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New Results on PID Controller Design of Discrete-time Systems via Pole Placement ⋆

Honghai Wang* Jianchang Liu* Xia Yu* Shubin Tan* Yu Zhang*

* *College of Information Science and Engineering, Northeastern University; State Key Laboratory of Synthetical Automation for Process Industries, Shenyang, 110819, China (e-mail: water.honghai@gmail.com(H. Wang), liujianchang@ise.neu.edu.cn(J. Liu), yuxia@ise.neu.edu.cn(X. Yu), tanshubin@ise.neu.edu.cn(S. Tan), zhangyu@ise.neu.edu.cn(Y. Zhang))*

Abstract: This paper considers the problem of proportional-integral-derivative (PID) controller design for a discrete-time system via pole placement. We first provide a method on the determination of the discrete-time PID gains which can assign a pair of dominant poles of a closed-loop system to the desired positions and locate the other poles inside a given smaller circle centered at the origin to guarantee the dominance of the assigned poles. The procedure for ascertaining the gains of PID controllers can be achieved in a straightforwardly computational way. Further, besides dominant pole placement, we propose a novel result on PID controller design according to some requirements of non-dominant pole distribution, which is a more interesting topic in the present paper.

Keywords: discrete-time system, proportional-integral-derivative (PID) control, dominant pole placement, non-dominant pole assignment, Hermite-Biehler Theorem.

1. INTRODUCTION

Computer-controlled systems are widely implemented in amounts of industrial precesses. As a foundation for the design of such systems, the analysis and synthesis of discrete-time systems have been widespread concerned (Åström and Wittenmark (2011)). It is well known that many performance indexes of a discrete-time closed-loop system directly depend on the locations of its poles (also known as the eigenvalues of the system). Thus, the studies on eigenvalue distribution and pole placement of a discrete-time system play an important role in its analysis and design.

Several decades ago, the classical Hermite-Biehler Theorem which can judge if the roots of a given real polynomial are all in the open left-half s-plane was provided. Then, Ho et al. (1999) developed some generalization of such a theorem applicable to the case of a real polynomial which is not necessarily Hurwitz. The Hermite-Biehler Theorem and its generalization were applied to solving the problem of PID stabilization of linear time-invariant (LTI) systems (Ho et al. (2004)) and LTI delay systems (Oliveira et al. (2009)), respectively. Further, based on the generalization of the Hermite-Biehler Theorem, some new methods on PID controller design via dominant pole assignment for LTI (delay) systems are proposed in Liu et al. (2015) and Wang et al. (2016).

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Combining with Tchebyshev polynomials (Pólya and Szegó (1976)), Keel and Bhattacharyya (2002) presented a generalization of Hermite-Biehler type result for Schur stability and applied it to the stabilizing P controller for discrete-time systems. Then, an extension of the result on eigenvlaue distribution presented in Keel and Bhattacharyya (2002) was given in Keel et al. (2003) where the problems of a "maximally" deadbeat design of PID controllers and the determination of the maximum delay in a closed-loop system under PID control were considered.

Pole placement, especially dominant pole placement, is one of the mainstream approaches in control system design since performance specifications of a closed-loop system can be usually determined by its dominant eigenvalue locations. Recently, a guarantee of dominance in the pole placement with discrete-PID controllers based on Nyquist plot applications was presented in Dincel and Söylemez (2014). According to their study, controller gains which achieve dominant pole placement can be obtained by partly drawing Nyquist plots. For a further research, it is expected to produce a new method to determine the set of controller gains via dominant pole placement in a straightforwardly computational way so that one could calculate the controller parameters directly according to the desired performance indexes. In addition to dominant pole placement, we also intend to give an approach to further control the non-dominant poles, which could improve the performance of the closed-loop system. All these facts motivate the present study.

In this paper, based on the discrete-time type of Hermite-Biehler Theorem in Keel et al. (2003) and the classical

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continuous-time type of Hermite-Biehler Theorem, some new results on PID controller design of discrete-time systems via dominant pole placement and non-dominant pole assignment are proposed. The main contributions of this paper lie as follows: i) Applying Hermite-Biehler type theorems develops the methods of controller design via pole placement for a discrete-time system. ii) The PID controller parameters to achieve dominant pole placement can be characterized by a direct calculation; iii) Besides the dominant pole placement, a further method is also produced to assign the non-dominant poles according to the requirement of quadrant in z-plane, which is a more interesting topic in the present paper.

2. PROBLEM STATEMENT

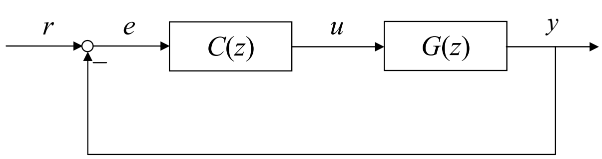

Fig. 1. Unit feedback control system

Consider a discrete unit feedback control system shown in Fig. 1, where $G(z)$ is a discrete-time plant described as

$$G(z) = \frac{N(z)}{D(z)} \tag{1}$$

and $C(z)$ is a discrete-time PID controller whose transfer function is

$$C(z) = k_p + k_i \cdot \frac{z}{z-1} + k_d \cdot \frac{z-1}{z}. \tag{2}$$

Here, $N(z)$ and $D(z)$ are the co-prime polynomials as

$$N(z) = d_m z^m + d_{m-1} z^{m-1} + \cdots + d_1 z + d_0,$$
$$D(z) = z^n + c_{n-1} z^{n-1} + \cdots + c_1 z + c_0,$$

where $d_0, d_1, \ldots, d_m$ and $c_0, c_1, \ldots, c_{n-1}$ are real numbers and $n \geq m$; k_p, k_i, and k_d are proportional, integral, and derivative gains, respectively.

