该公式是一个 2 阶公式. (1′)

解 1) 边值问题的五点差分格式为

$$\begin{cases} -\dfrac{1}{h_1^2}(u_{i+1,j}-2u_{ij}+u_{i-1,j})-\dfrac{1}{h_2^2}(u_{i,j+1}-2u_{ij}+u_{i,j-1})=x_iy_i, \\ \qquad\qquad\qquad 1\leqslant i\leqslant m-1, 1\leqslant j\leqslant n-1, \\ u_{i0}=x_i(1-x_i),\ u_{in}=0,\quad 0\leqslant i\leqslant m, \\ u_{0j}=0,\ u_{mj}=0,\qquad 1\leqslant j\leqslant n-1. \end{cases} \quad (3')$$

2) 当 $m=2, n=3$ 时, $h_1=1/2, h_2=1/3, x_0=0, x_1=1/2, x_2=1$, $y_0=0, y_1=1/3, y_2=2/3, y_3=1$. 由边界条件得 $u_{00}=0, u_{10}=0.5\times(1-0.5)=0.25, u_{20}=0, u_{i3}=0(i=0,1,2), u_{0j}=0, u_{2j}=0(j=1,2)$. (2′)

在五点差分格式中分别取 $m=2, n=3$ 得下面的方程组:

$$\begin{cases} 26u_{11}-9u_{12}=\dfrac{1}{6}+\dfrac{9}{4}=\dfrac{29}{12}, \\ -9u_{11}+26u_{12}=\dfrac{1}{3}, \end{cases} \quad (3')$$

解得

$$u_{11}=\frac{79}{714}\approx 0.1106,\quad u_{12}=\frac{73}{1428}\approx 0.0511,$$

即

$$u\left(\frac{1}{2},\frac{1}{3}\right)\approx 0.1106,\quad u\left(\frac{1}{2},\frac{2}{3}\right)\approx 0.0511. \quad (2')$$

2013 年秋季工学硕士研究生学位课程考试试题 (A)

解 由条件知

$$|e_r(x)|\leqslant \delta_1,\quad |e_r(y)|\leqslant \delta_2, \quad (1')$$

则有

$$|e(x)|\leqslant |x|\delta_1,\quad |e(y)|\leqslant |y|\delta_2, \quad (2')$$

因此

$$|e(z)|\approx |2x(\cos y)e(x)+x^2(-\sin y)e(y)| \quad (3')$$

$$\leqslant 2x^2|\cos y|\delta_1+x^2|\sin y||y|\delta_2. \quad (2')$$

解 1) 设 $f(x)=x^4-4x+1$, 则 $f'(x)=4x^3-4=4(x-1)(x^2+x+1)$. 当

$x<1$ 时, $f'(x)<0$, $f(x)$ 单调递减; 当 $x>1$ 时, $f'(x)>0$, $f(x)$ 单调增加. 注意到 $f(0)=1, f(1)=-2, f(2)=9$, 所以该方程有两个实根分别在区间 $(0,1)$ 和 $(1,2)$ 内. (4′)

2) 构造 Newton 迭代格式:

$$x_{k+1}=x_k-\frac{x_k^4-4x_k+1}{4x_k^3-4},\quad k=0,1,2,\cdots. \qquad (2')$$

取 $x_0=0.5$, 计算得 $x_1=0.2321, x_2=0.2510, x_3=0.2510$, 所以 $x_1^*\approx 0.25$;

取 $x_0=1.5$, 计算得 $x_1=1.4934, x_2=1.4934$, 所以 $x_2^*\approx 1.5$. (4′)

所以两个实根分别为 $x_1^*\approx 0.25$ 和 $x_2^*\approx 1.5$.

3. 解
$$\begin{bmatrix}3&2&6&-5\\5&-1&5&1\\4&-7&0&11\end{bmatrix}\xrightarrow{r_1\leftrightarrow r_2}\begin{bmatrix}5&-1&5&1\\3&2&6&-5\\4&-7&0&11\end{bmatrix} \qquad (1')$$

$$\xrightarrow[r_3-\frac{4}{5}r_1]{r_2-\frac{3}{5}r_1}\begin{bmatrix}5&-1&5&1\\0&\frac{13}{5}&3&-\frac{28}{5}\\0&-\frac{31}{5}&-4&\frac{51}{5}\end{bmatrix}\xrightarrow{r_2\leftrightarrow r_3}\begin{bmatrix}5&-1&5&1\\0&-\frac{31}{5}&-4&\frac{51}{5}\\0&\frac{13}{5}&3&-\frac{28}{5}\end{bmatrix} \qquad (4')$$

$$\xrightarrow{r_3+\frac{13}{31}r_2}\begin{bmatrix}5&-1&5&1\\0&-\frac{31}{5}&-4&\frac{51}{5}\\0&0&\frac{41}{31}&-\frac{41}{31}\end{bmatrix}, \qquad (2')$$

对应的线性方程组为

$$\begin{cases}5x_1-x_2+5x_3=1,\\ -\dfrac{31}{5}x_2-4x_3=\dfrac{51}{5},\\ \dfrac{41}{31}x_3=-\dfrac{41}{31},\end{cases}$$

