say M_x and M_y, such that

$$|x(n)| \le M_x < \infty \qquad |y(n)| \le M_y < \infty \tag{2.2.45}$$

for all n. If, for some bounded input sequence $x(n)$, the output is unbounded (infinite), the system is classified as unstable.

Example 2.2.7

Consider the nonlinear system described by the input–output equation

$$y(n) = y^2(n-1) + x(n)$$

As an input sequence we select the bounded signal

$$x(n) = C\delta(n)$$

where C is a constant. We also assume that $y(-1) = 0$. Then the output sequence is

$$y(0) = C, \quad y(1) = C^2, \quad y(2) = C^4, \quad \ldots, \quad y(n) = C^{2^n}$$

Clearly, the output is unbounded when $1 < |C| < \infty$. Therefore, the system is BIBO unstable, since a bounded input sequence has resulted in an unbounded output.

2.2.4 Interconnection of Discrete-Time Systems

Discrete-time systems can be interconnected to form larger systems. There are two basic ways in which systems can be interconnected: in cascade (series) or in parallel. These interconnections are illustrated in Fig. 2.21. Note that the two interconnected systems are different.

In the cascade interconnection the output of the first system is

$$y_1(n) = \mathcal{T}_1[x(n)] \tag{2.2.46}$$

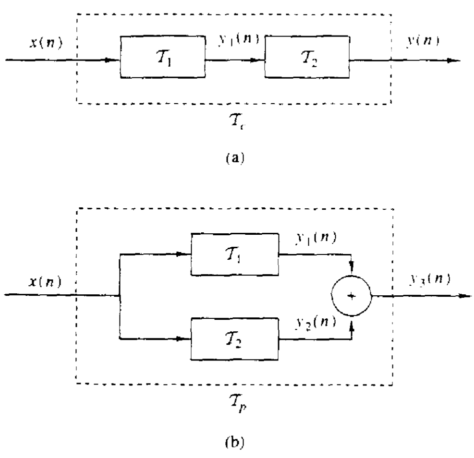


Figure 2.21 Cascade (a) and parallel (b) interconnections of systems.

and the output of the second system is

$$\begin{aligned} y(n) &= \mathcal{T}_2[y_1(n)] \\ &= \mathcal{T}_2\{\mathcal{T}_1[x(n)]\} \end{aligned} \tag{2.2.47}$$

We observe that systems $\mathcal{T}_1$ and $\mathcal{T}_2$ can be combined or consolidated into a single overall system

$$\mathcal{T}_c \equiv \mathcal{T}_2\mathcal{T}_1 \tag{2.2.48}$$

Consequently, we can express the output of the combined system as

$$y(n) = \mathcal{T}_c[x(n)]$$

In general, the order in which the operations $\mathcal{T}_1$ and $\mathcal{T}_2$ are performed is important. That is,

$$\mathcal{T}_2\mathcal{T}_1 \neq \mathcal{T}_1\mathcal{T}_2$$

for arbitrary systems. However, if the systems $\mathcal{T}_1$ and $\mathcal{T}_2$ are linear and time invariant, then (a) $\mathcal{T}_c$ is time invariant and (b) $\mathcal{T}_2\mathcal{T}_1 = \mathcal{T}_1\mathcal{T}_2$, that is, the order in which the systems process the signal is not important. $\mathcal{T}_2\mathcal{T}_1$ and $\mathcal{T}_1\mathcal{T}_2$ yield identical output sequences.

The proof of (a) follows. The proof of (b) is given in Section 2.3.4. To prove time invariance, suppose that $\mathcal{T}_1$ and $\mathcal{T}_2$ are time invariant; then

$$x(n-k) \xrightarrow{\mathcal{T}_1} y_1(n-k)$$

and

$$y_1(n-k) \xrightarrow{\mathcal{T}_2} y(n-k)$$

Thus

$$x(n-k) \xrightarrow{\mathcal{T}_c = \mathcal{T}_2\mathcal{T}_1} y(n-k)$$

and therefore, $\mathcal{T}_c$ is time invariant.

In the parallel interconnection, the output of the system $\mathcal{T}_1$ is $y_1(n)$ and the output of the system $\mathcal{T}_2$ is $y_2(n)$. Hence the output of the parallel interconnection is

$$\begin{aligned} y_3(n) &= y_1(n) + y_2(n) \\ &= \mathcal{T}_1[x(n)] + \mathcal{T}_2[x(n)] \\ &= (\mathcal{T}_1 + \mathcal{T}_2)[x(n)] \\ &= \mathcal{T}_p[x(n)] \end{aligned}$$

where $\mathcal{T}_p = \mathcal{T}_1 + \mathcal{T}_2$.

In general, we can use parallel and cascade interconnection of systems to construct larger, more complex systems. Conversely, we can take a larger system and break it down into smaller subsystems for purposes of analysis and implementation. We shall use these notions later, in the design and implementation of digital filters.

system to determine the formula for the output given any arbitrary input. This development is described in detail as follows.

2.3.2 Resolution of a Discrete-Time Signal into Impulses

Suppose we have an arbitrary signal $x(n)$ that we wish to resolve into a sum of unit sample sequences. To utilize the notation established in the preceding section, we select the elementary signals $x_k(n)$ to be

$$x_k(n) = \delta(n-k) \tag{2.3.7}$$

where k represents the delay of the unit sample sequence. To handle an arbitrary signal $x(n)$ that may have nonzero values over an infinite duration, the set of unit impulses must also be infinite, to encompass the infinite number of delays.

