3

The Z-Transform and Its Application to the Analysis of LTI Systems

Transform techniques are an important tool in the analysis of signals and linear time-invariant (LTI) systems. In this chapter we introduce the z-transform, develop its properties, and demonstrate its importance in the analysis and characterization of linear time-invariant systems.

The z-transform plays the same role in the analysis of discrete-time signals and LTI systems as the Laplace transform does in the analysis of continuous-time signals and LTI systems. For example, we shall see that in the z-domain (complex z-plane) the convolution of two time-domain signals is equivalent to multiplication of their corresponding z-transforms. This property greatly simplifies the analysis of the response of an LTI system to various signals. In addition, the z-transform provides us with a means of characterizing an LTI system, and its response to various signals, by its pole–zero locations.

We begin this chapter by defining the z-transform. Its important properties are presented in Section 3.2. In Section 3.3 the transform is used to characterize signals in terms of their pole–zero patterns. Section 3.4 describes methods for inverting the z-transform of a signal so as to obtain the time-domain representation of the signal. The one-sided z-transform is treated in Section 3.5 and used to solve linear difference equations with nonzero initial conditions. The chapter concludes with a discussion on the use of the z-transform in the analysis of LTI systems.

3.1 THE Z-TRANSFORM

In this section we introduce the z-transform of a discrete-time signal, investigate its convergence properties, and briefly discuss the inverse z-transform.

3.1.1 The Direct z-Transform

The z-transform of a discrete-time signal $x(n)$ is defined as the power series

$$X(z) \equiv \sum_{n=-\infty}^{\infty} x(n) z^{-n} \tag{3.1.1}$$

where z is a complex variable. The relation (3.1.1) is sometimes called the *direct z-transform* because it transforms the time-domain signal $x(n)$ into its complex-plane representation $X(z)$. The inverse procedure [i.e., obtaining $x(n)$ from $X(z)$] is called the *inverse z-transform* and is examined briefly in Section 3.1.2 and in more detail in Section 3.4.

For convenience, the z-transform of a signal $x(n)$ is denoted by

$$X(z) \equiv Z\{x(n)\} \tag{3.1.2}$$

whereas the relationship between $x(n)$ and $X(z)$ is indicated by

$$x(n) \overset{z}{\longleftrightarrow} X(z) \tag{3.1.3}$$

Since the z-transform is an infinite power series, it exists only for those values of z for which this series converges. The *region of convergence* (ROC) of $X(z)$ is the set of all values of z for which $X(z)$ attains a finite value. Thus any time we cite a z-transform we should also indicate its ROC.

We illustrate these concepts by some simple examples.

Example 3.1.1

Determine the z-transforms of the following *finite-duration* signals.

(a) $x_1(n) = \{1, 2, 5, 7, 0, 1\}$ 1+2×z^(-1)+5×z^(-2)+7×z^(-3)+z^(-5)

(b) $x_2(n) = \{1, 2, 5, 7, 0, 1\}$
$\uparrow$

(c) $x_3(n) = \{0, 0, 1, 2, 5, 7, 0, 1\}$

(d) $x_4(n) = \{2, 4, 5, 7, 0, 1\}$
$\uparrow$

(e) $x_5(n) = \delta(n)$

(f) $x_6(n) = \delta(n-k), k > 0$

(g) $x_7(n) = \delta(n+k), k > 0$

Solution From definition (3.1.1), we have

(a) $X_1(z) = 1 + 2z^{-1} + 5z^{-2} + 7z^{-3} + z^{-5}$, ROC: entire z-plane except $z = 0$

(b) $X_2(z) = z^2 + 2z + 5 + 7z^{-1} + z^{-3}$, ROC: entire z-plane except $z = 0$ and $z = \infty$

(c) $X_3(z) = z^{-2} + 2z^{-3} + 5z^{-4} + 7z^{-5} + z^{-7}$, ROC: entire z-plane except $z = 0$

(d) $X_4(z) = 2z^2 + 4z + 5 + 7z^{-1} + z^{-3}$, ROC: entire z-plane except $z = 0$ and $z = \infty$