The problem studied in this paper is to assign a pair of closed-loop eigenvalues (as dominant poles) to the desired positions and place the other eigenvalues inside the small circle $\mathcal{C}_\rho$ of radius r^v centered at the origin, where r is the distance between one of the assigned dominant poles and the origin and v is usually $3 \sim 5$ (Wang et al. (2009)). Besides dominant eigenvalue assignment, we also expect to place the non-dominant eigenvalues in some assigned regions, especially, locate all the non-dominant eigenvalues in the region including the positive real axis, the first quadrant, and the fourth quadrant of z-plane, which can improve the performance of closed-loop systems.

3. PRELIMINARY RESULTS

To achieve the purposes of this paper, we give some preliminary results here. Consider a polynomial as follows:

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0, \tag{3}$$

where z is a complex variable and a_i, $i = 0, 1, \ldots, n$, are real numbers. Letting $z = \rho e^{j\theta}$ and defining $x = -\cos\theta$ with $0 \leq \theta \leq \pi$, we have $z = -\rho x + j\rho\sqrt{1-x^2}$. According to the Tchebyshev polynomials described in Pólya and Szegó (1976), the polynomial (3) can be written as

$$\begin{aligned} P_c(x,\rho) :=& P(z)\mid_{z=-\rho x + j\rho\sqrt{1-x^2}} \\ =& U_P(x,\rho) + j\sqrt{1-x^2}\,T_P(x,\rho), \end{aligned} \tag{4}$$

where

$$\begin{aligned} U_P(x,\rho) &= a_n u_n(x,\rho) + \cdots + a_1 u_1(x,\rho) + a_0, \\ T_P(x,\rho) &= a_n t_n(x,\rho) + \cdots + a_1 t_1(x,\rho) \end{aligned}$$

with

$$u_k(x,\rho) = \rho^k u_k(x), t_k(x,\rho) = \rho^k t_k(x),\ k = 1, 2, \ldots, n.$$

Here, $u_k(x)$ and $t_k(x)$ are known as the Tchebyshev polynomials of the first kind and the second kind, respectively, and given as follows:

$$t_k(x) = -\frac{1}{k} \cdot \frac{du_k(x)}{dx},\ k = 1, 2, \ldots, n, \tag{5}$$

$$u_{k+1}(x) = -x u_k(x) - (1-x^2) t_k(x),\ k = 1, 2, \ldots, n. \tag{6}$$

Now, let

$$Q(z) = \frac{P_1(z)}{P_2(z)}, \tag{7}$$

where $P_1(z)$ and $P_2(z)$ are both polynomials. Denoting

$$P_1(z)\mid_{z=-\rho x + j\rho\sqrt{1-x^2}} = U_{P_1}(x,\rho) + j\sqrt{1-x^2}\,T_{P_1}(x,\rho),$$
$$P_2(z)\mid_{z=-\rho x + j\rho\sqrt{1-x^2}} = U_{P_2}(x,\rho) + j\sqrt{1-x^2}\,T_{P_2}(x,\rho),$$

we have

$$\begin{aligned} Q_c(x,\rho) :=& Q(z)\mid_{z=-\rho x + j\rho\sqrt{1-x^2}} \\ =& \frac{U(x,\rho) + j\sqrt{1-x^2}\,T(x,\rho)}{U_{P_2}^2(x,\rho) + (1-x^2) T_{P_2}^2(x,\rho)} \end{aligned} \tag{8}$$

with

$$\begin{aligned} U(x,\rho) &= U_{P_1}(x,\rho) U_{P_2}(x,\rho) + (1-x^2) T_{P_1}(x,\rho) T_{P_2}(x,\rho), \\ T(x,\rho) &= T_{P_1}(x,\rho) U_{P_2}(x,\rho) - (1-x^2) U_{P_1}(x,\rho) T_{P_2}(x,\rho). \end{aligned}$$

Definition 1. Let $-1 < x_1 < x_2 < \cdots < \omega_q < 1$ be the real and distinct zeros of $T(x,\rho)$ with odd multiplicities for $x \in (-1, 1)$ and suppose $T(x,\rho)$ has α roots at $x = -1$. Then, define an imaginary signature $\sigma_i(Q)$ of $Q(z)$ as

$$\begin{aligned} \sigma_i(Q) = \frac{1}{2}\mathrm{sgn}\left[T^\alpha(-1,\rho)\right]\{&\mathrm{sgn}[U(-1,\rho)] - 2\mathrm{sgn}[U(x_1,\rho)] \\ &+ \cdots + (-1)^q 2\mathrm{sgn}[U(x_q,\rho)] + (-1)^{q+1}\mathrm{sgn}[U(1,\rho)]\}, \end{aligned} \tag{9}$$

where $\mathrm{sgn}[x]$ is the standard signum function given by

$$\mathrm{sgn}[x] = \begin{cases} -1 & if\ x < 0, \\ 0 & if\ x = 0, \\ 1 & if\ x > 0. \end{cases}$$

Now, let us define a circle $\mathcal{C}_\rho$ of radius ρ centered at the origin and introduce a theorem on $\sigma_i(Q)$ below.

