Project: 4.3 The Rotating Top

May 1, 2019

1. **Explain the general theory underlying the motion of a rotating top, starting from equations (1)–(3). Demonstrate that your program can simulate the three main types of motion below and explain what initial conditions are required.**

   1. **Normal precession ($\dot{\phi}$ does not change sign)**
   2. **Retrograde motion ($\dot{\phi}$ changes sign)**
   3. **Motion with cusps (the border between 1 and 2)**

   You should obtain one hard copy of each type of motion.

   The motion of a rotating top can be captured by considering the rotational dynamics of a rigid body. Within this project, we consider a symmetric top from which we can obtain the Lagrangian:

$$L = \frac{1}{2}[\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta] + \frac{1}{2}C[\dot{\psi} + \dot{\phi}\cos\theta]^2 - \cos\theta.$$

   Here, as stated in the project, C is the ratio of the principle moments of inertia, i.e. $C = \frac{I_3}{I_1}$. WLOG, throughout this project I will set the value of $C = 0.5$ which will just change the energy value and not affect the motion since

   Now, starting from equations $(1) - (3)$, we have:

$$\frac{\partial L}{\partial \dot{\psi}} = \text{constant} \Rightarrow C[\dot{\psi} + \dot{\phi}\cos\theta] = \alpha, \tag{1}$$

$$\frac{\partial L}{\partial \dot{\phi}} = \text{constant} \Rightarrow \alpha\cos\theta + \dot{\phi}\sin^2\theta = \beta, \tag{2}$$

$$E = \frac{1}{2}[\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta] + \frac{1}{2}C[\dot{\psi} + \dot{\phi}\cos\theta]^2 + \cos\theta = \text{constant}. \tag{3}$$

   It is only necessary to consider the reduced energy:

$$E' = \frac{1}{2}[\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta] + \cos\theta = \text{constant}.$$

   This is constant since the omitted term can be replaced with $\alpha^2/2C$, which is also constant and hence does not change E' over time.

   We are considering the Euler Angles in this project to define the motion of the rotating top. The Euler angles are a set of generalised coordinates (ϕ, θ, ψ) for a rigid body rotating in a fixed frame and provide a useful analysis of the motion.

   When a top is spun with initial angular velocity $\dot{\psi}$ in the body frame, it spins around without toppling over due to the conservation of angular momentum. If we consider the torque vector about the pivot due to the mass of the top, it has direction perpendicular to the force vector due to gravity and the position vector of the centre of mass. This is effectively a vector in the $x - y$ plane in the space frame.

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$$

   The torque due to the top spinning acts in the z direction in the body frame and hence the net torque on the body causes the top to precess around the z axis in the space frame at a rate $\dot{\phi}$. Due to friction, this slows down the spin of the top and hence the torque due to gravity is greater causing the top to tilt more until the torque due to gravity causes the top to hit the ground.

For a sleeping top, this is when the angle $\theta = 0$, there will still be a precession rate as we will see later on in this project. This precession rate is dependent on $\dot{\psi}$ and C only.

The first thing to note about this motion is that ψ and ϕ are ignorable coordinates, hence we need not consider their values to define our motion. Specifying $\psi(0)$ will just give us the initial orientation of the top about the line of symmetry which will obviously have no effect on the motion. Specifying $\phi(0)$ will just give us an arbitrary starting point relative to the fixed space axis and hence we can set $\phi(0) = 0$, WLOG.

Equation (1) gives us our first constant of the motion which also enables us to be able to redefine $\dot{\psi}$ in terms of θ, $\dot{\phi}$ and a chosen constant α. From equation (1), we can also see that α and $\dot{\psi}$ are positively correlated, so as one quantity increases or decreases, the other does the same. Specifying alpha is equivalent to specifying $\dot{\psi}(0)$, given we also specify $\theta(0)$ and $\dot{\phi}(0)$. $\dot{\psi}(0)$ is the initial angular velocity with which the top is set in motion about the axis of symmetry. In this project it is most appropriate for me specify α to analyse the different motions of the symmetric top.

Equation (2) also allows us to redefine $\dot{\phi}$ in terms of θ and a constant β, however it is more useful for me to be able to define $\dot{\phi}$, which will then specify the constant β.

My initial conditions that need to be specified are $\theta, \dot{\theta}$ and $\dot{\phi}$ at time $t = 0$, as well as α.

As a spinning top rotates, there is a wobble or precession that is observed. This is defined to be the value of $\dot{\phi}$. Nutation is the rate of change of θ, i.e. $\dot{\theta}$. This is the observed rocking or swaying motion up and down.

My program works by numerically solving (using the Runge-Kutta method) four coupled first order differential equations. The differential equations being:

$$
\begin{aligned}
\dot{Y} &= Z, && \text{(I)}\\
\dot{Z} &= [X(X\cos Y - \alpha) + 1]\sin Y, && \text{(II)}\\
\dot{W} &= X, && \text{(III)}\\
\dot{X} &= \frac{Z\big(\alpha\sin^2 Y - 2\cos Y(\beta - \alpha\cos Y)\big)}{\sin^3 Y}, && \text{(IV)}
\end{aligned}
$$

where $Y = \theta, Z = \dot{\theta}, W = \phi,$ and $X = \dot{\phi}$.

Equation (IV) is derived from differentiating equation (4) given in the project. Adding this equation enables me to be able to choose the initial conditions of $\dot{\phi}$. My program can simulate the three main types of motion, shown below.

When θ is small, my program still works since in the limit as θ tends to 0, $\dot{\phi}$ converges. For $\theta = 0$, this is only possible if $\alpha = \beta$ since, from equation 2:

$$\alpha\cos(0) + \dot{\phi}\sin^2(0) = \beta \Rightarrow \alpha = \beta.$$

Hence this allows us to simplify the equation for $\dot{\phi}$ for the case of $\alpha = \beta$ and also $\alpha = -\beta$, which is the case that allows $\theta = \pi$.

$$
\begin{aligned}
\text{When } \alpha = \beta &\Rightarrow \dot{\phi} = \frac{\alpha}{1+\cos\theta}\\
\text{When } \alpha = -\beta &\Rightarrow \dot{\phi} = \frac{\alpha}{1-\cos\theta}
\end{aligned}
$$

This can then be differentated to get an equivalent version of equation (IV) to use in my program for the two cases above. This analysis also indicates that for $\alpha \neq \beta$, θ can never pass through $\theta = 0$ or π.

