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Discrete-Time Signals and Systems

In Chapter 1 we introduced the reader to a number of important types of signals and described the sampling process by which an analog signal is converted to a discrete-time signal. In addition, we presented in some detail the characteristics of discrete-time sinusoidal signals. The sinusoid is an important elementary signal that serves as a basic building block in more complex signals. However, there are other elementary signals that are important in our treatment of signal processing. These discrete-time signals are introduced in this chapter and are used as basis functions or building blocks to describe more complex signals.

The major emphasis in this chapter is the characterization of discrete-time systems in general and the class of linear time-invariant (LTI) systems in particular. A number of important time-domain properties of LTI systems are defined and developed, and an important formula, called the convolution formula, is derived which allows us to determine the output of an LTI system to any given arbitrary input signal. In addition to the convolution formula, difference equations are introduced as an alternative method for describing the input–output relationship of an LTI system, and in addition, recursive and nonrecursive realizations of LTI systems are treated.

Our motivation for the emphasis on the study of LTI systems is twofold. First, there is a large collection of mathematical techniques that can be applied to the analysis of LTI systems. Second, many practical systems are either LTI systems or can be approximated by LTI systems. Because of its importance in digital signal processing applications and its close resemblance to the convolution formula, we also introduce the correlation between two signals. The autocorrelation and crosscorrelation of signals are defined and their properties are presented.

2.1 DISCRETE-TIME SIGNALS

Discrete-time signals are defined only at certain specific values of time. The signal x (n) = e^n ; where n = 0, ± 1, ± 2,…provides an example of a discrete-time signal.

As we discussed in Chapter 1, a discrete-time signal $x(n)$ is a function of an independent variable that is an integer. It is graphically represented as in Fig. 2.1. It is important to note that a discrete-time signal is *not defined* at instants between

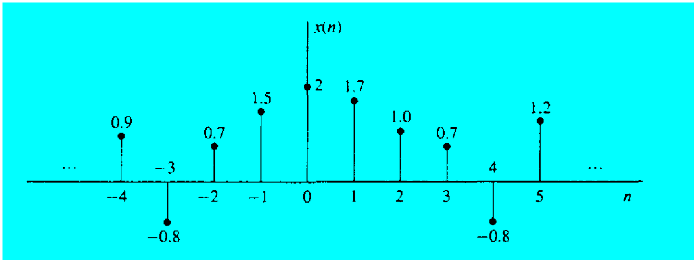


Figure 2.1 Graphical representation of a discrete-time signal.

two successive samples. Also, it is incorrect to think that $x(n)$ is equal to zero if n is not an integer. Simply, the signal $x(n)$ is not defined for noninteger values of n.

In the sequel we will assume that a discrete-time signal is defined for every integer value n for $-\infty < n < \infty$. By tradition, we refer to $x(n)$ as the "nth sample" of the signal even if the signal $x(n)$ is inherently discrete time (i.e., not obtained by sampling an analog signal). If, indeed, $x(n)$ was obtained from sampling an analog signal $x_a(t)$, then $x(n) \equiv x_a(nT)$, where T is the sampling period (i.e., the time between successive samples).

Besides the graphical representation of a discrete-time signal or sequence as illustrated in Fig. 2.1, there are some alternative representations that are often more convenient to use. These are:

1. Functional representation, such as

$$x(n) = \begin{cases} 1, & \text{for } n = 1, 3 \\ 4, & \text{for } n = 2 \\ 0, & \text{elsewhere} \end{cases} \tag{2.1.1}$$

2. Tabular representation, such as

n	...	−2	−1	0	1	2	3	4	5	...
$x(n)$	...	0	0	0	1	4	1	0	0	...

3. Sequence representation

An infinite-duration signal or sequence with the time origin ($n = 0$) indicated by the symbol $\uparrow$ is represented as

$$x(n) = \{\ldots 0, 0, \underset{\uparrow}{1}, 4, 1, 0, 0, \ldots\} \tag{2.1.2}$$

A sequence $x(n)$, which is zero for $n < 0$, can be represented as

$$x(n) = \{\underset{\uparrow}{0}, 1, 4, 1, 0, 0, \ldots\} \tag{2.1.3}$$

n= 0, 1, 2, 3, 4, 5,

The time origin for a sequence $x(n)$, which is zero for $n < 0$, is understood to be the first (leftmost) point in the sequence.

