Variable-, Fractional-Order RST/PID Controller Transient Characteristics Calculation

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***Abstract*—In the paper we propose a method of the variable-fractional-order (VFO) discrete-time linear system response calculation applied to the VFO RST/PID controllers response. To such controllers we cannot directly apply the commonly known one-sided $\mathcal{Z}$-Transform method. Treating the VFO-PID controller as a special case of the RST controller we show that using classical methods we can derive the dynamical properties of the VFO-PID controller.**

I. Introduction

In the last few decades, mathematical models based on the fractional calculus involving derivatives of any real orders (non-integer or integer) [17], [18], [8] have become a powerful mathematical tool in the modelling various physical, technical, biological, economical etc. phenomena [7], [1], [2], [5]. With more model parameters orders we can obtain better approximation with lower fractional orders.

In the paper we investigate the generalization of the fractional-order backward difference (FOBD): a variable-, fractional-order backward difference (VFOBD) [3]. Variable-order discrete-variable calculus as a relatively new concept may play an important role in better identification of real dynamical systems or sophisticated control strategies synthesis [1]. Out of many definitions of the VFOBDs we use the generalization of the Grünwald - Letnikov. Next, in a descending order we present its simplifications leading to the classical first-order backward one.

Almost 100 years old PID control is still dominating in industrial applications. Hence, there is a permanent effort to improve its action. One possible way is a generalization to the fractional-orders of differentiation and integration [19], [6], and even variable-, FO-PID ones [13], [14], [15]. The further generalization presents so called RST controllers. In this paper we demonstrate the possibility to analyse the transient responses of the VFO-RST/PID controllers using classical methods based on the one-sided $\mathcal{Z}$-Transform. The main result of the paper states that being able to evaluate the FORST/PID controller we can evaluate the response of the VFO-RST/PID controller.

The paper is organized as follows. After an introduction to the fractional calculus and presentation of the closed-loop system structure the main result is presented in Section III. The investigations are illustrated by several numerical examples.

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A. Variable-, fractional-order backward difference

We consider a discrete-variable bounded order function $\nu : \mathbb{Z} \to \mathbb{R}$ (OF). It is assusumed that $0 \leqslant \nu(l) \leqslant 1$. Next we define a discrete function of two variables: $k, l \in \mathbb{Z}$. Namely, function $a^{\nu(l)}(k)$ for a given order function ν. It plays a crucial role in the variable-, fractional–order backward difference (VFOBD) evaluation. It is called the „oblivion" or „decay" function. Let us use notation $\mathbb{N}_c = \{c, c+1, c+2, \ldots\}$.

Definition 1. *For $k, l \in \mathbb{Z}$ and a given order function $\nu(\cdot)$ with values $0 \leq \nu(l) \leqslant 1$ we define an oblivion function of two variables by its values $a^{[\nu(l)]}(k)$ in the following way:*

$$a^{\nu(l)}(k) = \\ = \begin{cases} 1 & \text{for} \quad k = 0 \\ (-1)^k \frac{\nu(l)[\nu(l)-1]\cdots[\nu(l)-k+1]}{k!} & \text{for} \quad k \in \mathbb{N}_1 \, . \end{cases} \tag{1}$$

The formula in Definition 1 is equivalent to the following

$$\begin{aligned} &a^{\nu(l)}(0) = 1 \\ &a^{\nu(l)}(k) = a^{\nu(l)}(k-1)\left[1 + \frac{\nu(l)-1}{k}\right] \text{ for } k \in \mathbb{N}_1 \, . \end{aligned} \tag{2}$$

Next we define the Grünwald–Letnikov variable–, fractional–order backward difference (VFOBD). For a discrete–variable bounded real–valued function $f(\cdot)$ defined over a discrete interval $[0, k]$ the VFOBD is defined as a sum (see for instance [4], [10], [11], [20]),

Definition 2. *The VFOBD with an order function ν, with values $\nu(k) \in [0, 1]$, is defined as a finite sum, provided that the series is convergent:*

$$\begin{aligned} {}_{k_0}\Delta_k^{\nu(k)} f(k) &= \sum_{i=0}^{k-k_0} a^{\nu(k)}(i) f(k-i) \\ = \begin{bmatrix} 1 & a^{\nu(k)}(1) & a^{\nu(k)}(2) & \cdots & a^{\nu(k)}(k_0) \end{bmatrix} &\begin{bmatrix} f(k) \\ f(k-1) \\ \vdots \\ f(k-k_0) \end{bmatrix} \end{aligned} \tag{3}$$