Theorem 2. Keel et al. (2003) Let $P_1(z)$ and $P_2(z)$ in (7) be real polynomials with $I(P_1)$ and $I(P_2)$ roots inside the circle $\mathcal{C}_\rho$ and no roots on it. Then, $\sigma_i(Q) = I(P_1) - I(P_2)$.

Next, we give another result which states the root distribution of a polynomial in the entire complex plane.

Definition 3. Ho et al. (1999) Let $\delta(\lambda)$ be a real polynomial of degree l with no zeros at the origin and define $\delta(j\omega) = \delta_r(\omega) + j\delta_i(\omega)$ by substituting $\lambda = j\omega$. Let $0 = \omega_0 < \omega_1 < \ldots < \omega_{p-1}$ be the real and distinct finite zeros of $\delta_i(\omega)$ with odd multiplicities, and define $\omega_p = \infty$. Then, the imaginary signature $\varrho_i(\delta)$ of $\delta(\lambda)$ is defined by

$$\varrho_i(\delta) = \begin{cases} \{\mathrm{sgn}[\delta_r(\omega_0)] - 2\mathrm{sgn}[\delta_r(\omega_1)] + 2\mathrm{sgn}[\delta_r(\omega_2)] - \cdots \\ + (-1)^{p-1}2\mathrm{sgn}[\delta_r(\omega_{p-1})] + (-1)^p \mathrm{sgn}[\delta_r(\omega_p)]\} \\ \cdot (-1)^{p-1}\mathrm{sgn}[\delta_i(\infty)], \qquad if\ l\ is\ even; \\ \{\mathrm{sgn}[\delta_r(\omega_0)] - 2\mathrm{sgn}[\delta_r(\omega_1)] + 2\mathrm{sgn}[\delta_r(\omega_2)] - \cdots \\ + (-1)^{p-1}2\mathrm{sgn}[\delta_r(\omega_{p-1})]\} \cdot (-1)^{p-1}\mathrm{sgn}[\delta_i(\infty)], \\ \qquad if\ l\ is\ odd. \end{cases} \tag{10}$$

Theorem 4. Ho et al. (1999) Let $L(\delta)$ and $R(\delta)$ denote the numbers of left-half plane roots and right-half plane roots of $\delta(\lambda)$, respectively. Then, $L(\delta) - R(\delta) = \varrho_i(\delta)$.

4. DISCRETE-TIME PID CONTROLLER DESIGN

In this section, we propose an approach on PID controller design of discrete-time systems via dominant eigenvalue assignment.

4.1 Determination of Dominant Eigenvalues

The transfer function of $C(z)$ in (2) can be written as

$$C(z) = \frac{K_p z + K_i + K_d z^2}{z(z-1)}, \tag{11}$$

where

$$\begin{bmatrix} K_p \\ K_i \\ K_d \end{bmatrix} = \begin{bmatrix} -1 & 0 & -2 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} k_p \\ k_i \\ k_d \end{bmatrix}. \tag{12}$$

Then, the characteristic function of the closed-loop system shown in Fig 1 is

$$\begin{aligned} \Delta(z, K_p, K_i, K_d) :=& z(z-1)D(z) \\ &+ (K_d z^2 + K_p z + K_i)N(z). \end{aligned} \tag{13}$$

Assuming that the positions of dominant poles in the z-plane are given as $z_{1,2} = a \pm jb$ and denoting $\mathcal{G}(z) = z(z-1)D(z)/N(z)$, we can give the expressions of K_i and K_d with respect to K_p, respectively, as follows:

$$K_i = X_{i1}K_p + X_{i2}, \tag{14}$$

$$K_d = X_{d1}K_p + X_{d2}, \tag{15}$$

where

$$X_{i1} = -\frac{a^2+b^2}{2a}, X_{i2} = \frac{a^2-b^2}{2ab}\Im\left[\mathcal{G}(z_1)\right] - \Re\left[\mathcal{G}(z_1)\right],$$

$$X_{d1} = -\frac{1}{2a}, \qquad X_{d2} = -\frac{1}{2ab}\Im\left[\mathcal{G}(z_1)\right].$$

The parameters K_p, K_i, and K_d satisfying the relationship (14) and (15) can exactly locate a pair of eigenvalues of the closed-loop system at $z = z_1$ and $z = z_2$, respectively.

