Assignment 2

February 4, 2023

1 Problem 1: Linear Regression from Scratch (30 points)

[1]:
```
# import the necessary packages
import numpy as np
from matplotlib import pyplot as plt
np.random.seed(100)
```

Let's generate some data points first, by the equation $y = x - 3$.

[2]:
```
x = np.random.randint(100, size=100)/30 - 2
X = x.reshape(-1, 1)

y = x + -3 + 0.3*np.random.randn(100)
```

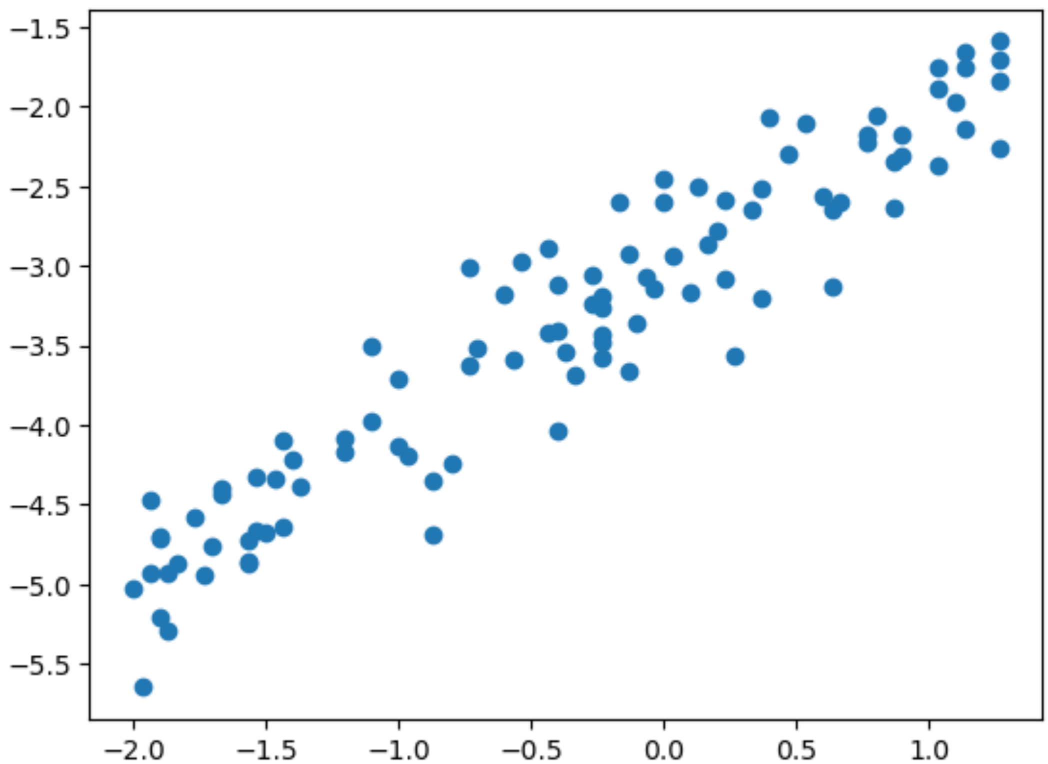
[3]:
```
x.shape, X.shape, y.shape
```

[3]:
```
((100,), (100, 1), (100,))
```

Let's then visualize the data points we just created.

[4]:
```
plt.scatter(X, y)
```

[4]:
```
<matplotlib.collections.PathCollection at 0x7fa9a2443d30>
```



1.1 1.1 Gradient of vanilla linear regression model (5 points)

In the lecture, we learn that the cost function of a linear regression model can be expressed as **Equation 1**:

$$J(\theta) = \frac{1}{2m} \sum_{i}^{m} \left(h_\theta \left(x^{(i)} \right) - y^{(i)} \right)^2$$

The gradient of it can be written as **Equation 2**:

$$\frac{\partial J(\theta)}{\partial \theta} = \frac{1}{m} [\sum_{i}^{m} \left(h_\theta \left(x^{(i)} \right) - y^{(i)} \right) x_j^{(i)}]$$

1.2 1.2 Gradient of vanilla regularized regression model (5 points)

After adding the L2 regularization term, the linear regression model can be expressed as **Equation 3**:

$$J(\theta) = \frac{1}{2m} \sum_{i}^{m} \left(h_\theta \left(x^{(i)} \right) - y^{(i)} \right)^2 + \frac{\lambda}{2m} \sum_{j}^{n} (\theta_j)^2$$