As the first special case of the defined above VFOBD for a constant order function $\nu(k) = \nu = const$ from (3) we get the fractional-order backward difference (FOBD). The second and the last special case of the defined above VFOBD for a constant integer order function $\nu(k) = \nu = n = const$ from we get (4) the integer-order backward difference (IOBD)

B. VFOBD, FOBD and IOBD fundamental properties

The VFOBD and FOBD fundamental properties like linearity and chain rule are given and proven in [10] and [16], respectively. Here, we concentrate on the one-sided $\mathcal{Z}$-Transform of the FOBD. It equals

$$\mathcal{Z}\{{}_{k_0}\Delta_k^{(\nu)} f\} = \left(1 - z^{-1}\right)^{\nu} \tag{4}$$

which is also valid for $\nu = n \in \mathbb{N}_1$.

C. Linear time-invariant difference equation

The typical linear time-invariant difference equation based on the backward difference (3) is of the form

$$\sum_{i=0}^{n} C_i y(k-i) = \sum_{j=0}^{m} D_j u(k-l-j) \tag{5}$$

where C_i, D_j are constant coefficients for $i = 0, 1, \cdots, n$, $j = 0, 1, \cdots, m$, $m \leq n$, $l \in \mathbb{N}_0$. Assuming zero initial conditions the one-sided $\mathbb{Z}$-Transform of the above equation is as follows

$$\sum_{i=0}^{n} C_i z^{-i} y(z^{-1}) = \sum_{j=0}^{m} D_j z^{-j-l} u(z^{-1}) \tag{6}$$

where $U(z^{-1}) = \mathcal{Z}\{u(k)\}$. Now we define a polynomial in a variable z^{-1}

$$C(z^{-1}) = \begin{bmatrix} 1 & z^{-1} & \cdots & z^{-n+1} & z^{-n} \end{bmatrix} \begin{bmatrix} C_0 \\ C_1 \\ \vdots \\ C_{n-1} \\ C_n \end{bmatrix} \tag{7}$$

Next we define an $(n+1) \times (n+1)$ matrix with its elements being the function ((1)) defined for $\nu = n \in \mathbb{N}_0$

$$\mathbf{E}_{n+1} = \begin{bmatrix} a^{(0)}(0) & a^{1)}(0) & \cdots & a^{(n-1)}(0) & a^{(n)}(0) \\ 0 & a^{(1)}(1) & \cdots & a^{(n-1)}(1) & a^{(n)}(1) \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & a^{(n-1)}(n-1) & a^{(n)}(n-1) \\ 0 & 0 & \cdots & 0 & a^{(n)}(n) \end{bmatrix} \tag{8}$$

The main property of the matrix (8) is stated by the following theorem.

Proposition 1. *For $n \in \mathbb{N}_0$ an inverse matrix* (8) *has the form*

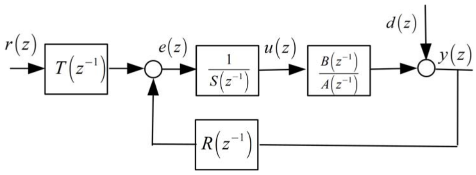

Fig. 1. Block diagram of the closed-loop system with RST controller

$$\left[\mathbf{E}_{n+1}\right]^{-1} = \begin{bmatrix} a^{(-1)}(0) & a^{(-1)}(1) & \cdots & a^{-(1)}(n-1) & a^{-1}(n) \\ 0 & a^{(-2)}(0) & \cdots & a^{(-2)}(n-2) & a^{(-2)}(n-1) \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & a^{(-n+1)}(0) & a^{(-n+1)}(1) \\ 0 & 0 & \cdots & 0 & a^{(-n)}(0) \end{bmatrix} \tag{9}$$

Proof. By direct calculation of a product of the matrices (7) and ((8)) we get the stated result. □