4.2 Dominant Eigenvalue Guarantee

Substituting (14) and (15) into (13), we have

$$\Delta(z, K_p) = \tilde{D}(z) + K_p\tilde{N}(z), \tag{16}$$

where

$$\tilde{D}(z) = z(z-1)D(z) + (X_{d2}z^2 + X_{i2})N(z),$$

$$\tilde{N}(z) = (X_{d1}z^2 + z + X_{i1})N(z).$$

Let us define

$$\begin{aligned} H(z, K_p) :=& \Delta(z, K_p)\tilde{N}(\rho^2 z^{-1}) \\ =& \tilde{D}(z)\tilde{N}(\rho^2 z^{-1}) + K_p\tilde{N}(z)\tilde{N}(\rho^2 z^{-1}). \end{aligned} \tag{17}$$

Combining with Tchebyshev representation and taking $z = -\rho x + j\rho\sqrt{1-x^2}$ into $H(z, K_p)$, we can get

$$\begin{aligned} H_c(x, \rho, K_p) :=& H(z, K_p) \mid_{z=-\rho x + j\rho\sqrt{1-x^2}} \\ =& U_H(x, \rho, K_p) + j\sqrt{1-x^2}T_H(x, \rho) \end{aligned} \tag{18}$$

with

$$\begin{aligned} U_H(x, \rho, K_p) =& [U_{\tilde{N}}^2(x, \rho) + (1-x^2)T_{\tilde{N}}^2(x, \rho)]K_p + \\ & U_{\tilde{D}}(x, \rho)U_{\tilde{N}}(x, \rho) + (1-x^2)T_{\tilde{D}}(x, \rho)T_{\tilde{N}}(x, \rho), \end{aligned} \tag{19}$$

$$T_H(x, \rho) = U_{\tilde{N}}(x, \rho)T_{\tilde{D}}(x, \rho) - T_{\tilde{N}}(x, \rho)U_{\tilde{D}}(x, \rho), \tag{20}$$

in which $U_{\tilde{D}}(x, \rho)$, $T_{\tilde{D}}(x, \rho)$, $U_{\tilde{N}}(x, \rho)$, and $T_{\tilde{N}}(x, \rho)$ are given by the following expressions:

$$\tilde{D}(z) \mid_{z=-\rho x + j\rho\sqrt{1-x^2}} = U_{\tilde{D}}(x, \rho) + j\sqrt{1-x^2}T_{\tilde{D}}(x, \rho),$$

$$\tilde{N}(z) \mid_{z=-\rho x + j\rho\sqrt{1-x^2}} = U_{\tilde{N}}(x, \rho) + j\sqrt{1-x^2}T_{\tilde{N}}(x, \rho).$$

Assume that $\tilde{N}(z)$ has no zeros on the circle $\mathcal{C}_\rho$ and denote by $O(\tilde{N})$ the numbers of its zeros outside the circle $\mathcal{C}_\rho$.

Theorem 5. Let $-1 < x_1 < x_2 < \cdots < x_q < 1$ be the real and distinct zeros of $T_H(x, \rho)$ with odd multiplicities for $x \in (-1, 1)$ and suppose $T_H(x, \rho)$ has α roots at $x = -1$. Then, for a fixed K_p, the characteristic function $\Delta(z, K_p)$ in (16) has a pair zeros at $z = z_{1,2}$ and all its other zeros are inside the circle $\mathcal{C}_\rho$ if and only if

$$\begin{aligned} & n + O(\tilde{N}) - (m+2) = \\ & \frac{1}{2}\mathrm{sgn}\left[T_H^\alpha(-1, \rho)\right]\{\mathrm{sgn}[U_H(-1, \rho, K_p)] \\ & - 2\mathrm{sgn}[U_H(x_1, \rho, K_p)] + \cdots + (-1)^q 2\mathrm{sgn}[U_H(x_q, \rho, K_p)] \\ & + (-1)^{q+1}\mathrm{sgn}[U_H(1, \rho, K_p)]\}. \end{aligned} \tag{21}$$

Proof. It is clear that the characteristic function $\Delta(z, K_p)$ has a pair zeros at $z = z_{1,2}$ and $\tilde{N}(\rho^2 z^{-1})$ has $O(\tilde{N})$ zeros inside the circle $\mathcal{C}_\rho$. Thus, all the other roots of $\Delta(z, K_p)$ except the assigned roots at $z = z_{1,2}$ are inside the circle $\mathcal{C}_\rho$ if and only if $H(z, K_p)$ in (17) has $n + O(\tilde{N})$ zeros in the interior of the circle $\mathcal{C}_\rho$. Moreover, it is sufficient that $H(z, K_p)$ has $m+2$ poles inside the circle $\mathcal{C}_\rho$ since $\tilde{N}(\rho^2 z^{-1})$ has $m+2$ poles at the origin point. Then, by Theorem 2, we have the result of this theorem. ∎

Remark 6. In order to obtain effective parameters of the controller, one can first determine the value of K_p by sweeping its allowable region and adopting Theorem 5. Then, the corresponding values of K_i and K_d can be calculated according to (14) and (15), respectively. Therefore, the allowable region of K_p is very critical. Such a region can be determined by the following lemma.

Lemma 7. For a fixed value of ρ, a necessary condition for (21) to be true is that $U_H(x, \rho, K_p)$ in (19) has at least $n + O(\tilde{N}) - (m+2)$ real and distinct roots with odd multiplicities for $x \in (-1, 1)$.