A finite-duration sequence can be represented as

$$x(n) = \{3, -1, -2, 5, \underset{\uparrow}{0}, 4, -1\} \tag{2.1.4}$$

n= -2, -1, 0, 1, 2, 3, 4

whereas a finite-duration sequence that satisfies the condition $x(n) = 0$ for $n < 0$ can be represented as

$$x(n) = \{\underset{\uparrow}{0}, 1, 4, 1\} \tag{2.1.5}$$

The signal in (2.1.4) consists of seven samples or points (in time), so it is called or identified as a seven-point sequence. Similarly, the sequence given by (2.1.5) is a four-point sequence.

2.1.1 Some Elementary Discrete-Time Signals

In our study of discrete-time signals and systems there are a number of basic signals that appear often and play an important role. These signals are defined below.

1. The *unit sample sequence* is denoted as $\delta(n)$ and is defined as

n=...-2,-1, 0,1,2,3...
x(n)=0,0, 1,0,0,0...

$$\delta(n) \equiv \begin{cases} 1, & \text{for } n = 0 \\ 0, & \text{for } n \neq 0 \end{cases} \tag{2.1.6}$$

In words, the unit sample sequence is a signal that is zero everywhere, except at $n = 0$ where its value is unity. This signal is sometimes referred to as a *unit impulse*. In contrast to the analog signal $\delta(t)$, which is also called a unit impulse and is defined to be zero everywhere except $t = 0$, and has unit area, the unit sample sequence is much less mathematically complicated. The graphical representation of $\delta(n)$ is shown in Fig. 2.2.

2. The *unit step signal* is denoted as $u(n)$ and is defined as

n=...-2,-1, 0,1,2,3...
x(n)=0,0, 1,1,1,1..

$$u(n) \equiv \begin{cases} 1, & \text{for } n \geq 0 \\ 0, & \text{for } n < 0 \end{cases} \tag{2.1.7}$$

Figure 2.3 illustrates the unit step signal.

3. The *unit ramp signal* is denoted as $u_r(n)$ and is defined as

n=...-2,-1, 0,1,2,3...
x(n)=0, 0,0,1,2,3...

$$u_r(n) \equiv \begin{cases} n, & \text{for } n \geq 0 \\ 0, & \text{for } n < 0 \end{cases} \tag{2.1.8}$$

This signal is illustrated in Fig. 2.4.

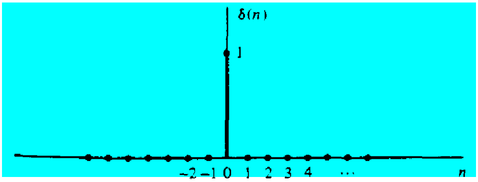


Figure 2.2 Graphical representation of the unit sample signal.

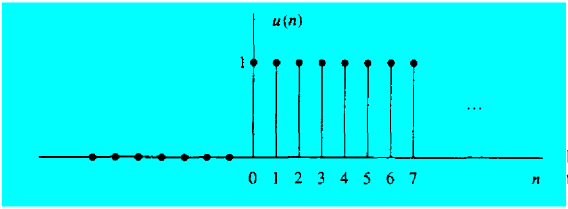


Figure 2.3 Graphical representation of the unit step signal.

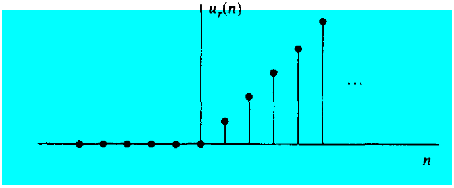


Figure 2.4 Graphical representation of the unit ramp signal.

4. The *exponential signal* is a sequence of the form

$$x(n) = a^n \qquad \text{for all } n \tag{2.1.9}$$

If the parameter a is real, then $x(n)$ is a real signal. Figure 2.5 illustrates $x(n)$ for various values of the parameter a.

