TRANSFORMASI FOURIER

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Mata Kuliah : Matematika Lanjut 2 (TI)

BAHASAN

- TRANSFORMASI FOURIER
- TRANSFORMASI COSINUS FOURIER
- TRANSFORMASI SINUS FOURIER
- SIFAT-SIFAT TRANSFORMASI FOURIER

TRANSFORMASI FOURIER

FAKULTAS MATEMATIKA DAN ILMU PENGETAHUAN ALAM
UNIVERSITAS INDONESIA

TRANSFORMASI FOURIER

Definisi Transformasi Fourier

Definisi

Fungsi $F(\alpha)$ disebut *transformasi Fourier* dari fungsi $f(x)$ dan ditulis

$$F(\alpha) = \mathbf{F}\{f(x)\},$$

bila dari (4), akan diperoleh berikut ini :

$$F(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u) e^{i\alpha u}\, du\,. \qquad (7)$$

TRANSFORMASI FOURIER

Definisi Transformasi Fourier Inverse

Sedangkan fungsi $f(x)$ disebut *transformasi Fourier inverse* dari fungsi $F(\alpha)$ dan ditulis

$$f(x) = \mathbf{F}^{-1}\{F(\alpha)\},$$

bila

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\alpha) e^{-i\alpha x} \, d\alpha. \qquad (8)$$

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Contoh Soal Transformasi Fourier

Carilah transformasi Fourier dari fungsi

$$f(x) = \begin{cases} 1 & ,\ bila \quad |x| < a \\ 0 & ,\ bila \quad |x| > a \end{cases}$$

$\longrightarrow f(x) = \begin{cases} 0 & ,\ bila \quad x < -a \\ 1 & ,\ bila \quad -a < x < a \\ 0 & ,\ bila \quad x > a \end{cases}$

di mana a konstanta positif. Gambarlah grafik dari $f(x)$ dan $F(\alpha) = \mathbf{F}\{f(x)\}$ tersebut.

$(|x|)^2 < (a)^2$

$x^2 < a^2$

$x^2 - a^2 < 0 \rightarrow < 0 \rightarrow -$

$(x-a)(x+a) = 0$

$x = a \vee x = -a$

$+ \quad - \quad +$

$-a \quad a$

$-a < x < a$

$(|x|)^2 > (a)^2$

$x^2 > a^2$

$x^2 - a^2 > 0$

$(x-a)(x+a) = 0$

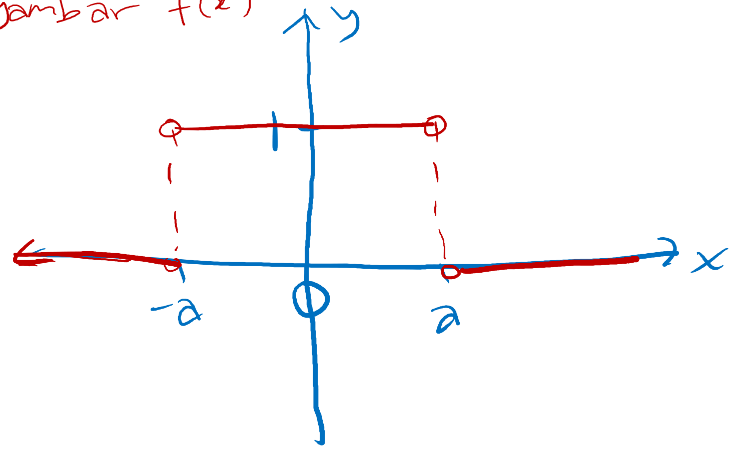
$x = a \vee x = -a$

$+ \quad - \quad +$

$-a \quad a$

$x < -a \vee x > a$

gambar $f(x)$



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Contoh Soal Transformasi Fourier

$$f(x) = \begin{cases} 0 & , x < -a \\ 1 & , -a < x < a \\ 0 & , x > a \end{cases}$$

jawab:

$$F(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u).e^{i\alpha u}\, du$$

$$F(\alpha) = \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{-a} 0.e^{i\alpha u}\, du + \int_{-a}^{a} 1.e^{i\alpha u}\, du + \int_{a}^{\infty} 0.e^{i\alpha u}\, du \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left(0 + \int_{-a}^{a} e^{i\alpha u}\, du + 0 \right)$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{i\alpha} \cdot \left[e^{i\alpha u} \right]_{-a}^{a}$$

