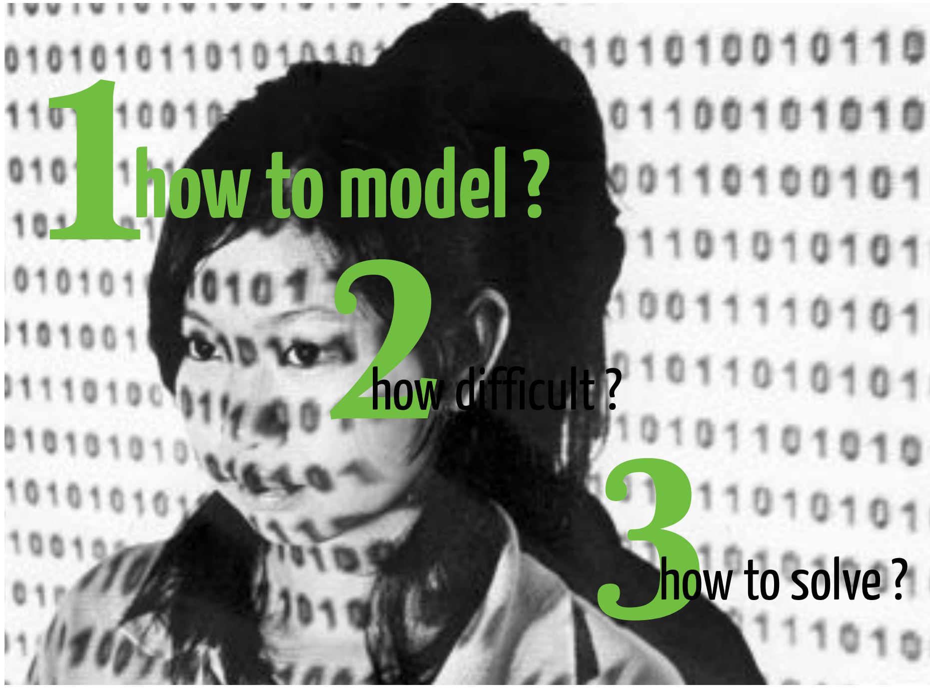
1 how to model ?

2 how difficult ?

3 how to solve ?

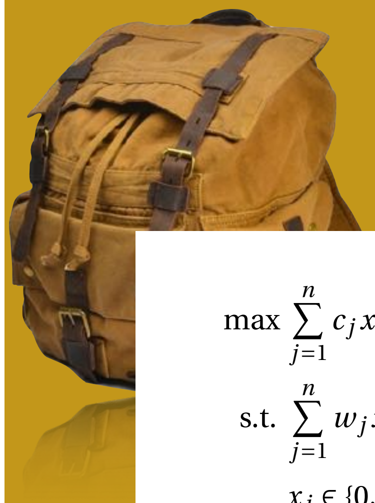


~~true~~ 1 or ~~false~~ 0

- is item j selected ? $x_j \in \{0,1\}$
- is item j associated to item i ? $y_{ij} \in \{0,1\}$
- is variable x greater than a ? $x \geq ay,\ y \in \{0,1\}$
- is constraint c satisfied ? $\ldots$

Integer Knapsack Problem

Input n items, value c_j and weight w_j for each item j, capacity K.
Output a maximum value subset of items whose total weight does not exceed K.



Integer Knapsack Problem

$$\max \sum_{j=1}^{n} c_j x_j$$

$$\text{s.t. } \sum_{j=1}^{n} w_j x_j \leq K$$

$$x_j \in \{0,1\} \qquad j = 1..n$$

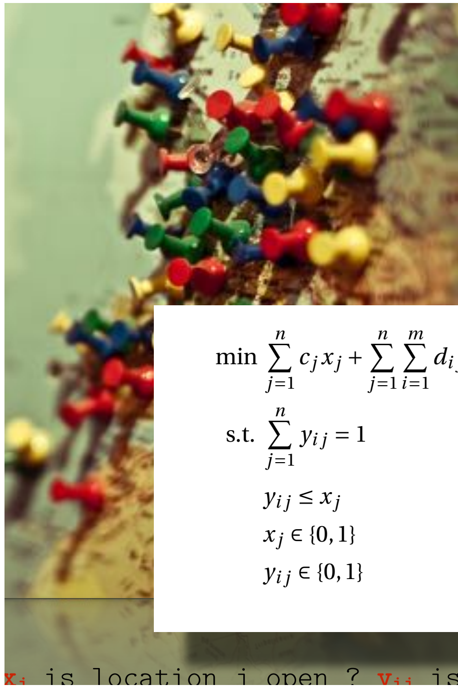
x_j is item j packed ?

logic with binaries

- either x or y $x + y = 1$
- if x then y $y \geq x$
- if x then f ≤ a $f \leq ax + M(1-x)$ "big M"
- at most 1 out of n $x_1 + \cdots + x_n \leq 1$
- at least k out of n $x_1 + \cdots + x_n \geq k$

Uncapacitated Facility Location Problem

Input n facility locations, m customers, cost c_j to open facility j, cost d_{ij} to serve customer i from facility j
Output a mimimum (opening and service) cost assignment of customers to facilities.



Uncapacitated Facility Location Problem

$$
\begin{aligned}
\min\ & \sum_{j=1}^{n} c_j x_j + \sum_{j=1}^{n}\sum_{i=1}^{m} d_{ij} y_{ij} & \\
\text{s.t. } & \sum_{j=1}^{n} y_{ij} = 1 & i = 1..m \\
& y_{ij} \le x_j & j = 1..n,\ i = 1..m \\
& x_j \in \{0,1\} & j = 1..n \\
& y_{ij} \in \{0,1\} & j = 1..n,\ i = 1..m
\end{aligned}
$$

x_j is location j open ? y_{ij} is customer i served from j ?

1||Cmax Scheduling Problem

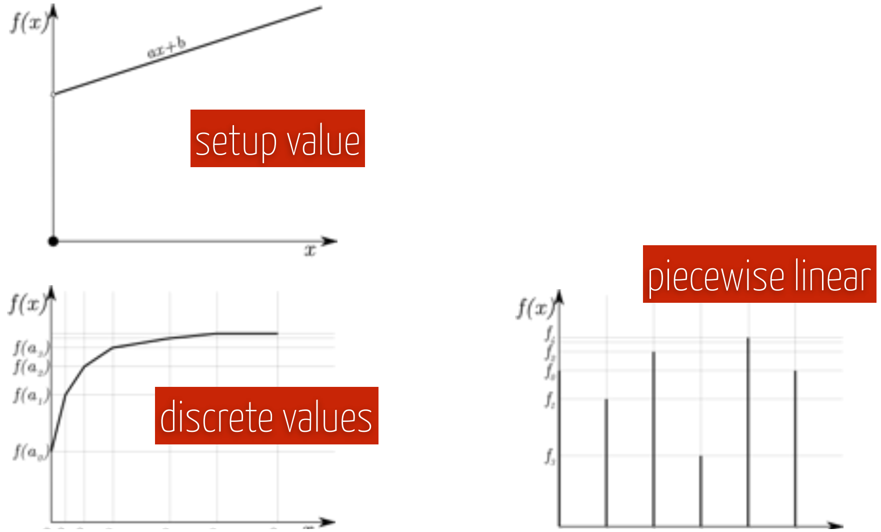
Input n tasks, duration pi for each task i, one machine
Output a minimal makespan schedule of the tasks on the machine without overlap

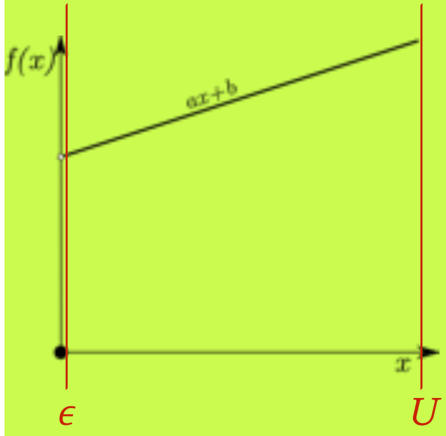
1||Cmax Scheduling Problem

$$
\begin{aligned}
\min\ & s_{n+1} & \\
\text{s.t. } & s_{n+1} \ge s_j + p_j & j = 1..n \\
& s_j - s_i \ge M x_{ij} + (p_i - M) & i, j = 1..n \\
& x_{ij} + x_{ji} = 1 & i, j = 1..n; i < j \\
& s_j \in \mathbb{Z}_+ & j = 1..n+1 \\
& x_{ij} \in \{0,1\} & i, j = 1..n
\end{aligned}
$$

x_{ij} does i precede j ? s_j starting time of j

non-linear functions



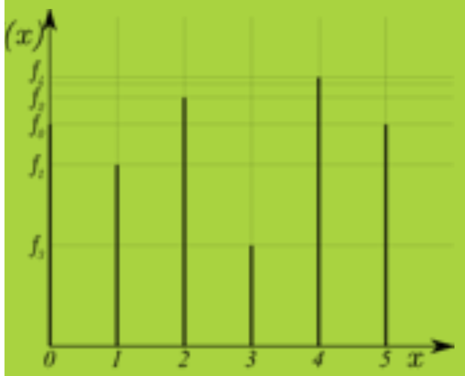


setup value

$f(x) = ax + b\delta$
$\epsilon\delta \leq x \leq U\delta$
$\delta \in \{0,1\}$

δ is x positive ?