(e) $X_5(z) = 1$[i.e., $\delta(n) \overset{z}{\longleftrightarrow} 1$], ROC: entire z-plane

(f) $X_6(z) = z^{-k}$[i.e., $\delta(n-k) \overset{z}{\longleftrightarrow} z^{-k}$], $k > 0$, ROC: entire z-plane except $z = 0$

(g) $X_7(z) = z^k$[*i.e.*, $\delta(n+k) \overset{z}{\longleftrightarrow} z^k$], $k > 0$, ROC: entire z-plane except $z = \infty$

From this example it is easily seen that the ROC of a *finite-duration signal* is the entire z-plane, except possibly the points $z = 0$ and/or $z = \infty$. These points are excluded, because $z^k (k > 0)$ becomes unbounded for $z = \infty$ and $z^{-k} (k > 0)$ becomes unbounded for $z = 0$.

From a mathematical point of view the z-transform is simply an alternative representation of a signal. This is nicely illustrated in Example 3.1.1, where we see that the coefficient of z^{-n}, in a given transform, is the value of the signal at time n. In other words, the exponent of z contains the time information we need to identify the samples of the signal.

In many cases we can express the sum of the finite or infinite series for the z-transform in a closed-form expression. In such cases the z-transform offers a compact alternative representation of the signal.

Example 3.1.2

UNIT STEP SIGNAL $u(n) \equiv \begin{cases} 1, & \text{for } n \geq 0 \\ 0, & \text{for } n < 0 \end{cases}$

Determine the z-transform of the signal

$$x(n) = (\tfrac{1}{2})^n u(n)$$

$u(n)=\{1,1,1,1......\}$

Solution The signal $x(n)$ consists of an infinite number of nonzero values

$$x(n) = \{1, (\tfrac{1}{2}), (\tfrac{1}{2})^2, (\tfrac{1}{2})^3, \ldots, (\tfrac{1}{2})^n, \ldots\}$$

The z-transform of $x(n)$ is the infinite power series

$$X(z) \equiv \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$X(z) = 1 + \tfrac{1}{2}z^{-1} + (\tfrac{1}{2})^2 z^{-2} + (\tfrac{1}{2})^n z^{-n} + \cdots$$

$$= \sum_{n=0}^{\infty} (\tfrac{1}{2})^n z^{-n} = \sum_{n=0}^{\infty} (\tfrac{1}{2}z^{-1})^n$$

This is an infinite geometric series. We recall that

$$1 + A + A^2 + A^3 + \cdots = \frac{1}{1 - A} \qquad \text{if } |A| < 1$$

Consequently, for $|\frac{1}{2}z^{-1}| < 1$, or equivalently, for $|z| > \frac{1}{2}$, $X(z)$ converges to

$|\frac{1}{2}z^{-1}| < 1$

$$X(z) = \frac{1}{1 - \frac{1}{2}z^{-1}} \qquad \text{ROC: } |z| > \tfrac{1}{2}$$

$|\frac{1}{2z}| < 1 \Rightarrow |z| > \frac{1}{2}$

We see that in this case, the z-transform provides a compact alternative representation of the signal $x(n)$.

Let us express the complex variable z in polar form as

$$z = re^{j\theta} \tag{3.1.4}$$

where $r = |z|$ and $\theta = \measuredangle z$. Then $X(z)$ can be expressed as

$$X(z)|_{z=re^{j\theta}} = \sum_{n=-\infty}^{\infty} x(n) r^{-n} e^{-j\theta n}$$

In the ROC of $X(z)$, $|X(z)| < \infty$. But

$$\begin{aligned} |X(z)| &= \left| \sum_{n=-\infty}^{\infty} x(n) r^{-n} e^{-j\theta n} \right| \\ &\leq \sum_{n=-\infty}^{\infty} |x(n) r^{-n} e^{-j\theta n}| = \sum_{n=-\infty}^{\infty} |x(n) r^{-n}| \end{aligned} \tag{3.1.5}$$

Hence $|X(z)|$ is finite if the sequence $x(n)r^{-n}$ is absolutely summable.