The initial conditions used for Figure 1 produce a constantly increasing ϕ. This is obtained by considering equation (4):

$$\dot{\phi} = \frac{\beta - \alpha\cos\theta}{\sin^2\theta}, \tag{4}$$

where we can see that the condition for $\dot{\phi} > 0,$ for all θ, requires that $0 < |\alpha| < \beta$, which is true since $\alpha = 1, \beta = 1.563232$. A similar condition of $\beta < -|\alpha| < 0$ can be obtained for the condition of $\dot{\phi} < 0,$ for all θ.

The initial conditions used for Figure 2 produce a varying value of $\dot{\phi}$ as θ changes $\big($since $\mathrm{sgn}(\dot{\phi}) = \mathrm{sgn}(\beta - \alpha\cos\theta)\big)$. This requires that $-|\alpha| < \beta < |\alpha|$. This results in the loops seen in the graph as the change in ϕ goes from being positive to negative, then positive and so on.

The initial conditions used for Figure 3 produce periodic cusps. To form these, the initial conditions of $\dot{\phi}(0) = \dot{\theta}(0) = 0$ is used. This forms a cusp since at this point there is no change in $\dot{\phi}$ or $\dot{\theta}$ since this is the intermediatary stage between the two previous cases. At time $t = 0$, the graph will start at a cusp and since it is a periodic function, this pattern will repeat as the system evolves in time.

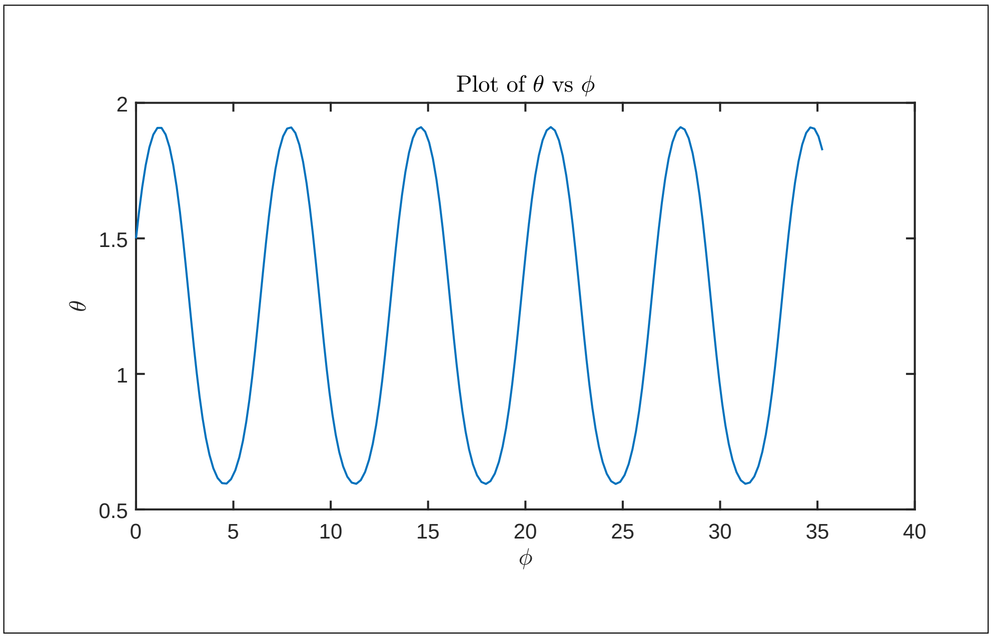

Figure 1: $\alpha = 1, \beta = 1.563232$, Initial Conditions: $[\theta(0), \dot{\theta}(0), \dot{\phi}(0)] = [1.5,\ 1,\ 1.5]$ - Wavy.

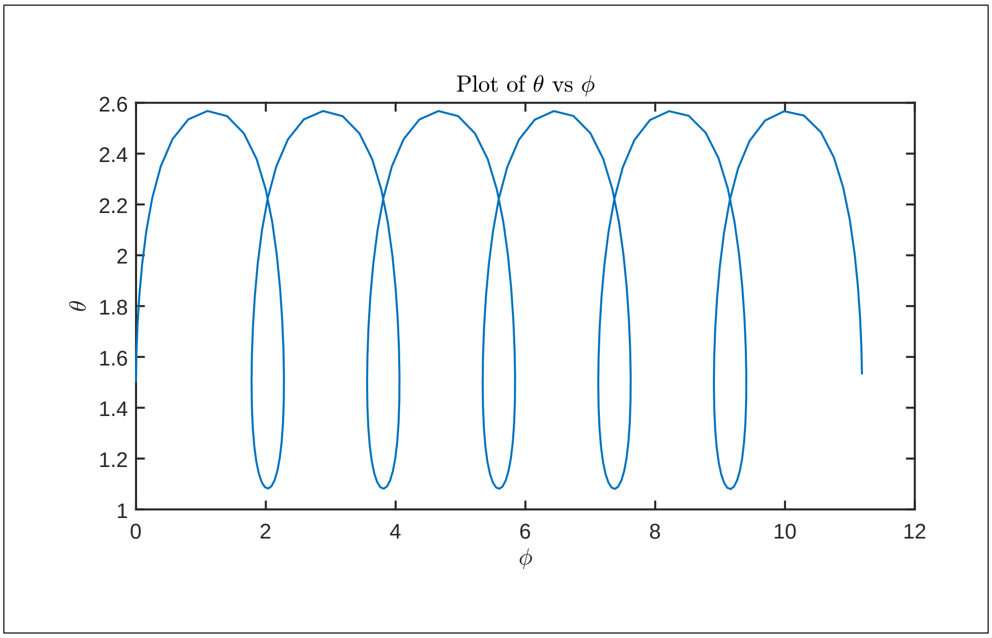

Figure 2: $\alpha = 1, \beta = 0.070737$, Initial Conditions: $[\theta(0), \dot{\theta}(0), \dot{\phi}(0)] = [1.5,\ 1,\ 0]$ - Loopy.