Special Ordered Set of type 1:
ordered set of variables, all zero except at most on

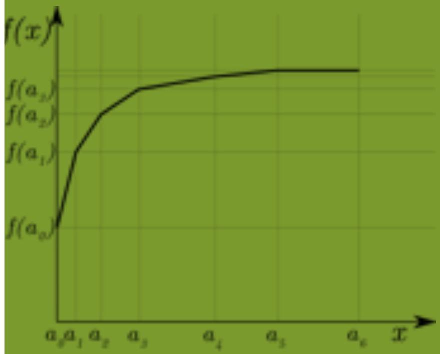


discrete values

$f(x) = \sum_i \delta_i f_i$
$\sum_i i\delta_i = x$
$\sum_i \delta_i \geq 1$
$\delta_i \in \{0,1\}\ i = 0..n$
SOS1($\boldsymbol{\delta}$)

δ_i is $x=i$ (and $f(x)=f_i$) ?

Special Ordered Set of type 2:
ordered set of variables, all zero except at most two consecutiv



piecewise linear

$f(x) = \sum_i \lambda_i f(a_i)$
$\sum_i a_i\lambda_i = x$
$\sum_i \lambda_i = 1$
$\lambda_i \in [0,1]\ i = 0..n$
SOS2($\boldsymbol{\lambda}$)

λ_i is $x=a_i$? (then $\lambda_i a_i+\lambda_{i+1}a_{i+1}$ in $[a_i,a_{i+1}]$ if $\lambda_i+\lambda_{i+1}=1$)

$$x_i = 5$$

to order i is the 5th item

to count 5 items are selected

to measure time task i starts at time 5

to measure space item i is located on floor 5

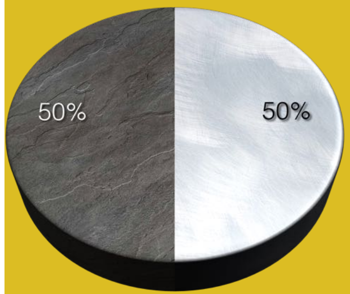
$$\simeq \delta_{i5} = 1$$

Binary Integer Linear Program (BIP) $\{0,1\}^n$

Integer Linear Program (IP) $\mathbb{Z}^n$

Mixed Integer Linear Program (MIP) $\mathbb{Z}^n \cup \mathbb{Q}^n$

Interlude 1

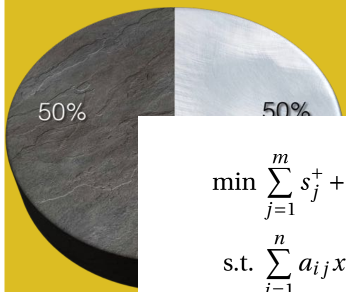


Market Split Problem

Input 1 company, 2 divisions, m products with availabilities d_j, n retailers with demands a_{ij} in each product j.

Output an assignment of the retailers to the divisions approaching a 50/50 production split.

Interlude 1

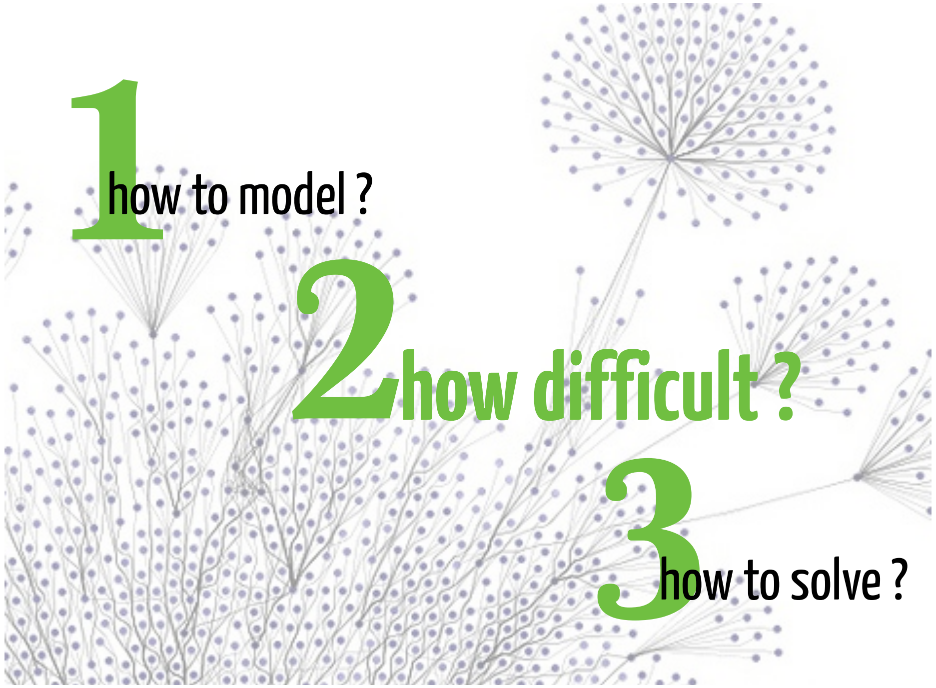


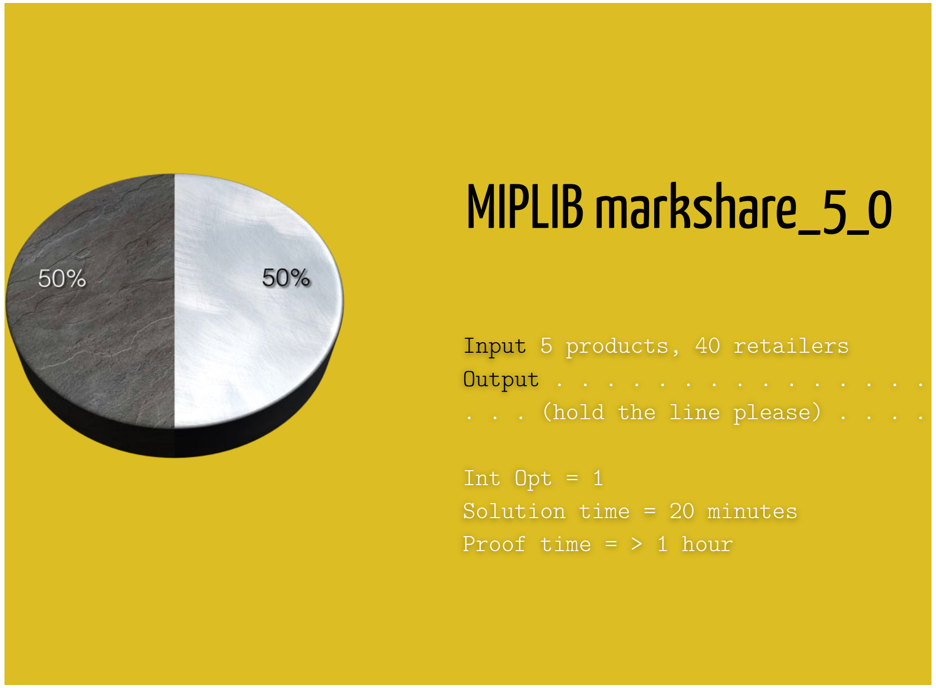
Market Split Problem

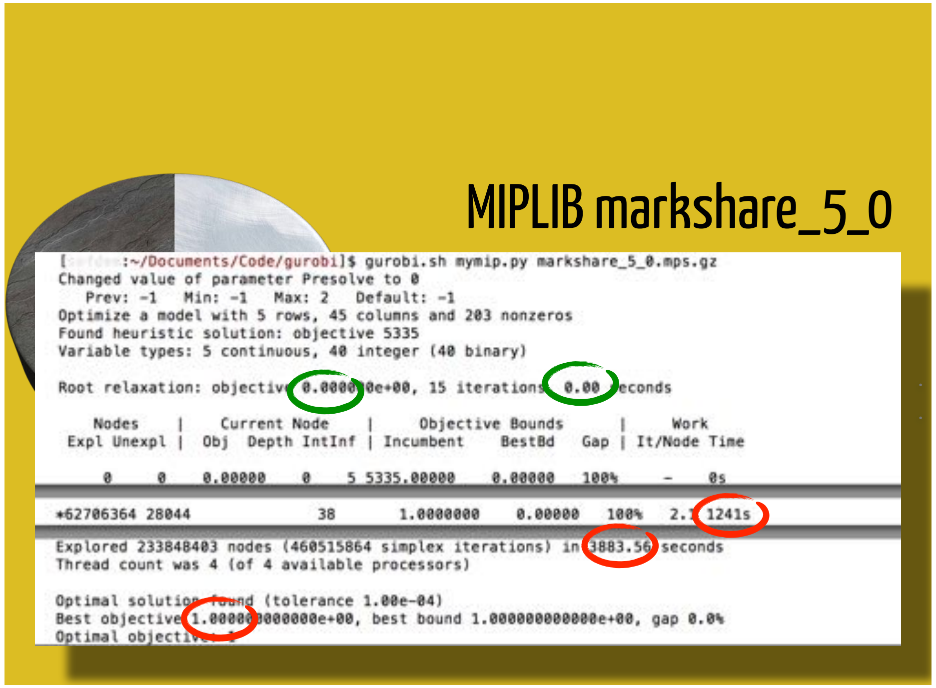
$$\begin{aligned}
\min \ & \sum_{j=1}^{m} s_j^+ + s_j^- \\
\text{s.t.} \ & \sum_{i=1}^{n} a_{ij} x_i + s_j^+ - s_j^- = \frac{d_j}{2} && j = 1..m \\
& x_i \in \{0,1\} && i = 1..n \\
& s_j^+ \geq 0, s_j^- \geq 0 && j = 1..m
\end{aligned}$$

