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STEADY-STATE ERROR-FREE RST-CONTROLLER DESIGN : A DOUBLE DIOPHANTINE EQUATION APPROACH

Eric Ostertag

LSIIT UPRES-A CNRS 7005
Ecole Nationale Supérieure de Physique de Strasbourg, Université de Strasbourg I
Parc d'Innovation, bd. Sébastien Brant, F-67400 ILLKIRCH, France
Fax:+33 3 88 65 54 89; e-mail : Eric.Ostertag@ensps.u-strasbg.fr

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Abstract

RST-controller design by pole placement has been introduced by several authors about fifteen years ago. However, in most of these works, the elimination of steady-state errors resulting from low-frequency inputs has been dealt with only for disturbance inputs, not for reference inputs. As to the latter input, a DC unity gain has been usually taken into account, thus insuring a zero input-output error for stepwise reference inputs, but not for signals of order higher than zero. In this paper, a new design method of RST-controllers is derived, which takes this error criterion into account by means of a *secondary* diophantine equation, in addition to the usual one.

1 Introduction

Among the numerous design methods of a discrete controller for linear SISO sampled-data systems, a very elegant method has been developed on the basis of pole placement, the formulation of which is often referred to as « RST-control » [1], [2]. Though the authors have dealt with the rejection of low-frequency disturbances, of an order eventually greater than zero, as will be briefly recalled in the next section, they have usually considered the steady-state error with respect to the reference input only for constant inputs (step functions). This is usually taken care of by introducing a constant factor in the desired transfer function of the closed loop, the so-called *model* transfer function, so as to have a DC input-output gain of one. Higher order inputs, such as ramp-type reference functions, are however not dealt with. According to those designs, also, this model transfer function is chosen entirely *a priori*, by assigning a desired behaviour to the closed loop, for instance that of a second order system, with the only restriction of the previously mentioned gain factor and the necessity to include in its numerator the zeros of the process lying outside the unit disc and its pure delays.

In this paper we derive a modified design method of RST-controllers, by which the cancellation of the steady-state error with respect to a reference input of any order is guaranteed. As a result, as will be seen, the numerator of the model transfer function cannot be chosen arbitrarily anymore, but is rather determined by a secondary diophantine equation.

2 Problem formulation

Let us assume that a discrete-time process, or *plant*, with control signal $u(k)$, measured output signal $y(k)$ and measurement and load disturbances $e(k)$ and $v(k)$ respectively, described in terms of z-transforms by

$$Y(z^{-1})=\frac{B(z^{-1})}{A(z^{-1})}U(z^{-1})+\frac{B(z^{-1})}{A(z^{-1})}V(z^{-1})+E(z^{-1}), \quad (1)$$

is controlled by an RST-structure (Fig.1) defined by the three polynomials $R(z^{-1})$, $S(z^{-1})$ and $T(z^{-1})$ such that the z-transform of the control law $u(k)$ is given by the following equation

$$U(z^{-1})=\frac{T(z^{-1})}{S(z^{-1})}W(z^{-1})-\frac{R(z^{-1})}{S(z^{-1})}Y(z^{-1}) \quad (2)$$

where $w(k)$ is the reference signal.

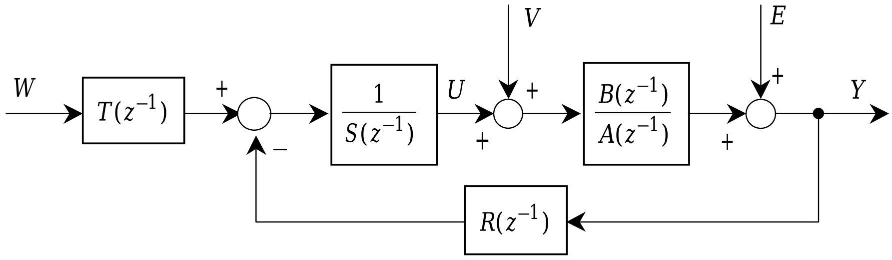

Fig. 1 : Discrete-time system controlled by an RST-regulator

The following relations hold in closed loop :

$$U = \frac{AT}{AS+BR}W - \frac{BR}{AS+BR}V - \frac{AR}{AS+BR}E \qquad (3)$$

and

$$Y = \frac{BT}{AS+BR}W + \frac{BS}{AS+BR}V + \frac{AS}{AS+BR}E \qquad (4)$$

Note that the choice has been made here to write all the equations in terms of negative powers of z and that the variable z^{-1} has been omitted in equations (3) and (4) in order to shorten the expressions, as will be done also in most of this paper unless there is any ambiguity.