Now suppose that we multiply the two sequences $x(n)$ and $\delta(n-k)$. Since $\delta(n-k)$ is zero everywhere except at $n = k$, where its value is unity, the result of this multiplication is another sequence that is zero everywhere except at $n = k$, where its value is $x(k)$, as illustrated in Fig. 2.22. Thus

$$x(n)\delta(n-k) = x(k)\delta(n-k) \tag{2.3.8}$$

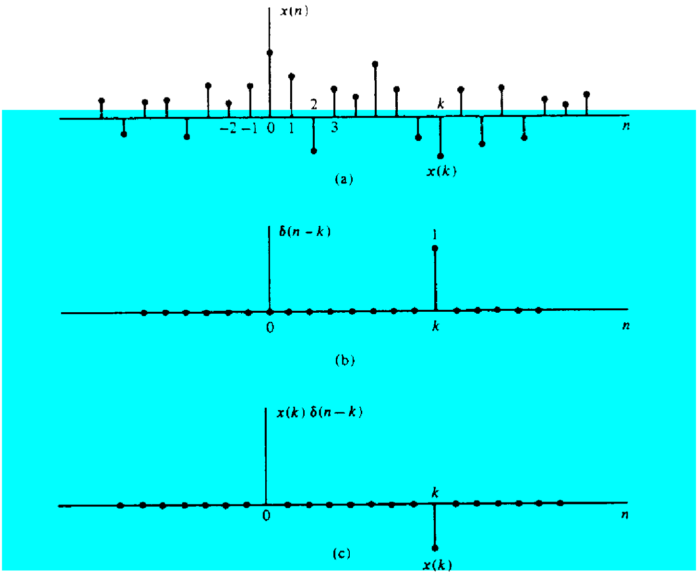


Figure 2.22 Multiplication of a signal $x(n)$ with a shifted unit sample sequence.

is a sequence that is zero everywhere except at $n = k$, where its value is $x(k)$. If we were to repeat the multiplication of $x(n)$ with $\delta(n-m)$, where m is another delay $(m \neq k)$, the result will be a sequence that is zero everywhere except at $n = m$, where its value is $x(m)$. Hence

$$x(n)\delta(n-m) = x(m)\delta(n-m) \tag{2.3.9}$$

In other words, each multiplication of the signal $x(n)$ by a unit impulse at some delay k, [i.e., $\delta(n-k)$], in essence picks out the single value $x(k)$ of the signal $x(n)$ at the delay where the unit impulse is nonzero. Consequently, if we repeat this multiplication over all possible delays, $-\infty < k < \infty$, and sum all the product sequences, the result will be a sequence equal to the sequence $x(n)$, that is,

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k) \tag{2.3.10}$$

We emphasize that the right-hand side of (2.3.10) is the summation of an infinite number of unit sample sequences where the unit sample sequence $\delta(n-k)$ has an amplitude value of $x(k)$. Thus the right-hand side of (2.3.10) gives the resolution of or decomposition of any arbitrary signal $x(n)$ into a weighted (scaled) sum of shifted unit sample sequences.

Example 2.3.1

Consider the special case of a finite-duration sequence given as

$$x(n) = \{2, \underset{\uparrow}{4}, 0, 3\}$$

Resolve the sequence $x(n)$ into a sum of weighted impulse sequences.

Solution Since the sequence $x(n)$ is nonzero for the time instants $n = -1, 0, 2$, we need three impulses at delays $k = -1, 0, 2$. Following (2.3.10) we find that

$$x(n) = 2\delta(n+1) + 4\delta(n) + 3\delta(n-2)$$

2.3.3 Response of LTI Systems to Arbitrary Inputs: The Convolution Sum

Having resolved an arbitrary input signal $x(n)$ into a weighted sum of impulses, we are now ready to determine the response of any relaxed linear system to any input signal. First, we denote the response $y(n, k)$ of the system to the input unit sample sequence at $n = k$ by the special symbol $h(n, k)$, $-\infty < k < \infty$. That is,

$$y(n, k) \equiv h(n, k) = \mathcal{T}[\delta(n-k)] \tag{2.3.11}$$

In (2.3.11) we note that n is the time index and k is a parameter showing the location of the input impulse. If the impulse at the input is scaled by an amount $c_k \equiv x(k)$, the response of the system is the correspondingly scaled output, that is,

$$c_k h(n, k) = x(k)h(n, k) \tag{2.3.12}$$

Finally, if the input is the arbitrary signal $x(n)$ that is expressed as a sum of weighted impulses, that is,

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k) \tag{2.3.13}$$

then the response of the system to $x(n)$ is the corresponding sum of weighted outputs, that is,

$$\begin{aligned} y(n) = \mathcal{T}[x(n)] &= \mathcal{T}\left[\sum_{k=-\infty}^{\infty} x(k)\delta(n-k)\right] \\ &= \sum_{k=-\infty}^{\infty} x(k)\mathcal{T}[\delta(n-k)] \\ &= \sum_{k=-\infty}^{\infty} x(k)h(n,k) \end{aligned} \tag{2.3.14}$$

Clearly, (2.3.14) follows from the superposition property of linear systems, and is known as the *superposition summation*.

We note that (2.3.14) is an expression for the response of a linear system to any arbitrary input sequence $x(n)$. This expression is a function of both $x(n)$ and the responses $h(n, k)$ of the system to the unit impulses $\delta(n-k)$ for $-\infty < k < \infty$. In deriving (2.3.14) we used the linearity property of the system but not its time-invariance property. Thus the expression in (2.3.14) applies to any relaxed linear (time-variant) system.

If, in addition, the system is time invariant, the formula in (2.3.14) simplifies considerably. In fact, if the response of the LTI system to the unit sample sequence $\delta(n)$ is denoted as $h(n)$, that is,

$$h(n) \equiv \mathcal{T}[\delta(n)] \tag{2.3.15}$$

then by the time-invariance property, the response of the system to the delayed unit sample sequence $\delta(n-k)$ is

$$h(n-k) = \mathcal{T}[\delta(n-k)] \tag{2.3.16}$$

Consequently, the formula in (2.3.14) reduces to

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \tag{2.3.17}$$

Now we observe that the relaxed LTI system is completely characterized by a single function $h(n)$, namely, its response to the unit sample sequence $\delta(n)$. In contrast, the general characterization of the output of a time-variant, linear system requires an infinite number of unit sample response functions, $h(n, k)$, one for each possible delay.

The formula in (2.3.17) that gives the response $y(n)$ of the LTI system as a function of the input signal $x(n)$ and the unit sample (impulse) response $h(n)$ is called a *convolution sum*. We say that the input $x(n)$ is convolved with the impulse response $h(n)$ to yield the output $y(n)$. We shall now explain the procedure for computing the response $y(n)$, both mathematically and graphically, given the input $x(n)$ and the impulse response $h(n)$ of the system.

Suppose that we wish to compute the output of the system at some time instant, say $n = n_0$. According to (2.3.17), the response at $n = n_0$ is given as

$$y(n_0) = \sum_{k=-\infty}^{\infty} x(k)h(n_0 - k) \tag{2.3.18}$$

Our first observation is that the index in the summation is k, and hence both the input signal $x(k)$ and the impulse response $h(n_0 - k)$ are functions of k. Second, we observe that the sequences $x(k)$ and $h(n_0 - k)$ are multiplied together to form a product sequence. The output $y(n_0)$ is simply the sum over all values of the product sequence. The sequence $h(n_0 - k)$ is obtained from $h(k)$ by, first, folding $h(k)$ about $k = 0$ (the time origin), which results in the sequence $h(-k)$. The folded sequence is then shifted by n_0 to yield $h(n_0 - k)$. To summarize, the process of computing the convolution between $x(k)$ and $h(k)$ involves the following four steps.

