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Signal-to-Noise Ratio Requirements for Discrete-time PID Controllers ⋆

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Abstract: In networked control a channel model is explicitly considered in discrete-time within the classic feedback loop scheme. It is well known by now that the channel model signal-to-noise ratio (SNR) happens to be an important parameter related to the overall stabilisability of the output feedback loop. In this paper we offer the rules for the design of discrete-time proportional-integral-derivative (PID) controllers for given closed loop bandwidth that quantifies the SNR requirement for the memoryless additive white Gaussian noise (AWGN) channel over the feedback path, which is a common used communication channel model. The design is centered on a PID controller structure due to its wide use in the industry. Since we focus on a PID controller structure, we then reduce the analysis to plant models of first and second order which are the structures more compatible with the PID controller design.

Keywords: Control over networks, Linear systems, Control under communication constraints, PID controllers, controllers design.

1. INTRODUCTION

Digital communication networks and wireless technology have created new possibilities and given a second air to the simple and effective PID controllers, now in the context of Networked control systems (NCS). However, this also creates new challenges such as how to adapt the controller design to deal with issues such as communication channels and the efficient use of energy Villanova and Visioli (2012). NCS are based on the basis that a nonideal communication link in the closed loop system. NCS have been object of sustained interest, with research area reviews already available in Antsaklis and Baillieul (2004), Nair et al. (2007) and J.Chen et al. (2011). One of the first results in NCS that caused a high impact was reported in Nair and Evans (2004) where a minimal rate for data transmission over a noiseless (error-free) channel is obtained for necessary and sufficient stability of the control loop as function of the plant linear time invariant (LTI) model unstable poles. The approach of the authors in Nair and Evans (2004) is from an information theory point of view and thus it applies to any possible feedback controller, be it LTI or nonlinear. A more control theory approach for NCS is proposed in Braslavsky et al. (2007) where an LTI framework for both continuous-time and discrete-time stability of feedback loops is offered based on the channel model signal-to-noise ratio (SNR). The fundamental limitation obtained in Braslavsky et al. (2007) is presented as an infimal lower bound on the channel SNR in order to ensure the stabilisation of an unstable plant by feedback over a memoryless additive white Gaussian noise (AWGN) communication channel such as the case depicted in Figure 1 for discrete-time.

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The disadvantage of the SNR framework proposed in Braslavsky et al. (2007) is that is limited to an LTI setting. However the linear setting links with the large body of linear control design techniques. More so the authors in Braslavsky et al. (2007) show that for a minimum phase delay-free plant stabilised with a feedback scheme, the obtained infimal SNR condition matches the minimal channel transmission data rate requirement from Nair and Evans (2004).

LTI limitations for NCS have been further researched for the feedback stabilisation problem (when $r_k = 0$), usually quantifying the H_2 norm of the closed loop transfer function between the channel signal input y_k and the channel noise process n_k. Thus an increasing body of knowledge has been growing on these LTI fundamental limitations for NCS due to the presence of a communication channel in the feedback path, see for example Martins and Dahleh (2008); Rojas et al. (2008); Silva et al. (2010); Ding et al. (2010); Silva and Pulgar (2013).

The contribution of this work is a set of simple control design rules for the control engineering practitioner that include the SNR limitation. This will extend the existing research results for LTI NCS. We consider for the controller mechanism a discrete-time PID controller, due to its ubiquitousness in the industry Astrom and Hagglund (1995). The communication channel model considered here is the memoryless AWGN channel, (see again Figure 1),

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which is characterised by its additive noise n_k and a power constraint P at the channel input y_k. We consider the two possible channel location settings, that is the one represented in Figure 1, and the control path setting depicted in Figure 2. Finally the plant models suitable for a PID controller are a first order model, a first order model with time-delay and a second order model and therefore are the ones we will study here. More complex plant models would require in general a controller structure more complex than a simple PID controller, in order to achieve closed-loop design specifications.

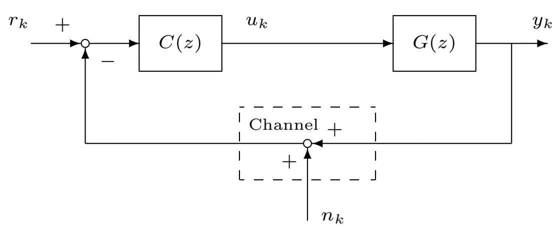

Fig. 1. LTI NCS in discrete-time with the communication channel over the feedback path.

The preliminary continuous-time counterpart results were communicated in the companion paper Rojas (2015).

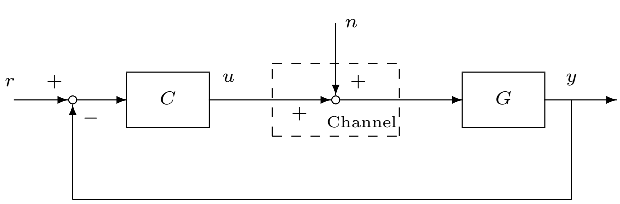

Fig. 2. LTI NCS in discrete-time with the communication channel over the control path.

