Chapter 1: Iterative methods for linear systems
Fall 2020

1.- Let $\|.\|$ be a norm on $\mathbb{R}^n$. If A is a real and non singular n-by-n matrix, show that $\|x\|_A = \|Ax\|$ is a norm on $\mathbb{R}^n$. Find the matrix norm induced by this norm.

2.- If $x \in \mathbb{R}^n$, prove that $\lim\limits_{p\to\infty} \|x\|_p = \|x\|_\infty$.

3.- If A is a n-by-n matrix, prove that

$$\|A\|_1 = \max_{1\le j\le n} \sum_{i=1}^{n} |a_{ij}|, \qquad \|A\|_\infty = \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}|.$$

4.- Let A be a real n-by-n matrix. If P and Q are (real) orthogonal n-by-n matrices, show that

$$\|A\|_2 = \|PAQ\|_2.$$

5.- Let A be a n-by-n matrix. If A is regular, prove that $A^T A$ is symmetric and positive definite. Is this true if A is singular?

6.- If A is a n-by-n matrix, prove that $\|A\|_2^2 \le \|A\|_1 \|A\|_\infty$.

7.- Let A be a symmetric matrix.

a) Prove that A is positive definite iff (if and only if) its eigenvalues are strictly positive.

b) Assume that A is strictly diagonally dominant (by rows or by columns), and with positive diagonal elements. Prove that A is positive definite.

8.- For any real squared matrix A we define

$$F(A) = \left[\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^2\right]^{\frac{1}{2}}.$$

a) Prove that $F(A)$ is a norm on the space of real matrices n-by-n. $F(A)$ is known as the *Frobenius norm* of A, and it is usually denoted by $\|A\|_F$.

b) Is there a norm $\|.\|_f$ on $\mathbb{R}^n$ such that $\|A\|_F = \max\limits_{\|x\|_f=1} \|Ax\|_f$? (hint: Consider $A = I$).

c) Prove that

$$\|A\|_2 \le \|A\|_F \le \sqrt{n}\,\|A\|_2.$$

Hint: $\|A\|_F^2 = \operatorname{trace}(A^T A)$.

d) Discuss if the following inequality is always true: $\|Ax\|_2 \le \|A\|_F \|x\|_2$.

9.- Given a linear system $Ax = b$, prove that if A is strictly diagonally dominant by columns, then the Jacobi method is convergent.

10.- Show that the Jacobi method converges for 2-by-2 symmetric positive definite systems.

11.- Let A be a real $n \times n$ matrix and $b \in \mathbb{R}^n$. We write $A = D + L + U$, where D is diagonal, L is strictly lower triangular and U is strictly upper triangular. Moreover, we suppose that $a_{ii} \neq 0$, $i = 1, \dots, n$. We want to solve the system $Ax = b$ by using the following iterative method:

$$x^{(k+1)} = B_\omega x^{(k)} + c_\omega, \qquad \omega > 0.$$

where

$$B_\omega = (1-\omega)I - \omega D^{-1}(L+U), \qquad c_\omega = \omega D^{-1} b.$$

a) Prove that if $(x^{(k)})_{k\geq 0}$ is convergent, then its limit $\bar{x}$ satisfies $A\bar{x} = b$. Moreover, λ is an eigenvalue of B_J (iteration matrix of the Jacobi method) iff $1 - \omega + \omega\lambda$ is an eigenvalue of B_ω. Conclude that if the Jacobi method is convergent, the defined iterative method converges if $0 < \omega \leq 1$.

b) Suppose that the Jacobi method is convergent and B_J has only real eigenvalues

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$$

Prove that the optimal value of ω is $\omega_{opt} = 2/(2 - \lambda_1 - \lambda_n)$. Which is the value of $\rho(B_{\omega_{opt}})$?

12.- We want to solve the linear system $Ax = b$, with matrix

$$A = \begin{pmatrix} 1 & a & a \\ a & 1 & a \\ a & a & 1 \end{pmatrix},$$

where a is a real parameter.

(a) For which values of a the matrix A is positive definite?

(b) For which values of a the Jacobi method applied to the system is convergent?

(c) Prove that, if $|a| \geq 1$, then the Gauss-Seidel method is divergent.

13.- Consider the system $Ax = b$, 4×4, with matrix

$$A = \begin{pmatrix} 2 & 0 & -1 & -1 \\ 0 & 2 & 0 & -1 \\ -1 & 0 & 2 & 0 \\ -1 & -1 & 0 & 2 \end{pmatrix}.$$

(a) Prove that the methods of Jacobi and Gauss-Seidel converge. Which is faster?

