Module 2 Sets, Relations & Functions

1.1.1. Motivation:

The present module deals with the relations and functions. In particular, the student should be able use the Concept of the relation and functions in higher mathematics & their project work. Two variables may be linked by some type of relationship. That brings us to the concept of relations. In contrast, a function defines how one variable depends on one or more other variables. There is nothing such as application of functions in our daily life. Like we use differentiation to find velocity, integration to find areas and volumes, Linear programming to get maximum profit, functions cannot be used like this but it makes our mathematics easy, it becomes really hard to explain mathematics without functions.

1.1.2. Syllabus:

Module2 Units	Content	Duration	Self study duration
1.2	Relation: Definition, different types of relation, their composition and inverses	2 lectures	5 Hrs
1.3	Function: Definition, different types of function, their composition and inverses.	2 lectures	5 Hrs

1.1.3. Weight age: 21 Marks

1.1.4. Prerequisite: Basics of logic and concept of Number system

1.1.5. Learning Objective:

1. Student shall be able to understand the concept of Relation and Function.
2. Student shall be able to understand the laws related to Relation and Function application.
3. Student shall be able to know different types of relations and to identify domain and range of a relation.
4. Student shall be able to learn pictorial presentation of relation and different types of relation and properties of relation.
5. Student shall be able to know different types of function.
6. Student shall be able to find composition and inverse of functions.

Lecture -01

Learning Objective:

1. Student shall be able to know different types of relations and to identify domain and range of it.

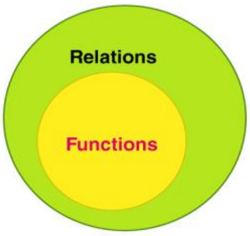
1.1 Relations

2.1.1. Introduction:

Relations and Functions is one of the most important topics in algebra. Relations and functions – these are the two different words having different meaning mathematically. One might get confused about their difference. Before we go deeper, let's understand the difference between both with a simple example.

An ordered pair, represents as (INPUT, OUTPUT):

Relation shows the relationship between INPUT and OUTPUT. Whereas a function is a relation which derives one OUTPUT for each given INPUT.

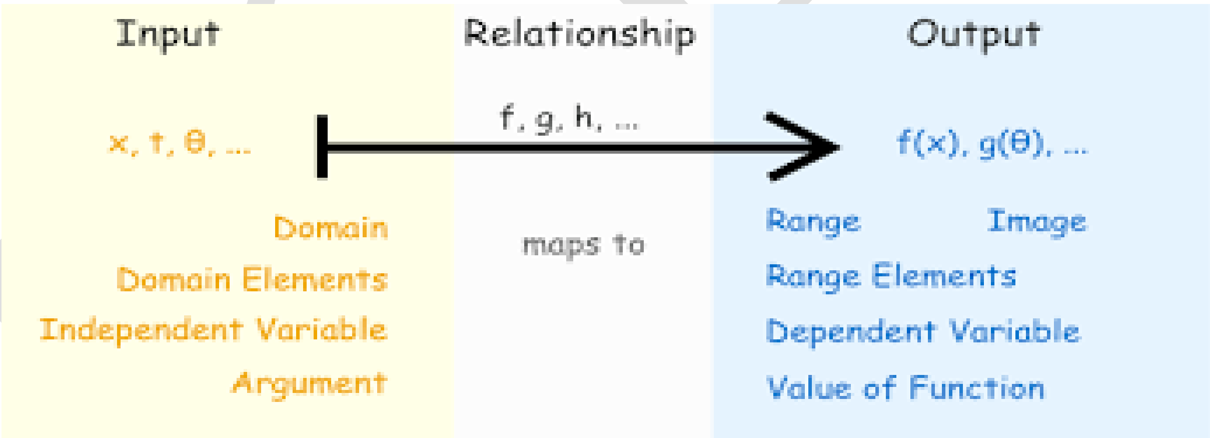


2.1.2. Key Definitions:

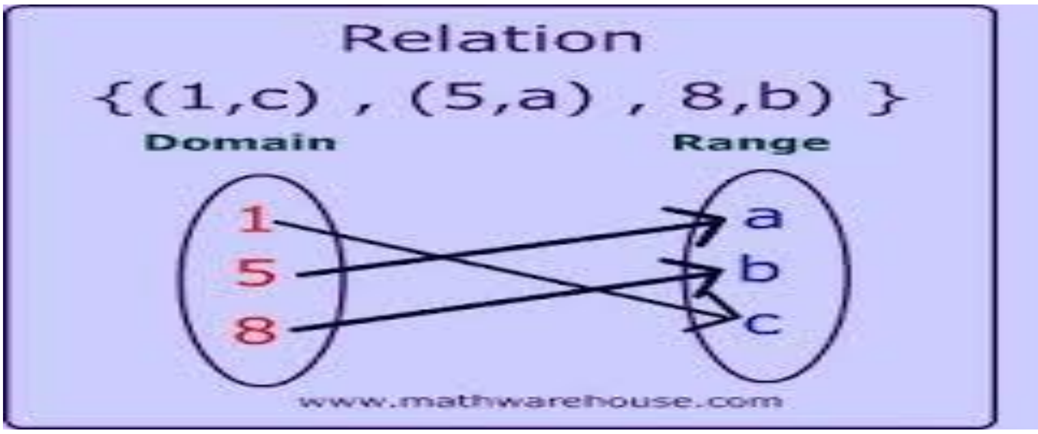
(1) Relation: A relation between two sets is a collection of ordered pairs containing one object from each set. If the object x is from the first set and the object y is from the second set, then the objects are said to be related if the ordered pair (x, y) is in the **relation**. In math, a relation is just a set of ordered pairs. It is a subset of the Cartesian product.

