Module 1 Set Theory

Lecture -01

1.1 Set Theory

1.1.1. Motivation: Set theory is a branch of mathematical logic that studies sets, which informally are collections of objects. The language of set theory can be used in the definitions of nearly all mathematical objects. Sets are important everywhere in mathematics because every field of mathematics uses or refers them in some way. They are important for building more complex mathematical structure.

The present also module deals with the relations and functions. In particular, the student should be able use the Concept of the relation and functions in higher mathematics & their project work. Two variables may be linked by some type of relationship. That brings us to the concept of relations. In contrast, a function defines how one variable depends on one or more other variables. There is nothing such as application of functions in our daily life. Like we use differentiation to find velocity, integration to find areas and volumes, Linear programming to get maximum profit, functions cannot be used like this but it makes our mathematics easy, it becomes really hard to explain mathematics without functions.

1.1.2. Syllabus:

Module2 Units	Content	Duration	Self study duration
1.1	Sets: Basic operations on sets, Cartesian products, disjoint union (sum), and power sets.	3 lectures	6 Hrs
1.2	Counting Principle	1 lectures	2 Hrs
1.3	Basic counting techniques – inclusion and exclusion, pigeon-hole principle	2 lectures	4 Hrs

1.1.3. Weight age: 21 Marks

1.1.4. Prerequisite: Basics of logic and concept of Number system

1.1.5. Learning Objective:

1. Student shall be able to understand the concept of sets.
2. Student shall be able to understand the laws related to set and their real life application.
3. Student shall be able to know different types of relations and to identify domain and range of a relation.
4. Student shall be able to learn pictorial presentation of relation and different types of relation and properties of relation.
5. Student shall be able to know different types of function.

6. Student shall be able to find composition and inverse of functions.

1.1.6. Key Notations:

(1) $\in$: belongs to
(2) $\cap$: intersaction
(3) $\subseteq$: subset
(4) $\subset$&$\supset$:proper subset
(5) { } : set
(6) U : universal set
(7) $\forall$: for every
(8) Z : set of integers
(9) Q : rational numbers
(10) $\cup$: union
(11) $\supseteq$: super set
(12) $\not\subset$: not a subset
(13) $\notin$: does not belong
(14) ϕ : empty set
(15) P (A) : power set
(16) : : such that
(17) R : set of real numbers
(18) A^c : Complement

1.1.7. Key Definitions:

(1) **Sets:** A set is any well-defined collection of objects called the elements or members of the set.

(2) **Null set:** The set that has no elements in it is denoted either by { } or the symbol ϕ and is called the empty set.

(3) **Equal sets:** Two sets A and B are said to be equal if they have the same elements and is written as A = B.

(4) **Universal Set:** A set which contains all the elements of other given sets is called a universal set and is denoted by U.

(5) **Finite set:** A set A is said to be finite if it has n distinct elements, where n ε N. Here n is called the cardinality of A and it is denoted by $|A|$. The set that is not finite is called **infinite**.

(6) **Power Set:** If A is a set, then the set of all subsets of A is called power set of A and it is denoted by P (A). If A has n elements, P(A) has 2^n elements. The power set of {1, 2, 3} is {{1, 2, 3}, {1, 2}, {1, 3}, {2, 3}, {1}, {2}, {3}, Ø}. The cardinality of the original set is 3, and the cardinality of the power set is $2^3 = 8$.

(7) **Venn Diagrams:** Pictorial representations of sets represented by closed figures are called set diagrams or Venn diagrams.

(8) **Subsets:** If any element of A is also an element of B, that is, if whenever $x\in A$ then $x\in B$, the A is called a subset of B or that A is contained in B and is written as $A \subseteq B$. If A is not a subset of B, it is written as A   B.

(9) **Countable sets:** A set is called countable (or countably infinite) if it has the same cardinality as N. Equivalently, a set A is countable if it can be enumerated in a sequence, i.e., if all of its elements can be listed as a sequence $a_1, a_2, \ldots$ .A set is called **uncountable** if it is infinite and not countable.

(10) **Cartesian Product of two sets:** If A and B are two non-empty sets, then their Cartesian product A × B is the set of all ordered pair of elements from A and B. $A\times B=\{(x,y):x\in A, y\in B\}$.

(11) **Duality:** If E is an equation in the set operations then equation obtained by replacing $\cup$ by $\cap$, $\cap$ by $\cup$, U by ϕ and ϕ by U is called the dual of E and is denoted by E^*.

