Poset

A relation R on a set A is said to be partial order if it is reflexive, antisymmetric and transitive. If R is a partial order relation on a set A then the set A with the partial order R is called partially ordered set or poset.

Ex:- let R be a set of real no. and a relation $\leq$ defined on R then prove that $(R, \leq)$ is a poset

Solution:-

1) Reflexive:- $\because a \leq a \ \forall a \in R$.

$\Rightarrow \leq$ is reflexive

2) Antisymmetric:-

let $a \leq b, b \leq a \Rightarrow a = b$

$\Rightarrow \leq$ is antisymmetric.

3) Transitive:- let $a \leq b, b \leq c \Rightarrow a \leq c$

$\Rightarrow \leq$ is transitive.

$\therefore \leq$ is partial order on R

$\therefore (R, \leq)$ is a poset

Ex:- Let z^+ be the set of +ive integers and a relation R defined on z^+ by aRb iff $a|b$ then prove that R is a partial order relation

Sol:-

1) Reflexive $\because a|a \ \forall a \in z^+$

$\therefore aRa \ \forall a \in z^+$

$\Rightarrow$ R is reflexive

ii) <u>Antisymmetric</u> :- let aRb and bRa.

$\Rightarrow a|b$ & $b|a$.

$\Rightarrow a=b \quad \forall\, a, b \in Z^+$

$\Rightarrow |$ is antisymmetric

iii) <u>Transitive</u> :-

let aRb and bRC.

$\Rightarrow a|b$ & $b|c$.

$\Rightarrow a|c$.

$\Rightarrow aRc$.

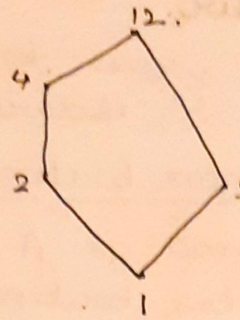
$\Rightarrow |$ is transitive

$\therefore |$ is a partial relation on Z^+

$(Z^+, |)$ is a poset.

Ex:- Draw the Hasse diagram of the following poset

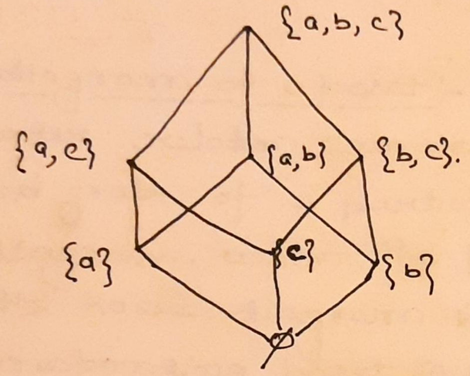
1) $A = \{1, 2, 3, 4, 12\}$ under partial order $|$



ii) $S = \{a, b, c\}$ & $A = P(S)$ under $\subseteq$.

Draw Hase diagram of the poset $(A, \subseteq)$.

$A = P(S) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}\ \{a,c\}\ \{b,c\}\ \{a,b,c\}\}$



Result:- If $\{A, R\}$ is a poset then (A, R^{-1}) is also a poset (Dual Poset)

Solution:-

Given R is a partial order on A.

To prove R^{-1} is also a partial order on A.

$aRa \ \forall a \in A$ ($\because$ R is reflexive).

$\Rightarrow aR^{-1}a \ \forall a \in A$

$\Rightarrow R^{-1}$ is reflexive.

$\Rightarrow aRb,\ bRa \Rightarrow a = b$

$\Rightarrow bR^{-1}a,\ aR^{-1}b \Rightarrow a = b$ (Hence it is antisymmetric).

aRb , $bRc \Rightarrow aRc$ (Hence R is transitive)

$\Rightarrow cR^{-1}b, bR^{-1}a \Rightarrow cR^{-1}a$

$\Rightarrow R^{-1}$ is transitive Hence.

R^{-1} is a partial order.

(A, R^{-1}) is a poset.

Definition:-

Comparable:- In a poset $(A, \leq)$ the element's a and b are said to be comparable if $a \leq b$ or $b \leq a$

In a poset every pair of elements need not be comparable

Linearly orderd set:- If every pair of element in a poset A is comparable then A is called linearly orderd set or chain or totally orderd set and partial order is called linear order.