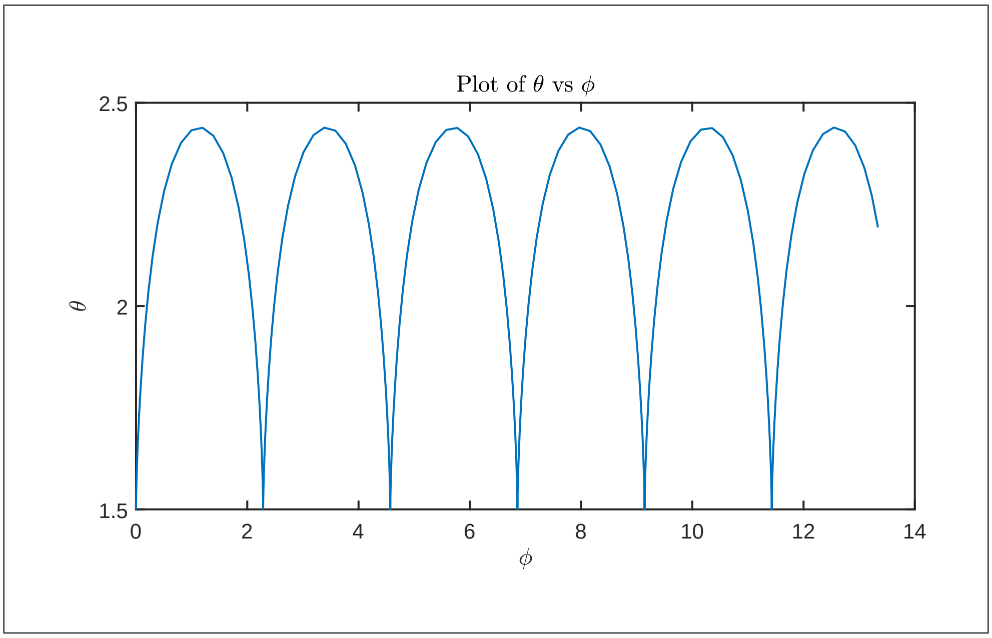

Figure 3: $\alpha = 1, \beta = 0.070737$, Initial Conditions: $[\theta(0), \dot{\theta}(0), \dot{\phi}(0)] = [1.5,\ 0,\ 0]$ - Cuspy.

2. **Choose values for α, θ such that there are 2 values of $\dot{\phi}$ that give a solution with constant θ, and show that your program can replicate the motion. Explain how these results fit the general theory.**

   If θ is constant, then $\dot{\theta} = 0 \Rightarrow \ddot{\theta} = 0$. Hence equation (5) in the project must be equal to zero.

$$\ddot{\theta} = [\dot{\phi}(\dot{\phi}\cos\theta - \alpha) + 1]\sin\theta = 0. \tag{5}$$

   For $\theta \neq 0$ or π, we require the following quadratic equation to be satisfied for $\ddot{\theta} = 0$.

$$\dot{\phi}^2(\cos\theta) + \dot{\phi}(-\alpha) + 1 = 0$$

The solution of this quadratic equation is:

$$\dot{\phi}_{\pm} = \frac{\alpha \pm \sqrt{\alpha^2 - 4\cos\theta}}{2\cos\theta}.$$

For example, choosing $\alpha = 1$ and $\theta = 1.5$ will give us a solution of $\dot{\phi}_+ = 13.053872$ and $\dot{\phi}_- = 1.082961$. These give us the initial conditions to start with since we choose: $\alpha = 1, \theta = 1.5, \dot{\theta} = 0, \dot{\phi} = \dot{\phi}_{\pm}$. The graphs below show the output of these initial conditions from my program.

When $\theta = 0$ or π, this is a special case of when θ is constant which will be investigated in question 3.

Having two values of $\dot{\phi}$ for which θ is constant agrees with the general theory since the initial $\dot{\psi}$ is related to α, θ and $\dot{\phi}$ from equation (1). Once we spin the top, for a fixed α (which is equivalent to a fixed $\dot{\psi}$) and θ, if we give the top a specific component of $\dot{\phi}$ equal to one of $\dot{\phi}_{\pm}$, then the top will spin with zero nutation. It happens to be that there exists two solutions, one with a fast precession, $\dot{\phi}_+$, and one with a slow precession, $\dot{\phi}_-$, for when θ is constant and one solution in the case of $\alpha = 4\cos\theta$.

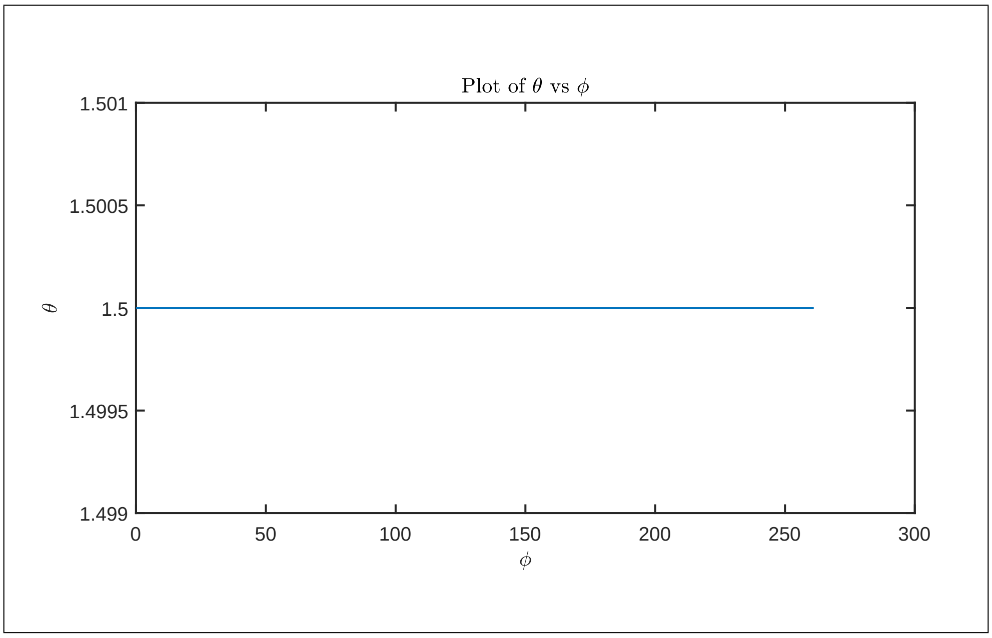

Figure 4: Plot of θ vs ϕ, with initial conditions: $\alpha = 1, \beta = 13.059291, [\theta(0), \dot{\theta}(0), \dot{\phi}(0)] = [1.5, 0, \dot{\phi}_+]$.