(12) **Proper subset:** If A and B are two sets, then A is called the proper subset of B if $A \subseteq B$ but $B \supseteq A$ i.e., $A \neq B$. The symbol '$\subset$' is used to denote proper subset. Symbolically, we write $A \subset B$.

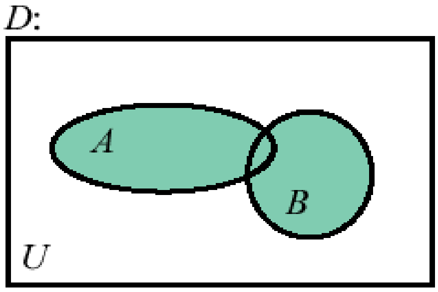
Note: i. No set is a proper subset of itself.
ii. Empty set is a proper subset of every set.

(13) **Operations on sets:**

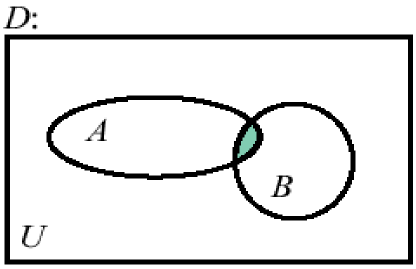
a. **Union**: If A and B are sets, then the set consisting of all elements that belong to A or B is said to be union and it is denote by $A\cup B$. Thus

$$A\cup B = \{x \mid x \varepsilon A \text{ or } x \varepsilon B\}$$

b. **Intersection**: If A and B are sets, then the set consisting of all elements that belong to both A and B is said to be intersection and it is denote by A ∩ B. Thus A∩B = {x | x ε A and x ε B}.



$A \cup B$

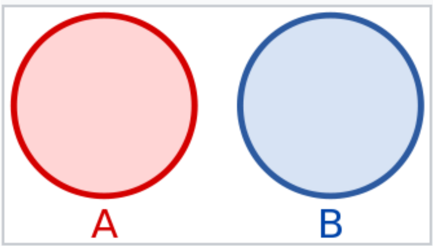


$A \cap B$

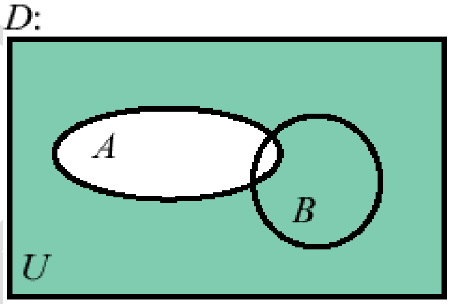
c. **Disjoint sets**: Two sets that have no common elements are called disjoint sets.

d. **Complement of B with respect to A:** If A and B are sets, then the set of all elements that belong to A but not in B is said to be complement of B with respect to A and it is denote by A – B. thus

A - B = {x | x ε A and x ∉ B}.



Two disjoint sets.



$\overline{A}$

e. **Symmetric difference**: If A and B are sets, then the set of all elements that belong to A or to B but not to both A and B is said to be symmetric difference and it is denote by A ⊕ B. Thus A⊕B = {x | (x ε A and x ∉ B) or (x ∉ A and x εB) } or A⊕B = (A-B)∪ (B-A). It is also represented by A∆B.

Sample Problem:

Q.1. Let U = {a, b, c, d, e, f, g, h, k}, A = {a, b, c, g), B = {d, e, f, g}, C = {a, c, f} and D = {f,h,k} compute the following: (a) $A \cup B$ (b) $A \cap C$ (c) $(A \cup B) - C$ (d) $\overline{A} \cap \overline{D}$ (e) $\overline{A \cup B}$ (f) $A \oplus B$ (g) $A \cup B \cup C$ (h) $A \cap B \cap C$ (i) $(A \cup B) \cap C$ (j) $\overline{A \cap B}$

Solution: (a) $A \cup B$ = {a, b, c, d, e, f, g}

(b) $A \cap C$ = {a, c}

(c) $(A \cup B) - C$ ={b, d, e, g}

(d) $\overline{A}$ = {d, e, f, h, k}, $\overline{D}$ = {a, b, c, d, e, g}, $\overline{A} \cap \overline{D}$ = {d, e}