2. PRELIMINARIES

2.1 Assumptions

Plant nominal model: Throughout the present work, if not stated otherwise, we assume that the plant model $G(z)$ is given by either

$$G_1(z)=\frac{K_p}{z-\rho},$$

with $K_p, \rho \in \mathbb{R}$. A first order model with time delay

$$G_2(z)=\frac{K_p}{z^N(z-\rho)},$$

with $N \in \mathbb{N}$, or a second order model defined as

$$G_3(z)=\frac{K_p z}{(z-\rho_1)(z-\rho_2)},$$

with $\rho_1, \rho_2 \in \mathbb{R}$. When no difference applies we will denote the plant model directly by $G(z)$.

Channel model: we consider here the memoryless AWGN channel model, see Figure 1. This type of channel is characterized by the channel input power $P > 0$ and its channel additive noise process n.

Channel additive noise process: The channel additive noise process n is a zero-mean i.i.d. Gaussian white noise process with variance equal to σ^2.

Setpoint assumptions: The setpoint signal is assumed to be a a zero-mean i.i.d. Gaussian white noise process with variance σ_r^2, see for example Li et al. (2009); Ding et al. (2009, 2010) for a similar setpoint assumption.

2.2 SNR Constrained Control

Consider the feedback loop in Figure 1 where the problem is to stabilise a continuous-time plant. From Figure 1 we observe that the closed loop relationships between the channel input u and the two exogenous signals r and n are given by $u=T(z)r-T(z)n$, where $T(z) = C(z)G(z)/(1 + C(z)G(z))$ is the closed loop complementary sensitivity function. The channel input is required to satisfy the power constraint $\mathcal{P} > \|y\|^2_{Pow}$, for some predetermined input power level $\mathcal{P} > 0$. We assume that the closed loop feedback system is stabilised, in the sense that for any distribution of initial conditions, the distribution of all closed loop signals in Figure 1 converges exponentially fast to a stationary distribution. Without loss of generality, we therefore consider the properties of the stationary distribution of the relevant signals. The power of the channel input signal then satisfies

$$\|y\|^2_{Pow}=\|T\|_2^2\sigma_r^2+\|T\|_2^2\sigma^2. \tag{1}$$

The channel input power constraint can be restated, from (1), as a constraint imposed on $\frac{\mathcal{P}}{\sigma^2}$ the channel SNR,

$$\frac{\mathcal{P}}{\sigma^2}>\|T\|_2^2\left(\frac{\sigma_r^2}{\sigma^2}+1\right), \tag{2}$$

On the other hand if we repeat a similar analysis for Figure 2 we obtain

$$\frac{\mathcal{P}}{\sigma^2}>\|SC\|_2^2\frac{\sigma_r^2}{\sigma^2}+\|T\|_2^2. \tag{3}$$

We thus observe that for the setting in Figure 1, the SNR limitation is uniquely defined by the controller design through the H_2 norms of $T(z)$ and $S(z)$.

2.3 Problem Definition

In the present paper we consider the effect of an imposed channel SNR as a parameter for the controller design. Thus the SNR requirement performance problem can be stated as:

Problem 1. **(Performance SNR Requirement).** *Design a proper rational function*

$$C(z)=K_c+\frac{K_iT_sz}{z-1}+\frac{K_dN(z-1)}{(1+NT_s)z-1}, \tag{4}$$

such that it stabilises the closed loop, and achieves a user required SNR constraint Γ_d imposed on the channel, whether it is located over the control or feedback path, which guarantees a user selected closed loop bandwidth.

We consider in the following sections the user defined SNR requirement Γ_d. We separate the presentation into the $G_1(z)$, $G_2(z)$ and $G_3(z)$ choices for plant models, in each case discussing in details the proportional controller, with $K_i = K_d = 0$, the proportional-integral controller, with $K_d = 0$, and finally the fully realizable proportional-integral-derivative controller structure in (4).

3. CONTROLLER DESIGN BASED ON AN SNR REQUIREMENT FOR $G_1(Z)$

3.1 Proportional Controller

We first consider the controller structure to be selected as a proportional controller, with $K_c \in \mathbb{R}$. The complementary sensitivity function for such a controller and the choice of $G_1(s)$ is given by $T_{1p}(z)=\frac{K_cK_p}{z-\rho+K_cK_p}$.

Theorem 2. When the AWGN channel is located over the feedback path, see Figure 1, then the SNR lower bound is given by

$$\frac{P}{\sigma^2} > \frac{K_c^2 K_p^2}{1-(\rho - K_c K_p)^2}\left(\frac{\sigma_r^2}{\sigma^2}+1\right). \tag{5}$$

Proof. From (2) and $T_{1p}(z)$ we have $\frac{P}{\sigma^2} > \left\| \frac{K_c K_p}{z-\rho+K_c K_p} \right\|_2^2 \left(\frac{\sigma_r^2}{\sigma^2}+1\right)$, which can be further developed resulting in (5).