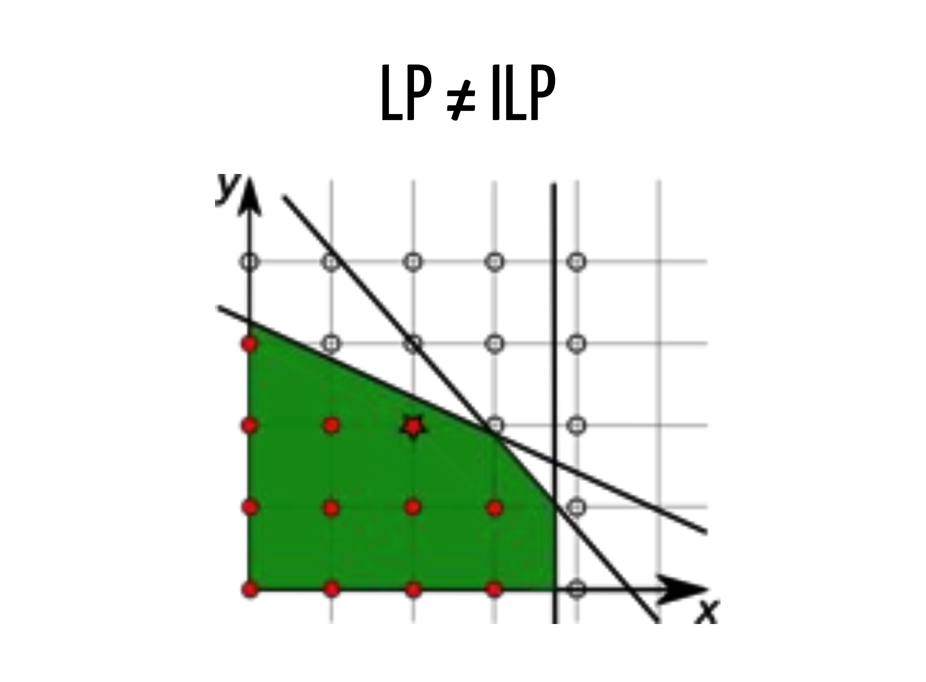
ns, m
es d_j, n
in each
e retailers
ng a 50/50

$_i$ is retailer i assigned to division 1 ?

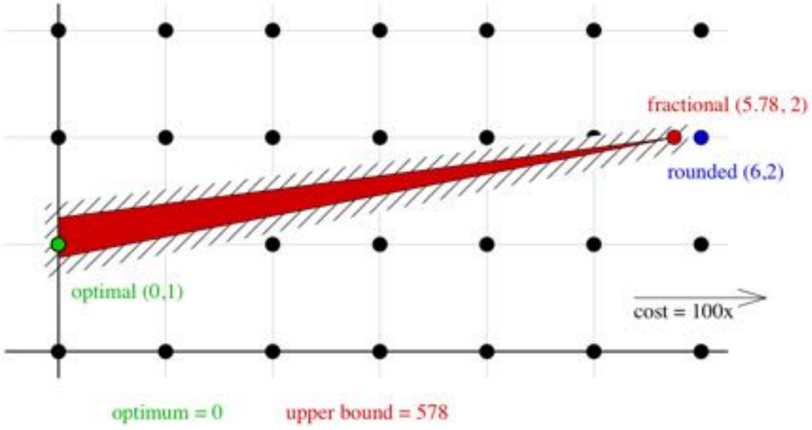

1 how to model ?
2 how difficult ?
3 how to solve ?


MIPLIB markshare_5_0
50%
50%
Input 5 products, 40 retailers
Output . . . .
. . . (hold the line please) . . . .
Int Opt = 1
Solution time = 20 minutes
Proof time = > 1 hour


MIPLIB markshare_5_0
:~/Documents/Code/gurobi]$ gurobi.sh mymip.py markshare_5_0.mps.gz
Changed value of parameter Presolve to 0
Prev: -1 Min: -1 Max: 2 Default: -1
Optimize a model with 5 rows, 45 columns and 203 nonzeros
Found heuristic solution: objective 5335
Variable types: 5 continuous, 40 integer (40 binary)
Root relaxation: objectiv 0.0000 0e+00, 15 iterations 0.00 econds
Nodes | Current Node | Objective Bounds | Work
Expl Unexpl | Obj Depth IntInf | Incumbent BestBd Gap | It/Node Time
0 0 0.00000 0 5 5335.00000 0.00000 100% - 0s
*62706364 28044 38 1.0000000 0.00000 100% 2.1 1241s
Explored 233848403 nodes (460515864 simplex iterations) in 3883.56 seconds
Thread count was 4 (of 4 available processors)
Optimal solution found (tolerance 1.00e-04)
Best objective 1.00000000000e+00, best bound 1.000000000000e+00, gap 0.0%
Optimal objective


LP ≠ ILP
y
x

round LP ≠ ILP



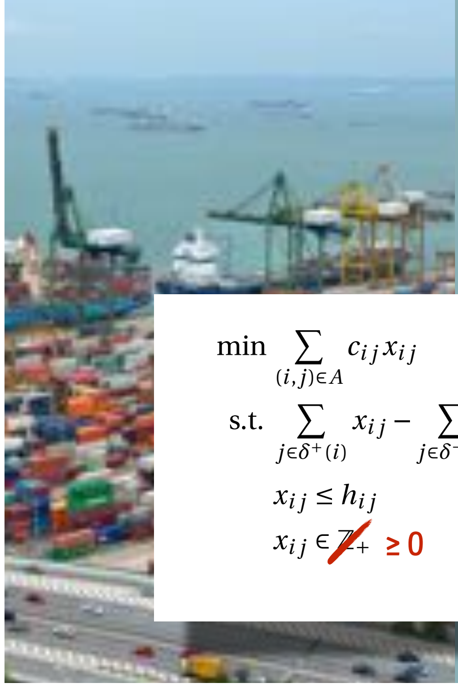
"ILP is NP-hard: I can't solve it !"

1||Cmax Scheduling Problem

$$
\begin{array}{lll}
\min & s_{n+1} \quad \boxed{= p_1 + \ldots + p_n} & \\
\text{s.t.} & s_{n+1} \geq s_j + p_j & j = 1..n \\
& s_j - s_i \geq M x_{ij} + (p_i - M) & i,j = 1..n \\
& x_{ij} + x_{ji} = 1 & i,j = 1..n; i < j \\
& s_j \in \cancel{\mathbb{Z}_+} \ \geq 0 & j = 1..n+1 \\
& x_{ij} \in \{0,1\} & i,j = 1..n
\end{array}
$$

Capacitated Transhipment Problem

Input digraph (V,A), demand or supply b_i at each node i, capacity h_{ij} and unit flow cost c_{ij} for each arc (i,j)
Output a mimimum cost integer flow to satisfy the demand



Capacitated Transhipment Problem

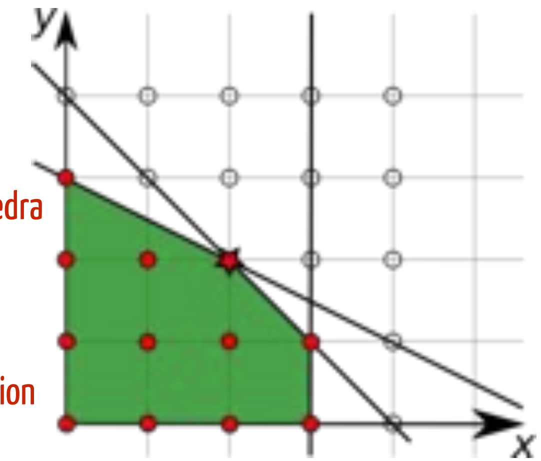
$$
\begin{aligned}
\min \quad & \sum_{(i,j)\in A} c_{ij} x_{ij} & \\
\text{s.t.} \quad & \sum_{j\in\delta^+(i)} x_{ij} - \sum_{j\in\delta^-(i)} x_{ij} = b_i & i \in V \\
& x_{ij} \le h_{ij} & (i,j) \in A \\
& x_{ij} \in \cancel{\mathbb{Z}_+} \ \geq 0 & (i,j) \in A
\end{aligned}
$$

demand or
i,
flow cost
integer
emand

x_{ij} flow on arc (i,j)

LP = ILP

integral polyhedra
=
convex hull
=
ideal formulation



totally unimodular matrix

(theory)

$$(P) = \max\{\ cx \mid Ax \le b, x \in \mathbb{Z}^n_+\ \}$$

- basic feasible solutions of the LP relaxation $(\bar{P})$ take the form: $\bar{x} = (\bar{x}_B, \bar{x}_N) = (B^{-1}b, 0)$ where B is a square submatrix of (A, I_m)
- Cramer's rule: $B^{-1} = B^*/det(B)$ where B^* is the adjoint matrix (made of products of terms of B)
- Proposition: if (P) has integral data (A, b) and if $det(B) = \pm 1$ then $\bar{x}$ is integral

Definition

A matrix A is totally unimodular (TU) if every square submatrix has determinant $+1$, -1 or 0.

Proposition

If A is TU and b is integral then any optimal solution of $(\bar{P})$ is integral.

totally unimodular matrix

(practice)

How to recognize TU ?

Sufficient condition

A matrix A is TU if

- all the coefficients are $+1$, -1 or 0
- each column contains at most 2 non-zero coefficient
- there exists a partition (M_1, M_2) of the set M of rows such that each column j containing two non zero coefficients satisfies $\sum_{i\in M_1} a_{ij} - \sum_{i \in M_2} a_{ij} = 0$.