Proof. If (21) holds, the number of the real, distinct, and odd multiple roots of $T_H(x, \rho)$ in (20) for $x \in (-1, 1)$ must be no less than $n+O(\tilde{N})-(m+2)-1$. When $T_H(x,\rho)$ has only $n+O(\tilde{N})-(m+2)-1$ real and distinct roots with odd multiplicities for $-1<x<1$, the expression (21) could be true only if $U_H(x,\rho,K_p)$ has at least $n+O(\tilde{N})-(m+2)$ real and distinct roots with odd multiplicities for $x\in(-1,1)$. This completes the proof of this lemma. ■

Remark 8. To ascertain the allowable region of K_p, an algorithm is given to determine the number of the real, distinct, and odd multiple roots of $U_H(x,\rho,K_p)$ in $x\in(-1,1)$ with respect to different values of K_p. Let $\mathcal{F}[K_p,f(x)]:=K_p-f(x)$, where

$$f(x)=\frac{U_{\tilde{D}}(x,\rho)U_{\tilde{N}}(x,\rho)+(1-x^2)T_{\tilde{D}}(x,\rho)T_{\tilde{N}}(x,\rho)}{U_{\tilde{N}}^2(x,\rho)+(1-x^2)T_{\tilde{N}}^2(x,\rho)}.$$

Step 1: Solve the equation $df(x)/dx=0$ in $(-1,1)$ and let $-1<\varsigma_1<\varsigma_2<\cdots<\varsigma_{g-1}<1$ be the real and distinct zeros with odd multiplicities. Define $\varsigma_0=-1$ and $\varsigma_g=1$.

Step 2: If $f'(-1)<0$, let $L_j:=f(\varsigma_i)$, $1\le j\le g+1$, which satisfy $L_1\le L_2\le\cdots\le L_{g+1}$; if $f'(-1)>0$, let L_j satisfy $L_1\ge L_2\ge\cdots\ge L_{g+1}$.

Step 3: Define

$$n_{L_j}=\begin{cases}0.5, & \textit{if } i=g \textit{ is odd},\\ -0.5, & \textit{if } i=0 \textit{ or } i=g \textit{ is even},\\ 1, & \textit{if } 1\le i\le g-1 \textit{ and } i \textit{ is odd},\\ -1, & \textit{if } 1\le i\le g-1 \textit{ and } i \textit{ is even}.\end{cases}$$

Step 4: Let $-\infty=\zeta_0<\zeta_1<\cdots<\zeta_{h-1}<\zeta_h=+\infty$ be constants such that for any $j=1,2,\ldots,g+1$, $\exists L_j=\zeta_k$ with $k=1,2,\ldots,h-1$. Then, for $1\le k\le h-1$, define $N_{\zeta_k}=\sum_{L_j=\zeta_k}n_{L_j}$ and $E_{\zeta_k}=\sum_{L_j=\zeta_k}\lceil n_{L_j}\rceil$, where $\lceil\cdot\rceil$ is a round up function. Besides, define $E_{\zeta_0}=N_{\zeta_0}=0$.

Step 5: Denote by $M(\zeta_k,\zeta_{k+1})$ and $M(\zeta_k)$ the number of the real and distinct roots of the function $U_H(x,\rho,K_p)$ with odd multiplicities for $-1<x<1$ when $\min\{\zeta_k,\zeta_{k+1}\}<K_p<\max\{\zeta_k,\zeta_{k+1}\}$ and $K_p=\zeta_k$, respectively. For $0\le k\le h-1$, we have

$$M(\zeta_k,\zeta_{k+1})=2\sum_{t=0}^{k}N_{\zeta_t},$$
$$M(\zeta_k)=M(\zeta_k,\zeta_{k+1})-E_{\zeta_k}.$$

Remark 9. For a set of (K_p,K_i,K_d) guaranteeing dominant pole placement, according to (12), we can get the corresponding parameters in $C(z)$ given by

$$\begin{bmatrix}k_p\\k_i\\k_d\end{bmatrix}=\begin{bmatrix}-1&-2&0\\1&1&1\\0&1&0\end{bmatrix}\begin{bmatrix}K_p\\K_i\\K_d\end{bmatrix}. \tag{22}$$

Now, we give an example to illustrate the procedure of the discrete-time PID controller design via dominant pole placement.

Example 10. Consider a plant

$$G(s)=\frac{-1.674s+2.41}{s^4+10s^3+33s^2+40s+16}.$$

The corresponding discrete-time transfer function including a zero-order hold by taking the sampling time 0.4 seconds can be given as

$$G(z)=\frac{-0.00561z^3+0.002711z^2+0.0123z+0.00103}{z^4-1.744z^3+1.031z^2-0.2361z+0.01832}.$$

For this discrete-time plant, it is expected to assign a pair of closed-loop poles at $z_{1,2}=0.8856\pm j0.1067$ and locate the other poles inside a circle $\mathcal{C}_\rho$ of radius $\rho=r^v$ centered at the origin by using a discrete-time PID controller, where v is chosen as 3.