$$F(\alpha) = \frac{1}{i\alpha\sqrt{2\pi}} \left(e^{i\alpha a} - e^{-i\alpha a} \right)$$

$$e^{i\theta} = \cos\theta + i \sin\theta$$

$$e^{i\alpha a} = \cos\alpha a + i \sin \alpha a$$

$$e^{-i\theta} = \cos\theta - i.\sin\theta$$

$$e^{-i\alpha a} = \cos\alpha a - i \sin\alpha a$$

$$F(\alpha) = \frac{1}{i\alpha\sqrt{2\pi}} \left(\cos\alpha a + i \sin\alpha a - (\cos\alpha a - i.\sin\alpha a) \right)$$

$$= \frac{1}{i\alpha\sqrt{2\pi}} \left(\cancel{\cos\alpha a} + i\sin\alpha a - \cancel{\cos\alpha a} + i \sin\alpha a \right)$$

$$F(\alpha) = \frac{1}{\cancel{i}\alpha\sqrt{2\pi}} \cdot 2\cancel{i}\sin\alpha a$$

$$= \frac{2}{\sqrt{2\pi}} \cdot \frac{\sin\alpha a}{\alpha}$$

$$= \frac{2}{\sqrt{2\pi}} \times \frac{\sqrt{2}}{\sqrt{2}} \cdot \frac{\sin\alpha a}{\alpha}$$

$$= \frac{\cancel{2}\sqrt{2}}{\cancel{2}\sqrt{\pi}} \cdot \frac{\sin\alpha a}{\alpha}$$

$$F(\alpha) = \sqrt{\frac{2}{\pi}} \cdot \frac{\sin\alpha a}{\alpha}$$

untuk $\alpha \neq 0$

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Contoh Soal Transformasi Fourier

$$F(\alpha) = F(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u).e^{i.0.u}\, du$$

$$F(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u)\, du$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{-a} 0\, du + \int_{-a}^{a} 1\, du + \int_{a}^{\infty} 0\, du \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left(0 + [u]_{-a}^{a} + 0 \right)$$

$$= \frac{1}{\sqrt{2\pi}} (a - (-a)) = \frac{1}{\sqrt{2\pi}} . 2a$$

$$= \frac{1}{\sqrt{2\pi}} \times \frac{\sqrt{2}}{\sqrt{2}} . 2a$$

$$F(0) = \frac{\sqrt{2}}{\cancel{2}\sqrt{\pi}} . \cancel{2}a = \sqrt{\frac{2}{\pi}} . a$$

Jadi Transformasi fourier

$$F(\alpha) = \begin{cases} \sqrt{\frac{2}{\pi}} . \frac{\sin \alpha a}{\alpha}, & \text{utk } \alpha \neq 0 \\ \sqrt{\frac{2}{\pi}} . a, & \text{utk } \alpha = 0 \end{cases}$$

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Contoh Soal Transformasi Fourier

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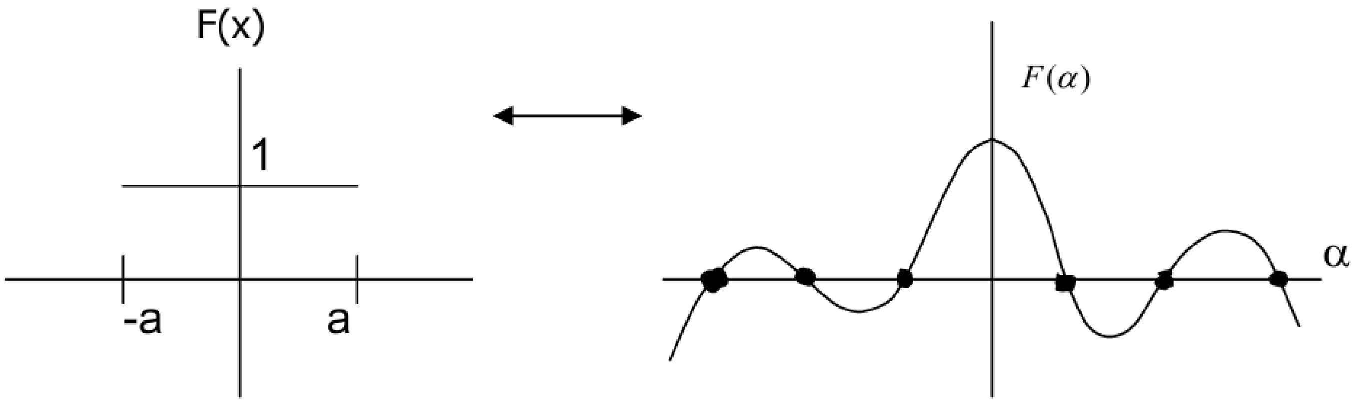
Jawaban Contoh Soal Transformasi Fourier