The problem of finding the ROC for $X(z)$ is equivalent to determining the range of values of r for which the sequence $x(n)r^{-n}$ is absolutely summable. To elaborate, let us express (3.1.5) as

$$\begin{aligned} |X(z)| &\leq \sum_{n=-\infty}^{-1} |x(n) r^{-n}| + \sum_{n=0}^{\infty} \left| \frac{x(n)}{r^n} \right| \\ &\leq \sum_{n=1}^{\infty} |x(-n) r^{n}| + \sum_{n=0}^{\infty} \left| \frac{x(n)}{r^n} \right| \end{aligned} \tag{3.1.6}$$

If $X(z)$ converges in some region of the complex plane, both summations in (3.1.6) must be finite in that region. If the first sum in (3.1.6) converges, there must exist values of r small enough such that the product sequence $x(-n)r^n$, $1 \leq n < \infty$, is absolutely summable. Therefore, the ROC for the first sum consists of all points in a circle of some radius r_1, where $r_1 < \infty$, as illustrated in Fig. 3.1a. On the other hand, if the second sum in (3.1.6) converges, there must exist values of r large enough such that the product sequence $x(n)/r^n$, $0 \leq n < \infty$, is absolutely summable. Hence the ROC for the second sum in (3.1.6) consists of all points outside a circle of radius $r > r_2$, as illustrated in Fig. 3.1b.

Since the convergence of $X(z)$ requires that both sums in (3.1.6) be finite, it follows that the ROC of $X(z)$ is generally specified as the annular region in the z-plane, $r_2 < r < r_1$, which is the common region where both sums are finite. This region is illustrated in Fig. 3.1c. On the other hand, if $r_2 > r_1$, there is no common region of convergence for the two sums and hence $X(z)$ does not exist.

The following examples illustrate these important concepts.

Example 3.1.3

Determine the z-transform of the signal

$$x(n) = \alpha^n u(n) = \begin{cases} \alpha^n, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

Solution From the definition (3.1.1) we have

1+αz^(-1)+α^2 z^(-2)+⋯

$$X(z) = \sum_{n=0}^{\infty} \alpha^n z^{-n} = \sum_{n=0}^{\infty} (\alpha z^{-1})^n$$

If $|\alpha z^{-1}| < 1$ or equivalently, $|z| > |\alpha|$, this power series converges to $1/(1 - \alpha z^{-1})$.

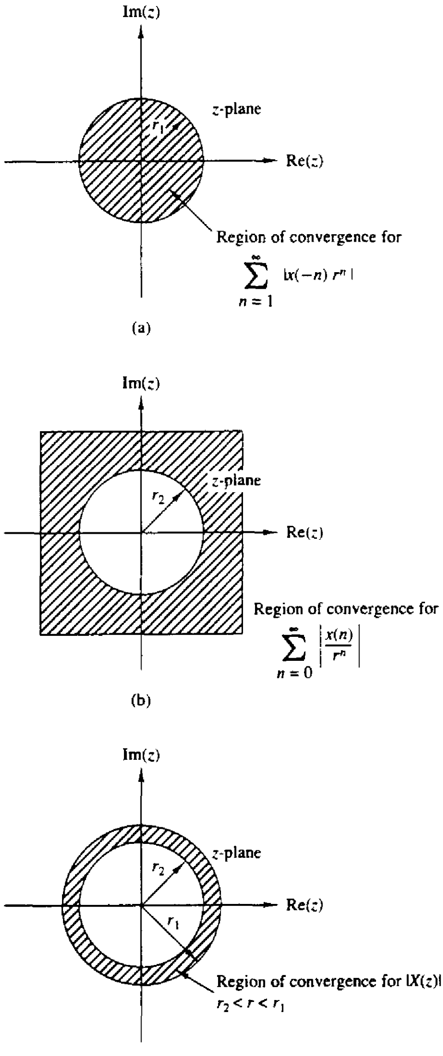


Figure 3.1 Region of convergence for $X(z)$ and its corresponding causal and anticausal components.

