle = \pi$

$$f_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k(\cos(kx) + b_k \sin(kx) \quad \text{on:}$$

$$\left\{\begin{array}{l} a_0 = \dfrac{\langle \frac{1}{2}, x\rangle}{\langle \frac{1}{2}, \frac{1}{2}\rangle} = \dfrac{\int_0^{2\pi} \frac{x}{2}\,dx}{\pi/2} = \dfrac{2\frac{x^2}{4}\Big|_0^{2\pi}}{\pi} = 2\pi \\[2ex] a_k = \dfrac{\langle \cos(kx), x\rangle}{\langle \cos(kx), \cos(kx)\rangle} = \dfrac{1}{\pi}\cdot\int_0^{2\pi} \cos(kx)\,x\,dx = 0 \\[2ex] b_k = \dfrac{\langle \sin(kx), x\rangle}{\langle \sin(kx), \sin(kx)\rangle} = \dfrac{1}{\pi}\cdot\int_0^{2\pi} \sin(kx)\,x\,dx \overset{\text{int. per parts}}{=} -\dfrac{2}{k} \end{array}\right.$$

$$\Downarrow$$

$$f_n(x) = \pi - \sum_{k=1}^{n} \frac{2}{k}\sin(kx)$$

b) $$\|f - f_n\|^2 = \|f\|^2 - \|f_n\|^2 = \frac{2}{3}\pi^3 - 4\pi\cdot\sum_{k=1}^{n}\frac{1}{k^2} = \pi\cdot\Big[\underbrace{\frac{2}{3}\pi^2 - 4\pi\sum_{k=1}^{n}\frac{1}{k^2}}_{\frac{4}{6}\pi^2\ (!)\ \to\ 0}\Big]$$

$$\Big[\sum_{n=1}^{\infty}\frac{1}{n^2} \to \frac{\pi^2}{6}\Big]\ (!)$$

$$\|f\|^2 = \int_0^{2\pi} x^2\,dx = \frac{x^3}{3}\Big|_0^{2\pi} = \frac{8\pi^3}{3}$$

$$\|f_n\|^2 = \langle f_n, f_n\rangle = \frac{a_0^2}{2}\langle 1/2, 1/2\rangle + \sum_{k=1}^{n} a_k^2 \langle\cos(kx), \cos(kx)\rangle +$$

$$+ \sum_{k=1}^{n} b_k^2 \cdot \langle \sin kx, \sin kx\rangle = \ldots = 2\pi^3 + 4\pi\cdot\sum_{k=1}^{n}\frac{1}{k^2}$$

Per tant $f_n \longrightarrow f$ en la norma $\|f\|^2 = \int_0^{2\pi} f(x)^2\,dx$

$\boxed{12}$ $\langle f,g\rangle = \sum_{i=0}^{m} f(x_i)\cdot g(x_i)$ $\quad x_i = -1+\frac{2i}{m}$, $\mathbb{R}_m[x]$

a) $m=4$ polinomis ortogonals de graus $0,1,2$.

$P_0(x)=1$

$P_1(x)=x-\delta_1 \qquad \langle P_0,P_1\rangle = \langle 1, x-\delta_1\rangle = 0$

$\langle 1,x\rangle - \langle 1,\delta_1\rangle = 0 \iff \langle 1,x\rangle - \delta_1\langle 1,1\rangle = 0 \iff$

$\iff \delta_1 = \frac{\langle 1,x\rangle}{\langle 1,1\rangle} = 0$

$$\boxed{\begin{array}{l}\langle 1,x\rangle = \sum_{i=0}^{4} 1\cdot\left(-1+\frac{2i}{4}\right) = 0 \\ \langle 1,1\rangle = \sum_{i=0}^{4} 1 = 5\end{array}}$$

Per tant $P_1(x) = x-0 = x$

Sempre considerem $\langle fg,h\rangle = \langle f,gh\rangle$. Pel teorema sobre polinomis ortogonals, podem usar la recurrència:

$P_2(x) = (x-\delta_2)\cdot P_1(x) - \gamma_2^2\, P_0(x) = x\cdot P_1(x) - \frac{1}{2}P_0(x) = \underline{\underline{x^2-1/2}}$

$\delta_2 = \frac{\langle xP_1,P_1\rangle}{\langle P_1,P_1\rangle} = 0$

$\gamma_2^2 = \frac{\langle P_1,P_1\rangle}{\langle P_0,P_0\rangle} = 1/2$

b)

x	-1.0	-0.5	0.0	0.5	1.0
$f(x)$	1.1	0.3	0.1	0.2	0.9

Aproximació de graus 1 i 2: (q_1 i q_2 respect.)

