Theory: Relations and Functions in Simple Terms

What is a Relation?

A relation is a connection or association between elements of two sets. In math, it's often shown as a set of ordered pairs (a, b), where the first element $a$is related to the second element b.

Example: Students and their roll numbers; each student is related to a unique roll number.

What is a Function?

A function is a special type of relation where each element in the first set (called the domain) is related to exactly one element in the second set (called the range).

Important: All functions are relations, but not all relations are functions.

1. Domain and Range

- **Domain**: The set of all possible inputs (first elements).
- **Range**: The set of all possible outputs (second elements).

2. Representation of Relations and Functions

- Ordered pairs: e.g., $\{(1,2), (3,4)\}$.
- Mapping diagram: Arrows pointing from domain elements to range elements.
- Graphs: Plotting pairs on coordinate axes.
- Equations: $f(x) = 2x + 3$defines a function rule.

3. Types of Relations

- **Reflexive**: every element relates to itself.
- **Symmetric**: if $a$relates to b, then $b$relates to a.
- **Transitive**: if $a$relates to $b$and $b$relates to c, then $a$relates to c.
- **Equivalence relation**: if relation is reflexive, symmetric, and transitive.

4. Types of Functions

- **One-one (Injective)**: Different inputs map to different outputs.
- **Onto (Surjective)**: Every element in the range is covered.
- **Bijective**: Both one-one and onto (perfect pairing).
- **Constant function**: Maps all inputs to the same output.
- **Identity function**: Maps each element to itself.

5. Special Functions

- Even function: $f(-x) = f(x)$(symmetrical graph about y-axis).
- Odd function: $f(-x) = -f(x)$(symmetrical about origin).

- Polynomial, exponential, logarithmic functions, etc..

6. Function Operations

- Addition, subtraction, multiplication, division of functions.
- Composition: $(f \circ g)(x) = f(g(x))$.

7. Inverse Functions

- Reverse the roles of inputs and outputs.
- f^{-1}exists only if $f$is bijective.

Examples

Example 1: Relation vs Function

Relation: $\{(2,3), (2,4)\}$(2 relates to both 3 and 4, so not a function).

Function: $\{(1,2), (2,3)\}$

Example 2: Domain and Range

For function $f(x) = x^2$:

- Domain: All real numbers.
- Range: Non-negative real numbers $[0, \infty)$.

Example 3: Even Function

$f(x) = x^2, f(-x) = (-x)^2 = x^2 = f(x)$, so even.

Example 4: Composition of Functions

If $f(x) = 2x + 1$and $g(x) = x^2$,

$$(f \circ g)(x) = f(g(x)) = f(x^2) = 2x^2 + 1$$

Example 5: Inverse Function

If $f(x) = 3x + 4$,
Solve $y = 3x + 4$for x:

$$f^{-1}(x) = \frac{x - 4}{3}$$

Practice Questions

Basic Level (Board Exam)

1. Define a relation with an example.

2. What makes a relation a function?
3. Find the domain and range of $f(x) = 2x + 5$.
4. Is the relation $\{(1,2), (2,2), (3,3)\}$a function?
5. Define an even function with an example.
6. What is the range of $f(x) = -x^2$?
7. Write the function rule for mapping $\{1 \to 3, 2 \to 5\}$.
8. Explain what an identity function is.
9. Define one-one function.
10. Define onto function.

Moderate Level (Bridging Board & JEE)

11. Determine whether the function $f(x) = \frac{1}{x}$is even, odd, or neither.
12. Find the inverse of $f(x) = 4x - 7$.
13. If $f(x) = x^2$and $g(x) = x + 1$, find $(f \circ g)(x)$.
14. State and prove whether $f(x) = x^3$is an odd function.
15. Find the domain of $f(x) = \sqrt{4 - x^2}$.
16. Show that the composition of two functions is a function.
17. Prove that $f(x) = \frac{2x+3}{x-1}$is one-one.
18. Define equivalence relation with an example.
19. Find the range of $f(x) = \frac{1}{x+2}$.
20. Give an example of a function that is neither one-one nor onto.

Advanced Level (JEE Main/Advanced, MCQ included)

21. (MCQ) Which of the following is a function?
    - (A) $\{(1,2), (2,3), (1,4)\}$
    - (B) $\{(1,2), (2,3), (3,2)\}$
    - (C) $\{(1,2), (1,2), (2,3)\}$
    - (D) $\{(2,3), (3,4), (3,5)\}$
22. (MCQ) The inverse of $f(x) = 5x + 9$is:
    - (A) $\frac{y-9}{5}$
    - (B) $5x - 9$
    - (C) $\frac{5y+9}{x}$

- (D) $\frac{x+9}{5}$

23. Let $f(x) = x^2$ and $g(x) = 3x + 1$. Find $(g \circ f)(2)$.
24. Prove that the function $f(x) = x^3 + x$ is one-one.
25. (MCQ) If $f(x) = x^2 - 4$, then $f(-2)$ equals:
    - (A) 0
    - (B) -8
    - (C) 4
    - (D) 8

Answers and Detailed Explanations

1. Relation is a set of ordered pairs showing association; example: $\{(1,2), (3,4)\}$.

2. Function requires each input to map to only one output.

3. Domain: $(-\infty, \infty)$, Range: $(-\infty, \infty)$.

4. Yes, it's a function since each input maps to one output.

5. Even function: $f(-x) = f(x)$, e.g., x^2.

6. Range is $(-\infty, 0]$ because $x^2 \geq 0$, so $-x^2 \leq 0$.

7. Function rule: For input x, output $2x + 1$ or given mapping specifically.

8. Identity function: $f(x) = x$.

9. One-one: different inputs have different outputs.

10. Onto: every element in range has a pre-image in domain.

11. $f(x) = \frac{1}{x}$ is neither even nor odd since $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$.

12. Inverse of $f(x) = 4x - 7$: $f^{-1}(x) = \frac{x+7}{4}$.

13. $(f \circ g)(x) = (g(x))^2 = (x + 1)^2 = x^2 + 2x + 1$.

14. $f(-x) = (-x)^3 = -x^3 = -f(x)$; proof that x^3 is odd.

15. Domain: $| x | \leq 2$ (because inside square root must be ≥ 0).

16. If g and f are functions, their composition is a function since each input in domain of g produces exactly one output, which is input to f.

17. $f(x)$ one-one because if $f(x_1) = f(x_2)$ implies $x_1 = x_2$.

18. Equivalence relation example: equality $=$ on real numbers.

19. Range $f(x) = \frac{1}{x+2}$ is all real numbers except 0.

20. Example: $f(x) = x^2$over integers not one-one (many x give same output) and not onto all integers.

21. Correct: (B) only.

22. Correct: (A) $\frac{y-9}{5}$.

23. $f(2) = 4$, $g(f(2)) = 3 \times 4 + 1 = 13$.

24. $f(x) = x^3 + x$is one-one; derivative positive for all x, so strictly increasing.

25. $f(-2) = (-2)^2 - 4 = 4 - 4 = 0$, correct (A).
