times, the wrong classification may lead to erroneous results, since some mathematical tools may apply only to deterministic signals while others may apply only to random signals. This will become clearer as we examine specific mathematical tools.

1.3 THE CONCEPT OF FREQUENCY IN CONTINUOUS-TIME AND DISCRETE-TIME SIGNALS

The concept of frequency is familiar to students in engineering and the sciences. This concept is basic in, for example, the design of a radio receiver, a high-fidelity system, or a spectral filter for color photography. From physics we know that frequency is closely related to a specific type of periodic motion called harmonic oscillation, which is described by sinusoidal functions. The concept of frequency is directly related to the concept of time. Actually, it has the dimension of inverse time. Thus we should expect that the nature of time (continuous or discrete) would affect the nature of the frequency accordingly.

1.3.1 Continuous-Time Sinusoidal Signals

A simple harmonic oscillation is mathematically described by the following continuous-time sinusoidal signal:

$$x_a(t) = A\cos(\Omega t + \theta), \quad -\infty < t < \infty \tag{1.3.1}$$

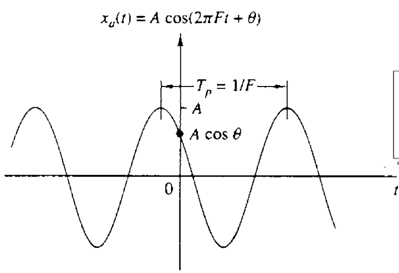
shown in Fig. 1.10. The subscript a used with $x(t)$ denotes an analog signal. This signal is completely characterized by three parameters: A is the *amplitude* of the sinusoid, Ω is the *frequency* in radians per second (rad/s), and θ is the *phase* in radians. Instead of Ω, we often use the frequency F in cycles per second or hertz (Hz), where

$$\Omega = 2\pi F \tag{1.3.2}$$

In terms of F, (1.3.1) can be written as

$$x_a(t) = A\cos(2\pi Ft + \theta), \quad -\infty < t < \infty \tag{1.3.3}$$

We will use both forms, (1.3.1) and (1.3.3), in representing sinusoidal signals.



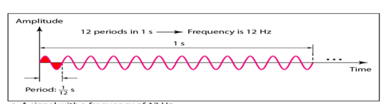


Figure 1.10 Example of an analog sinusoidal signal.

The analog sinusoidal signal in (1.3.3) is characterized by the following properties:

A1. For every fixed value of the frequency F, $x_a(t)$ is periodic. Indeed, it can easily be shown, using elementary trigonometry, that

$$x_a(t + T_p) = x_a(t)$$

where $T_p = 1/F$ is the fundamental period of the sinusoidal signal.

A2. Continuous-time sinusoidal signals with distinct (different) frequencies are themselves distinct.

A3. Increasing the frequency F results in an increase in the rate of oscillation of the signal, in the sense that more periods are included in a given time interval.

We observe that for $F = 0$, the value $T_p = \infty$ is consistent with the fundamental relation $F = 1/T_p$. Due to continuity of the time variable t, we can increase the frequency F, without limit, with a corresponding increase in the rate of oscillation.

The relationships we have described for sinusoidal signals carry over to the class of complex exponential signals

$$x_a(t) = Ae^{j(\Omega t+\theta)} \tag{1.3.4}$$

This can easily be seen by expressing these signals in terms of sinusoids using the Euler identity

$$e^{\pm j\phi} = \cos\phi \pm j\sin\phi \tag{1.3.5}$$

By definition, frequency is an inherently positive physical quantity. This is obvious if we interpret frequency as the number of cycles per unit time in a periodic signal. However, in many cases, only for mathematical convenience, we need to introduce negative frequencies. To see this we recall that the sinusoidal signal (1.3.1) may be expressed as

$\cos(\phi) = \frac{e^{j\phi}+e^{-j\phi}}{2}$

$$x_a(t) = A\cos(\Omega t + \theta) = \frac{A}{2}e^{j(\Omega t+\theta)} + \frac{A}{2}e^{-j(\Omega t+\theta)} \tag{1.3.6}$$

$\sin(\phi) = \frac{e^{j\phi}-e^{-j\phi}}{2j}$

which follows from (1.3.5). Note that a sinusoidal signal can be obtained by adding two equal-amplitude complex-conjugate exponential signals, sometimes called phasors, illustrated in Fig. 1.11. As time progresses the phasors rotate in opposite directions with angular frequencies $\pm\Omega$ radians per second. Since a *positive frequency* corresponds to counterclockwise uniform angular motion, a *negative frequency* simply corresponds to clockwise angular motion.