解得 $x_1=1, x_2=-1, x_3=-1$. (3′)

4. 解 1) Gauss-Seidel 迭代格式为

$$\begin{cases}x^{(k+1)}=\left(d_1-by^{(k)}\right)/a,\\ y^{(k+1)}=\left(d_2-cx^{(k+1)}-bz^{(k)}\right)/a,\\ z^{(k+1)}=\left(d_3-cy^{(k+1)}\right)/a.\end{cases} \qquad (3')$$

2) Gauss-Seidel 迭代格式矩阵的特征方程为

$$\begin{vmatrix} a\lambda & b & 0 \\ c\lambda & a\lambda & b \\ 0 & c\lambda & a\lambda \end{vmatrix} = a^3\lambda^3 - 2abc\lambda^2 = 0, \qquad (4')$$

求得 $\lambda_{1,2} = 0, \lambda_3 = \dfrac{2bc}{a^2}$, 则 Gauss-Seidel 迭代格式收敛的充要条件为

$$\left|\frac{2bc}{a^2}\right| < 1,$$

即 $2|bc| < a^2$. (3′)

5. 解 设 $p'(2) = m$, 在 [0,2] 上以 $p(0) = 1, p(2) = 3, p'(2) = m$ 为插值条件建立二次多项式得

$$p_2(x) = 1 + x + \frac{m-1}{2}x(x-2), \qquad (3')$$

在 [2,3] 上以 $p(2) = 3, p'(2) = m, p(3) = 5$ 为插值条件建立二次多项式得

$$\tilde{p}_2(x) = 3 + m(x-2) + (2-m)(x-2)(x-2). \qquad (3')$$

根据 $\int_0^2 p(x)\,\mathrm{d}x = 0$ 可得

$$\int_0^2 \left[1 + x + \frac{m-1}{2}x(x-2)\right] \mathrm{d}x = 0, \qquad (1')$$

即

$$\left[x + \frac{x^2}{2} + \frac{m-1}{2}\left(\frac{x^3}{3} - x^2\right)\right]\Bigg|_0^2 = 0,$$

求得 $m = 7$. 所以 (3′)

$$p(x) = \begin{cases} 1 + x + 3x(x-2), & x \in [0,2]; \\ 3 + 7(x-2) - 5(x-2)^2, & x \in [2,3]. \end{cases} \qquad (2')$$

6. 解 设 $p(x) = a + bx$. 由于 $f'(x) = -\sin x, f''(x) = -\cos x$, 则 $\left(0, \dfrac{\pi}{2}\right)$ 上 $f''(x) < 0$, 所以 $f(x) - p(x)$ 在 $\left(0, \dfrac{\pi}{2}\right)$ 上有三个交错偏差点为 $0, x_1, \dfrac{\pi}{2}$, 且满足 (2′)

$$\begin{cases} f(0) - p(0) = -[f(x_1) - p(x_1)] = f\left(\dfrac{\pi}{2}\right) - p\left(\dfrac{\pi}{2}\right), \\ f'(x_1) - p'(x_1) = 0, \end{cases}$$

即

$$\begin{cases} 1 - a = -(\cos x_1 - a - bx_1) = -a - \dfrac{\pi}{2}b, \\ -\sin x_1 = b, \end{cases} \qquad (3')$$

求得

$$b=-\frac{2}{\pi},\quad x_1=\arcsin\frac{2}{\pi},$$

$$a=\frac{1}{2}+\frac{1}{2}\cos\arcsin\frac{2}{\pi}+\frac{1}{\pi}\arcsin\frac{2}{\pi}=\frac{1}{2}+\frac{\sqrt{\pi^2-4}}{2\pi}+\frac{1}{\pi}\arcsin\frac{2}{\pi},$$

则

$$p(x)=\frac{1}{2}+\frac{\sqrt{\pi^2-4}}{2\pi}+\frac{1}{\pi}\arcsin\frac{2}{\pi}-\frac{2}{\pi}x, \tag{3'}$$

$$\max_{0\leqslant x\leqslant\frac{\pi}{2}}|f(x)-p(x)|=\|f-p\|_\infty=|f(0)-p(0)|=|1-a|$$

$$=\frac{\sqrt{\pi^2-4}}{2\pi}+\frac{1}{\pi}\arcsin\frac{2}{\pi}-\frac{1}{2}. \tag{2'}$$

7. **解**　1) 记 $x=\frac{a+b}{2}+\frac{b-a}{2}t$, 则

$$\int_a^b f(x)\,\mathrm{d}x=\int_{-1}^1 f\left(\frac{a+b}{2}+\frac{b-a}{2}t\right)\cdot\frac{b-a}{2}\,\mathrm{d}t,$$

