internal description of the system, we know exactly how the system building blocks are configured. In terms of such a realization, we can see that a system is *relaxed* at time $n = n_0$ if the outputs of all the *delays* existing in the system are zero at $n = n_0$ (i.e., all memory is *filled* with zeros).

2.2.3 Classification of Discrete-Time Systems

In the analysis as well as in the design of systems, it is desirable to classify the systems according to the general properties that they satisfy. In fact, the mathematical techniques that we develop in this and in subsequent chapters for analyzing and designing discrete-time systems depend heavily on the general characteristics of the systems that are being considered. For this reason it is necessary for us to develop a number of properties or categories that can be used to describe the general characteristics of systems.

We stress the point that for a system to possess a given property, the property must hold for every possible input signal to the system. If a property holds for some input signals but not for others, the system does not possess that property. Thus a counterexample is sufficient to prove that a system does not possess a property. However, to prove that the system has some property, we must prove that this property holds for every possible input signal.

Static versus dynamic systems. A discrete-time system is called *static* or memoryless if its output at any instant n depends at most on the input sample at the same time, but not on past or future samples of the input. In any other case, the system is said to be *dynamic* or to have memory. If the output of a system at time n is completely determined by the input samples in the interval from $n - N$ to $n(N \geq 0)$, the system is said to have *memory* of duration N. If $N = 0$, the system is static. If $0 < N < \infty$, the system is said to have *finite memory*, whereas if $N = \infty$, the system is said to have *infinite memory*.

The systems described by the following input–output equations

Static System or System without Memory

$$y(n) = ax(n) \tag{2.2.7}$$

$$y(n) = nx(n) + bx^3(n) \tag{2.2.8}$$

are both static or memoryless. Note that there is no need to store any of the past inputs or outputs in order to compute the present output. On the other hand, the systems described by the following input–output relations

Dynamic System or System with Memory

$$y(n) = x(n) + 3x(n-1) \tag{2.2.9}$$

$$y(n) = \sum_{k=0}^{n} x(n-k) \tag{2.2.10}$$

$$y(n) = \sum_{k=0}^{\infty} x(n-k) \tag{2.2.11}$$

are dynamic systems or systems with memory. The systems described by (2.2.9) and (2.2.10) have finite memory, whereas the system described by (2.2.11) has infinite memory.

We observe that static or memoryless systems are described in general by input–output equations of the form

$$y(n) = \mathcal{T}[x(n), n] \tag{2.2.12}$$

and they do not include delay elements (memory).

Time-invariant versus time-variant systems. We can subdivide the general class of systems into the two broad categories, time-invariant systems and time-variant systems. A system is called time-invariant if its input–output characteristics do not change with time. To elaborate, suppose that we have a system $\mathcal{T}$ in a relaxed state which, when excited by an input signal $x(n)$, produces an output signal $y(n)$. Thus we write

$$y(n) = \mathcal{T}[x(n)] \tag{2.2.13}$$

Now suppose that the same input signal is delayed by k units of time to yield $x(n-k)$, and again applied to the same system. If the characteristics of the system do not change with time, the output of the relaxed system will be $y(n-k)$. That is, the output will be the same as the response to $x(n)$, except that it will be delayed by the same k units in time that the input was delayed. This leads us to define a time-invariant or shift-invariant system as follows.

Definition. A relaxed system $\mathcal{T}$ is *time invariant* or *shift invariant* if and only if

$$x(n) \xrightarrow{\mathcal{T}} y(n)$$

implies that

$$x(n-k) \xrightarrow{\mathcal{T}} y(n-k) \tag{2.2.14}$$

for every input signal $x(n)$ and every time shift k.

To determine if any given system is time invariant, we need to perform the test specified by the preceding definition. Basically, we excite the system with an arbitrary input sequence $x(n)$, which produces an output denoted as $y(n)$. Next we delay the input sequence by same amount k and recompute the output. In general, we can write the output as

$$y(n, k) = \mathcal{T}[x(n-k)]$$

Now if this output $y(n, k) = y(n-k)$, for all possible values of k, the system is time invariant. On the other hand, if the output $y(n, k) \neq y(n-k)$, even for one value of k, the system is time variant.