For mathematical convenience, we use both negative and positive frequencies throughout this book. Hence the frequency range for analog sinusoids is $-\infty < F < \infty$.

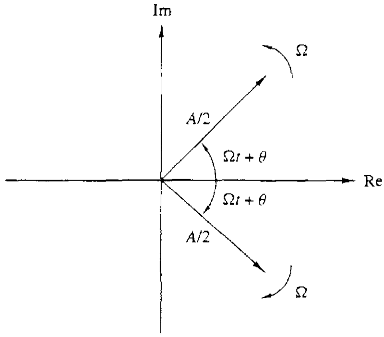


Figure 1.11 Representation of a cosine function by a pair of complex-conjugate exponentials (phasors).

1.3.2 Discrete-Time Sinusoidal Signals

A discrete-time sinusoidal signal may be expressed as

$$x(n) = A\cos(\omega n + \theta), \quad -\infty < n < \infty \tag{1.3.7}$$

where n is an integer variable, called the sample number, A is the *amplitude* of the sinusoid, ω is the *frequency* in radians per sample, and θ is the *phase* in radians.

If instead of ω we use the frequency variable f defined by

$$\omega \equiv 2\pi f \tag{1.3.8}$$

the relation (1.3.7) becomes

$$x(n) = A\cos(2\pi f n + \theta), \quad -\infty < n < \infty \tag{1.3.9}$$

The frequency f has dimensions of cycles per sample. In Section 1.4, where we consider the sampling of analog sinusoids, we relate the frequency variable f of a discrete-time sinusoid to the frequency F in cycles per second for the analog sinusoid. For the moment we consider the discrete-time sinusoid in (1.3.7) independently of the continuous-time sinusoid given in (1.3.1). Figure 1.12 shows a sinusoid with frequency $\omega = \pi/6$ radians per sample ($f = \frac{1}{12}$ cycles per sample) and phase $\theta = \pi/3$.

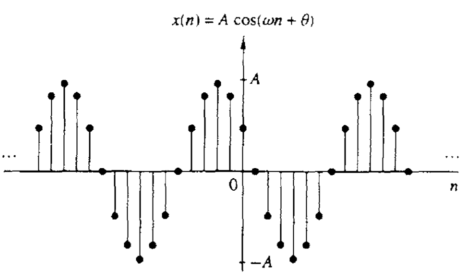


Figure 1.12 Example of a discrete-time sinusoidal signal ($\omega = \pi/6$ and $\theta = \pi/3$).

In contrast to continuous-time sinusoids, the discrete-time sinusoids are characterized by the following properties:

B1. *A discrete-time sinusoid is periodic only if its frequency f is a rational number.*

By definition, a discrete-time signal $x(n)$ is periodic with period $N(N > 0)$ if and only if

$$x(n+N) = x(n) \qquad \text{for all } n \tag{1.3.10}$$

The smallest value of N for which (1.3.10) is true is called the *fundamental period*.

The proof of the periodicity property is simple. For a sinusoid with frequency f_0 to be periodic, we should have $x(n) = \cos(2\pi f n + \theta)$

$$\cos[2\pi f_0(N+n) + \theta] = \cos(2\pi f_0 n + \theta)$$

This relation is true if and only if there exists an integer k such that

$$2\pi f_0 N = 2k\pi$$

or, equivalently,

$$f_0 = \frac{k}{N} \tag{1.3.11}$$

According to (1.3.11), a discrete-time sinusoidal signal is periodic only if its frequency f_0 can be expressed as the ratio of two integers (i.e., f_0 is rational).

To determine the fundamental period N of a periodic sinusoid, we express its frequency f_0 as in (1.3.11) and cancel common factors so that k and N are relatively prime. Then the fundamental period of the sinusoid is equal to N. Observe that a small change in frequency can result in a large change in the period. For example, note that $f_1 = 31/60$ implies that $N_1 = 60$, whereas $f_2 = 30/60$ results in $N_2 = 2$.

