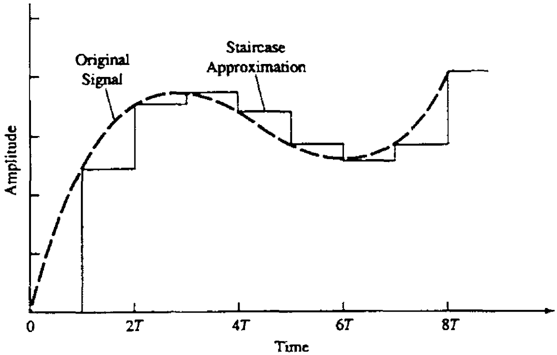

Figure 1.15 Zero-order hold digital-to-analog (D/A) conversion.

the accuracy, as measured by the number of bits, in the A/D conversion process. The factors affecting the choice of the desired accuracy of the A/D converter are cost and sampling rate. In general, the cost increases with an increase in accuracy and/or sampling rate.

1.4.1 Sampling of Analog Signals

There are many ways to sample an analog signal. We limit our discussion to *periodic* or *uniform sampling*, which is the type of sampling used most often in practice. This is described by the relation

Continuous Time Signal,xa(t)=A cos(Ωt+θ)=A cos(2πFt+θ)
Discrete Time Signal,x(n)=A cos(ωn+θ)=A cos(2πfn+θ)

$$x(n) = x_a(nT), \qquad -\infty < n < \infty \tag{1.4.1}$$

where $x(n)$ is the discrete-time signal obtained by "taking samples" of the analog signal $x_a(t)$ every T seconds. This procedure is illustrated in Fig. 1.16. The time interval T between successive samples is called the *sampling period* or *sample interval* and its reciprocal $1/T = F_s$ is called the *sampling rate* (samples per second) or the *sampling frequency* (hertz).

Periodic sampling establishes a relationship between the time variables t and n of continuous-time and discrete-time signals, respectively. Indeed, these variables are linearly related through the sampling period T or, equivalently, through the sampling rate $F_s = 1/T$, as

$$t = nT = \frac{n}{F_s} \tag{1.4.2}$$

As a consequence of (1.4.2), there exists a relationship between the frequency variable F (or Ω) for analog signals and the frequency variable f (or ω) for discrete-time signals. To establish this relationship, consider an analog sinusoidal signal of the form

$$x_a(t) = A\cos(2\pi F t + \theta) \tag{1.4.3}$$

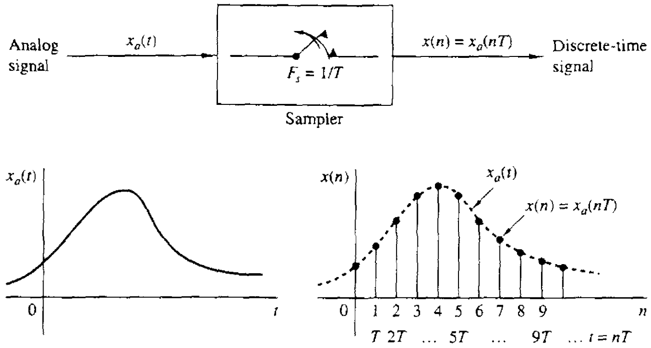

Figure 1.16 Periodic sampling of an analog signal.

Continuous Time Signal,xa(t)=A cos(Ωt+θ)=A cos(2πFt+θ)
Discrete Time Signal,x(n)=A cos(ωn+θ)=A cos(2πfn+θ)

which, when sampled periodically at a rate $F_s = 1/T$ samples per second, yields

$$\begin{aligned} x_a(nT) \equiv x(n) &= A\cos(2\pi FnT + \theta) \\ &= A\cos\left(\frac{2\pi nF}{F_s} + \theta\right) \end{aligned} \tag{1.4.4}$$

If we compare (1.4.4) with (1.3.9), we note that the frequency variables F and f are linearly related as

x(n) = A cos(2πfn +θ)

$$f = \frac{F}{F_s} \tag{1.4.5}$$

Fs=sampling frequency
F=frequency of analog signal
f=frequency of digital signal
=relative or normalized frequency

or, equivalently, as

$$\omega = \Omega T \tag{1.4.6}$$

The relation in (1.4.5) justifies the name *relative* or *normalized frequency*, which is sometimes used to describe the frequency variable f. As (1.4.5) implies, we can use f to determine the frequency F in hertz only if the sampling frequency F_s is known.