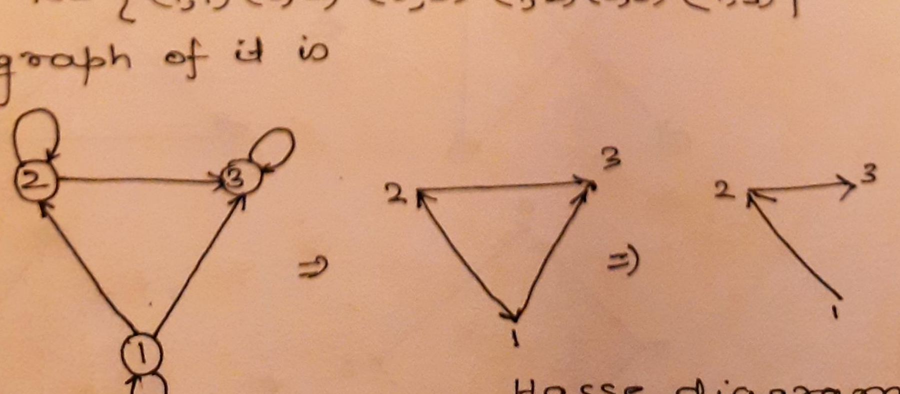
e.g.:- $(R, \leq)$ is linearly orderd set because every pair of real numbers are comparable. w.r.to this order pair.

Hasse diagram of a poset:- The diagraph of a partial order relation can be simplified and such simplified graph of a partial order relation is called Hasse diagram. when the partial order is a total order Its hasse diagram is a straight line and corresponding poset is called a chain.

consider $A = \{1, 2, 3\}$ a relation R on A defind by.

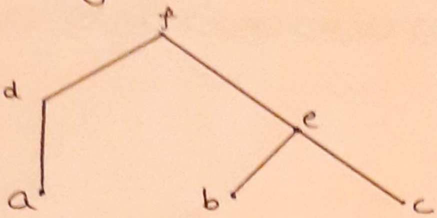
$$R = \{(1,1)\ (2,2)\ (3,3)\ (1,2)\ (2,3)\ (1,3)\}$$

diagraph of it is

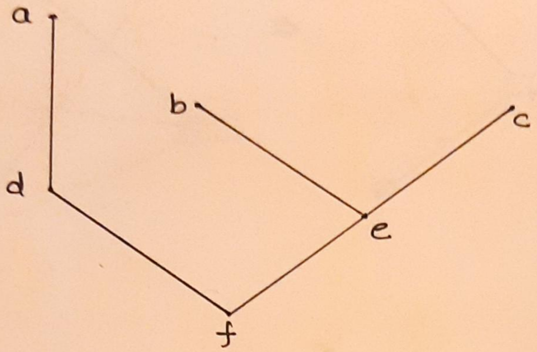


Hasse diagram

EXAMPLE :- Draw Hasse diagram of dual poset poset of the poset of the poset whose Hasse diagram is given.



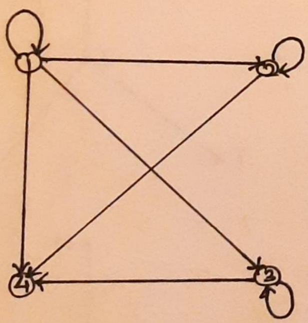
Solution :-



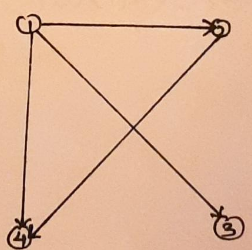
EXAMPLE :- Determine Hasse diagram of R

A = { 1, 2, 3, 4 } and R = { (1,1) (1,2) (1,3) (1,4) (2,2) (2,4) (3,3) (3,4) (4,4)