The characteristic function of the closed-loop system is given by (13), where $D(z)=z^4-1.744z^3+1.031z^2-0.2361z+0.01832$, $N(z)=-0.00561z^3+0.002711z^2+0.0123z+0.00103$, and the expressions of K_p, K_i, and K_d are shown in (12). Substituting $z=z_{1,2}$ into $\Delta(z,K_p,K_i,K_d)=0$, we have

$$K_i=-0.4492K_p-0.4969,$$
$$K_d=-0.5646K_p+1.0534.$$

It is clear that $r=0.8920$, and therefore $\rho=0.7097$. Then, according to (16)-(18) and combining with the Tchebyshev representation described in (4), we can obtain

$$\begin{aligned}U_H(x,0.7097,K_p)&=(0.8446x^5-1.5706x^4-8.3213x^3\\&\quad-2.6000x^2+9.4260x+6.1247)\times10^{-5}\times K_p\\&\quad+(-0.0019x^6+0.0029x^5+0.0182x^4+0.0150x^3\\&\quad-0.0071x^2-0.0118x-0.0033),\\T_H(x,0.7097)&=0.0189x^5-0.0285x^4-0.1741x^3\\&\quad-0.1724x^2-0.0318x+0.0139.\end{aligned}$$

The real zeros of $T_H(x,0.7097)$ with odd multiplicities in $(-1,1)$ are $x_1=-0.8673$ and $x_2=0.1937$. Moreover, by Lemma 7 and Remark 8, we obtain the allowable region of K_p which is $(-1.8216,8.7123)$. Grid this interval of K_p by $\mathcal{N}$, where $\mathcal{N}$ is a desired number of points, and set $\varepsilon=[8.7123-(-1.8216)]/(\mathcal{N}+1)$. Here, let us set $\mathcal{N}=199$ which results in $\varepsilon=0.5267$. Sweep the values of K_p at all these distinct points from $K_p=-1.8216$ to $K_p=8.7123$ and compute every $\varrho_i(H)$ with respect to the corresponding K_p by

$$\begin{aligned}\varrho_i(H)=&\frac{1}{2}\mathrm{sgn}\,[T_H(-1,0.7097)]\,\{\mathrm{sgn}[U_H(-1,0.7097,K_p)]\\&-2\mathrm{sgn}[U_H(-0.8673,0.7097,K_p)]\\&+2\mathrm{sgn}[U_H(0.1937,0.7097,K_p)]\\&-\mathrm{sgn}[U_H(1,0.7097,K_p)]\}.\end{aligned} \tag{23}$$

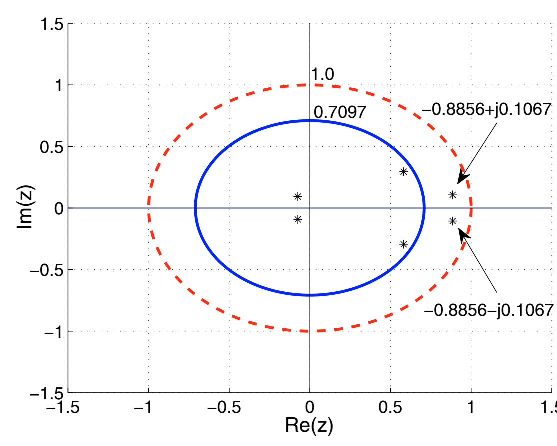

Fig. 2. The eigenvalue distribution for Example 10.

According to Theorem 5, all the values of K_p which can guarantee $\varrho_i(H) = 3$ meet the requirement of this example. Then, we can get that $K_p = -16.1090 + 0.5267k$, where $k = 0, 1, \ldots, 32$. Now, we choose $K_p = -11.3687$ in the effective K_p-region and calculate the corresponding values of K_i and K_d through (14) and (15). Then, by (22), the controller gains in $C(z)$ are $k_p = 2.1483$, $k_i = 0.7136$, and $k_d = 4.6102$. The eigenvalue distribution of the closed-loop system is shown in Fig. 2.

5. A FURTHER RESULT ON CONTROLLER DESIGN

In this section, on the basis of dominant eigenvalue assignment, we give a further result on PID controller design for the purpose of locating the non-dominant eigenvalues in some assigned regions.

Multiplying $\Delta(z, K_p)$ by $\tilde{N}(-z)$, we define

$$\begin{aligned} F(z, K_p) :=& \Delta(z, K_p)\tilde{N}(-z) \\ =& \tilde{D}(z)\tilde{N}(-z) + K_p\tilde{N}(z)\tilde{N}(-z). \end{aligned} \tag{24}$$

Substituting $z = j\omega$ into $F(z, K_p)$, we have

$$F(j\omega, K_p) = F_r(\omega, K_p) + jF_i(\omega), \tag{25}$$

where

$$\begin{aligned} F_r(\omega, K_p) =& [\tilde{N}_r^2(\omega) + \tilde{N}_i^2(\omega)]K_p + \tilde{D}_r(\omega)\tilde{N}_r(\omega) \\ &+ \tilde{D}_i(\omega)\tilde{N}_i(\omega), \end{aligned} \tag{26}$$

$$F_i(\omega) = \tilde{N}_r(\omega)\tilde{D}_i(\omega) - \tilde{N}_i(\omega)\tilde{D}_r(\omega), \tag{27}$$

and $\tilde{D}_r(\omega)$, $\tilde{D}_i(\omega)$, $\tilde{N}_r(\omega)$, and $\tilde{N}_i(\omega)$ are the real and imaginary parts of $\tilde{D}(j\omega)$ and $\tilde{N}(j\omega)$, respectively. For the zeros of $\Delta(z, K_p)$ inside the circle, let $L_{\mathcal{C}_\rho}(\Delta)$ and $R_{\mathcal{C}_\rho}(\Delta)$ be the numbers of those in the left-half z-plane and in the right-half z-plane, respectively. Moreover, denote the numbers of the left-half z-plane zeros and the right-half z-plane zeros of $\tilde{N}(z)$ by $L(\tilde{N})$ and $R(\tilde{N})$, respectively.