In math, a relation R defines the relationship between sets of values of ordered pairs.

e.g. If $A = \{1,2,3\}$ and we define R from A to A as $1\ R\ 2$, $1\ R\ 3$, $2\ R\ 3$, this is less than relation. This as a set of ordered pair $R = \{(1,2),(1,3),(2,3)\}$. This is subset of $A \times A$.



(2) Domain & range: The set of elements in the first set are called domain which is denoted by Dom (R).These elements are related to the element in another set, which is called range . range is denoted by Ran(R).



2.1.3. Key Notation:

(1) Relation: R

(2) Elements a and b of a set are related: ‘$a\ R\ b'$

(3) Elements a and b of a set are not related:$'a\ \cancel{R}\ b'$

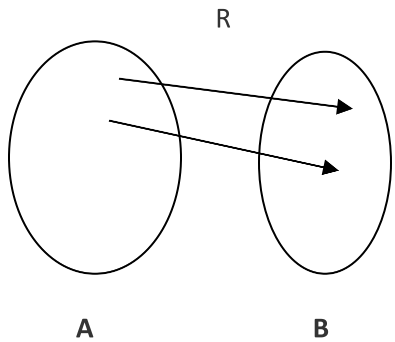
2.1.4. Important Formulae/ Theorems / Properties on Relations:

(1) Relation Representation

There are other ways too to write the relation, apart from set notation such as through mapping diagram, matrix, digraph etc.

(a) Mapping diagram of relation

Relation can be easily represented by a mapping diagram. For this we denote two sets A and B by some geometric figures such as circles and ellipses. Show the elements of A and B of the sets by dots inside these figures. We then draw an arrow from the element of A to the elements of B to which it is related. This is done by all elements of A which are related to the elements of B. This is the required mapping diagram of the relation R.



(b) Matrix of a relation

A relation between two finite sets can also be represented by a matrix.

If $A\ = \{a_1, a_2, a_3, \dots \dots a_m\}$, $B\ = \{b_1, b_2, b_3, \dots \dots b_n\}$ are finite sets containing m and n elements respectively and if R is a relation from A to B, we represent R by the matrix

$M_R = [m_{ij}]$ of order $m \times n$ which is defined by $m_{ij} = \begin{cases} 1 & if\ (a_i,\ b_j) \in R \\ 0 & if\ (a_i,\ b_j) \notin R \end{cases}$

To write the matrix M_R, we write the elements of A vertically on the left and the elements of B horizontally at the top outside the matrix. The matrix M_R is called the adjacency matrix of the relation R or simply the matrix of the relation. Conversely from a given matrix of a relation we can write the relation as a set.

Note: **(1)** The matrix whose elements are either 1 or 0 is called a Boolean Matrix.

(2) If a set has n elements then the number of binary relations that can be defined on A is 2^{n^2}.

e.g. Let R be the relation given $R = \{(1, a), (2, b)\}$ where $A = \{1,2,3\}, B = \{a, b\}$ then matrix of R is

$$M_R = \begin{array}{cc} & \begin{array}{cc} a & b \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \end{array}$$

(c) Diagraph of a relation:

A relation from A to A can also be represented pictorially as follows:
Draw a small circle for each element of A and write the element in that circle. These circles are called vertices. If the element a_i is related to the element a_j i.e. $a_i R a_j$ draw a straight line or an arc with an arrow in the direction from a_i to a_j. Such a straight line or an arc with an arrow is called an edge. If an element a_i is related to itself, we draw an arc with an arrowhead starting and ending at a_j. Such an arc is called a loop. The resulting pictorial representation is called directed graph or diagraph of R.
Thus, if a relation R on A is represented by a digraph then the edges and loops in the diagraph correspond to the pairs in R and the vertices correspond to the element of A.
Conversely if a diagraph is given, we can write the set of the corresponding relation.

(2) Types of Relations:

(i) **Void Relation:** As Φ ⊂ A x A, for any set A, so Φ is a relation on A, called the empty or void relation.

(ii) **Universal Relation:** Since, A x A ⊆ A x A, so A x A is a relation on A, called the universal relation.

(iii) **Identity Relation:** If every element of set A is related to itself only, it is called Identity relation.

$$I = \{(a, a), a \in A\}$$

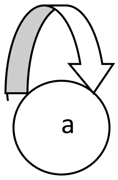
e.g. When we throw a dice, the total outcome we get is 36. i.e. (1, 1) (1, 2), (1, 3)…..(6, 6). From these, if we consider the relation (1, 1), (2, 2), (3, 3) (4, 4) (5, 5) (6, 6), it is an identity relation.

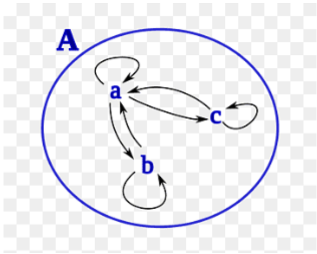
(iv) **Inverse Relation:** Let R be a relation from set A to set B. Then relation from B to A defined only if by b R^{-1}a if and only if a R b. i.e. R^{-1} = {(b, a): (a, b) ∈ R}.

e.g. If you throw two dice and if R = {(1, 2) (2, 3)}, R^{-1} = {(2, 1) (3, 2)}.