(e) $\overline{A \cup B}$={h, k}

(f) $A \oplus B$ =(A–B)∪(B–A) here (A–B)= {a, b, c} and (B–A)={d, e, f}, then

$A \oplus B$ = {a, b, c, d, e, f}

(g) $A \cup B \cup C = \{a, b, c, d, e, f, g\}$

(h) $A \cap B \cap C = \{\phi\}$

(i) $(A \cup B) \cap C = \{a, c, f\}$

(j) $A \cap B = \{g\}$, $\overline{A \cap B} = \{a, b, c, d, e, f, h, k\}$

Exercise 1

1. Let U={1,2,3,4,5,6,7,8,9}, A={1,2,3,4}, B={4,5,6,7}, C={1,3,6}, D={6,8,9}. Compute the following: (a) $A \cap B$ (b) $B \cap \overline{C}$ (b) $\overline{A} \cap \overline{C}$ (c) $A \cap (B \cup C)$ (d) $(A \cup B) \cap C$ (e) $A \oplus B$
2. Let the universal set be the set R of all real numbers and let $A = \{x \in R \mid 1 < x \leq 5\}$ and $B = \{x \in R \mid 3 \leq x \leq 8\}$. Find each of the following

   (i) $A \cup B$ (ii) $A \cap B$ (iii) A-B (iv) B-A

Let's check the take away from lecture

Choose the correct option from the following:

1. If A = {1, 3, 5, 7, 9,….}, B = {2, 4, 6, 8, 10…..}, $(A \cap B)$ is

   (a) {1,2,3,4,5,6,7,8,9,10,….} (b) $\{\phi\}$ (c) {1,3,5,7,….} (d) none of these

2. $A \cap \overline{A}$ is

   (a) U (b) ϕ (c) A (d) $\overline{A}$

Practice problems

1. If a finite set A has *n* elements, prove that the power set of A has 2^n elements.
2. Determine the sets A and B, given that

   (i) $A - B$ = {1,2,4}, $B - A$ = {7,8} and $A \cup B$ = {1,2,4,5,7,8,9}.

3. If A = {1,2,3} determine the power set of A.
4. Let U={1,2,3,4,5,6,7,8,9}, A={1,2,3,4}, B={4,5,6,7}, C={1,3,6}, D={6,8,9}. Compute the following: (a) $\overline{A} \cap \overline{D}$ (b) $\overline{A \cup C}$ (c) $B \cup \overline{D}$ (d) $A \cup B \cup C$ (e) $A \cap B \cap C$ (f) $\overline{A} \oplus \overline{C}$

Learning from the topic: Students will be able to understand the concept of sets.

Lecture -02

1.1. Set theory continued……….

Learning Objective:

1. Student shall be able to understand the laws of sets.

1.1.8. Important Formulae/ Theorems / Properties:

Theorems:

1. **Laws of idempotent :** a) A∪A=A b)A∩A=A
2. **Laws of commutativity:** **a)** A∪B=B∪A b)A∩B=B∩A
3. **Laws of associativity:** a) (A∪B) ∪ C=A∪(B∪C) b) (A∩B)∩C=A∩(B∩C)
4. **Laws of distributivity:** a) A∩(B∪C) =(A∩B) ∪ (A∩C)
   b) A∪(B∩C) = (A∪B) ∩ (A∪C)
5. **Properties of Complement**: a) $\overline{\overline{A}} = A$ b) $A \cup \overline{A} = U$
   c) $A \cap \overline{A} = \Phi$ (d) $\overline{\Phi} = U$ e) $\overline{U} = \Phi$ f) $\overline{A \cup B} = \overline{A} \cap \overline{B}$
   g) $\overline{A \cap B} = \overline{A} \cup \overline{B}$
6. **Properties of a Universal set:** a) $A \cup U = U$ b) $A \cap U = A$
7. **Properties of empty set:** a) $A \cup \Phi = A$ b) $A \cap \Phi = \Phi$

Formulae: (with proof / counter example)

1. (A∩B)C=AC∪BC
2. A−(B∪C)=(A−B)∩(A−C)
3. A−(B∪C)=A∩(B∪C)C
4. A×(B∪C)=(A×B)∪(A×C)
5. A×(B−C)=(A×B)−(A×C)
6. A×(B∩C)=(A×B)∩(A×C)
7. A−(B∩C)=(A−B)∪(A−C)
8. A⊕B=(A−B)∪(B−A)

