The Initial Value Theorem. If $x(n)$ is *causal* [i.e., $x(n) = 0$ for $n < 0$], then

$$x(0) = \lim_{z \to \infty} X(z) \tag{3.2.23}$$

Proof. Since $x(n)$ is causal, (3.1.1) gives

$$X(z) = \sum_{n=0}^{\infty} x(n)z^{-n} = x(0) + x(1)z^{-1} + x(2)z^{-2} + \cdots$$

Obviously, as $z \to \infty$, $z^{-n} \to 0$ since $n > 0$ and (3.2.23) follows.

All the properties of the z-transform presented in this section are summarized in Table 3.2 for easy reference. They are listed in the same order as they have been introduced in the text. The conjugation properties and Parseval's relation are left as exercises for the reader.

We have now derived most of the z-transforms that are encountered in many practical applications. These z-transform pairs are summarized in Table 3.3 for easy reference. A simple inspection of this table shows that these z-transforms are all *rational functions* (i.e., ratios of polynomials in z^{-1}). As will soon become apparent, rational z-transforms are encountered not only as the z-transforms of various important signals but also in the characterization of discrete-time linear time-invariant systems described by constant-coefficient difference equations.

3.3 RATIONAL Z-TRANSFORMS

As indicated in Section 3.2, an important family of z-transforms are those for which $X(z)$ is a rational function, that is, a ratio of two polynomials in z^{-1} (or z). In this section we discuss some very important issues regarding the class of rational z-transforms.

3.3.1 Poles and Zeros

The *zeros* of a z-transform $X(z)$ are the values of z for which $X(z) = 0$. The *poles* of a z-transform are the values of z for which $X(z) = \infty$. If $X(z)$ is a rational function, then

$$X(z) = \frac{N(z)}{D(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \cdots + a_N z^{-N}} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} \tag{3.3.1}$$

If $a_0 \neq 0$ and $b_0 \neq 0$, we can avoid the negative powers of z by factoring out the terms $b_0 z^{-M}$ and $a_0 z^{-N}$ as follows:

$$X(z) = \frac{N(z)}{D(z)} = \frac{b_0 z^{-M}}{a_0 z^{-N}} \frac{z^M + (b_1/b_0)z^{M-1} + \cdots + b_M/b_0}{z^N + (a_1/a_0)z^{N-1} + \cdots + a_N/a_0}$$

TABLE 3.3 SOME COMMON Z-TRANSFORM PAIRS

	Signal, $x(n)$	z-Transform, $X(z)$	ROC
1	$\delta(n)$	1	All z
2	$u(n)$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3	$a^n u(n)$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
4	$na^n u(n)$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
5	$-a^n u(-n-1)$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
6	$-na^n u(-n-1)$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
7	$(\cos\omega_0 n)u(n)$	$\dfrac{1-z^{-1}\cos\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\lvert z\rvert > 1$
8	$(\sin\omega_0 n)u(n)$	$\dfrac{z^{-1}\sin\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\lvert z\rvert > 1$
9	$(a^n\cos\omega_0 n)u(n)$	$\dfrac{1-az^{-1}\cos\omega_0}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$
10	$(a^n\sin\omega_0 n)u(n)$	$\dfrac{az^{-1}\sin\omega_0}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$

Since $N(z)$ and $D(z)$ are polynomials in z, they can be expressed in factored form as

$$X(z) = \frac{N(z)}{D(z)} = \frac{b_0}{a_0} z^{-M+N} \frac{(z-z_1)(z-z_2)\cdots(z-z_M)}{(z-p_1)(z-p_2)\cdots(z-p_N)}$$

$$X(z) = Gz^{N-M}\frac{\prod_{k=1}^{M}(z-z_k)}{\prod_{k=1}^{N}(z-p_k)} \tag{3.3.2}$$

where $G \equiv b_0/a_0$. Thus $X(z)$ has M finite zeros at $z = z_1, z_2, \ldots, z_M$ (the roots of the numerator polynomial), N finite poles at $z = p_1, p_2, \ldots, p_N$ (the roots of the denominator polynomial), and $|N-M|$ zeros (if $N > M$) or poles (if $N < M$) at the origin $z = 0$. Poles or zeros may also occur at $z = \infty$. A zero exists at $z = \infty$ if $X(\infty) = 0$ and a pole exists at $z = \infty$ if $X(\infty) = \infty$. If we count the poles and zeros at zero and infinity, we find that $X(z)$ has exactly the same number of poles as zeros.

