Project: 12.9 Differential Equations for Nonlinear Oscillators

May 1, 2019

Part 1

1. **Write a program (using the Runge-Kutta routine for example) to integrate this system from five initial conditions with $-2 \leq x(0) \leq 2$ and $-2 \leq \dot{x}(0) \leq 2$, plotting all five solutions on a single picture. (Plot $x(t)$ against $x(t)$.)**

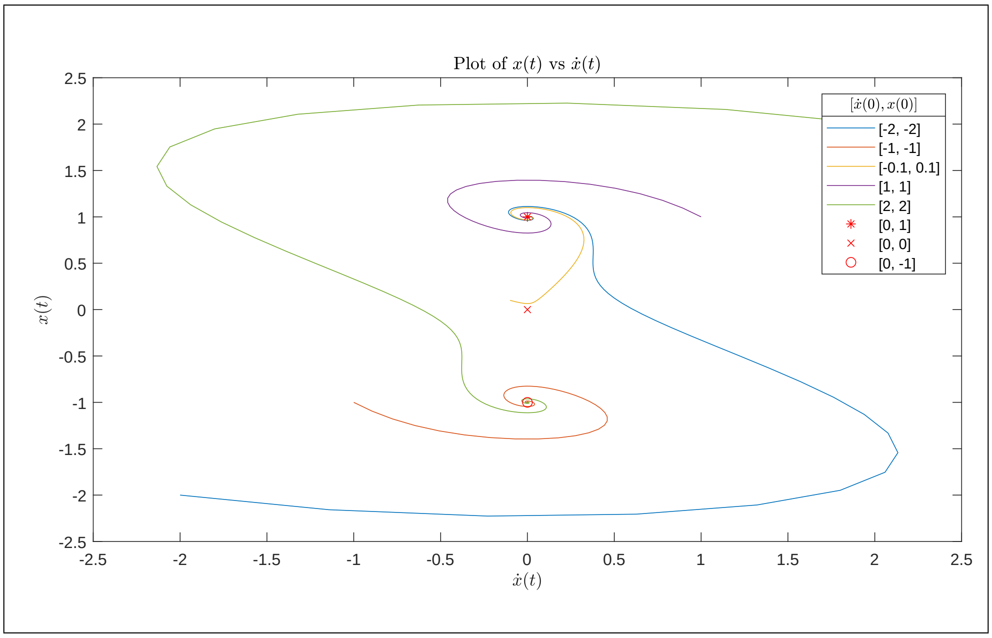

Figure 1: Plot for $a = 1, b = 0$ of $x(t)$ vs $\dot{x}(t)$ for $t \in [0, 20]$.

My program plots the phase-diagram of the coupled system:

$$\dot{z} = -az + x - x^3 + b\cos t$$
$$\dot{x} = z$$

In this case, $a = 1, b = 0$, which implies that we have stable fixed points at $[0, -1]$ and $[0, 1]$, with $[0, 0]$ being a saddle point. This agrees with the analytic analysis of the coupled system, considering the Jacobian at each fixed point.

2. **Test your program by running it with $b = 0$ at $a = -0.12$, $a = 0$ and $a = 0.12$ and show and describe the results. Comment on any special features of the case $a = b = 0$.**

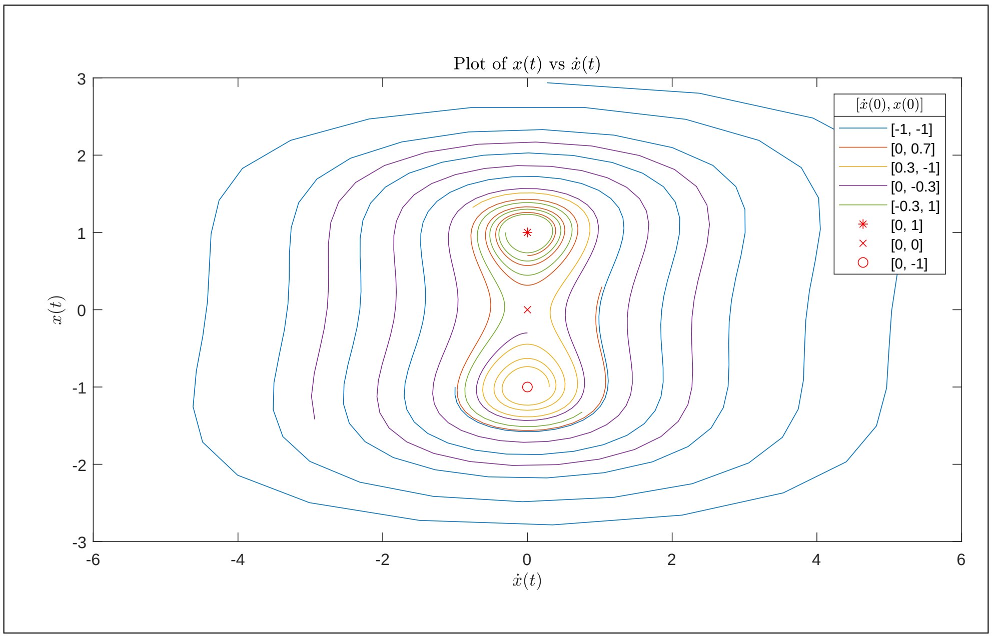

Figure 2: Plot for $a = -0.12, b = 0$ of $x(t)$ vs $\dot{x}(t)$ for $t \in [0, 20]$.

From Figure 2, we can see that there are 3 fixed points since when you use initial values $[\dot{x}(0), x(0)] = [0, -1], [0, 0], [0, 1]$, no trajectories are shown and both values of $\dot{x}(t)$ and $x(t)$ as t increases remains constant. Since the trajectories are pushed outwards, even when close to the points $[0, -1]$ and $[0, 1]$, this implies that these points are unstable fixed points. We can also see a saddle type shape forming near the point $[0, 0]$, which tells us this point must be a saddle point. All of these ideas can be confirmed analytically by considering the coupled system.

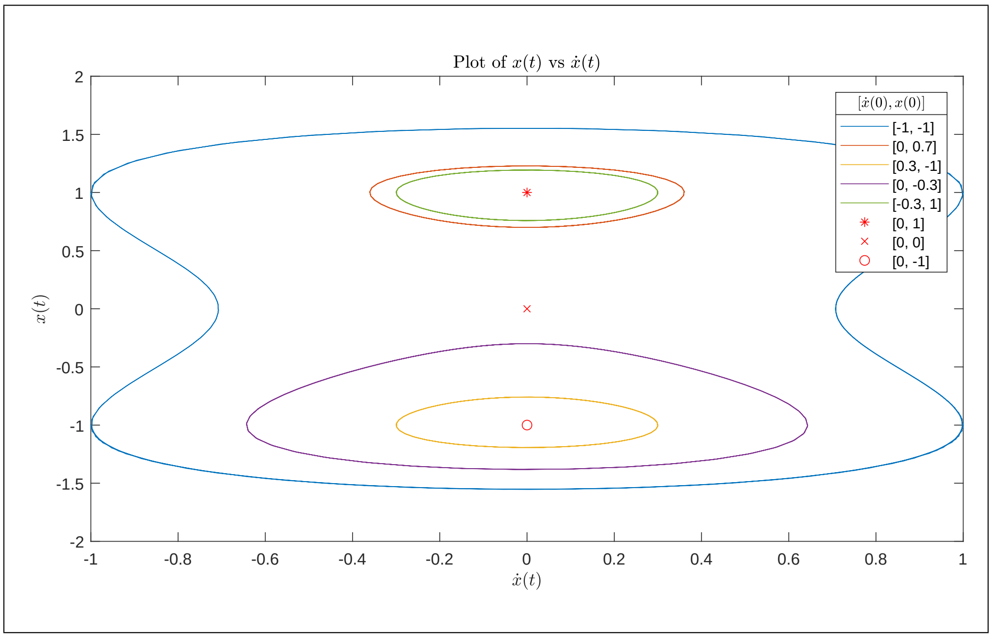

Figure 3: Plot for $a = 0, b = 0$ of $x(t)$ vs $\dot{x}(t)$ for $t \in [0, 20]$.

In the case of $a = 0$ (Figure 3), we can see that all the orbits are closed. Here both points $[0, -1]$ and $[0, 1]$ are centres and hence we get periodic orbits about them. The point $[0, 0]$

is still a saddle point as before. Furthermore, the larger orbit starting from $[-1,1]$ is also closed, which makes sense since if we consider the Poincaré index of a potential trajectory around all three fixed points, we get +1 (since a saddle is -1 and a centre is +1). This coupled system is also a Hamiltonian system, with the following Hamiltonian:

$$\left.\begin{array}{l} \dot{z} = x - x^3 = \frac{\partial H}{\partial x} \\ \dot{x} = z = -\frac{\partial H}{\partial z} \end{array}\right\} \implies H(z,x) = \tfrac{1}{2}x^2 - \tfrac{1}{4}x^4 - \tfrac{1}{2}z^2$$

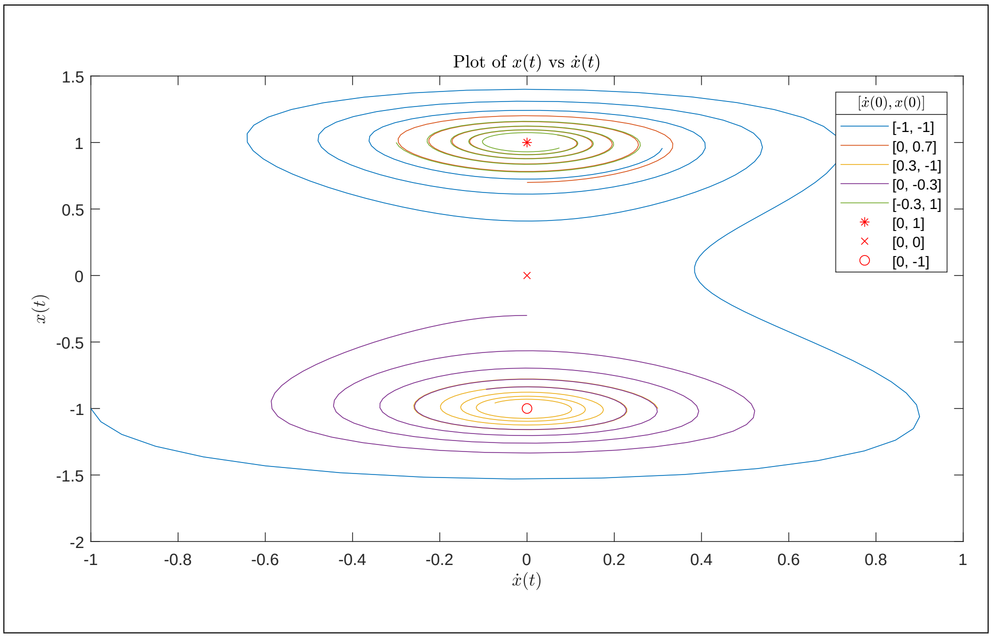

Figure 4: Plot for $a = 0.12, b = 0$ of $x(t)$ vs $\dot{x}(t)$ for $t \in [0, 20]$.

Figure 4 is the case where there are stable nodes at $[0,-1]$ and $[0,1]$ with a saddle point at $[0,0]$. As we can see from the plots, Figures $2-4$, as a changes from being positive to negative, the points $[0,-1]$ and $[0,1]$ change in their stability while $[0,0]$ remains as a saddle point. Hence we can see a as a bifurcation parameter with a bifurcation point at $a = 0$.

3. **Choose two initial conditions, one which tends towards each attractor, and adapt your program to integrate with these two sets of initial conditions only. Display your results.**

   I initially plotted the phase portrait starting at 5 uniform points along the line $\dot{x}(0) = 0$, going from -2 to 2. I let the program run for a large amount of time until I could see the features observed in Figure 5. We can see the periodic orbit exists in green and the strange attractor sits in the middle of the periodic orbit. Using this information, it was clear that any initial values within the strange attractor would definitely remain in the strange attractor. Similarly initial values beyond the periodic orbit would tend towards it. This is known as a limit cycle.

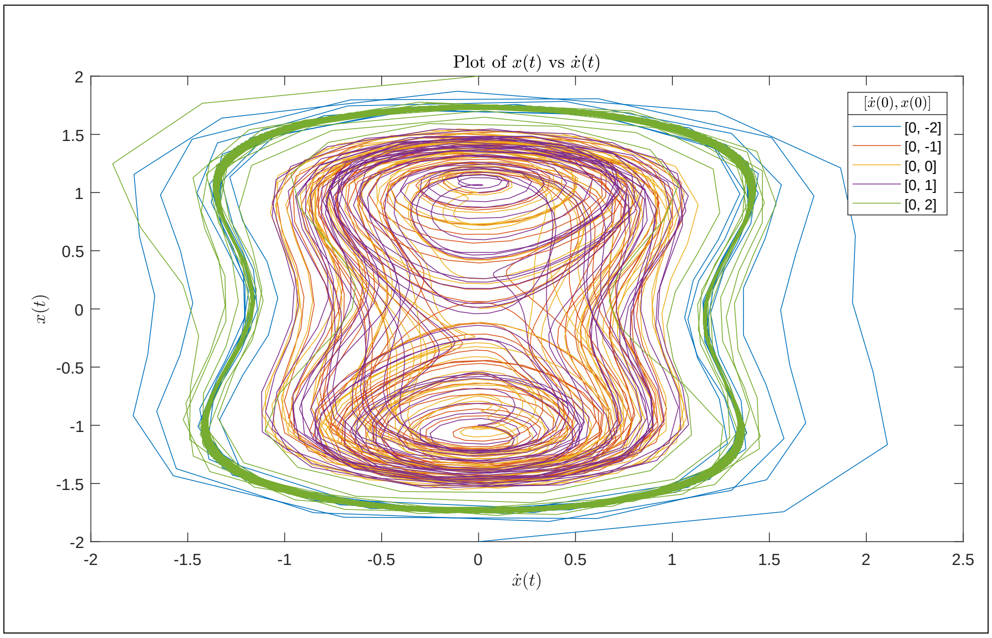

Figure 5: Plot for $a = 0.15, b = 0.3$ of $x(t)$ vs $\dot{x}(t)$ for $t \in [0, 300]$. This is for 5 generic initial values.

I then used the fact that I know where the periodic orbit is on the phase portrait to check which initial values would cause trajectories to remain on the periodic cycle. Sampling a few points I found one that worked, $[\dot{x}(0), x(0)] = [1.3, 0.5]$.

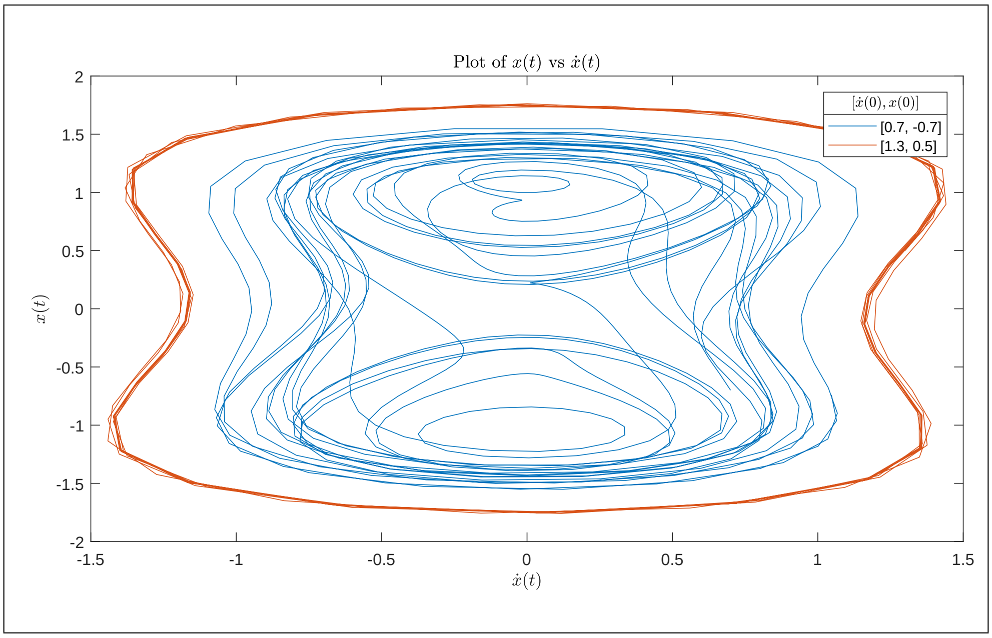

Figure 6: Plot for $a = 0.15, b = 0.3$ of $x(t)$ vs $\dot{x}(t)$ for $t \in [0, 188]$. These show initial values that tend towards the periodic orbit as well as the strange attractor.