When the parameter a is complex valued, it can be expressed as

$$a \equiv re^{j\theta}$$

where r and θ are now the parameters. Hence we can express $x(n)$ as

$$\begin{aligned} x(n) &= r^n e^{j\theta n} \\ &= r^n(\cos\theta n + j\sin\theta n) \end{aligned} \tag{2.1.10}$$

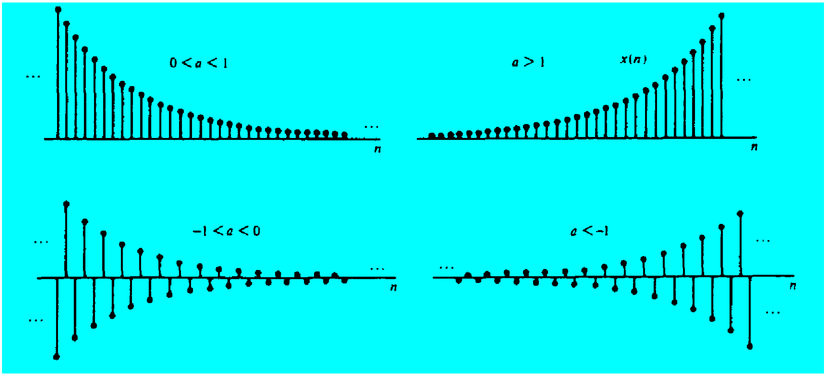


Figure 2.5 Graphical representation of exponential signals.

Since $x(n)$ is now complex valued, it can be represented graphically by plotting the real part

$$x_R(n) \equiv r^n \cos \theta n \tag{2.1.11}$$

as a function of n, and separately plotting the imaginary part

$$x_I(n) \equiv r^n \sin \theta n \tag{2.1.12}$$

as a function of n. Figure 2.6 illustrates the graphs of $x_R(n)$ and $x_I(n)$ for $r = 0.9$ and $\theta = \pi/10$. We observe that the signals $x_R(n)$ and $x_I(n)$ are a damped (decaying exponential) cosine function and a damped sine function. The angle variable θ is simply the frequency of the sinusoid, previously denoted by the (normalized) frequency variable ω. Clearly, if $r = 1$, the damping disappears and $x_R(n)$, $x_I(n)$, and $x(n)$ have a fixed amplitude, which is unity.

Alternatively, the signal $x(n)$ given by (2.1.10) can be represented graphically by the amplitude function

$$|x(n)| = A(n) \equiv r^n \tag{2.1.13}$$

and the phase function

$$\angle x(n) = \phi(n) \equiv \theta n \tag{2.1.14}$$

Figure 2.7 illustrates $A(n)$ and $\phi(n)$ for $r = 0.9$ and $\theta = \pi/10$. We observe that the phase function is linear with n. However, the phase is defined only over the interval $-\pi < \theta \leq \pi$ or, equivalently, over the interval $0 \leq \theta < 2\pi$. Consequently, by convention $\phi(n)$ is plotted over the finite interval $-\pi < \theta \leq \pi$ or $0 \leq \theta < 2\pi$. In other words, we subtract multiples of 2π from $\phi(n)$ before plotting. In one case, $\phi(n)$ is constrained to the range $-\pi < \theta \leq \pi$ and in the other case $\phi(n)$ is constrained to the range $0 \leq \theta < 2\pi$. The subtraction of multiples of 2π from $\phi(n)$ is equivalent to interpreting the function $\phi(n)$ as $\phi(n)$, modulo 2π. The graph for $\phi(n)$, modulo 2π, is shown in Fig. 2.7b.