We recall from Section 1.3.1 that the range of the frequency variable F or Ω for continuous-time sinusoids are

$$\begin{aligned} -\infty < F < \infty \\ -\infty < \Omega < \infty \end{aligned} \tag{1.4.7}$$

However, the situation is different for discrete-time sinusoids. From Section 1.3.2 we recall that

$$\begin{aligned} -\tfrac{1}{2} < f < \tfrac{1}{2} \\ -\pi < \omega < \pi \end{aligned} \tag{1.4.8}$$

By substituting from (1.4.5) and (1.4.6) into (1.4.8), we find that the frequency of the continuous-time sinusoid when sampled at a rate $F_s = 1/T$ must fall in the range

$$-\frac{1}{2T} = -\frac{F_s}{2} \le F \le \frac{F_s}{2} = \frac{1}{2T} \tag{1.4.9}$$

or, equivalently,

$$-\frac{\pi}{T} = -\pi F_s \le \Omega \le \pi F_s = \frac{\pi}{T} \tag{1.4.10}$$

These relations are summarized in Table 1.1.

TABLE 1.1 RELATIONS AMONG FREQUENCY VARIABLES

Continuous-time signals		Discrete-time signals
$\Omega = 2\pi F$ $\frac{\text{radians}}{\text{sec}}$ Hz		$\omega = 2\pi f$ $\frac{\text{radians}}{\text{sample}}$ $\frac{\text{cycles}}{\text{sample}}$
	$\xrightarrow{\omega = \Omega T,\ f = F/F_s}$	$-\pi \le \omega \le \pi$ $-\frac{1}{2} \le f \le \frac{1}{2}$
	$\xleftarrow{\Omega = \omega/T,\ F = f \cdot F_s}$	
$-\infty < \Omega < \infty$ $-\infty < F < \infty$		$-\pi/T \le \Omega \le \pi/T$ $-F_s/2 \le F \le F_s/2$

From these relations we observe that the fundamental difference between continuous-time and discrete-time signals is in their range of values of the frequency variables F and f, or Ω and ω. Periodic sampling of a continuous-time signal implies a mapping of the infinite frequency range for the variable F (or Ω) into a finite frequency range for the variable f (or ω). Since the highest frequency in a discrete-time signal is $\omega = \pi$ or $f = \frac{1}{2}$, it follows that, with a sampling rate F_s, the corresponding highest values of F and Ω are

$$\begin{aligned} F_{\max} &= \frac{F_s}{2} = \frac{1}{2T} \\ \Omega_{\max} &= \pi F_s = \frac{\pi}{T} \end{aligned} \tag{1.4.11}$$

Therefore, sampling introduces an ambiguity, since the highest frequency in a continuous-time signal that can be uniquely distinguished when such a signal is sampled at a rate $F_s = 1/T$ is $F_{\max} = F_s/2$, or $\Omega_{\max} = \pi F_s$. To see what happens to frequencies above $F_s/2$, let us consider the following example.

Example 1.4.1

The implications of these frequency relations can be fully appreciated by considering the two analog sinusoidal signals

$$\begin{aligned} x_1(t) &= \cos 2\pi(10)t \\ x_2(t) &= \cos 2\pi(50)t \end{aligned} \tag{1.4.12}$$

which are sampled at a rate $F_s = 40$ Hz. The corresponding discrete-time signals or sequences are

$$\begin{aligned} x_1(n) &= \cos 2\pi \left(\frac{10}{40}\right) n = \cos \frac{\pi}{2} n \\ x_2(n) &= \cos 2\pi \left(\frac{50}{40}\right) n = \cos \frac{5\pi}{2} n \end{aligned} \tag{1.4.13}$$

However, $\cos 5\pi n/2 = \cos(2\pi n + \pi n/2) = \cos \pi n/2$. Hence $x_2(n) = x_1(n)$. Thus the sinusoidal signals are identical and, consequently, indistinguishable. If we are given the sampled values generated by $\cos(\pi/2)n$, there is some ambiguity as to whether these sampled values correspond to $x_1(t)$ or $x_2(t)$. Since $x_2(t)$ yields exactly the same values as $x_1(t)$ when the two are sampled at $F_s = 40$ samples per second, we say that the frequency $F_2 = 50$ Hz is an *alias* of the frequency $F_1 = 10$ Hz at the sampling rate of 40 samples per second.

It is important to note that F_2 is not the only alias of F_1. In fact at the sampling rate of 40 samples per second, the frequency $F_3 = 90$ Hz is also an alias of F_1, as is the frequency $F_4 = 130$ Hz, and so on. All of the sinusoids $\cos 2\pi(F_1 + 40k)t$, $k = 1, 2, 3, 4, \ldots$ sampled at 40 samples per second, yield identical values. Consequently, they are all aliases of $F_1 = 10$ Hz.