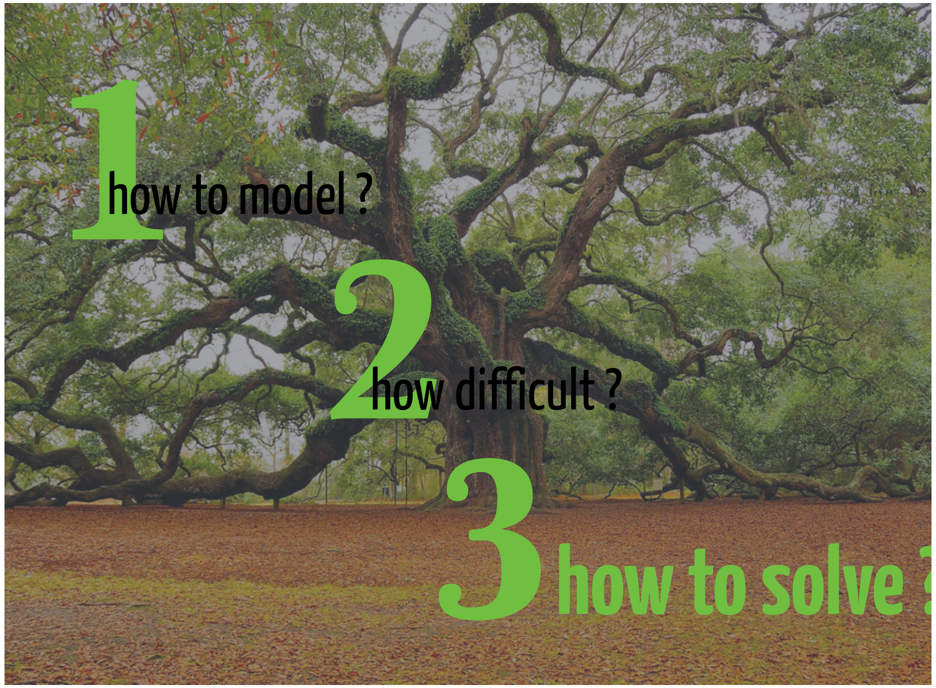
Proposition

A is TU $\iff$ A^t is TU $\iff$ (A, I_m) is TU
where A^t is the transpose matrix, I_m the identitiy matrix

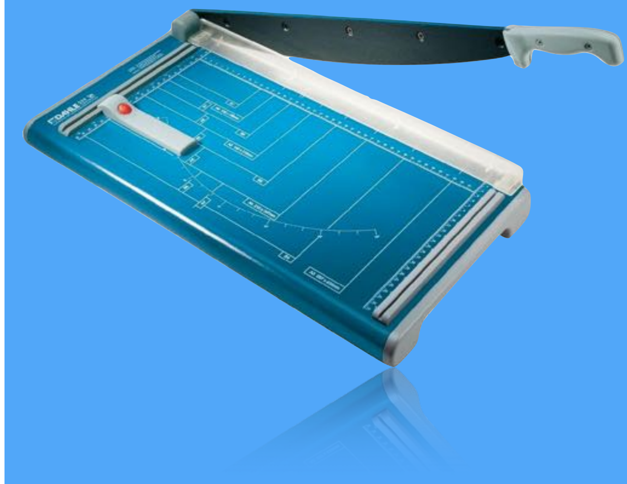
Interlude 2

Show that the **Transhipment** ILP is **ideal**

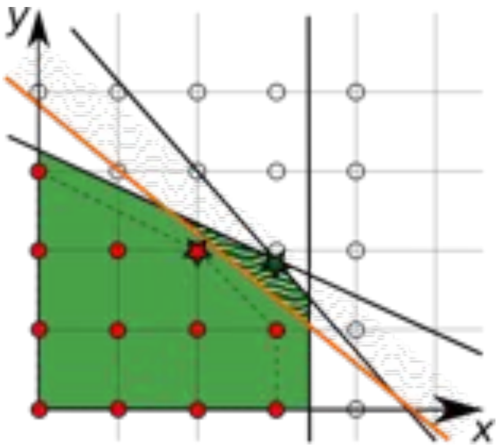
Show that the **Scheduling** ILP is **NOT ideal**



1. **Cuts** — compute an ideal formulation and solve the LP
2. **Branch&Bound** — enumerate solutions implicitely
3. **modern Branch&Cut** — mix up+presolve +heuristics
4. **decomposition methods** — (Branch&Price, Lagrangian, Benders)



Cutting Plane Algorithm



Cut valid inequality that separates the LP solution

Farkas Lemma any cut is a linear combination of the constraint

cutting plane algorithm

```
1. solve the LP relaxation (P), get x*
2. if x* is integral, STOP
3. find a cut for (P,x*) from a template T
```

templates

generic
Gomory Mixed Integer, Mixed Integer Rounding, Split, Chvátal-Gomory

structural
clique, cover, flow cover, zero half

problem-specific
subtour elimination (TSP), odd-set (matching)

ex 1 Mixed Integer Rounding

Combining constraints, then rounding leads to valid inequalities.

Let $u \in \mathbb{R}^m_+$, then the following inequalities are valid for (P):

- surrogate: $\sum_{j=1}^m u_j a_{ij} x_i \leq \sum_{j=1}^m u_j b_j$ (since $u \geq 0$)
- round off: $\sum_{j=1}^m \lfloor u_j a_{ij} \rfloor x_i \leq \sum_{j=1}^m u_j b_j$ (since $\lfloor u_j a_{ij} \rfloor \leq u_j a_{ij}$ and $x \geq 0$)
- Chvátal-Gomory: $\sum_{j=1}^m \lfloor u_j a_{ij} \rfloor x_i \leq \lfloor \sum_{j=1}^m u_j b_j \rfloor$ (since $e \in \mathbb{Z}$ and $e \leq f$ implies that $e \leq \lfloor f \rfloor$)

- CG inequalities form a generic class of valid inequalities: they apply to any IP

ex 2 Cover

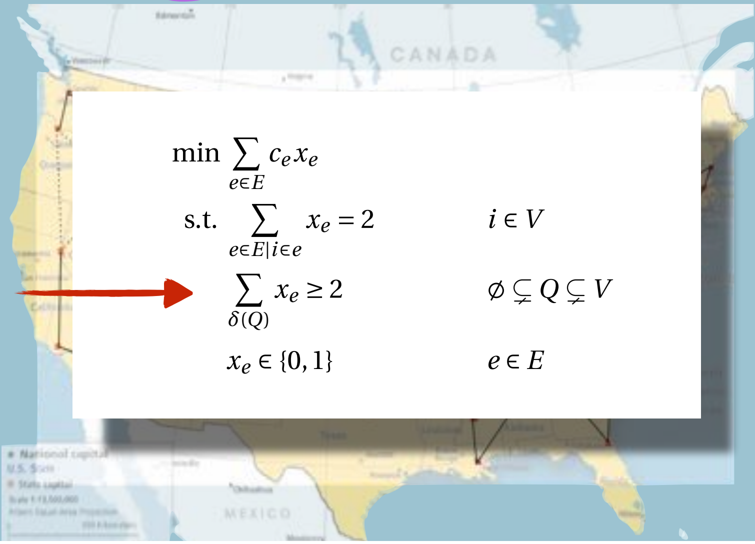
Cover inequalities

$$S = \{y \in \{0,1\}^7 | 11y_1 + 6y_2 + 6y_3 + 5y_4 + 5y_5 + 4y_6 + y_7 \leq 19\}$$

- (y_3, y_4, y_5, y_6) is a minimal cover for $11y_1 + 6y_2 + 6y_3 + 5y_4 + 5y_5 + 4y_6 + y_7 \leq 19$ as $6 + 5 + 5 + 4 > 19$ then $y_3 + y_4 + y_5 + y_6 \leq 3$ is a cover inequality
- we can derive a stronger valid inequality $y_1 + y_2 + y_3 + y_4 + y_5 + y_6 \leq 3$ by noting that y_1, y_2 has greater coefficients than any variable in the cover
- note furthermore that (y_1, y_i, y_j) is a cover $\forall i \neq j \in \{2,3,4,5,6\}$ then $2y_1 + y_2 + y_3 + y_4 + y_5 + y_6 \leq 3$ is also valid

