2.1 The Restricted Three-Body Problem

This project is related to the IA Dynamics and Relativity lecture course, but is self-contained.

1 Introduction

The problem of determining the motion of a number of gravitating bodies is a classical one. For two bodies there is a stable analytic solution, describing rotation of the bodies about their joint centre of mass. The problem for three bodies is not soluble analytically. Various simplifications have historically been considered, one of which is the 'restricted three-body problem' in which the third body is taken to be much smaller in mass than the other two and therefore does not affect their motion. The problem is then to solve for the motion of the third body under the influence of the gravitational field of the first two bodies.

It is convenient to transform to a rotating frame of reference in which the first two bodies appear stationary and the origin corresponds to their joint centre of mass. Scalings may be chosen so that the angular velocity of this frame is 1 and the distance between the two bodies is 1. The only parameter then appearing is the quantity μ defined such that the two masses are in the ratio $\mu : 1-\mu$ and are situated respectively at the points $(\mu - 1, 0)$ and $(\mu, 0)$. It is convenient to refer to these points as P_1 and P_2. The equation of motion for the third body, whose position at time t is $(x(t), y(t))$ may then be written as:

$$\ddot{x} - 2\dot{y} = -\frac{\partial \Omega}{\partial x}\,, \tag{1a}$$

$$\ddot{y} + 2\dot{x} = -\frac{\partial \Omega}{\partial y}\,, \tag{1b}$$

where

$$\Omega = -\tfrac{1}{2}\mu r_1^2 - \tfrac{1}{2}(1-\mu) r_2^2 - \frac{\mu}{r_1} - \frac{1-\mu}{r_2}\,, \tag{2}$$

with $r_1^2 = (x + 1 - \mu)^2 + y^2$ and $r_2^2 = (x - \mu)^2 + y^2$.

Despite the substantial restriction to the full three-body problem which this represents, it is not possible to solve the system (1a), (1b) and (2) analytically.

> **Question 1** Show from (1a) and (1b) that the quantity
>
> $$J = \tfrac{1}{2}\dot{x}^2 + \tfrac{1}{2}\dot{y}^2 + \Omega(x, y)$$
>
> is constant following the motion. Deduce that trajectories must be confined to the region
>
> $$\Omega(x, y) \leqslant \Omega(x_0, y_0) + \tfrac{1}{2}u_0^2 + \tfrac{1}{2}v_0^2\,, \tag{3}$$
>
> where x_0, y_0, u_0 and v_0 are the initial values of x, y, $\dot{x}$ and $\dot{y}$, respectively.

Programming Task Write a program to solve (1a), (1b) and (2) numerically, given suitable initial conditions on x, y, $\dot{x}$ and $\dot{y}$. You may use a black-box ODE solver such as the MATLAB routine ode45 which automatically adapts the time-step according to specified absolute and relative error tolerances (these may need to be adjusted). A fixed-step integration routine (as in the Ordinary Differential Equations core project) may require very small time-steps.

Whenever you write a computer program to find a numerical solution, it is necessary to check that the program is generating accurate results. Standard checks include (i) testing the program against known analytic solutions (if there are any), and (ii) varying the time-step or error tolerances. For this problem, the fact that J is constant provides not only a useful constraint on the behaviour of solutions, but also a possible check on the accuracy of a numerical solution.

2 Space travel

Assume that the third body is a spacecraft, with the first two bodies being co-orbiting planets of equal size, i.e. $\mu = 0.5$.

Question 2 Consider motion in the neighbourhood of P_2 say, so that the effect of P_1 may be ignored and (2) may be approximated by

$$\Omega = -\frac{1}{2r_2} \ . \tag{4}$$

Show that the system (1a), (1b) and (4) has analytic solutions with the spacecraft in a circular orbit of radius a about P_2, where a has any sufficiently small value for which the approximation (4) is valid.

Modify your program to solve (1a) and (1b) with Ω specified by (4) instead of (2). Demonstrate, for one value of a, that the modified program can *accurately* reproduce the analytic solutions.

Question 3 Now return to the original system (1a), (1b) and (2), with $\mu = 0.5$, and take initial conditions $x = 0.5$, $y = 0.2$, $\dot{x} = u_0$, $\dot{y} = 0$ with $u_0 = -1.0, -1.39, -1.53, -1.54, -1.69, -1.7$ and -1.9 in turn. For each case, use your programs to integrate from $t = 0$ to $t = 30$ and

(i) plot x against t,

(ii) display the trajectory in the (x, y) plane, and the forbidden region $\Omega(x, y) > J$ (shaded, say, using the MATLAB function `contourf`), on the same plot,

(iii) state the values of x and y at $t = 30$, giving a reasoned assessment of their accuracy.

Comment on the trajectories, and how these and the allowed region change as $|u_0|$ increases. Is the allowed region a useful guide to the size of the trajectory? What value of u_0 would be most suitable to travel from the neighbourhood of P_2 to the neighbourhood of P_1?

Suggestion: it *may* be instructive to try other values of u_0 and/or integrate further in time.

3 Lagrange points and asteroids

In this part of the project do not restrict attention to the case $\mu = 0.5$.

Question 4 By examining contour plots of Ω show that the system (1a), (1b) and (2) generally has five equilibrium points; three 'collinear Lagrange points' on the x-axis and two 'equilateral Lagrange points' at the third vertex of an equilateral triangle whose other two vertices are at P_1 and P_2. Display contour plots for three values of μ with the equilibrium points marked. Note that without loss of generality you may assume that $\mu \in (0, 0.5]$; why?

Suggestion: you may wish to use a black-box root-finder such as the MATLAB routine `fzero` to locate the collinear points accurately.

Numerically investigate the *linear* stability of the collinear Lagrange points, i.e. the stability to very small, formally *infinitesimal*, perturbations, by starting trajectories a small distance away from the equilibrium point and integrating forward in time. Display plots of some representative trajectories in the (x, y) plane, together with corresponding plots of x and y against t, to illustrate your results. Explain why you are satisfied that these computations, which start with small but necessarily finite perturbations, have captured the behaviour for infinitesimal perturbations.

What do you conclude about the linear stability of the collinear Lagrange points? Does it depend on μ? Confirm your numerical findings analytically by performing a linearised stability analysis about such points.

Suggestion: you may be able to deduce the necessary information about the second derivatives of Ω by considering the shapes of the contours, rather than by detailed calculation.

Question 5 Continue with a numerical investigation of the linear stability of the equilateral Lagrange points for parameter values $\mu = 0.01$, 0.025, 0.05, 0.1 and 0.5. Illustrate the results in your write-up with at least one trajectory picture, and corresponding plots of x and y against t, for each.

How do the stability properties change with μ? By further numerical experimentation find (to two significant figures) the critical value μ_c dividing values of μ for which the point is linearly stable from those for which it is unstable, and present numerical results in support of your conclusion. Confirm it by performing a linearised stability analysis, which this time will require calculation of the second derivatives of Ω. For the stable cases, what does this analysis indicate about the form of the motion?

Question 6 The Trojans are a group of asteroids observed at the Sun-Jupiter equilateral Lagrange point. For the Sun-Jupiter system $\mu = 9.54 \times 10^{-4}$. Is the persistence of the Trojans at this point consistent with your findings above? The Earth-Moon system has $\mu = 0.012141$ but no analogue of the Trojans is observed; can you suggest why?