In general, the sampling of a continuous-time sinusoidal signal

$$x_a(t) = A\cos(2\pi F_0 t + \theta) \tag{1.4.14}$$

with a sampling rate $F_s = 1/T$ results in a discrete-time signal

$$x(n) = A\cos(2\pi f_0 n + \theta) \tag{1.4.15}$$

where $f_0 = F_0/F_s$ is the relative frequency of the sinusoid. If we assume that $-F_s/2 \le F_0 \le F_s/2$, the frequency f_0 of $x(n)$ is in the range $-\frac{1}{2} \le f_0 \le \frac{1}{2}$, which is the frequency range for discrete-time signals. In this case, the relationship between F_0 and f_0 is one-to-one, and hence it is possible to identify (or reconstruct) the analog signal $x_a(t)$ from the samples $x(n)$.

On the other hand, if the sinusoids

$$x_a(t) = A\cos(2\pi F_k t + \theta) \tag{1.4.16}$$

where

$$F_k = F_0 + kF_s, \qquad k = \pm 1, \pm 2, \ldots \tag{1.4.17}$$

are sampled at a rate F_s, it is clear that the frequency F_k is outside the fundamental frequency range $-F_s/2 \le F \le F_s/2$. Consequently, the sampled signal is

$$\begin{aligned} x(n) \equiv x_a(nT) &= A\cos\left(2\pi \frac{F_0 + kF_s}{F_s} n + \theta\right) \\ &= A\cos(2\pi n F_0/F_s + \theta + 2\pi kn) \\ &= A\cos(2\pi f_0 n + \theta) \end{aligned}$$

Since $F_s/2$, which corresponds to $\omega = \pi$, is the highest frequency that can be represented uniquely with a sampling rate F_s, it is a simple matter to determine the mapping of any (alias) frequency above $F_s/2$ $(\omega = \pi)$ into the equivalent frequency below $F_s/2$. We can use $F_s/2$ or $\omega = \pi$ as the pivotal point and reflect or "fold" the alias frequency to the range $0 \leq \omega \leq \pi$. Since the point of reflection is $F_s/2$ $(\omega = \pi)$, the frequency $F_s/2$ $(\omega = \pi)$ is called the *folding frequency*.

Example 1.4.2

Consider the analog signal

$$x_a(t) = 3\cos 100\pi t \quad =3\cos(2\pi\times 50\times t)$$

(a) Determine the minimum sampling rate required to avoid aliasing.

(b) Suppose that the signal is sampled at the rate $F_s = 200$ Hz. What is the discrete-time signal obtained after sampling?

(c) Suppose that the signal is sampled at the rate $F_s = 75$ Hz. What is the discrete-time signal obtained after sampling?

(d) What is the frequency $0 < F < F_s/2$ of a sinusoid that yields samples identical to those obtained in part (c)?

Solution

(a) The frequency of the analog signal is $F = 50$ Hz. Hence the minimum sampling rate required to avoid aliasing is $F_s = 100$ Hz.

(b) If the signal is sampled at $F_s = 200$ Hz, the discrete-time signal is

$$x(n) = 3\cos\frac{100\pi}{200}n = 3\cos\frac{\pi}{2}n$$

(c) If the signal is sampled at $F_s = 75$ Hz, the discrete-time signal is

$$\begin{aligned} x(n) &= 3\cos\frac{100\pi}{75}n = 3\cos\frac{4\pi}{3}n \\ &= 3\cos\left(2\pi - \frac{2\pi}{3}\right)n \\ &= 3\cos\frac{2\pi}{3}n \quad =3\cos 2\pi\times 1/3\times n \quad =A(\cos 2\pi\times f\times n) \end{aligned}$$

(d) For the sampling rate of $F_s = 75$ Hz, we have

$f=F/FS$

$$F = fF_s = 75f$$

The frequency of the sinusoid in part (c) is $f = \frac{1}{3}$. Hence

$$F = 25 \text{ Hz}$$

Clearly, the sinusoidal signal

$$\begin{aligned} y_a(t) &= 3\cos 2\pi Ft \\ &= 3\cos 50\pi t \end{aligned}$$

sampled at $F_s = 75$ samples/s yields identical samples. Hence $F = 50$ Hz is an alias of $F = 25$ Hz for the sampling rate $F_s = 75$ Hz.

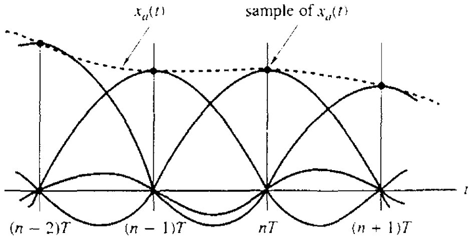

Figure 1.19 Ideal D/A conversion (interpolation).

formulas are primarily of theoretical interest. Practical interpolation methods are given in Chapter 9.