The procedure to get this last equality is called *lifting*

ex 3 Subtour for TSP



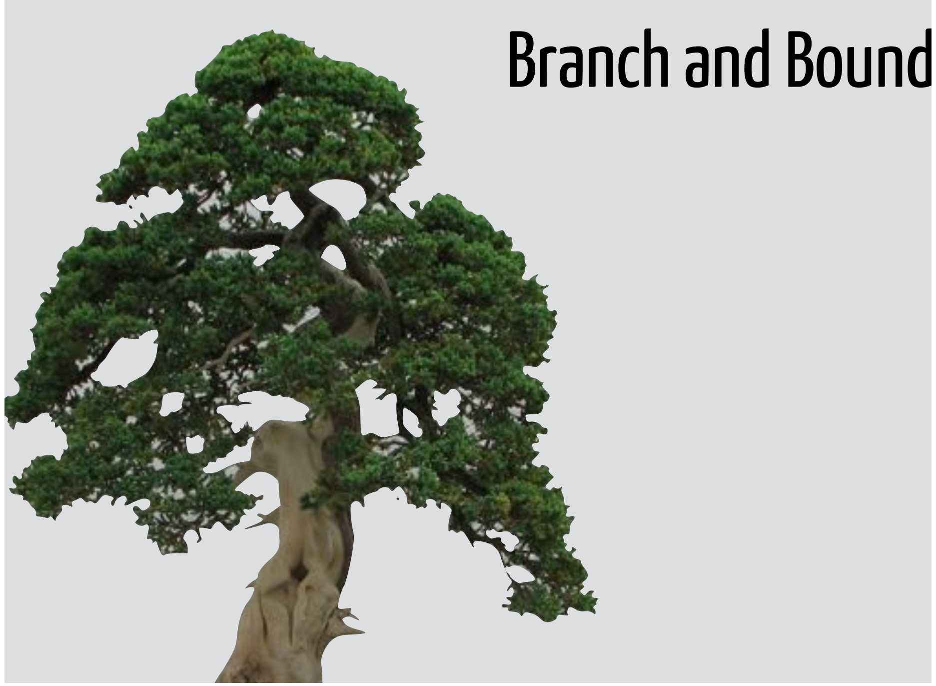
$$\begin{aligned} \min \sum_{e \in E} c_e x_e & \\ \text{s.t.} \sum_{e \in E | i \in e} x_e = 2 & \qquad i \in V \\ \sum_{\delta(Q)} x_e \geq 2 & \qquad \emptyset \subsetneq Q \subsetneq V \\ x_e \in \{0,1\} & \qquad e \in E \end{aligned}$$

ex 3 Subtour for TSP

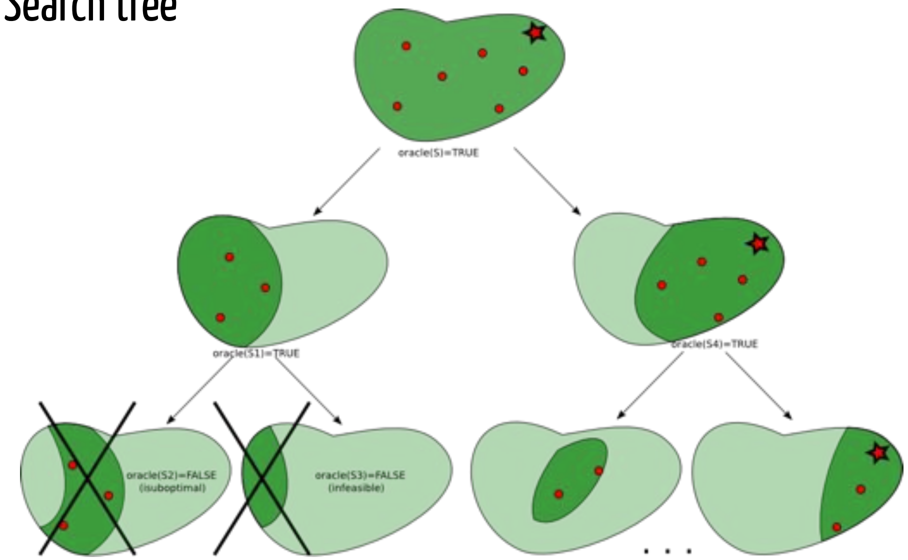
limits depending on the cut families

- the algorithm may stop prematurely
- the algorithm may not converge
- the algorithm may converge slowly
- the separation procedure may be NP-hard
- the LP grows
- the LP structure changes

Branch and Bound



Search tree

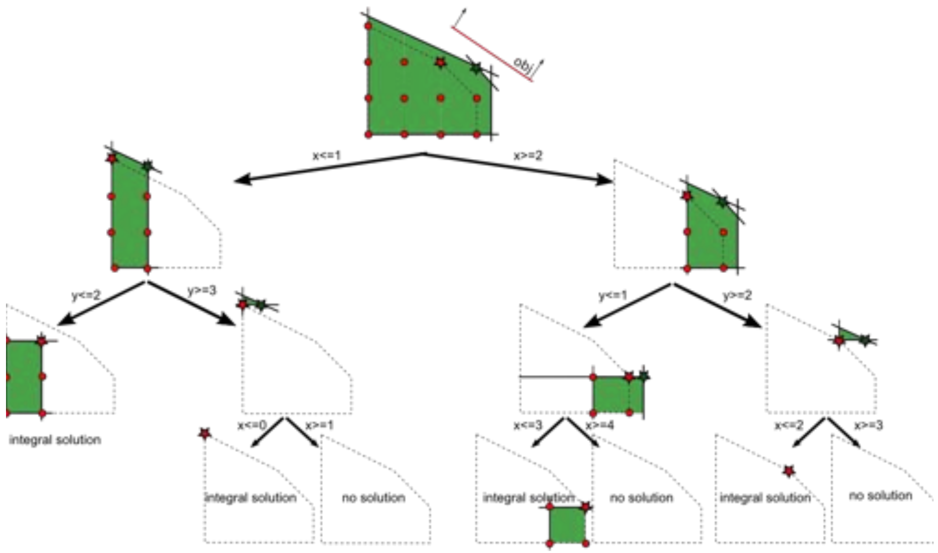


LP-based B&B

oracle(S) = FALSE iff either:

- LP is infeasible
- the fractional solution $\bar{x}$ is not better than the incumbent x^*
- $\bar{x}$ is integer (update x^*)

then prune node S



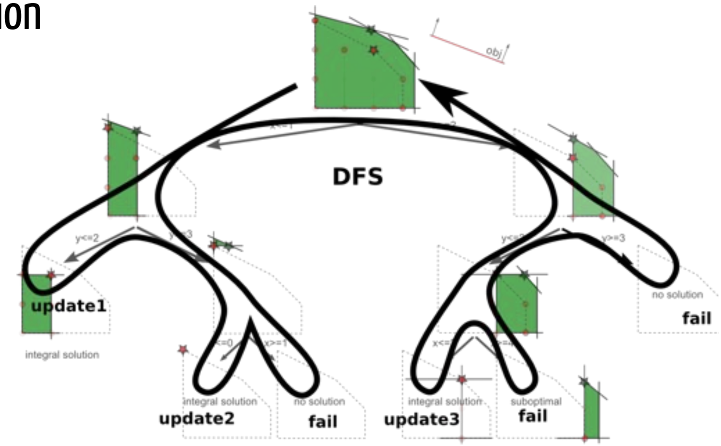
branching

node selection
which order to visit nodes ?

variable selection
how to separate nodes ?

constraint branching
alternative to variable branching

node selection

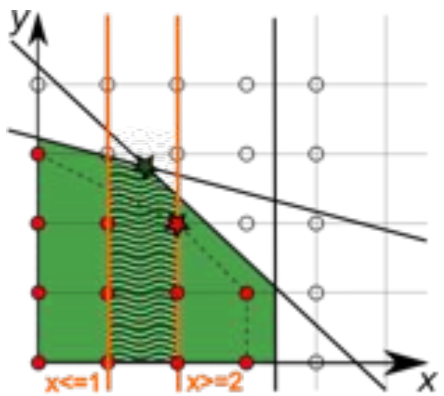


Best Bound First Search explore less nodes, manages larger trees

Depth First Search sensible to bad decisions at or near the root

DFS (up to n solutions) **+ BFS** (to prove optimality)

variable selection



most fractional easy to implement but not better than random

strong branching best improvement among all candidates (impractical)

pseudocost branching record previous branching success for each var (inaccurate at root)

reliability branching pseudocosts initialised with strong branching

onstraint branching

example: GUB dichotomy

- if (P) contains a GUB constraint $\sum_C x_i = 1, x \in \{0,1\}^n$
- choose $C' \subseteq C$ s.t. $0 < \sum_{C'} \bar{x}_i < 1$
- create two child nodes by setting either $\sum_{C'} x_i = 0$ or $\sum_{C'} x_i = 1$

- enforced by fixing the variable values
- leads to more balanced search trees

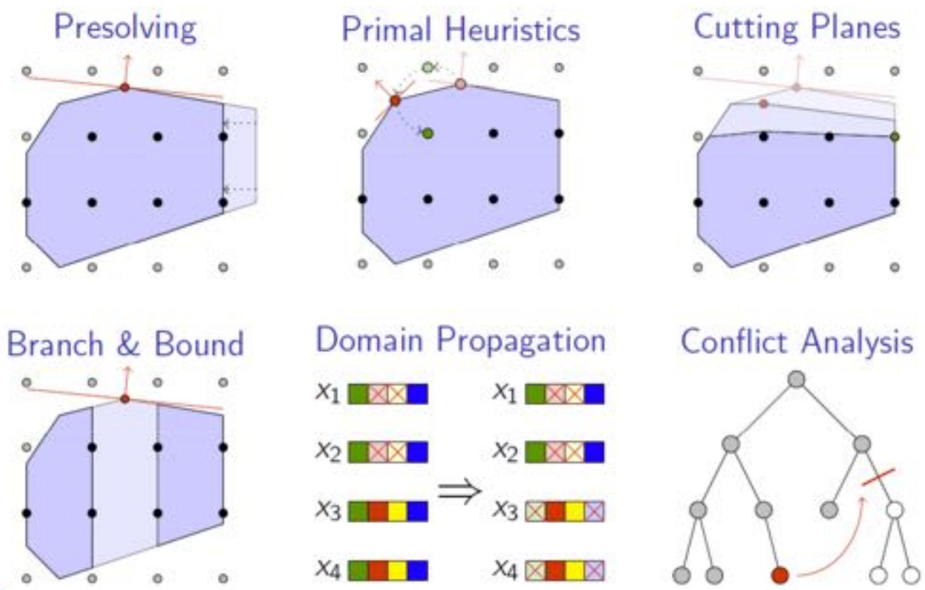
SOS1 branching in a facility location problem

choose a warehouse depending on its size/cost:

$$\begin{aligned} \text{COST} &= 100x_1 + 180x_2 + 320x_3 + 450x_4 + 600x_5 \\ \text{SIZE} &= 10x_1 + 20x_2 + 40x_3 + 60x_4 + 80x_5 \\ \text{(SOS1)} &: x_1 + x_2 + x_3 + x_4 + x_5 = 1 \end{aligned}$$