(v) **Reflexive / Irreflexive Relation:** A relation R on a set A is called reflexive If (a, a) ∈ R for all a ∈ A, (a, a). i. e. a R a for **all** a ∈ A.

A relation is call A relation R on a set A is called reflexive If (a, a) ∉ R, for all a ∈ A,(a, a) .i.e. a $\not R$ a for **all** a ∈ A.
In other words, R is reflexive if each element of A is related to itself and it is irreflexive if no element is related to itself.





Note: (1) Since in a reflexive relation every element is related to itself in the matrix M_R of a reflexive relation we should have all diagonal elements unity. Non diagonal element may be zero or unity.
(2) By the same reasoning in the matrix of an irreflexive relation, all diagonal elements are zero.
(3) In the digraph of a reflexive relation we get loop around each vertex. For irreflexive relation we get no loop around any vertex.
(4) The number of reflexive relations R on a set A with n elements is $2^{n(n-1)}$.

(vi) Symmetric / Asymmetric Relation: A relation R on a set A is called symmetric relation if (a,b) ∈ R then (b, a) ∈ R, for all a & b ∈ A.

i.e. if a R b then for symmetry we must have b R a.

e.g. Let A be the set of all males and R be the relation of "being a brother". This relation is symmetric because if a is a brother of b then b is also a brother of a.

A relation R on a set A is called asymmetric relation if (a,b) ∈ R then (b, a) ∉ R, for all a&b ∈ A.

i.e. if a R b then for asymmetry we must have b $\not R$ a.

e.g. If Z^+ is the set of all positive integers then > is asymmetric.

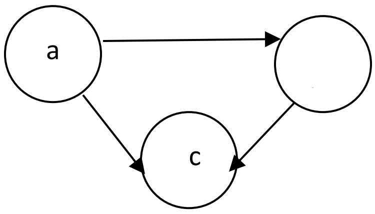
Note: (1) Matrix of symmetric relation M_R satisfies the property that if m_{ij} = 1then m_{ji} = 1and if m_{ij} = 0 then m_{ji} = 0. Hence transpose of M_R of a symmetric matrix is M_R and M_R must be a square matrix.
(2) While matrix of asymmetric relation M_R satisfies the property that if m_{ij} = 1then m_{ji} = 0 for all i and j. Hence we must have m_{ii} = 0. i.e. diagonal elements are zero and nondiagonal elements are not symmetrically placed.
(3) In the digraph of symmetric relation if there is an edge from I th vertex to the j th then there is also an edge from the j th vertex to the i th vertex. i.e. in a symmetric relation there are two lines connecting one vertex to another or there is no line at all. There is "both way traffic" or the line is "closed both ways".
(4) In the diagraph of an asymmetric relation we get all edges in "one direction" only. Also, there is no loop around any element.
(5) The number of symmetric relations on a set A with n elements is $2^{n(n+1)/2}$.

(vii) Transitive Relation: A relation R on a set A is called transitive relation If (a, b) ∈ R, (b,c) ∈ R, then (a, c) ∈ R, for all a, b, c ∈ A.

i.e. if a R b and b R c then for transitive we must have a R c.

A relation is non transitive if we can find a, b, c ∈ R such that a R b and b R c and still a $\not R$ c.

e.g. Let A be the set of all integers and let R be the "less than" relation. We know that if a<b, b<c then a<c. Hence R is transitive.



Note: (1) A relation is transitive if and only if its matrix M_R is such that if $m_{ij}=1$ and $m_{jk}=1$ then $m_{ik}=1$.

(2) In the diagraph of a transitive relation the edges form atleast one triangle.

Remark: A relation is either reflexive or irreflexive or neither. On the other hand relation cn be symmetric or asymmetric or transitive if there is even only one pair satisfying the conditions.

(viii) Equivalence Relation: A relation R on a set A is called equivalence relation if it is reflexive, symmetric and transitive.

e.g. Let R be the relation on the set of real numbers such that a R b if and only if a-b is an integer. Here R is an equivalence relation.

Note: (1) The matrix of an equivalence relation M_R has the following properties:

(i)all elements in leading diagonal are unity.

(ii) the elements 0,1 are symmetric wrt the leading diagonal.

(2) The diagraph of an equivalence relation has the following propertie.

(i) all vertices have loops around them.

(ii) vertices if connected then they are connected both ways.

(iii) edges of atleast three vertices form a triangle by ouble lines.

Sample Problems

Q.1. State the domain and range of the following relation.
R = {(2, –3), (4, 6), (3, –1), (6, 6), (2, 3)}

Solution: The above list of points, being a relationship between certain x's and certain y's, is a relation. The domain is all the x-values, and the range is all the y-values.

domain: {2, 3, 4, 6}   range: {–3, –1, 3, 6}

Q.2. State the domain and range of the following relation.

R = {(–3, 5), (–2, 5), (–1, 5), (0, 5), (1, 5), (2, 5)}

Solution: I'll just list the *x*-values for the domain and the *y*-values for the range:

domain: {–3, –2, –1, 0, 1, 2}   range: {5}

Q.3 Let A ={1, 3 } and B= { 2, 5 }. Then find the relation R from A to B via the inequality ``<''.