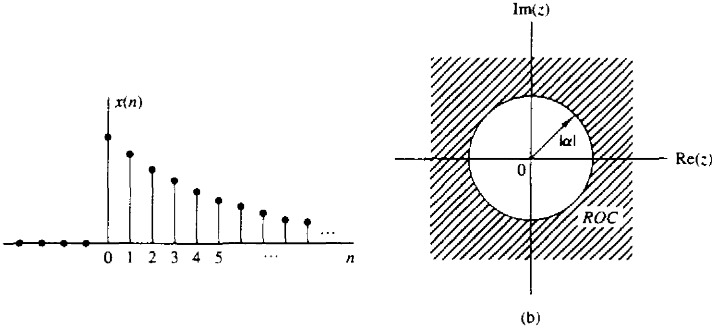


Figure 3.2 The exponential signal $x(n) = \alpha^n u(n)$ (a), and the ROC of its z-transform (b).

Thus we have the z-transform pair

$$x(n) = \alpha^n u(n) \overset{z}{\longleftrightarrow} X(z) = \frac{1}{1 - \alpha z^{-1}} \qquad \text{ROC: } |z| > |\alpha| \tag{3.1.7}$$

The ROC is the exterior of a circle having radius $|\alpha|$. Figure 3.2 shows a graph of the signal $x(n)$ and its corresponding ROC. Note that, in general, α need not be real.

If we set $\alpha = 1$ in (3.1.7), we obtain the z-transform of the unit step signal

$$x(n) = u(n) \overset{z}{\longleftrightarrow} X(z) = \frac{1}{1 - z^{-1}} \qquad \text{ROC: } |z| > 1 \tag{3.1.8}$$

Example 3.1.4

Determine the z-transform of the signal

$$x(n) = -\alpha^n u(-n-1) = \begin{cases} 0, & n \geq 0 \\ -\alpha^n, & n \leq -1 \end{cases}$$

Solution From the definition (3.1.1) we have

$$X(z) = \sum_{n=-\infty}^{-1} (-\alpha^n) z^{-n} = -\sum_{l=1}^{\infty} (\alpha^{-1} z)^l$$

where $l = -n$. Using the formula

$$A + A^2 + A^3 + \cdots = A(1 + A + A^2 + \cdots) = \frac{A}{1 - A}$$

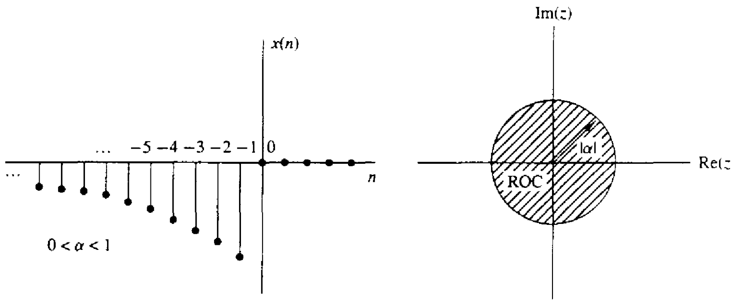
when $|A| < 1$ gives

$$X(z) = -\frac{\alpha^{-1} z}{1 - \alpha^{-1} z} = \frac{1}{1 - \alpha z^{-1}}$$

provided that $|\alpha^{-1} z| < 1$ or, equivalently, $|z| < |\alpha|$. Thus

$$x(n) = -\alpha^n u(-n-1) \overset{z}{\longleftrightarrow} X(z) = -\frac{1}{1 - \alpha z^{-1}} \qquad \text{ROC: } |z| < |\alpha| \tag{3.1.9}$$

The ROC is now the interior of a circle having radius $|\alpha|$. This is shown in Fig. 3.3.



(a) (b)

Figure 3.3 Anticausal signal $x(n) = -\alpha^n u(-n-1)$ (a), and the ROC of its z-transform (b).

