`QuasinormalModes.jl`: A Julia package for computing discrete eigenvalues of second order ODEs

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Summary

In General Relativity, when perturbing a black hole with an external field, or particle, the system relaxes by emitting damped gravitational waves, known as *quasinormal modes*. These are the characteristic frequencies of the black hole and gain the *quasi* prefix due to the fact that they have a real frequency, which represents the oscillation frequency of the response, and an imaginary frequency that represents the decay rate of said oscillations. In many cases, such perturbations can be described by a second order homogeneous ordinary differential equation (ODE) with discrete complex eigenvalues.

Determining these characteristic frequencies quickly and accurately for a large range of models is important for many practical reasons. It has been shown that the gravitational wave signal emitted at the final stage of the coalescence of two compact objects is well described by quasinormal modes ([Buonanno et al., 2007](#); [Seidel, 2004](#)). This means that if one has access to a database of quasinormal modes and of gravitational wave signals from astrophysical collision events, it is possible to characterize the remnant object using its quasinormal frequencies. Since there are many different models that aim to describe remnants, being able to compute the quasinormal frequencies for such models in a reliable way is paramount for confirming or discarding them.

Statement of need

`QuasinormalModes.jl` is a Julia package for computing the quasinormal modes of any General Relativity model whose perturbation equation can be expressed as second order homogeneous ODE. Not only that, the package can be used to compute the discrete eigenvalues of *any* second order homogeneous ODE (such as the energy eigenstates of the time independent Schrödinger equation) provided that these eigenvalues actually exist. The package features a flexible and user friendly API where the user simply needs to provide the coefficients of the problem ODE after incorporating boundary and asymptotic conditions on it. The user can also choose to use machine or arbitrary precision arithmetic for the underlying floating point operations involved and whether or not to do computations sequentially or in parallel using threads. The API also tries not to force any particular workflow on the users so that they can incorporate and adapt the existing functionality on their research pipelines without unwanted intrusions. Often user friendliness, flexibility and performance are treated as mutually exclusive, particularly in scientific applications. By using `Julia` as an implementation language, the package can have all of theses features simultaneously.

Another important motivation for using `Julia` and writing this package was the lack of generalist, free (both in the financial and license-wise sense) open source tools that serve the same purpose. More precisely, there are tools which are free and open source, but run on top of a proprietary paid and expensive software framework such as ([Jansen, 2017](#)) and ([Fortuna

41 & Vega, 2020), which are both excellent packages that aim to perform the same task as
42 `QuasinormalModes.jl` and can be obtained and modified freely but, unfortunately, require
43 the user to own a license of the proprietary `Wolfram Mathematica` CAS. Furthermore, their
44 implementations are limited to solve problems where the eigenvalues must appear in the ODE
45 as a polynomials of order p. While this is not prohibitively restrictive to most astrophysics
46 problems, it can be an important limitation in other areas. There are also packages that are
47 free and run on top of `Mathematica` but are not aimed at being general eigenvalue solvers at
48 all, such as (O'Toole et al., 2019), that can only compute modes of Schwarzschild and Kerr
49 black holes. Finally, the Python package (Stein, 2019) is open source and free but can only
50 compute Kerr quasinormal modes.

51 `QuasinormalModes.jl` fills the existing gap for free, open source tools that are able to
52 compute discrete eigenvalues (and in particular, quasinormal modes) efficiently for a broad
53 class of models and problems. The package was developed during the author's PhD research
54 where it is actively used for producing novel results that shall appear on the author's thesis.
55 It is also actively used in a collaborative research effort (of which the author is one of the
56 members) for computing quasinormal modes produced by perturbations with integer (but
57 different than 0) spins and semi-integer spins. These results are being contrasted with those
58 obtained by other methods and so far show excellent agreement with each other and with
59 literature results.

Underlying algorithm

61 `QuasinormalModes.jl` internally uses a relativity new numerical method called the Asymp-
62 totic Iteration Method (AIM). The method was introduced in (Ciftci et al., 2003) but the
63 actual implementation used in this package is based on the revision performed in (Cho et al.,
64 2012). The main purpose of the (AIM) is to solve the following general linear homogeneous
65 second order ODE:

$$y''(x) - \lambda_0(x)y'(x) - s_0(x)y(x) = 0, \tag{1}$$

66 where primes denote derivatives with respect to to the variable x (that is defined over some
67 interval that is not necessarily bounded), $\lambda_0(x) \neq 0$ and $s_0(x) \in C_\infty$. The method is based
68 upon the following theorem: let λ_0 and s_0 be functions of the variable $x \in (a, b)$ that are C_∞
69 on the same interval, the solution of the differential equation, Eq. (1), has the form

$$y(x) = \exp\left(-\int \alpha \mathrm{d}t\right) \times \left[C_2 + C_1 \int^x \exp\left(\int^t (\lambda_0(\tau) + 2\alpha(\tau))\mathrm{d}\tau\right) \mathrm{d}t\right] \tag{2}$$

70 if for some $n > 0$ the condition

$$\delta \equiv s_n \lambda_{n-1} - \lambda_n s_{n-1} = 0 \tag{3}$$

is satisfied, where

$$\lambda_k(x) \equiv \lambda'_{k-1}(x) + s_{k-1}(x) + \lambda_0(x)\lambda_{k-1}(x) \tag{4}$$
$$s_k(x) \equiv s'_{k-1}(x) + s_0\lambda_{k-1}(x) \tag{5}$$

71 where k is an integer that ranges from 1 to n.