令

$$g(t)=\frac{b-a}{2}f\left(\frac{a+b}{2}+\frac{b-a}{2}t\right),$$

则有

$$G(f)=\frac{b-a}{2}\left[f\left(\frac{a+b}{2}-\frac{b-a}{2}\frac{1}{\sqrt{3}}\right)+f\left(\frac{a+b}{2}+\frac{b-a}{2}\frac{1}{\sqrt{3}}\right)\right], \tag{3'}$$

$$I(f)-G(f)=\frac{1}{135}g^{(4)}(\xi)=\frac{1}{135}\frac{\mathrm{d}^4}{\mathrm{d}t^4}\left[\frac{b-a}{2}f\left(\frac{a+b}{2}+\frac{b-a}{2}t\right)\right]\Bigg|_{t=\xi}$$

$$=\frac{1}{135}\left(\frac{b-a}{2}\right)^5 f^{(4)}(\gamma),\quad \gamma\in(a,b). \tag{3'}$$

2) 记 $x_{i+\frac{1}{2}}=\frac{x_i+x_{i+1}}{2}$, 则

$$G_n(f)=\sum_{i=0}^{n-1}\frac{h}{2}\left[f\left(x_{i+\frac{1}{2}}-\frac{h}{2\sqrt{3}}\right)+f\left(x_{i+\frac{1}{2}}+\frac{h}{2\sqrt{3}}\right)\right], \tag{3'}$$

$$I(f)-G_n(f)$$

$$=\sum_{i=0}^{n-1}\left\{\int_{x_i}^{x_{i+1}}f(x)\,\mathrm{d}x-\frac{h}{2}\left[f\left(x_{i+\frac{1}{2}}-\frac{h}{2\sqrt{3}}\right)+f\left(x_{i+\frac{1}{2}}+\frac{h}{2\sqrt{3}}\right)\right]\right\}$$

$$=\sum_{i=0}^{n-1}\frac{1}{135}\left(\frac{h}{2}\right)^5 f^{(4)}(\gamma_i)$$

$$= \frac{b-a}{135} \cdot \frac{h^4}{2^5} \cdot \frac{1}{n} \sum_{i=0}^{n-1} f^{(4)}(\gamma_i), \quad \gamma_i \in (x_i, x_{i+1}), \tag{2'}$$

因为 $f \in C^4[a,b]$, 则必存在 m 和 M, 使得

$$m \leqslant \frac{1}{n} \sum_{i=0}^{n-1} f^{(4)}(\gamma_i) \leqslant M,$$

由连续函数的介值定理知, 存在 $\eta \in (a,b)$, 使得

$$f^{(4)}(\eta) = \frac{1}{n} \sum_{i=0}^{n-1} f^{(4)}(\gamma_i),$$

因此

$$I(f) - G_n(f) = \frac{b-a}{135} \left(\frac{h^4}{2^5}\right) f^{(4)}(\eta). \tag{1'}$$

8. 解 局部截断误差为

$$R_{i+1} = y(x_{i+1}) - y(x_i) - hf\left(x_i + \frac{h}{2}, y(x_i) + \frac{h}{2} f(x_i, y(x_i))\right) \tag{2'}$$

$$= y(x_i) + hy'(x_i) + \frac{h^2}{2} y''(x_i) + \frac{h^3}{6} y'''(x_i) + O(h^4) - y(x_i) \tag{1'}$$

$$\begin{aligned}&- h\left\{f(x_i, y(x_i)) + \frac{\partial f(x_i, y(x_i))}{\partial x} \cdot \frac{h}{2} + \frac{\partial f(x_i, y(x_i))}{\partial y} \cdot \frac{h}{2} y'(x_i)\right.\\
&+ \frac{1}{2}\left[\frac{\partial^2 f(x_i, y(x_i))}{\partial x^2} \cdot \left(\frac{h}{2}\right)^2 + 2\frac{\partial f(x_i, y(x_i))}{\partial x \partial y} \cdot \frac{h}{2} \cdot \frac{h}{2} y'(x_i)\right.\\
&\left.\left.+ \frac{\partial^2 f(x_i, y(x_i))}{\partial y^2} \cdot \left(\frac{h}{2} y'(x_i)\right)^2\right] + O(h^4)\right\}\end{aligned} \tag{3'}$$

$$\begin{aligned}&= \frac{h^2}{2}\left[y''(x_i) - \frac{\partial f(x_i, y(x_i))}{\partial x} - \frac{\partial f(x_i, y(x_i))}{\partial y} y'(x_i)\right]\\
&+ h^3\left[\frac{1}{6} y'''(x_i) - \frac{1}{8} \frac{\partial^2 f(x_i, y(x_i))}{\partial x^2} - \frac{1}{4} \frac{\partial^2 f(x_i, y(x_i))}{\partial x \partial y} y'(x_i)\right.\\
&\left.- \frac{1}{8} \frac{\partial^2 f(x_i, y(x_i))}{\partial y^2} \cdot (y'(x_i))^2\right] + O(h^4)\end{aligned} \tag{3'}$$

$$= h^3\left[\left(\frac{1}{6} - \frac{1}{8}\right) y'''(x_i) + \frac{1}{8} \frac{\partial f(x_i, y(x_i))}{\partial y} y''(x_i)\right] + O(h^4)$$

$$= h^3\left[\frac{1}{24} y'''(x_i) + \frac{1}{8} \frac{\partial f(x_i, y(x_i))}{\partial y} y''(x_i)\right] + O(h^4), \tag{2'}$$