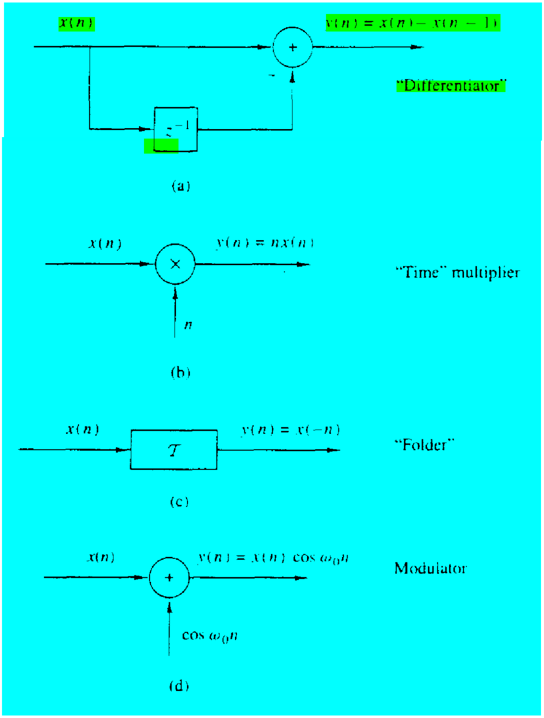

Figure 2.19 Examples of a time-invariant (a) and some time-variant systems (b)–(d).

Example 2.2.4

Determine if the systems shown in Fig. 2.19 are time invariant or time variant.

Solution

$y(n)=x(n)-x(n-1)$

(a) This system is described by the input–output equations

$$y(n) = \mathcal{T}[x(n)] = x(n) - x(n-1) \tag{2.2.15}$$

Now if the input is delayed by k units in time and applied to the system, it is clear from the block diagram that the output will be

$$y(n, k) = x(n-k) - x(n-k-1) \tag{2.2.16}$$

On the other hand, from (2.2.14) we note that if we delay $y(n)$ by k units in time, we obtain

$$y(n-k) = x(n-k) - x(n-k-1) \tag{2.2.17}$$

Since the right-hand sides of (2.2.16) and (2.2.17) are identical, it follows that $y(n, k) = y(n-k)$. Therefore, the system is time invariant.

(b) The input–output equation for this system is

$$y(n) = \mathcal{T}[x(n)] = nx(n) \tag{2.2.18}$$

The response of this system to $x(n-k)$ is

$$y(n,k) = nx(n-k) \tag{2.2.19}$$

Now if we delay $y(n)$ in (2.2.18) by k units in time, we obtain

$$\begin{aligned} y(n-k) &= (n-k)x(n-k) \\ &= nx(n-k) - kx(n-k) \end{aligned} \tag{2.2.20}$$

This system is time variant, since $y(n,k) \neq y(n-k)$.

(c) This system is described by the input–output relation

$$y(n) = \mathcal{T}[x(n)] = x(-n) \tag{2.2.21}$$

The response of this system to $x(n-k)$ is

$$y(n,k) = \mathcal{T}[x(n-k)] = x(-n-k) \tag{2.2.22}$$

Now, if we delay the output $y(n)$, as given by (2.2.21), by k units in time, the result will be

$$y(n-k) = x(-n+k) \tag{2.2.23}$$

Since $y(n,k) \neq y(n-k)$, the system is time variant.

(d) The input–output equation for this system is

$$y(n) = x(n)\cos\omega_0 n \tag{2.2.24}$$

The response of this system to $x(n-k)$ is

$$y(n,k) = x(n-k)\cos\omega_0 n \tag{2.2.25}$$

If the expression in (2.2.24) is delayed by k units and the result is compared to (2.2.25), it is evident that the system is time variant.

Linear versus nonlinear systems. The general class of systems can also be subdivided into linear systems and nonlinear systems. A linear system is one that satisfies the *superposition principle*. Simply stated, the principle of superposition requires that the response of the system to a weighted sum of signals be equal to the corresponding weighted sum of the responses (outputs) of the system to each of the individual input signals. Hence we have the following definition of linearity.

C(A+B)=A.C+B.C A=5KG,B=6KG,C=2; 2(5+6)=2.5+2.6=>22=22

Definition. A relaxed $\mathcal{T}$ system is linear if and only if

$$\mathcal{T}[a_1x_1(n) + a_2x_2(n)] = a_1\mathcal{T}[x_1(n)] + a_2\mathcal{T}[x_2(n)] \tag{2.2.26}$$

for any arbitrary input sequences $x_1(n)$ and $x_2(n)$, and any arbitrary constants a_1 and a_2.

Figure 2.20 gives a pictorial illustration of the superposition principle.