(b) Obtain an estimate of the number of iterates of the method of Jacobi needed to reduce the initial error by a factor of 10^{-8}.

(c) Write the iteration matrix of the SOR method in the case $\omega = 3/2$. Give all the non-zero components as irreducible fractions.

(d) Is it possible to get the optimal value of ω?

14.- Let $Ax = b$ be a linear system of dimension n, where A is a symmetric matrix such that $a_{ii} = d > 0$ for all $i = 1, \dots, n$.

a) Assume that the method of Jacobi is convergent (for any initial point) and prove that the method of Gauss-Seidel is also convergent.

From this point on, we will work with the following n-dimensional matrix:

$$A = \begin{pmatrix} 2 & 1 & 0 & \cdots & \cdots & 0 \\ 1 & 2 & 1 & 0 & & \vdots \\ 0 & 1 & 2 & 1 & & \vdots \\ \vdots & 0 & 1 & \ddots & \ddots & 0 \\ \vdots & & & \ddots & 2 & 1 \\ 0 & \cdots & \cdots & 0 & 1 & 2 \end{pmatrix}.$$

b) Show that the method of Gauss-Seidel is convergent. In this particular example, the convergence of Gauss-Seidel implies the convergence of Jacobi? If both are convergent, which one converges faster?

c) Consider now the linear system $(A + 2I)x = b$, where $\|b\|_\infty = 1$. If $x^{(0)} = 0$, how many iterates of the Jacobi method are needed to approximate the solution with an error (in sup norm) below 10^{-8}?

15.- Consider a linear system $Ax = b$, where A is symmetric.

a) Assume that there exist two values $\mu \in \mathbb{R} \setminus \{0\}$ and $\varepsilon > 0$ such that $\operatorname{Spec}(A) \subset \{z \in \mathbb{C} \,/\, |z - \mu| \le \varepsilon\}$. Prove that there exists a value $\varepsilon_0 > 0$ such that, if $\varepsilon < \varepsilon_0$, then A is strictly diagonally dominant (either by rows or by columns).

b) Assume that $\operatorname{Spec}(A) \subset \{z \in \mathbb{C} \,/\, |z - 1| \le \varepsilon\} \cup \{z \in \mathbb{C} \,/\, |z + 1| \le \varepsilon\}$. If ε is small enough, then derive an iterative method to solve the system and prove its convergence (hint: use A^2).

16.- Let $\|.\|$ be a matrix norm induced by a vector norm, and let A be a n-by-n real matrix such that $a_{ii} = 1$ for all $1 \le i \le n$.

a) Let us write $A = I - B$, and assume that $\|B\| < 1$. Prove that A is invertible and that $A^{-1} = I + B + B^2 + B^3 + \cdots$. Derive a bound for $\|A^{-1}\|$ in terms of $\|B\|$ (hint: show that $A(I + B + B^2 + \cdots + B^k) = I - B^{k+1}$).

b) Let us write $A = L + I + U$, where L and U denote the lower and upper parts of A. Assume that $\|L\| = 1$ and $\|U\| = u$. If u is small enough, prove that the Gauss-Seidel method (applied to $Ax = b$) is convergent (hint: prove that L is nilpotent).

17.- Consider the linear system $Ax = b$ of dimension n, where A is a symmetric and positive definite matrix. Let $\bar{x}$ denote the solution of this system.

a) Show that there exists a set of n vectors $\{v_1, \ldots, v_n\}$ (of $\mathbb{R}^n \backslash \{0\}$) such that $v_i^T A v_j = 0$ if $i \neq j$. Discuss the uniqueness.

b) Prove that the previously defined set $\{v_1, \ldots, v_n\}$ is a basis of $\mathbb{R}^n$.

From now on, let $\{v_1, \ldots, v_n\}$ be a set of vectors satisfying a).

Consider the following iterative method: given a set of vectors $\{p^{(k)}\}_k$, the iteration is based on the formula $x^{(k+1)} = x^{(k)} - \alpha_k p^{(k)}$, where α_k is the value of α minimizing $Q(x) = \frac{1}{2} x^T A x - b^T x$ on the line $x^{(k)} - \alpha p^{(k)}$, $\alpha \in \mathbb{R}$.

c) Let $x^{(0)} \in \mathbb{R}^n$ be an initial point. If we choose $p^{(0)} = v_1$, show that the first coordinate of $x^{(1)}$ and the first coordinate of $\bar{x}$ (both in the basis $\{v_1, \ldots, v_n\}$) coincide.

d) If we choose $p^{(0)} = v_i$ (i is between 1 and n), is there any coincidence between the coordinates of $x^{(1)}$ and $\bar{x}$?

e) Assume that, in the iterative scheme, we use $p^{(0)} = v_1$, $p^{(1)} = v_2$, ..., $p^{(n-1)} = v_n$, for the first n iterations. Is it possible to derive the value of the error $\|x^{(n)} - \bar{x}\|$?