For Figure 6, I used a my program with a step size $h = \frac{2\pi}{30} = \frac{\pi}{15}$. This then allows me to do question 4 much more comparatively since the step size is the same, and every 30 points in my vector output would equate to a time step of 2π.

4. **Repeat this numerical experiment with the same two initial conditions and the same parameter values, but this time, instead of drawing the whole solution, plot points (without joining them up) only when $t = 2n\pi$ $(n = 0, 1, 2, \ldots)$. Comment on the relationship between the two different ways (whole trajectories and points) of representing solutions.**

   Plotting these specific points in the phase plane allows us to see how the initial value points on the trajectories develop in time steps of 2π. Since the time variable is the argument of a periodic function, $\cos t$, the trajectory from a specific initial point in a time interval $t \in [0, 2\pi]$ will be different if after that time period, the trajectory ends up at a different point compared to the initial point. For a periodic orbit, we must have that the trajectory ends up at the same place after a period of 2π.

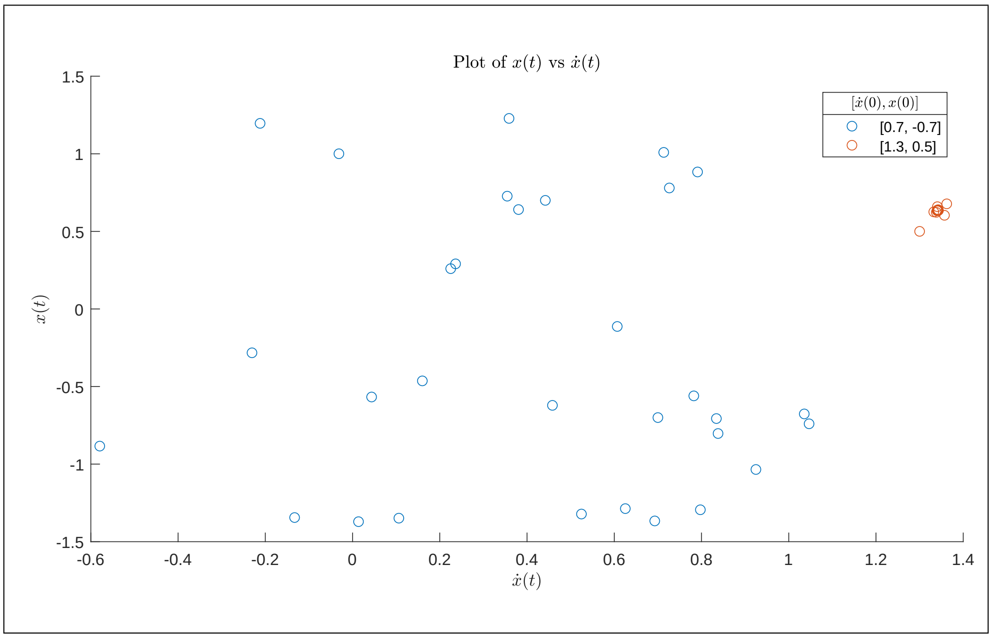

Figure 7: Plot for $a = 0.15, b = 0.3$ of $x(t)$ vs $\dot{x}(t)$ for $t \in [0, 188]$. These show only points where $t = 2n\pi$ $(n = 0, 1, 2, \ldots)$.

Here we can see exactly what we would expect, the red points represent the periodic orbit since they end up in approximately the same place after one time period of 2π. The blue points are scattered around which shows that after one time period, the trajectory does not end up in the same place which hence causes the formation of this strange attractor.

Plotting the whole trajectory gives a more visual understanding of the trajectories and how they look. It also allows us to clearly see the the two stable structures and where they exist in relation to each other. However, when only plotting the points at regular 2π intervals of time, it gives us a better idea of where the true periodic orbit exists. We can see that the red points converge towards the point $[1.341, 0.635]$ after each period which gives us the value of an initial condition that lies much closer to the periodic orbit. We can see from the following two plots (Figure 8 and Figure 9) that using the initial condition of $[x(t), \dot{x}(t)] = [1.341, 0.635]$, this gives a much more accurate idea of where the periodic orbit lies.

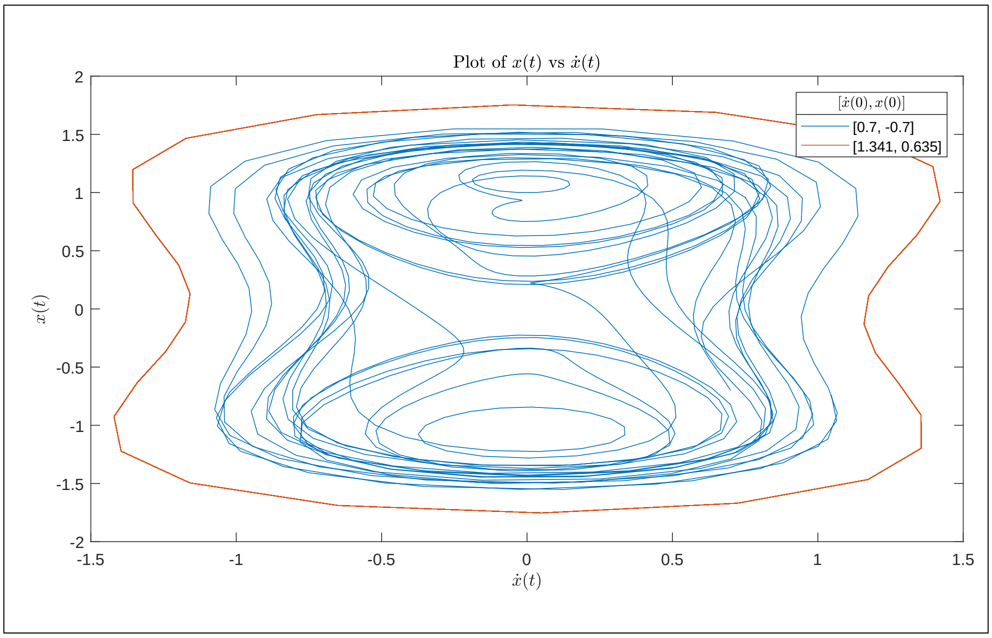

Figure 8: Plot for $a = 0.15, b = 0.3$ of $x(t)$ vs $\dot{x}(t)$ for $t \in [0, 188]$. These show only points where $t = 2n\pi$ $(n = 0, 1, 2, \ldots)$. The initial value of the periodic trajectory effectively lies on the same point after each time period.

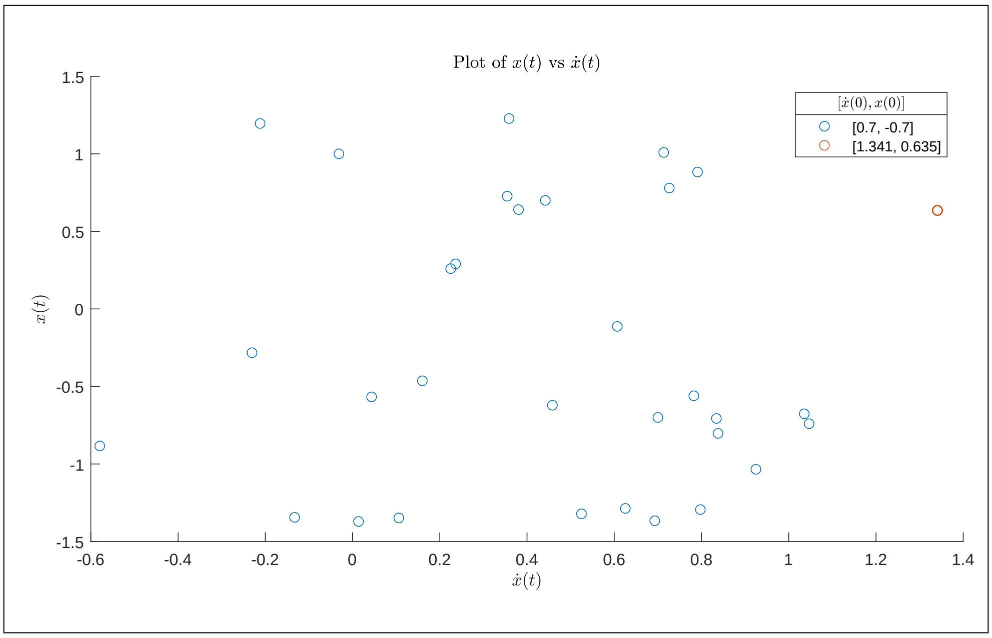

Figure 9: Plot for $a = 0.15, b = 0.3$ of $x(t)$ vs $\dot{x}(t)$ for $t \in [0, 188]$. These show initial values that lie effectively on the periodic orbit as well as the conditions that tend towards the strange attractor.

5. **Use this program to investigate in detail the behaviour of the system at different values of a between 0.1 and 0.5 (with $b = 0.3$ and a range of initial conditions), in particular the evolution of the strange attractor (when it exists). Show any pictures that seem interesting (four extra pictures are sufficient).**

Starting with $a = 0.1$, we can see that this has the same structure as when $a = 0.15$. There is a strange attractor in the middle with a periodic orbit surrounding it.

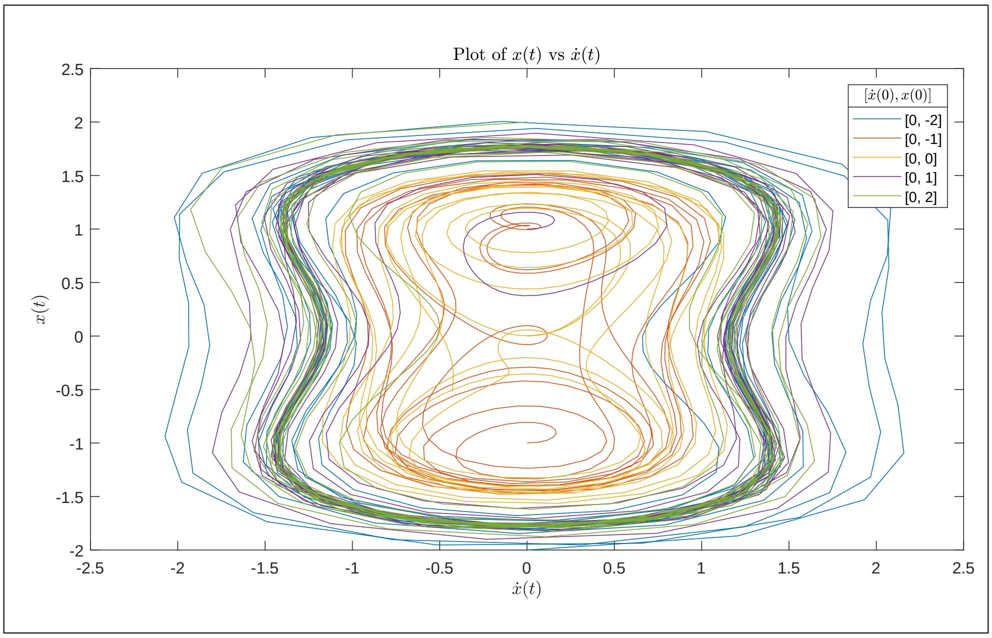

Figure 10: Plot for $a = 0.1, b = 0.3$ of $x(t)$ vs $\dot{x}(t)$ for $t \in [0, 94]$.

As we increase a passed 0.15, one value where the periodic orbit vanishes is at $a = 0.2$. Here we only have a strange attractor for which all trajectories will tend into.

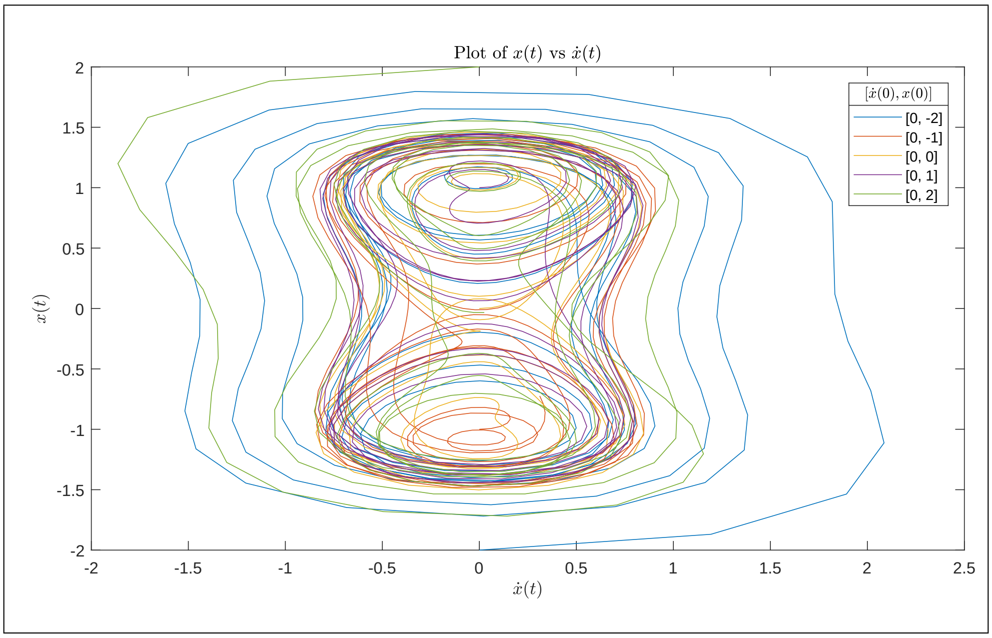

Figure 11: Plot for $a = 0.2, b = 0.3$ of $x(t)$ vs $\dot{x}(t)$ for $t \in [0, 94]$.

When a increases towards $a = 0.5$, this set of values that the strange attractor can take decreases and we observe that it gets smaller and more concentrated.