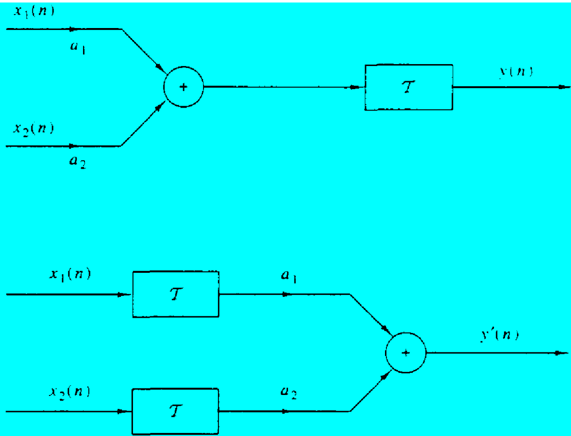

Figure 2.20 Graphical representation of the superposition principle. $\mathcal{T}$ is linear if and only if $y(n) = y'(n)$.

The superposition principle embodied in the relation (2.2.26) can be separated into two parts. First, suppose that $a_2 = 0$. Then (2.2.26) reduces to

$\mathcal{T}[a_1x_1(n) + a_2x_2(n)] = a_1\mathcal{T}[x_1(n)] + a_2\mathcal{T}[x_2(n)]$

$$\mathcal{T}[a_1x_1(n)] = a_1\mathcal{T}[x_1(n)] = a_1y_1(n) \tag{2.2.27}$$

where

$$y_1(n) = \mathcal{T}[x_1(n)]$$

The relation (2.2.27) demonstrates the *multiplicative* or *scaling property* of a linear system. That is, if the response of the system to the input $x_1(n)$ is $y_1(n)$, the response to $a_1x_1(n)$ is simply $a_1y_1(n)$. Thus any scaling of the input results in an identical scaling of the corresponding output.

Second, suppose that $a_1 = a_2 = 1$ in (2.2.26). Then

$$\begin{aligned} \mathcal{T}[x_1(n) + x_2(n)] &= \mathcal{T}[x_1(n)] + \mathcal{T}[x_1(n)] \\ &= y_1(n) + y_2(n) \end{aligned} \tag{2.2.28}$$

This relation demonstrates the *additivity property* of a linear system. The additivity and multiplicative properties constitute the superposition principle as it applies to linear systems.

The linearity condition embodied in (2.2.26) can be extended arbitrarily to any weighted linear combination of signals by induction. In general, we have

$$x(n) = \sum_{k=1}^{M-1} a_kx_k(n) \xrightarrow{\mathcal{T}} y(n) = \sum_{k=1}^{M-1} a_ky_k(n) \tag{2.2.29}$$

where

$$y_k(n) = \mathcal{T}[x_k(n)] \qquad k = 1, 2, \ldots, M-1 \tag{2.2.30}$$

We observe from (2.2.27) that if $a_1 = 0$, then $y(n) = 0$. In other words, a relaxed, linear system with zero input produces a zero output. If a system produces a nonzero output with a zero input, the system may be either nonrelaxed or nonlinear. If a relaxed system does not satisfy the superposition principle as given by the definition above, it is called *nonlinear*.

Example 2.2.5

Determine if the systems described by the following input–output equations are linear or nonlinear.

(a) $y(n) = nx(n)$ **(b)** $y(n) = x(n^2)$ **(c)** $y(n) = x^2(n)$
(d) $y(n) = Ax(n) + B$ **(e)** $y(n) = e^{x(n)}$

Solution

$$\mathcal{T}[a_1x_1(n) + a_2x_2(n)] = a_1\mathcal{T}[x_1(n)] + a_2\mathcal{T}[x_2(n)]$$

(a) For two input sequences $x_1(n)$ and $x_2(n)$, the corresponding outputs are

$$\begin{aligned} y_1(n) &= nx_1(n) \\ y_2(n) &= nx_2(n) \end{aligned} \tag{2.2.31}$$

A linear combination of the two input sequences results in the output

$$\begin{aligned} y_3(n) &= \mathcal{T}[a_1x_1(n) + a_2x_2(n)] = n[a_1x_1(n) + a_2x_2(n)] \\ &= a_1nx_1(n) + a_2nx_2(n) \end{aligned} \tag{2.2.32}$$

On the other hand, a linear combination of the two outputs in (2.2.31) results in the output

$$a_1y_1(n) + a_2y_2(n) = a_1nx_1(n) + a_2nx_2(n) \tag{2.2.33}$$

Since the right-hand sides of (2.2.32) and (2.2.33) are identical, the system is linear.

