CRA Based Control Of Incommensurate Fractional Order Systems

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***Abstract*— Shaping the transient response of incommensurate fractional order systems is a challenge. In this paper, incommensurate fractional order controllers are employed to control a two-order incommensurate fractional order system. An appropriate RST control structure is employed to construct a closed loop system with a commensurate fractional order transfer function. Then, the characteristic ratios of this transfer function are chosen to reach a low overshoot step response. The transient response speed is adjusted by selecting the appropriate generalized time constant. A numerical example is given to show the efficiency of the proposed method for shaping the transient response of an incommensurate fractional order system.**

Keywords—Incommensurate fractional order systems; RST controller; Characteristic ratios assignment; Transient response control

I. INTRODUCTION

Fractional order systems are a special case of infinite dimensional systems describing with non-integer differential equations [1]. Utilizing fractional order operators in their dynamics could increase the modelling accuracy. Therefore, proposing new control methods for these systems has been considered in the literature [2]. Few works have been reported for shaping the transient response specifications of fractional order systems [3]. In [4], some transient response specifications of a three-term fractional order system (like settling time and overshoot) have been calculated. In [5], a sufficient condition for overshoot existence in the step response of fractional order systems is derived. It is shown that overshoot in step response of a commensurate fractional order system with commensurate order between 1 and 2 is inevitable [6]. The transient response and the frequency response of a system are two related ways for describing its dynamics. In [7], some algebraic conditions are proposed for attaining a monotonically decreasing magnitude-frequency response in all-pole fractional order systems. This could lead to a non-overshooting step response.

One of the most common methods for shaping the transient response of dynamical systems is the Characteristic Ratios Assignment (CRA) method [8]. In this method, the characteristic ratios are adjusted to attain a low overshoot step response. Then, the transient response speed could be adjusted by the generalized time constant, separately. Several patterns have been proposed for the characteristic ratios for integer order systems in the literature [9, 10]. In [11], an alternative pattern for characteristic ratios assignment in commensurate fractional order systems with commensurate order between 0.5 and 1 has been proposed. The proposed pattern has been modified to cover any all-pole commensurate fractional order system [12]. The sensitivity of the transient response to the proposed characteristic ratios has been investigated, too [13]. A complementary polynomial has been employed to generalize the proposed pattern for non-minimum phase fractional-order plants [14]. In [15], CRA control of fractional order systems subject to control signal constraint using the CRA method has been studied. In [16], the moment matching and the CRA approach have been utilized to design fractional order PID (FOPID) controllers for commensurate fractional order systems.

All the previously published works about CRA for fractional order systems have been devoted to commensurate fractional order systems. On the other hand, some of the fractional order plants are not commensurate. This means that the fractional orders in their dynamics are not necessarily integer multiples of a definite order (called commensurate order). Shaping the transient response of incommensurate fractional order systems is difficult. In classical control, the feedback linearization method is a common method to control nonlinear plants. In this method, a nonlinear plant is linearized using an appropriate nonlinear controller. This means that the nonlinear terms in the plant are canceled by means of the similar nonlinear terms in the controller structure [17]. This idea is employed in this paper for shaping the transient response of an incommensurate fractional order plant. Incommensurate fractional order controllers (in an RST control structure) with appropriate fractional orders are designed to build a commensurate closed loop transfer function. Now, the CRA approach proposed in [11] is utilized to adjust the transient response specifications of the obtained commensurate fractional order transfer function. A numerical example is presented to show the efficiency of the proposed method.

This paper is arranged as follows. Basic concepts about the fractional calculus and the CRA method are given in Section II. In section III, the proposed CRA method for shaping the transient response of a special case of an incommensurate

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fractional order system is illustrated. Simulation results and conclusions are given in Section IV and V, respectively.

II. MATHEMATICAL PRELIMINARIES

In this section, the mathematical background of the fractional calculus and the CRA approach is presented.

A. Brief Review on Fractional Calculus

The Grunwald-Letnikov definition for the fractional order derivative of a function $f(t)$, called ${}_aD_t^\rho f(t)$, is given by [1]

$$ {}_aD_t^\rho f(t) = \lim_{h\to 0} h^{-\rho} \sum_{j=0}^{\left\lfloor \frac{t-a}{h} \right\rfloor} (-1)^j \binom{\rho}{j} f(t-jh) \tag{1}$$

where ρ is the fractional order.