1. *Folding*. Fold $h(k)$ about $k = 0$ to obtain $h(-k)$.
2. *Shifting*. Shift $h(-k)$ by n_0 to the right (left) if n_0 is positive (negative), to obtain $h(n_0 - k)$.
3. *Multiplication*. Multiply $x(k)$ by $h(n_0 - k)$ to obtain the product sequence $v_{n_0}(k) \equiv x(k)h(n_0 - k)$.
4. *Summation*. Sum all the values of the product sequence $v_{n_0}(k)$ to obtain the value of the output at time $n = n_0$.

We note that this procedure results in the response of the system at a single time instant, say $n = n_0$. In general, we are interested in evaluating the response of the system over all time instants $-\infty < n < \infty$. Consequently, steps 2 through 4 in the summary must be repeated, for all possible time shifts $-\infty < n < \infty$.

In order to gain a better understanding of the procedure for evaluating the convolution sum, we shall demonstrate the process graphically. The graphs will aid us in explaining the four steps involved in the computation of the convolution sum.

Example 2.3.2

The impulse response of a linear time-invariant system is

$$h(n) = \{1, \underset{\uparrow}{2}, 1, -1\} \tag{2.3.19}$$

Determine the response of the system to the input signal

$$x(n) = \{\underset{\uparrow}{1}, 2, 3, 1\} \tag{2.3.20}$$

Solution We shall compute the convolution according to the formula (2.3.17), but we shall use graphs of the sequences to aid us in the computation. In Fig. 2.23a we illustrate the input signal sequence $x(k)$ and the impulse response $h(k)$ of the system, using k as the time index in order to be consistent with (2.3.17).

The first step in the computation of the convolution sum is to fold $h(k)$. The folded sequence $h(-k)$ is illustrated in Fig. 2.23b. Now we can compute the output at $n = 0$, according to (2.3.17), which is

$$y(0) = \sum_{k=-\infty}^{\infty} x(k)h(-k) \tag{2.3.21}$$

Since the shift $n = 0$, we use $h(-k)$ directly without shifting it. The product sequence

$$v_0(k) \equiv x(k)h(-k) \tag{2.3.22}$$

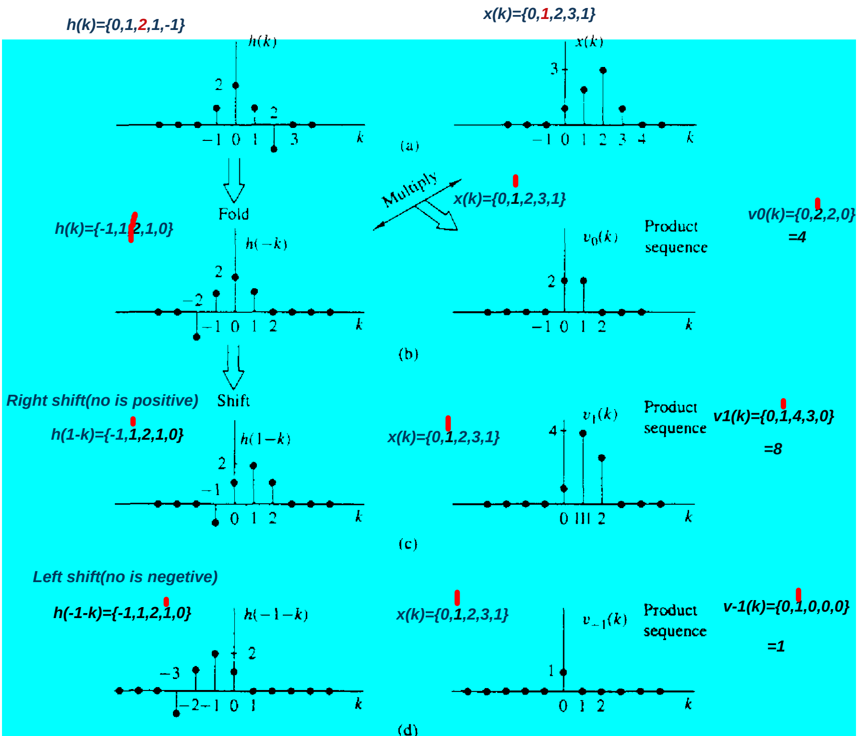


Figure 2.23 Graphical computation of convolution.

$$y(n) = \{\ldots, 0, 0, 1, \underset{\uparrow}{4}, 8, 8, 3, -2, -1, 0, 0, \ldots\}$$

is also shown in Fig. 2.23b. Finally, the sum of all the terms in the product sequence yields

$$y(0) = \sum_{k=-\infty}^{\infty} v_0(k) = 4$$

We continue the computation by evaluating the response of the system at $n = 1$. According to (2.3.17),

$$y(1) = \sum_{k=-\infty}^{\infty} x(k)h(1-k) \tag{2.3.23}$$

The sequence $h(1-k)$ is simply the folded sequence $h(-k)$ shifted to the right by one unit in time. This sequence is illustrated in Fig. 2.23c. The product sequence

$$v_1(k) = x(k)h(1-k) \tag{2.3.24}$$

is also illustrated in Fig. 2.23c. Finally, the sum of all the values in the product sequence yields

$$y(1) = \sum_{k=-\infty}^{\infty} v_1(k) = 8$$

In a similar manner, we obtain $y(2)$ by shifting $h(-k)$ two units to the right, forming the product sequence $v_2(k) = x(k)h(2-k)$ and then summing all the terms in the product sequence obtaining $y(2) = 8$. By shifting $h(-k)$ farther to the right, multiplying the corresponding sequence, and summing over all the values of the resulting product sequences, we obtain $y(3) = 3$, $y(4) = -2$, $y(5) = -1$. For $n > 5$, we find that $y(n) = 0$ because the product sequences contain all zeros. Thus we have obtained the response $y(n)$ for $n > 0$.