Examples 3.1.3 and 3.1.4 illustrate two very important issues. The first concerns the uniqueness of the z-transform. From (3.1.7) and (3.1.9) we see that the causal signal $\alpha^n u(n)$ and the anticausal signal $-\alpha^n u(-n-1)$ have identical closed-form expressions for the z-transform, that is,

$$Z\{\alpha^n u(n)\} = Z\{-\alpha^n u(-n-1)\} = \frac{1}{1-\alpha z^{-1}}$$

This implies that a closed-form expression for the z-transform does not uniquely specify the signal in the time domain. The ambiguity can be resolved only if in addition to the closed-form expression, the ROC is specified. In summary, *a discrete-time signal $x(n)$ is uniquely determined by its z-transform $X(z)$ and the region of convergence of $X(z)$.* In this text the term "z-transform" is used to refer to both the closed-form expression and the corresponding ROC. Example 3.1.3 also illustrates the point that *the ROC of a causal signal is the exterior of a circle of some radius r_2 while the ROC of an anticausal signal is the interior of a circle of some radius r_1.* The following example considers a sequence that is nonzero for $-\infty < n < \infty$.

Example 3.1.5

Determine the z-transform of the signal

$$x(n) = \alpha^n u(n) + b^n u(-n-1)$$

Solution From definition (3.1.1) we have

$$X(z) = \sum_{n=0}^{\infty} \alpha^n z^{-n} + \sum_{n=-\infty}^{-1} b^n z^{-n} = \sum_{n=0}^{\infty} (\alpha z^{-1})^n + \sum_{l=1}^{\infty} (b^{-1} z)^l$$

The first power series converges if $|\alpha z^{-1}| < 1$ or $|z| > |\alpha|$. The second power series converges if $|b^{-1} z| < 1$ or $|z| < |b|$.

In determining the convergence of $X(z)$, we consider two different cases.

Case 1 $|b| < |\alpha|$: In this case the two ROC above do not overlap, as shown in Fig. 3.4(a). Consequently, we cannot find values of z for which both power series converge simultaneously. Clearly, in this case, $X(z)$ does not exist.

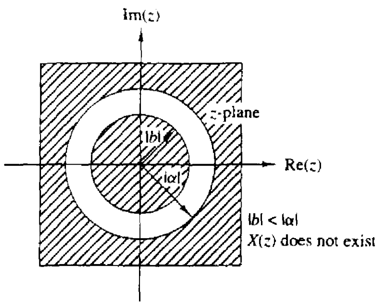
Case 2 $|b| > |\alpha|$: In this case there is a ring in the z-plane where both power series converge simultaneously, as shown in Fig. 3.4(b). Then we obtain

$$\begin{aligned} X(z) &= \frac{1}{1-\alpha z^{-1}} - \frac{1}{1-bz^{-1}} \\ &= \frac{b-\alpha}{\alpha + b - z - \alpha b z^{-1}} \end{aligned} \tag{3.1.10}$$

The ROC of $X(z)$ is $|\alpha| < |z| < |b|$.

This example shows that *if there is a ROC for an infinite duration two-sided signal, it is a ring (annular region) in the z-plane.* From Examples 3.1.1, 3.1.3, 3.1.4, and 3.1.5, we see that the ROC of a signal depends on both its duration (finite or infinite) and on whether it is causal, anticausal, or two-sided. These facts are summarized in Table 3.1.

One special case of a two-sided signal is a signal that has infinite duration on the right side but not on the left [i.e., $x(n) = 0$ for $n < n_0 < 0$]. A second case is a signal that has infinite duration on the left side but not on the