An all-pole incommensurate fractional order system could be represented as

$$M(s) = \frac{d_0}{d_n s^{v_n} + d_{n-1}s^{v_{n-1}} + \cdots + d_1 s^{v_1} + d_0}, \quad n>1 \tag{2}$$

where $v_i, i=1,\ldots,n$ are the fractional orders and $d_i, i=0,\ldots,n$ are the denominator coefficients. In transfer function (2), if $v_i = iv, i=1,\ldots,n$ are considered, then we have

$$M(s) = \frac{d_0}{d_n s^{nv} + d_{n-1}s^{(n-1)v} + \cdots + d_1 s^{v} + d_0}, \quad n>1 \tag{3}$$

where v is the commensurate order.

B. Charcteritic Ratios Assignment in Fractional-order Systems

The characteristic ratios for commensurate transfer function (3) are defined according to the following definition.

Definition 1. The characteristic ratios ($\alpha_k, k=1,\cdots,n-1$) and the generalized time constant (τ) for fractional order system (3) are defined as [11]

$$\alpha_i = \frac{d_i^2}{d_{i-1}d_{i+1}}, i=1,\cdots,n-1, \quad \tau = \left(\frac{d_1}{d_0}\right)^{\frac{1}{v}}. \tag{4}$$

The step response overshoot could be adjusted by the characteristic ratios while the step response speed could be adjusted by means of τ [11].

The denominator coefficients of (3) could be obtained in terms of the characteristic ratios and the generalized time constant as [11]

$$d_1 = \tau^v d_0, \quad d_i = \frac{d_0 \tau^{iv}}{\alpha_{i-1}\alpha_{i-2}^2\alpha_{i-3}^3 \cdots \alpha_1^{i-1}}, \quad i=2,3,\cdots,n. \tag{5}$$

To reach a step response with low overshoot (say 2%), the following alternative pattern for the characteristic ratios has been proposed [11]

$$\alpha_i = \begin{cases} -2\beta\cos(\pi v), & \text{if}: i=2k+1, \\ \dfrac{-2}{\beta\cos(\pi v)}, & \text{if}: i=2k, \end{cases} \quad , \ k\in N \tag{6}$$

where β could be determined through the following relation to attain a step response with 2% overshoot [11]

$$\beta = 1.254 v^{4.717} - 0.05652. \tag{7}$$

III. CRA METHOD FOR INCOMMENSURATE FRACTIIONAL ORDER SYSTEMS

In this section, CRA method is employed for shaping the transient response of special case of incommensurate fractional order plants. To attain this goal, incommensurate fractional order controllers are employed to construct a commensurate closed loop system. This is similar to the idea used for control of nonlinear plants in which nonlinear controllers are employed to eliminate the nonlinear terms in the plant dynamics. Then, the pattern (7) is utilized to assign the characteristic ratios of this transfer function.

Now, consider a two-order incommensurate fractional order system as

$$G(s) = \frac{b_0}{s^{v_2} + a_1 s^{v_1} + a_0}, \quad v_2 > 2v_1 \tag{8}$$

where b_0, a_1, a_0, v_2 and v_1 are arbitrary real numbers and $v_2 \neq kv_1, k\in N$. The condition $v_2 > 2v_1$ is necessary for the design procedure.

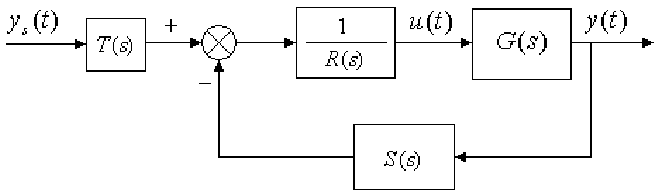

Fig. 1. The RST control structure.

Now, consider the RST control structure shown in Fig. 1. $y_s(t)$, $u(t)$ and $y(t)$ are the reference signal, the control signal and the output signal, respectively. The pseudo polynomials $R(s)$, $S(s)$ and $T(s)$ are considered as

$$R(s)=s^{\gamma_2}+r_1s^{\gamma_1}+r_0, \qquad T(s)=t_0$$
$$S(s)=s_2s^{v_2}+s_1s^{v_1}+s_{02}s^{2\lambda}+s_{01}s^{\lambda}+s_2's^{\gamma_2}+s_1's^{\gamma_1} \tag{9}$$