Next we wish to evaluate $y(n)$ for $n < 0$. We begin with $n = -1$. Then

$$y(-1) = \sum_{k=-\infty}^{\infty} x(k)h(-1-k) \tag{2.3.25}$$

Now the sequence $h(-1-k)$ is simply the folded sequence $h(-k)$ shifted one time unit to the left. The resulting sequence is illustrated in Fig. 2.23d. The corresponding product sequence is also shown in Fig. 2.23d. Finally, summing over the values of the product sequence, we obtain

$$y(-1) = 1$$

From observation of the graphs of Fig. 2.23, it is clear that any further shifts of $h(-1-k)$ to the left always results in an all-zero product sequence, and hence

$$y(n) = 0 \qquad \text{for } n \le -2$$

Now we have the entire response of the system for $-\infty < n < \infty$, which we summarize below as

$$y(n) = \{\ldots, 0, 0, 1, \underset{\uparrow}{4}, 8, 8, 3, -2, -1, 0, 0, \ldots\} \tag{2.3.26}$$

In Example 2.3.2 we illustrated the computation of the convolution sum, using graphs of the sequences to aid us in visualizing the steps involved in the computation procedure.

Before working out another example, we wish to show that the convolution operation is commutative in the sense that it is irrelevant which of the two sequences is folded and shifted. Indeed, if we begin with (2.3.17) and make a change in the variable of the summation, from k to m, by defining a new index $m = n - k$, then $k = n - m$ and (2.3.17) becomes

$$y(n) = \sum_{m=-\infty}^{\infty} x(n-m)h(m) \qquad (2.3.27)$$

Since m is a dummy index, we may simply replace m by k so that

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

$$y(n) = \sum_{k=-\infty}^{\infty} x(n-k)h(k) \qquad (2.3.28)$$

The expression in (2.3.28) involves leaving the impulse response $h(k)$ unaltered, while the input sequence is folded and shifted. Although the output $y(n)$ in (2.3.28) is identical to (2.3.17), the product sequences in the two forms of the convolution formula are not identical. In fact, if we define the two product sequences as

$$v_n(k) = x(k)h(n-k)$$

$$w_n(k) = x(n-k)h(k)$$

it can be easily shown that

$$v_n(k) = w_n(n-k)$$

and therefore,

$$y(n) = \sum_{k=-\infty}^{\infty} v_n(k) = \sum_{k=-\infty}^{\infty} w_n(n-k)$$

since both sequences contain the same sample values in a different arrangement.

Example 2.3.3

Determine the output $y(n)$ of a relaxed linear time-invariant system with impulse response

$$h(n) = a^n u(n), \; |a| < 1$$

when the input is a unit step sequence, that is,

$$x(n) = u(n)$$

Solution In this case both $h(n)$ and $x(n)$ are infinite-duration sequences. We use the form of the convolution formula given by (2.3.28) in which $x(k)$ is folded. The

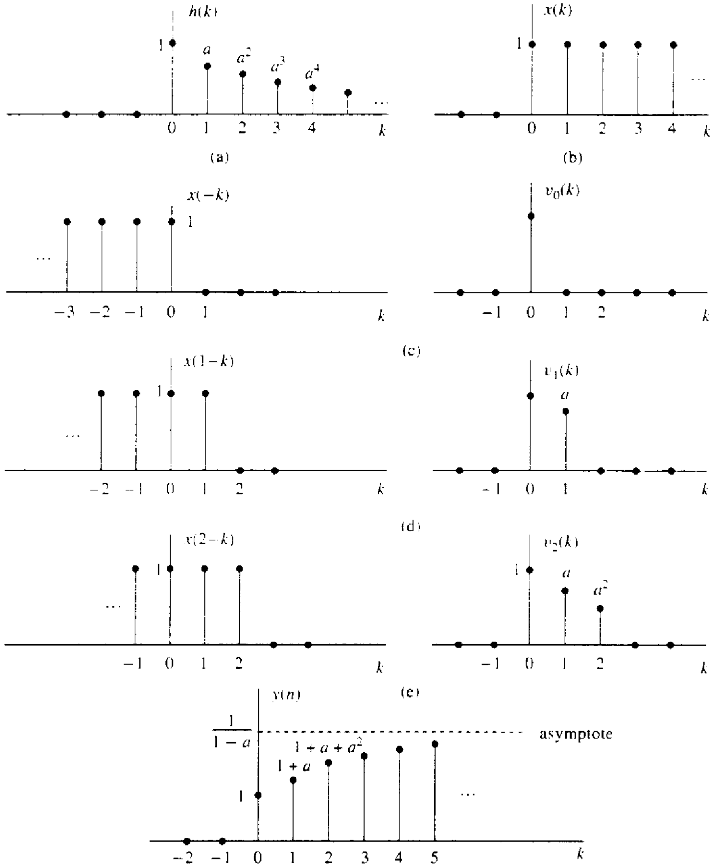


Figure 2.24 Graphical computation of convolution in Example 2.3.3.

sequences $h(k)$, $x(k)$, and $x(-k)$ are shown in Fig. 2.24. The product sequences $v_0(k)$, $v_1(k)$, and $v_2(k)$ corresponding to $x(-k)h(k)$, $x(1-k)h(k)$, and $x(2-k)h(k)$ are illustrated in Fig. 2.24c, d, and e, respectively. Thus we obtain the outputs

$$y(0) = 1$$

$$y(1) = 1 + a$$

$$y(2) = 1 + a + a^2$$

Clearly, for $n > 0$, the output is

$$\begin{aligned} y(n) &= 1 + a + a^2 + \cdots + a^n \\ &= \frac{1 - a^{n+1}}{1 - a} \end{aligned} \tag{2.3.29}$$

On the other hand, for $n < 0$, the product sequences consist of all zeros. Hence

$$y(n) = 0 \qquad n < 0$$

A graph of the output $y(n)$ is illustrated in Fig. 2.24f for the case $0 < a < 1$. Note the exponential rise in the output as a function of n. Since $|a| < 1$, the final value of the output as n approaches infinity is

$$y(\infty) = \lim_{n\to\infty} y(n) = \frac{1}{1-a} \tag{2.3.30}$$

To summarize, the convolution formula provides us with a means for computing the response of a relaxed, linear time-invariant system to any arbitrary input signal $x(n)$. It takes one of two equivalent forms, either (2.3.17) or (2.3.28), where $x(n)$ is the input signal to the system, h(n) is the impulse response of the system, and $y(n)$ is the *output* of the system in response to the input signal $x(n)$. The evaluation of the convolution formula involves four operations, namely: *folding* either the impulse response as specified by (2.3.17) or the input sequence as specified by (2.3.28) to yield either $h(-k)$ or $x(-k)$, respectively, *shifting* the folded sequence by n units in time to yield either $h(n-k)$ or $x(n-k)$, *multiplying* the two sequences to yield the product sequence, either $x(k)h(n-k)$ or $x(n-k)h(k)$, and finally *summing* all the values in the product sequence to yield the output $y(n)$ of the system at time n. The folding operation is done only once. However, the other three operations are repeated for all possible shifts $-\infty < n < \infty$ in order to obtain $y(n)$ for $-\infty < n < \infty$.