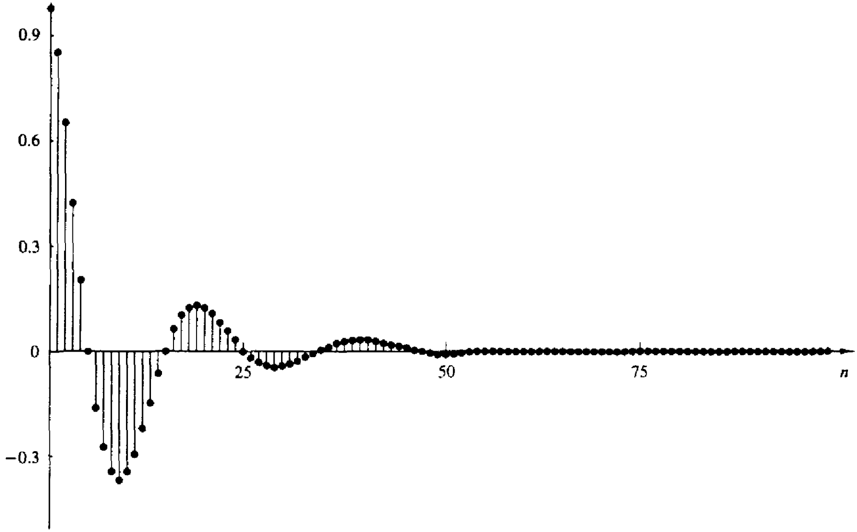
2.1.2 Classification of Discrete-Time Signals

The mathematical methods employed in the analysis of discrete-time signals and systems depend on the characteristics of the signals. In this section we classify discrete-time signals according to a number of different characteristics.

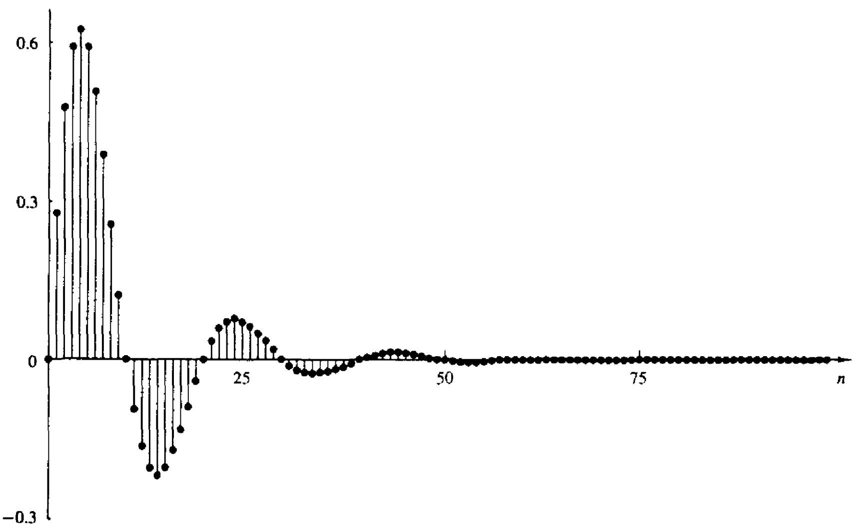
Energy signals and power signals. The energy E of a signal $x(n)$ is defined as

$$E \equiv \sum_{n=-\infty}^{\infty} |x(n)|^2 \tag{2.1.15}$$

We have used the magnitude-squared values of $x(n)$, so that our definition applies to complex-valued signals as well as real-valued signals. The energy of a signal can be finite or infinite. If E is finite (i.e., $0 < E < \infty$), then $x(n)$ is called an *energy*

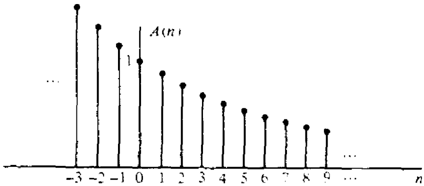


(a) Graph of $x_R(n) = (0.9)^n \cos \frac{\pi n}{10}$

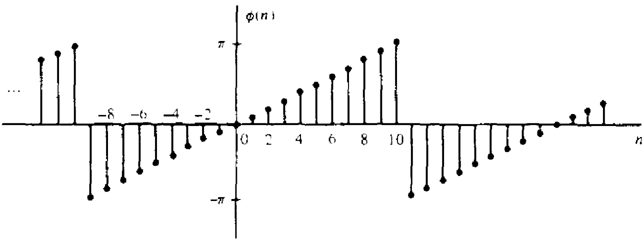


(b) Graph of $x_I(n) = (0.9)^n \sin \frac{\pi n}{10}$

Figure 2.6 Graph of the real and imaginary components of a complex-valued exponential signal.