where

$$\lambda=v_2-v_1,\ \gamma_1=2v_2-3v_1,\ \gamma_2=3v_2-4v_1. \tag{10}$$

The control signal $u(t)$ could be calculated as

$$u(t)=\frac{T(p)}{R(p)}y_s(t)-\frac{S(p)}{R(p)}y(t) \tag{11}$$

where p is the derivative operator. According to (11), the control signal $u(t)$ is realizable if the following inequalities are satisfied

$$\deg(T(s))\le\deg(R(s)), \qquad \deg(S(s))\le\deg(R(s)) \tag{12}$$

where $\deg(Q(s))$ is the maximum degree of a pseudo polynomial $Q(s)$. Consider that $v_2>2v_1$. Thus, $\max(\gamma_2,\gamma_1,\lambda,2\lambda,v,v_2)=\gamma_2$. This means that $\deg(S(s))=\deg(R(s))=\gamma_2$. Therefore, the condition (12) is fulfilled. This is the reason why the condition $v_2>2v_1$ is considered.

The following relations will be obtained according to (10)

$$v_2+\gamma_2=4\lambda,\ v_2+\gamma_1=v_1+\gamma_2=3\lambda,\ v_1+\gamma_1=2\lambda. \tag{13}$$

Now, according to (8), (9) and (13), the closed loop transfer function for the control structure shown in Fig.1 is obtained as

$$H(s)=\frac{Y(s)}{Y_s(s)}=\frac{b_0t_0}{P(s)} \tag{14}$$

where

$$\begin{aligned}P(s)=&s^{4\lambda}+(a_1+r_1)s^{3\lambda}+(a_1r_1+b_0s_{02})s^{2\lambda}+b_0s_{01}s^{\lambda}+\\&(a_0+b_0s_2')s^{\gamma_2}+(a_0r_1+b_0s_1')s^{\gamma_1}+(r_0+b_0s_2)s^{v_2}+\\&(a_1r_0+b_0s_1)s^{v_1}+a_0r_0\end{aligned}. \tag{15}$$

According to (15), the transfer function (14) is commensurate if the following relations are satisfied

$$a_0+b_0s_2'=a_0r_1+b_0s_1'=r_0+b_0s_2=a_1r_0+b_0s_1=0. \tag{16}$$

Now, the transfer function of the closed loop system becomes

$$H(s)=\frac{b_0t_0}{s^{4\lambda}+(a_1+r_1)s^{3\lambda}+(a_1r_1+b_0s_{02})s^{2\lambda}+b_0s_{01}s^{\lambda}+a_0r_0}. \tag{17}$$

The ideal closed loop transfer function ($H_d(s)$)is considered as

$$H_d(s)=\frac{d_0'}{s^{4\lambda}+d_3's^{3\lambda}+d_2's^{2\lambda}+d_1's^{\lambda}+d_0'}. \tag{18}$$

Consider that the denominator of the required closed loop transfer function could be considered a monic pseudo polynomial in the general case (it could be achieved by dividing nominator and denominator of (2) by d_n or $d_i'=\frac{d_i}{d_n}, i=0,\dots,n-1$).

Matching the transfer functions (17) and (18) gives

$$b_0t_0=d_0', a_1+r_1=d_3', a_1r_1+b_0s_{02}=d_2', b_0s_{01}=d_1', a_0r_0=d_0'. \tag{19}$$

Solving equations (16) and (19) yields the following controller parameters

$$\begin{aligned}&r_1=d_3'-a_1, r_0=\frac{d_0}{a_0}, t_0=\frac{d_0}{b_0}, s_2=\frac{-d_0}{a_0b_0}, s_1=\frac{-d_0a_1}{a_0b_0}\\&s_{02}=\frac{d_2-a_1(d_3-a_1)}{b_0}, s_{01}=\frac{d_1}{b_0}, s_2'=\frac{-a_0}{b_0}, s_1'=\frac{-a_0(d_3-a_1)}{b_0}.\end{aligned} \tag{20}$$

Consider that for realizing pseudo polynomials (9), approximation with high-order integer transfer functions could be employed.

Remark 1. The relation (10) for the fractional orders of the incommensurate controllers is not unique. Other combinations of the fractional orders may be utilized to reach a commensurate fractional order system.

Remark 2. The same idea used for plant (8) (with two fractional orders) could be utilized for any incommensurate fractional order plant with more fractional orders.

Remark 3. If $0.5<v_2-v_1\le 1$, then, the pattern (7) could be employed for characteristic ratios assignment of the closed loop transfer function. If $0<v_2-v_1\le 0.5$, then the method presented in [12] could be utilized.