Theorem 3. When the AWGN channel is located over the control path, see Figure 2, then the SNR lower bound is given by

$$\frac{P}{\sigma^2} > \frac{K_c^2 K_p^2}{1-(\rho - K_c K_p)^2}\left(\frac{\sigma_r^2}{\sigma^2}+1\right) + K_c^2 \frac{\sigma_r^2}{\sigma^2}. \tag{6}$$

Proof. From (3) and $T_{1p}(z)$ we have

$$\frac{P}{\sigma^2} > \left\| \frac{K_c(z-\rho)}{z-\rho+K_c K_p} \right\|_2^2 \frac{\sigma_r^2}{\sigma^2} + \left\| \frac{K_c K_p}{z-\rho+K_c K_p} \right\|_2^2,$$

which can be further developed resulting in (6).

Example 1. Consider for this example $\sigma^2 = \sigma_r^2 = 1$ and $K_p = 1$ and $\rho = \sqrt{2}$. We use Theorems 2, in blue, and 3, in dashed red, to predict the SNR for all stabilisng values of K_c which are in general in the set defined by $\left[\frac{\rho-1}{K_p}, \frac{\rho+1}{K_p}\right]$, see Figure 3. We can see that the possible

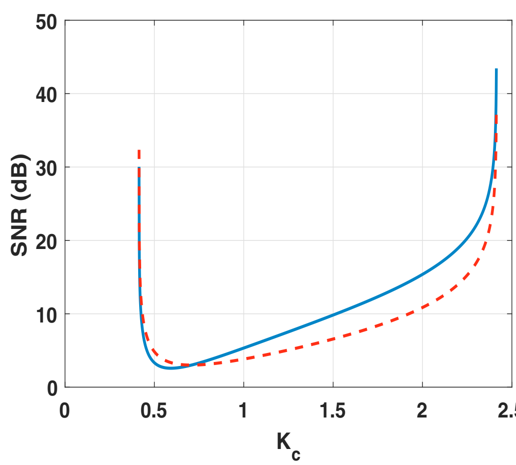

Fig. 3. Value of SNR in decibels for $G_1(z)$ in this example and as function of K_c, for the control path, blue line, and the feedback path, dashed redline.

values of the SNR that the user can decide through the value of K_c span an infimal value (different for each channel location) up to infinity. We thus explore in Figure 4 the resulting gain K_c predicted by Theorem 2 in blue, and predicted by Theorem 3 in dashed red. In Figure 5 we report the resulting infimal SNR in decibels for each case, again in blue for the control path case and dashed red line for the feedback path case. We observe the symmetry of the results, and also as predicted bu the infimal SNR expressions in equations (5) and (6), as $|\rho| \to 1$ for $|\rho| \geq 1$, then $K_c \to 0$ and for some value of K_c the infimal SNR over the control path will become lower that the infimal SNR for the feedback path. The precise K_c can be identified numerically to be for $|\rho| = 1.6405$ and an equal infimal SNR of 7.8367 (dB) for both channel locations. That is, there is no a priori preferred location for the communication channel. Of course, if $|\rho| \leq 1$ then the plant model $G_1(z)$ is stable, and thus the best policy

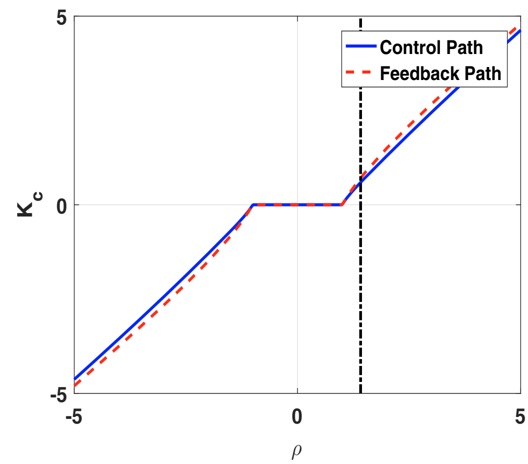

Fig. 4. Value of $C(z) = K_c$ that achieves the infimal SNR as function of ρ for $G_1(z)$.

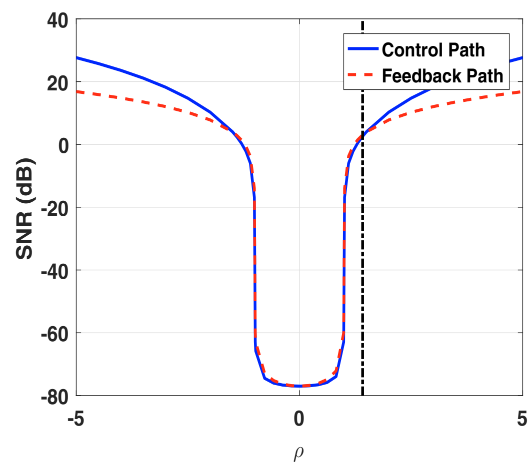

Fig. 5. Infimal SNR as function of ρ for $G_1(z)$.

in terms of channel SNR is not to control at all, which is explained by the infimal choice of $K_c = 0$ in Figure 4.