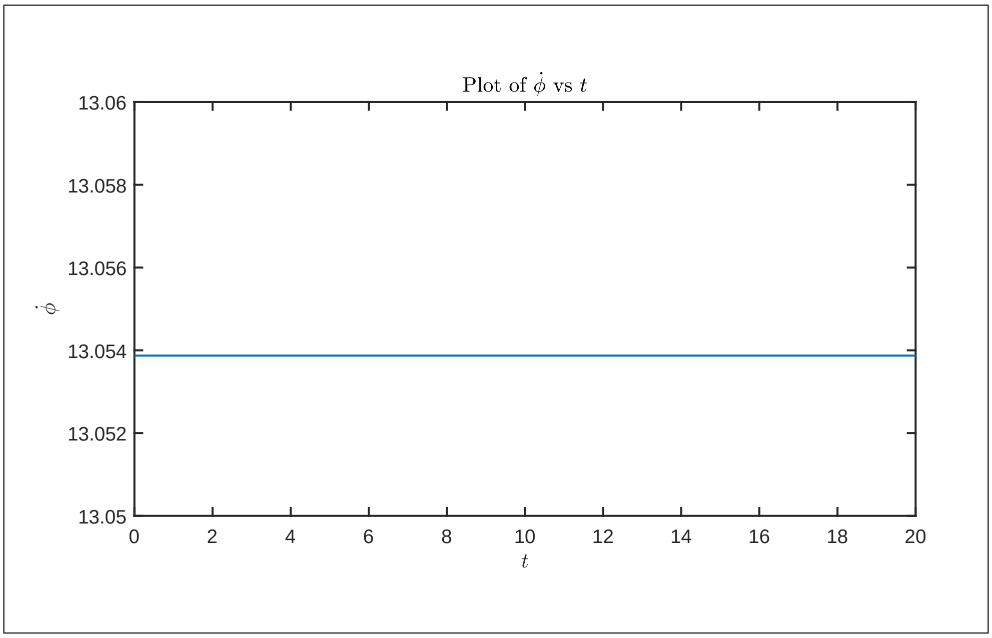

Figure 5: Plot of $\dot{\phi}$ vs t, with initial conditions $\alpha = 1, \beta = 13.059291, [\theta(0), \dot{\theta}(0), \dot{\phi}(0)] = [1.5, 0, \dot{\phi}_+]$.

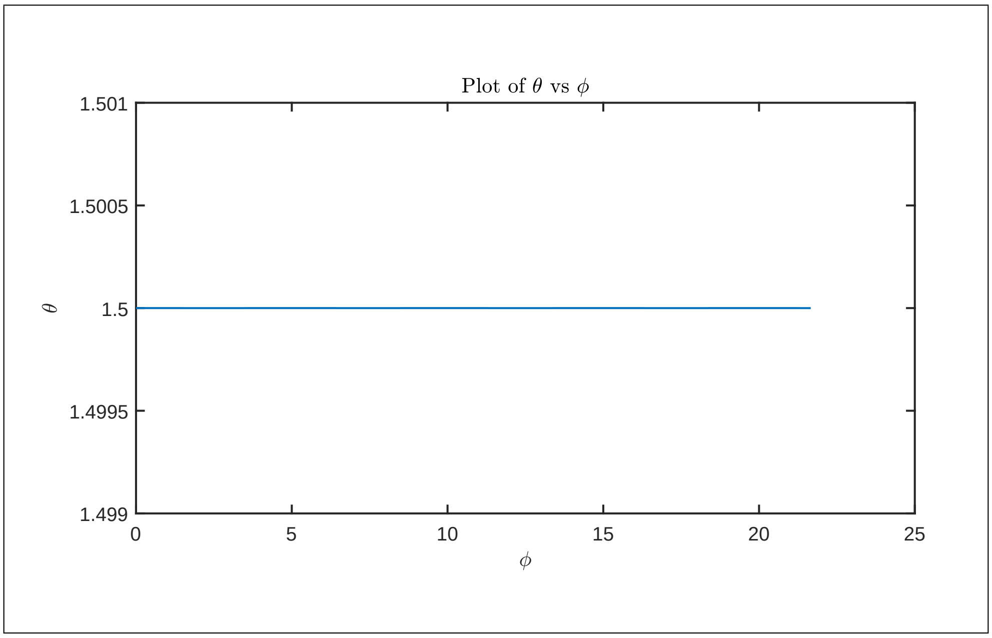

Figure 6: Plot of θ vs ϕ, with initial conditions $\alpha = 1, \beta = 1.148279$, $[\theta(0), \dot{\theta}(0), \dot{\phi}(0)] = [1.5,\ 0,\ \dot{\phi}_-]$.

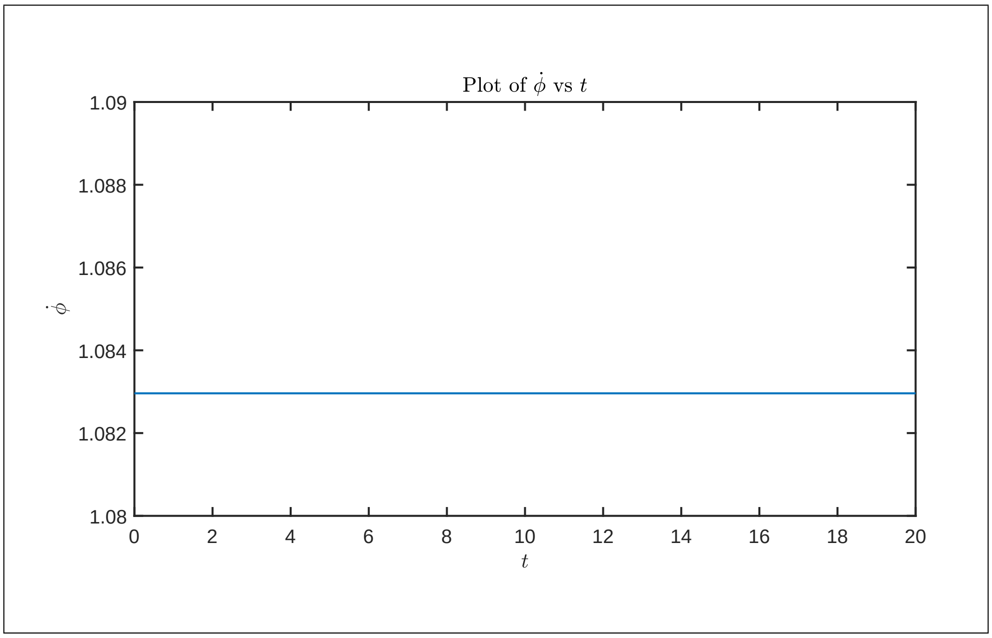

Figure 7: Plot of $\dot{\phi}$ vs t, with initial conditions $\alpha = 1, \beta = 1.148279$, $[\theta(0), \dot{\theta}(0), \dot{\phi}(0)] = [1.5,\ 0,\ \dot{\phi}_{-}]$.