Remark 4. For incommensurate fractional order plants with stable zeros, the zeros could be canceled through the pseudo polynomial $R(s)$ and the same procedure could be applied in this case, too. For non-minimum phase plants, the idea of the complementary polynomials presented in [14] could be employed.

IV. SIMULATION RESULTS

In this section, a numerical example is presented to show the effectiveness of the proposed method

Example 1. Consider the following incommensurate fractional order system

$$G(s) = \frac{20}{s^{1.5} + 0.25s^{\frac{2}{3}} + 10}. \quad (21)$$

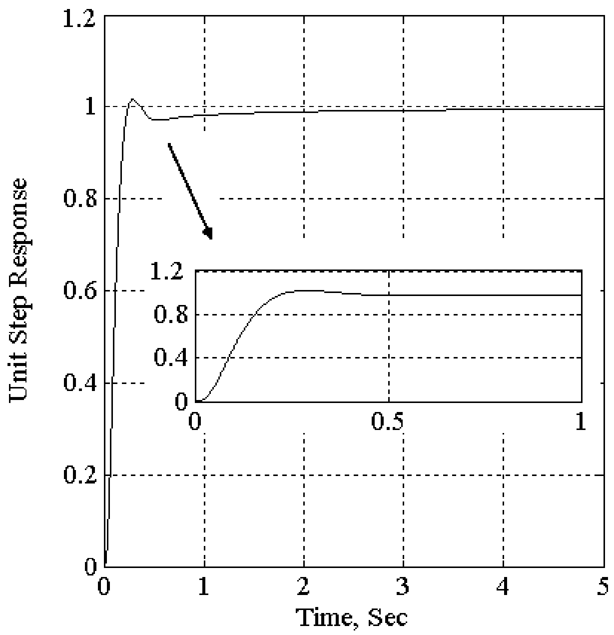

Fig. 2. The closed loop system unit step response for Example 1.

It is obvious that $(1.5)/(\frac{2}{3}) = 2.25$ is not an integer number. Thus, system (21) is incommensurate. According to (10), $\lambda = 0.833, \gamma_1 = 1, \gamma_2 = 1.8333$ are obtained. According to (6) and (7), the following characteristic ratios are obtained

$$\alpha_1 = 0.8212, \alpha_2 = 4.8709, \alpha_3 = 0.8212. \quad (22)$$

To reach $0.2\,\mathrm{sec}$ settling time, $\tau = 0.0712$ is selected (see [10] and [11] for choosing τ). The required closed loop transfer function is obtained as

$$H_d(s) = \frac{23246.1}{s^{3.3333} + 22.38s^{2.5} + 609.86s^{1.6667} + 3412.06s^{0.8333} + 23246.1}. \quad (23)$$

The control pseudo polynomials are obtained as

$$\begin{aligned} &R(s) = s^{1.8333} + 29.45s + 7212.5, \qquad T(s) = t_0 \\ &S(s) = -360.62s^{1.5} - 90.1558s^{0.6667} + 53.34s^{1.6667} \\ &+398.82s^{0.8333} - 0.5s^{1.8333} - 14.725s \end{aligned} \quad (24)$$

For transient response simulations, the FOTF MATLAB toolbox presented in [18] is employed. Fig. 2 show the unit step response for the closed loop system. The overshoot is about 1.6% and the settling time is $0.2\,\mathrm{sec}$. The control signal is shown in Fig.3. The control signal has acceptable values (the ratio between its maximum and final values is about 3.2).

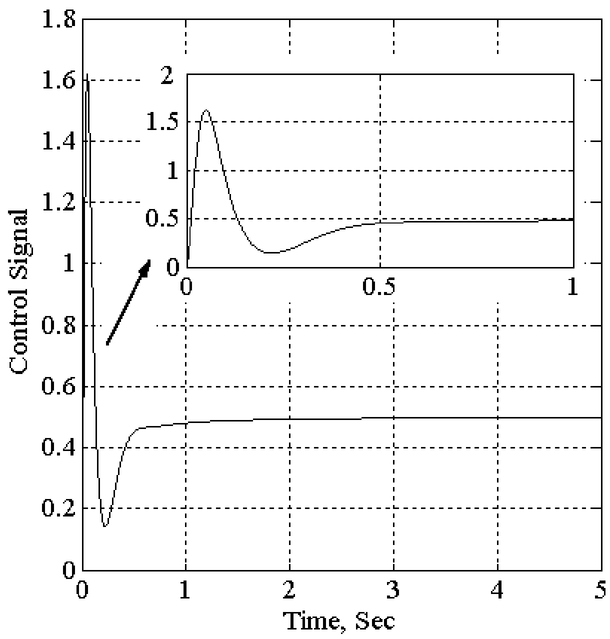

Fig. 3. The control signal for Example 1.