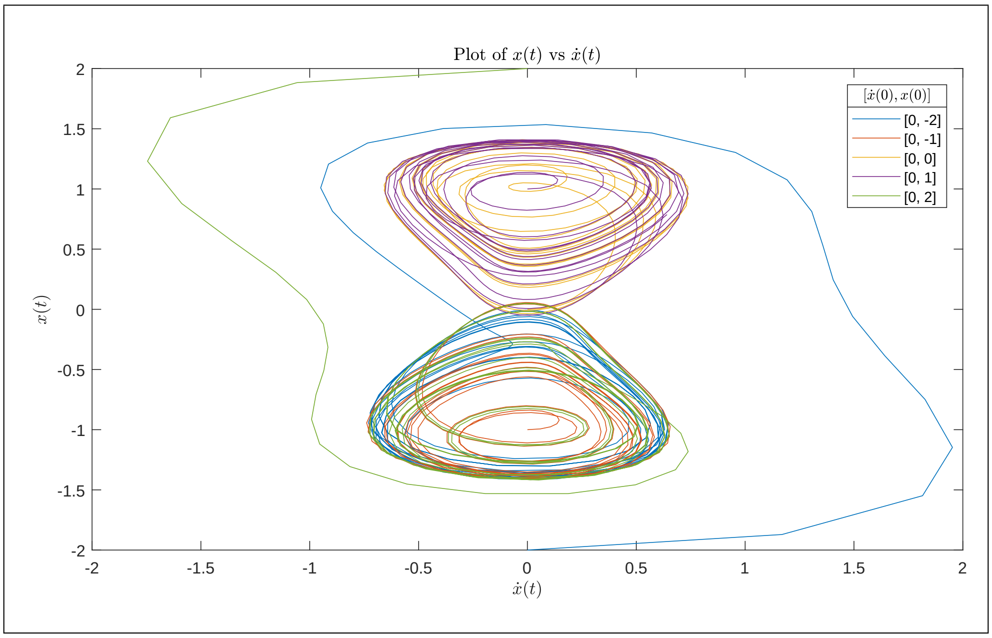

Figure 12: Plot for $a = 0.4, b = 0.3$ of $x(t)$ vs $\dot{x}(t)$ for $t \in [0, 94]$.

Finally when $a = 0.5$, the strange attractor structure is lost and instead we obtain two periodic orbits, or limit cycles. Trajectories inside the periodic orbits, as well as outside, tend towards the periodic orbit. There is one anomalous trajectory that leaves from the point [0,0] and enters into one of the periodic orbits. However this can be possible since the governing differential equation depends on the time variable as well. Hence even when this trajectory crosses the periodic orbit, it doesn't necessarily stay on it. We do see that for large time this trajectory still tends to the periodic orbit.

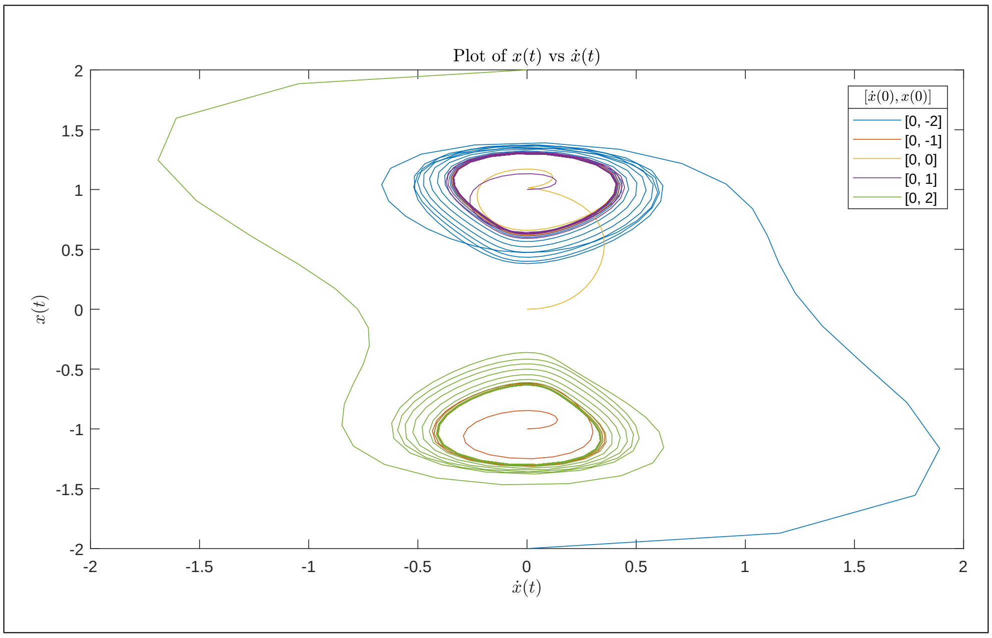

Figure 13: Plot for $a = 0.5, b = 0.3$ of $x(t)$ vs $\dot{x}(t)$ for $t \in [0, 94]$.

Part 2

6. **Run this program (with suitable choices of the initial conditions so that the pictures are as clear as possible) for a single value of (a, b) in each of the six regions, and show representative pictures for each region. You may like to set $b = \pm 1$ and vary a to find the different regions.**

7. **Describe the dynamics in each region, including the nature of any fixed points or other features, and the transition from one region to the next. What happens as you move through the boundaries between each region? There is no need to find c; you are only required to find an example of behaviour in each region.**

   I will answer both questions here together and analyse each region before showing the plot. To begin, we can consider which regions we expect there to exist fixed points as well as their nature depending on a and b. We start with the coupled differential equations:

$$\dot{z} = -(x^2 - b)z + ax - x^3$$
$$\dot{x} = z$$

   We have fixed points at $z = 0$ (where $z \equiv \dot{x}$) and $x = 0$ always. When a is positive, we have two extra fixed points at $x = \pm\sqrt{a}$. If we consider the Jacobian at each point we can analyse the local behaviour close to each fixed point.

$$J_{(z^*,x^*)} = \begin{pmatrix} \frac{\partial \dot{z}}{\partial z} & \frac{\partial \dot{z}}{\partial x} \\ \frac{\partial \dot{x}}{\partial z} & \frac{\partial \dot{x}}{\partial x} \end{pmatrix} \Bigg|_{(z^*,x^*)} = \begin{pmatrix} -(x^2 - b) & -2xz + a - 3x^2 \\ 1 & 0 \end{pmatrix} \Bigg|_{(z^*,x^*)}$$

   Where (z^*, x^*) is the fixed point we are condsidering. The point $(0, 0)$ has a Jacobian of:

$$J_{(0,0)} = \begin{pmatrix} b & a \\ 1 & 0 \end{pmatrix}$$

   Here we have $\mathrm{Det}(J_{(0,0)}) = -a$ and $\mathrm{Tr}(J_{(0,0)}) = b$. We have the eigenvalue equation:

$$\lambda^2 - b\lambda - a = 0$$

   For all different values of a and b, we either get a saddle point, stable or unstable focus (or node depending on the sign of $b^2 + 4a$, which is the discriminant of the eigenvalue equation). When a is positive we always get saddle points since the determinant in this case is negative for all $a > 0$.

   For a positive, we also get two extra fixed points at $(z, x) = (0, \pm\sqrt{a})$. The Jacobian at this fixed point is:

$$J_{(0,\pm\sqrt{a})} = \begin{pmatrix} b - a & -2a \\ 1 & 0 \end{pmatrix}$$

   In this case, we have $\mathrm{Det}(J_{(0,\pm\sqrt{a})}) = 2a$ and $\mathrm{Tr}(J_{(0,\pm\sqrt{a})}) = b - a$. We have the eigenvalue equation:

$$\lambda^2 - (b - a)\lambda + 2a = 0$$

   Since a is positive, we have that the determinant is always positive and hence we have either stable or unstable foci/nodes at $(z, x) = (0, \pm\sqrt{a})$, depending on the sign of the discriminant $(b - a)^2 - 8a$.

For Region 1, we consider $a = -1, b = -1$. We can see from the phase diagram that there is one fixed point which is a stable focus. We can also check and see that $b^2 + 4a < 0 \Rightarrow$ stable focus.

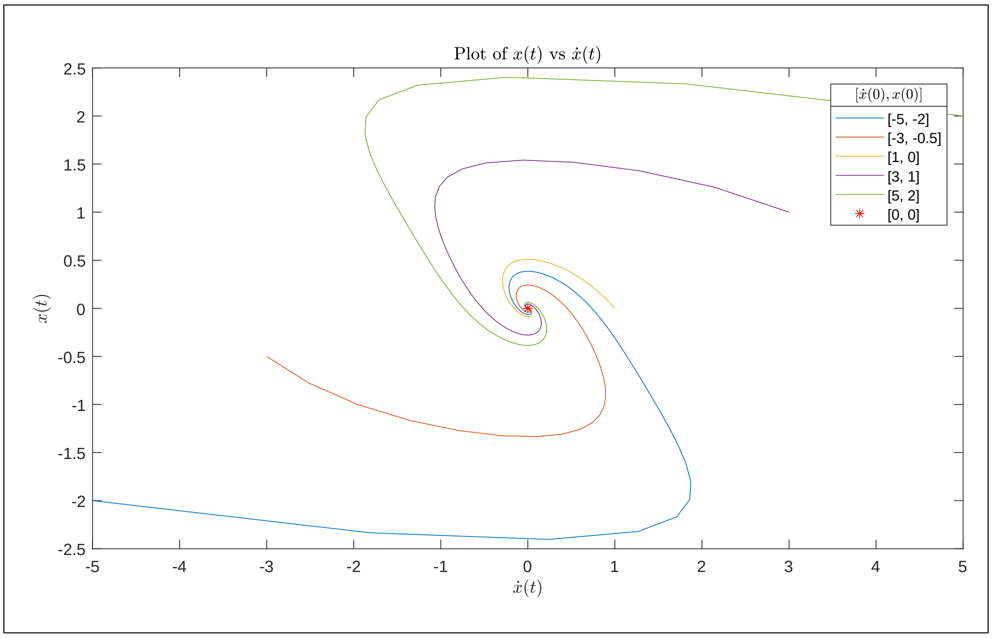

Figure 14: Region 1: Plot for $a = -1, b = -1$ of $x(t)$ vs $\dot{x}(t)$ for $t \in [0, 20]$.

At the boundary where $b = 0$, local to the fixed point at $[0, 0]$, we will have centres. As b increases (for a fixed negative a, say $a = -1$) from negative to positive, the fixed point at $[0, 0]$ goes from being a stable focus, to an unstable focus, and becoming a centre when $b = 0$. We can see b as a bifurcation variable, for a fixed a. Hence there is a supercritical Hopf bifurcation at $b = 0$.

For Region 2, we consider $a = -1, b = +1$. The fixed point is unstable, since trajectories leave the fixed point and also a focus since $b^2 + 4a < 0$. We also have a stable limit cycle that forms around this fixed point, where the trajectories inside and outside tend towards it.

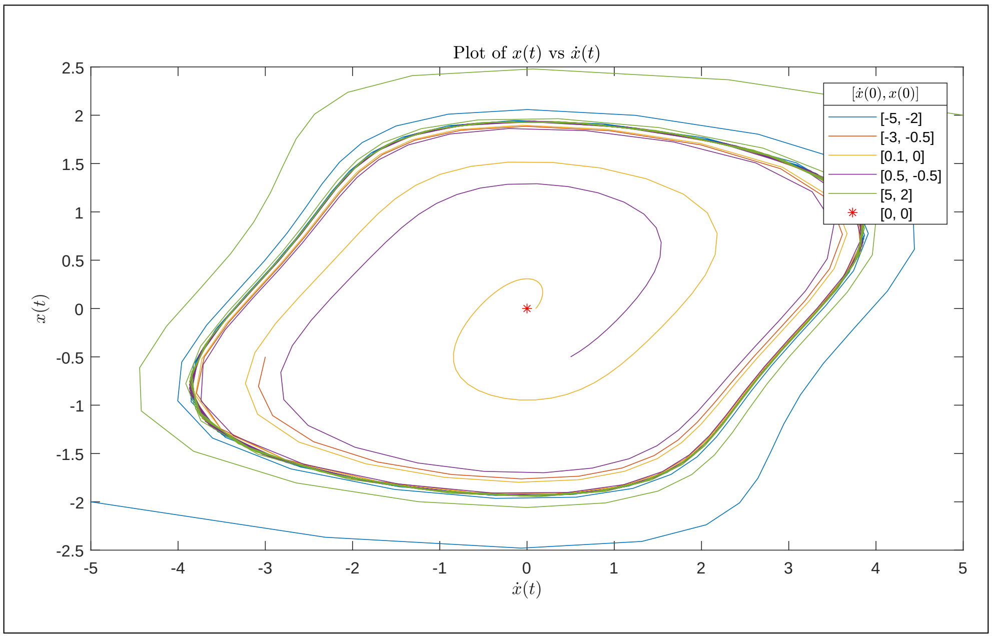

Figure 15: Region 2: Plot for $a = -1, b = +1$ of $x(t)$ vs $\dot{x}(t)$ for $t \in [0, 20]$.

When $a = 0$, we have a degenerate case where the Jacobian evaluated at the fixed point has a zero determinant. In this case we are at the boundary between saddle points and nodes/foci. For this case we expect to locally have a line of fixed points. This fixed points will be unstable since the eigenvalue $b > 0$. The trajectories will initially move parallel to the eigenvector associated with the non zero eigenvalue b. In this case, the associated eigenvector would be $\begin{pmatrix} 1 \\ 1/b \end{pmatrix}$.

For Region 3, we consider $a = 0.5, b = +1$. We always have a saddle at $(0, 0)$, from the analysis above. $(b - a)^2 - 8a < 0$ and so the other two fixed points are unstable foci from which trajectories tend into the stable limit cycle that surrounds it.

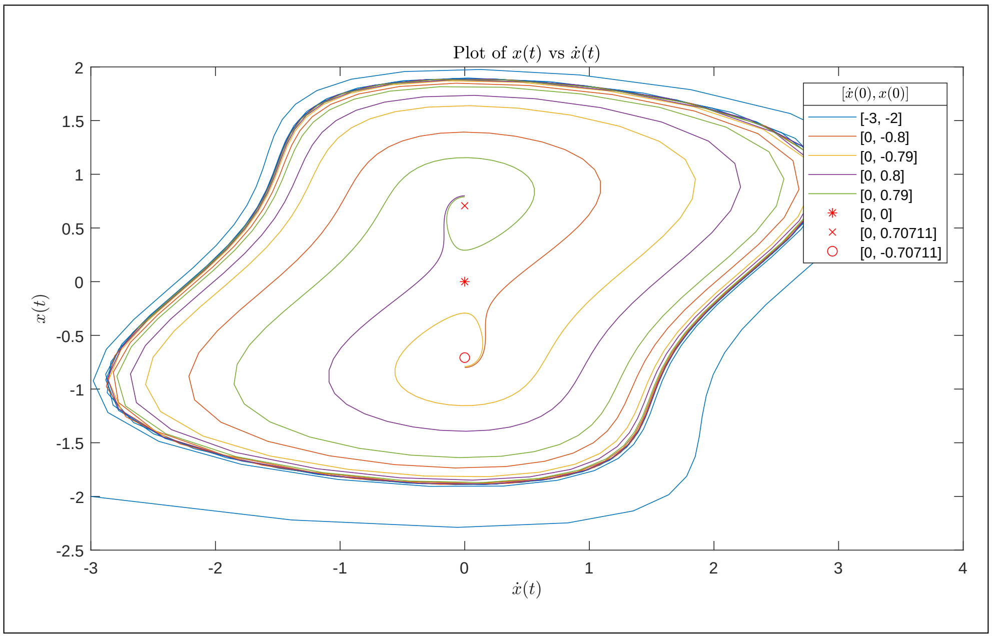

Figure 16: Region 3: Plot for $a = 0.5, b = +1$ of $x(t)$ vs $\dot{x}(t)$ for $t \in [0, 20]$.

At the boundary at $a = 1$, the foci at $(0, \pm\sqrt{a})$ become centres locally. As a increases, these foci become stable and trajectories close to it will tend inwards. Two unstable limit cycles are created around each stable foci from which trajectories inside tend inwards and trajectories outside tend towards the bigger limit cycle surrounding them both. In this case, we can view $-a$ as a bifurcation variable (for a fixed b) and see that at $a = 1$, we have a subcritical Hopf bifurcation, leading to the formation of the two unstable limit cycles. This occurs at both fixed points, $(0, \pm\sqrt{a})$. Figure 4 shows a better schematic of some of the trajectories.