1.1.9. Key Notations:

1. R= the set of all real numbers.
2. Z: the set of all integers positive or negative including zero.

Sample Problem:

Q.2 Prove that $\overline{A} \oplus \overline{B} = A \oplus B$

Solution: LHS $\overline{A} \oplus \overline{B} = (\overline{A} - \overline{B}) \cup (\overline{B} - \overline{A}) = (\overline{A} \cap \overline{\overline{B}}) \cup (\overline{B} \cap \overline{\overline{A}}) = (\overline{A} - B) \cup (\overline{B} - A)$

$$(B - A) \cup (A - B) = (A - B) \cup (B - A) = A \oplus B = RHS$$

Q.3 Using set laws, prove that $(A \cap \overline{B}) \cup (\overline{A} \cap B) \cup (\overline{A} \cap \overline{B}) = \overline{A} \cup \overline{B}$

Solution:

$$(A \cap \overline{B}) \cup (\overline{A} \cap B) \cup (\overline{A} \cap \overline{B}) = (A \cap \overline{B}) \cup \left[(\overline{A} \cap B) \cup (\overline{A} \cap \overline{B}) \right] = (A \cap \overline{B}) \cup \left[\overline{A} \cap (B \cup \overline{B}) \right]$$

$$= (A \cap \overline{B}) \cup \left(\overline{A} \cap U \right) = (A \cap \overline{B}) \cup \overline{A} = \overline{A} \cup (A \cap \overline{B}) = (\overline{A} \cup A) \cap (\overline{A} \cup \overline{B})$$

$$= U \cap (\overline{A} \cup \overline{B}) = \overline{A} \cup \overline{B} = RHS$$

Exercise 2

1. Using Venn diagram show that $P - \left(Q \bigcup R\right) = \left(P - Q\right) \bigcap \left(P - R\right)$.
2. Using Venn diagram show that $P - \left(Q \bigcap R\right) = \left(P - Q\right) \cup \left(P - R\right)$.
3. Using Venn diagram show that $P \bigcap \left(Q \oplus R\right) = \left(P \bigcap Q\right) \oplus \left(P \bigcap R\right)$.

4. For the sets A, B, C given that $A \cap B = A \cap C$ and $\bar{A} \cap B = \bar{A} \cap C$. Is it necessary that B = C? Justify.

Let's check the take away from lecture

Choose the correct option from the following:

1. Dual of $A = (A \cup B) \cap (A \cup \Phi)$ is

   (a) $A = (A \cap B) \cap (A \cup \Phi)$ (b) $A = (A \cap B) \cup (A \cap \Phi)$

   (c) $A = (A \cap B) \cup (A \cap U)$ (d) $A = (A \cap B) \cap (A \cap \Phi)$

2. If U={1,2,3,4,5,6,7,8,9,10}, A={1,4,7,10}, B={1,2,3,4,5}, C={2,4,6,8}, then A∩(BUC)

   (a) {1,2,3,4,5,7,10} (b) {1,4,7,10} (c) {1,2,3,4,5} (d) {1,4}

Practice Problems

1. Using Venn diagram, prove: $A \cap (B - C) = (A \cap B) - C$
2. Using Venn diagram, prove: $(A - B) \cap (A - C) = A - (B \cup C)$
3. A, B ,C are of subsets of universal set U, then prove that
   A X (B U C) = (A X B) U (A X C).
4. Prove that the following distributive law, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Learning from the topic: Students will be able to understand the laws of sets.

Lecture -03

1.1. Set theory continued……….

Learning Objective:

1. Student shall be able to understand the real life application of set concept.

Key Definitions:

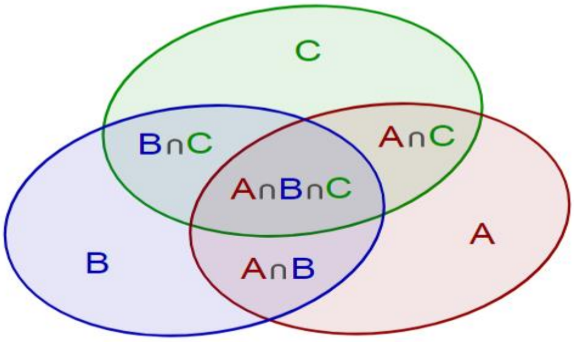
[1] The **principle** of **inclusion** and **exclusion** (PIE): It is a counting technique that computes the number of elements that satisfy at least one of several properties while guaranteeing that elements satisfying more than one property are not counted twice.