We can represent $X(z)$ graphically by a *pole–zero plot* (or *pattern*) in the complex plane, which shows the location of poles by crosses ($\times$) and the location of zeros by circles ($\circ$). The multiplicity of multiple-order poles or zeros is indicated by a number close to the corresponding cross or circle. Obviously, by definition, the ROC of a z-transform should not contain any poles.

$$For\ zeros = \frac{0}{Any\ thing} = 0$$

$$For\ poles = \frac{Any\ thing}{0} = \infty$$

Example 3.3.1

Determine the pole-zero plot for the signal

See Example 3.1.3 to find Z transformation

$$x(n) = a^n u(n) \qquad a > 0$$

Solution From Table 3.3 we find that Numerator and denominator is multiply by z

$$X(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z - a} \qquad \text{ROC: } |z| > a$$

Thus $X(z)$ has one zero at $z_1 = 0$ and one pole at $p_1 = a$. The pole-zero plot is shown in Fig. 3.7. Note that the pole $p_1 = a$ is not included in the ROC since the z-transform does not converge at a pole.

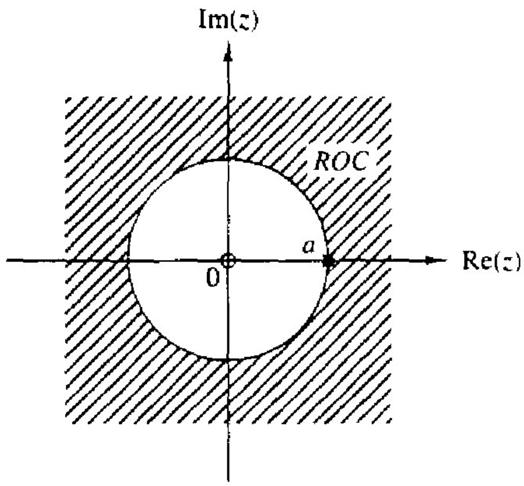

Figure 3.7 Pole-zero plot for the causal exponential signal $x(n) = a^n u(n)$.

Example 3.3.2

Determine the pole-zero plot for the signal

$$x(n) = \begin{cases} a^n, & 0 \leq n \leq M - 1 \\ 0, & \text{elsewhere} \end{cases}$$

where $a > 0$.

Solution From the definition (3.1.1) we obtain

$$X(z) = \sum_{n=0}^{M-1} (az^{-1})^n = \frac{1 - (az^{-1})^M}{1 - az^{-1}} = \frac{z^M - a^M}{z^{M-1}(z - a)}$$

Since $a > 0$, the equation $z^M = a^M$ has M roots at

$$z_k = ae^{j2\pi k/M} \qquad k = 0, 1, \ldots, M - 1$$

The zero $z_0 = a$ cancels the pole at $z = a$. Thus

$$X(z) = \frac{(z - z_1)(z - z_2)\cdots(z - z_{M-1})}{z^{M-1}}$$

which has $M - 1$ zeros and $M - 1$ poles, located as shown in Fig. 3.8 for $M = 8$. Note that the ROC is the entire z-plane except $z = 0$ because of the $M - 1$ poles located at the origin.

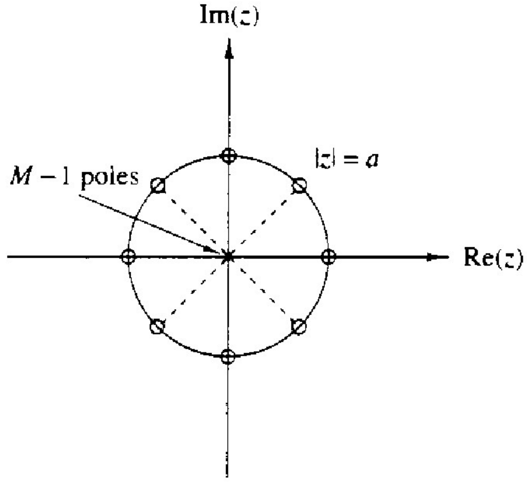

Figure 3.8 Pole–zero pattern for the finite-duration signal $x(n) = a^n$, $0 \le n \le M - 1 (a > 0)$, for $M = 8$.

Clearly, if we are given a pole–zero plot, we can determine $X(z)$, by using (3.3.2), to within a scaling factor G. This is illustrated in the following example.

Example 3.3.3

Determine the z-transform and the signal that corresponds to the pole–zero plot of Fig. 3.9.