For Region 4, we consider $a = \frac{9}{8}, b = +1$. $(b-a)^2 - 8a < 0$ so the we have two foci, both of which are stable since trajectories that start close tend inwards. We can see we must have two small limit cycles surrounding the two fixed points at $(0, \pm\sqrt{a})$, since the red and yellow trajectories tend inwards and the purple and green trajectories tend outwards. By the Poincaré Bendixson theorem (considering when we reverse time), there must exist an unstable limit cycle within a specific annulus surrounding the fixed point, where trajectories on the larger boundary of the annulus tend outwards and trajectories on the smaller boundary tend inwards, for all time.

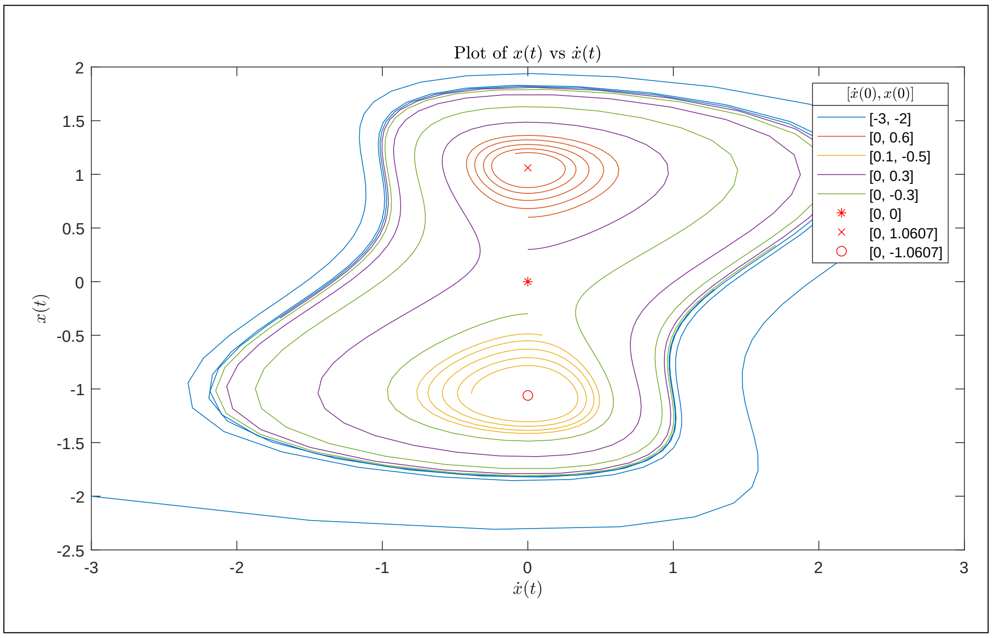

Figure 17: Region 4: Plot for $a = \frac{9}{8}, b = +1$ of $x(t)$ vs $\dot{x}(t)$ for $t \in [0, 20]$.

As we increase a from $a = \frac{9}{8}$, the two smaller unstable limit cycles get larger. There exists a critical point a_c, $\frac{9}{8} < a_c < \frac{5}{4}$, where these two limit cycles will meet. After this critical point, as a is increased further, the limit cycles effectively join together and we have one larger unstable limit cycle surrounded by the stable limit cycle from before. As a passes the point where $a = \frac{5}{4}$, these two unstable and stable limit cycles cancel each other out and hence no limit cycles exist. All trajectories now tend towards the stable foci at $(0, \pm\sqrt{a})$.

For Region 5, we consider $a = \frac{11}{8}, b = +1$. As before we have the condition that $(b - a)^2 - 8a < 0$ and so we have two stable foci again at $(0, \pm\sqrt{a})$. At this point, there are no limit cycles and instead all trajectories tend towards the stable foci at $(0, \pm\sqrt{a})$. The interesting thing to note about these trajectories is that they initially tend towards some attractor set around the fixed points, in a similar place to where the limit cycle was previously. However after passing through this attractor set, the trajectories tend towards the foci.

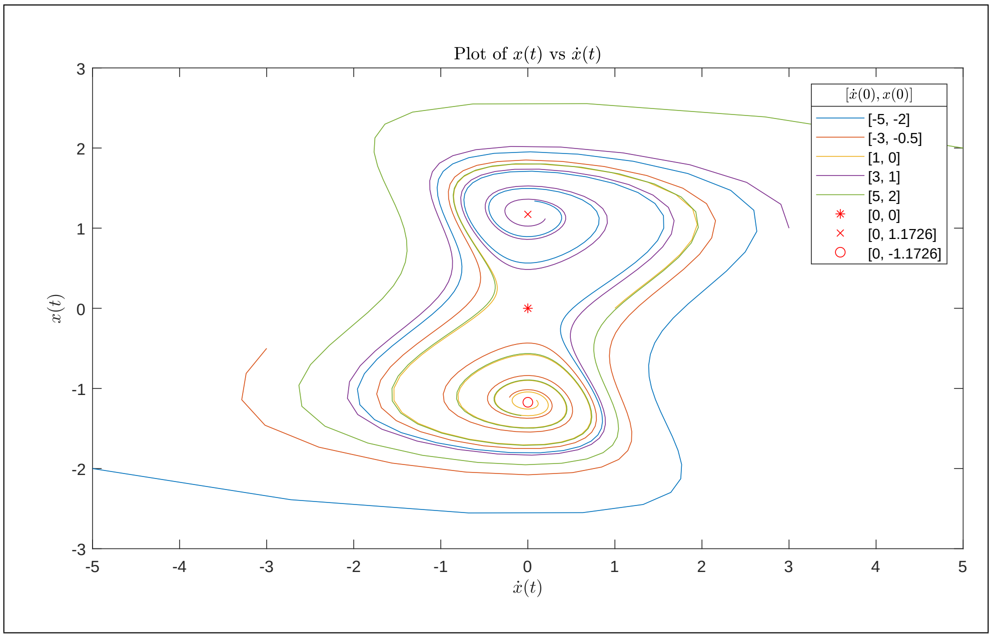

Figure 18: Region 5: Plot for $a = \frac{11}{8}, b = +1$ of $x(t)$ vs $\dot{x}(t)$ for $t \in [0, 20]$.

According to the project, if we consider a fixed $b = 1$, then when a passes some critical value $a = \frac{1}{c}$, where $0 < c < \frac{4}{5}$, the trajectories in the phase plane change. From Figure 18 and Figure 19, we can see the difference in what the phase portrait looks like. This is the point where this attractor set around the fixed points vanish and as a increases, we can see that the trajectories spiral into the fixed points directly.

For Region 6, we consider $a = \frac{15}{8}, b = +1$. $(b - a)^2 - 8a < 0$ again here, so the nature of the fixed points have not changed. We can see clearly from the trajectories how the analytic solutions agree with the phase portrait, where we get a saddle point in the middle and two stable foci above and below.

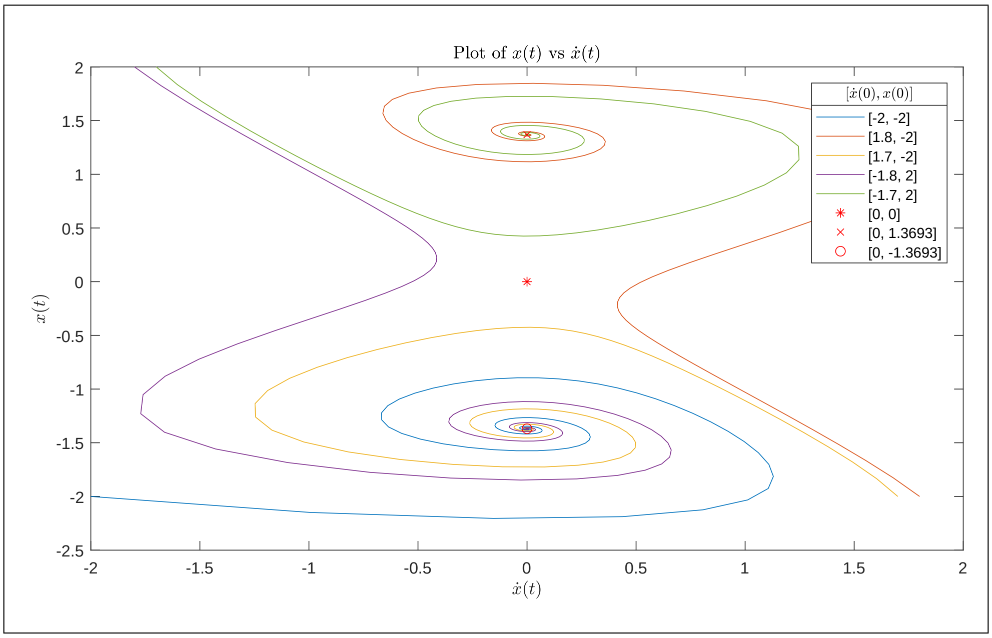

Figure 19: Region 6: Plot for $a = \frac{15}{8}, b = +1$ of $x(t)$ vs $\dot{x}(t)$ for $t \in [0, 20]$.

At the last boundary, where $a = 0$ again but $b = -1$, we get a very similar solution to the previous case when $a = 0$. We have a line of fixed points close to $(0, 0)$, where trajectories this time tend inward and it is a stable line of fixed points, since the eigenvalue $b < 0$. The associated eigenvector is the same, $\binom{1}{1/b}$, and since it depends on b, will have a different direction to before.

Part 3

8. **Locate the periodic orbit when $b = -0.001$ for $a = 1, a = 5$ and $a = 10$, giving a picture of the orbit by plotting $x(t)$ against $\dot{x}(t)$ at each time step.**

The coupled differential equations being considered in this part are:

$$\dot{x} = y - a\left(\frac{x^3}{3} - x\right)$$
$$\dot{y} = -x + 1 + b$$

There is a fixed point in the $x - y$ plane at $(x, y) = \left(1 + b, a\left(\frac{(1+b)^3}{3} - (1 + b)\right)\right)$.

In the $\dot{x} - x$ plane, there is one fixed point at $[\dot{x}(t), x(t)] = [0, 1 + b]$.

Since we know there is a Hopf bifurcation at $b = 0$, we can think about the nature of this Hopf bifurcation depending on where the periodic orbit lies, just above or below $b = 0$, as well as whether it is an attractor or not (i.e. stable limit cycle). We know there exists a periodic orbit at $b = -0.001$, which trajectories tend into, hence it is a stable limit cycle. This implies that we have a supercritical Hopf bifurcation. In the standard case for a supercritical Hopf bifurcation, we have our bifurcation variable being initially negative with a fixed point and then as it becomes positive, a stable limit cycle is produced. To make this analgous to the example we are considering in this question, we can easily just define a new bifurcation variable $\mu = -b$. This will agree with our standard notation of the supercritical Hopf Bifurcation.

When $a = 1$, by analysing where trajectories tend towards as well as tend away, I have found that a periodic orbit exists with initial values: $[\dot{x}(0), x(0)] = [0.09455, 1]$.

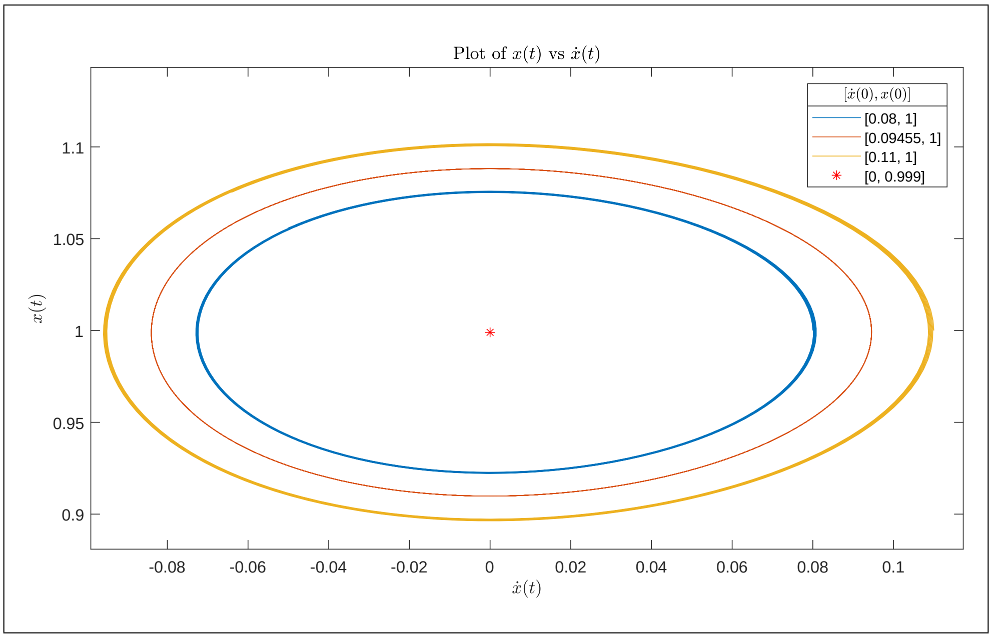

Figure 20: Plot of three trajectories for $a = 1, b = -0.001$. We can see that the yellow trajectory tends inwards while the blue trajectory tends out. The red trajectory represents the approximate location of the periodic orbit.

When $a = 5$, we have a periodic orbit starting at $[\dot{x}(0), x(0)] = [0.1235, 1]$.

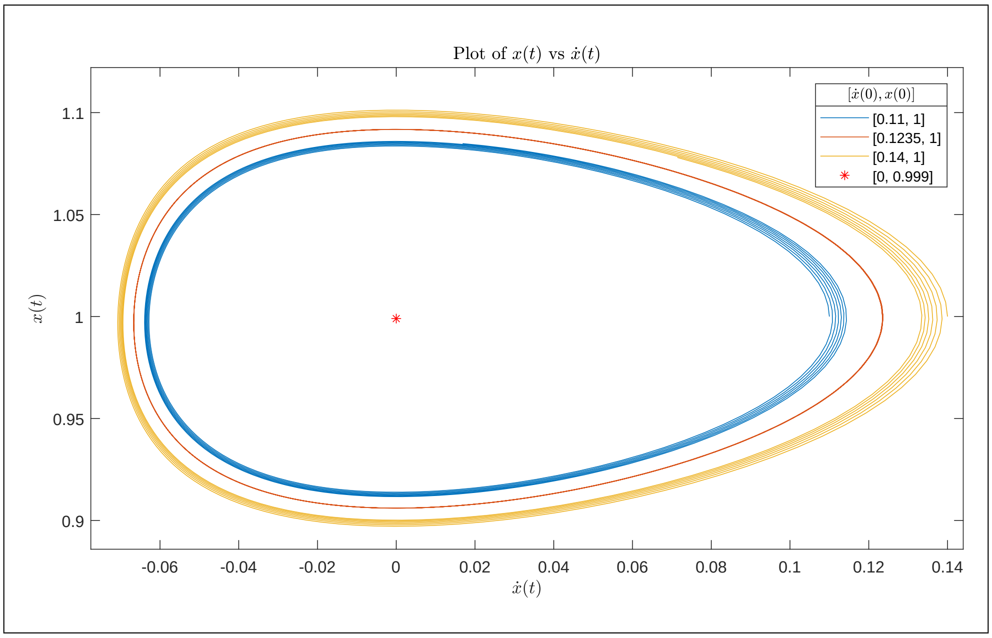

Figure 21: Plot of three trajectories for $a = 5, b = -0.001$. We can see that the yellow trajectory tends inwards while the blue trajectory tends out. The red trajectory represents the approximate location of the periodic orbit.

When $a = 10$, we have a periodic orbit starting at $[\dot{x}(0), x(0)] = [0.219, 1]$.