Now using the results of Proposition 1 we can write

$$\begin{aligned} C(z^{-1}) &= \begin{bmatrix} 1 & z^{-1} & \cdots & z^{-n} \end{bmatrix} \mathbf{E}_{n+1} \left[\mathbf{E}_{n+1}\right]^{-1} \begin{bmatrix} C_0 \\ C_1 \\ \vdots \\ C_n \end{bmatrix} \\ &= \begin{bmatrix} 1 & (1-z^{-1})^1 & \cdots & (1-z^{-1})^n \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix} \end{aligned} \tag{10}$$

where

$$\begin{bmatrix} c_0 & c_1 & \cdots & c_n \end{bmatrix}^{\mathrm{T}} = \left[\mathbf{E}_{n+1}\right]^{-1} \begin{bmatrix} C_0 & C_1 & \cdots C_n \end{bmatrix}^{\mathrm{T}} \tag{11}$$

In the light of formula (10) the inverse one-sided $\mathcal{Z}$-Transform of the left-hand side of equation (5) is

$$\mathcal{Z}^{-1}\{C(z^{-1}) y\left(z^{-1}\right)\} = \sum_{i=0}^{n} c_i {}_{k_0}\Delta_k^{(n)} y(k) \tag{12}$$

Similar procedure performed on the right-hand side of equation (5) gives

$$\mathcal{Z}^{-1}\{D(z^{-1}) u\left(z^{-1}\right)\} = \sum_{i=0}^{m} c_i {}_{k_0}\Delta_k^{(n)} u(k-l) \tag{13}$$

D. Closed-loop system with the RST discrete-time controller

The fundamental structure of the closed-loop system with the RST controller [9] is given in Fig.1 where $r(z), e(z), u(z), d(z), y(z)$ are reference, error, control, external disturbance and output signals, respectively.

$$R(z^{-1}) = R_0 + R_1 z^{-1} + \cdots + R_{n-1} z^{-n_r}, \quad (14)$$
$$S(z^{-1}) = S_0 + S_1 z^{-1} + \cdots + S_{n-1} z^{-n_s}, \quad (15)$$
$$T(z^{-1}) = T_0 + T_1 z^{-1} + \cdots + T_{n-1} z^{-n_t}, \quad (16)$$

are polynomials in z^{-1} describing the RST controller. Next,

$$B(z^{-1}) = z^{-l}(B_0 + B_1 z^{-1} + \cdots + B_m z^{-m}), \quad (17)$$
$$A(z^{-1}) = A_0 + A_1 z^{-1} + \cdots + A_n z^{-n}, \quad (18)$$

where $m \leq n, l \in \mathbb{N}_0$. It is easy to verify that the RST controller is described by an equation

$$S(z^{-1})u(z) = T(z^{-1})r(z) - R(z^{-1})y(z) \quad (19)$$

The closed-loop system equations are

$$e(z) = T(z^{-1})r(z) - R(z^{-1})y(z) \quad (20)$$
$$S(z^{-1})u(z) = e(z) \quad (21)$$
$$B(z^{-1})u(z) = A(z^{-1})d(z) + A(z^{-1})y(z) \quad (22)$$

Hence, the corresponding discrete transfer functions of the closed-loop system are as follows

$$y(z) = \frac{\frac{B(z^{-1})}{A(z^{-1})}\frac{T(z^{-1})}{S(z^{-1})}}{1 + \frac{B(z^{-1})}{A(z^{-1})}\frac{R(z^{-1})}{S(z^{-1})})} r(z) - \frac{1}{1 + \frac{B(z^{-1})}{A(z^{-1})}\frac{R(z^{-1})}{S(z^{-1})})} d(z) \quad (23)$$

1) RST controller special case - PID controller: For $T(z^{-1}) = R(z^{-1})$ from (23) we immediately get a simplified expression

$$y(z) = \frac{\frac{B(z^{-1})}{A(z^{-1})}\frac{R(z^{-1})}{S(z^{-1})}}{1 + \frac{B(z^{-1})}{A(z^{-1})}\frac{R(z^{-1})}{S(z^{-1})})} r(z) - \frac{1}{1 + \frac{B(z^{-1})}{A(z^{-1})}\frac{R(z^{-1})}{S(z^{-1})})} d(z) \quad (24)$$