(a) Graph of $A(n) = r^n$, $r = 0.9$



(b) Graph of $\phi(n) = \frac{\pi}{10} n$, modulo 2π plotted in the range $(-\pi, \pi]$

Figure 2.7 Graph of amplitude and phase function of a complex-valued exponential signal: (a) graph of $A(n) = r^n$, $r = 0.9$; (b) graph of $\phi(n) = (\pi/10)n$, modulo 2π plotted in the range $(-\pi, \pi]$.

signal. Sometimes we add a subscript x to E and write E_x to emphasize that E_x is the energy of the signal $x(n)$.

Many signals that possess infinite energy, have a finite average power. The average power of a discrete-time signal $x(n)$ is defined as

$$P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2 \tag{2.1.16}$$

If we define the signal energy of $x(n)$ over the finite interval $-N \le n \le N$ as

$$E_N \equiv \sum_{n=-N}^{N} |x(n)|^2 \tag{2.1.17}$$

then we can express the signal energy E as

$$E \equiv \lim_{N \to \infty} E_N \tag{2.1.18}$$

and the average power of the signal $x(n)$ as

$$P \equiv \lim_{N \to \infty} \frac{1}{2N+1} E_N \tag{2.1.19}$$

Clearly, if E is finite, $P = 0$. On the other hand, if E is infinite, the average power P may be either finite or infinite. If P is finite (and nonzero), the signal is called a *power signal*. The following example illustrates such a signal.

Example 2.1.1

Determine the power and energy of the unit step sequence. The average power of the unit step signal is $P \equiv \lim_{N\to\infty} \frac{1}{2N+1} E_N$ $E_N \equiv \sum_{n=-N}^{N} |x(n)|^2$

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=0}^{N} u^2(n)$$

$$= \lim_{N\to\infty} \frac{N+1}{2N+1} = \lim_{N\to\infty} \frac{1+1/N}{2+1/N} = \frac{1}{2}$$

Consequently, the unit step sequence is a power signal. Its energy is infinite.

Similarly, it can be shown that the complex exponential sequence $x(n) = Ae^{j\omega_0 n}$ has average power A^2, so it is a power signal. On the other hand, the unit ramp sequence is neither a power signal nor an energy signal.

Periodic signals and aperiodic signals. As defined on Section 1.3, a signal $x(n)$ is periodic with period $N (N > 0)$ if and only if

$$x(n + N) = x(n) \text{ for all } n \tag{2.1.20}$$

The smallest value of N for which (2.1.20) holds is called the (fundamental) period. If there is no value of N that satisfies (2.1.20), the signal is called *nonperiodic* or *aperiodic*.

We have already observed that the sinusoidal signal of the form

$$x(n) = A \sin 2\pi f_0 n \tag{2.1.21}$$

is periodic when f_0 is a rational number, that is, if f_0 can be expressed as

$$f_0 = \frac{k}{N} \tag{2.1.22}$$

where k and N are integers.

The energy of a periodic signal $x(n)$ over a single period, say, over the interval $0 \le n \le N-1$, is finite if $x(n)$ takes on finite values over the period. However, the energy of the periodic signal for $-\infty \le n \le \infty$ is infinite. On the other hand, the average power of the periodic signal is finite and it is equal to the average power over a single period. Thus if $x(n)$ is a periodic signal with fundamental period N and takes on finite values, its power is given by

$$P = \frac{1}{N} \sum_{n=0}^{N-1} |x(n)|^2 \tag{2.1.23}$$

Consequently, periodic signals are power signals.

Symmetric (even) and antisymmetric (odd) signals. A real-valued signal $x(n)$ is called symmetric (even) if

$$x(-n) = x(n) \tag{2.1.24}$$

On the other hand, a signal $x(n)$ is called antisymmetric (odd) if

$$x(-n) = -x(n) \tag{2.1.25}$$

We note that if $x(n)$ is odd, then $x(0) = 0$. Examples of signals with even and odd symmetry are illustrated in Fig. 2.8.