这是一个 2 阶公式. (1′)

9. 解 记

$$\Omega_h = \{(x_i, y_j) | 0 \leqslant i \leqslant M, 0 \leqslant j \leqslant N\}, \quad \omega_h = \{(i,j) | (x_i, y_j) \in \Omega_h\},$$

$$\mathring{\Omega}_h = \{(x_i, y_j) | 1 \leqslant i \leqslant M-1, 1 \leqslant j \leqslant N-1\}, \quad \mathring{\omega}_h = \left\{(i,j) | (x_i, y_j) \in \mathring{\Omega}_h\right\},$$

$$\Gamma_h = \Omega_h \backslash \mathring{\Omega}_h, \quad \gamma_h = \{(i,j) | (x_i, y_j) \in \Gamma_h\},$$

则求解该问题的五点差分格式为

$$\begin{cases} \dfrac{1}{h^2}(4u_{ij} - u_{i+1,j} - u_{i-1,j} - u_{i,j-1} - u_{i,j+1}) = 12x_i y_j, & (i,j) \in \mathring{\omega}_h, \\ u_{ij} = 0, & (i,j) \in \gamma_h, \end{cases} \quad (3')$$

注意到在边界上有 $u_{ij} = 0$, 则可得方程组

$$\begin{cases} 4u_{11} - u_{21} - u_{12} = \dfrac{4}{27}, \\ 4u_{12} - u_{11} - u_{22} = \dfrac{8}{27}, \\ 4u_{21} - u_{22} - u_{11} = \dfrac{8}{27}, \\ 4u_{22} - u_{12} - u_{21} = \dfrac{16}{27}, \end{cases} \quad (3')$$

或者

$$\begin{bmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{12} \\ u_{21} \\ u_{22} \end{bmatrix} = \begin{bmatrix} \frac{4}{27} \\ \frac{8}{27} \\ \frac{8}{27} \\ \frac{16}{27} \end{bmatrix} \quad (2')$$

解得

$$u_{11} = \frac{19}{162} \approx 0.117284, \quad u_{12} = \frac{26}{162} \approx 0.160494,$$

$$u_{21} = \frac{26}{162} \approx 0.160494, \quad u_{22} = \frac{37}{162} \approx 0.228395. \quad (2')$$

10. **证**　1) 因为 $p(x)$ 为 $f(x)$ 的以 $x_0, x_1, \cdots, x_n$ 为节点的 n 次插值多项式, 则 $R(x) = f(x) - p(x)$ 以 $x_0, x_1, \cdots, x_n$ 为零点. 在每一个小区间 $[x_j, x_{j+1}], j = 0, 1, \cdots, n-1$ 上应用罗尔中值定理, 可知至少存在 $z_{j+1} \in (x_j, x_{j+1}), j = 0, 1, \cdots, n-1$, 使

$$R'(z_j) = f'(z_j) - p'(z_j) = 0, \quad j = 1, 2, \cdots, n,$$

所以方程 $f'(x) - p'(x) = 0$ 至少存在 n 个互异的实根 $z_1, z_2, \cdots, z_n$.　(3′)

2) 因为 $f'(z_j) = p'(z_j), j = 1, 2, \cdots, n$, 则 $p'(x)$ 是 $f'(x)$ 以 $z_1, z_2, ... z_n$ 为插

值节点的 $(n-1)$ 次插值多项式, 由插值余项定理知, 存在 $\xi \in (a,b)$, 使得

$$f'(x) - p'(x) = \frac{f^{n+1}(\xi)}{n!}(x-z_1)(x-z_2)\cdots(x-z_n), \quad x \in [a,b]. \qquad (3')$$

2013 年秋季工学硕士研究生学位课程考试试题 (B)

1. **解** 因为 $S = 2\pi ab, V = \pi ab^2$, 又由条件得

$$|e(a)| \leqslant \frac{1}{2} \times 10^{-1}, \quad |e(b)| \leqslant \frac{1}{2} \times 10^{-2}, \qquad (2')$$

所以由

$$e(S) \approx 2\pi[b \cdot e(a) + a \cdot e(b)],$$

得

$$|e_r(S)| \leqslant \frac{2\pi}{S}(b|e(a)| + a|e(b)|)$$
$$= \frac{1}{a}|e(a)| + \frac{1}{b}|e(b)| \leqslant 0.215 \times 10^{-2}, \qquad (3')$$

由

$$e(V) \approx \pi[b^2 e(a) + 2ab \cdot e(b)],$$

得

$$|e_r(V)| \leqslant \frac{\pi}{V}[b^2|e(a)| + 2ab|e(b)|]$$
$$= \frac{1}{a}|e(a)| + \frac{2}{b}|e(b)| \leqslant 0.265 \times 10^{-2}. \qquad (3')$$