Further, we assume $n_r = 2, n_s = 1$. Then, from (24) we get

$$R(z^{-1}) = R_0 + R_1 z^{-1} + R_2 z^{-2}$$
$$= r_0 + r_1(1 - z^{-1})^1 + r_2(1 - z^{-1})^2, \quad (25)$$
$$S(z^{-1}) = S_0 + S_1 z^{-1} = s_0 + s_1(1 - z^{-1})^1, \quad (26)$$

where according to the algorithm described above we have

$$r_0 = R_0 + R_1 + R_2 \; r_1 = -2R_2 - R_1$$
$$r_2 = R_2 \quad s_0 = S_0 + S_1, \quad s_1 = -S_1, \quad (27)$$

Now we take $S_0 = -S_1$, so $s_0 = 0$ and next $s_1 = -1$. Then, the RST=RSR=PID is

$$\frac{R(z^{-1})}{S(z^{-1})} = \frac{r_0 + r_1(1 - z^{-1})^1 + r_2(1 - z^{-1})^2}{s_1(1 - z^{-1})^1}$$
$$= \frac{r_1}{s_1} + \frac{r_0}{s_1}(1 - z^{-1})^{-1} + \frac{r_2}{s_1}(1 - z^{-1})^1$$
$$= K_P + K_I(1 - z^{-1})^{-1} + K_D(1 - z^{-1})^1 \quad (28)$$

Having evaluated the discrete transfer function of the RST or PID controller and the closed loop system one should apply the inverse one-sided $\mathcal{Z}$-transform. One of the efficient numerical method is numerator and denominator polynomials division method is [16].

II. VFO-RST/PID CONTROLLER

Basing on the RST controller structure we generalize it is admitting a variability of orders σ_i, τ_i, ρ_i i.e. we consider orders as discrete-time functions $\sigma_i(k), \tau_i(k), \rho_i(k)$. Hence, the VFO-RST equation takes the form

$$\sum_{i=0}^{s_{n_s}} s_{ik_0} \Delta_k^{[\sigma_i(k)]} u(k)$$
$$\sum_{i=0}^{t_{n_t}} t_{ik_0} \Delta_k^{[\tau_i(k)]} r(k) - \sum_{i=0}^{r_{n_r}} r_{ik_0} \Delta_k^{[\rho_i(k)]} y(k) \quad (29)$$

where the order functions $\sigma_i(k) > \sigma_{i-1}(k)$ for $i = 0, 1, \cdots, n_s$, $\sigma_0 = 0$, $\rho_i(k) > \rho_{i-1}(k)$ for $i = 0, 1, \cdots, n_r$, $\sigma_0 = 0$, $\tau_i(k) > \tau_{i-1}(k)$ for $i = 0, 1, \cdots, n_t$, $\tau_0 = 0$.

III. FO-PID AND VFO-PID CONTROLLER TRANSIENT CHARACTERISTICS

Now in (29) we take: $s_{n_s} = 0$, $s_0 = 1$, $t_{n_t} = r_{n_r} = 2$, $t_i = r_i$ and denote $e(k) = r(k) - y(k)$. Then, we get

$$u(k) = \sum_{i=0}^{3} t_{ik_0} \Delta_k^{[\tau_i(k)]} e(k) \quad (30)$$

Next we assume $t_0 = K_P$, $\tau_0(k) = 0$, $t_1 = K_I$, $\tau_1(k) = -\mu(k)$, $t_2 = K_D$, $\tau_2(k) = \nu(k)$ with $0 < \mu(k) = \nu(k) \leq 1$ for $k \in \mathbb{N}_0$. For mentioned assumptions we get a description of the VFO-PID controller

$$u(k) = K_P e(k) + K_I \Delta_k^{[-\mu(k)]} e(k) + K_D \Delta_k^{[\nu(k)]} e(k) \quad (31)$$

Now we prove a proposition relating the VFO-PID controller to the solutions of the FO-PID controllers. First we define a set of solutions of equations

$$u_i(k) = K_P e(k) + K_I \Delta_k^{[-\mu(i)]} e(k) + K_D \Delta_k^{[\nu(i)]} e(k) \quad (32)$$

for $i, k \in \mathbb{N}_0$.

