Chapter 2: Eigenvalues and eigenvectors

Fall 2020

1.- Let A be an n-by-n matrix, and let λ be an eigenvalue of A.

a) If B is an n-by-n matrix such that $\lambda \notin \operatorname{Spec}(B)$, prove that $\|(B-\lambda I)^{-1}(A-B)\| \geq 1$ (hint: $\rho(A) \leq \|A\|$).

b) Use a) to prove the Gerschgorin Theorem (hint: choose B appropriately).

2.- Consider the matrix

$$A = \begin{pmatrix} -3.0 & 0.1 & 0.1 \\ 0.2 & 0.0 & -0.1 \\ -0.1 & 0.2 & 3.0 \end{pmatrix}.$$

a) Use the Gerschgorin Theorem to determine a region $\Omega \subset \mathbb{C}$ such that all the eigenvalues of A lie in Ω.

b) Can you conclude that all eigenvalues of A are real? If so, find an interval for each eigenvalue.

3.- Let A be a symmetric n-by-n matrix, and let us denote by λ one of its eigenvalues. If $\|.\|_F$ denotes the Frobenius norm and B is a matrix such that $\|B\|_F = 1$, show that there is an eigenvalue λ_ε of $A + \varepsilon B$ such that $|\lambda_\varepsilon - \lambda| = O(\varepsilon)$. Give an explicit bound on $|\lambda_\varepsilon - \lambda|$ (hint: show that, for any vector $v \in \mathbb{R}^n$, $\|v\|_1 \leq \sqrt{n}\,\|v\|_2$).

4.- Assume that A is a symmetric matrix. Let us select a fixed $x \in \mathbb{R}^n \setminus \{0\}$, and let us define the map $\lambda \in \mathbb{R} \mapsto f(\lambda) = \|Ax - \lambda x\|_2$. Prove that the Rayleigh quotient for the vector x,

$$\lambda_x = \frac{x^T A x}{x^T x},$$

is a local minimum of f (hint: consider $f(\lambda)^2$ and its derivatives with respect to λ).

5.- Is it possible to apply the power method to matrix A in Exercise 2? And the shifted power method? Explain how to use these two methods to compute all eigenvalues and eigenvectors of A.

6.- Consider the matrix

$$A = \begin{pmatrix} -1 & 1 & 1 \\ 1 & 8 & -1 \\ 1 & -1 & -2 \end{pmatrix}.$$

a) Prove that it has a unique eigenvalue of largest modulus, which is real.

b) Use the power method to determine this eigenvalue and its eigenvector.

c) Use the trace and determinant of the matrix to obtain all eigenvalues.

7.- Consider the matrix

$$A = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix}.$$

a) Apply the power method to A. Explain what happens.

b) Apply the inverse power method to A. Use the trace and determinant of A to obtain all of its eigenvalues.

8.- Let T be an n-by-n matrix of the form

$$T = \begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix},$$

where T_{11} and T_{22} are squared matrices.

a) Prove that $\operatorname{Spec}(T) = \operatorname{Spec}(T_{11}) \cup \operatorname{Spec}(T_{22})$.

b) Explain the relationship between the eigenvectors of T_{11} and T_{22} and the ones of T.

9.- Let A be an $n \times n$ matrix such that $A = U\Sigma V^\top$, where $U, V, \Sigma \in \mathbb{R}^{n\times n}$, U and V are orthogonal and $\Sigma = \operatorname{diag}(\sigma_1, \ldots, \sigma_n)$, $\sigma_i \geq 0$, for $1 \leq i \leq n$. Suppose that $\|A^T A - I\|_2 = \epsilon < 1$.

1. Prove that $1 - \epsilon \leq \sigma_i \leq 1 + \epsilon$, where $1 \leq i \leq n$.
2. Prove that there exists an orthogonal matrix Q such that $\|A - Q\|_2 \leq \epsilon$.
3. Prove that $1 - \epsilon \leq \rho(A) \leq 1 + \epsilon$.
4. We define the disks $D(q_{ii}, r_i)$, such that $r_i = \sum_{j=1, j\neq i}^{n} |q_{ij}|$, for $1 \leq i \leq n$, and $Q = (q_{ij})_{1\leq i,j\leq n}$. Prove that if there exists j such that $D(q_{jj}, r_j) \cap D(q_{ii}, r_i) = \emptyset$, for all $i \neq j$, $1 \leq i \leq n$, and $\det A > 0$, then 1 is a simple eigenvaule of Q.

10.- Let A be a symmetric matrix of dimension n, with $\operatorname{Spec}(A) = \{\lambda_1, \ldots, \lambda_n\}$.

a) Assume $\lambda_1 = \lambda_2$ and $|\lambda_1| > |\lambda_j|$ if $j > 2$, and study the behaviour of the power method in this case. Is it possible to compute the eigenvectors corresponding to λ_1 and λ_2?. If the power method converges, discuss the speed of convergence of the Rayleigh quotioent.

b) Discuss the same questions as in a) but for the case $\lambda_1 = -\lambda_2$, $|\lambda_1| > |\lambda_j|$ if $j > 2$.

11.- If A denotes a n-by-n matrix, let us denote by $A_{k,\alpha}$ the matrix obtained by multiplying row k of A by α and dividing then column k by α (of course, $1 \leq k \leq n$, $\alpha \in \mathbb{R} \setminus \{0\}$).

a) Show that $\operatorname{Spec}(A_{k,\alpha}) = \operatorname{Spec}(A)$.

