EEE424 Homework 2

(Clearly justify all answers.)

(Due 16 March 2023)

1- It is given that V is an N-dimensional inner product space (standard inner product), and S_2 be a subspace in V. $Q = \{\mathbf{p_1}, \mathbf{p_2}, \cdots \mathbf{p_M}\}$, $M < N$, is a basis for S_2. The problem is to find a solution to $\underset{\hat{\mathbf{y}}}{\operatorname{argmin}} ||\mathbf{y} - \hat{\mathbf{y}}||^2$, where $||\cdot||$ is the inner product induced norm, $\mathbf{y} \in V$ and $\hat{\mathbf{y}} \in S_2$.

a) Prove that the solution to the above problem must satisfy,

i- $(y - \hat{y}) \perp S_2$

ii- $(y - \hat{y}) \perp \hat{y}$

iii- $(y - \hat{y}) \perp \mathbf{p_j}, \quad j = 1, \cdots, M$

b) Derive the normal equations in terms of inner products $\langle p_i, p_j \rangle$ and $\langle y, p_j \rangle$ to solve for coefficients, c_i's where $\hat{\mathbf{y}} = \sum_{i=1}^{M} c_i \mathbf{p}_i$ using the facts given in (a).

c) $\mathbf{x}$ is a vector in S_1 which is an M-dimensional inner product space (standard inner product), and $\mathbf{A}$ is an $N \times M$ matrix, $N > M$, whose columns are linearly independent, representing a linear transform. Derive the left pseudoinverse $\mathbf{A}_{ps}$ of $\mathbf{A}$ which would yield $\hat{\mathbf{x}} = \mathbf{A}_{ps}\mathbf{y}$ such that $\hat{\mathbf{x}}$ is the solution to $\hat{\mathbf{x}} = \underset{\mathbf{x}}{\operatorname{argmin}} ||\mathbf{y} - \mathbf{A}\mathbf{x}||^2$

d) Let

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 1 \\ -1 & 1 \end{bmatrix} \qquad \mathbf{y} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$$

Find $\hat{\mathbf{x}} = \underset{\mathbf{x}}{\operatorname{argmin}} ||\mathbf{y} - \mathbf{A}\mathbf{x}||^2$.

e) Find $\hat{\mathbf{y}} = \mathbf{A}\hat{\mathbf{x}}$ and show that $(\mathbf{y} - \hat{\mathbf{y}})$ is orthogonal to $\hat{\mathbf{y}}$.

2- Functions $\phi_i(t)$, $i = 1, 2, 3$ are given as,

$$\phi_1(t) = \begin{cases} t & \text{if} \quad t \in [0, 1] \\ 0 & \text{else} \end{cases}$$

$$\phi_2(t) = \begin{cases} -1 & \text{if} \quad t \in [0, 2] \\ 0 & \text{else} \end{cases}$$

$$\phi_3(t) = \begin{cases} -t+1 & \text{if} \quad t \in [1,2] \\ 0 & \text{else} \end{cases}$$

a) Plot these functions. Are these functions orthogonal to each other?

b) Find the range space of $\{\phi_1(t), \phi_2(t), \phi_3(t)\}$. Select an arbitrary (c_1, c_2, c_3) (indicate you selection). Plot a typical function in that range space using your selected coefficients, i.e., plot $c_1\phi_1(t) + c_2\phi_2(t) + c_3\phi_3(t)$ for your c_i's.

c) Find the parameters, c_i's to minimize $\left|\left|y(t) - \sum_{i=1}^{3} c_i\phi_i(t)\right|\right|^2$, where

$$y(t) = \begin{cases} -1 & \text{if} \quad t \in [-1,1] \\ 2 & \text{if} \quad t \in [2,3] \\ 0 & \text{else} \end{cases}$$

Plot ($y(t)$ and $\hat{y}(t) = c_1\phi_1(t) + c_2\phi_2(t) + c_3\phi_3(t)$ using the optimum c_i's you found on the same graph.

d) Find and plot the error function $e(t) = y(t) - \hat{y}(t)$, and show that $e(t)$ is orthogonal to $\hat{y}(t)$.

