Chapter 1: Iterative methods for linear systems
Solutions

Exercise 15.-

a) Let us define $B = A - \mu I$. Note that B is symmetric, and if $\lambda \in \operatorname{Spec}(B)$ then $|\lambda| \leq \varepsilon$. Let Q be an orthogonal matrix such that $B = Q^T D Q$, where D is a (real) diagonal matrix. Note that $\|D\|_\infty \leq \varepsilon$ and that $\|B\|_\infty \leq \varepsilon \|Q^T\|_\infty \|Q\|_\infty$. If $q = \|Q^T\|_\infty \|Q\|_\infty$, we have that $\|B\|_\infty \leq \varepsilon q$, which implies that $\sum_j |b_{ij}| \leq \varepsilon q$ for any row i. Consider $A = B + \mu I$ (the idea is that if ε is small enough, the entries of B are going to be small and then A will be diagonally dominant). The condition to be satisfied by A to be strictly diagonally dominant (by rows) is

$$\sum_{j \neq i} |a_{ij}| = \sum_{j \neq i} |b_{ij}| < |a_{ii}| = |b_{ii} + \mu|, \quad \text{for any row } i.$$

So, we need that $\varepsilon q < |b_{ii} + \mu|$ for all i. As $|b_{ii} + \mu| \geq |\mu| - \varepsilon q$ (because $|b_{ii}| \leq \varepsilon q$), it is enough to require $\varepsilon < \varepsilon_0 = \frac{|\mu|}{2q}$ to have $\varepsilon q \leq |\mu| - \varepsilon q \leq |b_{ii} + \mu|$, which implies that A is strictly diagonally dominant (by rows). The dominance by columns follows immediately since A is symmetric.

b) The system of equations is equivalent to $A^2 x = Ab$. Let us see first that the eigenvalues of A^2 are close to 1. Let λ be an eigenvalue of A, $\lambda \in \operatorname{Spec}(A) \subset \mathbb{R}$. If $|\lambda - 1| \leq \varepsilon$ then $(\lambda - 1)^2 \leq \varepsilon^2$, which implies $\lambda^2 - 1 \leq \varepsilon^2 + 2(\lambda - 1)$ and $|\lambda^2 - 1| \leq \varepsilon^2 + 2|\lambda - 1| \leq \varepsilon^2 + 2\varepsilon \equiv \delta$. Analogously, if $|\lambda + 1| \leq \varepsilon$ we have that $|\lambda^2 - 1| \leq \varepsilon^2 + 2\varepsilon = \delta$. Note that we have proved that $\operatorname{Spec}(A^2) \subset \{z \in \mathbb{C} \ / \ |z - 1| \leq \delta\}$. Using a) we have that there exists δ_0 such that, if $\delta < \delta_0$ then A^2 is strictly diagonally dominant (note that $\delta < \delta_0$ holds if ε is small enough). Then, the system $A^2 x = Ab$ can be solved by Jacobi, Gauss-Seidel or SOR.