Solution: Here 1< 2, 1 <5, 3≮2, 3 <5 .

In other words, 3 pairs (1, 2), (1, 5) and (3, 5) observed the `<' relationship.
Hence by collecting them in R,

R= {(1, 2), (1, 5), (3 ,5)}

Exercise 4

1. Let S={1,2,3,4} and T={1,2,3,4}.In each of the following find all the pairs of S×T that belong to R: (a) R={ (x, y)/x ≥ y}, (b) R={ (x, y)/x > y}, (c) R={ (x, y)/x ≤y}, (d) R={ (x, y)/ x =y^2}.
2. If the relation R defines on A by "if x divides y then xRy" then find R and its digraph & matrix, where $A = \{2, 3, 4, 6, 8\}$.

Let's check the take away from lecture

Choose the correct option from the following:

1. If n/m means that n is a factor m, the relation '/' in z – {0} is
   (a) Reflexive and symmetric (b) Symmetric ad transitive
   (c) Reflexive, symmetric and transitive (d) Reflexive, transitive and not symmetric
2. Let A= {1, 2, 3}, B = {1, 3, 5}. If relation R from A to B is given by {(1, 3), (2, 5), (3, 3)} then R – 1 is
   (a) {(3, 3), (3, 1), (5, 3)} (b) {(1, 3), (2, 5), (3, 3)}
   (c) {(1, 3), (5, 2)} (d) None of these
3. Let R be a relation in N defined by R = {(x, y) : x + 2y = 8}. The range of R is
   (a) {2, 4, 6} (b) {1, 2, 3} (c) {1, 2, 3, 4, 6} (d) None of these

Practice Problem

1. Let X={1,2,3,4,5} and Y={1,2,3}. Graph the relation R = {(x, y)/ x > y} from a subset of X into Y. Give the domain and the range of the relation.
2. Graph the relation {(x, y)/x^2+y^2 = 25} in the set of real numbers.
3. Let R be a relation on X={1,2,3,4} defined by (x, y) ϵ R if x ≤ y, x,y ϵ X. State true or false whether domain and range of R are both equal to X.
4. Let A={1,2,3,4,6} and R be the relation on A defined by $(a, b) \in R$ if and only if a is a multiple of b. Write down R as a set of ordered pairs.

Learning from the topic: Students will be able to understand different types of relations and to identify domain and range of it.

Lecture -02

2.1. Relation continued…….

Learning Objective:

1. Student shall be able to learn pictorial presentation of relation and different types of relation and properties of relation.

2.1.4. Important Formulae/ Theorems / Properties on Relations: (continued….)

(vi) Symmetric / Asymmetric Relation: A relation R on a set A is called symmetric relation if (a,b) ∈ R then (b, a) ∈ R, for all a & b ∈ A.

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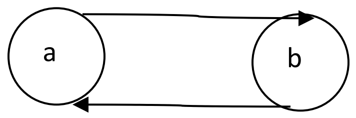

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Note: (1) Matrix of symmetric relation M_R satisfies the property that if $m_{ij} = 1$then $m_{ji} = 1$and if $m_{ij} = 0$ then $m_{ji} = 0$. Hence transpose of M_R of a symmetric matrix is M_R and M_R must be a square matrix.
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A relation is non transitive if we can find a, b, c ∈ R such that a R b and b R c and still a $\not R$ c.

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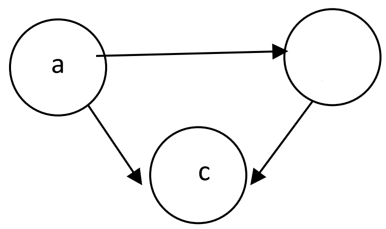

Note: (1) A relation is transitive if and only if its matrix M_R is such that if $m_{ij} =1$ and $m_{jk} =1$ then $m_{ik} =1$.

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Remark: A relation is either reflexive or irreflexive or neither. On the other hand relation cn be symmetric or asymmetric or transitive if there is even only one pair satisfying the conditions.

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e.g. Let R be the relation on the set of real numbers such that a R b if and only if a-b is an integer. Here R is an equivalence relation.

Note: (1) The matrix of an equivalence relation M_R has the following properties:

(i)all elements in leading diagonal are unity.

(ii) the elements 0,1 are symmetric wrt the leading diagonal.

(2) The diagraph of an equivalence relation has the following propertie.

(i) all vertices have loops around them.

(ii) vertices if connected then they are connected both ways.

(iii) edges of atleast three vertices form a triangle by ouble lines.

(ix) Circular relation: A relation is called circular if a R b, b R c hen c R a.

(x) Composition of Relation: Let R and S be two relations from sets A to B and B to C respectively, then we can define relation S o R from A to C such that (a, c) ∈ So R ⇔ ∃ b ∈ B such that (a, b) ∈ R and (b, c) ∈ S. This relation S o R is called the composition of R and S.
(i) $R \circ S \neq S \circ R$ (ii) $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$

Sample Problems

Q.4. Let A ={1,2} and B={1,2,3} and R be a relation from A to B such that (x, y) R iff x-y is even.

Determine R and verify Is 1R1 ? Is 2R3 ? Is 1R3 ?