Proposition 2. *For a given $e(k)$ and zero initial conditions the solution of equation (31) equals*

$$u(k) = \sum_{i=0}^{k} u_i(i)\delta(k - i) \quad (33)$$

Proof. For $k \in \mathbb{N}_0$ from (33) and (32) we have

$$u(k) = u_k(k) \quad (34)$$

Collecting all such equalities we get (31). This ends the proof. □

Comment 1. *The effect of the above Proposition can be explained as follows. We can treat the VFO-PID action as a consecutive change of VO-PID controllers by some key which connects a separate VO-PID controller to the main output for only one discrete-time instant. A block diagram of such a controller is presented in Fig. 2. Thus, for one "frozen" time instant l the controller $PI^{-\mu(l)}D^{\nu(l)}$ output is unequivocally characterized by the $PI^{-\mu(k)}D^{\nu(k)}$ output.*

Comment 2. *Every block in the VFO-PID controller can be described by the transfer function [12] $G_{VFO-PID}(z) = K_P + K_I(1-z^{-1})^{-\mu(i)} + K_D(1-z^{-1})^{\nu(i)}$ for $i = 0, 1, \cdots, k$. Hence, we can denote this set of discrete transfer functions as $G_{VFO-PID}(z, i)$. The proposed approach enables us to apply the one-sided $\mathcal{Z}$-transform to the VFO difference equations (29).*

The numerical example confirms the proposed explanation of the VFO-PID controller action.

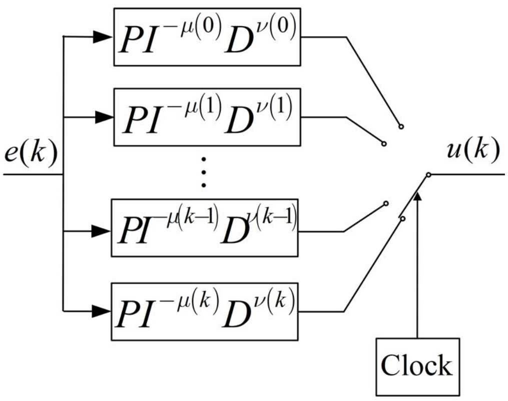

Fig. 2. VFOPID controller possible realization scheme.

A. Numerical example

Consider VFO-PID controller with $K_P = 3.1837$, $K_I = 0.1538$, $K_D = 1.5705$ and the order functions $\nu(k) = \mu(k) = 1.0 - 0.5e^{-0.1k}$. The plot of the order functions is presented in Fig. 3.

The unit step responses $u_i(k)$ evaluated for $i = 1, 2, \cdots, 10$ (in black) and $u(k)$ (in red) are presented in Fig. 4

Next we consider a case, in which the order functions are different $\nu(k) = 1.0 - 0.5e^{-0.1k}$ and $\mu(k) = 1.0 - 0.5e^{-0.01k}$. Considered order functions are plotted in Fig. 5.

Related plots are given in Fig.6.

Comment 3. *We can realize that the VFO-PID controller unit step response (red plot) jumps from the FO-PID controllers defined for constant orders $\nu(i)$ (blue plots) according to rule (33).*

The proposed approach in the VFO-RST/PID controller response evaluation suggests an inverse operation. Every plot

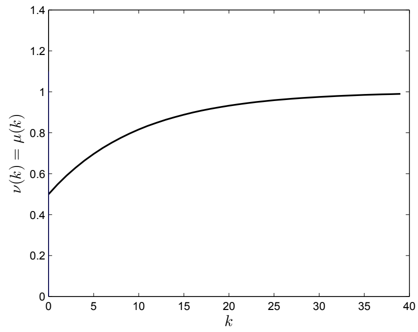

Fig. 3. Plot of the order functions.

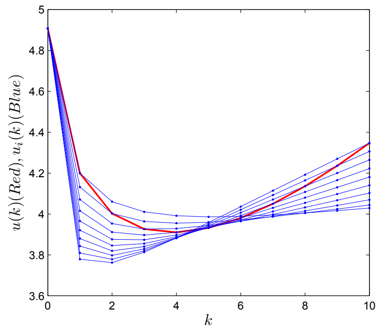

Fig. 4. VFO-PID unit step response.

$u_i(k)$ (in blue) in Figs. 4 and 6 for $k \in \mathcal{N}_0$ is related to a pair $0 \leqslant \nu(i), \mu(i) \leqslant 1$ where $i = const$.