(b) As in part (a), we find the response of the system to two separate input signals $x_1(n)$ and $x_2(n)$. The result is

$$\begin{aligned} y_1(n) &= x_1(n^2) \\ y_2(n) &= x_2(n^2) \end{aligned} \tag{2.2.34}$$

The output of the system to a linear combination of $x_1(n)$ and $x_2(n)$ is

$$y_3(n) = \mathcal{T}[a_1x_1(n) + a_2x_2(n)] = a_1x_1(n^2) + a_2x_2(n^2) \tag{2.2.35}$$

Finally, a linear combination of the two outputs in (2.2.36) yields

$$a_1y_1(n) + a_2y_2(n) = a_1x_1(n^2) + a_2x_2(n^2) \tag{2.2.36}$$

By comparing (2.2.35) with (2.2.36), we conclude that the system is linear.

(c) The output of the system is the square of the input. (Electronic devices that have such an input–output characteristic and are called square-law devices.) From our previous discussion it is clear that such a system is memoryless. We now illustrate that this system is nonlinear.

The responses of the system to two separate input signals are

$$y_1(n) = x_1^2(n)$$
$$y_2(n) = x_2^2(n) \tag{2.2.37}$$

The response of the system to a linear combination of these two input signals is

$$\begin{aligned} y_3(n) &= \mathcal{T}[a_1x_1(n) + a_2x_2(n)] \\ &= [a_1x_1(n) + a_2x_2(n)]^2 \\ &= a_1^2x_1^2(n) + 2a_1a_2x_1(n)x_2(n) + a_2^2x_2^2(n) \end{aligned} \tag{2.2.38}$$

On the other hand, if the system is linear, it would produce a linear combination of the two outputs in (2.2.37), namely,

$$a_1y_1(n) + a_2y_2(n) = a_1x_1^2(n) + a_2x_2^2(n) \tag{2.2.39}$$

Since the actual output of the system, as given by (2.2.38), is not equal to (2.2.39), the system is nonlinear.

(d) Assuming that the system is excited by $x_1(n)$ and $x_2(n)$ separately, we obtain the corresponding outputs

$$\begin{aligned} y_1(n) &= Ax_1(n) + B \\ y_2(n) &= Ax_2(n) + B \end{aligned} \tag{2.2.40}$$

A linear combination of $x_1(n)$ and $x_2(n)$ produces the output

$$\begin{aligned} y_3(n) &= \mathcal{T}[a_1x_1(n) + a_2x_2(n)] \\ &= A[a_1x_1(n) + a_2x_2(n)] + B \\ &= Aa_1x_1(n) + a_2Ax_2(n) + B \end{aligned} \tag{2.2.41}$$

On the other hand, if the system were linear, its output to the linear combination of $x_1(n)$ and $x_2(n)$ would be a linear combination of $y_1(n)$ and $y_2(n)$, that is,

$$a_1y_1(n) + a_2y_2(n) = a_1Ax_1(n) + a_1B + a_2Ax_2(n) + a_2B \tag{2.2.42}$$

Clearly, (2.2.41) and (2.2.42) are different and hence the system fails to satisfy the linearity test.

The reason that this system fails to satisfy the linearity test is not that the system is nonlinear (in fact, the system is described by a linear equation) but the presence of the constant B. Consequently, the output depends on both the input excitation and on the parameter $B \neq 0$. Hence, for $B \neq 0$, the system is not relaxed. If we set $B = 0$, the system is now relaxed and the linearity test is satisfied.

(e) Note that the system described by the input–output equation

$$y(n) = e^{x(n)} \tag{2.2.43}$$

is relaxed. If $x(n) = 0$, we find that $y(n) = 1$. This is an indication that the system is nonlinear. This, in fact, is the conclusion reached when the linearity test, is applied.

Causal versus noncausal systems. We begin with the definition of causal discrete-time systems.

Definition. A system is said to be *causal* if the output of the system at any time n [i.e., $y(n)$] depends only on present and past inputs [i.e., $x(n)$, $x(n-1)$, $x(n-2), \ldots$], but does not depend on future inputs [i.e., $x(n+1)$, $x(n+2), \ldots$]. In mathematical terms, the output of a causal system satisfies an equation of the form

$$y(n) = F[x(n), x(n-1), x(n-2), \ldots] \tag{2.2.44}$$

where $F[\cdot]$ is some arbitrary function.