Solution There are two zeros ($M = 2$) at $z_1 = 0$, $z_2 = r \cos \omega_0$ and two poles ($N = 2$) at $p_1 = re^{j\omega_0}$, $p_2 = re^{-j\omega_0}$. By substitution of these relations into (3.3.2), we obtain

$$X(z) = G\frac{(z - z_1)(z - z_2)}{(z - p_1)(z - p_2)} = G\frac{z(z - r\cos\omega_0)}{(z - re^{j\omega_0})(z - re^{-j\omega_0})} \qquad \text{ROC: } |z| > r$$

After some simple algebraic manipulations, we obtain

$$X(z) = G\frac{1 - rz^{-1}\cos\omega_0}{1 - 2rz^{-1}\cos\omega_0 + r^2z^{-2}} \qquad \text{ROC: } |z| > r$$

From Table 3.3 we find that

$$x(n) = G(r^n \cos\omega_0 n)u(n)$$

From Example 3.3.3, we see that the product $(z - p_1)(z - p_2)$ results in a polynomial with real coefficients, when p_1 and p_2 are complex conjugates. In

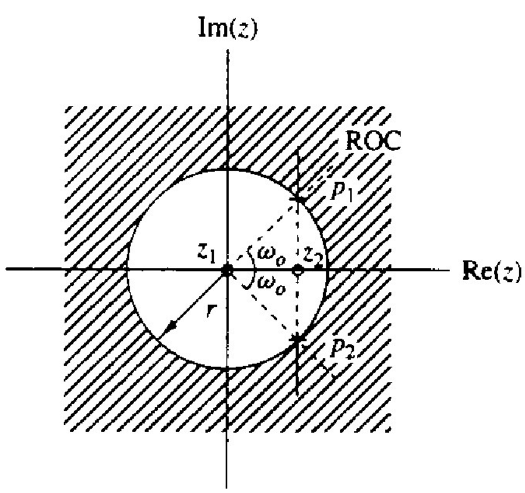

Figure 3.9 Pole-zero pattern for Example 3.3.3.

1. If $n \geq 0$, $f(z)$ has only zeros and hence no poles inside C. The only pole inside C is $z = a$. Hence

$$x(n) = f(z_0) = a^n \qquad n \geq 0$$

2. If $n < 0$, $f(z) = z^n$ has an nth-order pole at $z = 0$, which is also inside C. Thus there are contributions from both poles. For $n = -1$ we have

$$x(-1) = \frac{1}{2\pi j} \oint_C \frac{1}{z(z-a)} dz = \left.\frac{1}{z-a}\right|_{z=0} + \left.\frac{1}{z}\right|_{z=a} = 0$$

If $n = -2$, we have

$$x(-2) = \frac{1}{2\pi j} \oint_C \frac{1}{z^2(z-a)} dz = \left.\frac{d}{dz}\left(\frac{1}{z-a}\right)\right|_{z=0} + \left.\frac{1}{z^2}\right|_{z=a} = 0$$

By continuing in the same way we can show that $x(n) = 0$ for $n < 0$. Thus

$$x(n) = a^n u(n)$$

3.4.2 The Inverse z-Transform by Power Series Expansion

The basic idea in this method is the following: Given a z-transform $X(z)$ with its corresponding ROC, we can expand $X(z)$ into a power series of the form

$$X(z) = \sum_{n=-\infty}^{\infty} c_n z^{-n} \tag{3.4.7}$$

which converges in the given ROC. Then, by the uniqueness of the z-transform, $x(n) = c_n$ for all n. When $X(z)$ is rational, the expansion can be performed by long division.

To illustrate this technique, we will invert some z-transforms involving the same expression for $X(z)$, but different ROC. This will also serve to emphasize again the importance of the ROC in dealing with z-transforms.

Example 3.4.2

Determine the inverse z-transform of

$$X(z) = \frac{1}{1 - 1.5z^{-1} + 0.5z^{-2}}$$

when

(a) ROC: $|z| > 1$

(b) ROC: $|z| < 0.5$

Solution

(a) Since the ROC is the exterior of a circle, we expect $x(n)$ to be a causal signal. Thus we seek a power series expansion in negative powers of z. By dividing the numerator of $X(z)$ by its denominator, we obtain the power series

$$X(z) = \frac{1}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}} = 1 + \tfrac{3}{2}z^{-1} + \tfrac{7}{4}z^{-2} + \tfrac{15}{8}z^{-3} + \tfrac{31}{16}z^{-4} + \cdots$$

By comparing this relation with (3.1.1), we conclude that

$$x(n) = \{\underset{\uparrow}{1}, \tfrac{3}{2}, \tfrac{7}{4}, \tfrac{15}{8}, \tfrac{31}{16}, \ldots\}$$

Note that in each step of the long-division process, we eliminate the lowest-power term of z^{-1}.