Proposition 3. *A set of responses $u_i(k)$, $k \in \mathbb{N}_0$ related to constant order functions ν_i, μ_i, $i \in \mathbb{N}_{0,l}$ ie. $u_i(k, \nu_i, \mu_i)$ is defined. For $u(k)$ such that $\forall k \in \mathbb{N}_0, \exists i \in \mathbb{N}_{0,l} : u(k) = u_i(k)$ there exists a function of indexes $i(k) = \gamma(k)$ such that $u(k) = u\left(k\nu_{\gamma(k)}, \mu_{\gamma(k)}\right)$.*

Proof. The function $i(k)$ may be defined step by step. For $k = 0$ we find $u_i(0)$ such that $u(0) = u_i(0)$ and $\gamma(0) = i$. In a case that there are $i_1, i_2, \cdots, m$ as i we take any (we suggest taking the lowest value). Hence, we have two function γ values $\gamma(0)$. Continuing this procedure we define function γ. This means that $u(k) = u\left(k, \nu_{\gamma(k)}, \mu_{\gamma(k)}\right)$ for $k \in \mathbb{N}_0$. □

The essence of the Proposition 3 is supported by a numerical example.

B. Numerical example

Two responses denoted as $u_i(k, \nu_i, \mu_i)$ for $i = 1, 2$ evaluated for constant FOs pairs ν_i, μ_i are plotted in Fig. 7. First one is marked by black points (.) whereas second one by blue asterisks (*). Due to Proposition 3.

There can be defined a response $u(k, \nu(\gamma(k)), \mu(k, \nu(\gamma(k))))$ where

$$\gamma(k) = \begin{cases} 1 & \text{for} \quad \text{even k and 0} \\ 2 & \text{for} \quad \text{odd k} \end{cases} \tag{35}$$

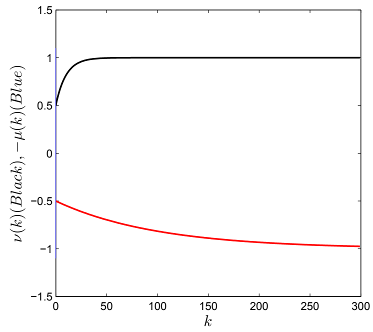

Fig. 5. Plot of the order functions.

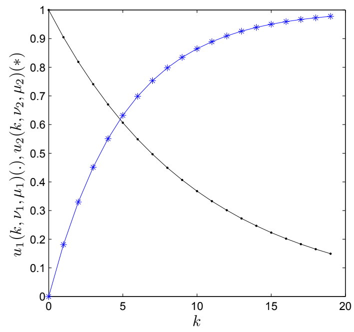

Fig. 7. Plot of u_1 and u_2

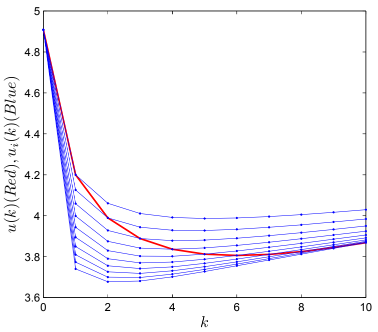

Fig. 6. Plot of the order functions.

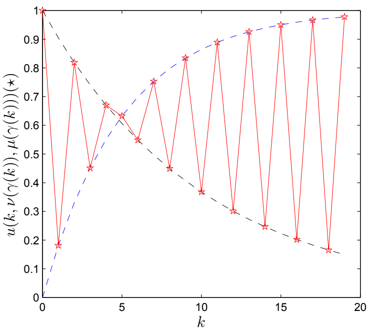

Fig. 8. Plot of $u(k)$.

Plot of $u(k)$ is given in Fig.8

C. Time-variant VFO-RST/PID controller

Similar arguments may be used in an explanation of the transient characteristic of the time-variant VFO-PID controller. Here, we assume the controller gains being the functions of time. Hence, from (31) we get

$$u(k) = \left[K_P(k) + K_I(k)\Delta_k^{[-\mu(k)]} + K_D(k)\Delta_k^{[\nu(k)]}\right] e(k) \tag{36}$$

where $K_{P_{min}} \leq K_P(k) \leq K_{P_{max}}$, $K_{I_{min}} \leq K_I(k) \leq K_{I_{max}}$ and $K_{D_{min}} \leq K_D(k) \leq K_{D_{max}}$.