Solution: (a) Give R by its explicit elements.
For any (x,y) pair in A ✕ B ={ (1,1), (1,2), (1,3), (2,1), (2,2), (2,3) }, we must check if xRy or if x-y is even.
Hence R={ (1,1), (1,3), (2,2)}.

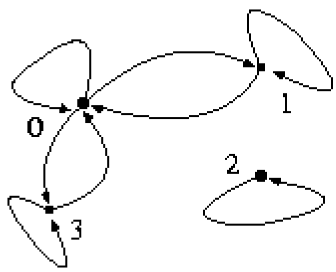
(b) Is 1R1 ? Is 2R3 ? Is 1R3 ?

Yes,	1R1	since (1,1) ∈ R.
No,	2R3	since (2,3) ∉ R.
Yes,	1R3	since (1,3) ∈ R.

Q.5 Let A={ 0,1,2,3} and a relation R on A be given by

R={ (0,0), (0,1), (0,3), (1,0), (1,1), (2,2), (3,0), (3,3) } .

Is R reflexive? symmetric? transitive?



Solution: Plotting the diagraph of R, we observe

(a) R is reflexive, i.e. there is a loop at each vertex.

(b) R is symmetric, i.e. the arrows joining a pair of different vertices always appear in a pair with opposite arrow directions.

(c) R is not transitive. This is because otherwise the arrow from 1 to 0 and arrow from 0 to 3 would imply the existence of an arrow from 1 to 3 (which doesn't exist). In other words (1,0) ∈R, (0,3) ∈R and (1,3)∉R imply R is not transitive.

Q.6 Show that the relation 'congruence modulo 3' [m ≡ n (mod 3)] over the set of integers is an equivalence relation.

Note: Let m, n and d be integers with d≠0. If d divides (m-n), denoted by d | (m-n), i.e. m-n=dk for some integer k, then we say m is congruent to n modulo d, written simply as m ≡n (mod d).

Solution: Here R is the relation of congruence modulo 3 on the set Z of all integers, i. e. m R n ⇔ m ≡ n (mod 3) ⇔3 | (m-n).

We just need to verify that R is reflexive, symmetric and transitive.

(a) Reflexive: for any n ∈ Z we have n R n because 3 divides n-n=0.

(b) Symmetric: for any m, n ∈ Z if m R n, i.e. m ≡ n (mod 3) then there exists k∈ Z such that m-n = 3k. This means n-m = 3(-k), i.e. n ≡ m (mod 3), implying finally n R m. For example, 7R4 is equivalent to 4R7 can be seen from 7R4 ⇔7 ≡ 4 (mod 3) ⇔ 7-4 =3 ×1 ⇔4-7 =3 ×(-1) ⇔ 4 ≡ 7 (mod 3) ⇔4R7 .

(c) Transitive: for any m, n, p ∈ Z, if m R n and n R p, then there exist r, s ∈Z such that m-n =3r and n-p=3s. Hence m-p=(m-n) + (n-p) = 3 (r + s), i.e. m R p. We thus conclude that R is an equivalence relation.

Exercise 2

1. Let A = (1,4,5) and R be relation on A defined by a R b if a+b≤6. Write R, M_R check for reflexive and symmetry.
2. Let $A = \{1,2,3\}$. Determine the nature of the following relations on A :(reflexive, symmetric, transitive)
   i. $R_1 = \{(1,2),(2,1),(1,3),(3,1)\}$.
   ii. $R_2 = \{(1,1),(2,2),(3,3),(2,3)\}$.
3. For $x, y \in Z, xRy\ if\ and\ if\ only\ 2x + 5y$ is divisible by 7 is R an equivalence relation ? Find equivalence relation.
4. Show that a relation is reflexive and circular iff it is an equivalence relation.

Let's check the take away from lecture

Choose the correct option from the following

1. If n/m means that n is a factor m, the relation '/' in z – {0} is
   (a) Reflexive and symmetric (b) Symmetric ad transitive
   (c) Reflexive, symmetric and transitive (d) Reflexive, transitive and not symmetric
2. Let A= {1, 2, 3}, B = {1, 3, 5}. If relation R from A to B is given by {(1, 3), (2, 5), (3, 3)} then R – 1 is
   (a) {(3, 3), (3, 1), (5, 3)} (b) {(1, 3), (2, 5), (3, 3)}
   (c) {(1, 3), (5, 2)} (d) None of these

Practice Problem

1. Let $A = \{1,2,3\}$. Determine the nature of the following relations on A:(reflexive, symmetric , transitive)
   i. $R_3 = \{(1,1),(2,2),(3,3)\}$.

ii. $R_4 = \{(1,1), (2,2), (3,3), (2,3), (3,2)\}$.
iii. $R_5 = \{(1,1), (2,3), (3,3)\}$.
iv. $R_6 = \{(2,3), (3,4), (2,4)\}$.
v. $R_7 = \{(1,3), (3,2)\}$.

2. Let m be a positive integer greater than 1. Let a R b if and only if m divides a-b i. e. R = {(a,b) | a ≡ b (mod m) }. Verify that R is an equivalence relation.
3. Let R be a binary operation on the set of positive integers such that

   R = {(a,b)| a-b is an odd positive integer}

   Is it an equivalence relation?

4. A relation R is defined on Z, the set of integers as a R b iff ab ≥ 0. Is it an equivalence relation?

Learning from the topic: Students will be able to understand types of function, composition of functions, recursively defined functions.