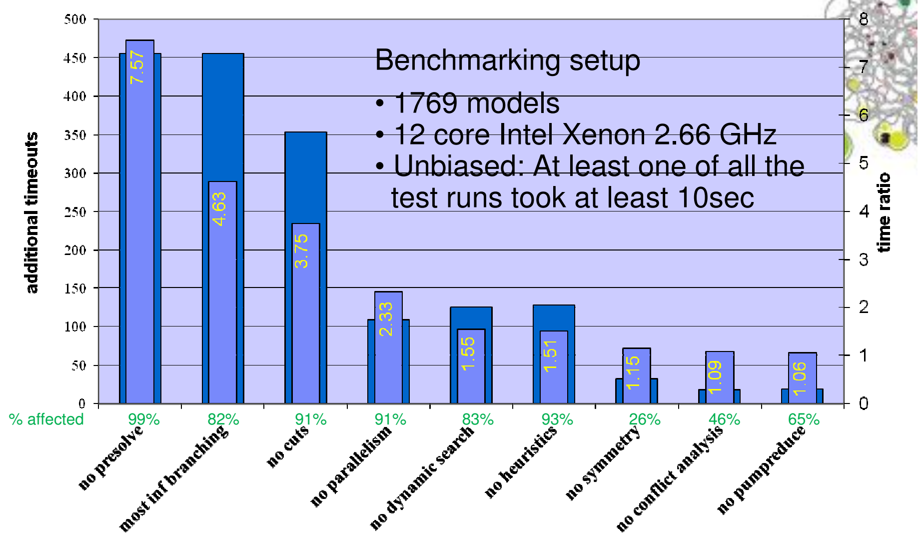
- let $\bar{x}_1 = 0.35$ and $\bar{x}_5 = 0.65$ in the LP solution then SIZE$= 55.5$
- choose $C' = \{1,2,3\}$ in order to model SIZE≤ 40 or SIZE≥ 60

Simplex
var branching
Preprocessing
Branch & Cut
Heuristics
Parallelism



Slide from Martin Grötschel Co@W Berlin 201

Component Impact CPLEX 12.5 Summary



CPLEX 11

.OG CPLEX generates its cuts in such a way that they
ubsequent cuts, ILOG CPLEX may add a constraint to

- Clique Cuts
- Cover Cuts
- Disjunctive Cuts
- Flow Cover Cuts
- Flow Path Cuts
- Gomory Fractional Cuts
- Generalized Upper Bound (GUB) Cover Cuts
- Implied Bound Cuts
- Mixed Integer Rounding (MIR) Cuts
- Adding Cuts and Re-Optimizing
- Counting Cuts
- Parameters Affecting Cuts

GUROBI 5.6

Parameter name	Purpose
Cuts	Global cut generation control
CliqueCuts	Clique cut generation
CoverCuts	Cover cut generation
FlowCoverCuts	Flow cover cut generation
FlowPathCuts	Flow path cut generation
GUBCoverCuts	GUB cover cut generation
ImpliedCuts	Implied bound cut generation
MIPSepCuts	MIP separation cut generation
MIRCuts	MIR cut generation
ModKCuts	Mod-k cut generation
NetworkCuts	Network cut generation
SubMIPCuts	Sub-MIP cut generation
ZeroHalfCuts	Zero-half cut generation
CutAggPasses	Constraint aggregation passes perform
CutPasses	Root cutting plane pass limit
GomoryPasses	Root Gomory cut pass limit

Preprocessing

reduce size

remove redundancies $x+y\leq 3$, *binaries*

substitute variables $x+y-z=0$

fix variables by duality $c_j\geq 0,\ A_j\geq 0 \Rightarrow x=x_{min}$

fix variables by probing $x=1$ *infeas* $\Rightarrow x=0$

strengthen LP relaxation

adjust bounds $2x+y\leq 1$, *binaries* $\Rightarrow x=0$

lift coefficients $2x-y\leq 1$, *binaries* $\Rightarrow x-y\leq 1$

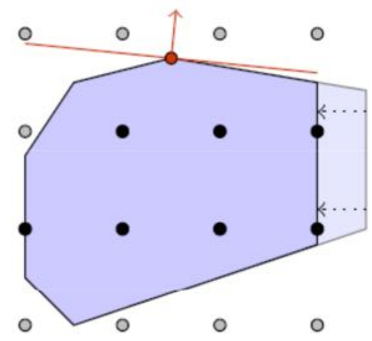
identify/exploit properties

detect implied integer $3x+y=7$, x *int* $\Rightarrow y$ *int*

build the conflict graph

detect disconnected components

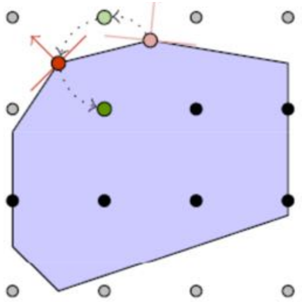
remove symmetries



MIPLIB markshare_5_0

```
[        :~/Documents/Code/gurobi]$ gurobi.sh mymip.py markshare_5_0.mps.gz
Changed value of parameter Presolve to 0
   Prev: -1  Min: -1  Max: 2  Default: -1
Optimize a model with 5 rows, 45 columns and 203 nonzeros
Found heuristic solution: objective 5335
Variable types: 5 continuous, 40 integer (40 binary)

Root relaxation: objective 0.000000e+00, 15 iterations, 0.00 seconds

    Nodes    |    Current Node    |     Objective Bounds      |     Work
 Expl Unexpl |  Obj  Depth IntInf | Incumbent    BestBd   Gap | It/Node Time

     0     0    0.00000    0    5 5335.00000    0.00000   100%     -    0s

*62706364 28044              38       1.0000000    0.00000   100%   2.1 1241s

Explored 233848403 nodes (460515864 simplex iterations) in 3883.5 seconds
Thread count was 4 (of 4 available processors)

Optimal solution found (tolerance 1.00e-04)
Best objective 1.000000000000e+00, best bound 1.000000000000e+00, gap 0.0%
Optimal objective: 1
```

```
[sofdem:~/Documents/Code/gurobi]$ gurobi.sh mymip.py markshare_5_0.mps.gz
Optimize a model with 5 rows, 45 columns and 203 nonzeros
Found heuristic solution: objective 5335
Presolve time: 0.00s
Presolved: 5 rows, 45 columns, 203 nonzeros
Variable types: 0 continuous, 45 integer (40 binary)

Root relaxation: objective 0.000000e+00, 15 iterations, 0.00 seconds

    Nodes    |    Current Node    |     Objective Bounds      |     Work
 Expl Unexpl |  Obj  Depth IntInf | Incumbent    BestBd   Gap | It/Node Time

     0     0    0.00000    0    5 5335.00000    0.00000   100%     -    0s
H    0     0                     320.0000000    0.00000   100%     -    0s
     0     0    0.00000    0    6  320.00000    0.00000   100%     -    0s
     0     0    0.00000    0    5  320.00000    0.00000   100%     -    0s
     0     0    0.00000    0    6  320.00000    0.00000   100%     -    0s
     0     0    0.00000    0    5  320.00000    0.00000   100%     -    0s
H    0     0                     239.0000000    0.00000   100%     -    0s
     0     0    0.00000    0    5  239.00000    0.00000   100%     -    0s
*   36     0               29      96.0000000    0.00000   100%   2.7    0s
*   99    32               34      58.0000000    0.00000   100%   2.1    0s
H  506   214                      53.0000000    0.00000   100%   1.9    0s
H30682   442                       1.0000000    1.00000  0.00%   2.1    0s

Cutting planes:
  Cover: 26

Explored 30682 nodes (65348 simplex iterations) in 0.70 seconds
Thread count was 4 (of 4 available processors)

Optimal solution found (tolerance 1.00e-04)
Best objective 1.000000000000e+00, best bound 1.000000000000e+00, gap 0.0%
Optimal objective: 1
```



rounding LP solution
diving at some nodes
local search in the incumbent neighbourhood