We wish to illustrate that any arbitrary signal can be expressed as the sum of two signal components, one of which is even and the other odd. The even signal component is formed by adding $x(n)$ to $x(-n)$ and dividing by 2, that is,

$$x_e(n) = \frac{1}{2}[x(n) + x(-n)] \tag{2.1.26}$$

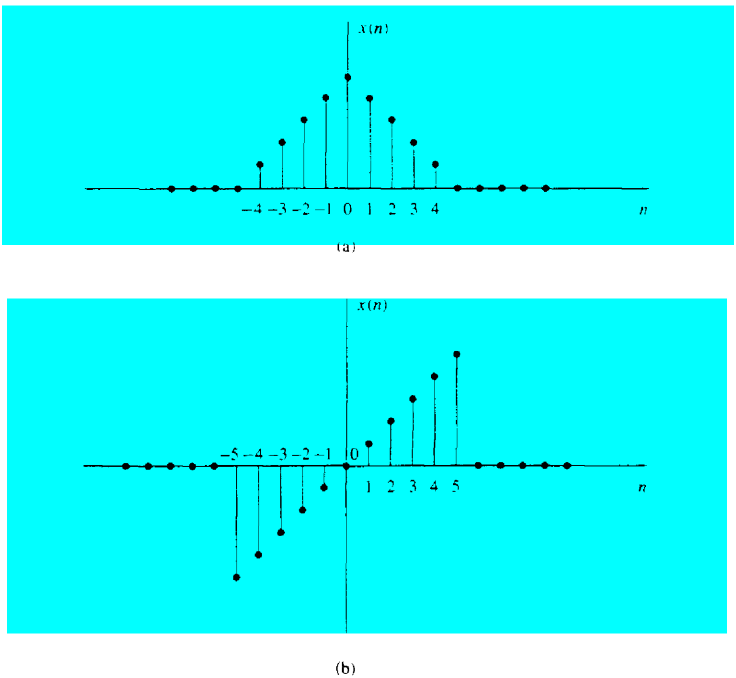


Figure 2.8 Example of even (a) and odd (b) signals.

Clearly, $x_e(n)$ satisfies the symmetry condition (2.1.24). Similarly, we form an odd signal component $x_o(n)$ according to the relation

$$x_o(n) = \frac{1}{2}[x(n) - x(-n)] \tag{2.1.27}$$

Again, it is clear that $x_o(n)$ satisfies (2.1.25); hence it is indeed odd. Now, if we add the two signal components, defined by (2.1.26) and (2.1.27), we obtain $x(n)$, that is,

$$x(n) = x_e(n) + x_o(n) \tag{2.1.28}$$

Thus any arbitrary signal can be expressed as in (2.1.28).

2.1.3 Simple Manipulations of Discrete-Time Signals

In this section we consider some simple modifications or manipulations involving the independent variable and the signal amplitude (dependent variable).

Transformation of the independent variable (time). A signal $x(n)$ may be shifted in time by replacing the independent variable n by $n-k$, where k is an integer. If k is a positive integer, the time shift results in a delay of the signal by k units of time. If k is a negative integer, the time shift results in an advance of the signal by $|k|$ units in time.

Example 2.1.2

A signal $x(n)$ is graphically illustrated in Fig. 2.9a. Show a graphical representation of the signals $x(n-3)$ and $x(n+2)$.

Solution The signal $x(n-3)$ is obtained by delaying $x(n)$ by three units in time. The result is illustrated in Fig. 2.9b. On the other hand, the signal $x(n+2)$ is obtained by advancing $x(n)$ by two units in time. The result is illustrated in Fig. 2.9c. Note that delay corresponds to shifting a signal to the right, whereas advance implies shifting the signal to the left on the time axis.

If the signal $x(n)$ is stored on magnetic tape or on a disk or, perhaps, in the memory of a computer, it is a relatively simple operation to modify the base by introducing a delay or an advance. On the other hand, if the signal is not stored but is being generated by some physical phenomenon in real time, it is not possible to advance the signal in time, since such an operation involves signal samples that have not yet been generated. Whereas it is always possible to insert a delay into signal samples that have already been generated, it is physically impossible to view the future signal samples. Consequently, in real-time signal processing applications, the operation of advancing the time base of the signal is physically unrealizable.

Another useful modification of the time base is to replace the independent variable n by $-n$. The result of this operation is a *folding* or a *reflection* of the signal about the time origin $n=0$.