We can now consider solving Problem 1 by imposing a $|\rho_c| < 1$, where $\rho_c = \rho - K_c K_p$ is the closed loop pole of $T_{1p}(z)$. We observe that such closed loop location will then define the closed loop bandwidth according to

$$\omega_{clbd} = \frac{1}{T_s}\cos^{-1}\left(\frac{-\rho_c^2+4\rho_c-1}{2\rho_c}\right)$$

and thus inform the selection of K_c, and the channel SNR as functions of
rho_c.

Corollary 4. For a design requirement given by a closed loop pole location at $|\rho_c| < 1$ the minimal SNR requirement Γ_d can be restated, in terms of the design parameter ρ_c, as

$$\Gamma_d = \frac{(\rho-\rho_c)^2}{1-\rho_c^2}\left(\frac{\sigma_r^2}{\sigma^2}+1\right) \tag{7}$$

for the feedback path channel location, and

$$\Gamma_d = \frac{(\rho-\rho_c)^2}{1-\rho_c^2}\left(\frac{(\rho-\rho_c)^2}{K_p^2}\frac{\sigma_r^2}{\sigma^2}+1\right) + \frac{(\rho-\rho_c)^2}{K_p^2}, \tag{8}$$

for the control path channel location.

Proof. From $T_{1p}(z)$ we have that the unique closed loop pole of the complementary sensitivity $T_{1p}(z)$ is located at

$\rho_c = \rho - K_c K_p$. By restating this last expression in terms of K_c we obtain $K_c = \frac{\rho - \rho_c}{K_p}$. Replacing the expression of K_c as function of ρ_c in (5) gives (7). Similarly, replacing it in (6) gives (8)

Notice, from expressions in (7) and (8) that the proposed SNRs will be zero if $\rho_c = \rho$. However if $|\rho| \geq 1$, in which case $G_1(z)$ would be unstable, the resulting SNR will never be zero since we require for the closed loop to be stable. Hence $|\rho_c|$ will always be less than one by design, avoiding the possibility of $\Gamma_d = 0$, independently of the channel location.

3.2 Proportional-Integral Controller

We now continue the exposition by considering a PI controller, see equation (4), with K_c and K_i in $\mathbb{R}^+$ and $K_d = 0$.

Theorem 5. When the AWGN channel is located over the feedback path, see Figure 1, for the plant model $G_1(z)$ and the controller selected to be a PI controller, then the SNR lower bound is given by

$$\frac{P}{\Phi} > \frac{2b_1 b_o a_1 - (b_o^2 + b_1^2)(a_o + 1)}{(a_o - 1)((a_o + 1)^2 - a_1^2)} \left(\frac{\sigma_r^2}{\sigma^2} + 1 \right), \tag{9}$$

with $b_1 = K_p(K_c + T_s K_i)$, $b_o = -K_p K_c$, $a_1 = -1 - \rho + K_p(K_c + K_i T_s)$, $a_o = (\rho - K_p K_c)$.

Proof. As before we obtain first the complementary sensitivity for the proposed choice of plant, controller model which results in

$$T_{1pi}(z) = \frac{K_p((K_c + T_s K_i)z - K_c)}{z^2 + (-1 - \rho + K_p(K_c + K_i T_s))z + \rho - K_p K_c}. \tag{10}$$

From this and (2) we have $\frac{P}{\sigma^2} > \|T_{1pi}\|_2^2 \left(\frac{\sigma_r^2}{\sigma^2} + 1 \right)$, which, when further developed, results in (9).

Remark 6. We observe from Theorem 5 that as K_i tends to zero then $a_1 \to -(1 + a_o)$ and $b_1 \to -b_o$, thus the expression on the RHS of (9) becomes an indeterminate form. However, upon using L'Hôpital's rule with respect to K_i, we then recover the expression in (5).

Theorem 7. When the AWGN channel is located over the control path, see Figure 2, then the SNR lower bound is given by

$$\frac{P}{\sigma^2} > \frac{2b_1 b_o a_1 - (b_o^2 + b_1^2)(a_o + 1)}{(a_o - 1)((a_o + 1)^2 - a_1^2)} + \left(\frac{2(c_1 - a_1 c_2)(c_o - a_o c_2)a_1 - ((c_o - a_o c_2)^2 + (c_1 - a_1 c_2)^2)(a_o + 1)}{(a_o - 1)((a_o + 1)^2 - a_1^2)} + c_2^2 \right) \frac{\sigma_r^2}{\sigma^2}, \tag{11}$$

with $c_2 = K_c + K_i T_s$, $c_1 = -\rho(K_c + K_i T_s)$, $c_o = \rho K_c$, and b_1, b_o, a_1, a_o as in Theorem 5.