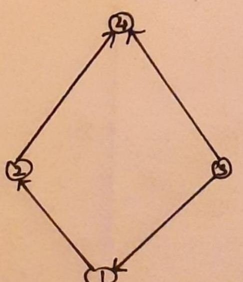
Solution :-



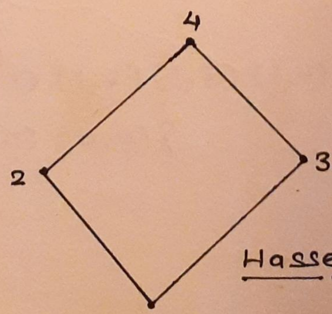
$\Rightarrow$



$\Rightarrow$



$\Rightarrow$

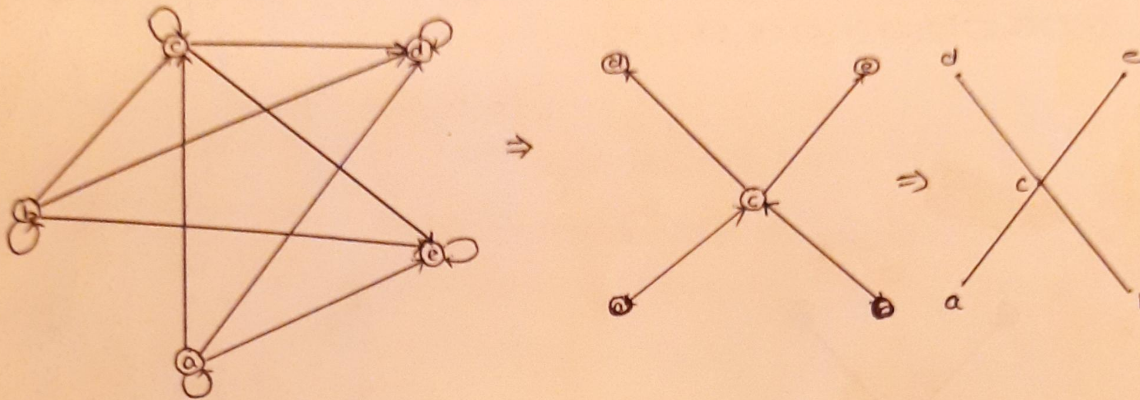


Hasse diagram

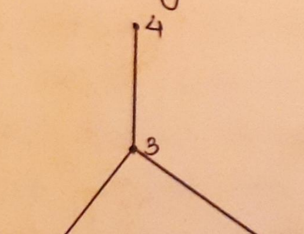
EXAMPLE:- Let $A = \{a, b, c, d, e\}$

$R = \{(a,a)\ (a,c)\ (a,d)\ (a,e)\ (b,b)\ (b,c)\ (b,d)\ (b,e)$
$(c,c)\ (c,d)\ (c,e)\ (d,d)\ (e,e)\}$

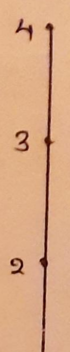
Solution:-



EXAMPLE:- Describle the order pairs in the relation defind by the Hasse diagram on $A = \{1, 2, 3, 4\}$ in the figure's



$R = \{(1,1)\ (1,3)\ (1,4)\ (2,2)\ (2,3)\ (2,4)\ (3,3)\ (3,4)\ (4,4)\}$



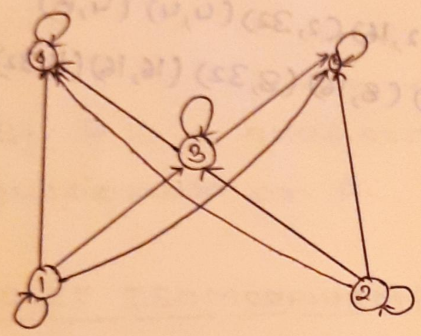
$R = \{(1,1)\ (1,2)\ (1,3)\ (1,4)\ (2,2)\ (2,3)\ (2,4)\ (3,3)$
$(3,4)\ (4,4)\}$

EXAMPLE:- Determine the Hasse diagram of the relation on $A = \{1, 2, 3, 4, 5\}$

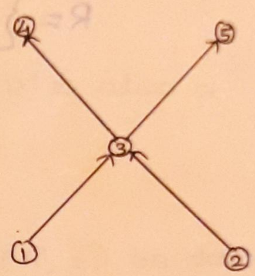
$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

$\therefore$ $R = \{(1,1)\ (1,3)\ (1,4)\ (1,5)\ (2,2)\ (2,3)\ (2,4)\ (2,5)\ (3,3)\ (3,4)$
$(3,5)\ (4,4)\ (5,5)\}$

diagraph.



Hasse diagram $\Rightarrow$



EXAMPLE:- Consider the partial order of divisibility on set A

Draw Hasse diagram of the poset and determine which poset are linearly ordered

(i) $A = \{1, 2, 3, 5, 6, 10, 15, 30\}$

(ii) $A = \{2, 4, 8, 16, 32\}$

Solution:-

1) $A = \{1, 2, 3, 5, 6, 10, 15, 30\}$

$R = \{(1,1)\ (1,2)\ (1,3)\ (1,5)\ (1,6)\ (1,10)\ (1,15)\ (1,30)$
$(2,2)\ (2,6)\ (2,10)\ (2,30)\ (3,3)\ (3,6)\ (3,15)\ (3,30)$
$(5,5)\ (5,10)\ (5,15)\ (5,30)\ (6,6)\ (6,30)\ (10,10)$
$(10,30)\ (15,15)\ (15,30)\}$

Hasse diagram

Non Linearly ordered

ii) If $A = \{2, 4, 8, 16, 32\}$

∴ Partial order relation

$R = \{(2,2)(2,4)(2,8)(2,16)(2,32)(4,4)(4,8)$
$(4,16)(4,32)(8,8)(8,16)(8,32)(16,16)(16,32)$
$(32,32)\}$.

is the Hasse diagram.