Example 1.4.3

Consider the analog signal =3 cos2π×25×t+10 cos2π×150×t-cos2π×50×t

$$x_a(t) = 3\cos 50\pi t + 10\sin 300\pi t - \cos 100\pi t$$

What is the Nyquist rate for this signal?

Solution The frequencies present in the signal above are

$$F_1 = 25 \text{ Hz}, \qquad F_2 = 150 \text{ Hz}, \qquad F_3 = 50 \text{ Hz}$$

Thus $F_{max} = 150$ Hz and according to (1.4.19),

$$F_s > 2F_{max} = 300 \text{ Hz}$$

The Nyquist rate is $F_N = 2F_{max}$. Hence

$$F_N = 300 \text{ Hz}$$

Discussion It should be observed that the signal component $10\sin 300\pi t$, sampled at the Nyquist rate $F_N = 300$, results in the samples $10\sin\pi n$, which are identically zero. In other words, we are sampling the analog sinusoid at its zero-crossing points, and hence we miss this signal component completely. This situation would not occur if the sinusoid is offset in phase by some amount θ. In such a case we have $10\sin(300\pi t + \theta)$ sampled at the Nyquist rate $F_N = 300$ samples per second, which yields the samples

$$\begin{aligned} 10\sin(\pi n + \theta) &= 10(\sin\pi n\cos\theta + \cos\pi n\sin\theta) \\ &= 10\sin\theta\cos\pi n \\ &= (-1)^n 10\sin\theta \end{aligned}$$

Thus if $\theta \neq 0$ or π, the samples of the sinusoid taken at the Nyquist rate are not all zero. However, we still cannot obtain the correct amplitude from the samples when the phase θ is unknown. A simple remedy that avoids this potentially troublesome situation is to sample the analog signal at a rate higher than the Nyquist rate.

Example 1.4.4

Consider the analog signal

$$x_a(t) = 3\cos 2000\pi t + 5\sin 6000\pi t + 10\cos 12{,}000\pi t$$

(a) What is the Nyquist rate for this signal?

(b) Assume now that we sample this signal using a sampling rate $F_s = 5000$ samples/s. What is the discrete-time signal obtained after sampling?

(c) What is the analog signal $y_a(t)$ we can reconstruct from the samples if we use ideal interpolation?

Solution

(a) The frequencies existing in the analog signal are

$$F_1 = 1 \text{ kHz}, \qquad F_2 = 3 \text{ kHz}, \qquad F_3 = 6 \text{ kHz}$$

Thus $F_{max} = 6$ kHz, and according to the sampling theorem,

$$F_s > 2F_{max} = 12 \text{ kHz}$$

The Nyquist rate is

$$F_N = 12 \text{ kHz}$$

(b) Since we have chosen $F_s = 5$ kHz, the folding frequency is

$$\frac{F_s}{2} = 2.5 \text{ kHz}$$

and this is the maximum frequency that can be represented uniquely by the sampled signal. By making use of (1.4.2) we obtain

$$\begin{aligned} x(n) &= x_a(nT) = x_a\left(\frac{n}{F_s}\right) \\ &= 3\cos 2\pi(\tfrac{1}{5})n + 5\sin 2\pi(\tfrac{3}{5})n + 10\cos 2\pi(\tfrac{6}{5})n \\ &= 3\cos 2\pi(\tfrac{1}{5})n + 5\sin 2\pi(1-\tfrac{2}{5})n + 10\cos 2\pi(1+\tfrac{1}{5})n \\ &= 3\cos 2\pi(\tfrac{1}{5})n + 5\sin 2\pi(-\tfrac{2}{5})n + 10\cos 2\pi(\tfrac{1}{5})n \end{aligned}$$

Finally, we obtain

$$x(n) = 13\cos 2\pi(\tfrac{1}{5})n - 5\sin 2\pi(\tfrac{2}{5})n$$

The same result can be obtained using Fig. 1.17. Indeed, since $F_s = 5$ kHz, the folding frequency is $F_s/2 = 2.5$ kHz. This is the maximum frequency that can be represented uniquely by the sampled signal. From (1.4.17) we have $F_0 = F_k - kF_s$. Thus F_0 can be obtained by subtracting from F_k an integer multiple of F_s such that $-F_s/2 \le F_0 \le F_s/2$. The frequency F_1 is less than $F_s/2$ and thus it is not affected by aliasing. However, the other two frequencies are above the folding frequency and they will be changed by the aliasing effect. Indeed,

$$\begin{aligned} F_2' &= F_2 - F_s = -2 \text{ kHz} \\ F_3' &= F_3 - F_s = 1 \text{ kHz} \end{aligned}$$

From (1.4.5) it follows that $f_1 = \frac{1}{5}$, $f_2 = -\frac{2}{5}$, and $f_3 = \frac{1}{5}$, which are in agreement with the result above.