Proof. From (3) and (10) we have

$$\frac{P}{\sigma^2} > \left\| \frac{K_p((K_c + T_s K_i)z - K_c)}{z^2 + (-1 - \rho + K_p(K_c + K_i T_s))z + \rho - K_p K_c} \right\|_2^2 + \left\| \frac{(K_c + K_i T_s)z^2 - (K_c + \rho(K_c + K_i T_s))z + \rho K_c}{z^2 + (-1 - \rho + K_p(K_c + K_i T_s))z + \rho - K_p K_c} \right\|_2^2 \frac{\sigma_r^2}{\sigma^2},$$

which can be further developed resulting in (11).

We now solve Problem 1 by imposing a closed loop pole such that $|\rho_c| < 1$ with multiplicity two, and thus obtain from comparison to the characteristic polynomial of (10), expressions for K_c and K_i as functions of ρ_c,

$$K_p = \frac{\rho - \rho_c^2}{K_p}, \quad K_i = \frac{(\rho_c - 1)^2}{K_p T_s}.$$

From this, Γ_d as function of ρ_C can be obtained from Theorem 5 or Theorem 7.

3.3 Proportional-Integral-Derivative Controller

For the controller structure proposed in (4) and the plant model $G_1(z)$ we follow similar steps, and thus omit them to avoid needless repetitions. The resulting infimal SNR bound when the AWGN channel is located over the feedback path as function of the controller parameter K_p, K_i, K_d, T_s and N is given then by

$$\frac{P}{\sigma^2} > \left(\frac{\sigma_r^2}{\sigma^2} + 1 \right) \cdot \left(\frac{2(b_1 - a_1 b_2)(b_o - a_o b_2)a_1 - ((b_o - a_o b_2)^2 + (b_1 - a_1 b_2)^2)(a_o + 1)}{(a_o - 1)((a_o + 1)^2 - a_1^2)} + b_2^2 \right), \tag{12}$$

where

$$b_2 = \frac{K_c K_p (1/N + T_s) + K_i K_p T_s (1/N + T_s) + K_d K_p}{(1/N + T_s) + K_p(K_c(1/N + T_s) + K_i T_s (1/N + T_s) + K_d)}$$

$$b_1 = \frac{-K_c K_p (2/N + T_s) - K_i K_p T_s / N - 2K_d K_p}{(1/N + T_s) + K_p(K_c(1/N + T_s) + K_i T_s (1/N + T_s) + K_d)}$$

$$b_o = \frac{K_c K_p / N + K_d K_p}{(1/N + T_s) + K_p(K_c(1/N + T_s) + K_i T_s (1/N + T_s) + K_d)}$$

$$a_1 = \frac{-(1/N + T_s) - 1/N + K_p(K_c(-2/N - T_s) - K_i T_s / N - 2K_d)}{(1/N + T_s) + K_p(K_c(1/N + T_s) + K_i T_s (1/N + T_s) + K_d)}$$

$$a_o = \frac{1/N + K_p(K_c / N + K_d)}{(1/N + T_s) + K_p(K_c(1/N + T_s) + K_i T_s (1/N + T_s) + K_d)}$$

If we now impose a closed loop solution at ρ_c with multiplicity three, and consider N as a user defined parameter, then the remaining controller parameters will be $K_d = \frac{3(\alpha\rho_c - \beta)\alpha\beta\rho_c + \beta^3 - \alpha^2\beta\rho_c - \alpha^2\beta\rho_c - \alpha^2\rho_c^3 + \alpha\beta\rho_c}{(\beta - \alpha)^2}$, $K_c = \frac{-\alpha\rho_c^3 + \beta\rho_c - K_d}{\beta}$, and $K_i = \frac{-2\alpha\rho_c + \alpha + \beta - K_d - \alpha K_c}{\alpha T_s}$ with $\alpha = (1/N + T_s)$ and $\beta = 1/N$. This last results conclude the section devoted to the study of the SNR imposed by the choice of plant model $G_1(z)$ and a PID controller structure and its variants.

4. CONTROLLER DESIGN BASED ON AN SNR REQUIREMENT FOR $G_2(Z)$

In this section we treat the case of $G_2(z)$, in the same order proposed in the previous section, that is for the P, PI and PID controller structures.