The first step of the usual pole placement design consists then in choosing arbitrarily a desired closed-loop transfer function or *model* [1-4],

$$F_m(z^{-1}) = \frac{B_m(z^{-1})}{A_m(z^{-1})} \qquad (5)$$

or eventually only its poles [2], thus the polynomial $A_m(z^{-1})$. The unique restriction imposed usually to that arbitrary choice is that $B_m(z^{-1})$ must contain the zeros of the process lying outside the unit disc, also abusively (though commonly) called its "unstable" zeros, and its pure delays. There is additionally a DC-gain requirement, which will be discussed in Section 3. The choice of writing all polynomials in terms of negative powers of z is here of advantage, since a delay of d sampling times, which appears in the numerator $B(z^{-1})$ of the plant as a factor z^{-d}, is then nothing else than a zero of order d of the plant, located at infinity, thus pertaining to the group of its "unstable" zeros. Note that it may also be advisable to include into $B_m(z^{-1})$ some *extra* zeros of the process, such as poorly damped (though "stable") zeros or "stable" zeros with a negative real part. Letting

$$B = B^+B^- \qquad (6)$$

B_m must be of the form

$$B_m = B^-B_m^{'} \qquad (7)$$

where B^- contains all the "unstable" and extra zeros and pure delays of the plant and $B_m^{'}$ is still arbitrary.

Substituting (7) in (5) and equating the closed loop transfer function for vanishing load and measurement disturbances, $\frac{Y(z)}{W(z)} = \frac{BT}{AS+BR}$, to its expression given by (5) gives $\frac{B^+B^-T}{B^+(AS^{'}+B^-R)} = \frac{B^-B_m^{'}}{A_m}$, where S has been factored as

$$S = B^+S^{'} \qquad (8)$$

in order to cancel the "stable" zeros of the plant. This in turn yields the following system of equations :

$$AS^{'} + B^-R = A_0A_m \qquad (9)$$

$$T = A_0B_m^{'} \qquad (10)$$

where A_0 is an observer polynomial, which contains the closed loop modes that will not be excited from the reference input, or can contain the stable poles of the plant if their cancellation is desired [1]. The second and last step of the design consists then in solving the diophantine equation (9), which will be referred to hereafter as the *primary diophantine equation*, for the two unknown polynomials $S^{'}$ and R, of the smallest possible degree in z^{-1}, and deriving S and T from (8) and (10) respectively.

If the specifications for the closed loop require that constant or low frequency disturbances of some order n should be cancelled in the steady state, an appropriate number of integral terms is introduced into the loop by means of the S polynomial according to the number l of such terms already present in the plant, so that their total number amounts to $n+1$. By letting

$$S^{'} = (1-z^{-1})^{n+1-l}S_1^{'} \qquad (11)$$

equation (9) becomes then :

$$(1-z^{-1})^{n+1}A^{+}A^{-}S_1^{'}+B^{-}R=A_0A_m \tag{12}$$

where $A(z^{-1})$ has been factored in its integral terms, stable and unstable parts according to :

$$A(z^{-1})=(1-z^{-1})^{l}A^{+}A^{-} \tag{13}$$

3 Input error cancellation

In order to ensure an input-output gain of one for the closed loop in the steady state, which is a very common requirement, a constant gain factor is usually introduced into B_m such that $B_m(1)=A_m(1)$. This does not, however, make the steady-state error vanish in case of a reference input of higher order than a step function, even though the loop might already have a sufficient number of integrators. This would only occur if the polynomials T and R were identical, which is not the case in the general RST-design.

In the general case, assuming furthermore zero disturbances e and v, the error $\mathcal{E}$ between reference and measured output signals is given, according to equation (4), by :

$$\mathcal{E}(z^{-1})=W-Y=\left(1-\frac{BT}{AS+BR}\right)W$$
$$=\frac{AS+B(R-T)}{AS+BR}W=\frac{AS^{'}+B^{-}(R-T)}{AS^{'}+B^{-}R}W$$

after simplification by B^{+}. Choosing a reference of the form $w(t)=t^{m}$, having thus a z-transform

$$W(z)=\frac{W_1}{(1-z^{-1})^{m+1}}, \tag{14}$$

where W_1 is some polynomial in z^{-1}, yields :

$$\mathcal{E}(z^{-1})=\frac{AS^{'}+B^{-}(R-T)}{AS^{'}+B^{-}R}\,\frac{W_1}{(1-z^{-1})^{m+1}}$$