72 Provided that the theorem is satisfied we can find both the eigenvalues and eigenvectors of
73 the second order ODE using, respectively, Eq. (3) and Eq. (2). Due to the recursive nature
74 of Eq.(4) and Eq.(5), to compute the quantization condition, Eq.(3), using n iterations the
75 n-th derivatives of λ_0 and s_0 must be computed multiple times. To address this issue, (Cho
76 et al., 2012) proposed the use of a Taylor expansion of both λ and s around a point ξ where
77 the AIM is to be performed. This improved version is implemented in `QuasinormalModes.jl`

Benchmark

To show `QuasinormalModes.jl` in action, this section provides a simple benchmark where the fundamental quasinormal mode ($n = \ell = m = s = 0$) of a Schwarzschild black hole was computed using 16 threads on a Intel(R) Core(TM) i9-7900X @ 3.30GHz CPU with 256 bit precision floating point numbers.
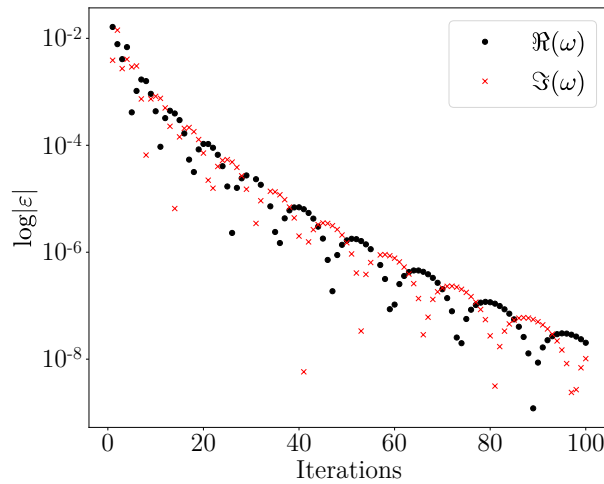

Figure 1: Error convergence for the fundamental Schwarzschild quasinormal mode as a function of the number of AIM iterations.
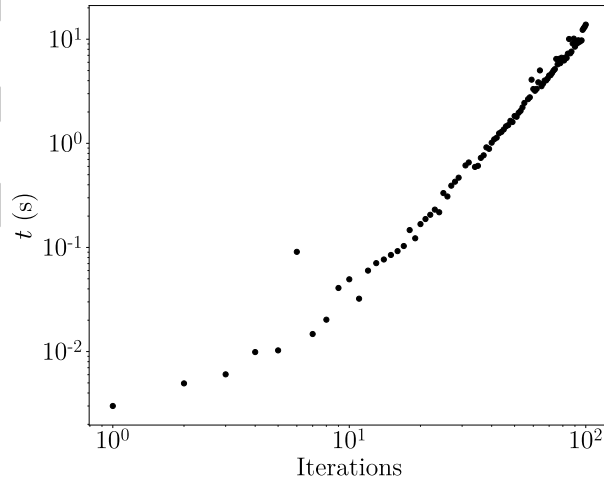

Figure 2: Time taken to compute the fundamental Schwarzschild quasinormal mode as a function of the number of AIM iterations on a logarithmic scale.

In order to quantify the rate at which the method converges to the correct results, the error measure ε was defined as follows: given the computed real and imaginary quasinormal frequencies, denoted respectively by ω_R and ω_I, and the reference frequencies given in (Berti et al., 2009), denoted respectively as $\overline{\omega}_R$ and $\overline{\omega}_I$ we have that $|\varepsilon| = |\omega_{R,I} - \overline{\omega}_{R,I}|$. In Figure 1 ε is plotted as a function of the number of iterations for both the real and imaginary frequencies. We can see that, as the number of iterations increases, the error in the computed values rapidly decreases.

90 In Figure 2, the time required to perform a certain number of iterations is plotted in logarithmic
91 scale. Each data point is the arithmetic mean time obtained after 10 runs of the algorithm
92 for a given number of iterations. The time measurement takes care to exclude the overhead
93 induced at "startup" due to Julia's JIT compilation. Even tough the time taken to perform
94 a certain number of iterations increase with a power law, the time scale required to achieve
95 highly accurate results is still around 10s. This time would be even smaller if one chooses to
96 use built-in floating point types instead of arbitrary precision numbers.

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