4.1 Proportional Controller

For the P controller and the plant model $G_2(z)$ the complementary sensitivity results in

$$T_{2p}(z) = \frac{K_c K_p}{z^N (z - \rho) K_c K_p}.$$

With the above explicit expression for the complementary sensitivity, we can then obtain the general expression for the infimal SNR as function of the plant and controller parameters when the communication channel is over the feedback path

$$\frac{P}{\sigma^2} > \left(\frac{K_c^2 K_p^2}{-K_c^2 K_p^2 + 1 + \rho^2 \left(\frac{K_c K_p - 1}{K_c K_p + 1} \right)} \right) \left(\frac{\sigma_r^2}{\sigma^2} + 1 \right), \tag{13}$$

for N=1. The case for $N = 2$, on the other hand, reports an SNR of

$$\frac{P}{\sigma^2} > \frac{-K_c^2 K_p^2 (1 + \rho K_c K_p - K_c^2 K_p^2)}{(K_c^2 K_p^2 + \rho^2 - 1)(1 + \rho K_c K_p - K_c^2 K_p^2) - 2\rho^2 K_p K_c (\rho - K_p K_c)} \left(\frac{\sigma_r^2}{\sigma^2} + 1 \right), \tag{14}$$

When the communication channel is located over the control path the reported SNRs are for $N = 1$

$$\frac{P}{\sigma^2} > K_c^2 \frac{\sigma_r^2}{\sigma^2} + \left(\frac{K_c^4 K_p^2}{-K_c^2 K_p^2 + 1 + \rho^2 \left(\frac{K_c K_p - 1}{K_c K_p + 1} \right)} \right) \left(\frac{\sigma_r^2}{\sigma^2} + \frac{1}{K_c^2} \right), \tag{15}$$

and for $N = 2$

$$\frac{P}{\sigma^2} > K_c^2 \frac{\sigma_r^2}{\sigma^2} + \frac{-K_c^4 K_p^2 (1+\rho K_c K_p - K_c^2 K_p^2)}{(K_c^2 K_p^2 + \rho^2 - 1)(1+\rho K_c K_p - K_c^2 K_p^2) - 2\rho^2 K_p K_c (\rho - K_p K_c)} \left(\frac{\sigma_r^2}{\sigma^2} + \frac{1}{K_c}^2\right), \qquad (16)$$

Higher values of N can be explored numerically. The previous expressions suggest that, even for the simple proportional controller, we face a rich SNR behaviour as N increases, which can be hard to predict. Thus, it is probably advised to consider for $G_2(z)$ the use of a Smith Predictor scheme, see Figure 6, for the control path location.

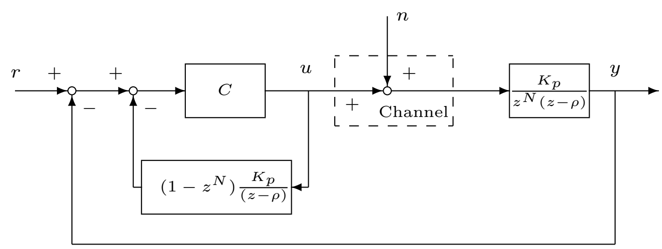

Fig. 6. LTI NCS in discrete-time with the communication channel over the control path and a SMith Predictor scheme applied to the controller.

The algebra behind the Smith Predictor allows for the following complementary sensitivity $T_{2p}(z) = \frac{K_c K_p}{z^N (z - \rho + K_c K_p)}$, thus the effect of the delay $1/z^N$ does not modify the H_2 norm and we recover the simpler case studied for the plant model $G_1(z)$. The same applies if a Smith Predictor scheme is used when the channel is located over the feedback path. We thus omit for $G_2(z)$ the PI and PID controller selections since they match the results already presented for $G_1(z)$.

5. CONTROLLER DESIGN BASED ON AN SNR REQUIREMENT FOR $G_3(Z)$

We conclude this work by discussing the infimal SNR obtained by imposing the performance requirement imposed by Problem 1, for the plant model $G_3(z)$ when we consider alternatively a P, PI, and PID structure.

5.1 Proportional Controller

For the P controller case we obtain the following SNR limitation

$$\frac{P}{\sigma^2} > \frac{-K_c^2 K_p^2 (\rho_1\rho_2 + K_c K_p + 1)}{(\rho_1\rho_2 + K_c K_p - 1)((\rho_1\rho_2 + K_c K_p + 1)^2 - (\rho_1+\rho_2)^2)} \left(\frac{\sigma_r^2}{\sigma^2} + 1\right), \qquad (17)$$

when the channel is located over the feedback path. This is effectively, as predicted in (2), the H_2 norm of the complementary sensitivity given by

$$T_{3p}(z) = \frac{K_c K_p}{z^2 - (\rho_1+\rho_2) z + \rho_1\rho_2 + K_c K_p}.$$

If the communication channel is located over the control path, in line with what predicted in (3) we obtain the following SNR

$$\frac{P}{\sigma^2} > K_c^2 \frac{\sigma_r^2}{\sigma^2} + \frac{-K_c^2 K_p^2 (\rho_1\rho_2 + K_c K_p + 1)}{(\rho_1\rho_2 + K_c K_p - 1)((\rho_1\rho_2 + K_c K_p + 1)^2 - (\rho_1+\rho_2)^2)} \left(\frac{\sigma_r^2}{\sigma^2} + 1\right). \qquad (18)$$