18.- Consider $(A_0 + \delta A_1)x = b$, where

$$A_0 = \begin{pmatrix} 1 & 0 & 0 & \cdots & \cdots & 0 \\ 1 & 1 & 0 & 0 & & \vdots \\ 0 & 1 & 1 & 0 & & \vdots \\ \vdots & 0 & 1 & \ddots & \ddots & 0 \\ \vdots & & & \ddots & 1 & 0 \\ 0 & \cdots & \cdots & 0 & 1 & 1 \end{pmatrix}, \qquad A_1 = \begin{pmatrix} 1 & 1 & 1 & \cdots & \cdots & 1 \\ 1 & 1 & 1 & 1 & & \vdots \\ 1 & 1 & 1 & 1 & & \vdots \\ \vdots & 1 & 1 & \ddots & \ddots & 1 \\ \vdots & & & \ddots & 1 & 1 \\ 1 & \cdots & \cdots & 1 & 1 & 1 \end{pmatrix},$$

are matrices of dimension n, δ is a real number such that $0 < \delta < \frac{1}{n}$, and $b = (1, \ldots, 1)^T$.

a) Is A_0 a regular matrix? If so, write A_0^{-1} explicitly.

b) Prove that $(A_0 + \delta A_1)x = b$ has a unique solution.

Consider the iterative method

$$\left.\begin{aligned} x_1^{(k+1)} &= -\delta \sum_{j=1}^{n} x_j^{(k)} + 1, \\ x_i^{(k+1)} &= -x_{i-1}^{(k+1)} - \delta \sum_{j=1}^{n} x_j^{(k)} + 1, \quad i = 2, \ldots, n \end{aligned}\right\} \quad k = 0, 1, 2, \ldots$$

c) Show that this method converges to the solution of $(A_0 + \delta A_1)x = b$.

d) If $x^{(0)} = 0$ and $\delta = \frac{1}{2n}$, how many iterates are needed to provide the solution with an error in $\|.\|_\infty$ below 10^{-12}?

19.- Assume that $Ax = b$ is a linear system of dimension n such that $a_{ii} \neq 0$ for $i = 1, \dots, n$. Consider the iterative method defined by

For $i = n, n-1, \dots, 2, 1$,

$$x_i^{(k+1)} = \frac{1}{a_{ii}} \left[-\sum_{j=1}^{i-1} a_{ij} x_j^{(k)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k+1)} + b_i \right].$$

Note that this is like the Gauss-Seidel method but in reverse order (going from the last equation to the first).

a) Write this method in matrix form $x^{(k+1)} = Bx^{(k)} + c$, giving an explicit expression for the matrix B and the vector c.

b) Prove that this method converges if A is strictly diagonally dominant by rows.

c) Let us write $A = D + U + \delta L$, where D is diagonal, U is upper triangular and L is lower triangular. Assume that $\|L\|_\infty = \|U\|_\infty = 1$ and that $\min\{|a_{11}|, \dots, |a_{nn}|\} \geq 1$ (note that we are not assuming that A is diagonally dominant). Show that, if $|\delta| < \frac{1}{n}$, the method converges.

20.- Let us consider a linear system $Ax = b$ of dimension n, where

$$A = \begin{pmatrix} 8 & -1 & 0 & \cdots & 0 & 0 \\ 3 & 8 & -1 & 0 & & 0 \\ 0 & 3 & 8 & -1 & & \vdots \\ \vdots & 0 & 3 & \ddots & \ddots & 0 \\ 0 & & & \ddots & 8 & -1 \\ 0 & 0 & \cdots & 0 & 3 & 8 \end{pmatrix}.$$

We want to solve it using the Gauss-Seidel iteration, $x^{(k+1)} = B_{GS} x^{(k)} + c$.

a) Find a bound for $\|B_{GS}\|_\infty$. Is the Gauss-Seidel method convergent? If it is convergent, does it converge faster than the method of Jacobi?