Tenim 5 punts $\longrightarrow$ $m=4$ (perquè comença a 0)

Aleshores $q_1 = c_0 P_0 + c_1 P_1$ on

$q_2 = c_0 P_0 + c_1 P_1 + c_2 P_2$

$$c_i = \frac{\langle f, P_i \rangle}{\langle P_i, P_i \rangle} \quad i = 0, 1, 2 \quad \text{(perquè són ortogonals)}$$

$= \sum_{i=1}^{5} f(x_i) P_0(x_i) = 1{,}1 \cdot 1 - 0{,}3 \cdot 1 + 0{,}1 \cdot 1 + 0{,}2 \cdot 1 + 0{,}9 \cdot 1 = 2{,}6$

Resultats: $\langle f, P_0 \rangle = 2.6 \quad \langle f, P_1 \rangle = -0.25 \quad \langle f, P_2 \rangle = 0.825$

$\langle P_0, P_0 \rangle = 5 \quad \langle P_1, P_1 \rangle = \frac{5}{2} \quad \langle P_2, P_2 \rangle = 7/8$

$$c_0 = 0.52, \quad c_1 = -0.1, \quad c_2 = \frac{33}{35} \simeq 0.9429\ldots$$

$\boxed{13}$

x	0	0.2	0.4	0.6	0.8	1
$f(x)$	0.76	1.5	0.9	-0.3	-0.38	0.76

període 1. (OK, $f(0) = f(1) = 0.76$)

Ajustar a una f del tipus $a + b\sin(2\pi x) + c\cdot\cos(2\pi x)$

Trobar millor aprox mínim quadràtica:

6 punts ≡ 5 punts $\rightarrow m=4$

Producte escalar: el últim punt no ens cal perquè és repetit

$$\langle f, g \rangle = \sum_{i=0}^{4} f(x_j)\cdot g(x_j)$$

$x_j = \frac{j}{5}$ si considerem $g(\theta) = f\left(\frac{\theta}{2\pi}\right)$

$$g(\theta + 2\pi) = f\left(\frac{\theta + 2\pi}{2\pi}\right) = f\left(\frac{\theta}{2\pi} + 1\right) \overset{\text{periòdica}}{=} g(\theta)$$

Si $g^*(\theta) = f^*\left(\frac{\theta}{2H}\right)$ és la millor aprox de g

$g(\theta) = f\left(\frac{\theta}{2H}\right)$ llavors f^* és la millor aprox a f

$\varphi_0(\theta) = 1/2$, $\varphi_1(\theta) = \cos\theta$, $\varphi_2(\theta) = \sin\theta$

$$\langle \varphi_i, \varphi_j \rangle_m = \begin{cases} 0 & \text{si } i \neq j \\ \frac{m+1}{4} & \text{si } i = j = 0 \\ \frac{m+1}{2} & \text{si } i = j > 0 \end{cases}$$

$f^*\left(\frac{\theta}{2K}\right) = a + b\sin(\theta) + c\cos(\theta)$, ... càlculs... sol:

$a \simeq 0.496$

$b \simeq 0.2482$

$c \simeq 0.9973$

comentari

$\boxed{14}$ $\langle f, g \rangle = \int_{-1}^{1} f(x)\, g(x)\, dx + \int_{-1}^{1} f'(x) \cdot g'(x)\, dx$

És prod escalar, en general no compleix

$\langle fg, h \rangle \neq \langle f, gh \rangle$ per tant no podem

calcular polinomis ortogonals per recurrència,

hem d'imposar condicions per calcular

$\boxed{15}$ comentari

TABLA EQUIVALENCIAS

ESTE DOC		LISTA EJ. CAMPUS
2	⟶	3
3	⟶	4
4	⟶	5
5	⟶	6
6	⟶	8
7	⟶	9
8	⟶	10
9	⟶	11
10	⟶	12
11	⟶	13
12	⟶	14
13	⟶	15
14	⟶	17 → Examen 19/19 resolt Drive
15	⟶	18