Linearly ordered

Quassi Order :- A relation R on A is called quassi order if it is transitive and irreflexive.

Ex :- Let $A = \{x \mid x$ is a real no. and $-5 \le x \le 20\}$. S.T. Usual relation $<$ is an quasi order on A

irreflexive.

$\because x \nless x \;\; \forall x \in A$

$\therefore <$ is irreflexive.

Transitive. let $x < y$, $y < z$. $\therefore x < z$.

$\therefore <$ is transitive.

$\therefore <$ is quasi order.

If R is a quasi order on A then R^{-1} is also a quasiorder on A.

POSET ISOMORPHISM :- Let $(A, \le)$ and $(A', \le')$ be two poset's and $f : A \to A'$ be a one-one correspondence between A to A' then f is said to be isomorphic. if $\forall a, b \in A$ $a \le b \Rightarrow f(a) \le f(b)$ in A'

Ex :- let $A = \{1, 2, 3, 6\}$ under division.

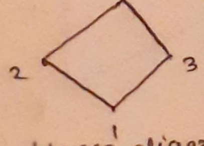
$A' = P\{a, b\} = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$ under $\subseteq$.

Show that $(A, |)$ is isomorphic to $(A', \subseteq)$

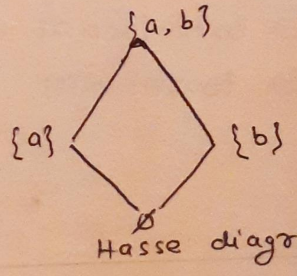
Solution :-

$A = \{1, 2, 3, 6\}$.

Hase diagram



Hasse diagram of A



Hasse diagram of A'

$f(1) = \emptyset$

$f(2) = \{a\}$

$f(3) = \{b\}$

$f(6) = \{a, b\}$.

Isomorphism between two poset's means.

I) their Hasse diagram's are absolutely identical.

II) Their is a bijection among the elements of two posets.

III) The partial order's are preserved if $a \leq b$ iff $f(a) \leq f(b)$

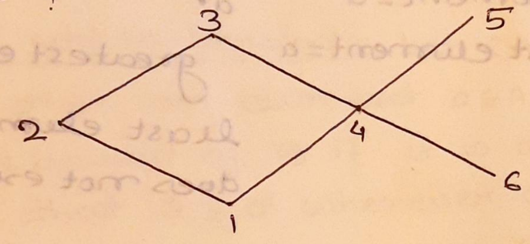
EXTREMAL ELEMENTS OF A POSET:-

Maximal Element:- let $(A, \leq)$ be a poset and $a \in A$ then the element a is said to be maximal element of A if there is no b such that $a \leq b$

Minimal Element:- Let $(A, \leq)$ be a poset then an element $a \in A$ is said to be minimal element of A if there is no b such that $b \leq a$.

1) Determine maximal and minimal element of the poset?

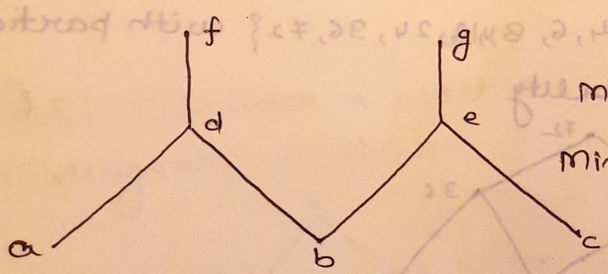
1)



Maximal element = 3, 5

Minimal element = 1, 6.

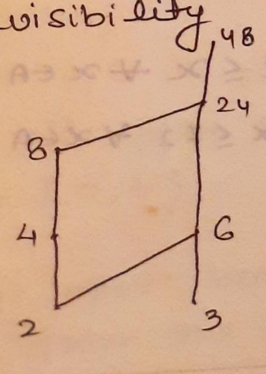
2)



Maximal element = f, g.

Minimal element = a, b, c.