The inclusion–exclusion principle (also known as the sieve principle) is an equation relating the sizes of two sets and their union. It states that if A and B are two (finite) sets, then

$$|A \cup B| = |A| + |B| - |A \cap B|$$

The meaning of the statement is that the number of elements in the union of the two sets is the sum of the elements in each set, respectively, minus the number of elements that are in both. Similarly, for three sets A, B and C

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|$$



This can be seen by counting how many times each region in the figure to the right is included in the right hand side. More generally, for finite sets $A_1, \ldots, A_n$ one has the identity

$$|\cup_{i=1}^{n} A_i| = \sum_{k=1}^{n}(-1)^{k+1}\left(\sum_{1\le i_1<\cdots<i_k\le n}\left|A_{i_1} \cap \ldots . . \cap A_{i_k}\right|\right)$$A

Sample Problem:

Q.1. In a town there are 2000 literate persons. Of them 60% read newspaper A, 55% read newspaper B and 20% read neither A nor B. How many persons read (i) both the newspapers A and B? (ii) only one newspaper?

Solution: Let there be 100 literate persons in the town. Given that 20 of them do not read any newspaper. This means 100-20 = 80 read A or B or both.

Since 60 read A, 80 - 60 = 20 read only B. Since 55 read B, 50 - 55 = 25 read A only.

80 – (20+25) =35 read both.

Number of persons reading both newspapers = $2000 \times \frac{35}{100} = 700$

Q.2 Find the number of positive integers n where 1≤n≤100 and n is not divisible by 2, 3 or 5.

Solution: Let us denote the sets by A, B, C respectively. Then we have

n(A)=50, n(B)=33, n(C)=20, $n(A \cap B)$ =16, $n(A \cap C)$ =10, $n(B \cap C)$ =6, $n(A \cap B \cap C)$ =3

Number of integers divisible by 2 or 3 or 5 is

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$$
=50+33+20-16-6-10+3=74

Number which are not divisible by 2 or 3 or 5 = $n(S) - n(A \cup B \cup C)$ =100 – 74 = 26

Exercise 3

1. It is known that at the university 60% of the professors play tennis, 50% of them play bridge,70% jog, 20% play tennis and bridge, 30% play tennis and jog, 40% play bridge and jog. If someone claimed that 20% of the professor's jog and play bridge and tennis, would you believe this claim? Why?
2. 30 cars are assembled in a factory. The options available were a radio, AC and white-wall tyres. It is known that 15 of the cars have radio, 8 of them AC and 6 of them white-wall tyres. 3 of them have all 3 options. Find how many cars don't have any options at all.

3. 75 Children went to an amusement park where they can ride on the merry-go-round, roller coaster and ferris wheel. It is known that 20 of them have taken all 3 rides, and 55 of them have taken at least two of the 3 rides. Each ride costs 0.50 Rs and the total receipt of the amusement park was 70 Rs. Determine the number of children who did not try any of the rides.

Let's check the take away from lecture

Choose the correct option from the following:

1. Is this $A\times(B\cup C)=(A\times B)\cup(A\times C)$ correct?
   (a) Yes (b) No (c) don't know
2. Is this $P-(Q\cup R)=(P-Q)\cap(P-R)$ correct?
   (a) Yes (b) No (c) don't know

Practice Problems

1. A menu in the restaurant read like the following:

   Group A : Tomato soup, Manchow soup, Spring soup, Roasted papad
   Group B: Almond Ice-creame, Chowmin, Gobhi Paratha
   Group C: Sweet corn, French fries, Fried rice, Dal-rice, salad
   Group D: Coffee, Tea, Milk
   1) Suppose you select one course from each group without omition or substitution. How many complete different four course dinner you can make out of this menu?
   2) Suppose you select one course from each group A, B and D and two courses from group C without omition or substitution. How many different dinner you can make out of this menu?
2. A survey of a sample of 25 new cars being sold by an auto-dealer was conducted to see which of the three options: air-conditioning, radio and power windows were already installed. The survey found: 15 had air-conditioning, 12 had radio, 11 had power windows, 5 had air-conditioning and power windows, 9 had air-conditioning and radio, 4 had radio and power windows and 3 had all the three options. Find the number of cars that had :
   (a) only power windows (b) only one of the options (c) atleast one option (d) none of the options (e) air-conditioning and radio but not power windows.
3. A survey of 500 television viewers of a sports channel produced the following information: 285 watch cricket, 195 watch hockey, 115 watch football, 45 watch cricket and football, 70 watch cricket and hockey, 50 watch hockey and football and 50 do not watch any of the three kinds of games.