2. **解** 因为 $f'(x) = 5(x^2+2)(x+\sqrt{2})(x-\sqrt{2})$, 所以 $f(x)$ 在 $(0, \sqrt{2})$ 上单调减, 在 $(\sqrt{2}, +\infty)$ 上单调增, 又 $f(0) = -1$, $f(2) < 0$, $f(3) > 0$, 所以方程 $f(x) = 0$ 有唯一正根, 且在区间 $[2,3]$ 内. (4′)

用 Newton 迭代格式求解:

$$x_{k+1} = x_k - \frac{x_k^5 - 20x_k - 1}{5x_k^4 - 20}, \quad k = 0, 1, \cdots, \qquad (4')$$

取 $x_0 = 2.5$, 计算得 $x_1 = 2.23387, x_2 = 2.13866, x_3 = 2.12722, x_4 = 2.12706$, 因此 $x^* \approx 2.127$. (2′)

3. **解**
$$\begin{bmatrix} 1 & -2 & 1 & 2 \\ 3 & 0 & -4 & -3 \\ -10 & 2 & 2 & -2 \end{bmatrix} \xrightarrow{r_1 \leftrightarrow r_3} \begin{bmatrix} -10 & 2 & 2 & -2 \\ 3 & 0 & -4 & -3 \\ 1 & -2 & 1 & 2 \end{bmatrix} \qquad (2')$$

$$\xrightarrow[r_3+\frac{1}{10}r_1]{r_2+\frac{3}{10}r_1}\begin{bmatrix}-10 & 2 & 2 & -2\\ 0 & \frac{3}{5} & -\frac{17}{5} & -\frac{18}{5}\\ 0 & -\frac{9}{5} & \frac{6}{5} & \frac{9}{5}\end{bmatrix}\xrightarrow{r_2\leftrightarrow r_3}\begin{bmatrix}-10 & 2 & 2 & -2\\ 0 & -\frac{9}{5} & \frac{6}{5} & \frac{9}{5}\\ 0 & \frac{3}{5} & -\frac{17}{5} & -\frac{18}{5}\end{bmatrix}$$

$$\xrightarrow{r_3+\frac{1}{3}r_2}\begin{bmatrix}-10 & 2 & 2 & -2\\ 0 & -\frac{9}{5} & \frac{6}{5} & \frac{9}{5}\\ 0 & 0 & -3 & -3\end{bmatrix} \qquad (5')$$

等价的三角形方程组为

$$\begin{cases}-10x_1+2x_2+2x_3=-2,\\ -\frac{9}{5}x_2+\frac{6}{5}x_3=\frac{9}{5},\\ -3x_3=-3,\end{cases}$$

求得 $x_1=\frac{1}{3}, x_2=-\frac{1}{3}, x_3=1.$ (3′)

4. **解**　Jacobi 迭代矩阵 $\boldsymbol{J}$ 的特征方程为

$$\begin{vmatrix}a\lambda & 1 & 3\\ 1 & a\lambda & 2\\ -3 & 2 & a\lambda\end{vmatrix}=0,$$

展开得

$$a\lambda(a^2\lambda^2+4)=0,$$

求得

$$\lambda_1=0,\quad \lambda_{2,3}=\pm\frac{2}{|a|}\mathrm{i}, \qquad (5')$$

所以 $\rho(\boldsymbol{J})=\left|\frac{2}{a}\right|$. 当 $|a|>2$，即 $\rho(\boldsymbol{J})<1$ 时，Jacobi 迭代收敛. (5′)

5. **证**　设 λ 是 $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}$ 的主特征值，对应的特征向量为 $\boldsymbol{x}\neq\boldsymbol{0}$，则

$$\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{x}=\lambda\boldsymbol{x}, \qquad (2')$$

两边取范数得

$$|\lambda|\|\boldsymbol{x}\|_\infty=\|\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{x}\|_\infty\leqslant\|\boldsymbol{A}^{\mathrm{T}}\|_\infty\|\boldsymbol{A}\|_\infty\|\boldsymbol{x}\|_\infty, \qquad (2')$$

又因为 $\boldsymbol{A}^{\mathrm{T}}=\boldsymbol{A}$，所以

$$|\lambda|\leqslant\|\boldsymbol{A}\|_\infty^2,$$

即

$$\|\boldsymbol{A}\|_2 = \sqrt{\lambda} \leqslant \|\boldsymbol{A}\|_\infty. \tag{2'}$$

6. **解** 使用 $H'(b)=0,\ H'(c)=2, H''(c)=1$ 构造 2 次插值多项式 $H'(x)$, 列差商表如下:

x_k	$H'(x_k)$	$H'[x_k,x_{k+1}]$	$H'[x_k,x_{k+1},x_{k+2}]$
b	0	$\dfrac{2}{c-b}$	$\dfrac{1}{c-b}-\dfrac{2}{(c-b)^2}$
c	2	1	
c	2		

(3′)

可得

$$H'(x)=\frac{2}{c-b}(x-b)+\left[\frac{1}{c-b}-\frac{2}{(c-b)^2}\right](x-b)(x-c), \tag{3'}$$