2) let $A = \{2, 3, 4, 6, 8, 24, 48\}$ with partial order of divisibility



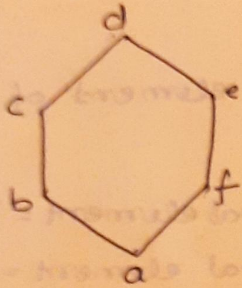
maximal element = 48.

minimal element = 2, 3.

GREATEST ELEMENT :- An element $a \in A$ is said to be greatest element of A if $x \leq a \; \forall x \in A$

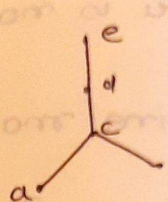
Least element :- An element $a \in A$ is said to be the least element of A if $a \leq x \; \forall x \in A$

EXAMPLE :- Determine the greatest and least elements if they exist in the poset



greatest element = d

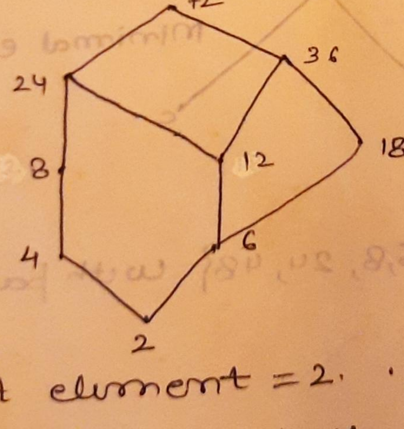
Least element = a



greatest element = e

least element does not exist.

EXAMPLE :- $A = \{2, 4, 6, 8, 12, 18, 24, 36, 72\}$ with partial order of divisibility



Least element = 2. $\because 2 \leq x \; \forall x \in A$

Greatest element = 72 $\quad x \leq 72 \; \forall x \in A$.

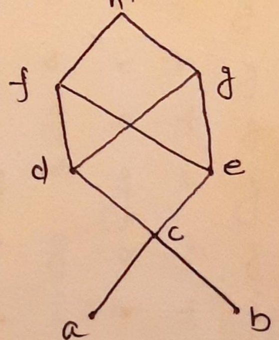
Lower Bound:- Let $(A, \leq)$ be a poset and $B \subseteq A$. then an element $a \in A$ is said to be lower bound of B if $a \leq b \ \forall \ b \in B$

Upper bound:- Let $(A, \leq)$ be a poset and $B \subseteq A$. then an element $a \in A$ is said to be an upper bound of B if $b \leq a \ \forall \ b \in B$.

Greatest Lower bound:- Let $(A, \leq)$ be a poset and $B \subseteq A$ then an element $a \in A$ is said to be greatest Lower bound of B if it is a lower bound of B s.t. $a' \leq a$ whenever a' is Lower bound of B.

Least upper bound:- Let $(A, \leq)$ be a poset and $B \subseteq A$ then an element $a \in A$ is said to be least upperbound of B if it is an upper bound of B. such that $a \leq a'$ whenever a' is the upperbound of B.

EXAMPLE:- Consider a poset $A = \{a, b, c, d, e, f, g, h\}$ where hasse diagram is shown.



Find all upper and lower bounds of subset of A.

I) $B_1 = \{a, b\}$ II) $B_2 = \{c, d, e\}$

Sol:- I) upper bound of $B_1 = \{c, d, e, f, g, h\}$

from the figure there is no Lower bound of B_1

greatest lower bound does not exist

& least upperbound: c

ii) Upper bound's of $B_2 = \{f, g, h\}$.

Lower bound's $B_2 = \{c, a, b\}$.

greatest Lower bound = c.

Least upper bound does not exist.

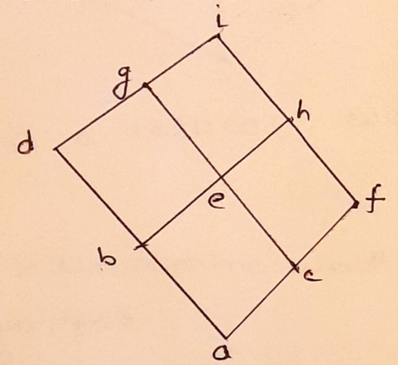
LATTICES :-

A Lattice is a poset $(L, \leq)$ in which every subset $\{a, b\}$ consisting two elements a, b has least upperbound and greatest lower bound.

lub of a and b = $a \vee b$ (called a join b)

glb of a and b = $a \wedge b$ (called a meet b)

EXAMPLE :- Determine weather the Hasse diagram represent a Lattice



Lets make the composition table.