The application of the final limit theorem shows then that, for the steady-state error $\varepsilon^{*}(\infty)=\lim_{z\to 1}\left[(1-z^{-1})\mathcal{E}(z^{-1})\right]$ to vanish, it is mandatory that $(1-z^{-1})^{m+1}$ divides the expression $AS^{'}+B^{-}(R-T)$. On the other hand, noting from (7) and (10) that $B^{-}T=A_0B_m$ and subtracting this equation from (9), the following result is obtained :

$$AS^{'}+B^{-}(R-T)=A_0(A_m-B_m) \tag{15}$$

Since obviously A_0 and $(1-z^{-1})$ are coprime, (15) implies then that $(1-z^{-1})^{m+1}$ must divide the second term of its right side :

$$A_m-B_m=(1-z^{-1})^{m+1}L(z^{-1}) \tag{16}$$

where $L(z^{-1})$ is some yet unknown polynomial, to be determined.

The first consequence of (16) is that $B_m^{'}$, and thus B_m, can no longer be chosen arbitrarily. Indeed, B^{-} being imposed as previously by the “unstable” zeros and pure delays of the plant, $L(z^{-1})$ and $B_m^{'}$ will now both be obtained by solving the following equation derived directly from (16) :

$$(1-z^{-1})^{m+1}L+B^{-}B_m^{'}=A_m \tag{17}$$

which will be called hereafter *auxiliary (or secondary) diophantine equation*.

The case of the DC input-output unity gain, corresponding to $m=0$, is embedded in this derivation, since it results from equation (16) that, for $m\geq 0$, $A_m(1)=B_m(1)$.

4 Numerical example

Assume a process is given by

$$G(z)=\frac{2z^{-1}(1+2z^{-1})}{(1-z^{-1})(1-0.3z^{-1})}$$

This plant is to be controlled so that the closed loop has the following characteristic polynomial :

$$A_m(z^{-1})=1-0.7417z^{-1}+0.2020z^{-2}$$

and so that it has zero steady-state error in response to a ramp reference signal.

According to (6), we factor the numerator of G as follows :

$$B^{+}=1$$
$$B^{-}=2z^{-1}(1+2z^{-1})$$

the plant having two “unstable” zeros, one at $z=-2$ and one at infinity. Since here $m=1$ the auxiliary diophantine equation (17) becomes :

$$(1-z^{-1})^{2}L+2z^{-1}(1+2z^{-1})B_m^{'}=1-0.7417z^{-1}+0.2020z^{-2}$$

which gives the following lowest degree solution :

$B_m^{'}(z^{-1}) = 0.2609 - 0.1841\, z^{-1}$ and $L(z^{-1}) = 1 + 0.7366\, z^{-1}$

Note that, at the difference with many authors [1-4], the polynomial B_m is not chosen arbitrarily, but results from equation (7) :

$$B_m = B^- B_m^{'} = 0.5217\, z^{-1} + 0.6752\, z^{-2} - 0.7366\, z^{-3}$$

which in turn yields the transfer function that the closed loop will have. No disturbance rejection being specified here, the primary diophantine equation to be used is then (9), which writes with a choice of $A_0 = 1$:

$$(1 - z^{-1})(1 - 0.3z^{-1})S^{'} + 2z^{-1}(1 + 2z^{-1}) = 1 - 0.7417\, z^{-1} + 0.2020\, z^{-2}$$

and has the following solution :

$R(z^{-1}) = 0.1031 - 0.0264\, z^{-1}$ and $S(z^{-1}) = 1 + 0.3521\, z^{-1}$

From (10) we derive then :

$$T(z^{-1}) = B_m^{'}(z^{-1}) = 0.2609 - 0.1841\, z^{-1}$$

The corresponding response to a unit ramp is plotted in Fig. 2 (circles).

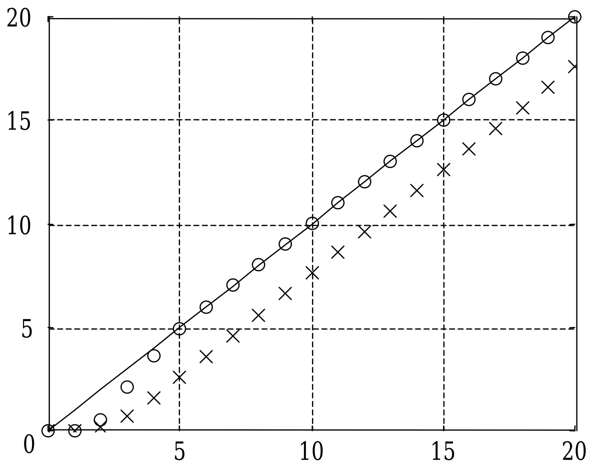

Fig. 2: Ramp response of example system:
circles = design with double diophantine equation
crosses = design with insertion of integrator in $S(z^{-1})$