将上式在 $[a,x]$ 上积分, 并注意到 $H(a)=4$, 得

$$\begin{aligned}H(x)=&4+\frac{(x-b)^2-(a-b)^2}{c-b}+\left[\frac{1}{c-b}-\frac{2}{(c-b)^2}\right]\\&\cdot\left[\frac{1}{3}(x^3-a^3)-\frac{b+c}{2}(x-a)+bc(x-a)\right].\end{aligned} \tag{6'}$$

7. **解** 设 $z=\dfrac{1}{y}$, 则经验函数为

$$z=a+bx^2. \tag{2'}$$

令 $\varphi_0(x)=1,\ \varphi_1(x)=x^2$, 则

$$\begin{gathered}\boldsymbol{\varphi_0}=(1,1,1,1)^{\mathrm{T}},\quad \boldsymbol{\varphi_1}=(0,1,4,9)^{\mathrm{T}},\quad \boldsymbol{z}=(1,2,2,4)^{\mathrm{T}},\\ (\boldsymbol{\varphi_0},\boldsymbol{\varphi_0})=4,\quad (\boldsymbol{\varphi_0},\boldsymbol{\varphi_1})=14,\quad (\boldsymbol{\varphi_1},\boldsymbol{\varphi_1})=98,\\ (\boldsymbol{z},\boldsymbol{\varphi_0})=9,\quad (\boldsymbol{z},\boldsymbol{\varphi_1})=46,\end{gathered} \tag{3'}$$

可得正规方程组为

$$\begin{cases}4a+14b=9,\\ 14a+98b=46,\end{cases} \tag{3'}$$

解得

$$a=\frac{238}{196}\approx 1.214,\quad b=\frac{58}{196}\approx 0.296,$$

所以

$$y=\frac{1}{1.214+0.296x^2}. \tag{2'}$$

8. 解　2 次插值多项式为

$$p_2(x)=\frac{\left(x-\frac{3}{2}\right)(x-2)}{\left(1-\frac{3}{2}\right)(1-2)}f(1)+\frac{(x-1)(x-2)}{\left(\frac{3}{2}-1\right)\left(\frac{3}{2}-2\right)}f\left(\frac{3}{2}\right)$$

$$+\frac{(x-1)\left(x-\frac{3}{2}\right)}{(2-1)\left(2-\frac{3}{2}\right)}f(2), \qquad (4')$$

代入积分公式得

$$A_0=2\int_0^3\left(x-\frac{3}{2}\right)(x-2)\,\mathrm{d}x=\frac{9}{2},$$

$$A_1=-4\int_0^3(x-1)(x-2)\,\mathrm{d}x=-6,$$

$$A_2=2\int_0^3(x-1)\left(x-\frac{3}{2}\right)\mathrm{d}x=\frac{9}{2}, \qquad (4')$$

该求积公式为插值型的, 代数精度至少为 2.

又当 $f(x)=x^3$ 时, 左 $=\frac{81}{4}$, 右 $=\frac{81}{4}$;

当 $f(x)=x^4$ 时, 左 $=\frac{243}{5}$, 右 $=\frac{369}{8}\neq$ 左, 所以公式代数精度为 3.　(4′)

9. 解　局部截断误差为

$$\begin{aligned}R_{i+1}=&y(x_{i+1})-c_0y(x_i)-c_1y(x_{i-1})\\&-h[d_0f(x_i,y(x_i))+d_1f(x_{i+1},y(x_{i+1}))]\\=&y(x_i)+hy'(x_i)+\frac{h^2}{2}y''(x_i)+\frac{h^3}{6}y'''(x_i)+\frac{h^4}{24}y^{(4)}(x_i)+O(h^5)\\&-c_0y(x_i)-c_1\left[y(x_i)-hy'(x_i)+\frac{h^2}{2}y''(x_i)-\frac{h^3}{6}y'''(x_i)\right.\\&\left.+\frac{h^4}{24}y^{(4)}(x_i)+O(h^5)\right]-hd_0y'(x_i)\\&-hd_1\left[y'(x_i)+hy''(x_i)+\frac{h^2}{2}y'''(x_i)+\frac{h^3}{6}y^{(4)}(x_i)+O(h^4)\right]\\=&(1-c_0-c_1)y(x_i)+(1+c_1-d_0-d_1)hy'(x_i)\\&+\left(\frac{1}{2}-\frac{c_1}{2}-d_1\right)h^2y''(x_i)+\left(\frac{1}{6}+\frac{c_1}{6}-\frac{d_1}{2}\right)h^3y'''(x_i)\\&+\left(\frac{1}{24}-\frac{c_1}{24}-\frac{d_1}{6}\right)h^4y^{(4)}(x_i)+O(h^5),\end{aligned} \qquad (4')$$

要使求解公式至少具有 3 阶精度, 当且仅当

$$\begin{cases} 1 - c_0 - c_1 = 0, \\ 1 + c_1 - d_0 - d_1 = 0, \\ \dfrac{1}{2} - \dfrac{c_1}{2} - d_1 = 0, \\ \dfrac{1}{6} + \dfrac{c_1}{6} - \dfrac{d_1}{2} = 0, \end{cases}$$