2.3.4 Properties of Convolution and the Interconnection of LTI Systems

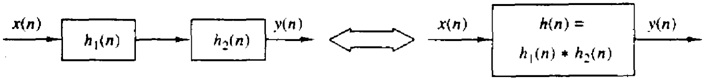
In this section we investigate some important properties of convolution and interpret these properties in terms of interconnecting linear time-invariant systems. We should stress that these properties hold for every input signal.

It is convenient to simplify the notation by using an asterisk to denote the convolution operation. Thus

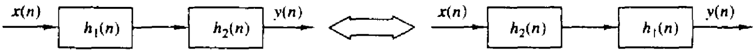
$$y(n) = x(n) * h(n) \equiv \sum_{k=-\infty}^{\infty} x(k)h(n-k) \tag{2.3.31}$$

In this notation the sequence following the asterisk [i.e., the impulse response $h(n)$] is folded and shifted. The input to the system is $x(n)$. On the other hand, we also showed that

$$y(n) = h(n) * x(n) \equiv \sum_{k=-\infty}^{\infty} h(k)x(n-k) \tag{2.3.32}$$



(a)



(b)

Figure 2.26 Implications of the associative (a) and the associative and commutative (b) properties of convolution.

Example 2.3.4

Determine the impulse response for the cascade of two linear time-invariant systems having impulse responses

$$h_1(n) = (\tfrac{1}{2})^n u(n)$$

and

$$h_2(n) = (\tfrac{1}{4})^n u(n)$$

Solution To determine the overall impulse response of the two systems in cascade, we simply convolve $h_1(n)$ with $h_2(n)$. Hence

$$h(n) = \sum_{k=-\infty}^{\infty} h_1(k)h_2(n-k)$$

where $h_2(n)$ is folded and shifted. We define the product sequence

$$\begin{aligned} v_n(k) &= h_1(k)h_2(n-k) \\ &= (\tfrac{1}{2})^k(\tfrac{1}{4})^{n-k} \end{aligned}$$

which is nonzero for $k \geq 0$ and $n-k \geq 0$ or $n \geq k \geq 0$. On the other hand, for $n < 0$, we have $v_n(k) = 0$ for all k, and hence

$$h(n) = 0, n < 0$$

For $n \geq k \geq 0$, the sum of the values of the product sequence $v_n(k)$ over all k yields

$$\begin{aligned} h(n) &= \sum_{k=0}^{n} (\tfrac{1}{2})^k(\tfrac{1}{4})^{n-k} \\ &= (\tfrac{1}{4})^n \textstyle\sum_{k=0}^{n} 2^k \\ &= (\tfrac{1}{4})^n(2^{n+1}-1) \\ &= (\tfrac{1}{2})^n[2-(\tfrac{1}{2})^n], n \geq 0 \end{aligned}$$

2.3.5 Causal Linear Time-Invariant Systems

In Section 2.2.3 we defined a causal system as one whose output at time n depends only on present and past inputs but does not depend on future inputs. In other words, the output of the system at some time instant n, say $n = n_0$, depends only on values of $x(n)$ for $n \leq n_0$.

In the case of a linear time-invariant system, causality can be translated to a condition on the impulse response. To determine this relationship, let us consider a linear time-invariant system having an output at time $n = n_0$ given by the convolution formula

$$y(n_0) = \sum_{k=-\infty}^{\infty} h(k)x(n_0 - k)$$

Suppose that we subdivide the sum into two sets of terms, one set involving present and past values of the input [i.e., $x(n)$ for $n \leq n_0$] and one set involving future values of the input [i.e., $x(n)$, $n > n_0$]. Thus we obtain

$$\begin{aligned} y(n_0) &= \sum_{k=0}^{\infty} h(k)x(n_0 - k) + \sum_{k=-\infty}^{-1} h(k)x(n_0 - k) \\ &= [h(0)x(n_0) + h(1)x(n_0 - 1) + h(2)x(n_0 - 2) + \cdots] \\ &\quad + [h(-1)x(n_0 + 1) + h(-2)x(n_0 + 2) + \cdots] \end{aligned}$$

We observe that the terms in the first sum involve $x(n_0)$, $x(n_0 - 1)$, ..., which are the present and past values of the input signal. On the other hand, the terms in the second sum involve the input signal components $x(n_0+1)$, $x(n_0+2)$, ... Now, if the output at time $n = n_0$ is to depend only on the present and past inputs, then, clearly, the impulse response of the system must satisfy the condition

$$h(n) = 0 \qquad n < 0 \tag{2.3.38}$$

Since $h(n)$ is the response of the relaxed linear time-invariant system to a unit impulse applied at $n = 0$, it follows that $h(n) = 0$ for $n < 0$ is both a necessary and a sufficient condition for causality. Hence an *LTI system is causal if and only if its impulse response is zero for negative values of n.*

Since for a causal system, $h(n) = 0$ for $n < 0$, the limits on the summation of the convolution formula may be modified to reflect this restriction. Thus we have the two equivalent forms

$$y(n) = \sum_{k=0}^{\infty} h(k)x(n-k) \tag{2.3.39}$$

$$= \sum_{k=-\infty}^{n} x(k)h(n-k) \tag{2.3.40}$$

As indicated previously, causality is required in any real-time signal processing application, since at any given time n we have no access to future values of the input signal. Only the present and past values of the input signal are available in computing the present output.

It is sometimes convenient to call a sequence that is zero for $n < 0$, a *causal sequence*, and one that is nonzero for $n < 0$ and $n > 0$, a *noncausal sequence*. This terminology means that such a sequence could be the unit sample response of a causal or a noncausal system, respectively.