The gredient of it can be written as **Equation 4**:

$$\frac{\partial J(\theta)}{\partial \theta} = \frac{1}{m}[\sum_{i}^{m}\left(h_\theta\left(x^{(i)}\right) - y^{(i)}\right)x_j^{(i)} + \lambda\theta_j]$$

Parameter update is as follows:

$$\longrightarrow$$

$$\theta_j := \theta_j - \alpha(\frac{1}{m}[\sum_{i}^{m}\left(h_\theta\left(x^{(i)}\right) - y^{(i)}\right)x_j^{(i)} + \lambda\theta_j])$$

1.3 1.3 Implement the cost function of a regularized regression model (5 points)

Please implement the cost function of a regularized regression model according to the above equations.

[5]:
```
def objective_function(m, y_residual, l2_regularization=False,␣
↪lambda_value=None, W=None):
        if l2_regularization:
                (y_residual.dot(y_residual.T) / (2*m)) + (lambda_value/(2*m) *␣
↪W.dot(W.T))
        return y_residual.dot(y_residual.T) / (2*m)
```

1.4 1.4 Implement the gradient of the cost function of a regularized regression model (5 points)

Please implement the gradient of the cost function of a regularized regression model according to the above equations.

[6]:
```
def calculate_gradient(m, y_residual, X, l2_regularization=False,␣
↪lambda_value=None, W=None):
       if l2_regularization:
              return (1 / m) * (X.T.dot(y_residual) + (lambda_value * W))
       return (1 / m) * (X.T.dot(y_residual))
```

[7]:
```
def regularized_linear_regression(X, y, alpha=0.01, lambda_value=1, epochs=30):
    """
    :param x: feature matrix
    :param y: target vector
    :param alpha: learning rate (default:0.01)
    :param lambda_value: lambda (default:1)
    :param epochs: maximum number of iterations of the
           linear regression algorithm for a single run (default=30)
    :return: weights, list of the cost function changing overtime
    """
```

```
    m = np.shape(X)[0]  # total number of samples
    n = np.shape(X)[1]  # total number of features

    X = np.concatenate((np.ones((m, 1)), X), axis=1)
    W = np.random.randn(n + 1, )

    # stores the updates on the cost function (loss function)
    cost_history_list = []

    # iterate until the maximum number of epochs
    for current_iteration in np.arange(epochs):  # begin the process

        # compute the dot product between our feature 'X' and weight 'W'
        y_estimated = X.dot(W)

        # calculate the difference between the actual and predicted value
        error = y_estimated - y

##############################################################################################
 ↪
######################################## Begin of Question 1.3␣
 ↪################################################
##############################################################################################

        ##### Please write down your code here:####

        # calculate the cost (MSE) (Equation 1)
        cost_without_regularization = objective_function(m=m, y_residual=error,␣
 ↪l2_regularization=False)

        ##### Please write down your code here:####

        # regularization term
        reg_term = (lambda_value / (2 * m)) * W.dot(W.T)

        # calculate the cost (MSE) + regularization term (Equation 3)
        cost_with_regularization = cost_without_regularization + reg_term

##############################################################################################
 ↪
######################################## End of Question 1.3␣
 ↪################################################
##############################################################################################
 ↪
```

```
####################################################################################
↪
######################################## Begin of Question 1.4␣
↪################################################
####################################################################################
↪

        ##### Please write down your code here:####

        # calculate the gradient of the cost function with regularization term␣
↪(Equation )
        gradient = calculate_gradient(m=m, y_residual=error, X=X,␣
↪l2_regularization=True, lambda_value=lambda_value, W=W)

        # Now we have to update our weights
        W = W - alpha * gradient

####################################################################################
↪
######################################## End of Question 1.4␣
↪##################################################
####################################################################################
↪


        # keep track the cost as it changes in each iteration
        cost_history_list.append(cost_with_regularization)

    # Let's print out the cost
    print(f"{lambda_value=}")
    print(f"Cost with regularization: {cost_with_regularization}")
    print(f"Mean square error: {cost_without_regularization}")

    return W, cost_history_list
```