If, on the contrary, the design of the controller is made by introducing a second integrator into the loop, by means of the polynomial $S(z^{-1})$ as in equation (11) of Section 2, and determining then the polynomials R and S from the primary diophantine equation (12) after having chosen arbitrarily B_m with the restrictions of satisfying (7) and ensuring that $B_m(1) = A_m(1)$, the following solution is obtained :

$$R(z^{-1}) = 0.3285 - 0.3194\, z^{-1} + 0.0676\, z^{-2}$$

$$S(z^{-1}) = (1 - z^{-1})(1 + 0.9014\, z^{-1}) = 1 - 0.09863 z^{-1} - 0.9014\, z^{-2}$$

$$T(z^{-1}) = 0.07672$$

As was expected, this control law *does not* cancel the steady state error in response to a ramp input, as is illustrated by the crosses in Fig. 2.

5 Embedding of other controller design algorithms

The case of the more « classical » cascade compensator, represented by a transfer function $C(z)$ inserted into the direct path of a unity-feedback control loop (Fig. 3), can be considered as a subclass of the RST-structure, in which :

$$R = T \tag{18}$$

so that :

$$C(z) = \frac{R(z^{-1})}{S(z^{-1})} = \frac{T(z^{-1})}{S(z^{-1})} \tag{19}$$

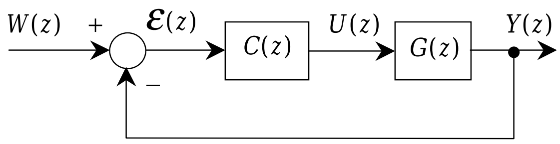

Fig. 3 : Control loop with cascade regulator and unity feedback

The usual pole-zero cancellation presented in early textbooks on digital control [5] and developed initially by Zdan [6], consists in selecting a regulator which cancels the stable poles and zeros of the process, adds the required number of integrators in the loop, according to the order m of the steady-state error to be cancelled and the number l of such poles already present in the process, and introduces additional terms so as to match the number of specifications to be satisfied. Note that, due to the simple structure of the control loop in this case, the order of the steady-state errors to be cancelled with respect to the reference input and with respect to disturbances is necessarily the same, so that here $n = m$. If the process is put as above in factored form :

$$G(z) = \frac{B^+(z^{-1})B^-(z^{-1})}{(1 - z^{-1})^l A^+(z^{-1})A^-(z^{-1})}$$

the regulator's expression is then of the form

$$C(z) = \frac{1}{(1 - z^{-1})^{m+1-l}} \frac{A^+}{B^+} \frac{\Delta_1}{\Delta_2}$$

where Δ_1 and Δ_2 are unknown polynomials in z^{-1} of smallest possible degree, to be determined. They are obtained by equating the closed loop characteristic polynomial to the denominator A_m of the desired transfer function, thus by solving

$$(1-z^{-1})^{m+1}A^-\Delta_2 + B^-\Delta_1 = A_m \tag{20}$$

If in (12) the substitution $A_0 = A^+$ is made, since it is desired here to cancel the stable process poles, as well as $R = A^+B_m^{'}$ which results from the substitution of (18) into (10) under use of the previous relation, it appears readily after division by A^+ that (20) and (12) are the same, with $\Delta_1(z^{-1}) = B_m^{'}(z^{-1})$ and $\Delta_2(z^{-1}) = S_1^{'}(z^{-1})$.

A subclass of the previous one is the *finite time settling controller*. The aim of this design method is that the sampled error signal be cancelled in a finite number of sampling steps, for a given reference signal $w(t) = t^m$. Introducing again the closed loop transfer function $F_m = \dfrac{Y}{W} = \dfrac{B_m}{A_m}$, the z-transform of the sampled error signal is given by

$$\mathcal{E}(z^{-1}) = (1-F_m)W(z^{-1}) = \left(1-\frac{B_m}{A_m}\right)W = \frac{A_m - B_m}{A_m}W$$