$\vee$	a	b	c	d	e	f	g	h	i
a	a	b	a	d	e	f	g	h	i
b	b	b	e	d	e	h	g	h	i
c	c	e	c	g	e	f	g	h	i
d	d	d	g	d	g	i	g	i	i
e	e	e	e	g	e	h	g	h	i
f	f	h	f	i	h	f	i	h	i
g	g	g	g	g	g	i	g	i	i
h	h	h	h	i	h	h	i	h	i
i	i	i	i	i	i	i	i	i	i

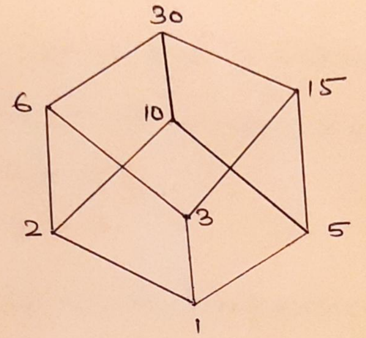
$\wedge$	a	b	c	d	e	f	g	h	i
a	a	a	a	a	a	a	a	a	a
b	a	b	a	b	b	a	b	b	b
c	a	a	c	a	c	c	c	c	c
d	a	b	a	d	b	a	d	b	d
e	a	b	c	b	e	c	e	e	e
f	a	a	c	a	c	f	c	f	f
g	a	b	c	d	e	c	g	e	g
h	a	b	c	b	e	f	e	h	h
i	a	b	c	d	e	f	g	h	i

Since from composition table we find that $\forall a, b \in A$. $a \vee b$, $a \wedge b \in A$ $\therefore$ Poset A is a Lattice.

A set D_n is a set of all positive integer's which are divisor's of n always form a Lattice.

Example :- Prove that D_{30} is a Lattice.

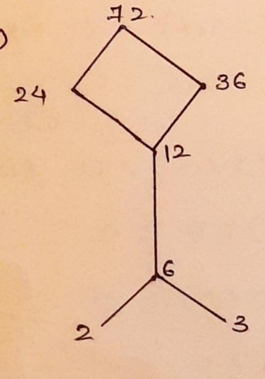
Solution :- $D_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$.



This is a Lattice $\because$ every pair has a LUB & glb.

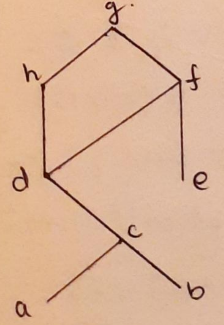
EXAMPLE :- Determine weather the following is a Lattice are not.

a)



$\because$ $\{2, 3\}$ has no GLB it is not a Lattice.

b)



Here $\{c, e\}$ has no glb hence it is not a Lattice.

Note :-

i) $a \le a \vee b$ and $b \le a \vee b$

ii) $a \le c,\ b \le c \Rightarrow a \vee b \le c$ then $a \vee b$ is LUB of a & b

iii) $a \wedge b \le a \quad a \wedge b \le b$

iv) $c \le a,\ c \le b \Rightarrow c \le a \wedge b$ then $a \wedge b$ is GLB of a & b

Theorem :- Let L be a Lattice then

I) Indempotent properties

a) $a \vee a = a$

b) $a \wedge a = a$

II) Commutative properties

a) $a \vee b = b \vee a$

b) $a \wedge b = b \wedge a$

III) Associative properties

a) $a \vee (b \vee c) = (a \vee b) \vee c$

b) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$

Proof :-

a) $a \vee (b \vee c) = (a \vee b) \vee c$

$\because a \le a \vee (b \vee c)$ & $(b \vee c) \le a \vee (b \vee c)$

$b \le (b \vee c)$ and $c \le b \vee c$

(by transitive property)

$\therefore b \le a \vee (b \vee c)$ & $c \le a \vee (b \vee c)$

$\because a \le a \vee (b \vee c)$ and $b \le a \vee (b \vee c)$

$(a \vee b) \le a \vee (b \vee c)$

$\Rightarrow (a \vee b) \vee c \le a \vee (b \vee c)$.

Similarly we can find

$a \vee (b \vee c) \le (a \vee b) \vee c$.

By antisymmetric relation

$a \vee (b \vee c) = (a \vee b) \vee c$.

Absorption Properties:-

a) $a \vee (a \wedge b) = a$

b) $a \wedge (a \vee b) = a$

Proof:-

a) $a \leq a \vee (a \wedge b)$ — ①

$a \leq a$ (by reflexive) $a \wedge b \leq a$ (by definition)

$\Rightarrow a \vee (a \wedge b) \leq a$. — ②

$\Rightarrow$ from ① & ②

$a \vee (b \wedge a) = a$.

b) $a \wedge (a \vee b) \leq a$. (by definition)

$a \leq (a \vee b)$ & $a \leq a$.

