第六章作业题 6 的证明

题目 证明下面的公式至少是 3 阶的:

$$
\begin{cases}
y_{i+1} = y_i + \dfrac{h}{6}(k_1 + 4k_2 + k_3) \\
k_1 = f(x_i, y_i) \\
k_2 = f(x_i + \dfrac{1}{2}h, y_i + \dfrac{1}{2}hk_1) \\
k_3 = f(x_i + h, y_i - hk_1 + 2hk_2)
\end{cases}
$$

1. 首先写出局部截断误差的定义,

$$
\begin{cases}
R_{i+1} = y(x_{i+1}) - y(x_i) - \dfrac{h}{6}(K_1 + 4K_2 + K_3) \\
K_1 = f(x_i, y(x_i)) \\
K_2 = f(x_i + \dfrac{1}{2}h, y(x_i) + \dfrac{1}{2}hK_1) \\
K_3 = f(x_i + h, y(x_i) - hK_1 + 2hK_2)
\end{cases}
$$

2. 要证明公式至少 3 阶的,则要证明 $R_{i+1} = O(h^4)$.

3. $y(x_{i+1})$ 只需展开为:

$$y(x_{i+1}) = y(x_i) + hy'(x_i) + \frac{h^2}{2}y''(x_i) + \frac{h^3}{6}y'''(x_i) + O(h^4).$$

4. $K_1 = f(x_i, y(x_i)) = y'(x_i)$.

5. K_2, K_3 则只要展开到 $O(h^3)$,即要 h^2 及以下部分的系数.

6. K_2 的展开为:

$$\begin{aligned}
K_2 &= f(x_i+\frac{h}{2}, y(x_i)+\frac{1}{2}hy'(x_i)) \\
&= f(x_i, y(x_i)) + \frac{h}{2}\left(f_x + f_y y'(x)\right) \\
&\quad + \frac{h^2}{8}\left(f_{xx} + 2f_{xy}y'(x_i) + f_{yy}(y'(x_i))^2\right) + O(h^3) \\
&= y'(x_i) + \frac{h}{2}y''(x_i) + \frac{h^2}{8}\left(y'''(x_i) - y''(x_i)f_y\right) + O(h^3).
\end{aligned}$$

7. 为了计算 K_3 ,需要计算

$$\Delta y = -hK_1 + 2hK_2 = -hy'(x_i) + 2hy'(x_i) + h^2y''(x_i) + O(h^3).$$

$$\Delta y = hy'(x_i) + h^2y''(x_i) + O(h^3).$$

8. K_3 的展开为:

$$\begin{aligned}
K_3 &= f(x_i+h, y(x_i) - hK_1 + 2hK_2) \\
&= f(x_i, y(x_i)) + hf_x + \Delta y f_y + \frac{1}{2}\left(f_{xx}h^2 + 2h\Delta y f_{xy} + f_{yy}\Delta y^2\right) + O(h^3) \\
&= y'(x_i) + hy''(x_i) + h^2y''(x_i)f_y + \frac{1}{2}\Big[f_{xx}h^2 + 2f_{xy}h\left(hy'(x_i) + O(h^2)\right) \\
&\quad + f_{yy}\left(y'(x_i)h + O(h^2)\right)^2\Big] + O(h^3) \\
&= y'(x_i) + hy''(x_i) + h^2y''(x_i)f_y \\
&\quad + \frac{h^2}{2}\left(f_{xx} + 2f_{xy}y'(x_i) + f_{yy}y'(x_i)^2\right) + O(h^3) \\
&= y'(x_i) + hy''(x_i) + h^2y''(x_i)f_y + \frac{h^2}{2}\left(y'''(x_i) - y''(x_i)f_y\right) + O(h^3)
\end{aligned}$$

9. $K_1 + 4K_2 + K_3 = 6y'(x_i) + 3hy'(x_i) + h^2y'''(x_i) + O(h^3)$.

10. 代入到 R_{i+1} 的表达式即可得到 $R_{i+1} = O(h^4)$,证明完成.