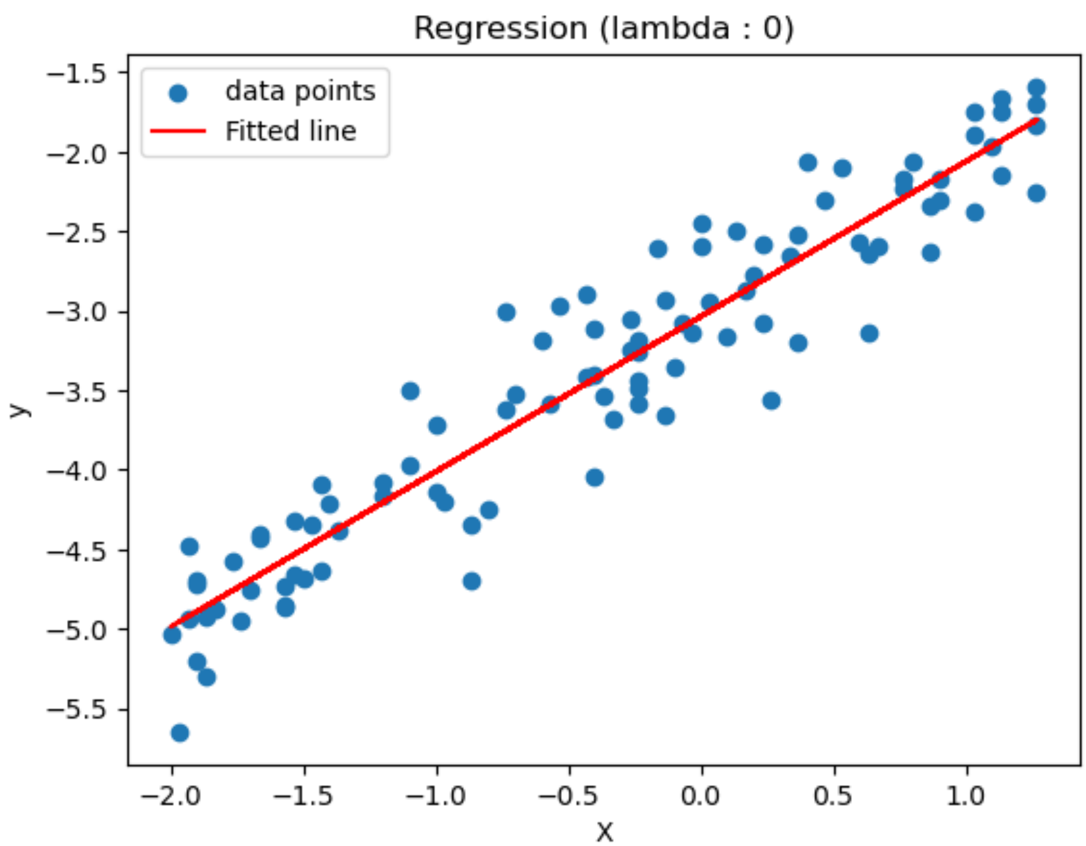
Run the following code to train your model.

Hint: If you have the correct code written above, the cost should be 0.5181222986588751 when $\lambda = 10$.