$\Rightarrow a \leq a \wedge (a \vee b)$

$\Rightarrow a = a \wedge (a \vee b)$

Sublattice:- Let $(L, \leq)$ be the Lattice a nonempty subset S of L is called sublattice of L if $\forall a, b \in S$ $a \vee b \in S$ & $a \wedge b \in S$.

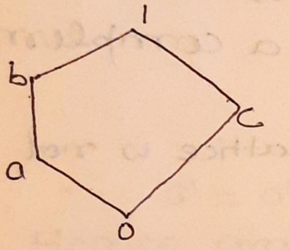
Bounded Lattice:- Let $(L, \leq)$ be a Lattice it is said to be the bounded Lattice if it has a greatest element 1 and the Least element 0

Distributive Lattice:- A Lattice L is called distributive Lattice if $\forall$ $a, b, c \in L$ It follows distributive properties

I) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

II) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$.

Example:- Show that the Lattice pictured in the figure is non distributive.



$a \vee (b \wedge c) = a \vee 0 = a$

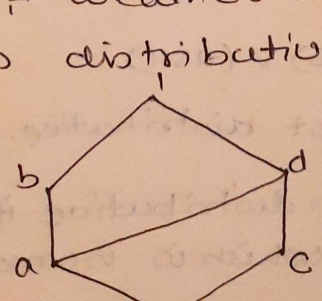
$(a \vee b) \wedge (a \vee c) = b \wedge 1 = b$

$\because a \neq b$

$\Rightarrow a \vee (b \wedge c) \neq (a \vee b) \wedge (a \vee c)$

$\therefore$ Lattice is not distributive.

Example:- weather the following Lattice shown in figure is distributive or not.



$a, b, c \in L$

$a \vee (b \wedge c) = a \vee 0$

$= a$

$(a \vee b) \wedge (a \vee c) = b \wedge d = a$.

$\Rightarrow a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.

$a \wedge (b \vee c) = a \wedge 1 = a$

$(a \wedge b) \vee (a \wedge c) = a \vee 0 = a$

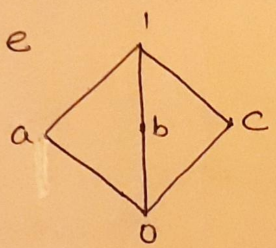
$\Rightarrow a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

$\Rightarrow$ Lattice is distributive Lattice.

Complement of an element:- Let L be a bounded Lattice with greatest element 1 and Least element 0 and $a \in L$ an element $a' \in L$ is said to be complement of a if $a \vee$ and $a \wedge a' = 0$

Complemented Lattice:- A Lattice is said to be complemented if it is bounded and every element of L has a complement.

(**) **Example**:- Show that the Lattice is not distributive



Solution:-

$a \vee (b \wedge c) = a \vee 0 = a$

& $(a \vee b) \vee (a \vee c) = 1 \vee 1 = 1$

$\because a \neq 1$

$\Rightarrow a \vee (b \wedge c) = (a \vee b) \vee (a \vee c)$

Hence it is not distributive.

Note:- A Lattice L is non distributive iff it is contain's a sublattice which is isomorphic to the Lattice in (*) & (**)

EXAMPLE:- If L is a bounded distributive Lattice and the complement exist then it is unique.

Proof:- let a' & a'' be two complement of $a \in L$ then by definition

$$a \vee a' = 1 \quad \text{and} \quad a \wedge a' = 0$$

$$a \vee a'' = 1 \quad \text{and} \quad a \wedge a'' = 0$$

$$a' = a' \vee 0 = a' \vee (a \wedge a'')$$

$$= (a' \vee a) \wedge (a' \wedge a'') \quad \text{(distributive)}$$

$$= 1 \wedge (a' \wedge a'')$$

$$= a' \wedge a'' \quad -(1)$$

$$a'' = a'' \vee 0 = a'' \vee (a \wedge a')$$

$$= (a'' \vee a) \wedge (a'' \vee a')$$

$$= 1 \wedge (a'' \vee a')$$

$$= a'' \vee a'$$

$$= a' \vee a''$$

$$\therefore \quad a' = a''$$

Hence complement is unique

Boolian Algebra:- A boolian algebra is a lattice which is boundid distributive and complemented.