Definition 11. Let $0 = \omega_0 < \omega_1 < \ldots < \omega_{p-1}$ be the non-negative, real, and distinct finite zeros of $F_i(\omega)$ with odd multiplicities, and define $\omega_p = \infty$. Then, the imaginary signature $\varrho_i(F)$ of $F(z)$ is defined by

$$\varrho_i(F) = \begin{cases} \{\mathrm{sgn}[F_r(\omega_0, K_p)] - 2\mathrm{sgn}[F_r(\omega_1, K_p)] + \cdots \\ \quad + (-1)^p \mathrm{sgn}[F_r(\omega_p, K_p)]\} \cdot (-1)^{p-1}\mathrm{sgn}[F_i(\infty)], \\ \quad if\ m+n\ is\ even; \\ \{\mathrm{sgn}[F_r(\omega_0, K_p)] - 2\mathrm{sgn}[F_r(\omega_1, K_p)] + \cdots \\ \quad + (-1)^{p-1} 2\mathrm{sgn}[F_r(\omega_{p-1}, K_p)]\} \cdot (-1)^{p-1}\mathrm{sgn}[F_i(\infty)], \\ \quad if\ m+n\ is\ odd. \end{cases} \tag{28}$$

Theorem 12. For a fixed K_p satisfying the condition (21),

$$\varrho_i(F) = L_{\mathcal{C}_\rho}(\Delta) - R_{\mathcal{C}_\rho}(\Delta) - [L(\tilde{N}) - R(\tilde{N})] - 2\mathrm{sgn}[a].$$

Proof. We can easily determine that $\tilde{N}(-z)$ has $R(\tilde{N})$ zeros and $L(\tilde{N})$ zeros in the left-half z-plane and in the right-half z-plane, respectively. In addition, the difference between the zero number of $\Delta(z, K_p)$ in the left-half z-plane and that in the right-half z-plane is $L_{\mathcal{C}_\rho}(\Delta) - R_{\mathcal{C}_\rho}(\Delta) - 2\mathrm{sgn}[a]$. Then, according to (24) and Theorem 4, we have the result of this theorem. ■

According to the relationship between pole map and natural response of a closed-loop system described in K. Ogata (1976), it is better to locate the non-dominant poles on the positive real axis, in the first quadrant, or in the fourth quadrant of z-plane. Based on Theorem 12, it is not difficult to give a corollary.

Corollary 13. For a fixed K_p satisfying the condition (21), the characteristic function $\Delta(z, K_p)$ in (16) has a pair eigenvalues at $z = z_{1,2}$ and all its other eigenvalues are in the interior of the circle $\mathcal{C}_\rho$ and are located in the region including the positive real axis, the first quadrant, and the fourth quadrant of z-plane if and only if

$$\varrho_i(F) = -n - [L(\tilde{N}) - R(\tilde{N})] - 2\mathrm{sgn}[a].$$

Proof. Since the degree of $\Delta(z, K_p)$ in z is $n + 2$ and the number of the dominant poles which are outside the circle $\mathcal{C}_\rho$ is 2, it is clear that the number of the non-dominant eigenvalues of $\Delta(z, K_p)$ is n. Then, $\Delta(z, K_p)$ in (16) has a pair eigenvalues at $z = z_{1,2}$ and all its other eigenvalues are in the interior of the circle $\mathcal{C}_\rho$ and are located in the region including the positive real axis, the first quadrant, and the fourth quadrant of z-plane if and only if $L_{\mathcal{C}_\rho}(\Delta) = 0$ and $R_{\mathcal{C}_\rho}(\Delta) = n$. Thus, according to Theorem 12, we have the result of this corollary. ■

Remark 14. Here, the allowable region of K_p consists of all the values of K_p that meet the condition (21). When a effective value of K_p is determined, the corresponding values of K_i and K_d can be ascertained from (14) and (15). The gains (k_p, k_i, k_d) can be therefore obtained according to (22).

Below we give an example to describe the process of the discrete-time PID controller design according to the requirement of the non-dominant pole assignment.