(b) In this case the ROC is the interior of a circle. Consequently, the signal $x(n)$ is anticausal. To obtain a power series expansion in positive powers of z, we perform the long division in the following way:

$$\begin{array}{r|l}
 & 2z^2 + 6z^3 + 14z^4 + 30z^5 + 62z^6 + \cdots \\
\hline
\frac{1}{2}z^{-2} - \frac{3}{2}z^{-1} + 1 & 1 \\
 & \underline{1 - 3z + 2z^2} \\
 & \quad 3z - 2z^2 \\
 & \quad \underline{3z - 9z^2 + 6z^3} \\
 & \qquad 7z^2 - 6z^3 \\
 & \qquad \underline{7z^2 - 21z^3 + 14z^4} \\
 & \qquad\quad 15z^3 - 14z^4 \\
 & \qquad\quad \underline{15z^3 - 45z^4 + 30z^5} \\
 & \qquad\qquad 31z^4 - 30z^5
\end{array}$$

Thus

$$X(z) = \frac{1}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}} = 2z^2 + 6z^3 + 14z^4 + 30z^5 + 62z^6 + \cdots$$

In this case $x(n) = 0$ for $n \geq 0$. By comparing this result to (3.1.1), we conclude that

$$x(n) = \{\cdots 62, 30, 14, 6, 2, 0, \underset{\uparrow}{0}\}$$

We observe that in each step of the long-division process, the lowest-power term of z is eliminated. We emphasize that in the case of anticausal signals we simply carry out the long division by writing down the two polynomials in "reverse" order (i.e., starting with the most negative term on the left).

From this example we note that, in general, the method of long division will not provide answers for $x(n)$ when n is large because the long division becomes tedious. Although, the method provides a direct evaluation of $x(n)$, a closed-form solution is not possible, except if the resulting pattern is simple enough to infer the general term $x(n)$. Hence this method is used only if one wished to determine the values of the first few samples of the signal.

Example 3.4.3

Determine the inverse z-transform of

$$X(z) = \log(1 + az^{-1}) \qquad |z| > |a|$$

Solution Using the power series expansion for $\log(1 + x)$, with $|x| < 1$, we have

$$X(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} a^n z^{-n}}{n}$$

Thus

$$x(n) = \begin{cases} (-1)^{n+1} \dfrac{a^n}{n}, & n \geq 1 \\ 0, & n \leq 0 \end{cases}$$

Expansion of irrational functions into power series can be obtained from tables.

3.4.3 The Inverse z-Transform by Partial-Fraction Expansion

In the table lookup method, we attempt to express the function $X(z)$ as a linear combination

$$X(z) = \alpha_1 X_1(z) + \alpha_2 X_2(z) + \cdots + \alpha_K X_K(z) \tag{3.4.8}$$

where $X_1(z), \ldots, X_K(z)$ are expressions with inverse transforms $x_1(n), \ldots, x_K(n)$ available in a table of z-transform pairs. If such a decomposition is possible, then $x(n)$, the inverse z-transform of $X(z)$, can easily be found using the linearity property as

$$x(n) = \alpha_1 x_1(n) + \alpha_2 x_2(n) + \cdots + \alpha_K x_K(n) \tag{3.4.9}$$

This approach is particularly useful if $X(z)$ is a rational function, as in (3.3.1). Without loss of generality, we assume that $a_0 = 1$, so that (3.3.1) can be expressed as

$$X(z) = \frac{N(z)}{D(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}} \tag{3.4.10}$$

Note that if $a_0 \neq 1$, we can obtain (3.4.10) from (3.3.1) by dividing both numerator and denominator by a_0.

A rational function of the form (3.4.10) is called *proper* if $a_N \neq 0$ and $M < N$. From (3.3.2) it follows that this is equivalent to saying that the number of finite zeros is less than the number of finite poles.

An improper rational function ($M \geq N$) can always be written as the sum of a polynomial and a proper rational function. This procedure is illustrated by the following example.

Example 3.4.4

Express the improper rational transform

$$X(z) = \frac{1 + 3z^{-1} + \frac{11}{6}z^{-2} + \frac{1}{3}z^{-3}}{1 + \frac{5}{6}z^{-1} + \frac{1}{6}z^{-2}}$$

in terms of a polynomial and a proper function.