V. CONCLUSION

Control of a two-order incommensurate fractional order system using appropriate incommensurate fractional order controllers is the main idea of this paper. The controller pseudo polynomials in an RST control structure are selected such that a commensurate closed loop system is obtained. This is similar to the idea utilized in feedback-linearized controllers. For shaping the transient response of the plant, the closed loop transfer function is matched with a desired transfer function obtained from the CRA approach. Generalizing this idea for any incommensurate fractional order system with arbitrary number of orders could be considered as a future work. Moreover, the robustness analysis of the proposed controller against the plant parameters variations could be verified, later.

REFERENCES

[1] I. Podlubny,Fractional Differential Equations. Academic Press, New York, 1999.

[2] C.A. Monje, Y.Q. Chen, B.M. Vinagre, D. Xue, and V. Feliu, Fractional-order Systems and Controls. Springer-Verlag, London, 2010.

[3] M.S. Tavazoei, "Time response analysis of fractional-order control systems: A survey on recent results," Fract. Calc. Appl. Anal., vol. 17, pp. 440-461, June 2014.

[4] F. Merrikh-Bayat and M. Karimi-Ghartemani, "Some properties of three-term fractional order system," Fract. Calc. Appl. Anal., vol. 11, pp. 317 – 328, September 2008.

[5] M.S. Tavazoei, "Overshoot in the step response of fractional-order control systems," J. Process Contr., vol. 22, pp. 90-94, January 2012.

[6] M.S. Tavazoei, "On Monotonic and nonmonotonic step responses in fractional order systems," IEEE T. Circuits-II, vol. 58, pp. 447 – 451, July 2011.

[7] M.S. Tavazoei, "Algebraic conditions for monotonicity of magnitude-frequency responses in all-pole fractional order systems," IET Control Theory A., vol. 8, pp. 1091-1095, August 2014.

[8] P. Naslin, Essentials of Optimal Control. Technical Publishers, Boston, 1969.

[9] S. Manabe, "Coefficient diagram method," Proceedings of the 14th IFAC symposium on Automatic Control in Aerospace, Seoul, Korea, pp. 199-210, August 1998.

[10] Y.C. Kim, L.H. Keel, and S.P. Bhattacharyya, "Transient response control via characteristic ratio assignment," IEEE T. Automat. Contr., vol. 48, pp. 2238-2244, December 2003.

[11] M. Tabatabaei and M. Haeri, "Characteristic ratio assignment in fractional order systems," ISA T., vol. 49, pp. 470-478, October 2010.

[12] M. Tabatabaei and M. Haeri, "Characteristic ratio assignment in fractional order systems (case $0 < v \leq 0.5$)," T. I. Meas. Control, vol. 35, pp. 360-374, May 2013.

[13] M. Tabatabaei and M. Haeri, "Sensitivity analysis of CRA based controllers in fractional order systems," Signal Process., vol. 92, pp. 2040-2055, September 2012.

[14] M. Tabatabaei and M. Haeri, "CRA based Control of non-minimum phase fractional order systems," Proceedings of the International Conference on Control Applications, Dubrovnik, Croatia, pp. 855 – 858, October 2012.

[15] M. Tabatabaei and M. Haeri, "CRA based control of fractional order systems subject to control signal constraint,' Proceedings of the 11th International Conference on Control, Automation and Systems, Gyeonggi-do, Korea, pp. 268-272, October 2011.

[16] M. Tabatabaei and M. Haeri, "Design of fractional order proportional-integral-derivative controller based on moment matching and characteristic ratio assignment method,' P. I. Mech. Eng. I-J. Sys., vol. 225, pp. 1040-1053, December 2011.

[17] J-J. E. Slotine and W. Li, Applied Nonlinear Control. Prentice Hall, USA, 1991.

[18] Y.Q. Chen, I. Petras, and D. Xue, "Fractional order control: a tutorial," Proceedings of the American Control Conference, St. Louis, MO., USA, pp. 1397-1411, June 2009.