If the input to a causal linear time-invariant system is a causal sequence [i.e., if $x(n) = 0$ for $n < 0$], the limits on the convolution formula are further restricted. In this case the two equivalent forms of the convolution formula become

$$y(n) = \sum_{k=0}^{n} h(k)x(n-k) \tag{2.3.41}$$

$$= \sum_{k=0}^{n} x(k)h(n-k) \tag{2.3.42}$$

We observe that in this case, the limits on the summations for the two alternative forms are identical, and the upper limit is growing with time. Clearly, the response of a causal system to a causal input sequence is causal, since $y(n) = 0$ for $n < 0$.

Example 2.3.5

Determine the unit step response of the linear time-invariant system with impulse response

$$h(n) = a^n u(n) \qquad |a| < 1$$

Solution Since the input signal is a unit step, which is a causal signal, and the system is also causal, we can use one of the special forms of the convolution formula, either (2.3.41) or (2.3.42). Since $x(n) = 1$ for $n \geq 0$, (2.3.41) is simpler to use. Because of the simplicity of this problem, one can skip the steps involved with sketching the folded and shifted sequences. Instead, we use direct substitution of the signals sequences in (2.3.41) and obtain

$$y(n) = \sum_{k=0}^{n} a^k$$

$$= \frac{1 - a^{n+1}}{1 - a}$$

and $y(n) = 0$ for $n < 0$. We note that this result is identical to that obtained in Example 2.3.3. In this simple case, however, we computed the convolution algebraically without resorting to the detailed procedure outlined previously.

2.3.6 Stability of Linear Time-Invariant Systems

As indicated previously, stability is an important property that must be considered in any practical implementation of a system. We defined an arbitrary relaxed system as BIBO stable if and only if its output sequence $y(n)$ is bounded for every bounded input $x(n)$.

The condition in (2.3.43) implies that the impulse response $h(n)$ goes to zero as n approaches infinity. As a consequence, the output of the system goes to zero as n approaches infinity if the input is set to zero beyond $n > n_0$. To prove this, suppose that $|x(n)| < M_x$ for $n < n_0$ and $x(n) = 0$ for $n \geq n_0$. Then, at $n = n_0 + N$, the system output is

$$y(n_0 + N) = \sum_{k=-\infty}^{N-1} h(k)x(n_0 + N - k) + \sum_{k=N}^{\infty} h(k)x(n_0 + N - k)$$

But the first sum is zero since $x(n) = 0$ for $n \geq n_0$. For the remaining part, we take the absolute value of the output, which is

$$\begin{aligned} |y(n_0 + N)| &= \left|\sum_{k=N}^{\infty} h(k)x(n_0 + N - k)\right| \leq \sum_{k=N}^{\infty} |h(k)||x(n_0 + N - k)| \\ &\leq M_x \sum_{k=N}^{\infty} |h(k)| \end{aligned}$$

Now, as N approaches infinity,

$$\lim_{N\to\infty} \sum_{k=N}^{\infty} |h(n)| = 0$$

and hence

$$\lim_{N\to\infty} |y(n_0 + N)| = 0$$

This result implies that any excitation at the input to the system, which is of a finite duration, produces an output that is "transient" in nature; that is, its amplitude decays with time and dies out eventually, when the system is stable.

Example 2.3.6

Determine the range of values of the parameter a for which the linear time-invariant system with impulse response

$$h(n) = a^n u(n)$$

is stable.

Solution First, we note that the system is causal. Consequently, the lower index on the summation in (2.3.43) begins with $k = 0$. Hence

$$\sum_{k=0}^{\infty} |a^k| = \sum_{k=0}^{\infty} |a|^k = 1 + |a| + |a|^2 + \cdots$$

Clearly, this geometric series converges to

$$\sum_{k=0}^{\infty} |a|^k = \frac{1}{1 - |a|}$$

provided that $|a| < 1$. Otherwise, it diverges. Therefore, the system is stable if $|a| < 1$. Otherwise, it is unstable. In effect, $h(n)$ must decay exponentially toward zero as n approaches infinity for the system to be stable.

Example 2.3.7

Determine the range of values of a and b for which the linear time-invariant system with impulse response

$$h(n) = \begin{cases} a^n, & n \geq 0 \\ b^n, & n < 0 \end{cases}$$

is stable.

Solution This system is noncasual. The condition on stability given by (2.3.43) yields

$$\sum_{n=-\infty}^{\infty} |h(n)| = \sum_{n=0}^{\infty} |a|^n + \sum_{n=-\infty}^{-1} |b|^n$$

From Example 2.3.6 we have already determined that the first sum converges for $|a| < 1$. The second sum can be manipulated as follows:

$$\sum_{n=-\infty}^{-1} |b|^n = \sum_{n=1}^{\infty} \frac{1}{|b|^n} = \frac{1}{|b|}\left(1 + \frac{1}{|b|} + \frac{1}{|b|^2} + \cdots\right)$$

$$= \beta(1 + \beta + \beta^2 + \cdots) = \frac{\beta}{1-\beta}$$

where $\beta = 1/|b|$ must be less than unity for the geometric series to converge. Consequently, the system is stable if both $|a| < 1$ and $|b| > 1$ are satisfied.

2.3.7 Systems with Finite-Duration and Infinite-Duration Impulse Response

Up to this point we have characterized a linear time-invariant system in terms of its impulse response $h(n)$. It is also convenient, however, to subdivide the class of linear time-invariant systems into two types, those that have a finite-duration impulse response (FIR) and those that have an infinite-duration impulse response (IIR). Thus an FIR system has an impulse response that is zero outside of some finite time interval. Without loss of generality, we focus our attention on causal FIR systems, so that

$$h(n) = 0 \qquad n < 0 \text{ and } n \geq M$$

The convolution formula for such a system reduces to

$$y(n) = \sum_{k=0}^{M-1} h(k)x(n-k)$$

A useful interpretation of this expression is obtained by observing that the output at any time n is simply a weighted linear combination of the input signal samples $x(n), x(n-1), \ldots, x(n-M+1)$. In other words, the system simply weights, by the values of the impulse response $h(k)$, $k = 0, 1, \ldots, M-1$, the most recent M signal samples and sums the resulting M products. In effect, the system acts as a *window* that views only the most recent M input signal samples in forming the output. It neglects or simply "forgets" all prior input samples [i.e., $x(n-M)$, Now, (2.5.22) represents a recursive realization of the FIR system. The structure of this recursive realization of the moving average system is illustrated in Fig. 2.37.