D. Numerical example

Consider FO-PID controller with $K_P = 3.1837$, $K_D = 1.5705$ and the orders $\nu = 0.5, \mu = 0.75$. The integration gain is a function of a discrete-time K_I

$$K_I(k) = \begin{cases} K_{Imin} + k\frac{K_{Imax} - K_{Imin}}{L} & \text{for} \quad 0 \leq k \leq L \\ K_{Imax} & \text{for} \quad k > L \end{cases} \tag{37}$$

where $L = 30$, $K_{Imin} = 0.1$, $K_{Imax} = 0.2$. The plots of the integration gain (in black), FO-PID controllers with constant gains (in blue) and the time-variant FO-PID controller (in red) are given in Figs. 9 and 10, respectively.

Finally, we analyse the case where the integration gain is defined as $K_I(k) = 1 - e^{-0.7(k-1)}$. Related plots are given in Figs.11 and 12, respectively.

IV. Conclusions

There are still open problems some of which which we mention below

- The stability conditions for a linear discrete-time closed-loop system with VFO-PID or more general VFO-RST controller are unknown. There are only some indications drawn from many simulations of the closed-loop system stability with such controllers.
- There is a very promising application of the considered VFO-RST controllers to the robust systems synthesis.
- The considered in the papers VFO-RST/PID controllers can be applied in the control systems with non-linear plants. The variability of controller orders may reduce the non-linearities of the closed-loop systems.

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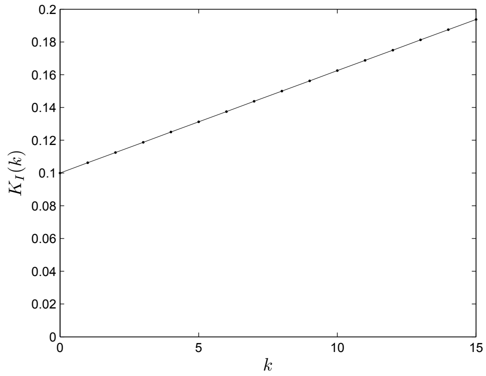

Fig. 9. Plot of the integration gain.

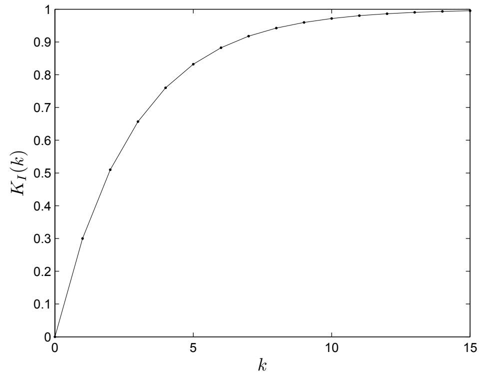

Fig. 11. Plot of the integration gain $K_I(k)$.

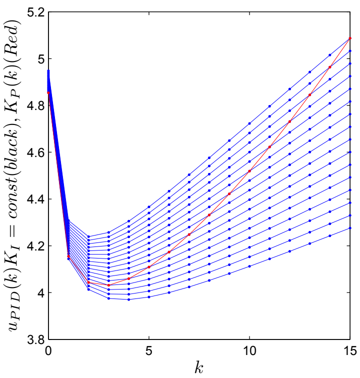

Fig. 10. Unit step responses of the FO-PID controllers for $K_I(k)$ and the time-variant FO-PID.

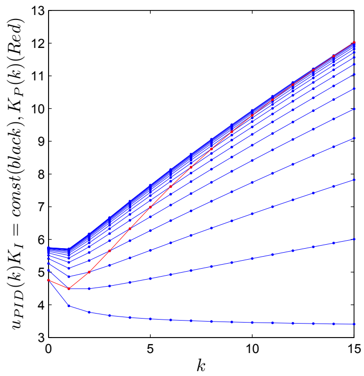

Fig. 12. Unit step responses of the FO-PID controllers for $K_I(k)$ and the time-variant FO-PID.

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