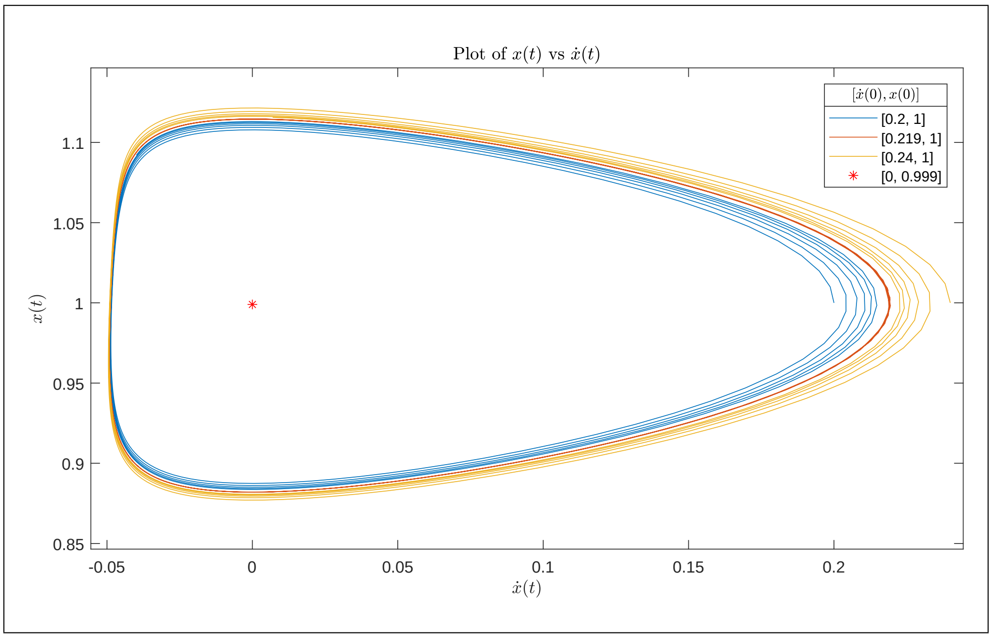

Figure 22: Plot of three trajectories for $a = 10, b = -0.001$. We can see that the yellow trajectory tends inwards while the blue trajectory tends out. The red trajectory represents the approximate location of the periodic orbit.

9. **Investigate the evolution of the periodic orbit for $b \in [-0.1, 0)$ at each of these values of a, commenting on any unusual behaviour that you observe at particular values of b.**

   For $a = 1$, we can start with the periodic orbit at $b = -0.001$ from above. As we slowly decrease b, the fixed point at the centre of the periodic orbit will move since it is a function of b (the fixed point lies at $(0, 1 + b)$. The periodic orbit itself increases in size in this range. The periodic orbit remains as a stable limit cycle with trajectories inside and outside tending towards it. Testing values of b within $[-0.1, 0)$ tells us that the periodic orbit does not vanish in this range.

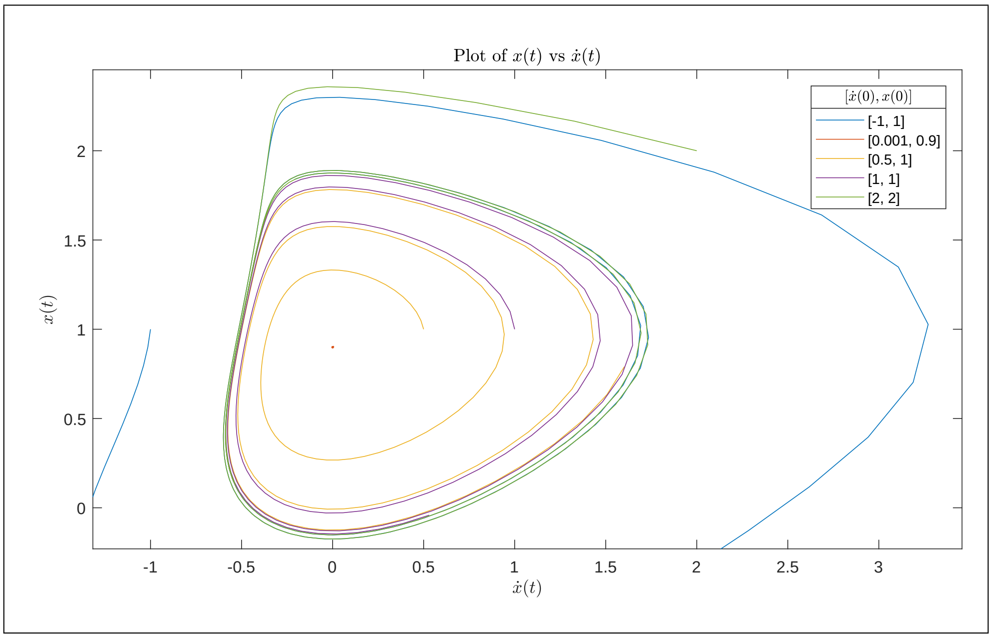

Figure 23: Plot of five trajectories for $a = 1, b = -0.1$. We can see that their still exists a periodic orbit since the yellow trajectory tends outward while the blue tends inwards.

For $a = 5$, the periodic orbit gets stretched in the positive $\dot{x}$ direction as b decreases from -0.001. There is a critical value of b_{5c}, $-0.0055 \leq b_{5c} \leq -0.005$. For $b \leq b_{5c}$, the periodic orbit vanishes and all trajectories tend outwards from the fixed point.

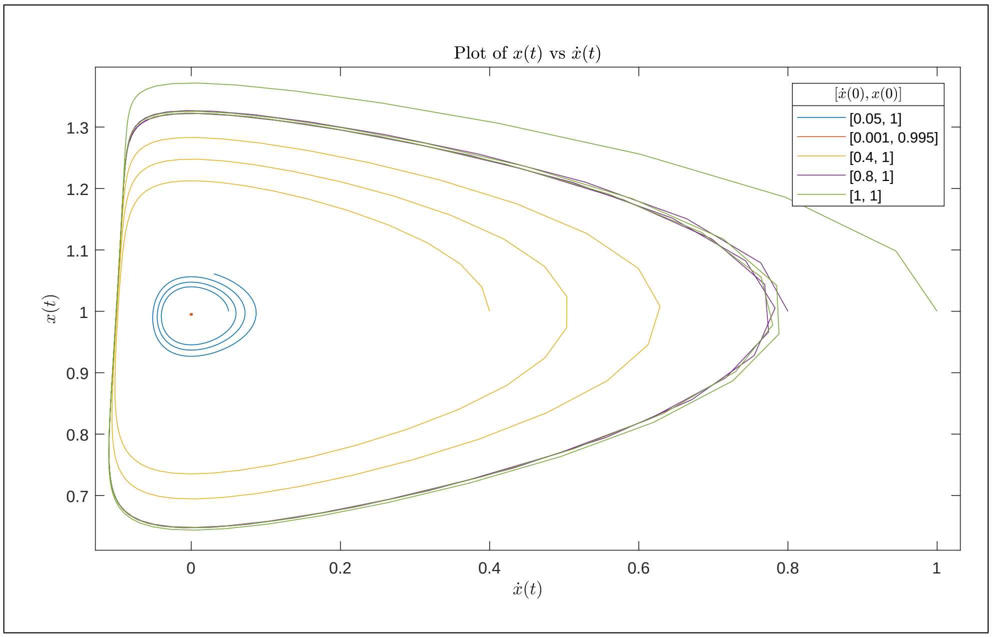

Figure 24: Plot of five trajectories for $a = 10, b = -0.00125$. We can see that their still exists a periodic orbit since the yellow trajectory tends outward while the blue tends inwards.

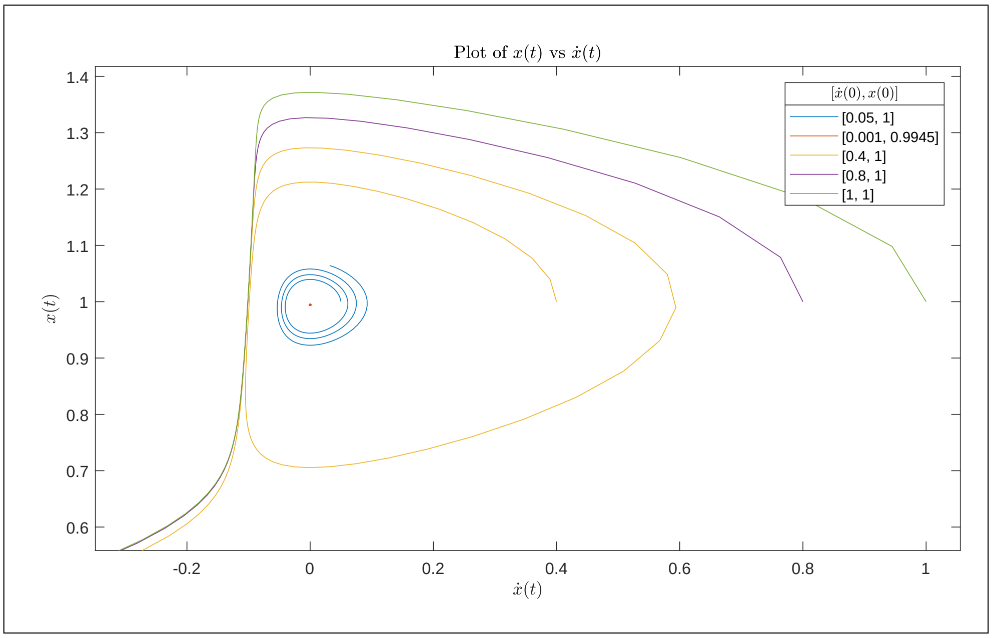

Figure 25: Plot of five trajectories for $a = 10, b = -0.0013$. We can see that there is no periodic orbit since the trajectories tend away from the fixed point.

When $a = 10$, the periodic orbit gets stretched in the positive $\dot{x}$ direction as b decreases from -0.001, faster than the case for $a = 5$. There is a critical value of b_{10c}, $-0.0013 \leq b_{10c} \leq -0.00125$. For $b \leq b_{10c}$, the periodic orbit vanishes and all trajectories tend outwards from the fixed point.

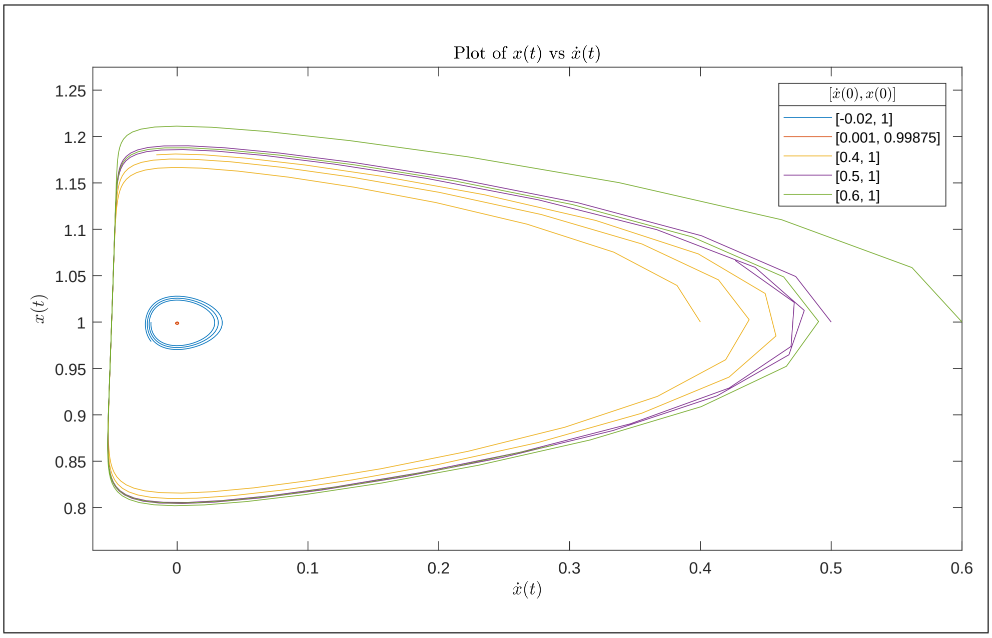

Figure 26: Plot of five trajectories for $a = 10, b = -0.00125$. We can see that their still exists a periodic orbit since the yellow trajectory tends outward while the blue tends inwards.

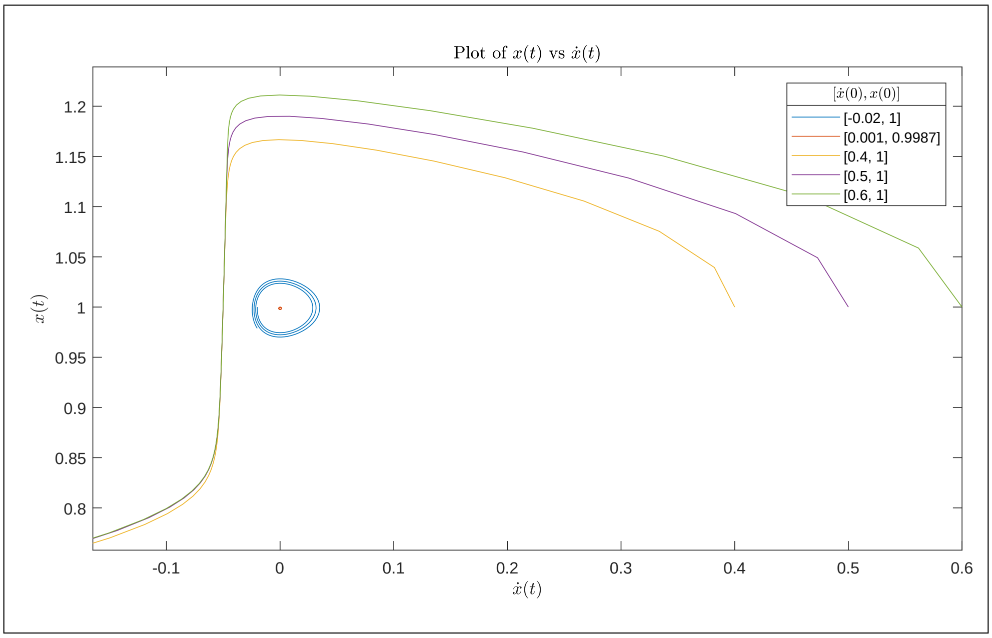

Figure 27: Plot of five trajectories for $a = 10, b = -0.0013$. We can see that there is no periodic orbit since the trajectories tend away from the fixed point.

10. **Consider also the appearance of the orbit in the $x-y$ plane. Can you explain its shape for large a? Explain what numerical difficulties can arise in calculating such an orbit.**

The plot of the orbit in the $x - y$ plane is effectively a transformation of coordinates from the $\dot{x} - x$ plane. The transformation being the following:

$$y = \dot{x} + a\left(\frac{x^3}{3} - x\right).$$

The difference for the $x - y$ plane plot is that the fixed point is now a function of both b *and* a. i.e. $(x, y) = \left(1 + b, a\left(\frac{(1+b)^3}{3} - (1+b)\right)\right)$.

Hence as a increases, the fixed point will move significantly. Specifically for $b \in [-0.1, 0)$, we have that the y coordinate of the fixed point is negative and also proportional to a.

The fixed point will move much more negative in the y direction as a increases. The fixed point still remains stable as you can see the trajectories tending towards the fixed point.