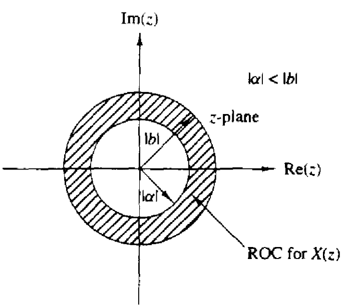
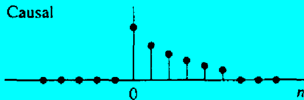
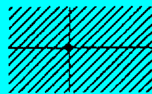
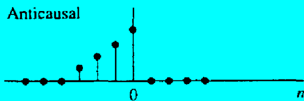
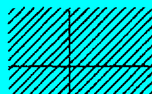
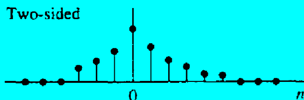
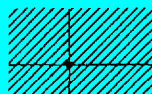
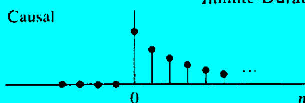
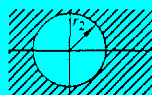
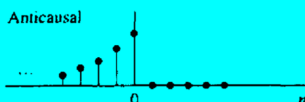
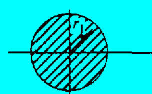
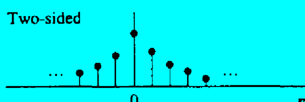
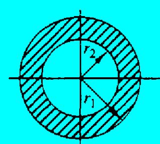


Figure 3.4 ROC for z-transform in Example 3.1.5.

TABLE 3.1 CHARACTERISTIC FAMILIES OF SIGNALS WITH THEIR CORRESPONDING ROC

Signal		ROC	
	Finite-Duration Signals		
Causal			Entire z-plane except $z = 0$
Anticausal			Entire z-plane except $z = \infty$
Two-sided			Entire z-plane except $z = 0$ and $z = \infty$
	Infinite-Duration Signals		
Causal			$\lvert z\rvert > r_2$
Anticausal			$\lvert z\rvert < r_1$
Two-sided			$r_2 < \lvert z\rvert < r_1$

right [i.e., $x(n) = 0$ for $n > n_1 > 0$]. A third special case is a signal that has finite duration on both the left and right sides [i.e., $x(n) = 0$ for $n < n_0 < 0$ and $n > n_1 > 0$]. These types of signals are sometimes called *right-sided, left-sided,* and *finite-duration two-sided,* signals, respectively. The determination of the ROC for these three types of signals is left as an exercise for the reader (Problem 3.5).

Finally, we note that the z-transform defined by (3.1.1) is sometimes referred to as the *two-sided* or *bilateral z-transform*, to distinguish it from the *one-sided* or *unilateral z-transform* given by

$$X^{+}(z) = \sum_{n=0}^{\infty} x(n)z^{-n} \tag{3.1.11}$$

The one-sided z-transform is examined in Section 3.5. In this text we use the expression z-transform exclusively to mean the two-sided z-transform defined by (3.1.1). The term "two-sided" will be used only in cases where we want to resolve any ambiguities. Clearly, if $x(n)$ is causal [i.e., $x(n) = 0$ for $n < 0$], the one-sided and two-sided z-transforms are equivalent. In any other case, they are different.

3.1.2 The Inverse z-Transform

Often, we have the z-transform $X(z)$ of a signal and we must determine the signal sequence. The procedure for transforming from the z-domain to the time domain is called the *inverse z-transform*. An inversion formula for obtaining $x(n)$ from $X(z)$ can be derived by using the *Cauchy integral theorem*, which is an important theorem in the theory of complex variables.

To begin, we have the z-transform defined by (3.1.1) as

$$X(z) = \sum_{k=-\infty}^{\infty} x(k)z^{-k} \tag{3.1.12}$$

Suppose that we multiply both sides of (3.1.12) by z^{n-1} and integrate both sides over a closed contour within the ROC of $X(z)$ which encloses the origin. Such a contour is illustrated in Fig. 3.5. Thus we have

$$\oint_C X(z)z^{n-1}\,dz = \oint_C \sum_{k=-\infty}^{\infty} x(k)z^{n-1-k}\,dz \tag{3.1.13}$$

where C denotes the closed contour in the ROC of $X(z)$, taken in a counterclockwise direction. Since the series converges on this contour, we can interchange the order of integration and summation on the right-hand side of (3.1.13). Thus

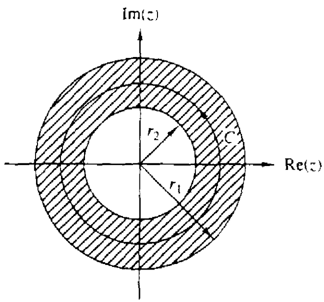


Figure 3.5 Contour C for integral in (3.1.13).