Let us define

$$A = \begin{pmatrix} 2.14 & -0.10 & 0.00 \\ -0.10 & 4.34 & 0.20 \\ 0.00 & 0.20 & 4.48 \end{pmatrix}.$$

b) Use Gerschgorin Theorem to give an approximation (with an error bound) to the smallest eigenvalue of A.

c) Choosing suitable values of k and α, use a) to produce a better (that is, with smaller error bound) approximation to the smallest eigenvalue.

d) Discuss if it is possible to use the same technique as in b) to improve the error bound of the remaining eigenvalues.

12.- Let A be a $n \times n$ matrix.

a) Assume that i) the value $\delta = \min_{i \neq j}\{|a_{ii} - a_{jj}|\}$ is different from zero; and ii) there exist n real values λ_j $(j = 1, \ldots, n)$ such that $\|Ae_j - \lambda_j e_j\|_\infty < \varepsilon < \dfrac{\delta}{2n}$, where $\{e_1, \ldots, e_n\}$ is the standard basis of $\mathbb{R}^n$.

a.1) Does A have n different real eigenvalues?

a.2) Bound the difference between the values $\lambda_1, \ldots, \lambda_n$ and the eigenvalues of A.

b) Assume now that A is a symmetric matrix and that we (only) know that the first vector of the standard basis, e_1, satisfies $\|Ae_1 - \lambda e_1\|_2 < \varepsilon$ for a given value λ. Show that, if ε is small enough, the matrix A has an eigenvalue near λ. Bound the distance between this eigenvalue and λ.

13.- Let A be a squared matrix of dimension 11, with $\text{Spec}\,(A) = \{\lambda_1, \ldots, \lambda_{11}\} \subset \mathbb{R}$. We assume that there exist 11 vectors of $\mathbb{R}^{11}$, $v_1, \ldots, v_{11}$, such that

- $Av_i = \lambda_i v_i$, $i = 1, \ldots, 11$.
- $v_i^T v_i = 1$, $i = 1, \ldots, 11$.
- $|v_i^T v_j| \leq 10^{-2}$ si $i, j = 1, \ldots, 11$, $i \neq j$.

a) Show that $v_1, \ldots, v_{11}$ are a basis of $\mathbb{R}^{11}$.

b) Let us define C as the matrix of the change of variables diagonalizing A. Show that the condition number of C in $\|.\|_2$ norm, $k_2(C) = \|C^{-1}\|_2 \|C\|_2$, satisfies $k_2(C) \leq \frac{\sqrt{11}}{3}$. Hint: Show that, if E and F are squared matrices, then $\text{Spec}\,(EF) = \text{Spec}\,(FE)$.

c) If we define $A_\varepsilon = A + \varepsilon B$, where $\|B\|_F = 1$ and ε is small enough, give a bound on the distance between the eigenvalues of A_ε and A.

14.- Consider the matrix

$$A = \begin{pmatrix} 7 & 6 \\ 3 & 4 \end{pmatrix}.$$

a) Compute the QR factorization of A.

b) Use the QR iteration to compute the eigenvalues of A, with an error lower than 10^{-6}.

15.- Use the QR factorization to solve, in the least squares sense, the overdetermined system

$$\begin{pmatrix} 2 & 3 \\ 1 & 3 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \\ 4 \end{pmatrix}.$$

16.- Let $\lambda_1 < \lambda_2 < \ldots < \lambda_n$ be real numbers and $\varepsilon > 0$. We consider $D = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$. We want to study the eigenvalues of the matrix $A_\varepsilon = D + \varepsilon B$ where $\operatorname{diag}(B) = 0$ and $\|B\|_F = 1$, for ε small.

a) Prove that, if ε is small enough, for each $j \in \{1, \ldots, n\}$ there exist $\lambda_\varepsilon^j \in \operatorname{Spec}(A_\varepsilon)$ and a constant $\kappa_1 > 0$ such that $|\lambda_\varepsilon^j - \lambda_j| \leq \kappa_1 \varepsilon$.

b) Prove that, if ε is small enough, for each $j \in \{1, \ldots, n\}$ there exist $\lambda_\varepsilon^j \in \operatorname{Spec}(A_\varepsilon)$ and a constant $\kappa_2 > 0$ such that $|\lambda_\varepsilon^j - \lambda_j| \leq \kappa_2 \varepsilon^{3/2}$.

c) Is it possible to improve the bound of the previous item? That is, are there constants $\kappa_3 > 0$ and $\alpha > 3/2$ such that $|\lambda_\varepsilon^j - \lambda_j| \leq \kappa_3 \varepsilon^\alpha$?

Hint: Multiply or divide rows/columns by suitable values.

17.- Let A, C and D be $n \times n$ matrices such that $AC = CD$, and D is diagonal.

a) Suppose that $C = Q + \epsilon E$ where Q is orthogonal, $\|E\|_F = 1$ and $|\epsilon| < 1$. Prove that C is invertible and $\kappa_2(C) = \|C\|_2 \|C^{-1}\|_2 \leq \frac{1+|\epsilon|}{1-|\epsilon|}$.

b) Let $C = QR$ be the QR factorization of C. If $\|R - I\|_F < 1$, prove that C is invertible and find an upper bound of $\kappa_2(C)$ depending on $\|R - I\|_F$.

c) Let $C = QR$ be as in b), but now $R = I + U$, where U is upper triangular withe zeros in the diagonal. See that C is also invertible and find an upper bound of $\kappa_2(C)$ depending on $\|U\|_F$.

d) If we define $A_\delta = A + \delta B$, where $\|B\|_F = 1$, C as in a), and $|\delta|$ small enough, give a bound of the distance between the eigenvalues of A and A_δ. Why do we need $|\delta|$ to be small enough?