B2. *Discrete-time sinusoids whose frequencies are separated by an integer multiple of 2π are identical.*

To prove this assertion, let us consider the sinusoid $\cos(\omega_0 n + \theta)$. It easily follows that

$$\cos[(\omega_0 + 2\pi)n + \theta] = \cos(\omega_0 n + 2\pi n + \theta) = \cos(\omega_0 n + \theta) \tag{1.3.12}$$

As a result, all sinusoidal sequences

$$x_k(n) = A\cos(\omega_k n + \theta), \qquad k = 0, 1, 2, \ldots \tag{1.3.13}$$

where

$$\omega_k = \omega_0 + 2k\pi, \qquad -\pi \le \omega_0 \le \pi$$

are *indistinguishable* (i.e., *identical*). On the other hand, the sequences of any two sinusoids with frequencies in the range $-\pi \le \omega \le \pi$ or $-\frac{1}{2} \le f \le \frac{1}{2}$ are distinct. Consequently, discrete-time sinusoidal signals with frequencies $|\omega| \le \pi$ or $|f| \le \frac{1}{2}$

$x1(n) = \cos\frac{\pi}{2}n$

$x2\ (n) = \cos\frac{5\pi}{2}n$

Both are identical because $x2 = \cos\frac{5\pi}{2}n = \cos\left(2\pi n + \frac{\pi}{2}n\right) = \cos\frac{\pi}{2}n$

are unique. Any sequence resulting from a sinusoid with a frequency $|\omega| > \pi$, or $|f| > \frac{1}{2}$, is identical to a sequence obtained from a sinusoidal signal with frequency $|\omega| < \pi$. Because of this similarity, we call the sinusoid having the frequency $|\omega| > \pi$ an *alias* of a corresponding sinusoid with frequency $|\omega| < \pi$. Thus we regard frequencies in the range $-\pi \le \omega \le \pi$, or $-\frac{1}{2} \le f \le \frac{1}{2}$ as unique and all frequencies $|\omega| > \pi$, or $|f| > \frac{1}{2}$, as aliases. The reader should notice the difference between discrete-time sinusoids and continuous-time sinusoids, where the latter result in distinct signals for Ω or F in the entire range $-\infty < \Omega < \infty$ or $-\infty < F < \infty$.

B3. *The highest rate of oscillation in a discrete-time sinusoid is attained when $\omega = \pi$ (or $\omega = -\pi$) or, equivalently, $f = \frac{1}{2}$ (or $f = -\frac{1}{2}$).*

To illustrate this property, let us investigate the characteristics of the sinusoidal signal sequence

$$x(n) = \cos\omega_0 n$$

when the frequency varies from 0 to π. To simplify the argument, we take values of $\omega_0 = 0, \pi/8, \pi/4, \pi/2, \pi$ corresponding to $f = 0, \frac{1}{16}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}$, which result in periodic sequences having periods $N = \infty, 16, 8, 4, 2$, as depicted in Fig. 1.13. We note that the period of the sinusoid decreases as the frequency increases. In fact, we can see that the rate of oscillation increases as the frequency increases.

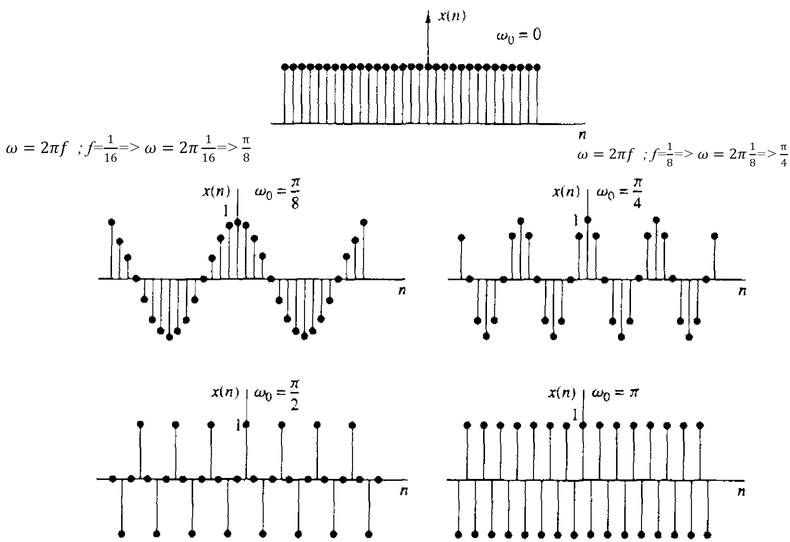


Figure 1.13 Signal $x(n) = \cos\omega_0 n$ for various values of the frequency ω_0.

(a) Determine how this table of values can be used to obtain values of harmonically related sinusoids having the same phase.

(b) Determine how this table can be used to obtain sinusoids of the same frequency but different phase.