From now on, we assume that $b = e_n$.

b) Give a bound, in sup norm, for the solution of the linear system.

c) If we choose $x_0 = 0$, compute how many Gauss-Seidel iterations are needed to approximate the solution with an error below 10^{-12} in sup norm.

d) Show that, if the m first components of $x^{(k-1)}$ are zero, then the $m-1$ first components of $x^{(k)}$ are also zero. If we denote by ℓ the amount of iterations found in c), how many components of $x^{(\ell)}$ are zero?

21.- We consider the linear system $Ax = b$, where

$$A = \begin{pmatrix} 1 & -1 & & & & \\ & 1 & -1 & & & \\ & & 1 & \ddots & & \\ & & & \ddots & -1 & \\ & & & & 1 & -1 \\ a & & & & & 1 \end{pmatrix},$$

is an $n \times n$ matrix and b is a real vector of dimension n. The elements corresponding to the blanks are zero

a) Prove that the previous system has a unique solution if $a \neq -1$.

b) Study the convergence of the Jacobi method depending on the real parameter a.

c) Study the convergence of the Gauss-Seidel method as a function of the real parameter a,

d) If for a value of a, both Jacobi and Gauss-Seidel methods converge, is it possible to say which of them converges faster?

22.- Let $u,\ v \in \mathbb{R}^n$ be such that $\|u\|_2 = \frac{1}{4}$ and $\|v\|_2 = 1$, and we consider the matrix $A = \frac{3}{2}I - uv^T$.

a) Compute $\operatorname{Spec}(A)$ and prove that $\det A \neq 0$.

We want to solve the linear system $Ax = b$ using an iterative method $x^{(k+1)} = Bx^{(k)} + c$.

b) If $B = -\frac{1}{2}I + uv^T$ i $c = b$, prove that the iterative scheme converges to the solution of $Ax = b$.

c) If $\|b\|_2 = 1$ and $x^{(0)} = 0$, how many iterates are needed to guarantee that the error, in norm the $\|.\|_2$, is less than 10^{-12} ?

23.- Consider the linear system $Ax = b$ of dimension $n + 1$, where $n = 10^6$ and

$$A = \begin{pmatrix} 4 & -1 & 0 & \cdots & 0 & -1 \\ -1 & 5 & -1 & 0 & & 0 \\ 0 & -1 & 4 & -1 & & \vdots \\ \vdots & 0 & -1 & \ddots & \ddots & 0 \\ 0 & & & \ddots & 5 & -1 \\ -1 & 0 & \cdots & 0 & -1 & 4 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 \\ \frac{n-1}{n} \\ \frac{n-2}{n} \\ \vdots \\ \frac{1}{n} \\ 0 \end{pmatrix}.$$

a) Given $\omega \in \mathbb{R} \setminus \{0\}$, let us consider the iterative scheme $x^{(k+1)} = (I - \omega A)x^{(k)} + \omega b$.

a.1) Prove that, if this scheme is convergent, its limit is the only solution of $Ax = b$.

a.2) Give an interval of values of ω for which the scheme is convergent.

a.3) Which value of ω gives the fastest convergence? Using this value of ω and $x^{(0)} = 0$, compute the number of iterations needed to obtain an approximation to the solution with an error below 10^{-12} in $\|.\|_\infty$.

b) Assume that we have an approximate solution x^* satisfying $\|Ax^* - b\|_\infty < 10^{-12}$. If $\bar{x}$ denotes the exact solution, bound the error $\|\bar{x} - x^*\|_\infty$ (hint: to bound $\|A^{-1}\|_\infty$, write $A = D(I - B)$, $D = \text{diag}(A)$, and use that, if $\|B\| < 1$, $(I - B)^{-1} = I + B + B^2 + B^3 + \cdots$).

24.- a) Let

$$A = \begin{pmatrix} 1 & a & b \\ -a & 1 & 0 \\ -b & 0 & 1 \end{pmatrix}$$

be a real 3×3 matrix. Under what conditions on the constants a and b do we have convergence for the Gauss-Seidel method applied to the system $Ax = y$, for any initial condition?

b) Consider the matrix

$$A = \begin{pmatrix} I_m & S \\ -S^T & I_n \end{pmatrix},$$

where I_n is the n-dimensional identity matrix and S is a $m \times n$ matrix. Give conditions over the eigenvalues of $S^T S$ for which the Gauss-Seidel method is convergent for all initial condition.

c) Now, we suppose that

$$A = \begin{pmatrix} I_m & S \\ S^T & I_n \end{pmatrix},$$

Prove that if A is positive definite then $\rho(S^T S) < 1$.