解得

$$c_0 = \frac{4}{5}, \quad c_1 = \frac{1}{5}, \quad d_0 = \frac{4}{5}, \quad d_1 = \frac{2}{5}, \tag{4'}$$

所以局部截断误差为

$$R_{i+1} = -\frac{1}{30}h^4 y^{(4)}(x_i) + O(h^5). \tag{2'}$$

该公式为 2 步 3 阶格式. (2′)

10. **解** 1) 在节点 (x_i, t_k) 处考虑方程

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(x_i, t_k) - \dfrac{\partial^2 u}{\partial x^2}(x_i, t_k) = f(x_i, t_k), & 1 \leqslant i \leqslant M-1,\ 1 \leqslant k \leqslant N-1, \\ u(x_i, t_0) = \varphi(x_i), & 1 \leqslant i \leqslant M-1 \\ u(x_i, t_1) = \Psi(x_i), & 1 \leqslant i \leqslant M-1, \\ u(0, t_k) = \alpha(t_k),\ u(1, t) = \beta(t_k), & 0 \leqslant k \leqslant N, \end{cases}$$

其中

$$\begin{aligned} \Psi(x_i) =& \varphi(x_i) + \tau\psi(x_i) + \frac{\tau^2}{2}[\varphi''(x_i) + f(x_i, t_0)] \\ &+ \frac{\tau^3}{6}\frac{\partial^3 u}{\partial t^3}(x_i, \eta_i), \quad \eta_i \in (0, \tau). \end{aligned} \tag{3'}$$

用差商代替导数得

$$\begin{aligned} &\frac{1}{\tau^2}[u(x_i, t_{k+1}) - 2u(x_i, t_k) + u(x_i, t_{k-1})] \\ &- \frac{1}{2h^2}[u(x_{i+1}, t_{k+1}) - 2u(x_i, t_{k+1}) + u(x_{i-1}, t_{k+1}) \\ &+ u(x_{i+1}, t_{k-1}) - 2u(x_i, t_{k-1}) + u(x_{i-1}, t_{k-1})] = f(x_i, t_k) + R_{ik}, \end{aligned}$$

其中

$$\begin{aligned} R_{ik} =& \frac{\tau^2}{12}\frac{\partial^4 u(x_i, \eta_i^k)}{\partial t^4} - \frac{\tau^2}{2}\frac{\partial^4 u(x_i, \tilde{\eta}_i^k)}{\partial x^2 \partial t^2} - \frac{h^2}{24}\frac{\partial^4 u(\xi_i^{k+1}, t_{k+1})}{\partial x^4} \\ &- \frac{h^2}{24}\frac{\partial^4 u(\xi_i^{k-1}, t_{k-1})}{\partial x^4}, \quad t_{k-1} \leqslant \eta_k, \tilde{\eta}_k \leqslant t_{k+1},\ x_{i-1} \leqslant \xi_i^{k-1}, \xi_i^{k+1} \leqslant x_{i+1} \end{aligned}$$

为截断误差. (2′)

略去截断误差 R_{ik}, 并用 $u_i^k \approx u(x_i, t_k)$, 得下面的差分格式:

$$\begin{cases} \dfrac{1}{\tau^2}(u_i^{k+1} - 2u_i^k + u_i^{k-1}) - \dfrac{1}{2h^2}(u_{i+1}^{k+1} - 2u_i^{k+1} + u_{i-1}^{k+1} + u_{i+1}^{k-1} - 2u_i^{k-1} + u_{i-1}^{k-1} \\ = f(x_i, t_k), \qquad 1 \leqslant i \leqslant M-1, 1 \leqslant k \leqslant N-1, \\ u_i^0 = \varphi(x_i),\ u_i^1 = \Phi(x_i), \quad 1 \leqslant i \leqslant M-1, \\ u_0^k = \alpha(t_k),\ u_M^k = \beta(t_k), \quad 0 \leqslant k \leqslant N, \end{cases}$$

其中

$$\Phi(x_i) = \varphi(x_i) + \tau\psi(x_i) + \frac{\tau^2}{2}[\varphi''(x_i) + f(x_i, t_0)].$$

2) 记 $s = \dfrac{\tau}{h}, p = 1 + s^2$, 则上述差分格式可写为下面的向量和矩阵形式:

$$\begin{bmatrix} p & -\frac{1}{2}s^2 & & & \\ -\frac{1}{2}s^2 & p & -\frac{1}{2}s^2 & & \\ & \ddots & \ddots & \ddots & \\ & & -\frac{1}{2}s^2 & p & -\frac{1}{2}s^2 \\ & & & -\frac{1}{2}s^2 & p \end{bmatrix} \begin{bmatrix} u_1^{k+1} \\ u_2^{k+1} \\ \vdots \\ u_{M-2}^{k+1} \\ u_{M-1}^{k+1} \end{bmatrix}$$