3. **Investigate the stability of a sleeping top (i.e., one that spins with $\theta = 0$) by giving the motion a small disturbance; specifically, use initial conditions $\theta = 0$ and $\dot{\theta}$ non-zero but small. Explain clearly the possible types of subsequent motion and give an appropriate criterion for stability. Obtain a rough estimate for the critical value of α, which should be independent of C, and show examples of the motion just above and below the critical value. Is your estimate consistent with the theoretical predictions?**

   When $\theta = 0$, this forces the condition of $\alpha = \beta$ as explained above. When $\theta = 0$, the top spins at a uniform rate $\dot{\psi}$ with uniform precession $\dot{\phi}$. Since the line of symmetry is aligned with the z axis in the space frame, there will be no visual indication of the precession.

   To consider stability of the sleeping top, we can consider the equation for the reduced energy:

$$E' = \frac{1}{2}[\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta] + \cos\theta$$

$$E' = \frac{1}{2}\dot{\theta}^2 + \underbrace{\frac{1}{2}\left(\frac{\beta - \alpha\cos\theta}{\sin\theta}\right)^2 + \cos\theta,}_{V_{\text{eff}}(\theta)}$$

   where $V_{\text{eff}}(\theta)$ is the effective potential. In the case of $\alpha = \beta$, we get a simplified version of this effective potential:

$$V_{\text{eff}}(\theta) = \frac{\alpha^2(1 - \cos\theta)}{2(1 + \cos\theta)} + \cos\theta$$

If we consider expanding this to second order in θ, we get:

$$\begin{aligned} V_{\text{eff}}(\theta) &\approx \frac{\alpha^2}{2}\left(\frac{\frac{\theta^2}{2}}{2-\frac{\theta^2}{2}}\right)+1-\frac{\theta^2}{2} \\ &\approx \frac{\alpha^2\theta^2}{8}+1-\frac{\theta^2}{2}, \\ V''_{\text{eff}}(\theta) &\approx \frac{\alpha^2}{4}-1 \end{aligned}$$

Hence the condition for $\theta = 0$ to be stable is equivalent to the turning point of $V_{\text{eff}}(\theta)$ being a minimum. This occurs iff:

$$V''_{\text{eff}}(\theta) > 0 \Rightarrow \alpha^2 > 4$$

Since these values are approximations, we have an estimate for the critical value of α, being $\alpha_c = 2$.

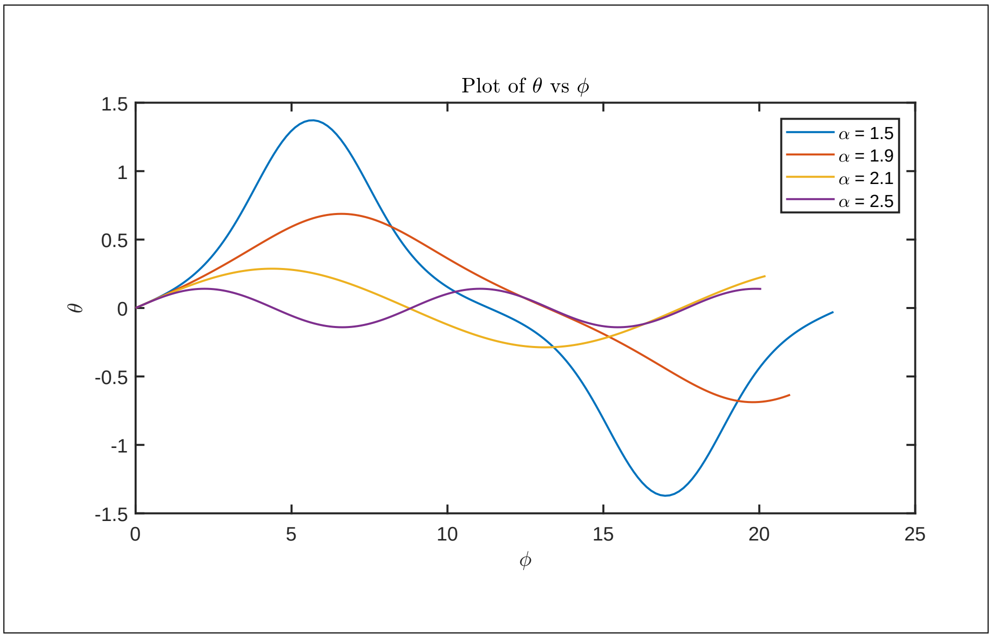

Figure 8: Plot of θ vs ϕ, with initial conditions $[\theta(0), \dot{\theta}(0), \dot{\phi}(0)] = [0, 0.1, 1]$, for varying values of α ($\alpha = 1.5, 1.9, 2.1, 2.5$).

Figure 8 shows the plot for two stable solutions with ($\alpha^2 > 2$) vs two unstable solution ($\alpha^2 < 2$). This is consistent with theoretical predictions since if we spin the top with a large $\dot{\psi}$, which is equivalent to a large α, we expect that the top stay spinning, hence stable, for a longer time. As friction slows down the top, and hence reduces α, the top passes the unstable condition of $\alpha^2 < 2$ and will then wobble more and eventually topple over when the excursions of the top pass a value of θ such that the side of the spinning top makes contact with the floor. This can only happen when $\alpha^2 < 2$ since this allows for θ to vary much more.