Learning from the topic: Students will be able to understand the real-life application of set concept.

Lecture -04

1.4 Pigeonhole Principal

Learning Objective:

1. Student shall be able to learn Pigeonhole Principal and counting problems related to it.

1.4.1. Important Formulae/ Theorems / Properties:

Theorems:

(1) Pigeonhole Principle:

Theorem: If n pigeons are assigned to m Pigeonholes and m < n then at least one Pigeonhole contains two or more pigeons.

(2) The extended Pigeonhole Principle:

Theorem : If there are m Pigeonholes and more than 2m, 3m, …. Pigeons, then at least one Pigeonhole will have more than 2, 3,…..pigeons respectivily.

Sample Problem:

Q.1. If eight persons are chosen from any group, show that at least two of them will have the same birthday.

Solution: There are seven days in a week. These 7 days are 7 pigeonholes and 8 persons from the group are 8 pigeons. Since there are more pigeons than the pigeonholes, at least two persons will have the same birthday.

Q.2. If any 5 numbers are chosen from 1 to 8, show that the sum of two of them will be 9.

Solution: Let us construct 4 sets each containing two numbers out of the above 8 numbers such that the sum of them is 9 as follows: (1,8), (2,7), (3,6), (4,5). Each of the 5 numbers chosen from 1 to 8 must belong to one of these 4 sets. Since there are four sets and we have chosen 5 numbers, the pigeonhole principle states that two of the chosen numbers must belong to the same set.

Q.3. Show that in a group of 50 students at least 5 are born in the same month.

Solution: Consider the months as pigeonholes and students as pigeons. Consider even distribution, if there are 48 students, (48/12=4) 4 are born in each month. Since there are 50 students at least 5 are born in the same month.

Exercise 4

1. If five points are taken in a square of side 2 units, show that atleast two of them are no more than $\sqrt{2}$ units apart.
2. Consider a regular hexagon whose sides are of length 1 unit. If any seven points are chosen in the region bounded by the hexagon, show that at least two of them will be no more than 1 unit apart.
3. Consider an equilateral triangle whose sides are of length 3 units. If ten points are chosen lying on or inside the triangle, then shoe that at least two of them are no kore than 1 unit apart.
4. Show that if any ten positive integers are chosen, two of them will have the same remainder when divided by 9.

Let's check the take away from lecture

Choose the correct option from the following:

1. If 7 colors are used to paint 50 bicycles at least how many bicycles will be of the same color?
   (a) 2 (b) 7 (c) 8 (d) 6
2. If nine persons are chosen from any group, then how many of them will have the same birthday ?
   (a) least two (b) least three (c) least four (d) don't know

Practice Problems

1. Show that if 7 numbers are chosen from 1 to 12 then two of them will add upto 13.
2. Show that among any group of 5 integers (not necessarily consecutive), there are two with the same remainder when divided by 4.

3. If a fair die is thrown 7 times show that it will show one of the numbers 1,2,3,4,5 and 6 atleast twice.
4. 4. If a number has 11digits then show that atleast two digits are repeated.
5. What is the least number of people required to guarantee that atleast two of them are born (i) on the same day, (ii) in the same month?

Learning from the topic: Student will be able to learn Pigeonhole Principal and counting problems related to it.

Learning Resources

1. D. S. Malik and M. K. Sen , "Discrete Mathematical Structures", Thomson.
2. C. L. Liu and D. P. Mohapatra, "Elements of Discrete Mathematics", SiE Edition, Tata McGraw-Hill.
3. Ralph P. Grimaldi, B. V. Ramana, " Discrete and Combinatorial Mathematics" Fifth Edision, Pearson Education.
4. Bernard Kolman, Robert C. Busby, Sharon Cutler Ross, Nadeem-ur-Rehman, "Discrete Mathematical Structures" Pearson Education.
5. Kenneth H. Rosen, "Discrete Mathematics and its Applications", Tata McGraw-Hill.

Online resource : 1) https://www.sanfoundry.com/discrete-mathematics-

2) https://swayam.gov.in