(3.1.13) becomes

$$\oint_C X(z)z^{n-1}dz = \sum_{k=-\infty}^{\infty} x(k)\oint_C z^{n-1-k}dz \tag{3.1.14}$$

Now we can invoke the Cauchy integral theorem, which states that

$$\frac{1}{2\pi j}\oint_C z^{n-1-k}\, dz = \begin{cases} 1, & k = n \\ 0, & k \neq n \end{cases} \tag{3.1.15}$$

where C is any contour that encloses the origin. By applying (3.1.15), the right-hand side of (3.1.14) reduces to $2\pi j x(n)$ and hence the desired inversion formula

$$x(n) = \frac{1}{2\pi j}\oint_C X(z)z^{n-1}\, dz \tag{3.1.16}$$

Although the contour integral in (3.1.16) provides the desired inversion formula for determining the sequence $x(n)$ from the z-transform, we shall not use (3.1.16) directly in our evaluation of inverse z-transforms. In our treatment we deal with signals and systems in the z-domain which have rational z-transforms (i.e., z-transforms that are a ratio of two polynomials). For such z-transforms we develop a simpler method for inversion that stems from (3.1.16) and employs a table lookup.

3.2 PROPERTIES OF THE Z-TRANSFORM

The z-transform is a very powerful tool for the study of discrete-time signals and systems. The power of this transform is a consequence of some very important properties that the transform possesses. In this section we examine some of these properties.

In the treatment that follows, it should be remembered that when we combine several z-transforms, the ROC of the overall transform is, at least, the intersection of the ROC of the individual transforms. This will become more apparent later, when we discuss specific examples.

Linearity. If

$$x_1(n) \overset{z}{\longleftrightarrow} X_1(z)$$

and

$$x_2(n) \overset{z}{\longleftrightarrow} X_2(z)$$

then

$$x(n) = a_1x_1(n) + a_2x_2(n) \overset{z}{\longleftrightarrow} X(z) = a_1X_1(z) + a_2X_2(z) \tag{3.2.1}$$

for any constants a_1 and a_2. The proof of this property follows immediately from the definition of linearity and is left as an exercise for the reader.

The linearity property can easily be generalized for an arbitrary number of signals. Basically, it implies that the z-transform of a linear combination of signals is the same linear combination of their z-transforms. Thus the linearity property helps us to find the z-transform of a signal by expressing the signal as a sum of elementary signals, for each of which, the z-transform is already known.

Example 3.2.1

Determine the z-transform and the ROC of the signal

$$x(n) = [3(2^n) - 4(3^n)]u(n) \quad x(n) = 3.2^n u(n) - 4.3^n u(n)$$

Solution If we define the signals $\qquad x(n) = a_1 x_1(n) + a_2 x_2(n)$

$$x_1(n) = 2^n u(n)$$

and

$$x_2(n) = 3^n u(n)$$

then $x(n)$ can be written as

$$x(n) = 3x_1(n) - 4x_2(n)$$

According to (3.2.1), its z-transform is $X(z) = a_1 X_1(z) + a_2 X_2(z)$

$$X(z) = 3X_1(z) - 4X_2(z)$$

From (3.1.7) we recall that

$$\alpha^n u(n) \overset{z}{\longleftrightarrow} \frac{1}{1 - \alpha z^{-1}} \qquad \text{ROC: } |z| > |\alpha| \tag{3.2.2}$$

By setting $\alpha = 2$ and $\alpha = 3$ in (3.2.2), we obtain

$$x_1(n) = 2^n u(n) \overset{z}{\longleftrightarrow} X_1(z) = \frac{1}{1 - 2z^{-1}} \qquad \text{ROC: } |z| > 2$$

$$x_2(n) = 3^n u(n) \overset{z}{\longleftrightarrow} X_2(z) = \frac{1}{1 - 3z^{-1}} \qquad \text{ROC: } |z| > 3$$