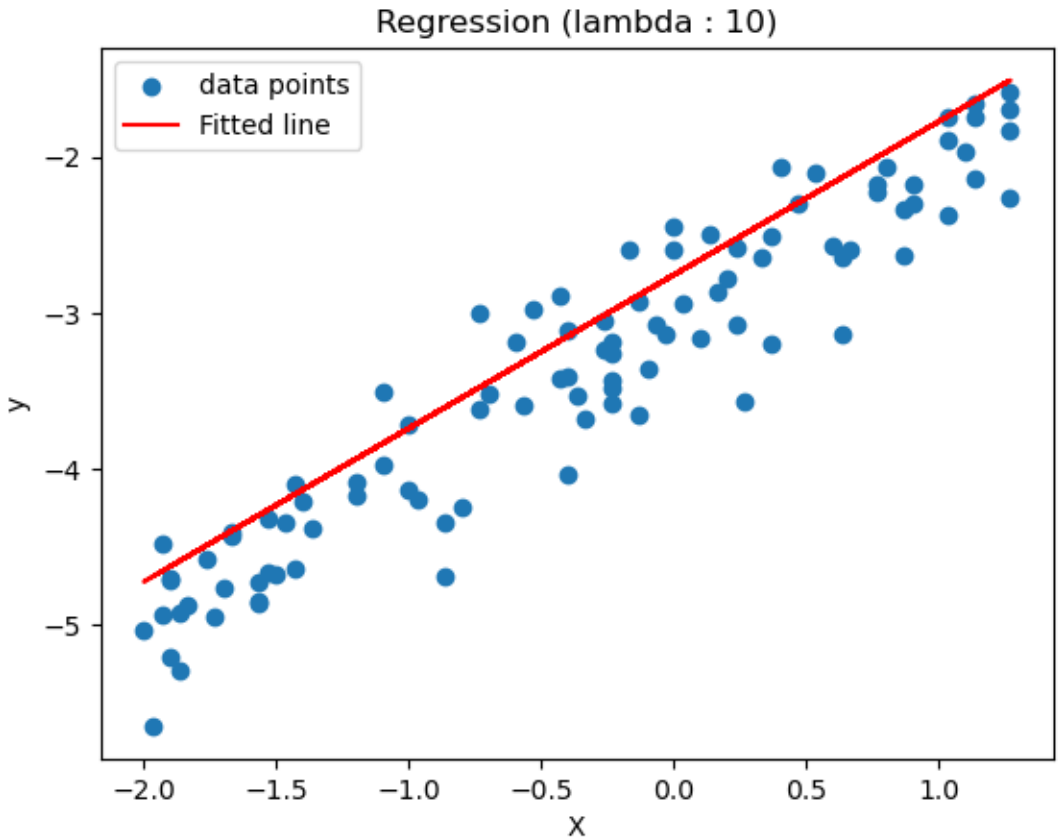
[8]:
```
lambda_list = [0, 10, 100, 1000, 10000]
for lambda_ in lambda_list:
    # calls regression function with different values of lambda
    weight, _ = regularized_linear_regression(X, y, alpha=0.01,
                                 lambda_value=lambda_, epochs=1000)

    fitted_line = np.dot(X, weight[1]) + weight[0]
```

```python
    plt.scatter(X, y, label='data points')
    plt.plot(X, fitted_line, color='r', label='Fitted line')
    plt.xlabel("X")
    plt.ylabel("y")
    plt.title(f"Regression (lambda : {lambda_})")
    plt.legend()
    plt.show()
```

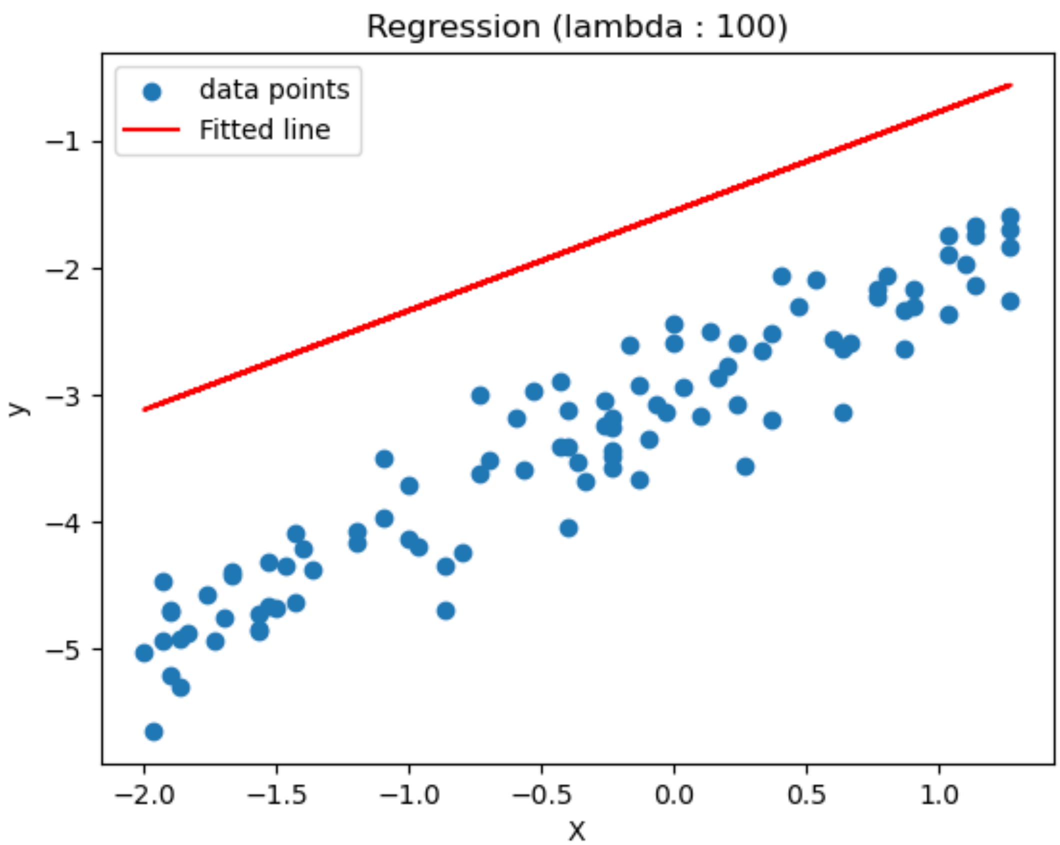
```
lambda_value=0
Cost with regularization: 0.05165888565058273
Mean square error: 0.05165888565058273
```