The above assumption is equivalent to the statement that $\mathcal{E}(z^{-1})$ is a finite series, in other words a polynomial in z^{-1} . Rewriting equation (15) by taking (18) into account as well as the present assumption on A_0 yields $A^+(A_m - B_m) = AS^{'}$, which with the help of (11) and (13) gives, after division by A^+ :

$$A_m - B_m = (1-z^{-1})^l A^- S^{'} = (1-z^{-1})^{m+1}A^-S_1^{'}$$

Using (14) the following expression is finally obtained :

$$\mathcal{E}(z^{-1}) = \frac{A^-S_1^{'}W_1}{A_m} \tag{21}$$

Since A_m can obviously not divide any of the numerator polynomials of (21), the only way for the division of (21) to end up with a finite polynomial is to let

$$A_m = 1 \tag{22}$$

which yields for the closed loop transfer function :

$$F_m = B_m = B^-B_m^{'} \tag{23}$$

and for the z-transform of the sampled error :

$$\mathcal{E}(z^{-1}) = A^-S_1^{'}W_1 \tag{24}$$

It is worthwhile to note that (23) illustrates the well known conclusion that, for the design of a finite time settling regulator, the closed loop transfer function has all its poles at the origin of the z-plane and contains in its numerator the "unstable" zeros of the process.

From (19), (8) and (11) the expression of the series controller becomes then :

$$C(z) = \frac{1}{(1-z^{-1})^{m+1-l}}\frac{A^+}{B^+}\frac{B_m^{'}}{S_1^{'}} \tag{25}$$

and the z-transform of the resulting control law is then :

$$U(z) = C(z)\mathcal{E}(z^{-1}) = \frac{A^-A^+B_m^{'}W_1}{(1-z^{-1})^{m+1-l}B^+} \tag{26}$$

By peeling the onion off once more, one more controller comes out, the *dead-beat* controller, which is again a subclass of the previous one.

In the dead-beat design it is required that, for a given reference signal $w(t) = t^m$, the output of the control system reaches its steady-state value, not only at the sampling times but also between them, in a finite number of steps. For the system to reach such a state when the time $t(k)$ grows to infinity, it is mandatory that the sequence $u(k)$ becomes constant or, from the finite value theorem, that $U(z)$ contains at most one pole at $z=1$. Setting then $U(z) = \dfrac{U_1(z^{-1})}{(1-z^{-1})}$, where U_1 is a polynomial in z^{-1}, into (26) induces the two following consequences :

(i) for the identification term by term of the two sides of (26) to be possible, it is necessary that $l \ge m$, which means that the plant must have at least m integrators ;

(ii) since $U_1 = \dfrac{(1-z^{-1})^{l-m}A^-A^+B_m^{'}W_1}{B^+}$ is a polynomial in z^{-1}, B^+ must divide $B_m^{'}$:

$$B_m^{'} = B^+M$$

which in turn implies, from (23) and (6), that :

$$F_m = B\,M$$

in other words, that the closed loop transfer function contains now all the process zeros, and not only its "unstable" ones. As is well known, condition **(i)** must be satisfied for such a design to be feasible and condition **(ii)** makes it of the dead-beat type.

6 Conclusion

The main contribution of this paper has been to add a new step to the design of an RST-controller, by which the cancellation of steady-state errors *of orders superior to zero*, with respect to the *reference signal*, is achieved. This was obtained by solving an auxiliary diophantine equation, the solution of which imposes the numerator of the desired closed loop transfer function. It has been highlighted that the usual design step, consisting in the introduction of the required number of additional integrators into the loop, by means of the S polynomial, *does not* cancel *those* steady state errors, though ensuring rejection of disturbances.

It has been shown also, that the RST-formulation of discrete controllers is general enough to contain, as subclasses, the more conventional cascade-regulator designs, such as the design according to Zdan, the finite time settling controller and the dead-beat controller, thus structuring and simplifying greatly the presentation of this material to the students.

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References

[1] K. Åström and B. Wittenmark, *Computer Controlled Systems : Theory and Design,* Englewood Ciffs, NJ : Prentice-Hall, (1984).

[2] I. Landau, *Identification et commande des systèmes*, Hermès, Paris, (1988).

[3] R. Longchamp, *Commande numérique de systèmes dynamiques*, chap. 10, Presses polytechniques et universitaires romandes, Lausanne, (1995).

[4] M. Rivoire et J.-L. Ferrier, *Automatique : commande par calculateur, identification* (Tome 3), chap. 17, Eyrolles, Paris, (1997).

[5] Y. Sévely, *Systèmes et asservissements linéaires échantillonnés*, Dunod, Paris, (1973).

[6] W. Zdan, « Digital Computers for Stabilizing Control Systems », **Convention IRE**, part. 4, (1961).