Primal Heuristics

accelerate the search a little
appeal to the practitioner a lot

limits

- highly heuristic (branching decisions, cut generation)
- floating-point errors and optimality tolerance (0.01%)
- generic features
- less effective on general integers (ex: scheduling)
- hard to model (and solve) non-linear structures
- NP-hard

how to tune modern solvers

play with Gurobi

```
Root relaxation: objective 0.000000e+00, 15 iterations, 0.00 seconds

    Nodes    |    Current Node    |     Objective Bounds      |     Work
 Expl Unexpl |  Obj  Depth IntInf | Incumbent    BestBd   Gap | It/Node Time

     0     0    0.00000    0    5 5335.00000    0.00000   100%     -    0s
H    0     0                     320.0000000    0.00000   100%     -    0s
     0     0    0.00000    0    6  320.00000    0.00000   100%     -    0s
     0     0    0.00000    0    5  320.00000    0.00000   100%     -    0s
     0     0    0.00000    0    6  320.00000    0.00000   100%     -    0s
     0     0    0.00000    0    5  320.00000    0.00000   100%     -    0s
H    0     0                     239.0000000    0.00000   100%     -    0s
     0     0    0.00000    0    5  239.00000    0.00000   100%     -    0s
*   36     0               29      96.0000000    0.00000   100%   2.7    0s
*   99    32               34      58.0000000    0.00000   100%   2.1    0s
H  506   214                      53.0000000    0.00000   100%   1.9    0s
H30682   442                       1.0000000    1.00000  0.00%   2.1    0s
```

use as a heuristic

set a time limit
`MIPFocus=1`
`ImproveStartGap=0.1`

```
Root relaxation: objective 0.000000e+00, 15 iterations, 0.00 seconds

    Nodes    |    Current Node    |     Objective Bounds      |     Work
 Expl Unexpl |  Obj  Depth IntInf | Incumbent    BestBd   Gap | It/Node Time

     0     0    0.00000    0    5 5335.00000    0.00000   100%     -    0s
H    0     0                    320.0000000    0.00000   100%     -    0s
     0     0    0.00000    0    6  320.00000    0.00000   100%     -    0s
     0     0    0.00000    0    5  320.00000    0.00000   100%     -    0s
     0     0    0.00000    0    6  320.00000    0.00000   100%     -    0s
     0     0    0.00000    0    5  320.00000    0.00000   100%     -    0s
H    0     0                    239.0000000    0.00000   100%     -    0s
     0     0    0.00000    0    5  239.00000    0.00000   100%     -    0s
*   36     0              29     96.0000000    0.00000   100%   2.7    0s
*   99    32              34     58.0000000    0.00000   100%   2.1    0s
H  506   214                     53.0000000    0.00000   100%   1.9    0s
H30682   442                      1.0000000    1.00000  0.00%   2.1    0s
```

change the LP solver

if nbIteration(node) ≥ nbIteration(root)/2

`NodeMethod=2`


```
Root relaxation: objective 0.000000e+00, 15 iterations, 0.00 seconds

    Nodes    |    Current Node    |     Objective Bounds      |     Work
 Expl Unexpl |  Obj  Depth IntInf | Incumbent    BestBd   Gap | It/Node Time

     0     0    0.00000    0    5 5335.00000    0.00000   100%     -    0s
H    0     0                    320.0000000    0.00000   100%     -    0s
     0     0    0.00000    0    6  320.00000    0.00000   100%     -    0s
     0     0    0.00000    0    5  320.00000    0.00000   100%     -    0s
     0     0    0.00000    0    6  320.00000    0.00000   100%     -    0s
     0     0    0.00000    0    5  320.00000    0.00000   100%     -    0s
H    0     0                    239.0000000    0.00000   100%     -    0s
     0     0    0.00000    0    5  239.00000    0.00000   100%     -    0s
*   36     0              29     96.0000000    0.00000   100%   2.7    0s
*   99    32              34     58.0000000    0.00000   100%   2.1    0s
H  506   214                     53.0000000    0.00000   100%   1.9    0s
H30682   442                      1.0000000    1.00000  0.00%   2.1    0s
```


supply a feasible solution

if built-in heuristics fail

```
PumpPasses,MinRelNodes,ZeroObjNodes
model.read('initSol.mst')
model.cbSetSolution(vars, newSol)
```


```
Root relaxation: objective 0.000000e+00, 15 iterations, 0.00 seconds

    Nodes    |    Current Node    |     Objective Bounds      |     Work
 Expl Unexpl |  Obj  Depth IntInf | Incumbent    BestBd   Gap | It/Node Time

     0     0    0.00000    0    5 5335.00000    0.00000   100%     -    0s
H    0     0                    320.0000000    0.00000   100%     -    0s
     0     0    0.00000    0    6  320.00000    0.00000   100%     -    0s
     0     0    0.00000    0    5  320.00000    0.00000   100%     -    0s
     0     0    0.00000    0    6  320.00000    0.00000   100%     -    0s
     0     0    0.00000    0    5  320.00000    0.00000   100%     -    0s
H    0     0                    239.0000000    0.00000   100%     -    0s
     0     0    0.00000    0    5  239.00000    0.00000   100%     -    0s
*   36     0              29     96.0000000    0.00000   100%   2.7    0s
*   99    32              34     58.0000000    0.00000   100%   2.1    0s
H  506   214                     53.0000000    0.00000   100%   1.9    0s
H30682   442                      1.0000000    1.00000  0.00%   2.1    0s
```


tighten the model

if the bound stagnates

```
Cuts=3
Presolve=3
model.cbCut(lhs, sense, rhs)
```

http://www.gurobi.com/

/documentation/6.0/reference-manual/mip_models

/resources/seminars-and-videos

/documentation/6.0/refman/parameter_guidelines.html you know your problem better than your solver does

improve your model

14 hours

$$\min \sum_{j=1}^{n} c_j x_j + \sum_{j=1}^{n}\sum_{i=1}^{m} d_{ij} y_{ij}$$

$$\text{s.t.} \sum_{j=1}^{n} y_{ij} = 1 \qquad i = 1..m$$

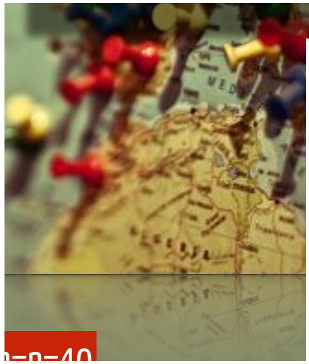
$$\boxed{\sum_{i=1}^{m} y_{ij} \le m x_j \qquad j = 1..n}$$

$$x_j \in \{0,1\} \qquad j = 1..n$$

$$y_{ij} \in \{0,1\} \qquad j = 1..n,\ i = 1..m$$

apacitated
ty Location
roblem

Input n facility locations, m



2 seconds

$$\min \sum_{j=1}^{n} c_j x_j + \sum_{j=1}^{n}\sum_{i=1}^{m} d_{ij} y_{ij}$$

$$\text{s.t.} \sum_{j=1}^{n} y_{ij} = 1 \qquad i = 1..m$$

$$y_{ij} \le x_j \qquad j = 1..n,\ i = 1..m$$

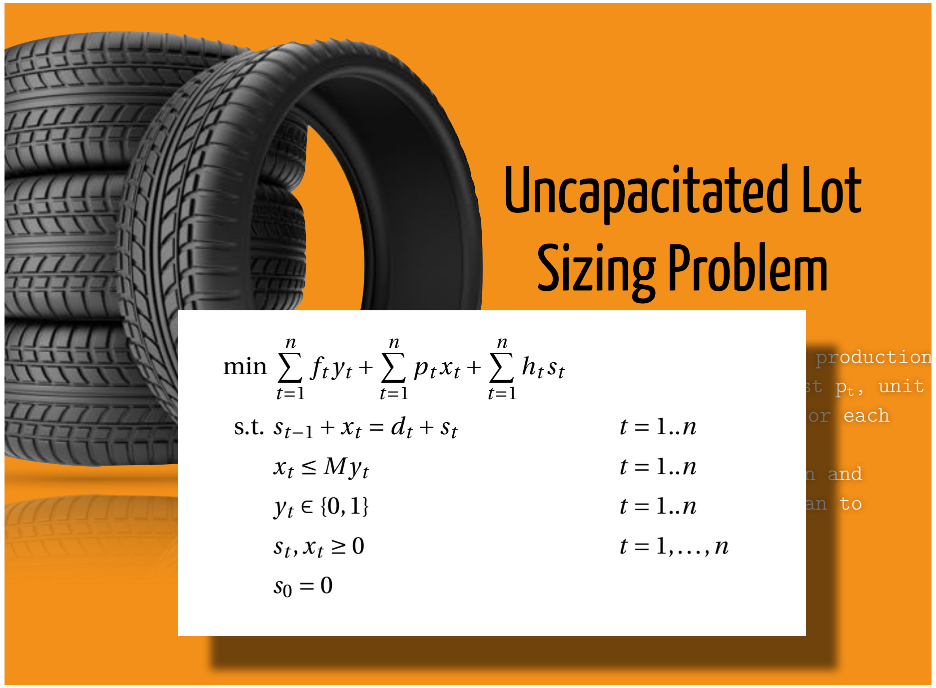
$$x_j \in \{0,1\} \qquad j = 1..n$$

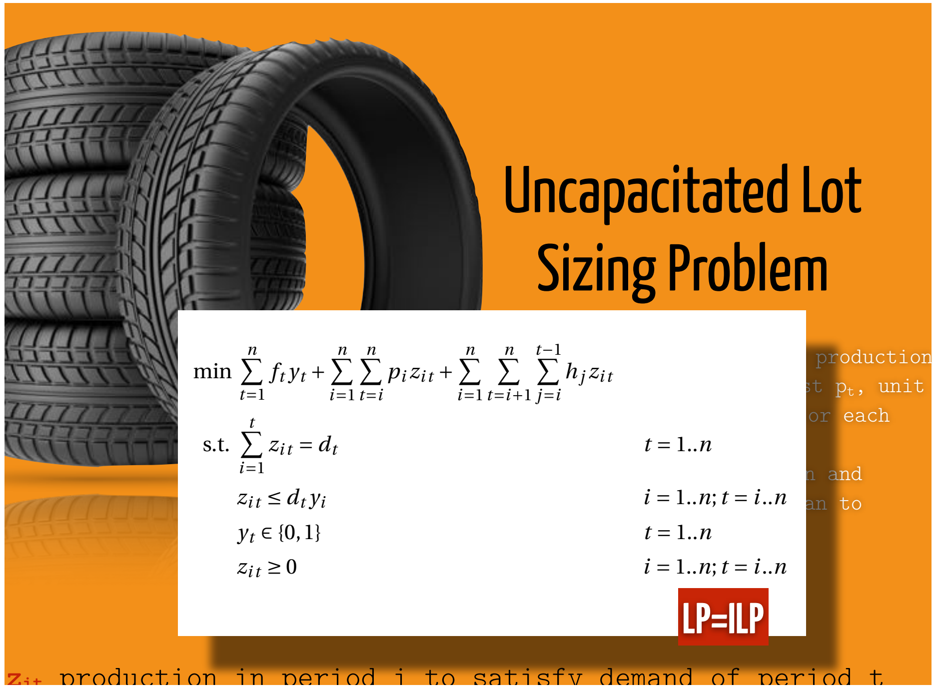
$$y_{ij} \in \{0,1\} \qquad j = 1..n,\ i = 1..m$$