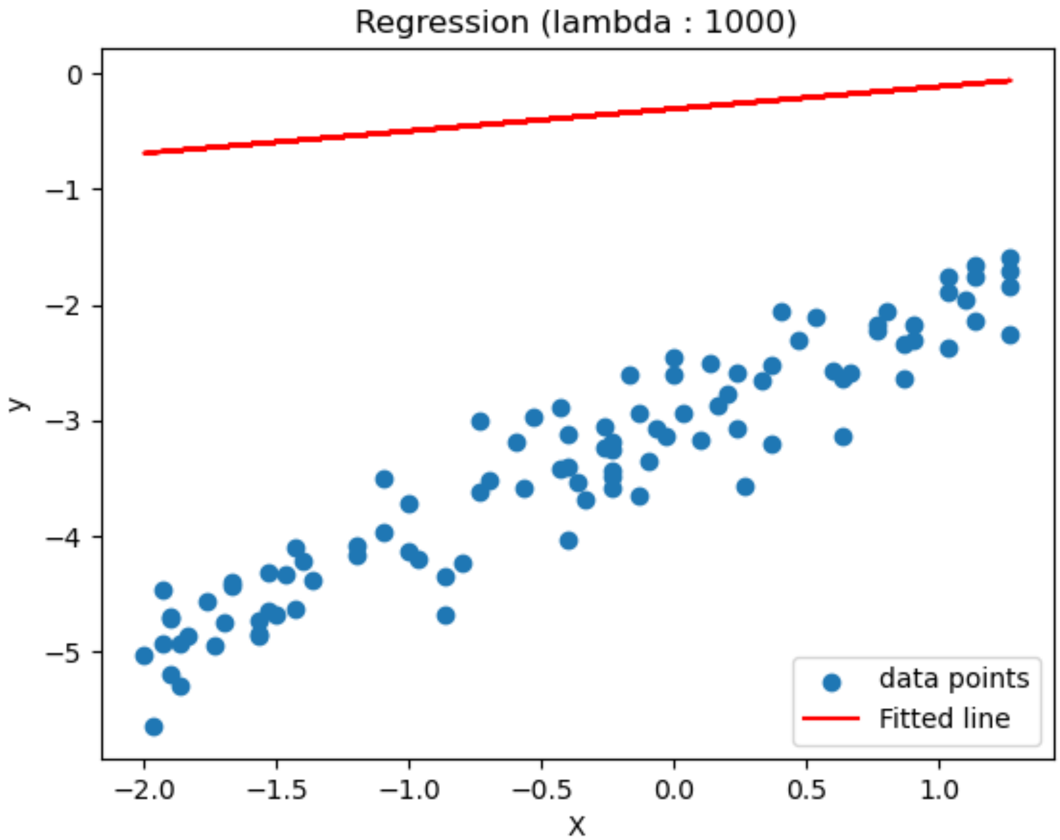

```
lambda_value=10
Cost with regularization: 0.5181225049184746
Mean square error: 0.08982014821513136
```