4. **Take α, θ very small (both 0.01 say) with initial conditions $\dot{\theta} = \dot{\phi} = 0$. What happens? Give a physical interpretation. Now change α to zero, and explain your results.**

When $\alpha = \theta(0) = 0.01$, this induces a very slow precession in the top. The initial condition represents spinning the top at a very slow speed at an initial angle of $\theta = 0.01$. The results imply that during the first 6 seconds of motion, the top starts to fall over, $\theta \to \pi$, however due to the induced precession, it doesn't fall directly down but turns with $\dot{\phi}$. Because of the slow initial $\dot{\psi}$, we expect that torque due to gravity will have a more significant affect than the torque due to the top spinning. Therefore the top will fall down away from $\theta = 0$. As it falls, due to the slight precession in the top, θ never becomes greater than π and instead gets very close to $\theta = \pi$. My program shows some quantities blowing up as time increases further. This is because as $\theta \to \pi, \sin\theta \to 0$. Since $\sin\theta$ appears in the denominator of the differential equations governing the motion, there is a massive increase in the value of $\dot{\phi}$ as well as E'. For these initial conditions, θ can never be equal to π however in this case we can see it gets extremely close which causes a significant increase in these quantities that are dependent on θ.

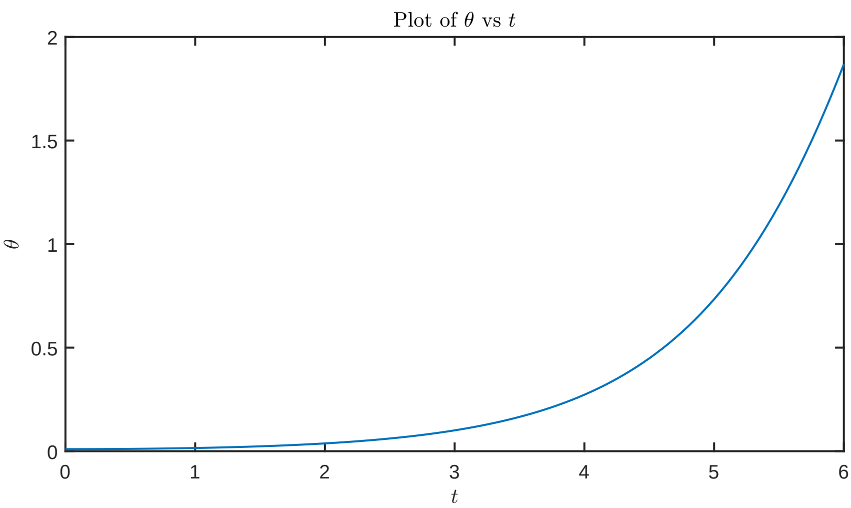

Figure 9: Plot of θ vs t, with initial conditions $[\theta(0), \dot{\theta}(0), \dot{\phi}(0)] = [0.01, 0, 0]$, for $\alpha = 0.01$.

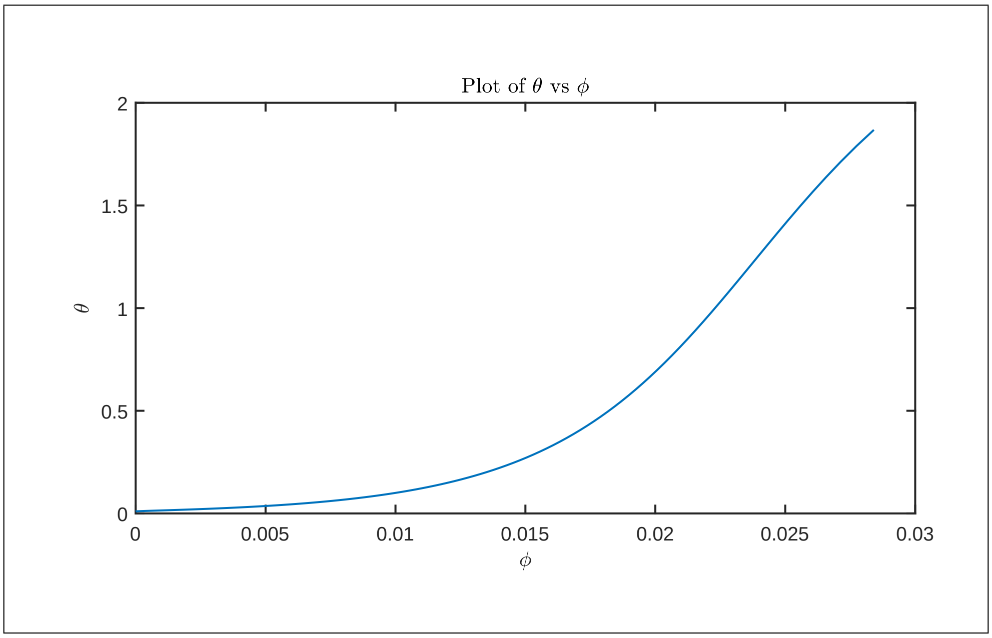

Figure 10: Plot of θ vs ϕ, with initial conditions $[\theta(0), \dot{\theta}(0), \dot{\phi}(0)] = [0.01, 0, 0]$, for $\alpha = 0.01$.

When $\alpha = 0$ and $\dot{\phi}(0) = 0$, this implies that $\beta = 0$. Hence $\dot{\phi}(\theta) = 0$ for all θ. Therefore there is zero precession (Figure 11) in this case which is a simulation of a simple pendulum. Since we start $\theta(0) = 0.01$, the only way this will evolve is due to the torque due to gravity. $\dot{\theta}(0) = 0$, so there is no initial push given to the system, hence by the conservation of energy, the top will traverse an arc of a circle and reach the point that is at $\theta = 2\pi - 0.01$, before going back to the starting point again (Figure 12).

One thing to also note is that θ has increased beyond the normal range of $[0, \pi]$. This makes sense in this context since θ is measured as increasing as you move away from the positive z-axis in the space frame. If we now think about the analogous real life set up of what the motion in this question is simulating, we can consider a pendulum (made of a rod attached to a pivot with a mass at the other end). If it starts with the rod at the highest position, with the mass at the top, displaced slightly at an angle of $\theta = 0.01$, then the mass would fall to the ground but constrained to move in the arc of the circle traced out by the end of the rod as it moves around the pivot. Neglecting friction, we expect that the rod will make an almost complete circle stopping at $\theta = 2\pi - 0.01$ due to the conservation of energy. This is outside the normal range, however in this context it makes sense that since the system is not spinning, θ is able to increase further.