If a system does not satisfy this definition, it is called *noncausal*. Such a system has an output that depends not only on present and past inputs but also on future inputs.

It is apparent that in real-time signal processing applications we cannot observe future values of the signal, and hence a noncausal system is physically unrealizable (i.e., it cannot be implemented). On the other hand, if the signal is recorded so that the processing is done off-line (nonreal time), it is possible to implement a noncausal system, since all values of the signal are available at the time of processing. This is often the case in the processing of geophysical signals and images.

Example 2.2.6 Causal=Present+Past Input Non-Causal=Present+Past+Future Input
Causal=Present Input Non-Causal=Future Input

Determine if the systems described by the following input–output equations are causal or noncausal.

(a) $y(n) = x(n) - x(n-1)$ **(b)** $y(n) = \sum_{k=-\infty}^{n} x(k)$ **(c)** $y(n) = ax(n)$

(d) $y(n) = x(n) + 3x(n+4)$ **(e)** $y(n) = x(n^2)$ **(f)** $y(n) = x(2n)$ y(2)=x(4)=>Future input

(g) $y(n) = x(-n)$ y(-1)=x(1)=>Future input

y(-1)=x(1)=>Future input

Solution The systems described in parts (a), (b), and (c) are clearly causal, since the output depends only on the present and past inputs. On the other hand, the systems in parts (d), (e), and (f) are clearly noncausal, since the output depends on future values of the input. The system in (g) is also noncausal, as we note by selecting, for example, $n = -1$, which yields $y(-1) = x(1)$. Thus the output at $n = -1$ depends on the input at $n = 1$, which is two units of time into the future.

Stable versus unstable systems. Stability is an important property that must be considered in any practical application of a system. Unstable systems usually exhibit erratic and extreme behavior and cause overflow in any practical implementation. Here, we define mathematically what we mean by a stable system, and later, in Section 2.3.6, we explore the implications of this definition for linear, time-invariant systems.

Definition. An arbitrary relaxed system is said to be bounded input–bounded output (BIBO) stable if and only if every bounded input produces a bounded output.

The conditions that the input sequence $x(n)$ and the output sequence $y(n)$ are bounded is translated mathematically to mean that there exist some finite numbers, say M_x and M_y, such that

$$|x(n)| \le M_x < \infty \qquad |y(n)| \le M_y < \infty \tag{2.2.45}$$

for all n. If, for some bounded input sequence $x(n)$, the output is unbounded (infinite), the system is classified as unstable.

Example 2.2.7

Consider the nonlinear system described by the input–output equation

$$y(n) = y^2(n-1) + x(n)$$

As an input sequence we select the bounded signal

$$x(n) = C\delta(n)$$

where C is a constant. We also assume that $y(-1) = 0$. Then the output sequence is

$$y(0) = C, \quad y(1) = C^2, \quad y(2) = C^4, \quad \ldots, \quad y(n) = C^{2^n}$$

Clearly, the output is unbounded when $1 < |C| < \infty$. Therefore, the system is BIBO unstable, since a bounded input sequence has resulted in an unbounded output.

2.2.4 Interconnection of Discrete-Time Systems

Discrete-time systems can be interconnected to form larger systems. There are two basic ways in which systems can be interconnected: in cascade (series) or in parallel. These interconnections are illustrated in Fig. 2.21. Note that the two interconnected systems are different.

In the cascade interconnection the output of the first system is

$$y_1(n) = \mathcal{T}_1[x(n)] \tag{2.2.46}$$

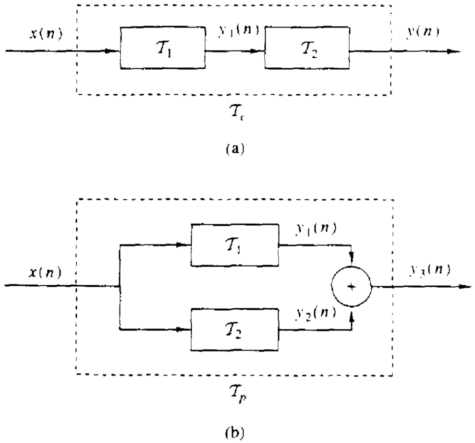

Figure 2.21 Cascade (a) and parallel (b) interconnections of systems.