Lecture -03

Learning Objective:

1. Student shall be able to know different types of function.

1.1 Function

1.1.1 Introduction:
Introduction: All functions are relations, but not all relations are functions. One of the most important concepts in mathematics is function. It is one of the oldest and richest concept. Function is a group of statement that together perform a certain task. Functions are basically used to avoid rewriting of same code again and again. One can use functions to compare the two routes. One can find application of functions everywhere e.g. a simple road trip driving the car.

1.1.2 Key Definition:

(1) Domain, Co-domain, image and Range of a function

Let f : A → B be a function, then

i. A is called **Domain**.
ii. B is called **Codomain**.
iii. f is a function, ∴ ∀ x ∊ A ∃ unique y ε B. Then y is called f-**image** of x.
iv. A subset of co-domain is said to be **range** of f if it contains all images ∀ x ∊ A.

(2) Injective (one-one) Function: Let f : A → B be a function, then f is said to be injective (one-one). A function ***f*** is **injective** if and only if whenever ***f(x) = f(y)***, ***x = y***.

e.g. $f(x) = x+5$ from the set of real numbers $\mathbb{R}$ to $\mathbb{R}$ is an injective function.
But $f(x) = x^2$ from the set of real numbers $\mathbb{R}$ to $\mathbb{R}$ is not an injective function because $f(2) = 4$ and $f(-2) = 4$. This is against the definition $f(x) = f(y)$, $x = y$, because $f(2) = f(-2)$ *but* $2 \neq -2$. Note that if we made it from the set of natural numbers $\mathbb{N}$ to $\mathbb{N}$ then it is injective, because $f(2) = 4$**,** there is no f(-2), because -2 is not a natural number'

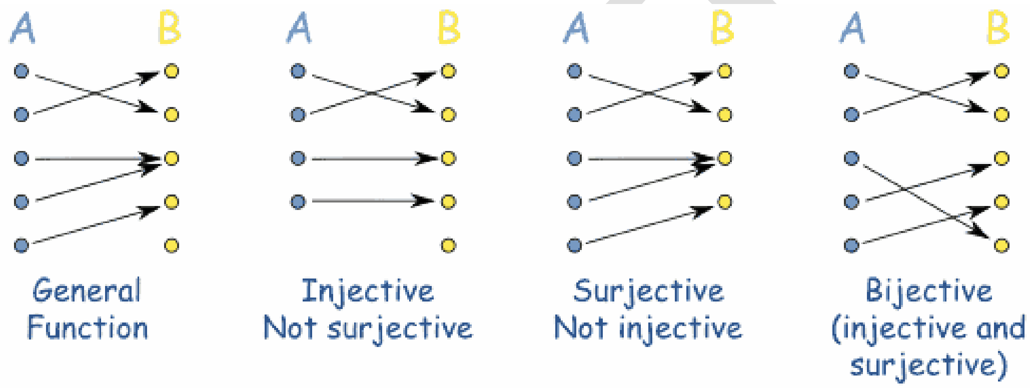
(3) Surjective (onto) Function: Let f : A → B be a function, then f is said to be surjective (onto). A function *f* (from set *A* to *B*) is surjective if and only for every *y* in *B*, there is at least one *x* in *A* such that *f*(*x*) **=** *y*, in other words *f* is surjective if and only if *f(A)* = *B*. So, every element of the range corresponds to at least one member of the domain.

e.g. The function $f(x) = 2x$ from the set of natural numbers $\mathbb{N}$ to the set of non-negative even numbers is a surjective function. However, $f(x) = 2x$ from the set of natural numbers $\mathbb{N}$ to $\mathbb{N}$ is not surjective, because, nothing in $\mathbb{N}$ can be mapped to *3* by this function.

(4) Bijective (one to one correspondence) Function: Let $f : A \rightarrow B$ be a function, then f is said to be bijective if f is injective and surjective both. A function *f* (from set *A* to *B*) is bijective if, for every *y* in *B*, there is exactly one *x* in *A* such that $f(x) = y$.

Alternatively, *f* is bijective if it is a **one-to-one correspondence** between those sets, in other words both **injective and surjective.**

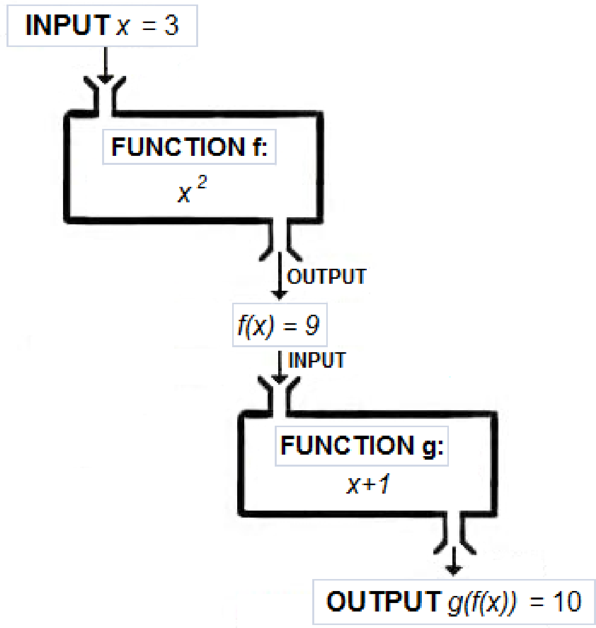
e.g. The function $f(x) = x^2$ from the set of positive real numbers to positive real numbers is injective and surjective. Thus it is also bijective. But not from the set of real numbers because you could have, both $f(2) = 4$ and $f(-2) = 4$.