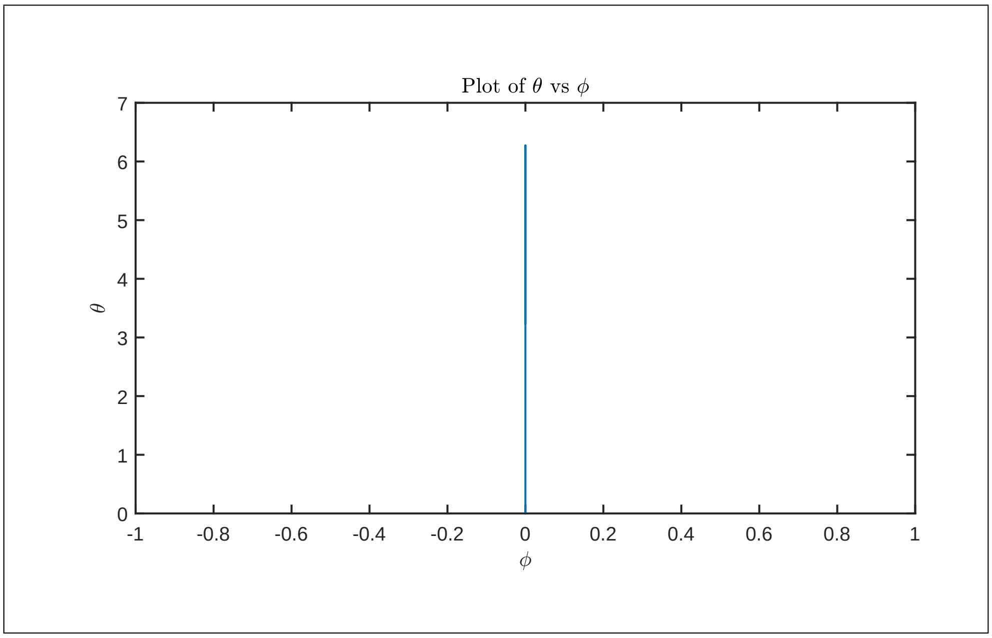

Figure 11: Plot of θ vs ϕ, with initial conditions $[\theta(0), \dot{\theta}(0), \dot{\phi}(0)] = [0.01, 0, 0]$, for $\alpha = 0$.