Solution

(a) Let $x_k(n)$ denote the sinusoidal signal sequence

$$x_k(n) = \sin\left(\frac{2\pi nk}{N} + \theta\right)$$

This is a sinusoid with frequency $f_k = k/N$, which is harmonically related to $x(n)$. But $x_k(n)$ may be expressed as

$$\begin{aligned} x_k(n) &= \sin\left[\frac{2\pi(kn)}{N} + \theta\right] \\ &= x(kn) \end{aligned}$$

Thus we observe that $x_k(0) = x(0)$, $x_k(1) = x(k)$, $x_k(2) = x(2k)$, and so on. Hence the sinusoidal sequence $x_k(n)$ can be obtained from the table of values of $x(n)$ by taking every kth value of $x(n)$, beginning with $x(0)$. In this manner we can generate the values of all harmonically related sinusoids with frequencies $f_k = k/N$ for $k = 0, 1, \ldots, N-1$.

(b) We can control the phase θ of the sinusoid with frequency $f_k = k/N$ by taking the first value of the sequence from memory location $q = \theta N/2\pi$, where q is an integer. Thus the initial phase θ controls the starting location in the table and we wrap around the table each time the index (kn) exceeds N.

1.4 ANALOG-TO-DIGITAL AND DIGITAL-TO-ANALOG CONVERSION

Most signals of practical interest, such as speech, biological signals, seismic signals, radar signals, sonar signals, and various communications signals such as audio and video signals, are analog. To process analog signals by digital means, it is first necessary to convert them into digital form, that is, to convert them to a sequence of numbers having finite precision. This procedure is called *analog-to-digital (A/D) conversion*, and the corresponding devices are called *A/D converters (ADCs)*.

Conceptually, we view A/D conversion as a three-step process. This process is illustrated in Fig. 1.14.

1. *Sampling*. This is the conversion of a continuous-time signal into a discrete-time signal obtained by taking "samples" of the continuous-time signal at discrete-time instants. Thus, if $x_a(t)$ is the input to the sampler, the output is $x_a(nT) \equiv x(n)$, where T is called the *sampling interval*.
2. *Quantization*. This is the conversion of a discrete-time continuous-valued signal into a discrete-time, discrete-valued (digital) signal. The value of each

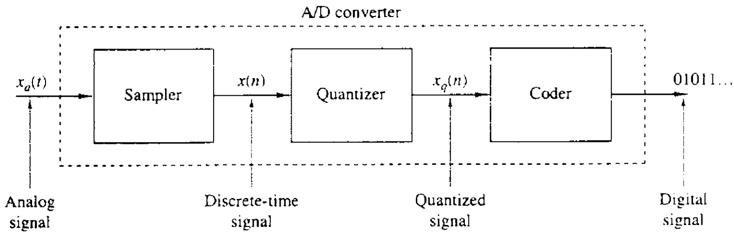


Figure 1.14 Basic parts of an analog-to-digital (A/D) converter.

signal sample is represented by a value selected from a finite set of possible values. The difference between the unquantized sample $x(n)$ and the quantized output $x_q(n)$ is called the quantization error.

3. *Coding.* In the coding process, each discrete value $x_q(n)$ is represented by a b-bit binary sequence.

Although we model the A/D converter as a sampler followed by a quantizer and coder, in practice the A/D conversion is performed by a single device that takes $x_a(t)$ and produces a binary-coded number. The operations of sampling and quantization can be performed in either order but, in practice, sampling is always performed before quantization.

In many cases of practical interest (e.g., speech processing) it is desirable to convert the processed digital signals into analog form. (Obviously, we cannot listen to the sequence of samples representing a speech signal or see the numbers corresponding to a TV signal.) The process of converting a digital signal into an analog signal is known as *digital-to-analog (D/A) conversion.* All D/A converters "connect the dots" in a digital signal by performing some kind of interpolation, whose accuracy depends on the quality of the D/A conversion process. Figure 1.15 illustrates a simple form of D/A conversion, called a zero-order hold or a staircase approximation. Other approximations are possible, such as linearly connecting a pair of successive samples (linear interpolation), fitting a quadratic through three successive samples (quadratic interpolation), and so on. Is there an optimum (ideal) interpolator? For signals having a *limited frequency content* (finite bandwidth), the sampling theorem introduced in the following section specifies the optimum form of interpolation.

Sampling and quantization are treated in this section. In particular, we demonstrate that sampling does not result in a loss of information, nor does it introduce distortion in the signal if the signal bandwidth is finite. In principle, the analog signal can be reconstructed from the samples, provided that the sampling rate is sufficiently high to avoid the problem commonly called *aliasing.* On the other hand, quantization is a noninvertible or irreversible process that results in signal distortion. We shall show that the amount of distortion is dependent on