If we now impose a closed loop pole, with multiplicity two, at $z = \rho_c$, the controller parameter is then defined as $K_c = \frac{\rho_c^2 - \rho_1\rho_2}{K_p}$, and $\rho_c = \frac{\rho_1+\rho_2}{2}$. The closed loop pole ρ_c is therefore not free to be chosen by the user, but forced by the requirement of the complementary sensitivity. This is highlighting that a proportional controller is not suitable for the control of a second order plant model if the user wishes complete freedom for the controller design to achieve the performance objectives. The resulting SNRs for both cases are given by

$$\frac{P}{\sigma^2} > \frac{-(\rho_1-\rho_2)^4((\rho_1+\rho_2)^2+4)}{((\rho_1+\rho_2)^2-4)^3} \left(\frac{\sigma_r^2}{\sigma^2} + 1\right), \qquad (19)$$

and

$$\frac{P}{\sigma^2} > \frac{((\rho_1-\rho_2)^4)}{16 K_p^2} \frac{\sigma_r^2}{\sigma^2} + \frac{-(\rho_1-\rho_2)^4((\rho_1+\rho_2)^2+4)}{((\rho_1+\rho_2)^2-4)^3} \left(\frac{\sigma_r^2}{\sigma^2} + 1\right). \qquad (20)$$

Example 8. In this example we explore the expression in (19). The result is presented in Figure 7 for $\sigma_r^2 = \sigma^2 = 1$. We observe from this figure, that as expected, the line

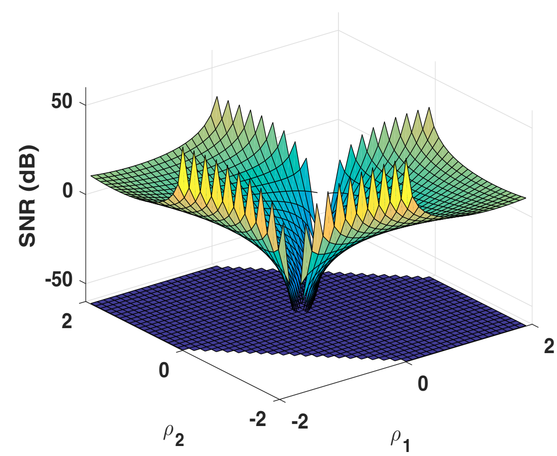

Fig. 7. Infimal SNR as function of ρ_1 and ρ_2.

$\rho_1 = \rho_2$ reports an SNR of zero. On the other hand in the range explored for ρ_1 and ρ_2 of minus 2 and plus 2, not all combinations of the open loop poles ρ_1 and ρ_2 result in a viable SNR (that is a positive SNR). Only the combinations insides the dark area in Figure 7. In fact, we observe that this is explained by the expression $(\rho_1+\rho_2)^2 - 4$ in the denominator of (19) which has to be negative in order for the overall SNR to be positive.

5.2 Proportional-Integral Controller

We consider now the use of a PI controller structure together with plant model $G_3(z)$. The complementary sensitivity resulting from such selection is given by

$$T_{3pi}(z) = \frac{K_p((K_c + K_i T_s) z - K_c)}{z^3 - (\rho_1+\rho_2+1) z^2 + (\rho_1\rho_2 + \rho_1 + \rho_2 + K_p(K_c + K_i T_s)) z - \rho_1\rho_2 - K_c K_p}.$$

If we impose a closed loop pole at $z = \rho_c$ with multiplicity 3, together with equation (2), we obtain the following SNR lower bound when the channel is located over the feedback path

$$\mathit{scriptstyle}\frac{P}{\sigma^2} > \left[\sum_{l=1}^{3}\sum_{p=1}^{3}\sum_{\substack{q=\max\\ \{0,\ l-p\}}}^{l-1} \binom{l-1}{q}\binom{p+q-1}{l-1} r_l \bar{r}_p \right.$$
$$\left. (-1)^{p+q-1} \frac{(\bar{\rho}_c)^q}{(\rho_c)^l} \left(\frac{\rho_c}{\rho_c\bar{\rho}_c - 1}\right)^{p+q}\right] \left(\frac{\sigma_r^2}{\sigma^2} + 1\right), \qquad (21)$$

where $r_1=0, r_2=b_1, r_3=b_o+\rho_c b_1$, and $b_o=-K_c K_p,\ b_1=K_p(K_c+K_i T_s)$. On the other hand , when the channel is located over the control path, together with equation (3) , we obtain the following lower bound for the channel SNR

