Chapter 1: Iterative methods for linear systems
Solutions

Exercise 22.-

a) Let $\{v_1, \ldots, v_{n-1}\}$ be a basis of the space of vectors that are orthogonal to v. To compute $\mathrm{Spec}\,(A)$, we distinguish two cases:

i) $\langle u, v\rangle \neq 0$. This implies that the set $\{v_1, \ldots, v_{n-1}, u\}$ is a basis of $\mathbb{R}^n$. Note that, for $j = 1, \ldots, n-1$, we have $Av_j = \frac{3}{2}v_j - u(v^T v_j) = \frac{3}{2}v_j$, which means that $\frac{3}{2}$ is an eigenvalue of multiplicity $n-1$. Moreover, $Au = \frac{3}{2}u - u(v^T u) = (\frac{3}{2} - \langle u, v\rangle)u$, which implies that $\lambda_n = \frac{3}{2} - \langle u, v\rangle$ is also an eigenvalue. As $|\langle u, v\rangle| \leq \|u\|_2\|v\|_2 \leq \frac{1}{4}$, we have that $|\lambda_n| \geq \frac{3}{2} - \frac{1}{4} = \frac{5}{4}$. Hence, $\det A = (\frac{3}{2})^{n-1}\lambda_n$ and, as $\lambda_n \neq 0$, we conclude that $\det A \neq 0$.

ii) $\langle u, v\rangle = 0$. In this case we consider a slightly different basis: we select $v_1 = u$ and $v_2, \ldots, v_{n-1}$ are chosen to complete a basis of the orthogonal space of v. Then, $\{v_1 = u, v_2 \ldots, v_{n-1}, v\}$ is a basis of $\mathbb{R}^n$. As before, $Av_j = \frac{3}{2}v_j$ and this shows that $\frac{3}{2}$ is an eigenvalue of multiplicity $n-1$. Moreover, as $Av = \frac{3}{2}v - \langle v, v\rangle\, u = \frac{3}{2}v - u$ and, recalling that $Au = \frac{3}{2}u$, it is clear that the matrix A, in the basis $\{v_1 = u, v_2 \ldots, v_{n-1}, v\}$, is all zero except for the values $\frac{3}{2}$ in the diagonal and the value -1 in the $(1, n)$ entry. Therefore, $\det(A) = (\frac{3}{2})^n \neq 0$.

b) Let us see first that, if the iterations converge, they converge to the solution of the linear system: if $x^{(k)} \to \bar{x}$, then $\bar{x} = B\bar{x} + b$, which is equivalent to $(I - B)\bar{x} = b$ and, as $I - B = A$, we have that $A\bar{x} = b$. To see the convergence we note that, as $B = I - A$, the eigenvalues of B are of the form $1 - \lambda_j$, where λ_j are the eigenvalues of A. We have seen that the eigenvalues of A are $\frac{3}{2}$ or the value $\lambda_n = \frac{3}{2} - \langle u, v\rangle$. Hence, the eigenvalues of B are either $1 - \frac{3}{2} = \frac{1}{2}$ or $1 - \lambda_b = \frac{1}{2} - \langle u, v\rangle$ that, using $|\langle u, v\rangle| \leq \|u\|_2\|v\|_2 \leq \frac{1}{4}$, has modulus strictly less than 1. Hence, as $\rho(B) < 1$, the method converges.

c) To bound $\|B\|_2$ we first note that $\|B\|_2 = \| -\frac{1}{2}I + uv^T\|_2 \leq \frac{1}{2} + \|uv^T\|_2$. Then, using that $u^T u = \frac{1}{16}$ we obtain $\|uv^T\|_2^2 = \rho((uv^T)^T(uv^T)) = \frac{1}{16}\rho(vv^T) = \frac{1}{16}$ which implies $\|uv^T\|_2 = \frac{1}{4}$. Hence, $\|B\|_2 \leq \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$. To obtain the number of iterates needed, we use the formula

$$\|x^{(k)} - \bar{x}\|_2 \leq \frac{\beta^k}{1-\beta}\|x^{(1)} - x^{(0)}\|_2.$$

Here, as $\beta = \frac{3}{4}$, $x^{(0)} = 0$ and $x^{(1)} = b$, we have that $\|x^{(k)} - \bar{x}\|_2 \leq 4(\frac{3}{4})^k < 10^{-12}$, which implies $k \geq 101$.