(5) Inverse Function: Let $f : A \rightarrow B$ be a function, then f is said to be invertible if $f^{-1} : B \rightarrow A$ is a function.

If f is a function from *X* to *Y* then an **inverse function** for f, denoted by f^{-1}, is a function in the opposite direction, from *Y* to *X*, with the property that a round trip (a composition) returns each element to itself. Not every function has an inverse; those that do are called **invertible**. The inverse function exists if and only if f is a bijection.

(6) Composite Function: Let $f : A \rightarrow B$ and $g: B \rightarrow C$ be two functions, then gof is said to be composite function of f and g. The **function composition** of two or more functions takes the output of one or more functions as the input of others. The functions $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ can be *composed* by first applying f to an argument *x* to obtain $y = f(x)$ and then applying *g* to *y* to obtain $z = g(y)$. The composite function formed in this way from general f and *g* may be written $g \circ f : X \rightarrow Z,\ and\ \ (g \circ f)x \longrightarrow g(f(x))$.



(7) **Identity Function:** The identity function I(x) is the function which assigns every real number x to the same real number x. The identity function is trivially idempotent, i.e. I(I(x))=x. Thus u function over a set X that maps each element to itself is called the **identity function** for X, and typically denoted by I_X. Each set has its own identity function, so the subscript cannot be omitted unless the set can be inferred from context. Under composition, an identity function is "neutral": if f is any function from X to Y, then $(f \circ I)(x) = f,\ (I \circ f)(x) = f$.

(8) **Explicit Function:** "Explicit" is when the function shows how to go directly from x to y, such as: $y = x^3 - 3$. *When you know x, you can find.* That is the classic y = f(x) style.

(9) **Implicit Function:** "Implicit" is when it is not given directly such as: $x^2 - 3xy + y^3 = 0$. *When you know x, how do you find y?* It may be hard (or impossible!) to go directly from x to y.

Sample Problems

Q.1. Define function, if f converts a temperature in degrees Celsius *C* to degrees Fahrenheit *F*.

Solution: The function converting degrees Fahrenheit to degrees Celsius would be a suitable f^{-1}.

$$f(C) = \frac{9}{5}C + 32 \quad \text{and} \quad f^{-1}(F) = \frac{5}{9}\,(F - 32).$$

Exercise 3

1. Given f:Z->Z. Find whether function is one-to-one and whether it is onto. If the function is not onto, determine range f(Z). f(x)=x+7.
2. Define Bijective function. Show that the function f: R - {2/5} -> R – {4/5} defined by f(x) = (4x+3)/(5x-3) is a bijection.

Let's check the take away from lecture

Choose the correct option from the following

1. A function is said to be ------ if and only if f(a) = f(b) implies that a = b for all a and b in the domain of f.

a) One-to-many b) One-to-one c) Many-to-many d) Many-to-one

2. The function f(x)=x+1 from the set of integers to itself is onto. Is it True or False?

a)True b) False

Practice Problems

1. Explain injective, surjective and bijective functions with example.
2. Let A = {1,2,3} and B={1,2,3,4,5}. F={(1,1),(2,3),(3,4)} is a one-to-one function from A to B. g={(1,1),(2,3),(3,3)} is a function but not one-to-one. Why?
3. Given f:Z->Z. Find whether function is one-to-one and whether it is onto. If the function is not onto, determine range f(Z). $f(x)=x^2+x$.
4. A={1,2,3,4}, B={1,2,3,4,5,6}.

   How many functions are there from A to B. How many are one-to-one and how many onto?

Learning from the topic: Students will be able to know different types of function

Lecture -04

Learning Objective:

1. Student shall be able to find composition and inverse of functions.

1.1. Function Continued……….

Exercise 4

1. Let f: R→R , $f(x) = x^2 - 1$, $g(x) = 4x^2 + 2$ find fo(gof), go(fog).

2. If A = {1, 2, 3} and f_1, f_2, f_3, f_4 be functions from A to A given by

   $f_1 = \{(1, 2), (2, 3), (3, 1)\}$

   $f_2 = \{(1, 2), (2, 1), (3, 3)\}$

   $f_3 = \{(1, 1), (2, 2), (3, 1)\}$

   $f_4 = \{(1, 1), (2, 2), (3, 3)\}$

   Compute $f_1 \circ f_2$; $f_2 \circ f_1$; $f_1 \circ f_2 \circ f_3$; $f_4 \circ f_4$.

3. Functions f, g and h are functions from N to N defined as $f(n) = n + 1$, $g(n) = 2n$ and $h(n) = \begin{cases} 0 \text{ if n is even} \\ 1 \text{ if n is odd} \end{cases}$. Find fog, fof, gof, goh, hog, (fog)oh .

4. Let f, g, h be the functions define as follows:

   $f = \{(a_1,b_1), (a_2,b_3), (a_3,b_2),(a_4,b_2)\}$

   $g = \{(b_1,c_1), (b_2,c_3), (b_3, c_3),(b_4, c_3)\}$

   $h = \{(c_1,d_1), (c_2,d_2), (c_3, d_3)\}$

   Find (i) $g\circ f$, $h\circ(g\circ f)$, $(h\circ g)\circ f$, h^{-1}

   (ii) Identify onto and one-one function for 3 of them.