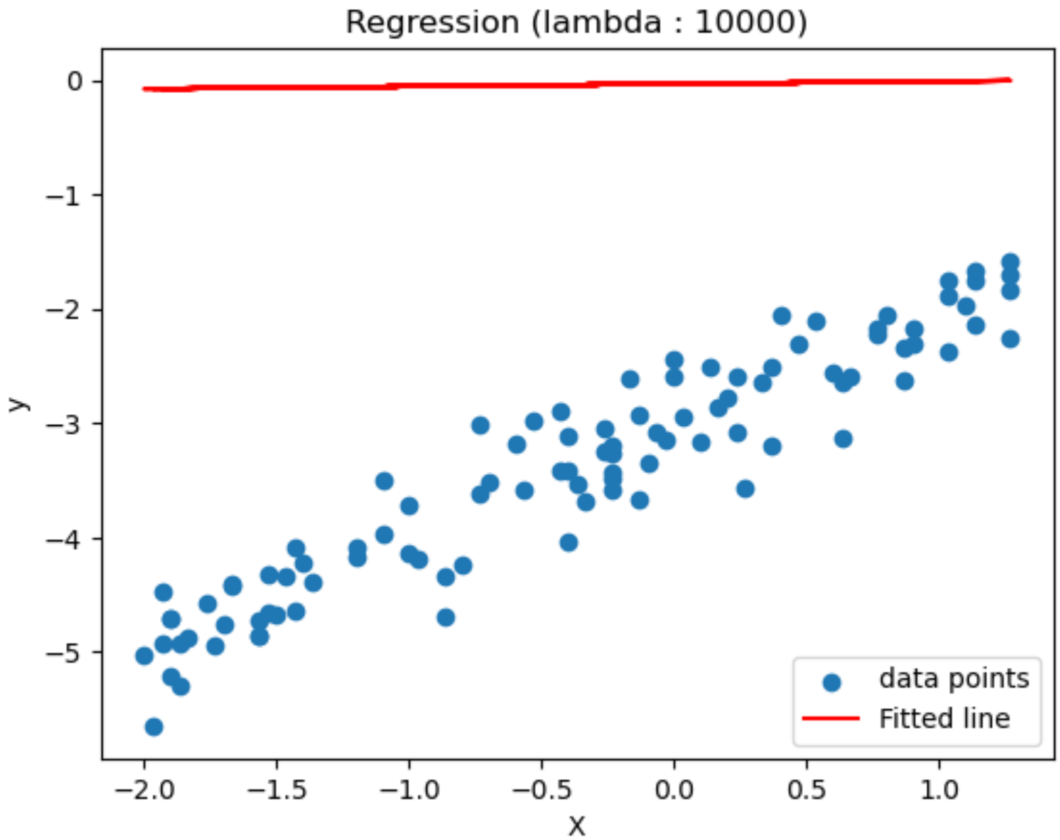

```
lambda_value=100
Cost with regularization: 2.7931724887400255
Mean square error: 1.2785107029715972
```




```
lambda_value=1000
Cost with regularization: 5.591464362606628
Mean square error: 4.946888025066497
```




```
lambda_value=10000
Cost with regularization: 6.242695626933973
Mean square error: 6.161442583355813
```



1.5 1.5 Analyze your results (10 points)

According to the above figures, what's the best choice of λ?

Why the regressed line turns to be flat as we increase λ?

Your answer:

1.5a - The best λ value is 0. This is not too surprising because regularization is intended to smooth out a fit best line. Since this data is very linear, a simple regression model with no curves or sharp edges can fit the data well, and there is no need for regularization. Actually, regularization made the model perform significantly worse.

1.5b - In the end of each epoch, the weights get updated as such:

$$\theta_j := \theta_j - \alpha(\frac{1}{m}[\sum_{i}^{m}\left(h_\theta\left(x^{(i)}\right) - y^{(i)}\right)x_j^{(i)} + \lambda\theta_j])$$

- When λ becomes very high, the regularization term completely dominates the weight update equation. And so all of the θ_1 parameters are very high values that are about the same since the non-regularization portion of the update equation become insiginficant. The nearly identical θ_1 values result in a flat-like slope.

2 Problem 2: Getting familiar with PyTorch (30 points)

[9]:
```
import mltools as ml
import torch
import numpy as np
import matplotlib.pyplot as plt
```

2.0.1 2.1

Load the “ data/curve80.txt ” data set, and split it into 75% / 25% training/test. We will use degree=5 for all the polynomial features.