$$\frac{P}{\sigma^2}>\Bigg[c_3^2+\sum_{l=1}^{3}\sum_{p=1}^{3}\sum_{\substack{q=\max\\ \{0,\ l-p\}}}^{l-1}\binom{l-1}{q}\binom{p+q-1}{l-1}t_l\bar{t}_p (-1)^{p+q-1}\frac{(\bar{\rho}_c)^q}{(\rho_c)^l}\Big(\frac{\rho_c}{\rho_c\bar{\rho_c}-1}\Big)^{p+q}\Bigg]\left(\frac{\sigma_r^2}{\sigma^2}\right)$$
$$\sum_{l=1}^{3}\sum_{p=1}^{3}\sum_{\substack{q=\max\\ \{0,\ l-p\}}}^{l-1}\binom{l-1}{q}\binom{p+q-1}{l-1}r_l\bar{r}_p (-1)^{p+q-1}\frac{(\bar{\rho}_c)^q}{(\rho_c)^l}\Big(\frac{\rho_c}{\rho_c\bar{\rho_c}-1}\Big)^{p+q}, \quad (22)$$

where

$$t_1=c_2+3\rho_c c_3, t_2=c_1+2\rho_c c_2, t_3=c_o+\rho_c c_1+\rho_c^2 c_2+\rho_c^3 c_3, \quad (23)$$

and $c_o=-\rho_1\rho_2 K_c,$, $c_1=\rho_1\rho_2(K_c+K_i T_s)+K_c(\rho_1+\rho_2)$, $c_2=-(K_c+(\rho_1+\rho_2)(K_c+K_i T_s))$, $c_3=K_c+K_i T_s$. The controller parameter, when imposing the closed loop pole at $z=\rho_c$ with multiplicity 3, are

$$K_c=\frac{\rho_1\rho_2-\rho_c^3}{K_p},$$
$$K_i=\frac{\rho_c^3+3\rho_c^2-2\rho_1\rho_2-\rho_1-\rho_2}{K_p T_s}.$$

Similar to the P controller case, for the PI we also have that the closed loop pole $z=\rho_c$ with multiplicity 3 is not free to be chosen by the user, but it is instead defined by $\rho_c=\rho_1+\rho_2+1$.

5.3 Proportional-Integral-Derivative Controller

We conclude this section by considering a PID controller structure for plant model $G_3(z)$. We omit the SNR expressions and just recapitulate that for these choices the resulting complementary sensitivity is then defined as

$$T_{3pid}(z)=\frac{1}{\frac{(z-\rho_1)(z-\rho_2)(z-1)(\alpha z-\beta)}{K_p(K_c\alpha+K_i T_s\alpha+K_d)z^2-K_p(K_c(\alpha+\beta)+K_i T_s\beta+2K_d)z+K_c K_p\beta+K_d K_p}+1}.$$

We observe that, by imposing a stable closed loop pole at $z=\rho_c$ with multiplicity 4, then the N parameter in (4) is fixed at $N=\frac{2+\rho_1+\rho_2-4\rho_c}{T_s(4\rho_c-\rho_1\rho_2-1)}$. It can be proved that if we require the PID controller to be independently stable then $N\in\mathbb{R}^+$ and $\frac{1+\rho_1+\rho_2}{4}<\rho_c<\frac{2+\rho_1+\rho_2}{4}$, or if $N\in\mathbb{R}^-$ and $N<\frac{-1}{2T_s}$, then $\frac{1+\rho_1+\rho_2}{4}<\rho_c<\frac{3+\rho_1+\rho_2}{4}$. That is the choice for ρ_c is then limited, if not plainly unfeasible, depending on the values of the poles of $G_3(z)$. If we lift any stability condition of the PID and choose N directly, the other controller parameter can be obtained by solving the following set of equations

$$\beta+(\rho_1+\rho_2)(\alpha+\beta)+\alpha\rho_1\rho_2+K_p(\alpha K_c+\alpha K_i T_s+K_d)=6\alpha\rho_c^2$$
$$\beta(\rho_1+\rho_2)+\rho_1\rho_2(\alpha+\beta)+K_p K_c(\alpha+\beta)+\beta K_p K_i T_s+2K_p K_d=4\alpha\rho_c^3$$
$$\beta\rho_1\rho_2+\beta K_p K_c+K_p K_d=\alpha\rho_c^4$$

where $\alpha=(1/N+T_s)$ and $\beta=1/N$.

6. CONCLUSIONS

In the present work we presented the infimal SNR required for a memoryless AWGN channel located over the feedback path or over the control path in a control feedback loop. We considered all combinations between the three Â´proposed plant models (first order model, first order model with delay and second order model) and the three proposed control structures (P, PI and PID). The objective for the controller design was a given closed loop bandwidth defined by means of a closed loop pole at ρ_c, with adequate multiplicity. As a result we could then quantify the SNR requirement for the memoryless AWGN channel as a function of ρ_c and the plant model parameters. Future work should consider the case of more complex, albeit more realistic, communication channel models.

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