Ex:- Let L be a distributive lattice show that if there exist an element a if.

$$a \wedge x = a \wedge y \text{ and } a \vee x = a \vee y \text{ then } x = y.$$

Proof:-

$$\begin{aligned}
x &= x \vee (x \wedge a) \\
&= (x \vee x) \wedge (x \vee a) \\
&= x \wedge (x \vee a). \\
&= x \wedge (a \vee y). \\
&= (x \wedge a) \vee (x \wedge y) \\
&= (a \wedge y) \vee (x \wedge y). \\
&= y \wedge (a \vee x) \\
&= y \wedge (a \vee y) \\
&= y
\end{aligned}$$

Example:- Show that for all $x \in B$ (Boolian algebra) the complement of the complement of x if exist is x i.e. $(\overline{\overline{x}}) = x$ if $\overline{x}$ exist

Proof:- Consider

$$\begin{aligned}
(\overline{\overline{x}}) &= (\overline{\overline{x}}) \wedge 1 \\
&= (\overline{\overline{x}}) \wedge (x \vee \overline{x}) \\
&= (\overline{\overline{x}} \wedge x) \vee (\overline{\overline{x}} \wedge \overline{x}) \\
&= (\overline{\overline{x}} \wedge x) \vee 0 \\
&= 0 \vee (\overline{\overline{x}} \wedge x) \\
&= (x \wedge \overline{x}) \vee (\overline{\overline{x}} \wedge x) \\
&= x \wedge (\overline{x} \vee \overline{\overline{x}}) \\
&= x \wedge 1 \\
&= x
\end{aligned}$$

EXAMPLE:- If n is a positive integer and p^2 divide's n where p is a primenumber then show that D_n is not a boolian algebra.

Solution:- Since $p^2 | n \Rightarrow \exists$ a positive integer q s.t.

$\frac{n}{p^2} = q$ i.e. $n = p^2 q$

Since p^2 is divisor of $n \Rightarrow p$ is also a divisor of n i.e. p is an element of D_n

It's complement $\bar{p}$ is also an element of D_n (If D_n is a boolian algebra)

Hence GCD $(p, \bar{p}) = 1$ & L.C.M. $(p, \bar{p}) = n$

$\therefore p\bar{p} = n$ so that $\bar{p} = \frac{n}{p} = \frac{p^2 q}{p} = pq$

This show's that

GCD $(p, pq) = 1$ which is impossible

Since p and pq has p as a common divisor

$\therefore D_n$ Cannot be a boolean algebra.

EXAMPLE:- Show that in a boolian algebra for any a, b $\quad b \wedge (a \vee (a' \wedge (b \vee b'))) = b$

Solution:-

$b \wedge (a \vee (a' \wedge (b \vee b')))$

$= b \wedge (a \vee (a' \wedge 1))$ $\qquad$ (as $b \vee b' = 1$)

$= b \wedge (a \vee a')$ $\qquad$ $(a' \wedge 1 = a')$

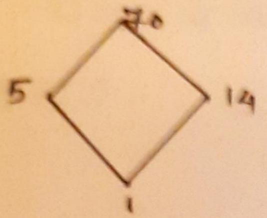
$= b \wedge 1$ $\qquad$ $(a \vee a' = 1)$

$= b$

Hence Proved

EXAMPLE:- Let $B = \{1, 5, 14, 70\}$ Let $a \vee b =$ l.c.m. of $\{a, b\}$ and $a \wedge b =$ g.c.d. $\{a, b\}$ $\bar{a} = 70/a$ Prove that $(B, \vee, \wedge, ^-, 0, 1)$ is a boolian algebra.

Solution:- i) The complete table of $a \vee b$ and $a \wedge b$ are given below which show's that B is a Lattice.



$\wedge$	1	5	14	70
1	1	1	1	1
5	1	5	1	5
14	14	1	14	14
70	1	5	14	70

$\vee$	1	5	14	70
1	1	5	14	70
5	5	5	70	70
14	14	70	14	70
70	70	70	70	70

ii) From the elementry properties of g.c.d. & l.c.m. we observe that the operation $\vee$ and $\wedge$ distribute over each other

iii) Its greatest element 1 is 70 and last element 0 is 1

iv) Further the complement are given by.

$a = \quad 1 \quad 5 \quad 14 \quad 70$

$\bar{a} = \quad 70 \quad 14 \quad 5 \quad 1$

$\therefore (L, \wedge, \vee, -, 0, 1)$ is a complemented distributive Lattice

$\therefore$ It is a booliam algebra.