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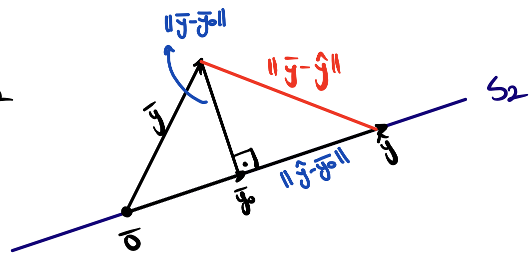
EEE424 Digital Signal Processing
Homework 2

1. It is given that V is an N-dimensional inner product space (standard inner product), and S_2 be a subspace in V. $Q = \{\mathbf{p_1}, \mathbf{p_2}, \cdots \mathbf{p_M}\}$, $M < N$, is a basis for S_2. The problem is to find a solution to $\underset{\hat{\mathbf{y}}}{\operatorname{argmin}} ||\mathbf{y} - \hat{\mathbf{y}}||^2$, where $||\cdot||$ is the inner product induced norm, $\mathbf{y} \in V$ and $\hat{\mathbf{y}} \in S_2$.

a) Prove that the solution to the above problem must satisfy,

1.a.i) $(y - \hat{y}) \perp S_2$

Let $\bar{y}_0 \in V_2$, $\bar{y} - \bar{y}_0 \perp S_2$



$$\underset{\hat{y}}{\operatorname{argmin}} \|\bar{y}-\hat{y}\|^2 = \underset{\hat{y}}{\operatorname{argmin}} \Big[\|\hat{y}-\bar{y}_0\|^2 + \underbrace{\|\bar{y}-\bar{y}_0\|^2}_{\text{Constant}} \Big] = \underset{\hat{y}}{\operatorname{argmin}} \|\hat{y}-\bar{y}_0\| = \bar{y}_0$$

$\Rightarrow$ Therefore $\hat{y} = \bar{y}_0$ and since $\bar{y} - \bar{y}_0 \perp S_2$ $\boxed{(\bar{y}-\hat{y}) \perp S_2}$

The proof is complete ■

1.a.ii) $(y - \hat{y}) \perp \hat{y}$

Since $\hat{y} \in S_2$ and $(\bar{y}-\hat{y}) \perp S_2$. $\boxed{(\bar{y}-\hat{y}) \perp \hat{y}}$ is also correct and prooved ■

1.a.iii) $(y - \hat{y}) \perp \mathbf{p_j}$, $j = 1, \cdots, M$

Since $\bar{p}_j \in S_2$ for $j = 0, 1, \ldots, M$

and $(\bar{y}-\hat{y}) \perp S_2$

then, $\boxed{(\bar{y}-\hat{y}) \perp p_j}$ is also correct and prooved ■

1.b) b) Derive the normal equations in terms of inner products $\langle p_i, p_j \rangle$ and $\langle y, p_j \rangle$ to solve for coefficients, c_i's where $\hat{\mathbf{y}} = \sum_{i=1}^{M} c_i \mathbf{p}_i$ using the facts given in (a).

$\bar{y} - \hat{y} = \bar{y} - \sum_{i=1}^{M} c_i \cdot \bar{p_i}$. By using the equation from 1.a.iii)

$\Rightarrow \left(\bar{y} - \sum_{i=1}^{M} c_i \bar{p_i}\right) \perp \bar{p_j}$ for $j = 1, \dots, M$

$\Rightarrow \langle (\hat{y} - \sum_{i=1}^{M} c_i \bar{p_i}), \bar{p_j} \rangle = 0$ for $j = 1, \dots, M$

$\langle \hat{y}, \bar{p_j} \rangle = \sum_{i=1}^{M} c_i \langle \bar{p_i}, \bar{p_j} \rangle$ for $j = 1, \dots, M$

$$\langle \hat{y}, \bar{p_j} \rangle = \begin{array}{c c} & \xrightarrow{i} \\ j \downarrow & \underbrace{\begin{bmatrix} & & \\ & \langle \bar{p_i}, \bar{p_j} \rangle & \\ & & \end{bmatrix}}_{\bar{\bar{R}}} \end{array} \cdot \underbrace{\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_i \\ \vdots \\ c_M \end{bmatrix}}_{\bar{c}} \Rightarrow \bar{p} = \bar{\bar{R}} \cdot \bar{c} \quad , \boxed{\bar{c} = \bar{\bar{R}}^{-1} \cdot \bar{p}}$$

↳ Normal eqn's for the given set.