Uncapacitated Lot Sizing Problem

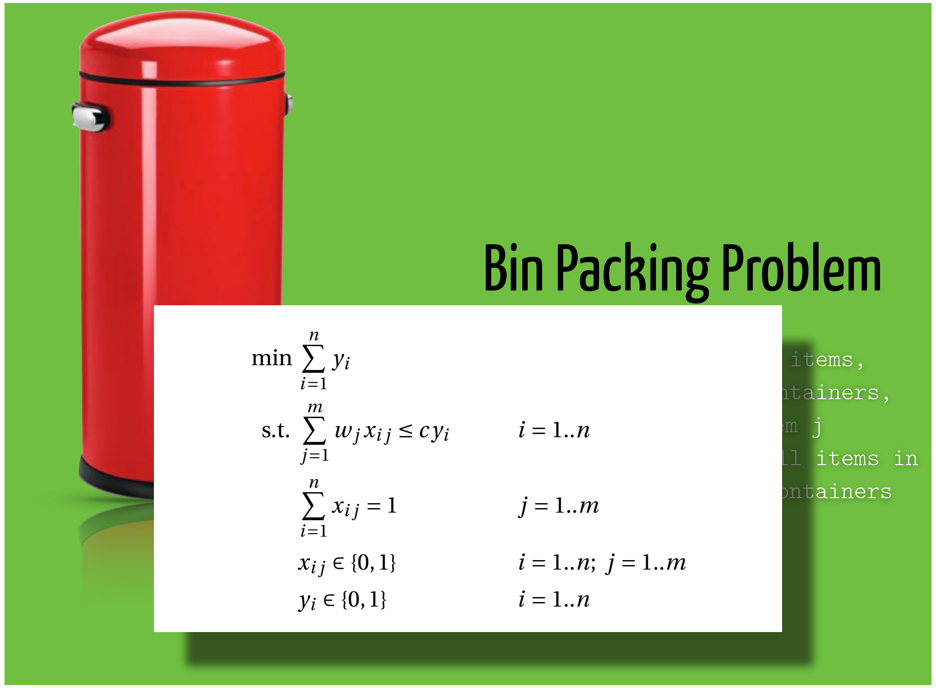
Input n time periods, fixed production cost f_j, unit production cost p_t, unit storage cost h_t, demand d_t for each period t

Output a mimimum (production and storage) cost production plan to satisfy the demand


Uncapacitated Lot Sizing Problem
$\min \sum_{t=1}^{n} f_t y_t + \sum_{t=1}^{n} p_t x_t + \sum_{t=1}^{n} h_t s_t$
s.t. $s_{t-1} + x_t = d_t + s_t$ $t = 1..n$
$x_t \leq M y_t$ $t = 1..n$
$y_t \in \{0,1\}$ $t = 1..n$
$s_t, x_t \geq 0$ $t = 1,\dots,n$
$s_0 = 0$


Uncapacitated Lot Sizing Problem
$\min \sum_{t=1}^{n} f_t y_t + \sum_{i=1}^{n}\sum_{t=i}^{n} p_i z_{it} + \sum_{i=1}^{n}\sum_{t=i+1}^{n}\sum_{j=i}^{t-1} h_j z_{it}$
s.t. $\sum_{i=1}^{t} z_{it} = d_t$ $t = 1..n$
$z_{it} \leq d_t y_i$ $i = 1..n; t = i..n$
$y_t \in \{0,1\}$ $t = 1..n$
$z_{it} \geq 0$ $i = 1..n; t = i..n$
LP=ILP
z_{it} production in period i to satisfy demand of period t


Bin Packing Problem
Input n containers, m items,
capacity c for all containers,
weight w_j for each item j
Output a packing of all items in
a mimimum number of containers


Bin Packing Problem
$\min \sum_{i=1}^{n} y_i$
s.t. $\sum_{j=1}^{m} w_j x_{ij} \leq c y_i$ $i = 1..n$
$\sum_{i=1}^{n} x_{ij} = 1$ $j = 1..m$
$x_{ij} \in \{0,1\}$ $i = 1..n;\ j = 1..m$
$y_i \in \{0,1\}$ $i = 1..n$

Bin Packing Problem

$$
\begin{aligned}
\min \ & \sum_{s \in \mathcal{S}} x_s \\
\text{s.t. } & \sum_{s \in \mathcal{S}} a_{js} x_s = 1 & j = 1..n \\
& x_s \in \{0,1\} & s \in \mathcal{S}
\end{aligned}
$$

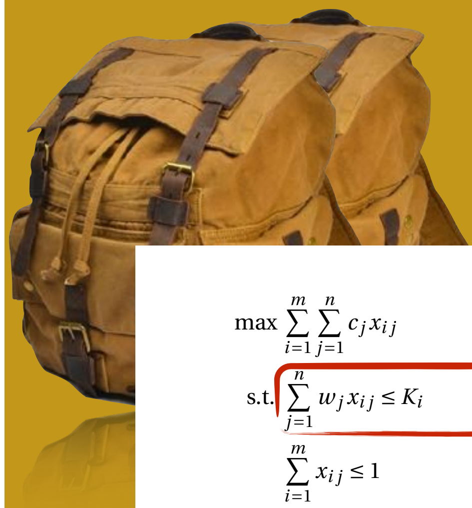
delayed column generation

$\mathcal{S}$ all the possible arrangements of items in a bin

Multi 0-1 Knapsack Problem

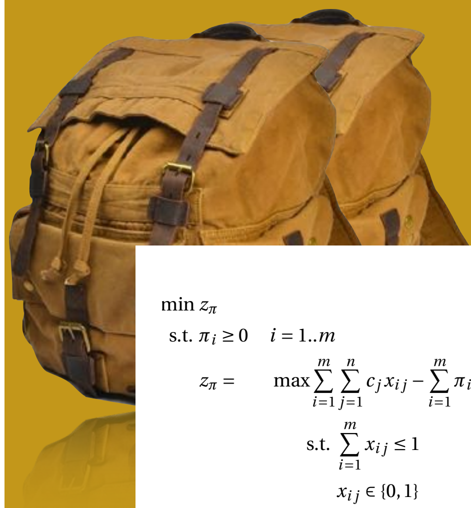
Input n items, m bins, value c_j and weight w_j for each item j, capacity K_i for each bin i.

Output a maximum value subset of items packed in the bins.



Multi 0-1 Knapsack Problem

$$
\begin{aligned}
\max \ & \sum_{i=1}^{m} \sum_{j=1}^{n} c_j x_{ij} \\
\text{s.t. } & \sum_{j=1}^{n} w_j x_{ij} \le K_i & i = 1..m \\
& \sum_{i=1}^{m} x_{ij} \le 1 & j = 1..n \\
& x_{ij} \in \{0,1\} & j = 1..n, i = 1..m
\end{aligned}
$$



Multi 0-1 Knapsack Problem

$$
\begin{aligned}
& \min z_\pi \\
& \text{s.t. } \pi_i \ge 0 \quad i = 1..m \\
& z_\pi = \max \sum_{i=1}^{m} \sum_{j=1}^{n} c_j x_{ij} - \sum_{i=1}^{m} \pi_i \Big(\sum_{j=1}^{n} w_j x_{ij} - K_i \Big) \\
& \qquad \text{s.t. } \sum_{i=1}^{m} x_{ij} \le 1 \quad j = 1..n \\
& \qquad x_{ij} \in \{0,1\} \quad j = 1..n, i = 1..m
\end{aligned}
$$

lagrangian relaxation

performance

maintainability

MIP advantages

transparency

extensibility

constraint programming

non-linear programming

but if all else fails

SAT

metaheuristics

chterberg T., Berthold T., Hendel G. (2012) Rounding and propagation heuristics for MIP. In OR Proceedings 2011, 71-76

xby R. (2012) A brief history of LP and MIP computation. Documenta Mathematica, 107-121.

ornuéjols G. (2008) Valid inequalities for MILP. Mathematical Programming, 112(1), 3-44.

lotz E., Newman A. (2013) Practical guidelines for solving difficult MILP. Surveys in ORMS, 18(1), 18-32.

nderoth J., Ralphs T. (2005) Noncommercial software for MILP. IP: theory and practice, 3, 253-303.

nderoth J., Lodi A. (2010) MILP software. Wiley encyclopedia of ORMS.

nderoth J., Savelsbergh M. (1999) A computational study of search strategies for MIP. INFORMS JoC, 11(2), 173-187.

odi A. (2010) MIP computation. In 50 Years of IP 1958-2008, 619-645.

ewman A., Weiss M. (2013) A survey of linear and mixed-integer optimization tutorials. INFORMS ToE, 14(1) 26-38.

avelsbergh M. (1994) Preprocessing and probing techniques for MIP. ORSA Journal on Computing, 6(4), 445-454.

anderbeck F., Wolsey L. (2010) Reformulation and decomposition of IP. In 50 Years of IP 1958-2008, 431-502.

Volsey L. (1998) Integer Programming. Wiley