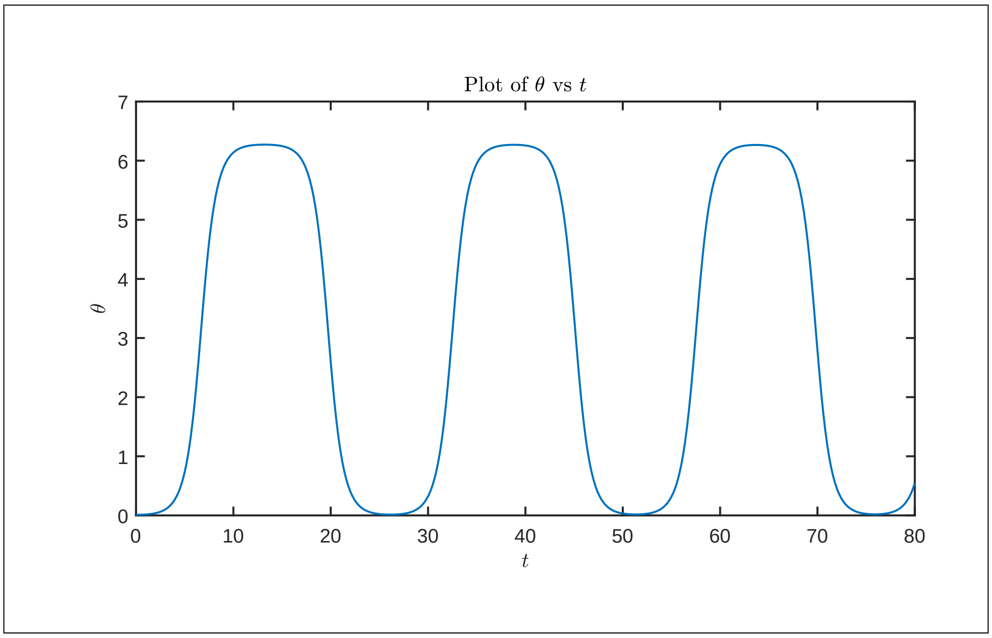

Figure 12: Plot of θ vs t, with initial conditions $[\theta(0), \dot{\theta}(0), \dot{\phi}(0)] = [0.01, 0, 0]$, for $\alpha = 0$.

```
function [t_vector, Y_vector, Z_vector, W_vector, X_vector] = RungeKutta4(N,h,
    f_1,f_2,f_3,f_4,Initial)
% Runge Kutta method to solve 3 first order Ordinary Differential Equations
% of the form y'=f_1(t,y,z,w) and z'=f_2(t,y,z,w,x) and w'=f_3(t,y,z,w,x)
% and x'=f_4(t,x,y,w,x)

% Input f(x,y), N is the number of iterations, h is the step length, x_0 is
% the initial x value and Y_0 is the corresponding initial y
% value.(x_0,Y_0) gives us a point on the solution curve.
syms t y z w x
f_1(t,y,z,w,x)= f_1;
%This is (dy/dt) = z = f_1(t,y,z,w,x)
f_2(t,y,z,w,x)= f_2;
%This is (dz/dt) = y'' = f_2(t,y,z,w,x)
f_3(t,y,z,w,x)= f_3;
%This is (dw/dt) = x = f_3(t,y,z,w,x)
f_4(t,y,z,w,x)= f_4;
%This is (dx/dt) = w'' = f_4(t,y,z,w,x)

t_0=Initial(1);
Y_0=Initial(2); %Theta
Z_0=Initial(3); %Thetadot
W_0=0; %Phi
X_0=Initial(4); %Phidot

count=1;
t_1=t_0;
Z_1=Z_0;
Y_1=Y_0;
W_1=W_0;
X_1=X_0;
Y_2=[Y_1,Z_1,W_1,X_1]';
%Initial Values

t_vector = zeros(1,N+1);
Y_vector = zeros(1,N+1);
Z_vector = zeros(1,N+1);
W_vector = zeros(1,N+1);
X_vector = zeros(1,N+1);
for count=1:N
    k_11=double(h*f_1(t_1,Y_1,Z_1,W_1,X_1));
    k_12=double(h*f_2(t_1,Y_1,Z_1,W_1,X_1));
    k_13=double(h*f_3(t_1,Y_1,Z_1,W_1,X_1));
    k_14=double(h*f_4(t_1,Y_1,Z_1,W_1,X_1));

    k_21=double(h*f_1(t_1+(1/2)*h,Y_1+(1/2)*k_11,Z_1+(1/2)*k_12,W_1+(1/2)*k_13,
        X_1+(1/2)*k_14));
    k_22=double(h*f_2(t_1+(1/2)*h,Y_1+(1/2)*k_11,Z_1+(1/2)*k_12,W_1+(1/2)*k_13,
        X_1+(1/2)*k_14));
    k_23=double(h*f_3(t_1+(1/2)*h,Y_1+(1/2)*k_11,Z_1+(1/2)*k_12,W_1+(1/2)*k_13,
        X_1+(1/2)*k_14));
    k_24=double(h*f_4(t_1+(1/2)*h,Y_1+(1/2)*k_11,Z_1+(1/2)*k_12,W_1+(1/2)*k_13,
        X_1+(1/2)*k_14));

    k_31=double(h*f_1(t_1+(1/2)*h,Y_1+(1/2)*k_21,Z_1+(1/2)*k_22,W_1+(1/2)*k_23,
        X_1+(1/2)*k_24));
    k_32=double(h*f_2(t_1+(1/2)*h,Y_1+(1/2)*k_21,Z_1+(1/2)*k_22,W_1+(1/2)*k_23,
        X_1+(1/2)*k_24));
```

```
    k_33=double(h*f_3(t_1+(1/2)*h,Y_1+(1/2)*k_21,Z_1+(1/2)*k_22,W_1+(1/2)*k_23,X_1+(1/2)*k_24));
    k_34=double(h*f_4(t_1+(1/2)*h,Y_1+(1/2)*k_21,Z_1+(1/2)*k_22,W_1+(1/2)*k_23,X_1+(1/2)*k_24));

    k_41=double(h*f_1(t_1+h,Y_1+k_31,Z_1+k_32,W_1+k_33,X_1+k_34));
    k_42=double(h*f_2(t_1+h,Y_1+k_31,Z_1+k_32,W_1+k_33,X_1+k_34));
    k_43=double(h*f_3(t_1+h,Y_1+k_31,Z_1+k_32,W_1+k_33,X_1+k_34));
    k_44=double(h*f_4(t_1+h,Y_1+k_31,Z_1+k_32,W_1+k_33,X_1+k_34));

    t_vector(1,count)= t_1;
    Y_vector(1,count)= Y_1;
    Z_vector(1,count)= Z_1;
    W_vector(1,count)= W_1;
    X_vector(1,count)= X_1;

    k_1=[k_11,k_12,k_13,k_14]';
    k_2=[k_21,k_22,k_23,k_24]';
    k_3=[k_31,k_32,k_33,k_34]';
    k_4=[k_41,k_42,k_43,k_44]';

    Y_3=Y_2+(1/6)*(k_1+2*k_2+2*k_3+k_4);

    t_1=t_0+count*h;

    Y_2=Y_3;
    Y_1=Y_2(1,1);
    Z_1=Y_2(2,1);
    W_1=Y_2(3,1);
    X_1=Y_2(4,1);
    count=count+1;
end
t_vector(1,N+1)=t_1;
Y_vector(1,N+1)=Y_1;
Z_vector(1,N+1)=Z_1;
W_vector(1,N+1)=W_1;
X_vector(1,N+1)=X_1;
end
```

```matlab
%Using Theta Double Dot and Phi double Dot
tic
syms t y z w x

N=120;
h=0.1;
alpha = 0.01;
C = 0.5;
InitialVals = [0,0.01,0,0];
%[Time (t), Theta (Y), Thetadot (Z), Phidot (X)] Phi = 0 (W)
beta = alpha*cos(InitialVals(2))+InitialVals(4)*((sin(InitialVals(2)))^2)

E = double(0.5*[((InitialVals(3))^2)+((InitialVals(4))^2)*((sin(InitialVals(2)))^2)]+cos(InitialVals(2)));

f_1(t,y,z,w,x) = z;
f_2(t,y,z,w,x) = ((x*(x*cos(y)-alpha))+1)*sin(y);
f_3(t,y,z,w,x) = x;
if alpha == beta
    f_4(t,y,z,w,x) = ((alpha*sin(y))*z)/((1+cos(y))^2);
elseif alpha == -beta
    f_4(t,y,z,w,x) = -((alpha*sin(y))*z)/((1-cos(y))^2);
else
    f_4(t,y,z,w,x) = (( ((alpha*sin(y))*sin(y)) - (2*cos(y)*(beta-alpha*cos(y))))/((sin(y))^3))*z;
end
[t_vector, Y_vector, Z_vector, W_vector, X_vector]=RungeKutta4(N,h,f_1,f_2,f_3,f_4,InitialVals);

E_vector=zeros(1,size(t_vector,2));

for i=1:size(t_vector,2)
    E_vector(i)=double(0.5*[((Z_vector(i))^2)+(((X_vector(i))^2)*((sin(Y_vector(i)))^2))]+cos(Y_vector(i)));
end

Eaverage = sum(E_vector)/(size(E_vector,2));

figure(1)
plot(t_vector,Y_vector);
title('Plot of $\theta$ vs $t$','interpreter','latex')
xlabel('$t$','interpreter','latex')
ylabel('$\theta$','interpreter','latex')

figure(2)
plot(t_vector,X_vector);
title('Plot of $\dot{\phi}$ vs $t$','interpreter','latex')
xlabel('$t$','interpreter','latex')
ylabel('$\dot{\phi}$','interpreter','latex')

figure(3)
plot(W_vector,Y_vector);
title('Plot of $\theta$ vs $\phi$','interpreter','latex')
xlabel('$\phi$','interpreter','latex')
ylabel('$\theta$','interpreter','latex')

figure(4)
plot(t_vector, E_vector);
title('Plot of Energy ($E$) vs $t$','interpreter','latex')
xlabel('$t$','interpreter','latex')
```

```
ylabel('Energy ($E$)','interpreter','latex')
toc
```