1.c) $\mathbf{x}$ is a vector in S_1 which is an M-dimensional inner product space (standard inner product), and $\mathbf{A}$ is an $N \times M$ matrix, $N > M$, whose columns are linearly independent, representing a linear transform. Derive the left pseudoinverse $\mathbf{A}_{ps}$ of $\mathbf{A}$ which would yield $\hat{\mathbf{x}} = \mathbf{A}_{ps}\mathbf{y}$ such that $\hat{\mathbf{x}}$ is the solution to $\hat{\mathbf{x}} = \underset{\mathbf{x}}{\operatorname{argmin}} ||\mathbf{y} - \mathbf{A}\mathbf{x}||^2$

Let $\bar{\bar{A}} = [\bar{a_1} \ \bar{a_2} \ \cdots \ \bar{a_M}]$,

$\Rightarrow \hat{x} = \underset{\bar{x}}{\operatorname{argmin}} \| \bar{y} - \bar{\bar{A}} \cdot \bar{x} \|^2$, $\hat{y} = \bar{\bar{A}} \cdot \hat{x}$. Then the minimization becomes $\underset{\hat{y}}{\operatorname{argmin}} \| \bar{y} - \hat{y} \|^2$ where $\hat{y} \in R(\bar{\bar{A}})$. By using the relations proved in previous parts:

$\Rightarrow (\bar{y} - \hat{y}) \perp R(\bar{\bar{A}})$. $\hat{y} = \sum_{i=1}^{M} \hat{x_i} \cdot \bar{a_i} \Rightarrow \langle \bar{y} - \sum_{i=1}^{M} \hat{x_i} \cdot \bar{a_i} , \bar{a_j} \rangle = 0$ for $j = 1, 2, \dots, M$

$\Rightarrow \langle \bar{y}, \bar{a_j} \rangle = \sum_{i=1}^{M} \hat{x_i} \cdot \langle \bar{a_i}, \bar{a_j} \rangle$ for $j = 1, 2, \dots, M$. → <u>The Normal Eqn</u>

$$j \downarrow \underbrace{\begin{bmatrix} \bar{a_1}^{*T} \\ \bar{a_2}^{*T} \\ \vdots \\ \bar{a_M}^{*T} \end{bmatrix}}_{\bar{\bar{A}}_{adj}} \cdot \bar{y} = \underbrace{\begin{bmatrix} \bar{a_1}^{*T} \\ \bar{a_2}^{*T} \\ \vdots \\ \bar{a_M}^{*T} \end{bmatrix}}_{\bar{\bar{A}}_{adj}} \cdot \underbrace{[\bar{a_1} \ \bar{a_2} \ \cdots \ \bar{a_M}]}_{\bar{\bar{A}}} \cdot \hat{x}$$

$$\bar{\bar{A}}_{adj} \cdot \bar{y} = \bar{\bar{A}}_{adj} \cdot \bar{\bar{A}} \cdot \hat{x} \Rightarrow \hat{x} = \underbrace{(\bar{\bar{A}}_{adj} \cdot \bar{\bar{A}})^{-1} \cdot \bar{\bar{A}}_{adj}}_{\bar{\bar{A}}_{ps}} \cdot \bar{y}$$

$$\boxed{\bar{\bar{A}}_{ps} = (\bar{\bar{A}}_{adj} \cdot \bar{\bar{A}})^{-1} \cdot \bar{\bar{A}}_{adj}}$$

Since $\bar{\bar{A}}$ has LI columns → $\bar{\bar{A}}_{ps}$ definitely exists

1.d)

d) Let

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 1 \\ -1 & 1 \end{bmatrix} \qquad \mathbf{y} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$$