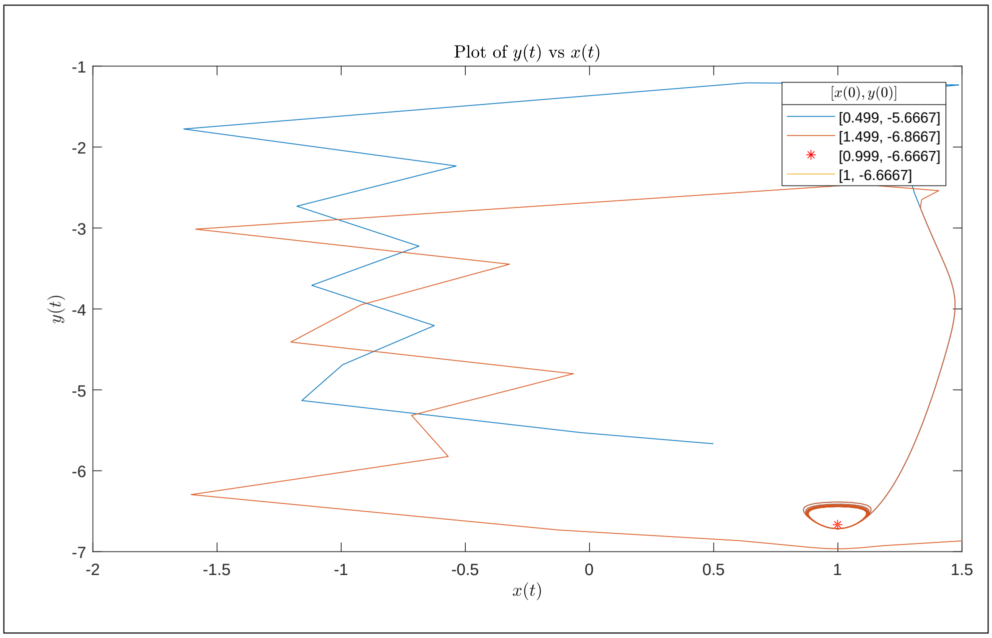

Figure 28: Plot of three trajectories for $a = 10, b = -0.001$ in the $x - y$ plane. We can see how the fixed point is sensitive to both a and b. Since b is small, it is really a that has the most significant effect on the fixed point location. For large a, we can see how the trajectories move relatively horizontally between time steps, but also travel along the curve defined below.

We can see that the periodic orbit has gotten much smaller and is concentrated very close to the fixed point. This seems to be one consequence of a larger a.

We can also see that the trajectories go horizontally quite fast as discussed below, but also go along the curve on the right side. This curve would represent the function:

$$y = a\left(\frac{x^3}{3} - x\right) \Rightarrow \dot{x} = 0$$

The dynamics on this curve would be a result of the sign of $\dot{y} = -x + 1 + b$.

As we increase a further, the dominant trajectory direction is the $\dot{x}$ direction. If $\dot{x}$ is large compared to $\dot{y}$, then this implies that over a set time period, there will be a much greater change in x compared to y. Hence we expect to see trajectories moving horizontally very fast, left when $\left(\frac{x^3}{3} - x\right) > 0$ and right when $\left(\frac{x^3}{3} - x\right) < 0$. As an example, we can see in Figure 29 how the trajectories move sharply horizontally in directions according to the sign of $\dot{x}$.

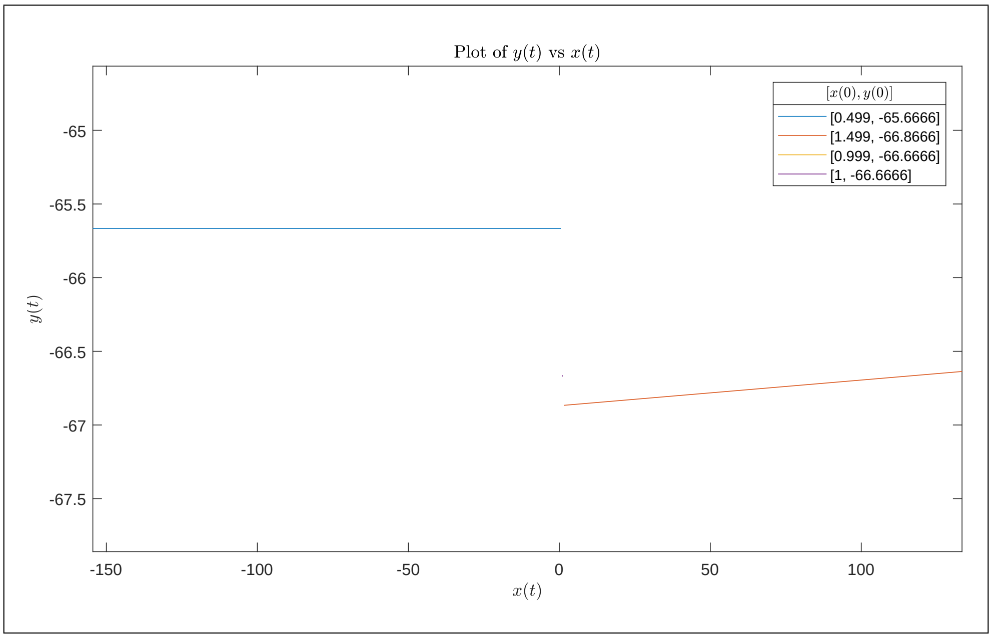

Figure 29: Plot of four trajectories for $a = 100, b = -0.001$ in the $x - y$ plane. We can see the trajectories further away from the fixed point are approximately horizontal.

In terms of the periodic orbit, we expect that it must lie close to the fixed point. For say $a = 10$, the trajectories outside tend inwards and trajectories close to the fixed point tend outwards, implying there exists a stable limit cycle, i.e. a periodic orbit. As a increases however, the trajectories seem to tend outwards close to the fixed point and also further away from the fixed point. So if a periodic orbit exists, then it must be extremely large since this is where the trajectories would head to.