The intersection of the ROC of $X_1(z)$ and $X_2(z)$ is $|z| > 3$. Thus the overall transform $X(z)$ is

$$X(z) = \frac{3}{1 - 2z^{-1}} - \frac{4}{1 - 3z^{-1}} \qquad \text{ROC: } |z| > 3$$

Example 3.2.2

Determine the z-transform of the signals

(a) $x(n) = (\cos \omega_0 n)u(n)$

(b) $x(n) = (\sin \omega_0 n)u(n)$

Solution

(a) By using Euler's identity, the signal $x(n)$ can be expressed as

$$x(n) = (\cos \omega_0 n)u(n) = \tfrac{1}{2}e^{j\omega_0 n}u(n) + \tfrac{1}{2}e^{-j\omega_0 n}u(n)$$

Thus (3.2.1) implies that

$$X(z) = \tfrac{1}{2}Z\{e^{j\omega_0 n}u(n)\} + \tfrac{1}{2}Z\{e^{-j\omega_0 n}u(n)\}$$

If we set $\alpha = e^{\pm j\omega_0}(|\alpha| = |e^{\pm j\omega_0}| = 1)$ in (3.2.2), we obtain

$$e^{j\omega_0 n}u(n) \overset{z}{\longleftrightarrow} \frac{1}{1 - e^{j\omega_0}z^{-1}} \qquad \text{ROC: } |z| > 1$$

and

$$e^{-j\omega_0 n}u(n) \overset{z}{\longleftrightarrow} \frac{1}{1 - e^{-j\omega_0}z^{-1}} \qquad \text{ROC: } |z| > 1$$

Thus

$$X(z) = \frac{1}{2}\frac{1}{1 - e^{j\omega_0}z^{-1}} + \frac{1}{2}\frac{1}{1 - e^{-j\omega_0}z^{-1}} \qquad \text{ROC: } |z| > 1$$

After some simple algebraic manipulations we obtain the desired result, namely,

$$(\cos\omega_0 n)u(n) \overset{z}{\longleftrightarrow} \frac{1 - z^{-1}\cos\omega_0}{1 - 2z^{-1}\cos\omega_0 + z^{-2}} \qquad \text{ROC: } |z| > 1 \qquad (3.2.3)$$

(b) From Euler's identity,

$$x(n) = (\sin\omega_0 n)u(n) = \frac{1}{2j}[e^{j\omega_0 n}u(n) - e^{-j\omega_0 n}u(n)]$$

Thus

$$X(z) = \frac{1}{2j}\left(\frac{1}{1 - e^{j\omega_0}z^{-1}} - \frac{1}{1 - e^{-j\omega_0}z^{-1}}\right) \qquad \text{ROC: } |z| > 1$$

and finally,

$$(\sin\omega_0 n)u(n) \overset{z}{\longleftrightarrow} \frac{z^{-1}\sin\omega_0}{1 - 2z^{-1}\cos\omega_0 + z^{-2}} \qquad \text{ROC: } |z| > 1 \qquad (3.2.4)$$

Time shifting. If

$$x(n) \overset{z}{\longleftrightarrow} X(z)$$

then

$$x(n-k) \overset{z}{\longleftrightarrow} z^{-k}X(z) \qquad (3.2.5)$$

The ROC of $z^{-k}X(z)$ is the same as that of $X(z)$ except for $z = 0$ if $k > 0$ and $z = \infty$ if $k < 0$. The proof of this property follows immediately from the definition of the z-transform given in (3.1.1)

The properties of linearity and time shifting are the key features that make the z-transform extremely useful for the analysis of discrete-time LTI systems.

Example 3.2.3

By applying the time-shifting property, determine the z-transform of the signals $x_2(n)$ and $x_3(n)$ in Example 3.1.1 from the z-transform of $x_1(n)$.

Solution It can easily be seen that

$$x_2(n) = x_1(n+2)$$