By using the equation derived in part 1.c . $\underset{\bar{x}}{\text{argmax}} \|\bar{y} - \bar{\bar{A}}\bar{x}\|^2 = (\bar{\bar{A}}_{adj} . \bar{\bar{A}})^{-1} . \bar{\bar{A}} . \bar{y}$

$$\hat{x} = (\bar{\bar{A}}_{adj} . \bar{\bar{A}})^{-1} . \bar{\bar{A}}_{adj} . \bar{y}$$

$$\bar{\bar{A}}_{adj} . \bar{\bar{A}} = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 1 & 1 \end{bmatrix} . \begin{bmatrix} 1 & -1 \\ 2 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 3 \end{bmatrix}$$

$$(\bar{\bar{A}}_{adj} . \bar{\bar{A}})^{-1} = \begin{bmatrix} 1/6 & 0 \\ 0 & 1/3 \end{bmatrix}$$

$$\hat{x} = \underbrace{\begin{bmatrix} 1/6 & 0 \\ 0 & 1/3 \end{bmatrix} . \begin{bmatrix} 1 & 2 & -1 \\ -1 & 1 & 1 \end{bmatrix}} . \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1/6 & 1/3 & -1/6 \\ -1/3 & 1/3 & 1/3 \end{bmatrix} . \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$\boxed{\hat{x} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}} \quad \text{which is } \underset{\bar{x}}{\text{argmin}} \|\bar{y} - \bar{\bar{A}}\bar{x}\|^2$$

1.e) e) Find $\hat{\mathbf{y}} = \mathbf{A}\hat{\mathbf{x}}$ and show that $(\mathbf{y} - \hat{\mathbf{y}})$ is orthogonal to $\hat{\mathbf{y}}$.

$$\hat{x} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \quad \bar{\bar{A}} = \begin{bmatrix} 1 & -1 \\ 2 & 1 \\ -1 & 1 \end{bmatrix}, \quad \hat{y} = \begin{bmatrix} 1 & -1 \\ 2 & 1 \\ -1 & 1 \end{bmatrix} . \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \quad \bar{y} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$$

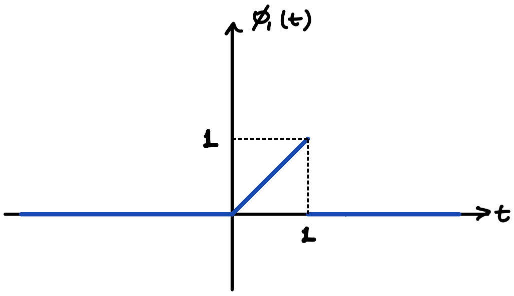
$$\bar{y} - \hat{y} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}, \quad \hat{y} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \quad \langle \bar{y} - \hat{y}, \hat{y} \rangle = 2.1 + 0.(-1) + 2.(-1) = 0$$

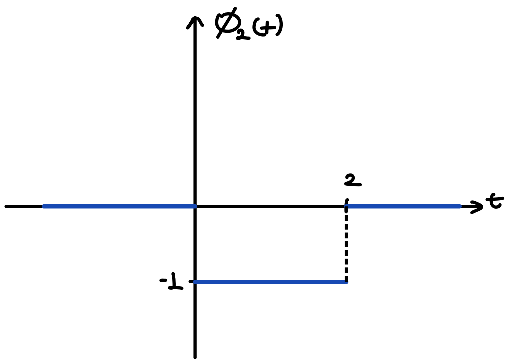
$(\bar{y} - \hat{y}) \perp \hat{y}$ as expected.

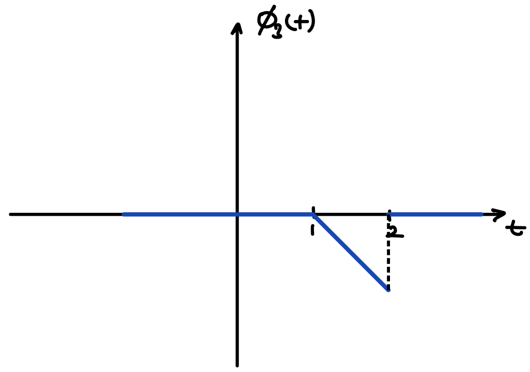
2.