$$F(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u) e^{i\alpha u}\, du = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} 1 \cdot e^{i\alpha u}\, du$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{1}{i\alpha} e^{i\alpha u} \Big|_{-a}^{a} \right] = \frac{1}{\sqrt{2\pi}} \left[\frac{1}{i\alpha} (e^{ia\alpha} - e^{-ia\alpha}) \right]$$

$$= \frac{2}{\alpha\sqrt{2\pi}} \left[\frac{e^{ia\alpha} - e^{-ia\alpha}}{2i} \right] = \frac{\sqrt{4}}{\sqrt{2\pi}} \frac{\sin(a\alpha)}{\alpha}, \alpha \neq 0,$$

$$= \sqrt{\frac{4}{2\pi}} \frac{\sin(a\alpha)}{\alpha} = \sqrt{\frac{2}{\pi}} \frac{\sin(a\alpha)}{\alpha}$$

$$F(0) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} 1\, dx = \frac{2a}{\sqrt{2\pi}} = \frac{\sqrt{4}}{\sqrt{2\pi}} a = \sqrt{\frac{2}{\pi}}\, \mathrm{a}$$

Jadi,

$$F(\alpha) = \begin{cases} \sqrt{\frac{2}{\pi}} \frac{\sin(a\alpha)}{\alpha}, & bila\ \alpha \neq 0 \\ \sqrt{\frac{2}{\pi}}\, \mathrm{a}, & bila\ \alpha = 0. \end{cases}$$

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TRANSFORMASI FOURIER

Definisi Transformasi Cosinus Fourier dan Invers nya

TRANSFORMASI COSINUS FOURIER

$f(x) \to F(\alpha)$

Bila f(x) fungsi genap, buktikan bahwa :

$$\boxed{F_c(\alpha) = \mathbf{F}\{f(x)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty f(u)\cos(\alpha u)\, du,}$$

dan

Invers Transformasi cos Fourier

$$f(x) = \mathbf{F}^{-1}\{F_c(\alpha)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(\alpha)\cos(\alpha x)\, d\alpha.$$

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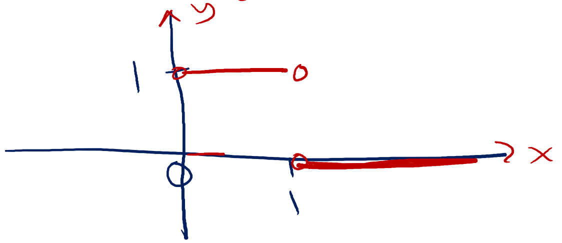
Contoh soal Transformasi Cosinus Fourier

1. Carilah transformasi cosinus fourier dari

$$f(x) = \begin{cases} 1, & bila \quad 0 < x < 1 \\ 0, & bila \quad x > 1. \end{cases}$$

Sketsa grafik



$$F_c(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(u).\cos(\alpha u)\,du$$

$$= \sqrt{\frac{2}{\pi}} \left(\int_0^1 1.\cos(\alpha u)\,du + \int_1^\infty 0.\cos(\alpha u)\,du \right)$$

$$= \sqrt{\frac{2}{\pi}} \cdot \left(\frac{1}{\alpha} \left[\sin \alpha u \right]_0^1 + 0 \right)$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{1}{\alpha} (\sin \alpha - \sin 0)$$

$$F_c(\alpha) = \sqrt{\frac{2}{\pi}} \cdot \frac{1}{\alpha} (\sin \alpha)$$

$$F_c(\alpha) = \sqrt{\frac{2}{\pi}} \cdot \frac{\sin \alpha}{\alpha} \quad \text{utk } \alpha \neq 0$$

jika $\alpha = 0$

$$F_c(\alpha) = F_c(0) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(u).\cos(0.u)\,du$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty f(u)\,du$$

$$= \sqrt{\frac{2}{\pi}} \left(\int_0^1 1\,du + \int_1^\infty 0\,du \right)$$

$$= \sqrt{\frac{2}{\pi}} \left(u \right)_0^1 = \sqrt{\frac{2}{\pi}} (1 - 0) = \sqrt{\frac{2}{\pi}}$$