In summary, we can think of the terms FIR and IIR as general characteristics that distinguish a type of linear time-invariant system, and of the terms *recursive* and *nonrecursive* as descriptions of the structures for realizing or implementing the system.

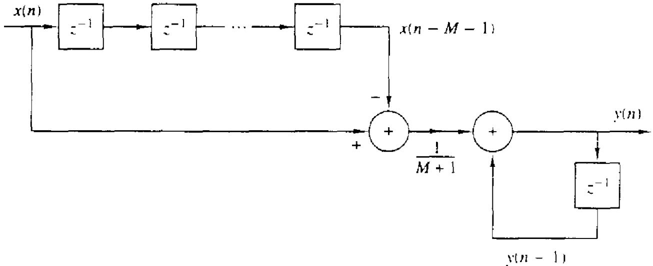


Figure 2.37 Recursive realization of an FIR moving average system.

2.6 CORRELATION OF DISCRETE-TIME SIGNALS

A mathematical operation that closely resembles convolution is correlation. Just as in the case of convolution, two signal sequences are involved in correlation. In contrast to convolution, however, our objective in computing the correlation between the two signals is to measure the degree to which the two signals are similar and thus to extract some information that depends to a large extent on the application. Correlation of signals is often encountered in radar, sonar, digital communications, geology, and other areas in science and engineering.

To be specific, let us suppose that we have two signal sequences $x(n)$ and $y(n)$ that we wish to compare. In radar and active sonar applications, $x(n)$ can represent the sampled version of the transmitted signal and $y(n)$ can represent the sampled version of the received signal at the output of the analog-to-digital (A/D) converter. If a target is present in the space being searched by the radar or sonar, the received signal $y(n)$ consists of a delayed version of the transmitted signal, reflected from the target, and corrupted by additive noise. Figure 2.38 depicts the radar signal reception problem.

We can represent the received signal sequence as

$$y(n) = \alpha x(n - D) + w(n) \tag{2.6.1}$$

where α is some attenuation factor representing the signal loss involved in the round-trip transmission of the signal $x(n)$, D is the round-trip delay, which is

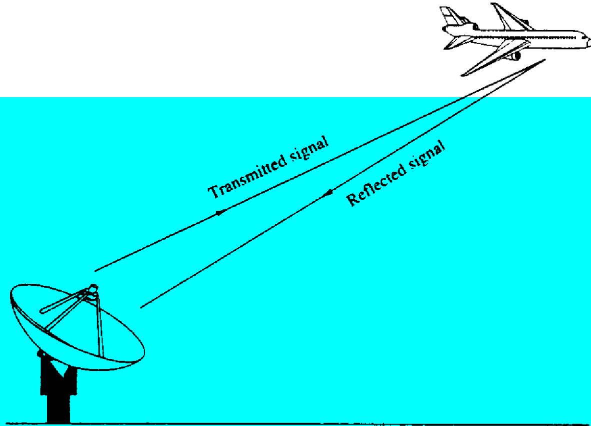


Figure 2.38 Radar target detection.

assumed to be an integer multiple of the sampling interval, and $w(n)$ represents the additive noise that is picked up by the antenna and any noise generated by the electronic components and amplifiers contained in the front end of the receiver. On the other hand, if there is no target in the space searched by the radar and sonar, the received signal $y(n)$ consists of noise alone.

Having the two signal sequences, $x(n)$, which is called the reference signal or transmitted signal, and $y(n)$, the received signal, the problem in radar and sonar detection is to compare $y(n)$ and $x(n)$ to determine if a target is present and, if so, to determine the time delay D and compute the distance to the target. In practice, the signal $x(n-D)$ is heavily corrupted by the additive noise to the point where a visual inspection of $y(n)$ does not reveal the presence or absence of the desired signal reflected from the target. Correlation provides us with a means for extracting this important information from $y(n)$.

Digital communications is another area where correlation is often used. In digital communications the information to be transmitted from one point to another is usually converted to binary from, that is, a sequence of zeros and ones, which are then transmitted to the intended receiver. To transmit a 0 we can transmit the signal sequence $x_0(n)$ for $0 \le n \le L-1$, and to transmit a 1 we can transmit the signal sequence $x_1(n)$ for $0 \le n \le L-1$, where L is some integer that denotes the number of samples in each of the two sequences. Very often, $x_1(n)$ is selected to be the negative of $x_0(n)$. The signal received by the intended receiver may be represented as

$$y(n) = x_i(n) + w(n) \qquad i = 0, 1 \qquad 0 \le n \le L-1 \tag{2.6.2}$$

where now the uncertainty is whether $x_0(n)$ or $x_1(n)$ is the signal component in $y(n)$, and $w(n)$ represents the additive noise and other interference inherent in any communication system. Again, such noise has its origin in the electronic components contained in the front end of the receiver. In any case, the receiver knows the possible transmitted sequences $x_0(n)$ and $x_1(n)$ and is faced with the task of comparing the received signal $y(n)$ with both $x_0(n)$ and $x_1(n)$ to determine which of the two signals better matches $y(n)$. This comparison process is performed by means of the correlation operation described in the following subsection.

2.6.1 Crosscorrelation and Autocorrelation Sequences

Suppose that we have two real signal sequences $x(n)$ and $y(n)$ each of which has finite energy. The *crosscorrelation* of $x(n)$ and $y(n)$ is a sequence $r_{xy}(l)$, which is defined as

$$r_{xy}(l) = \sum_{n=-\infty}^{\infty} x(n)y(n-l) \qquad l = 0, \pm 1, \pm 2, \dots \tag{2.6.3}$$

or, equivalently, as

$$r_{xy}(l) = \sum_{n=-\infty}^{\infty} x(n+l)y(n) \qquad l = 0, \pm 1, \pm 2, \dots \tag{2.6.4}$$

The index l is the (time) shift (or *lag*) parameter and the subscripts xy on the crosscorrelation sequence $r_{xy}(l)$ indicate the sequences being correlated. The order of the subscripts, with x preceding y, indicates the direction in which one sequence is shifted, relative to the other. To elaborate, in (2.6.3), the sequence $x(n)$ is left unshifted and $y(n)$ is shifted by l units in time, to the right for l positive and to the left for l negative. Equivalently, in (2.6.4), the sequence $y(n)$ is left unshifted and $x(n)$ is shifted by l units in time, to the left for l positive and to the right for l negative. But shifting $x(n)$ to the left by l units relative to $y(n)$ is equivalent to shifting $y(n)$ to the right by l units relative to $x(n)$. Hence the computations (2.6.3) and (2.6.4) yield identical crosscorrelation sequences.