2.a) Plot these functions. Are these functions orthogonal to each other?

$$\phi_1(t) = \begin{cases} t & \text{if} \quad t \in [0,1] \\ 0 & \text{else} \end{cases} \qquad \phi_2(t) = \begin{cases} -1 & \text{if} \quad t \in [0,2] \\ 0 & \text{else} \end{cases} \qquad \phi_3(t) = \begin{cases} -t+1 & \text{if} \quad t \in [1,2] \\ 0 & \text{else} \end{cases}$$







- $\langle \phi_1(t), \phi_2(t) \rangle = \int_0^1 -t \, dt = \underline{\underline{-\frac{1}{2}}}$
- $\langle \phi_2(t), \phi_3(t) \rangle = \int_1^2 t-1 \, dt = \frac{t^2}{2} - t \Big|_1^2 = \underline{\underline{\frac{1}{2}}}$
- $\langle \phi_1(t), \phi_3(t) \rangle = \underline{\underline{0}}$

$\phi_1(t) \perp \phi_3(t)$

$\phi_2(t)$ is orthogonal neither $\phi_1(t)$ nor $\phi_3(t)$.

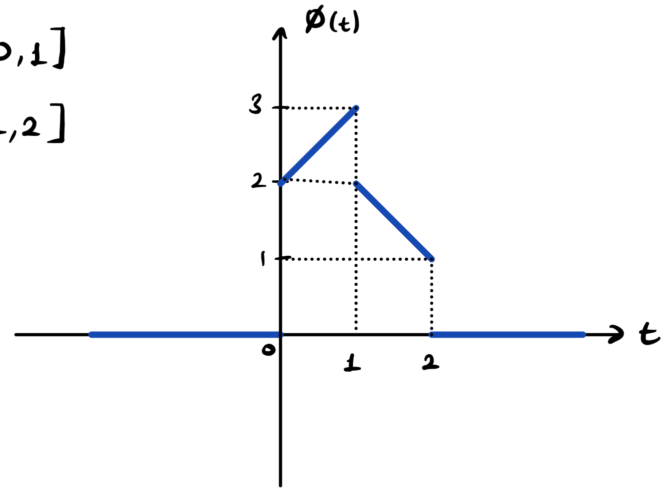
2.b) Find the range space of $\{\phi_1(t), \phi_2(t), \phi_3(t)\}$. Select an arbitrary (c_1, c_2, c_3) (indicate you selection). Plot a typical function in that range space using your selected coefficients, i.e., plot $c_1\phi_1(t) + c_2\phi_2(t) + c_3\phi_3(t)$ for your c_i's.

$$\phi(t) = c_1 \cdot \phi_1(t) + c_2 \cdot \phi_2(t) + c_3 \cdot \phi_3(t) \in R(\{\phi_1(t), \phi_2(t), \phi_3(t)\}) \quad \forall \, c_1, c_2, c_3 \in \mathbb{R}$$

$$\phi(t) = \begin{cases} c_1 \cdot t - c_2 & t \in [0,1] \\ -c_3 \cdot t - c_2 + c_3 & t \in [1,2] \\ 0 & \text{else} \end{cases} \qquad \forall \, c_1, c_2, c_3 \in \mathbb{R}$$

$$\phi(t) \in R(\{\phi_1(t), \phi_2(t), \phi_3(t)\})$$

Let say $\bar{c} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, then $\phi(t) = \begin{cases} t+2 & t \in [0,1] \\ 3-t & t \in [1,2] \\ 0 & \text{else} \end{cases}$



2.c)

c) Find the parameters, c_i's to minimize $\left\| y(t) - \sum_{i=1}^{3} c_i \phi_i(t) \right\|^2$, where

$$y(t) = \begin{cases} -1 & \text{if} \quad t \in [-1,1] \\ 2 & \text{if} \quad t \in [2,3] \\ 0 & \text{else} \end{cases}$$

Plot $(y(t)$ and $\hat{y}(t) = c_1\phi_1(t) + c_2\phi_2(t) + c_3\phi_3(t)$ using the optimum c_i's you found on the same graph.