The numerical difficulties when calculating an orbit for a large a would be that the time step size, despite being small, will still lead to large jumps in the phase portrait due to the trajectories moving very fast. Hence you would need to reduce the time step size to see how the trajectories travel on a more appropriate scale. A reduced time step when solving the coupled differential equation would result in a much longer time for the program to run. Furthermore, if a is large enough, then there may not necessarily be any appropriate small time step size which would be able to resolve where a periodic orbit would exist.

```
%Script to produce 5 graphs with inital conditions of x and xdot between -2
%and 2
tic
syms t x z

count=0;
N = 200;
h = 0.1;
a = 0.12;
b = 0;

InitialValues=[-1, 0.7,   -1, -0.3,     1, 1, 0, -1;
                 -1,    0, 0.3,     0, -0.3, 0, 0, 0];
%Top row is initial condition of x. Bottom row is initial condition of xdot

t_matrix=zeros(size(InitialValues,2),N+1);
X_matrix=zeros(size(InitialValues,2),N+1);
Z_matrix=zeros(size(InitialValues,2),N+1);

for i=1:size(InitialValues,2)
    Initial = [0,InitialValues(1,i),InitialValues(2,i)];
    %[Initial t, Initial x, Initial xdot]

    f_1(t,x,z) = z;
    %This is (dx/dt) = z = f_1(t,x,z)
    f_2(t,x,z) = -a*z+x-x^3+b*cos(t);
    %This is (dz/dt) = x'' = f_2(t,x,z)

    [t_vector, X_vector, Z_vector] = RungeKutta2(N,h,f_1,f_2,Initial);

t_matrix(i,:)=t_vector;
X_matrix(i,:)=X_vector;
Z_matrix(i,:)=Z_vector;

end

figure(1)
if all(X_matrix(1,:) == X_matrix(1,1)) && all(Z_matrix(1,:) == Z_matrix(1,1))
    plot(Z_matrix(1,1),X_matrix(1,1),'r+')
else
    plot(Z_matrix(1,:),X_matrix(1,:));
end
title('Plot of $x(t)$ vs $\dot{x}(t)$','interpreter','latex')
xlabel('$\dot{x}(t)$','interpreter','latex')
ylabel('$x(t)$','interpreter','latex')

hold

for i=2:size(InitialValues,2)
    if all(X_matrix(i,:) == X_matrix(i,1)) && all(Z_matrix(i,:) == Z_matrix(i,1)
        ) && count == 0
         plot(Z_matrix(i,1),X_matrix(i,1),'r*')
         count = 1
    elseif all(X_matrix(i,:) == X_matrix(i,1)) && all(Z_matrix(i,:) == Z_matrix(
        i,1)) && count == 1
         plot(Z_matrix(i,1),X_matrix(i,1),'rx')
         count = 2
    elseif all(X_matrix(i,:) == X_matrix(i,1)) && all(Z_matrix(i,:) == Z_matrix(
```

```
            i,1)) && count == 2
            plot(Z_matrix(i,1),X_matrix(i,1),'ro')
            count = 3
        else
            plot(Z_matrix(i,:),X_matrix(i,:));
        end
end

if size(InitialValues,2) == 8
    leg = legend(['[' num2str(InitialValues(2,1)) ', ' num2str(InitialValues(1,1)) ']'],['[' num2str(InitialValues(2,2)) ', ' num2str(InitialValues(1,2)) ']'],['[' num2str(InitialValues(2,3)) ', ' num2str(InitialValues(1,3)) ']'],['[' num2str(InitialValues(2,4)) ', ' num2str(InitialValues(1,4)) ']'],['[' num2str(InitialValues(2,5)) ', ' num2str(InitialValues(1,5)) ']'],['[' num2str(InitialValues(2,6)) ', ' num2str(InitialValues(1,6)) ']'],['[' num2str(InitialValues(2,7)) ', ' num2str(InitialValues(1,7)) ']'],['[' num2str(InitialValues(2,8)) ', ' num2str(InitialValues(1,8)) ']'],'Location','northeast')
    title(leg,'[$\dot{x}(0), x(0)]$','interpreter','latex')
elseif size(InitialValues,2) == 7
    leg = legend(['[' num2str(InitialValues(2,1)) ', ' num2str(InitialValues(1,1)) ']'],['[' num2str(InitialValues(2,2)) ', ' num2str(InitialValues(1,2)) ']'],['[' num2str(InitialValues(2,3)) ', ' num2str(InitialValues(1,3)) ']'],['[' num2str(InitialValues(2,4)) ', ' num2str(InitialValues(1,4)) ']'],['[' num2str(InitialValues(2,5)) ', ' num2str(InitialValues(1,5)) ']'],['[' num2str(InitialValues(2,6)) ', ' num2str(InitialValues(1,6)) ']'],['[' num2str(InitialValues(2,7)) ', ' num2str(InitialValues(1,7)) ']'],'Location','northeast')
    title(leg,'[$\dot{x}(0), x(0)]$','interpreter','latex')
elseif size(InitialValues,2) == 6
    leg = legend(['[' num2str(InitialValues(2,1)) ', ' num2str(InitialValues(1,1)) ']'],['[' num2str(InitialValues(2,2)) ', ' num2str(InitialValues(1,2)) ']'],['[' num2str(InitialValues(2,3)) ', ' num2str(InitialValues(1,3)) ']'],['[' num2str(InitialValues(2,4)) ', ' num2str(InitialValues(1,4)) ']'],['[' num2str(InitialValues(2,5)) ', ' num2str(InitialValues(1,5)) ']'],['[' num2str(InitialValues(2,6)) ', ' num2str(InitialValues(1,6)) ']'],'Location','northeast')
    title(leg,'[$\dot{x}(0), x(0)]$','interpreter','latex')
elseif size(InitialValues,2) == 5
    leg = legend(['[' num2str(InitialValues(2,1)) ', ' num2str(InitialValues(1,1)) ']'],['[' num2str(InitialValues(2,2)) ', ' num2str(InitialValues(1,2)) ']'],['[' num2str(InitialValues(2,3)) ', ' num2str(InitialValues(1,3)) ']'],['[' num2str(InitialValues(2,4)) ', ' num2str(InitialValues(1,4)) ']'],['[' num2str(InitialValues(2,5)) ', ' num2str(InitialValues(1,5)) ']'],'Location','northeast')
    title(leg,'[$\dot{x}(0), x(0)]$','interpreter','latex')
elseif size(InitialValues,2) == 4
    leg = legend(['[' num2str(InitialValues(2,1)) ', ' num2str(InitialValues(1,1)) ']'],['[' num2str(InitialValues(2,2)) ', ' num2str(InitialValues(1,2)) ']'],['[' num2str(InitialValues(2,3)) ', ' num2str(InitialValues(1,3)) ']'],['[' num2str(InitialValues(2,4)) ', ' num2str(InitialValues(1,4)) ']'],'Location','northeast')
    title(leg,'[$\dot{x}(0), x(0)]$','interpreter','latex')
elseif size(InitialValues,2) == 3
    leg = legend(['[' num2str(InitialValues(2,1)) ', ' num2str(InitialValues(1,1)) ']'],['[' num2str(InitialValues(2,2)) ', ' num2str(InitialValues(1,2)) ']'],['[' num2str(InitialValues(2,3)) ', ' num2str(InitialValues(1,3)) ']'],'Location','northeast')
    title(leg,'[$\dot{x}(0), x(0)]$','interpreter','latex')
elseif size(InitialValues,2) == 2
```

```
    leg = legend(['[' num2str(InitialValues(2,1)) ', ' num2str(InitialValues
        (1,1)) ']'],['[' num2str(InitialValues(2,2)) ', ' num2str(InitialValues
        (1,2)) ']'],'Location','northeast')
    title(leg,'[$\dot{x}(0), x(0)]$','interpreter','latex')
elseif size(InitialValues,2) == 1
    leg = legend(['[' num2str(InitialValues(2,1)) ', ' num2str(InitialValues
        (1,1)) ']'],'Location','northeast')
    title(leg,'[$\dot{x}(0), x(0)]$','interpreter','latex')
else
    disp('Adjust number of initial values');
end

hold

toc
```

```
%Script to produce 2 graphs with inital conditions:
% xdot(0) =  0.7 x(0) =  -0.7,
% xdot(0) = 1.3, x(0) =  0.5.
tic
syms t x z

count=0;
N = 450;
h = (2*pi)/30;
a = 0.2;
b = 0.3;
InitialValues=[-2, -1, 0, 1, -2; %y
               0, 0, 0, 0, 0];    %x
%Top row is initial condition of x. Bottom row is initial condition of xdot

t_matrix=zeros(size(InitialValues,2),N+1);
X_matrix=zeros(size(InitialValues,2),N+1);
Z_matrix=zeros(size(InitialValues,2),N+1);

for i=1:size(InitialValues,2)
    Initial = [0,InitialValues(1,i),InitialValues(2,i)];
    %[Initial t, Initial x, Initial xdot]

    f_1(t,x,z) = z;
    %This is (dx/dt) = z = f_1(t,x,z)
    f_2(t,x,z) = -a*z+x-x^3+b*cos(t);
    %This is (dz/dt) = x'' = f_2(t,x,z)

    [t_vector, X_vector, Z_vector] = RungeKutta2(N,h,f_1,f_2,Initial);

t_matrix(i,:)=t_vector;
X_matrix(i,:)=X_vector;
Z_matrix(i,:)=Z_vector;

end

figure(1)
if all(X_matrix(1,:) == X_matrix(1,1)) && all(Z_matrix(1,:) == Z_matrix(1,1))
    plot(Z_matrix(1,1),X_matrix(1,1),'r+')
else
    plot(Z_matrix(1,:),X_matrix(1,:));
end
title('Plot of $x(t)$ vs $\dot{x}(t)$','interpreter','latex')
xlabel('$\dot{x}(t)$','interpreter','latex')
ylabel('$x(t)$','interpreter','latex')

hold

for i=2:size(InitialValues,2)
    if all(X_matrix(i,:) == X_matrix(i,1)) && all(Z_matrix(i,:) == Z_matrix(i,1)) && count == 0
        plot(Z_matrix(i,1),X_matrix(i,1),'r*')
        count = 1
    elseif all(X_matrix(i,:) == X_matrix(i,1)) && all(Z_matrix(i,:) == Z_matrix(i,1)) && count == 1
        plot(Z_matrix(i,1),X_matrix(i,1),'rx')
        count = 2
    elseif all(X_matrix(i,:) == X_matrix(i,1)) && all(Z_matrix(i,:) == Z_matrix(i,1)) && count == 2
        plot(Z_matrix(i,1),X_matrix(i,1),'ro')
```

```
        count = 3
    else
        plot(Z_matrix(i,:),X_matrix(i,:));
    end
end

if size(InitialValues,2) == 8
    leg = legend(['[' num2str(InitialValues(2,1)) ', ' num2str(InitialValues(1,1)) ']'],['[' num2str(InitialValues(2,2)) ', ' num2str(InitialValues(1,2)) ']'],['[' num2str(InitialValues(2,3)) ', ' num2str(InitialValues(1,3)) ']'],['[' num2str(InitialValues(2,4)) ', ' num2str(InitialValues(1,4)) ']'],['[' num2str(InitialValues(2,5)) ', ' num2str(InitialValues(1,5)) ']'],['[' num2str(InitialValues(2,6)) ', ' num2str(InitialValues(1,6)) ']'],['[' num2str(InitialValues(2,7)) ', ' num2str(InitialValues(1,7)) ']'],['[' num2str(InitialValues(2,8)) ', ' num2str(InitialValues(1,8)) ']'],'Location','northeast')
    title(leg,'[$\dot{x}(0), x(0)]$','interpreter','latex')
elseif size(InitialValues,2) == 7
    leg = legend(['[' num2str(InitialValues(2,1)) ', ' num2str(InitialValues(1,1)) ']'],['[' num2str(InitialValues(2,2)) ', ' num2str(InitialValues(1,2)) ']'],['[' num2str(InitialValues(2,3)) ', ' num2str(InitialValues(1,3)) ']'],['[' num2str(InitialValues(2,4)) ', ' num2str(InitialValues(1,4)) ']'],['[' num2str(InitialValues(2,5)) ', ' num2str(InitialValues(1,5)) ']'],['[' num2str(InitialValues(2,6)) ', ' num2str(InitialValues(1,6)) ']'],['[' num2str(InitialValues(2,7)) ', ' num2str(InitialValues(1,7)) ']'],'Location','northeast')
    title(leg,'[$\dot{x}(0), x(0)]$','interpreter','latex')
elseif size(InitialValues,2) == 6
    leg = legend(['[' num2str(InitialValues(2,1)) ', ' num2str(InitialValues(1,1)) ']'],['[' num2str(InitialValues(2,2)) ', ' num2str(InitialValues(1,2)) ']'],['[' num2str(InitialValues(2,3)) ', ' num2str(InitialValues(1,3)) ']'],['[' num2str(InitialValues(2,4)) ', ' num2str(InitialValues(1,4)) ']'],['[' num2str(InitialValues(2,5)) ', ' num2str(InitialValues(1,5)) ']'],['[' num2str(InitialValues(2,6)) ', ' num2str(InitialValues(1,6)) ']'],'Location','northeast')
    title(leg,'[$\dot{x}(0), x(0)]$','interpreter','latex')
elseif size(InitialValues,2) == 5
    leg = legend(['[' num2str(InitialValues(2,1)) ', ' num2str(InitialValues(1,1)) ']'],['[' num2str(InitialValues(2,2)) ', ' num2str(InitialValues(1,2)) ']'],['[' num2str(InitialValues(2,3)) ', ' num2str(InitialValues(1,3)) ']'],['[' num2str(InitialValues(2,4)) ', ' num2str(InitialValues(1,4)) ']'],['[' num2str(InitialValues(2,5)) ', ' num2str(InitialValues(1,5)) ']'],'Location','northeast')
    title(leg,'[$\dot{x}(0), x(0)]$','interpreter','latex')
elseif size(InitialValues,2) == 4
    leg = legend(['[' num2str(InitialValues(2,1)) ', ' num2str(InitialValues(1,1)) ']'],['[' num2str(InitialValues(2,2)) ', ' num2str(InitialValues(1,2)) ']'],['[' num2str(InitialValues(2,3)) ', ' num2str(InitialValues(1,3)) ']'],['[' num2str(InitialValues(2,4)) ', ' num2str(InitialValues(1,4)) ']'],'Location','northeast')
    title(leg,'[$\dot{x}(0), x(0)]$','interpreter','latex')
elseif size(InitialValues,2) == 3
    leg = legend(['[' num2str(InitialValues(2,1)) ', ' num2str(InitialValues(1,1)) ']'],['[' num2str(InitialValues(2,2)) ', ' num2str(InitialValues(1,2)) ']'],['[' num2str(InitialValues(2,3)) ', ' num2str(InitialValues(1,3)) ']'],'Location','northeast')
    title(leg,'[$\dot{x}(0), x(0)]$','interpreter','latex')
elseif size(InitialValues,2) == 2
    leg = legend(['[' num2str(InitialValues(2,1)) ', ' num2str(InitialValues(1,1)) ']'],['[' num2str(InitialValues(2,2)) ', ' num2str(InitialValues
```

```
        (1,2)) ']'],'Location','northeast')
    title(leg,'[$\dot{x}(0), x(0)]$','interpreter','latex')
elseif size(InitialValues,2) == 1
    leg = legend(['[' num2str(InitialValues(2,1)) ', ' num2str(InitialValues
        (1,1)) ']'],'Location','northeast')
    title(leg,'[$\dot{x}(0), x(0)]$','interpreter','latex')
else
    disp('Adjust number of initial values');
end

hold

toc
```

```
%Script to produce 2 graphs with inital conditions:
% xdot(0) =  0.7 x(0) =  -0.7,
% xdot(0) = 1.3, x(0) =  0.5.
tic
syms t x z

count=0;
N = 450;
h = (2*pi)/30;
a = 0.1;
b = 0.3;
InitialValues=[-2,  -1, 0,  1, 2; %y
                 0, 0, 0, 0, 0];  %x
                %x
%Top row is initial condition of x. Bottom row is initial condition of xdot

t_matrix=zeros(size(InitialValues,2),N+1);
X_matrix=zeros(size(InitialValues,2),N+1);
Z_matrix=zeros(size(InitialValues,2),N+1);

for i=1:size(InitialValues,2)
    Initial = [0,InitialValues(1,i),InitialValues(2,i)];
    %[Initial t, Initial x, Initial xdot]

    f_1(t,x,z) = z;
    %This is (dx/dt) = z = f_1(t,x,z)
    f_2(t,x,z) = -a*z+x-x^3+b*cos(t);
    %This is (dz/dt) = x'' = f_2(t,x,z)

    [t_vector, X_vector, Z_vector] = RungeKutta2(N,h,f_1,f_2,Initial);

t_matrix(i,:)=t_vector;
X_matrix(i,:)=X_vector;
Z_matrix(i,:)=Z_vector;

end

t_scatter=zeros(size(InitialValues,2),floor((N+1)/30));

for i=1:size(InitialValues,2)
    for j=1:30:N+1 %This step size depends on the how much we divide the 2pi in the step size!
        k=1+(j-1)/30;
        t_scatter(i,k)=t_matrix(i,j);
    end
end

X_scatter=zeros(size(InitialValues,2),floor((N+1)/30));

for i=1:size(InitialValues,2)
    for j=1:30:N+1 %This step size depends on the how much we divide the 2pi in the step size!
        k=1+(j-1)/30;
        X_scatter(i,k)=X_matrix(i,j);
    end
end

Z_scatter=zeros(size(InitialValues,2),floor((N+1)/30));

for i=1:size(InitialValues,2)
```

```matlab
    for j=1:30:N+1 %This step size depends on the how much we divide the 2pi in the step size!
        k=1+(j-1)/30;
        Z_scatter(i,k)=Z_matrix(i,j);
    end
end

figure(1)
if all(X_scatter(1,:) == X_scatter(1,1)) && all(Z_scatter(1,:) == Z_scatter(1,1))
    plot(Z_scatter(1,1),X_scatter(1,1),'r+')
else
    scatter(Z_scatter(1,:),X_scatter(1,:));
end
title('Plot of $x(t)$ vs $\dot{x}(t)$','interpreter','latex')
xlabel('$\dot{x}(t)$','interpreter','latex')
ylabel('$x(t)$','interpreter','latex')

hold

for i=2:size(InitialValues,2)
    if all(X_scatter(i,:) == X_scatter(i,1)) && all(Z_scatter(i,:) == Z_scatter(i,1)) && count == 0
        plot(Z_scatter(i,1),X_scatter(i,1),'r*')
        count = 1
    else
        scatter(Z_scatter(i,:),X_scatter(i,:));
    end
end

if size(InitialValues,2) == 8
    leg = legend(['[' num2str(InitialValues(2,1)) ', ' num2str(InitialValues(1,1)) ']'],['[' num2str(InitialValues(2,2)) ', ' num2str(InitialValues(1,2)) ']'],['[' num2str(InitialValues(2,3)) ', ' num2str(InitialValues(1,3)) ']'],['[' num2str(InitialValues(2,4)) ', ' num2str(InitialValues(1,4)) ']'],['[' num2str(InitialValues(2,5)) ', ' num2str(InitialValues(1,5)) ']'],['[' num2str(InitialValues(2,6)) ', ' num2str(InitialValues(1,6)) ']'],['[' num2str(InitialValues(2,7)) ', ' num2str(InitialValues(1,7)) ']'],['[' num2str(InitialValues(2,8)) ', ' num2str(InitialValues(1,8)) ']'],'Location','northeast')
    title(leg,'[$\dot{x}(0), x(0)]$','interpreter','latex')
elseif size(InitialValues,2) == 7
    leg = legend(['[' num2str(InitialValues(2,1)) ', ' num2str(InitialValues(1,1)) ']'],['[' num2str(InitialValues(2,2)) ', ' num2str(InitialValues(1,2)) ']'],['[' num2str(InitialValues(2,3)) ', ' num2str(InitialValues(1,3)) ']'],['[' num2str(InitialValues(2,4)) ', ' num2str(InitialValues(1,4)) ']'],['[' num2str(InitialValues(2,5)) ', ' num2str(InitialValues(1,5)) ']'],['[' num2str(InitialValues(2,6)) ', ' num2str(InitialValues(1,6)) ']'],['[' num2str(InitialValues(2,7)) ', ' num2str(InitialValues(1,7)) ']'],'Location','northeast')