5. Functions f and g are defined as $f : R \to R$, $g: R \to R$ $f(x) = 2x + 3$, $g(x) = 3x - 4$. Find fog , f^{-1}, g^{-1} and verify that $(fog)^{-1} = g^{-1}of^{-1}$.

6. A function $f : R-\left\{\frac{7}{3}\right\} \to R-\left\{\frac{4}{3}\right\}$ is defined by $f(x) = \frac{4x-5}{3x-7}$, Prove that f is bijective and find the rule for f^{-1} .

Let’s check the take away from lecture

Choose the correct option from the following

1. Let f and g be the function from the set of integers to itself, defined by f(x) = 2x + 1 and g(x) = 3x + 4. Then the composition of f and g is ________.

   a) 6x + 9 b) 6x + 7 c) 6x + 6 d) 6x + 8

2. The inverse of function f(x) = x3 + 2 is __________.

   a) f -1 (y) = (y – 2) 1/2 b) f -1 (y) = (y – 2) 1/3 c) f -1 (y) = (y) 1/3 d) f -1 (y) = (y – 2)

3. The function f(x) = x3 is bijection from R to R. Is it True or False?

   a) True b) False

Practice Problems

1. Let f, g, h:R -> R is defined as $f(x)=x^3$, $g(x)=4x^2+1$ and h(x)=7x-1. Find rule of defining (hog)of, go(hof).
2. Let $A = \{1,2,3,4\}, B = \{a,b,c\}$ and $C = \{w,x,y,z\}$ with $f: A \to B$ and $g: B \to C$ given by $f = \{(1,a),(2,a),(3,b),(4,c)\}$ and $g = \{(a,x),(b,y),(c,z)\}$. Find gof.
3. Let f and g be functions from R to R defined by $f(x) = ax + b$ and $g(x) = 1 - x + x^2$. If $(gof)x = 9x^2 - 9x + 3$, determine a, b.
4. Let f and g be functions from R to R defined by $f(x) = ax + b$ and $g(x) = cx + d$. What relationship must be satisfied by a, b, c, d if $gof = fog$?
5. Let f, g, h be functions from Z to Z defined by $f(x) = x - 1$, $g(x) = 3x$, $h(x) = \begin{cases} 0, if\ x\ is\ even \\ 1, if\ x\ is\ odd \end{cases}$.
   Determine $(fo(goh))(x)$ and $((fog)oh)(x)$ and verify that $fo(goh) = (fog)oh$.
6. Consider the function $f: R \to R, g: R \to R$ defined by $f(x) = 2x + 5$ and $g(x) = \frac{1}{2}(x - 5)$. Prove that g is an inverse of f.
7. Functions f and g are defined as $f : R \to R$, $g: R \to R$ $f(x) = 3x - 4$, $g(x) = 2x + 3$. Find fog , f^{-1}, g^{-1} and verify that $(fog)^{-1} = g^{-1}of^{-1}$.

1. Let $A = B = C = R$, and $f: A \to B$ and $g: B \to C$ be defined by
   $$f(a) = 2a + 1, g(b) = \frac{1}{3}b, \forall a \in A, \forall b \in B.$$
   Compute gof and show that goh is invertible. What is $(gof)^{-1}$.

Learning from the topic: Students will be able to find composition and inverse of functions.

Learning Outcomes:

1. Know: Student should be able to

(a) Laws of set theory and teir applications

(b) Define various types of relations and functions.

(c) Solve different types of relations and functions.

2. Comprehend: Student should be able to

(a) Identify operations and laws of set theory.

(b) Identify properties of relations, Find composition of relations, domain and range of a relation, pictorial representation of relation.

(c) Identify properties of functions. Find composition of functions, domain and range of a functions.

3. Apply, analyze and synthesize: Student should be able to

(a) solve real life problems based on set theory

(b) distinguish between relations and functions.

(c) find compositon and inverse of the relation and function.

Self-Assessment

Name of student: **Class &Div:** **Roll No:**

1. Are you able to **identify and solve** problems based on set theory ?

(a) Yes (b) No

2. Are you able to distinguish the different types of relations and functions?

(a) Yes (b) No

3. Are you able to find the **composition of relations and functions**?

(a) Yes (b) No

4. Do you understand the difference between **compositions of relations and functions**?

(a) Yes (b) No

5. Will you able to solve the **pictorial representation of relation, properties of relation, partial ordering relation?**

(a) Yes (b) No

6. Do you understand this module?

(a) Fully understood (b) Partially understood (c) Not at all

Learning Resources

1. Higher Engineering Mathematics by Grewal B. S. 38th edition, Khanna Publication 2005.
2. Advanced Engineering Mathematics by Kreyszig E. 9th edition, John Wiley.

3. A Text Book of Applied Mathematics Vol. I & II by P.N. Wartilar & J.N. Wartikar, Pune, Vidyarthi Griha Prakashan, Pune.
4. Modern Digital Electronics by R. P. Jain 8th edition, Tata McGraw Hill.
5. C. L. Liu and D. P. Mohapatra, “Elements of Discrete Mathematics”, SiE Edition, Tata McGraw-Hill.