C_i's that minimize $\| y(t) - \sum_{i=1}^{3} c_i \phi_i(t) \|^2$ can be found by using the previous results.

$\bar{\bar{\Phi}} = [\phi_1(t) \quad \phi_2(t) \quad \phi_3(t)]$. Then, $\bar{\bar{\Phi}}_{adj} \cdot \bar{\bar{\Phi}} \cdot \bar{c} = \bar{\bar{\Phi}}_{adj} \cdot y(t)$

$$\bar{c} = (\bar{\bar{\Phi}}_{adj} \cdot \bar{\bar{\Phi}})^{-1} \cdot \bar{\bar{\Phi}} \cdot y(t)$$

- $\bar{\bar{\Phi}}_{adj} \cdot \bar{\bar{\Phi}} = \begin{bmatrix} \phi_1(t) \\ \phi_2(t) \\ \phi_3(t) \end{bmatrix} \cdot [\phi_1(t) \quad \phi_2(t) \quad \phi_3(t)]$

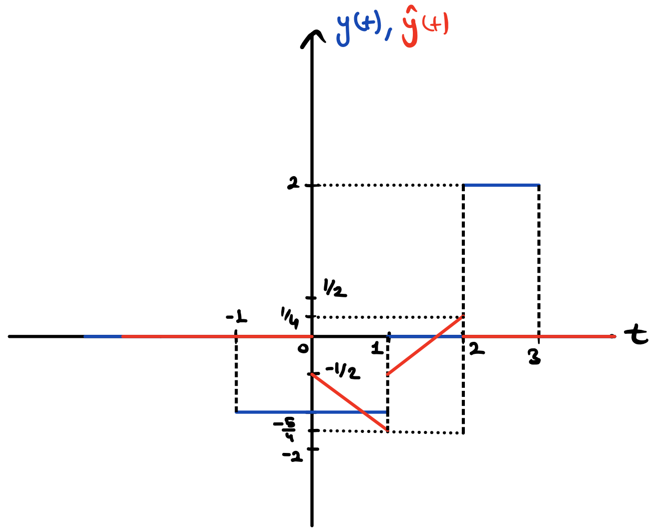
$$= \begin{bmatrix} \langle \phi_1(t), \phi_1(t) \rangle & \langle \phi_2(t), \phi_1(t) \rangle & \langle \phi_3(t), \phi_1(t) \rangle \\ \langle \phi_1(t), \phi_2(t) \rangle & \langle \phi_2(t), \phi_2(t) \rangle & \langle \phi_3(t), \phi_2(t) \rangle \\ \langle \phi_1(t), \phi_3(t) \rangle & \langle \phi_2(t), \phi_3(t) \rangle & \langle \phi_3(t), \phi_3(t) \rangle \end{bmatrix} = \begin{bmatrix} 1/3 & -1/2 & 0 \\ -1/2 & 2 & -1/2 \\ 0 & -1/2 & 1/3 \end{bmatrix}$$

$$(\bar{\bar{\Phi}}_{adj} \cdot \bar{\bar{\Phi}})^{-1} = \left(\begin{bmatrix} 1/3 & -1/2 & 0 \\ -1/2 & 2 & 1/2 \\ 0 & 1/2 & 1/3 \end{bmatrix} \right)^{-1} = \frac{\begin{bmatrix} 5/12 & 1/6 & -1/4 \\ 1/6 & 1/9 & -1/6 \\ -1/4 & -1/6 & \frac{5}{12} \end{bmatrix}^T}{\frac{1}{3} \cdot \left(\frac{2}{3} - \frac{1}{4}\right) + \frac{1}{2}\left(-\frac{1}{6}\right)} = \frac{\begin{bmatrix} 5/12 & 1/6 & -1/4 \\ 1/6 & 1/9 & -1/6 \\ -1/4 & -1/6 & \frac{5}{12} \end{bmatrix}}{\frac{5}{36} + \frac{1}{2} \cdot \left(-\frac{1}{6}\right)} = \frac{\begin{bmatrix} 5/12 & 1/6 & -1/4 \\ -1/6 & 1/9 & -1/6 \\ -1/4 & -1/6 & \frac{5}{12} \end{bmatrix}}{\frac{1}{18}} = \begin{bmatrix} 15/2 & 3 & -9/2 \\ 3 & 2 & -3 \\ -9/2 & -3 & 15/2 \end{bmatrix} = (\bar{\bar{\Phi}}_{adj} \cdot \bar{\bar{\Phi}})^{-1}$$