    title(leg,'[$\dot{x}(0), x(0)]$','interpreter','latex')
elseif size(InitialValues,2) == 6
    leg = legend(['[' num2str(InitialValues(2,1)) ', ' num2str(InitialValues(1,1)) ']'],['[' num2str(InitialValues(2,2)) ', ' num2str(InitialValues(1,2)) ']'],['[' num2str(InitialValues(2,3)) ', ' num2str(InitialValues(1,3)) ']'],['[' num2str(InitialValues(2,4)) ', ' num2str(InitialValues(1,4)) ']'],['[' num2str(InitialValues(2,5)) ', ' num2str(InitialValues(1,5)) ']'],['[' num2str(InitialValues(2,6)) ', ' num2str(InitialValues(1,6)) ']'],'Location','northeast')
    title(leg,'[$\dot{x}(0), x(0)]$','interpreter','latex')
```

```
elseif size(InitialValues,2) == 5
    leg = legend(['[' num2str(InitialValues(2,1)) ', ' num2str(InitialValues(1,1)) ']'],['[' num2str(InitialValues(2,2)) ', ' num2str(InitialValues(1,2)) ']'],['[' num2str(InitialValues(2,3)) ', ' num2str(InitialValues(1,3)) ']'],['[' num2str(InitialValues(2,4)) ', ' num2str(InitialValues(1,4)) ']'],['[' num2str(InitialValues(2,5)) ', ' num2str(InitialValues(1,5)) ']'],'Location','northeast')
    title(leg,'[$\dot{x}(0), x(0)]$','interpreter','latex')
elseif size(InitialValues,2) == 4
    leg = legend(['[' num2str(InitialValues(2,1)) ', ' num2str(InitialValues(1,1)) ']'],['[' num2str(InitialValues(2,2)) ', ' num2str(InitialValues(1,2)) ']'],['[' num2str(InitialValues(2,3)) ', ' num2str(InitialValues(1,3)) ']'],['[' num2str(InitialValues(2,4)) ', ' num2str(InitialValues(1,4)) ']'],'Location','northeast')
    title(leg,'[$\dot{x}(0), x(0)]$','interpreter','latex')
elseif size(InitialValues,2) == 3
    leg = legend(['[' num2str(InitialValues(2,1)) ', ' num2str(InitialValues(1,1)) ']'],['[' num2str(InitialValues(2,2)) ', ' num2str(InitialValues(1,2)) ']'],['[' num2str(InitialValues(2,3)) ', ' num2str(InitialValues(1,3)) ']'],'Location','northeast')
    title(leg,'[$\dot{x}(0), x(0)]$','interpreter','latex')
elseif size(InitialValues,2) == 2
    leg = legend(['[' num2str(InitialValues(2,1)) ', ' num2str(InitialValues(1,1)) ']'],['[' num2str(InitialValues(2,2)) ', ' num2str(InitialValues(1,2)) ']'],'Location','northeast')
    title(leg,'[$\dot{x}(0), x(0)]$','interpreter','latex')
elseif size(InitialValues,2) == 1
    leg = legend(['[' num2str(InitialValues(2,1)) ', ' num2str(InitialValues(1,1)) ']'],'Location','northeast')
    title(leg,'[$\dot{x}(0), x(0)]$','interpreter','latex')
else
    disp('Adjust number of initial values');
end

hold

toc
```

```matlab
%Script to produce 5 graphs with inital conditions of x and xdot between -2
%and 2
tic
syms t x z

count=0;
N = 200;
h = 0.1;
a = 15/8;
b = 1;

% InitialValues=[-1, 0, 1; %y
%                0 ,0 ,0]; %x

InitialValues=[-2, -0.5, 0, 1, 2, 0, sqrt(a), -sqrt(a) ; %y
               -5, -3, 1, 3 ,5 ,0,0,0]; %x
%Top row is initial condition of x. Bottom row is initial condition of xdot

t_matrix=zeros(size(InitialValues,2),N+1);
X_matrix=zeros(size(InitialValues,2),N+1);
Z_matrix=zeros(size(InitialValues,2),N+1);

for i=1:size(InitialValues,2)
    Initial = [0,InitialValues(1,i),InitialValues(2,i)];
    %[Initial t, Initial x, Initial xdot]

    f_1(t,x,z) = z;
    %This is (dx/dt) = z = f_1(t,x,z)
    f_2(t,x,z) = -(x^2-b)*z+a*x-x^3;
    %This is (dz/dt) = x'' = f_2(t,x,z)

    [t_vector, X_vector, Z_vector] = RungeKutta2(N,h,f_1,f_2,Initial);

t_matrix(i,:)=t_vector;
X_matrix(i,:)=X_vector;
Z_matrix(i,:)=Z_vector;

end

figure(1)
if all(X_matrix(1,:) == X_matrix(1,1)) && all(Z_matrix(1,:) == Z_matrix(1,1))
    plot(Z_matrix(1,1),X_matrix(1,1),'r+')
else
    plot(Z_matrix(1,:),X_matrix(1,:));
end
title('Plot of $x(t)$ vs $\dot{x}(t)$','interpreter','latex')
xlabel('$\dot{x}(t)$','interpreter','latex')
ylabel('$x(t)$','interpreter','latex')

hold

for i=2:size(InitialValues,2)
    if all(X_matrix(i,:) == X_matrix(i,1)) && all(Z_matrix(i,:) == Z_matrix(i,1)) && count == 0
        plot(Z_matrix(i,1),X_matrix(i,1),'r*')
        count = 1
    elseif all(X_matrix(i,:) == X_matrix(i,1)) && all(Z_matrix(i,:) == Z_matrix(i,1)) && count == 1
        plot(Z_matrix(i,1),X_matrix(i,1),'rx')
        count = 2
```

```
    elseif all(X_matrix(i,:) == X_matrix(i,1)) && all(Z_matrix(i,:) == Z_matrix(i,1)) && count == 2
        plot(Z_matrix(i,1),X_matrix(i,1),'ro')
        count = 3
    else
        plot(Z_matrix(i,:),X_matrix(i,:));
    end
end

if size(InitialValues,2) == 8
    leg = legend(['[' num2str(InitialValues(2,1)) ', ' num2str(InitialValues(1,1)) ']'],['[' num2str(InitialValues(2,2)) ', ' num2str(InitialValues(1,2)) ']'],['[' num2str(InitialValues(2,3)) ', ' num2str(InitialValues(1,3)) ']'],['[' num2str(InitialValues(2,4)) ', ' num2str(InitialValues(1,4)) ']'],['[' num2str(InitialValues(2,5)) ', ' num2str(InitialValues(1,5)) ']'],['[' num2str(InitialValues(2,6)) ', ' num2str(InitialValues(1,6)) ']'],['[' num2str(InitialValues(2,7)) ', ' num2str(InitialValues(1,7)) ']'],['[' num2str(InitialValues(2,8)) ', ' num2str(InitialValues(1,8)) ']'],'Location','northeast')
    title(leg,'[$\dot{x}(0), x(0)]$','interpreter','latex')
elseif size(InitialValues,2) == 7
    leg = legend(['[' num2str(InitialValues(2,1)) ', ' num2str(InitialValues(1,1)) ']'],['[' num2str(InitialValues(2,2)) ', ' num2str(InitialValues(1,2)) ']'],['[' num2str(InitialValues(2,3)) ', ' num2str(InitialValues(1,3)) ']'],['[' num2str(InitialValues(2,4)) ', ' num2str(InitialValues(1,4)) ']'],['[' num2str(InitialValues(2,5)) ', ' num2str(InitialValues(1,5)) ']'],['[' num2str(InitialValues(2,6)) ', ' num2str(InitialValues(1,6)) ']'],['[' num2str(InitialValues(2,7)) ', ' num2str(InitialValues(1,7)) ']'],'Location','northeast')
    title(leg,'[$\dot{x}(0), x(0)]$','interpreter','latex')
elseif size(InitialValues,2) == 6
    leg = legend(['[' num2str(InitialValues(2,1)) ', ' num2str(InitialValues(1,1)) ']'],['[' num2str(InitialValues(2,2)) ', ' num2str(InitialValues(1,2)) ']'],['[' num2str(InitialValues(2,3)) ', ' num2str(InitialValues(1,3)) ']'],['[' num2str(InitialValues(2,4)) ', ' num2str(InitialValues(1,4)) ']'],['[' num2str(InitialValues(2,5)) ', ' num2str(InitialValues(1,5)) ']'],['[' num2str(InitialValues(2,6)) ', ' num2str(InitialValues(1,6)) ']'],'Location','northeast')
    title(leg,'[$\dot{x}(0), x(0)]$','interpreter','latex')
elseif size(InitialValues,2) == 5
    leg = legend(['[' num2str(InitialValues(2,1)) ', ' num2str(InitialValues(1,1)) ']'],['[' num2str(InitialValues(2,2)) ', ' num2str(InitialValues(1,2)) ']'],['[' num2str(InitialValues(2,3)) ', ' num2str(InitialValues(1,3)) ']'],['[' num2str(InitialValues(2,4)) ', ' num2str(InitialValues(1,4)) ']'],['[' num2str(InitialValues(2,5)) ', ' num2str(InitialValues(1,5)) ']'],'Location','northeast')
    title(leg,'[$\dot{x}(0), x(0)]$','interpreter','latex')
elseif size(InitialValues,2) == 4
    leg = legend(['[' num2str(InitialValues(2,1)) ', ' num2str(InitialValues(1,1)) ']'],['[' num2str(InitialValues(2,2)) ', ' num2str(InitialValues(1,2)) ']'],['[' num2str(InitialValues(2,3)) ', ' num2str(InitialValues(1,3)) ']'],['[' num2str(InitialValues(2,4)) ', ' num2str(InitialValues(1,4)) ']'],'Location','northeast')
    title(leg,'[$\dot{x}(0), x(0)]$','interpreter','latex')
elseif size(InitialValues,2) == 3
    leg = legend(['[' num2str(InitialValues(2,1)) ', ' num2str(InitialValues(1,1)) ']'],['[' num2str(InitialValues(2,2)) ', ' num2str(InitialValues(1,2)) ']'],['[' num2str(InitialValues(2,3)) ', ' num2str(InitialValues(1,3)) ']'],'Location','northeast')
    title(leg,'[$\dot{x}(0), x(0)]$','interpreter','latex')
```

```
elseif size(InitialValues,2) == 2
    leg = legend(['[' num2str(InitialValues(2,1)) ', ' num2str(InitialValues(1,1)) ']'],['[' num2str(InitialValues(2,2)) ', ' num2str(InitialValues(1,2)) ']'],'Location','northeast')
    title(leg,'[$\dot{x}(0), x(0)]$','interpreter','latex')
elseif size(InitialValues,2) == 1
    leg = legend(['[' num2str(InitialValues(2,1)) ', ' num2str(InitialValues(1,1)) ']'],'Location','northeast')
    title(leg,'[$\dot{x}(0), x(0)]$','interpreter','latex')
else
    disp('Adjust number of initial values');
end

hold

toc
```

```
%Script to produce 5 graphs with inital conditions of x and xdot between -2
%and 2
tic
syms t x z

count=0;
N = 800;
h = 0.05;
a = 1;
b = -0.001;

InitialValues = [ 1,    1,  1;
                 -0.5,  0,  0.5];

x_vector = InitialValues(1,:);
temp = a.*((((x_vector).^3)./3)-x_vector);
y_vector = InitialValues(2,:) + temp;

Initial_ydot = [x_vector; y_vector];
%Top row is initial condition of x. Bottom row is initial condition of xdot

t_matrix=zeros(size(Initial_ydot,2),N+1);
X_matrix=zeros(size(Initial_ydot,2),N+1);
Z_matrix=zeros(size(Initial_ydot,2),N+1);

for i=1:size(InitialValues,2)
    Initial = [0,Initial_ydot(1,i),Initial_ydot(2,i)];
    %[Initial t, Initial x, Initial xdot]

    f_1(t,x,z) = z - a*((x^3/3)-x);
    %This is (dx/dt) = z = f_1(t,x,z)
    f_2(t,x,z) = -x + 1 + b;
    %This is (dz/dt) = x'' = f_2(t,x,z)

    [t_vector, X_vector, Z_vector] = RungeKutta2(N,h,f_1,f_2,Initial);

Z_vector = Z_vector - a.*((((X_vector).^3)./3) - X_vector);

t_matrix(i,:)=t_vector;
X_matrix(i,:)=X_vector;
Z_matrix(i,:)=Z_vector;

end

figure(1)
if all(X_matrix(1,:) == X_matrix(1,1)) && all(Z_matrix(1,:) == Z_matrix(1,1))
    plot(Z_matrix(1,1),X_matrix(1,1),'r+')
else
    plot(Z_matrix(1,:),X_matrix(1,:));
end
title('Plot of $x(t)$ vs $\dot{x}(t)$','interpreter','latex')
xlabel('$\dot{x}(t)$','interpreter','latex')
ylabel('$x(t)$','interpreter','latex')

hold

for i=2:size(InitialValues,2)
    if all(X_matrix(i,:) == X_matrix(i,1)) && all(Z_matrix(i,:) == Z_matrix(i,1)
        ) && count == 0
         plot(Z_matrix(i,1),X_matrix(i,1),'r*')
```

```
        count = 1
    elseif all(X_matrix(i,:) == X_matrix(i,1)) && all(Z_matrix(i,:) == Z_matrix(i,1)) && count == 1
        plot(Z_matrix(i,1),X_matrix(i,1),'rx')
        count = 2
    elseif all(X_matrix(i,:) == X_matrix(i,1)) && all(Z_matrix(i,:) == Z_matrix(i,1)) && count == 2
        plot(Z_matrix(i,1),X_matrix(i,1),'ro')
        count = 3
    else
        plot(Z_matrix(i,:),X_matrix(i,:));
    end
end

if size(InitialValues,2) == 8
    leg = legend(['[' num2str(InitialValues(2,1)) ', ' num2str(InitialValues(1,1)) ']'],['[' num2str(InitialValues(2,2)) ', ' num2str(InitialValues(1,2)) ']'],['[' num2str(InitialValues(2,3)) ', ' num2str(InitialValues(1,3)) ']'],['[' num2str(InitialValues(2,4)) ', ' num2str(InitialValues(1,4)) ']'],['[' num2str(InitialValues(2,5)) ', ' num2str(InitialValues(1,5)) ']'],['[' num2str(InitialValues(2,6)) ', ' num2str(InitialValues(1,6)) ']'],['[' num2str(InitialValues(2,7)) ', ' num2str(InitialValues(1,7)) ']'],['[' num2str(InitialValues(2,8)) ', ' num2str(InitialValues(1,8)) ']'],'Location','northeast')
    title(leg,'[$\dot{x}(0), x(0)]$','interpreter','latex')
elseif size(InitialValues,2) == 7
    leg = legend(['[' num2str(InitialValues(2,1)) ', ' num2str(InitialValues(1,1)) ']'],['[' num2str(InitialValues(2,2)) ', ' num2str(InitialValues(1,2)) ']'],['[' num2str(InitialValues(2,3)) ', ' num2str(InitialValues(1,3)) ']'],['[' num2str(InitialValues(2,4)) ', ' num2str(InitialValues(1,4)) ']'],['[' num2str(InitialValues(2,5)) ', ' num2str(InitialValues(1,5)) ']'],['[' num2str(InitialValues(2,6)) ', ' num2str(InitialValues(1,6)) ']'],['[' num2str(InitialValues(2,7)) ', ' num2str(InitialValues(1,7)) ']'],'Location','northeast')
    title(leg,'[$\dot{x}(0), x(0)]$','interpreter','latex')
elseif size(InitialValues,2) == 6
    leg = legend(['[' num2str(InitialValues(2,1)) ', ' num2str(InitialValues(1,1)) ']'],['[' num2str(InitialValues(2,2)) ', ' num2str(InitialValues(1,2)) ']'],['[' num2str(InitialValues(2,3)) ', ' num2str(InitialValues(1,3)) ']'],['[' num2str(InitialValues(2,4)) ', ' num2str(InitialValues(1,4)) ']'],['[' num2str(InitialValues(2,5)) ', ' num2str(InitialValues(1,5)) ']'],['[' num2str(InitialValues(2,6)) ', ' num2str(InitialValues(1,6)) ']'],'Location','northeast')
    title(leg,'[$\dot{x}(0), x(0)]$','interpreter','latex')
elseif size(InitialValues,2) == 5
    leg = legend(['[' num2str(InitialValues(2,1)) ', ' num2str(InitialValues(1,1)) ']'],['[' num2str(InitialValues(2,2)) ', ' num2str(InitialValues(1,2)) ']'],['[' num2str(InitialValues(2,3)) ', ' num2str(InitialValues(1,3)) ']'],['[' num2str(InitialValues(2,4)) ', ' num2str(InitialValues(1,4)) ']'],['[' num2str(InitialValues(2,5)) ', ' num2str(InitialValues(1,5)) ']'],'Location','northeast')
    title(leg,'[$\dot{x}(0), x(0)]$','interpreter','latex')
elseif size(InitialValues,2) == 4
    leg = legend(['[' num2str(InitialValues(2,1)) ', ' num2str(InitialValues(1,1)) ']'],['[' num2str(InitialValues(2,2)) ', ' num2str(InitialValues(1,2)) ']'],['[' num2str(InitialValues(2,3)) ', ' num2str(InitialValues(1,3)) ']'],['[' num2str(InitialValues(2,4)) ', ' num2str(InitialValues(1,4)) ']'],'Location','northeast')
    title(leg,'[$\dot{x}(0), x(0)]$','interpreter','latex')
elseif size(InitialValues,2) == 3
```

```
    leg = legend(['[' num2str(InitialValues(2,1)) ', ' num2str(InitialValues
        (1,1)) ']'],['[' num2str(InitialValues(2,2)) ', ' num2str(InitialValues
        (1,2)) ']'],['[' num2str(InitialValues(2,3)) ', ' num2str(InitialValues
        (1,3)) ']'],'Location','northeast')
    title(leg,'[$\dot{x}(0), x(0)]$','interpreter','latex')
elseif size(InitialValues,2) == 2
    leg = legend(['[' num2str(InitialValues(2,1)) ', ' num2str(InitialValues
        (1,1)) ']'],['[' num2str(InitialValues(2,2)) ', ' num2str(InitialValues
        (1,2)) ']'],'Location','northeast')
    title(leg,'[$\dot{x}(0), x(0)]$','interpreter','latex')
elseif size(InitialValues,2) == 1
    leg = legend(['[' num2str(InitialValues(2,1)) ', ' num2str(InitialValues
        (1,1)) ']'],'Location','northeast')
    title(leg,'[$\dot{x}(0), x(0)]$','interpreter','latex')
else
    disp('Adjust number of initial values');
end

hold

toc
```

```
function [t_vector, X_vector, Z_vector] = RungeKutta2(N,h,f_1,f_2,Initial)
% Runge Kutta method to solve first order Ordinary Differential Equations
% of the form y'=f_1(x,y,z) and z'=f_2(x,y,z)
% Input f(x,y), N is the number of iterations, h is the step length, x_0 is
% the initial x value and Y_0 is the corresponding initial y
% value.(t_0,X_0) gives us a point on the solution curve.
syms t
syms x
syms z
f_1(t,x,z)= f_1;
%This is (dx/dt) = z = f_1(t,x,z)
f_2(t,x,z)= f_2;
%This is (dz/dt) = x'' = f_2(t,x,z)
t_0=Initial(1);
X_0=Initial(2);
Z_0=Initial(3);
count=1;
t_1=t_0;
Z_1=Z_0;
X_1=X_0;
X_2=[X_1,Z_1]';
t_vector = zeros(1,N+1);
X_vector = zeros(1,N+1);
Z_vector = zeros(1,N+1);
for count=1:N
    k_11=double(h*f_1(t_1,X_1,Z_1));
    k_12=double(h*f_2(t_1,X_1,Z_1));
    k_21=double(h*f_1(t_1+(1/2)*h,X_1+(1/2)*k_11,Z_1+(1/2)*k_12));
    k_22=double(h*f_2(t_1+(1/2)*h,X_1+(1/2)*k_11,Z_1+(1/2)*k_12));
    k_31=double(h*f_1(t_1+(1/2)*h,X_1+(1/2)*k_21,Z_1+(1/2)*k_22));
    k_32=double(h*f_2(t_1+(1/2)*h,X_1+(1/2)*k_21,Z_1+(1/2)*k_22));
    k_41=double(h*f_1(t_1+h,X_1+k_31,Z_1+k_32));
    k_42=double(h*f_2(t_1+h,X_1+k_31,Z_1+k_32));

    t_vector(1,count)= t_1;
    X_vector(1,count)= X_1;
    Z_vector(1,count)= Z_1;

    k_1=[k_11,k_12]';
    k_2=[k_21,k_22]';
    k_3=[k_31,k_32]';
    k_4=[k_41,k_42]';

    Y_3=X_2+(1/6)*(k_1+2*k_2+2*k_3+k_4);

    t_1=t_0+count*h;
    X_2=Y_3;
    X_1=X_2(1,1);
    Z_1=X_2(2,1);
    count=count+1;
end
t_vector(1,N+1)=t_1;
X_vector(1,N+1)=X_1;
Z_vector(1,N+1)=Z_1;
end
```