Example 15. Consider the discrete-time plant $G(z)$ given in (23). Besides the requirement of dominant pole placement shown in Example 10, it is also expected to locate all the non-dominant poles in the region including the positive real axis, the first quadrant, and the fourth quadrant of z-plane.eigenvalue

According to (24)-(27), we have

$$\begin{aligned} F_r(\omega, K_p) =& (1.0032\omega^{10} + 6.1841\omega^8 + 12.4419\omega^6 \\ &+ 1.0147\omega^4 + 2.9927\omega^2 + 0.0214) \times 10^{-5} \times K_p \\ &+ (-0.0016\omega^{10} - 0.0101\omega^8 - 0.0125\omega^6 - 0.0043\omega^4 \\ &+ 2.3281 \times 10^{-4}\omega^2 + 2.3682 \times 10^{-7}), \\ F_i(\omega) =& 0.0032\omega^{11} + 0.0126\omega^9 + 0.0115\omega^7 + 0.0012\omega^5 \\ &- 0.0015\omega^3 + 9.0039 \times 10^{-6}\omega. \end{aligned}$$

The non-negative real zeros of $F_i(\omega)$ with odd multiplicities are $\omega_0 = 0$, $\omega_1 = 0.0786$, and $\omega_2 = 0.5193$. Sweep the eventually effective K_p-region given in Example 10 and calculate the corresponding values of $\varrho_i(F)$ described in (28). By Corollary 13, all the values of K_p resulting in $\varrho_i(F) = -5$ can achieve the objective of this example. Then, we can obtain $K_p = -3.4683 + 0.5267k$, where $k = 0, 1, 2, 3, 4$. Now, we choose $K_p = -2.4149$ in this K_p-region and compute the corresponding parameters K_i and K_d from (14) and (15). Finally, the controller gains in $C(z)$ can be obtained as $k_p = 1.2391$, $k_i = 0.5899$, and $k_d = 0.5879$ according to the expression (22). The eigenvalue distribution of the closed-loop system is illustrated in Fig. 3.

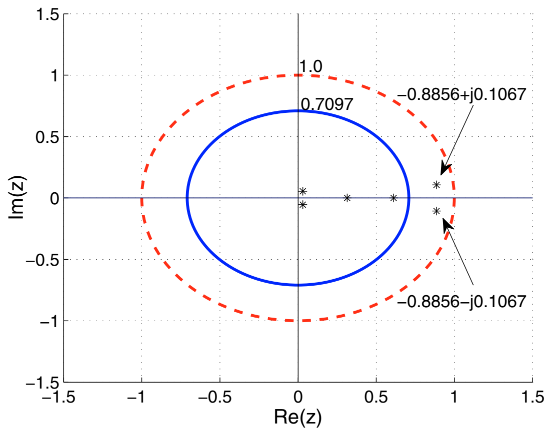

Fig. 3. The eigenvalue distribution for Example 15.

As a pair of dominant poles, $z_{1,2} = 0.8856 \pm j0.1067$ can make the step response of a closed-loop system close to that the overshoot and the settling time $t_s(2\%)$ are 5% and 15.12s, respectively, according to control theory.

Now, we choose $K_p = -16.1090$ and $K_p = -3.4683$. The corresponding (k_p, k_i, k_d) values are $(2.6296, 0.7791, 6.7397)$ and $(1.3461, 0.6044, 1.0611)$, respectively. The step response plots of the closed-loop system by using these controller gains are shown in Fig. 4. It is seen that the latter gains make the performances of the closed-loop system approach to the desired ones much more. Furthermore, they can improve the entire response plot.

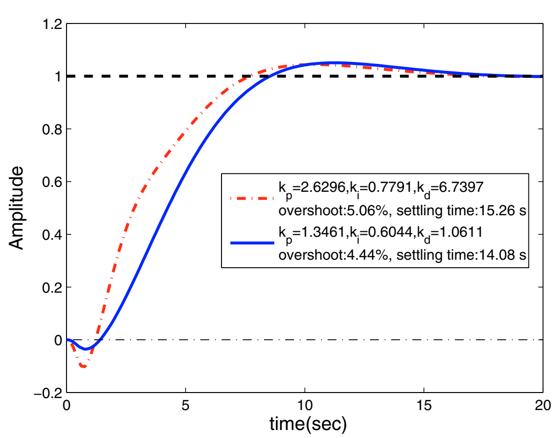

Fig. 4. The step response plots for Examples 10 and 15.

6. CONCLUSION

In this paper, based on the discrete-time type of Hermite-Biehler Theorem in Keel et al. (2003), a new method on the PID controller design for a discrete-time system via dominant pole placement is produced. This approach can assign a pair of dominant poles of a closed-loop system to the desired positions and locate the other poles inside a given smaller circle centered at the origin. The procedure can be characterized by a straightforward computation. Further, by using the generalization of the Hermite-Biehler Theorem in Ho et al. (1999), this paper proposes a novel result on the determination of the PID controller gains according to the non-dominant pole assignment in addition to the dominant pole placement. Especially, we provide an approach to calculate the controller parameters which can guarantee that all the non-dominant poles are placed in the region including the positive real axis, the first quadrant, and the fourth quadrant of z-plane. Numerical examples are also given to illustrate the effectiveness of the presented conclusions.

In our future work, we intend to filter the derivative term of a PID controller, which will certainly complicate the problem because of the addition of one more parameter in the controller. However, considering the practical problems impels us to do such an effort. Moreover, besides pole placement, we will also consider the effect of zero locations of the closed-loop system when designing its controller in our next-step work.

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