- $\bar{\bar{\Phi}}_{adj}(t)\cdot y(t) = \begin{bmatrix} \langle \phi_1(t), y(t)\rangle \\ \langle \phi_2(t), y(t)\rangle \\ \langle \phi_3(t), y(t)\rangle \end{bmatrix} = \begin{bmatrix} \int_0^1 (-t)\,dt \\ \int_0^1 dt \\ 0 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix}$

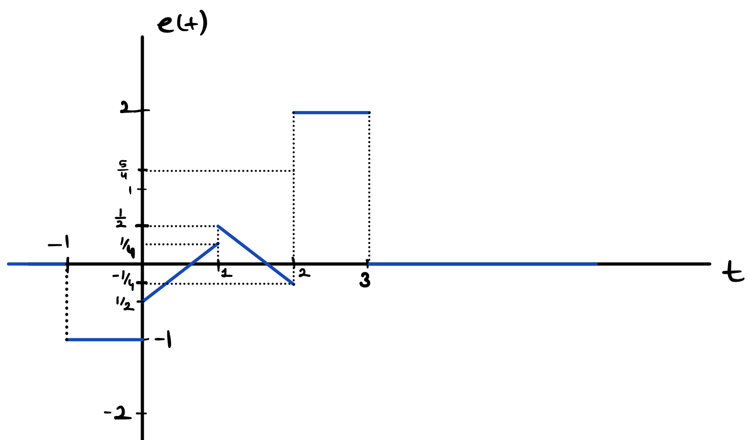
- $\bar{c} = (\bar{\bar{\Phi}}_{adj}\cdot\bar{\bar{\Phi}})^{-1}\cdot(\bar{\bar{\Phi}}_{adj}\cdot\bar{y}) = \begin{bmatrix} 15/2 & 3 & -9/2 \\ 3 & 2 & -3 \\ -9/2 & -3 & 15/2 \end{bmatrix}\cdot\begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -3/4 \\ 1/2 \\ -3/4 \end{bmatrix} = \bar{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$

- $\hat{y}(t) = \begin{cases} -\frac{3}{4}t - \frac{1}{2} & t\in[0,1] \\ \frac{3}{4}t - \frac{5}{4} & t\in[1,2] \\ 0 & \text{else} \end{cases}$



2.d) Find and plot the error function $e(t) = y(t) - \hat{y}(t)$, and show that $e(t)$ is orthogonal to $\hat{y}(t)$.

$$e(t) = y(t) - \hat{y}(t) = \begin{cases} -1 & \text{if } t\in[-1,0] \\ \frac{3}{4}t - \frac{1}{2} & \text{if } t\in[0,1] \\ \frac{5}{4} - \frac{3t}{4} & \text{if } t\in[1,2] \\ 2 & \text{if } t\in[2,3] \\ 0 & \text{else} \end{cases}$$



- $\langle e(t), \hat{y}(t)\rangle = -\int_0^1 \frac{9}{16}t^2 - \frac{1}{4}\,dt + \int_1^2 \frac{-9t^2}{16} + \frac{15t}{8} - \frac{25}{16}\,dt$

$$= \left(-\frac{3t^3}{16} + \frac{t}{4}\right)\Big|_0^1 + \left(\frac{-3t^3}{16} + \frac{15t^2}{16} - \frac{25t}{16}\right)\Big|_1^2 = \frac{1}{16} + \left(\frac{-24}{16} + \frac{60}{16} - \frac{50}{16} + \frac{3}{16} - \frac{15}{16} + \frac{25}{16}\right) = \frac{1}{16} + \left(-\frac{1}{16}\right) = \underline{\underline{0}}, \quad \boxed{e(t) \perp \hat{y}(t)}$$

as prooved.