Example 3.4.5

Determine the partial-fraction expansion of the proper function

$$X(z) = \frac{1}{1 - 1.5z^{-1} + 0.5z^{-2}} \tag{3.4.16}$$

Solution First we eliminate the negative powers, by multiplying both numerator and denominator by z^2. Thus

$$X(z) = \frac{z^2}{z^2 - 1.5z + 0.5}$$

The poles of $X(z)$ are $p_1 = 1$ and $p_2 = 0.5$. Consequently, the expansion of the form (3.4.15) is

$$\frac{X(z)}{z} = \frac{z}{(z-1)(z-0.5)} = \frac{A_1}{z-1} + \frac{A_2}{z-0.5} \tag{3.4.17}$$

A very simple method to determine A_1 and A_2 is to multiply the equation by the denominator term $(z-1)(z-0.5)$. Thus we obtain

$$z = (z-0.5)A_1 + (z-1)A_2 \tag{3.4.18}$$

Now if we set $z = p_1 = 1$ in (3.4.18), we eliminate the term involving A_2. Hence

$$1 = (1 - 0.5)A_1$$

Thus we obtain the result $A_1 = 2$. Next we return to (3.4.18) and set $z = p_2 = 0.5$, thus eliminating the term involving A_1, so we have

$$0.5 = (0.5 - 1)A_2$$

and hence $A_2 = -1$. Therefore, the result of the partial-fraction expansion is

$$\frac{X(z)}{z} = \frac{2}{z-1} - \frac{1}{z-0.5} \tag{3.4.19}$$

The example given above suggests that we can determine the coefficients A_1, $A_2, \ldots, A_N$, by multiplying both sides of (3.4.15) by each of the terms $(z - p_k)$, $k = 1, 2, \ldots, N$, and evaluating the resulting expressions at the corresponding pole positions, $p_1, p_2, \ldots, p_N$. Thus we have, in general,

$$\frac{(z-p_k)X(z)}{z} = \frac{(z-p_k)A_1}{z-p_1} + \cdots + A_k + \cdots + \frac{(z-p_k)A_N}{z-p_N} \tag{3.4.20}$$

Consequently, with $z = p_k$, (3.4.20) yields the kth coefficient as

$$A_k = \left.\frac{(z-p_k)X(z)}{z}\right|_{z=p_k} \qquad k = 1, 2, \ldots, N \tag{3.4.21}$$

Example 3.4.6

Determine the partial-fraction expansion of

$$X(z) = \frac{1 + z^{-1}}{1 - z^{-1} + 0.5z^{-2}} \tag{3.4.22}$$

following transform pair (see Table 3.3) is quite useful:

$$Z^{-1}\left\{\frac{pz^{-1}}{(1-pz^{-1})^2}\right\} = np^n u(n) \tag{3.4.35}$$

provided that the ROC is $|z| > |p|$. The generalization to the case of poles with higher multiplicity is left as an exercise for the reader.

Example 3.4.8

Determine the inverse z-transform of

$$X(z) = \frac{1}{1 - 1.5z^{-1} + 0.5z^{-2}}$$

if

(a) ROC: $|z| > 1$

(b) ROC: $|z| < 0.5$

(c) ROC: $0.5 < |z| < 1$

Solution This is the same problem that we treated in Example 3.4.2. The partial-fraction expansion for $X(z)$ was determined in Example 3.4.5. The partial-fraction expansion of $X(z)$ yields

$$X(z) = \frac{2}{1 - z^{-1}} - \frac{1}{1 - 0.5z^{-1}} \tag{3.4.36}$$

To invert $X(z)$ we should apply (3.4.28) for $p_1 = 1$ and $p_2 = 0.5$. However, this requires the specification of the corresponding ROC.

(a) In case when the ROC is $|z| > 1$, the signal $x(n)$ is causal and both terms in (3.4.36) are causal terms. According to (3.4.28), we obtain

$$x(n) = 2(1)^n u(n) - (0.5)^n u(n) = (2 - 0.5^n)u(n) \tag{3.4.37}$$

which agrees with the result in Example 3.4.2(a).

(b) When the ROC is $|z| < 0.5$, the signal $x(n)$ is anticausal. Thus both terms in (3.4.36) result in anticausal components. From (3.4.28) we obtain

$$x(n) = [-2 + (0.5)^n]u(-n-1) \tag{3.4.38}$$

(c) In this case the ROC $0.5 < |z| < 1$ is a ring, which implies that the signal $x(n)$ is two-sided. Thus one of the terms corresponds to a causal signal and the other to an anticausal signal. Obviously, the given ROC is the overlapping of the regions $|z| > 0.5$ and $|z| < 1$. Hence the pole $p_2 = 0.5$ provides the causal part and the pole $p_1 = 1$ the anticausal. Thus

$$x(n) = -2(1)^n u(-n-1) - (0.5)^n u(n) \tag{3.4.39}$$

Example 3.4.9

Determine the causal signal $x(n)$ whose z-transform is given by

$$X(z) = \frac{1 + z^{-1}}{1 - z^{-1} + 0.5z^{-2}}$$