Chapter 1: Iterative methods for linear systems
Solutions

Exercise 23.-

a) $x^{(k+1)} = (I - \omega A)x^{(k)} + \omega b$.

a.1) If we denote $\bar{x}$ to the limit of $\{x_n\}_n$, then $\bar{x} = (I - \omega A)\bar{x} + \omega b \Leftrightarrow A\bar{x} = b$.

a.2) Let us compute the range of values of ω for which we have $\|I - \omega A\|_\infty < 1$. It is clear that $\|I - \omega A\|_\infty = \max\{|1 - 4\omega| + 2|\omega|, |1 - 5\omega| + 2|\omega|\}$. If $\omega < 0$ then one has $|1 - 4\omega| > 1$ which implies $\|I - \omega A\|_\infty > 1$. So, we are restricted to positive values of ω. There are several ways of studying the set of (positive) values of ω for which the two conditions $|1-4\omega|+2\omega < 1$, $|1-5\omega|+2\omega < 1$ hold, but a simple one is to to draw the graphics of the functions $f_1(x) = |1 - 4x| + 2x$ and $f_2(x) = |1 - 5x| + 2x$ for $x > 0$ (see Fig. 1). Note that, as $\omega > 0$ the condition $|1 - 4\omega| + 2\omega < 1$ is satisfied when $4\omega - 1 + 2\omega < 1$ which implies $\omega < \frac{1}{3}$. The condition $|1-5\omega|+2\omega < 1$ is satisfied when $5\omega-1+2\omega < 1$ which implies $\omega < \frac{2}{7}$. As $\frac{2}{7} < \frac{1}{3}$, the condition $\|I - \omega A\|_\infty < 1$ is satisfied when $0 < \omega < \frac{2}{7}$.

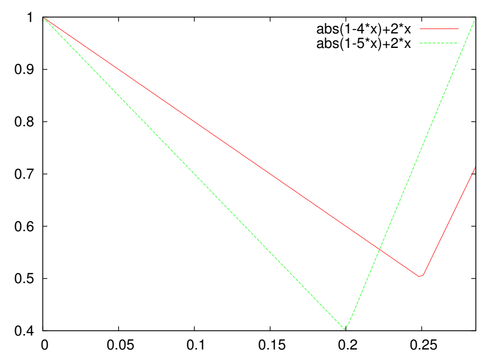

Figure 1: Graphics of f_1 and f_2.

A second option is to look for a set of values of ω such that the spectral radius of the iteration matrix $I - \omega A$ is lower than 1. This can be done using the Gerschgorin Theorem to show that the eigenvalues of A belong to the interval $[2, 7]$. Then, as $\lambda \in \text{Spec}\,(A) \Leftrightarrow 1 - \omega\lambda \in \text{Spec}\,(I - \omega A)$, let us ask for the condition $|1 - \omega\lambda| < 1$. It is clear that to have $1 - \omega\lambda < 1$ it is necessary to choose $\omega > 0$. Moreover, to have $-(1 - \omega\lambda) < 1$ for all $\lambda \in [2, 7]$ it is enough to ask for $\omega < \frac{2}{7}$. Hence, the range $0 < \omega < \frac{2}{7}$ guarantees that $\rho(I - \omega A)$ is lower than 1.

a.3) The minimum value of $\|I-\omega A\|_\infty$ is given by the intersection of the two graphs of Fig. 1, and it is easy to see that corresponds to $\omega = \frac{2}{9}$. For this value of ω, we have that $\beta = \|I-\omega A\|_\infty = \frac{5}{9}$. Then, as $x^{(0)} = 0$, $x^{(1)} = \omega b$ and $\|b\|_\infty = 1$, we have

$$\|x^{(k)} - \bar{x}\|_\infty \leq \frac{\beta^k}{1-\beta}\|x^{(1)} - x^{(0)}\|_\infty = \frac{1}{2}\left(\frac{5}{9}\right)^k < 10^{-12},$$

which implies $k \geq 46$.

b) Let us define $r = Ax^* - b$. If $A\bar{x} = b$, we have $A(x^* - \bar{x}) = r$, or $Ae = r$ ($e = x^* - \bar{x}$ is the error of the solution). Then, $\|e\|_\infty = \|A^{-1}r\|_\infty \leq \|A^{-1}\|_\infty\|r\|_\infty$. On the other hand, writing $A = D(I - B)$ we have that $A^{-1} = (I - B)^{-1}D^{-1}$ and, using that $\|D^{-1}\|_\infty = \frac{1}{4}$, $\|B\|_\infty = \frac{1}{2}$, it follows that

$$\|A^{-1}\|_\infty \leq \frac{1}{4}(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots) = \frac{1}{2}.$$

Then, $\|e\|_\infty \leq \frac{1}{2} \times 10^{-12}$.