If we reverse the roles of $x(n)$ and $y(n)$ in (2.6.3) and (2.6.4) and therefore reverse the order of the indices xy, we obtain the crosscorrelation sequence

$$r_{yx}(l) = \sum_{n=-\infty}^{\infty} y(n)x(n-l) \tag{2.6.5}$$

or, equivalently,

$$r_{yx}(l) = \sum_{n=-\infty}^{\infty} y(n+l)x(n) \tag{2.6.6}$$

By comparing (2.6.3) with (2.6.6) or (2.6.4) with (2.6.5), we conclude that

$$r_{xy}(l) = r_{yx}(-l) \tag{2.6.7}$$

Therefore, $r_{yx}(l)$ is simply the folded version of $r_{xy}(l)$, where the folding is done with respect to $l = 0$. Hence, $r_{yx}(l)$ provides exactly the same information as $r_{xy}(l)$, with respect to the similarity of $x(n)$ to $y(n)$.

Example 2.6.1

Determine the crosscorrelation sequence $r_{xy}(l)$ of the sequences

$$x(n) = \{\ldots, 0, 0, 2, -1, 3, \underset{\uparrow}{7}, 1, 2, -3, 0, 0, \ldots\}$$

$$y(n) = \{\ldots, 0, 0, 1, -1, 2, -2, \underset{\uparrow}{4}, 1, -2, 5, 0, 0, \ldots\}$$

Solution Let us use the definition in (2.6.3) to compute $r_{xy}(l)$. For $l = 0$ we have

$$r_{xy}(0) = \sum_{n=-\infty}^{\infty} x(n)y(n)$$

The product sequence $v_0(n) = x(n)y(n)$ is

$$v_0(n) = \{\ldots, 0, 0, 2, 1, 6, -14, \underset{\uparrow}{4}, 2, 6, 0, 0, \ldots\}$$

and hence the sum over all values of n is

$$r_{xy}(0) = 7$$

For $l > 0$, we simply shift $y(n)$ to the right relative to $x(n)$ by l units, compute the product sequence $v_l(n) = x(n)y(n-l)$, and finally, sum over all values of the product sequence. Thus we obtain

$$r_{xy}(1) = 13, \quad r_{xy}(2) = -18, \quad r_{xy}(3) = 16, \quad r_{xy}(4) = -7$$

$$r_{xy}(5) = 5, \quad r_{xy}(6) = -3, \quad r_{xy}(l) = 0, \quad l \geq 7$$

For $l < 0$, we shift $y(n)$ to the left relative to $x(n)$ by l units, compute the product sequence $v_l(n) = x(n)y(n-l)$, and sum over all values of the product sequence. Thus we obtain the values of the crosscorrelation sequence

$$r_{xy}(-1) = 0, \quad r_{xy}(-2) = 33, \quad r_{xy}(-3) = -14, \quad r_{xy}(-4) = 36$$

$$r_{xy}(-5) = 19, \quad r_{xy}(-6) = -9, \quad r_{xy}(-7) = 10, \quad r_{xy}(l) = 0, \ l \leq -8$$

Therefore, the crosscorrelation sequence of $x(n)$ and $y(n)$ is

$$r_{xy}(l) = \{10, -9, 19, 36, -14, 33, 0, \underset{\uparrow}{7}, 13, -18, 16, -7, 5, -3\}$$

The similarities between the computation of the crosscorrelation of two sequences and the convolution of two sequences is apparent. In the computation of convolution, one of the sequences is folded, then shifted, then multiplied by the other sequence to form the product sequence for that shift, and finally, the values of the product sequence are summed. Except for the folding operation, the computation of the crosscorrelation sequence involves the same operations: shifting one of the sequences, multiplication of the two sequences, and summing over all values of the product sequence. Consequently, if we have a computer program that performs convolution, we can use it to perform crosscorrelation by providing

With $y(n) = x(n)$, this relation results in the following important property for the autocorrelation sequence

$$r_{xx}(l) = r_{xx}(-l) \tag{2.6.19}$$

Hence the autocorrelation function is an even function. Consequently, it suffices to compute $r_{xx}(l)$ for $l \geq 0$.

Example 2.6.2

Compute the autocorrelation of the signal

$$x(n) = a^n u(n), 0 < a < 1$$

Solution Since $x(n)$ is an infinite-duration signal, its autocorrelation also has infinite duration. We distinguish two cases.

If $l \geq 0$, from Fig. 2.39 we observe that

$$r_{xx}(l) = \sum_{n=l}^{\infty} x(n)x(n-l) = \sum_{n=l}^{\infty} a^n a^{n-l} = a^{-l}\sum_{n=l}^{\infty}(a^2)^n$$

Since $a < 1$, the infinite series con erges and we obtain

$$r_{xx}(l) = \frac{1}{1-a^2}a^l \qquad l \geq 0$$

For $l < 0$ we have

$$r_{xx}(l) = \sum_{n=0}^{\infty} x(n)x(n-l) = a^{-l}\sum_{n=0}^{\infty}(a^2)^n = \frac{1}{1-a^2}a^{-l} \qquad l < 0$$

But when l is negative, $a^{-l} = a^{|l|}$. Thus the two relations for $r_{xx}(l)$ can be combined into the following expression:

$$r_{xx}(l) = \frac{1}{1-a^2}a^{|l|} \qquad -\infty < l < \infty \tag{2.6.20}$$

The sequence $r_{xx}(l)$ is shown in Fig. 2.42(d). We observe that

$$r_{xx}(-l) = r_{xx}(l)$$

and

$$r_{xx}(0) = \frac{1}{1-a^2}$$

Therefore, the normalized autocorrelation sequence is

$$\rho_{xx}(l) = \frac{r_{xx}(l)}{r_{xx}(0)} = a^{|l|} \qquad -\infty < l < \infty \tag{2.6.21}$$

2.6.3 Correlation of Periodic Sequences

In Section 2.6.1 we defined the crosscorrelation and autocorrelation sequences of energy signals. In this section we consider the correlation sequences of power signals and, in particular, periodic signals.

Let $x(n)$ and $y(n)$ be two power signals. Their crosscorrelation sequence is defined as

$$r_{xy}(l) = \lim_{M\to\infty}\frac{1}{2M+1}\sum_{n=-M}^{M} x(n)y(n-l) \tag{2.6.22}$$