[10]:
```
data = np.genfromtxt("data/curve80.txt")
X = data[:,0]
X = np.atleast_2d(X).T # code expects shape (M,N) so make sure i t s␣
↪2-dimensional
Y = data[:,1] # doesnt matter for Y
Xtr,Xte,Ytr,Yte = ml.splitData(X,Y,0.75) # split data set 75/25

degree = 5
XtrP = ml.transforms.fpoly(Xtr, degree=degree, bias=False)
XtrP,params = ml.transforms.rescale(XtrP)
```

Transform numpy arrays to tensor.

[11]:
```
XtrP_tensor = torch.from_numpy(XtrP)
Ytr_tensor = torch.from_numpy(Ytr.reshape(-1,1))

XtrP_tensor = XtrP_tensor.float()
Ytr_tensor = Ytr_tensor.float()
```

Make sure the XtrP_tensor has the shape of (60, 5) while Ytr_tensor has the shape of (60, 1).

[12]:
```
assert XtrP_tensor.shape == torch.Size([60, 5])
assert Ytr_tensor.shape == torch.Size([60, 1])
XtrP_tensor.shape, Ytr_tensor.shape
```

[12]:
```
(torch.Size([60, 5]), torch.Size([60, 1]))
```

2.0.2 2.2

Initialize our linear regressor.

[13]:
```
linear_regressor = torch.nn.Linear(in_features=XtrP_tensor.shape[1],␣
↪out_features=Ytr_tensor.shape[1])
```

2.0.3 2.3

Set up the criterion and optimizer.

[14]:
```
criterion = torch.nn.MSELoss()
optimizer = torch.optim.SGD(linear_regressor.parameters(), lr=0.1)
epochs = 100000
```

2.0.4 2.4

Training the regressor using gradient descent.

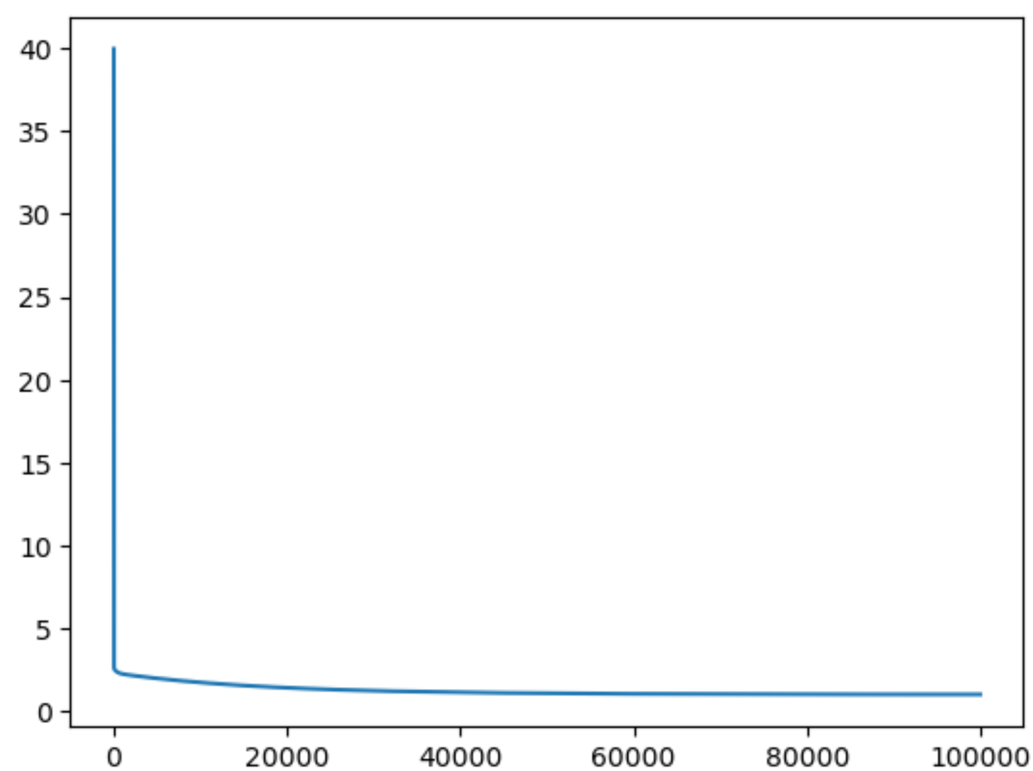
[15]:
```
loss_record = []
for _ in range(epochs):
        optimizer.zero_grad() # set gradient to zero
        pred_y = linear_regressor(XtrP_tensor)
        loss = criterion(pred_y, Ytr_tensor)  # calculate loss function
        loss.backward() # backpropagate gradient
        loss_record.append(loss.item())
        optimizer.step()  # update the parameters in the linear regressor
```

2.0.5 2.5

Plot the loss v.s. epochs. Show the plot here.