Jadi Transformasi cos fourier adl

$$F_c(\alpha) = \begin{cases} \sqrt{\frac{2}{\pi}} \cdot \frac{\sin \alpha}{\alpha} & \text{utk } \alpha \neq 0 \\ \sqrt{\frac{2}{\pi}} & \text{utk } \alpha = 0 \end{cases}$$

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TRANSFORMASI FOURIER

Contoh soal Transformasi Cosinus Fourier

2. Carilah transformasi cosinus Fourier dari fungsi $f(x) = e^{-x}, x \geq 0.$

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Contoh soal Transformasi Cosinus Fourier

Solusi

$$F_c(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(u)\cos(\alpha u)\, du = \sqrt{\frac{2}{\pi}}\left[\int_0^\infty e^{-u}\cos(\alpha u)du\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\lim_{p\to\infty}\int_0^p e^{-u}\cos(\alpha u)du\right] = \sqrt{\frac{2}{\pi}}\left[\lim_{p\to\infty}\left\{\frac{e^{-u}}{(-1)^2+\alpha^2}\left((-1)\cos(\alpha u)+\alpha\sin(\alpha)u\right)\Big|_0^p\right\}\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\lim_{p\to\infty}\left\{\frac{e^{-p}}{1+\alpha^2}\left(-\cos(\alpha p)+\alpha\sin(\alpha p)\right)\right\} - \frac{e^0}{1+\alpha^2}\left(-\cos 0+\alpha\sin 0\right)\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[0-\frac{1}{1+\alpha^2}(-1+0)\right] = \sqrt{\frac{2}{\pi}}\left[\frac{1}{1+\alpha^2}\right]$$

Jadi,

$$F_c(\alpha) = \sqrt{\frac{2}{\pi}}\left[\frac{1}{1+\alpha^2}\right]$$

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Contoh soal Transformasi Cosinus Fourier

TRANSFORMASI SINUS FOURIER

TRANSFORMASI FOURIER

Definisi Transformasi Sinus Fourier dan Invers nya

TRANSFORMASI SINUS FOURIER

Definisi

Fungsi $F_s(\alpha)$ disebut *transformasi sinus Fourier* dari fungsi $f(x)$ dan ditulis

$$F_s(\alpha) = \mathbf{F}_s\{f(x)\},$$

bila

$$F_s(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(u)\sin(\alpha u)\, du\,.$$

Sedangkan fungsi $f(x)$ disebut *transformasi sinus Fourier inverse* dari fungsi $F_s(\alpha)$ dan ditulis

$$f(x) = \mathbf{F}_s^{-1}\{F_s(\alpha)\},$$

bila

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(\alpha)\sin(\alpha x)\, d\alpha$$

mengingat $F_s(\alpha)$ adalah fungsi ganjil yaitu $F_s(-\alpha) = -F_s(\alpha)$ untuk tiap α ,
di mana f(x) adalah **Transformasi Sinus Fourier** (***Fourier Sine Transform***)

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Contoh soal Transformasi Sinus Fourier

1. Carilah transformasi sinus fourier dari

$$f(x) = \begin{cases} 1, & bila \quad 0 < x < 1 \\ 0, & bila \quad x > 1. \end{cases}$$

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Contoh soal Transformasi Sinus Fourier

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Contoh soal Transformasi Sinus Fourier

Solusi

$$F_S(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(u)\sin(\alpha u)\, du = \sqrt{\frac{2}{\pi}} \left[\int_0^1 1 \cdot \sin(\alpha u u)\, du + \int_1^\infty 0 \cdot \sin(\alpha u u)\, du \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[-\frac{1}{\alpha} \cos(\alpha u) \Big|_0^1 \right] = \sqrt{\frac{2}{\pi}} \left[-\frac{1}{\alpha} (\cos\alpha - \cos 0) \right],$$

$$F_S(\alpha) = \sqrt{\frac{2}{\pi}} \left[-\frac{1}{\alpha} (\cos\alpha - 1) \right] = \frac{1 - \cos\alpha}{\alpha} \sqrt{\frac{2}{\pi}}, \quad \alpha \neq 0$$

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Contoh soal Transformasi Sinus Fourier

2. Carilah transformasi sinus fourier dari $f(x) = e^{-x}, x \geq 0.$

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Contoh soal Transformasi Sinus Fourier

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Contoh soal Transformasi Sinus Fourier

SIFAT-SIFAT TRANSFORMASI FOURIER

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Sifat-sifat Transformasi Fourier