[16]:
```
plt.plot(range(epochs), (loss_record))
```

[16]:
```
[<matplotlib.lines.Line2D at 0x7fa9a65ac3a0>]
```

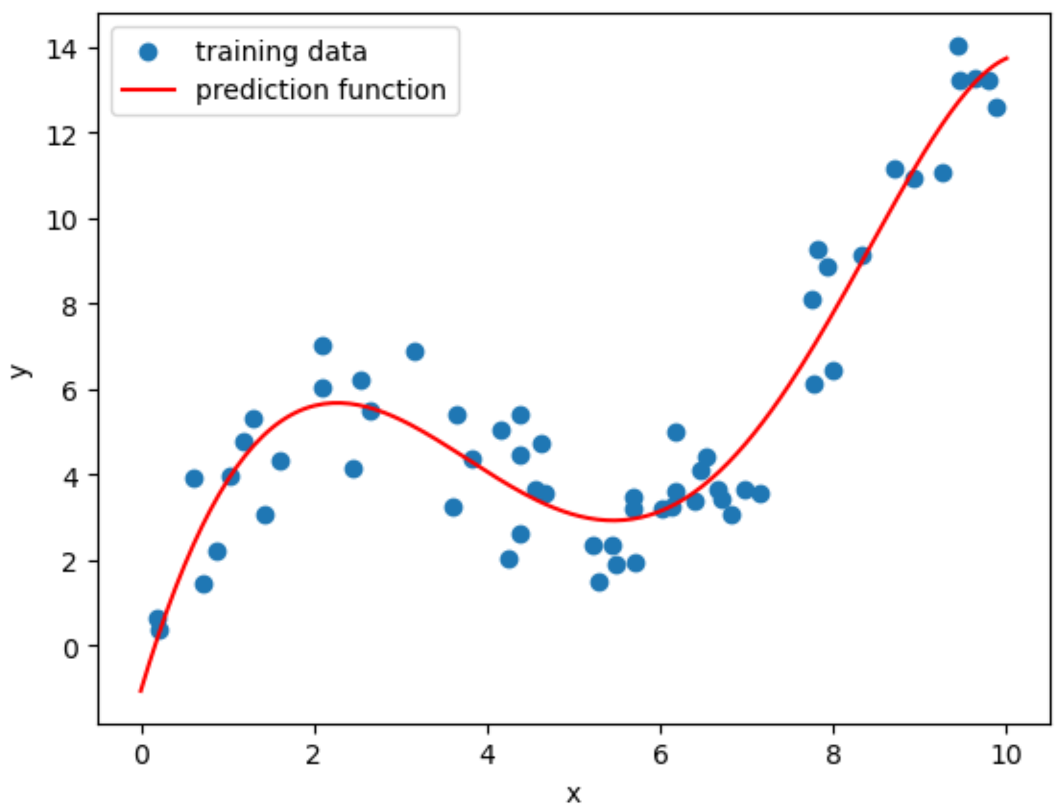


2.0.6 2.6

Visualize the trained linear regressor.

[17]:
```
xs = np.linspace(0,10,200)
xs = xs[:,np.newaxis]
xsP, _ = ml.transforms.rescale(ml.transforms.
↪fpoly(xs,degree=degree,bias=False), params)
xsP_tensor = torch.from_numpy(xsP).float()
ys = linear_regressor(xsP_tensor)

plt.scatter(Xtr,Ytr,label="training data")
plt.plot(xs,ys.detach().numpy(),label="prediction function",color ='red')
plt.xlabel('x')
plt.ylabel('y')
plt.legend()
```

[17]:
```
<matplotlib.legend.Legend at 0x7fa9a66f68e0>
```



Statement of Collaboration

- Sheldon Gu
    - Discussed how backpropagation methods works in PyTorch.
- Lanny Wang
    - Discussed about the math of how a gradient is derived in general for multivariable functions.