Dalam hal ini digunakan notasi

$$f(x) \leftrightarrow F(\alpha)$$

untuk menunjukkan pasangan transformasi

$$F(\alpha) = \mathbf{F}\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i\alpha x} dx$$

$$f(x) = \mathbf{F}^{-1}\{F(\alpha)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\alpha) e^{-i\alpha x} d\alpha$$

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Sifat-sifat Transformasi Fourier

Sifat-sifat Elementer

1. **Linieritas**

Bila $f_1(x) \leftrightarrow F_1(\alpha)$ dan $f_2(x) \leftrightarrow F_2(\alpha)$, maka

$$a_1 f_1(x) + a_2 f_2(x) \leftrightarrow a_1 F_1(\alpha) + a_2 F_2(\alpha),\ a_1, a_2 \text{ konstanta.}$$

2. **Time-shifting**

Bila $f(x) \leftrightarrow F(\alpha)$, maka

$$f\,(x - x_0) \leftrightarrow F\,(\alpha)\, e^{i\alpha x_0}.$$

3. **Frequency-shifting**

Bila $f(x) \leftrightarrow F(\alpha)$, maka

$$f\,(x) e^{-i\alpha_0 x} \leftrightarrow F\,(\alpha - \alpha_0).$$

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Sifat-sifat Transformasi Fourier

4. **Scaling**

Bila $f(x) \leftrightarrow F(\alpha)$, maka untuk konstanta a yang bernilai nyata (real) dan tidak sama dengan nol berlaku

$$f\,(ax) \leftrightarrow \frac{1}{|a|} F\,(\frac{\alpha}{a})\,.$$

5. **Time-reversal**

Bila $f(x) \leftrightarrow F(\alpha)$, maka

$$f\,(-x) \leftrightarrow -F\,(-\alpha)\,.$$

6. **Simetri**

Bila $f(x) \leftrightarrow F(\alpha)$, maka

$$F(x) \leftrightarrow f(-\alpha).$$

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Sifat-sifat Transformasi Fourier

Contoh-contoh

1. Buktikan sifat linieritas di atas.

Solusi

$$\mathbf{F}\,[a_1 f_1(x) + a_2 f_2(x)] = \int_{-\infty}^{\infty} [a_1 f_1(x) + a_2 f_2(x)] e^{-i\omega x}\, dx = a_1 \int_{-\infty}^{\infty} f_1(x)\, e^{-i\omega x} dx + a_2 \int_{-\infty}^{\infty} f_2(x)\, e^{-i\omega x} dx$$
$$= a_1 \mathbf{F}\,[f_1(x)] + a_2 \mathbf{F}\,[f_2(x)],$$

di mana a_1, a_2 kostanta.

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Sifat-sifat Transformasi Fourier

2. Buktikan sifat frequency-shifting di atas.

Solusi

$$\mathbf{F}\,[f(x)e^{i\alpha_0 x}] = \int_{-\infty}^{\infty}[f(x)e^{i\alpha_0 x}]\,e^{-i\alpha x}\;dx = \int_{-\infty}^{\infty} f\,(x)\,e^{-i(\alpha-\alpha_0)x}dx = \mathbf{F}(\alpha-\alpha_0).$$

Assignment 3

1. Buktikan sifat-sifat time-shifting, scaling, time-reversal, dan simetri di atas.

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Latihan Soal Transformasi Fourier

1. Carilah transformasi Fourier dari fungsi

$$f(x) = \begin{cases} \dfrac{1}{2a}, & bila \quad |x| < a, \\ 0 \quad , & bila \quad |x| > a, \end{cases}$$

di mana a konstanta positif.

2. Carilah transformasi Fourier dari fungsi

$$f(x) = \begin{cases} 1 - x^2, & bila \quad |x| < 1 \\ 0 \quad , & bila \quad |x| > 1. \end{cases}$$

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Latihan Soal Transformasi Fourier

1. Carilah transformasi sinus Fourier dari fungsi-fungsi :

(a) $f(x)=e^{-x}$, $x \geq 0$

(b) $f(x)=e^{-2x}$, $x \geq 0$.

2 . Carilah transformasi cosinus Fourier dari fungsi

$$f(x)=\begin{cases} 2, & bila \quad 0<x<2 \\ 0, & bila \quad x>2 \end{cases}$$

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Latihan Soal Transformasi Fourier

SELESAI

TERIMA KASIH