$$= 2\begin{bmatrix} u_1^k \\ u_2^k \\ \vdots \\ u_{M-2}^k \\ u_{M-1}^k \end{bmatrix} + \begin{bmatrix} -p & \frac{1}{2}s^2 & & & \\ \frac{1}{2}s^2 & -p & \frac{1}{2}s^2 & & \\ & \ddots & \ddots & \ddots & \\ & & \frac{1}{2}s^2 & -p & \frac{1}{2}s^2 \\ & & & \frac{1}{2}s^2 & -p \end{bmatrix} \begin{bmatrix} u_1^{k-1} \\ u_2^{k-1} \\ \vdots \\ u_{M-2}^{k-1} \\ u_{M-1}^{k-1} \end{bmatrix}$$

$$+ \begin{bmatrix} \tau^2 f(x_1, t_k) + \frac{1}{2}s^2(\alpha(t_{k-1}) + \alpha(t_{k+1})) \\ \tau^2 f(x_2, t_k) \\ \vdots \\ \tau^2 f(x_{M-2}, t_k) \\ \tau^2 f(x_{M-1}, t_k) + \frac{1}{2}s^2(\beta(t_{k-1}) + \beta(t_{k+1})) \end{bmatrix}, 1 \leqslant k \leqslant N-1.$$

2013 年秋季工学硕士研究生学位课程考试试题 (C)

1. **解** 由条件得

$$|e(x_1)| \leqslant \frac{1}{2} \times 10^{-3}, \quad |e(x_2)| \leqslant \frac{1}{2} \times 10^{-3}, \tag{2'}$$

因此有

$$\left|e\left(\frac{x_1}{x_2}\right)\right| \approx \left|\frac{x_2 e(x_1) - x_1 e(x_2)}{x_2^2}\right| \leqslant \frac{x_2|e(x_1)| + x_1|e(x_2)|}{x_2^2}$$

$$\leqslant 0.88654 \times 10^{-5}. \tag{3'}$$

$$\left|e_r\left(\frac{x_1}{x_2}\right)\right| = \left|\frac{e\left(\frac{x_1}{x_2}\right)}{\frac{x_1}{x_2}}\right| \leqslant 0.16138 \times 10^{-1}. \tag{3'}$$

2. **解** 记 $f(x) = x^3 - x - 1$, 则 $f(x)$ 在 $[1,2]$ 上连续, 且 $f(1) \cdot f(2) = -1 \times (8 - 2 - 1) < 0$, 因此方程 $f(x) = 0$ 在 $[1,2]$ 上至少有一个实根. 又当 $x \in [1,2]$ 时, 有 $f'(x) = 3x^2 - 1 \geqslant 3 - 1 > 0$, 即函数 $f(x)$ 在 $[1,2]$ 上单调增, 所以方程 $f(x) = 0$ 在 $[1,2]$ 上存在唯一实根, 记为 x^*. (4′)

下面用迭代法求方程的根.

方法 1: 构造 Newton 迭代格式

$$x_{k+1} = x_k - \frac{x_k^3 - x_k - 1}{3x_k^2 - 1}, \quad k = 0, 1, \cdots, \tag{3'}$$

取 $x_0 = 1.5$, 计算得 $x_1 = 1.3478$, $x_2 = 1.3252$, $x_3 = 1.3248$. 因为$|x_3 - x_2| = 0.0004 < 0.5 \times 10^{-3}$, 所以 $x^* \approx 1.3248$. (3′)

方法 2: 构造简单迭代格式

$$x_{k+1} = \sqrt[3]{x_k + 1}, \quad k = 0, 1, \cdots, \tag{3'}$$

取 $x_0 = 1.5$, 计算得 $\boxed{x_1 = 1.3572,}$ $x_2 = 1.3309$, $x_3 = 1.3259$, $x_4 = 1.3249$, $x_5 = 1.3248$. 因为 $|x_5 - x_4| = 0.0001 < 0.5 \times 10^{-3}$, 所以 $x^* \approx 1.3248$. (3′)

3. **解**
$$\begin{bmatrix} 3 & 1 & 2 & 4 \\ 1 & 0 & 1 & 1 \\ 4 & 1 & 0 & 5 \end{bmatrix} \xrightarrow{r_1 \leftrightarrow r_3} \begin{bmatrix} 4 & 1 & 0 & 5 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 2 & 4 \end{bmatrix} \tag{2'}$$

$$\xrightarrow[r_3-\frac{3}{4}r_1]{r_2-\frac{1}{4}r_1}\begin{bmatrix}4 & 1 & 0 & 5\\ 0 & -\frac{1}{4} & 1 & -\frac{1}{4}\\ 0 & \frac{1}{4} & 2 & \frac{1}{4}\end{bmatrix}\xrightarrow{r_3+r_2}\begin{bmatrix}4 & 1 & 0 & 5\\ 0 & -\frac{1}{4} & 1 & -\frac{1}{4}\\ 0 & 0 & 3 & 0\end{bmatrix}, \quad (6')$$

回代求得 $x_1 = 1, x_2 = 1, x_3 = 0.$ (2′)

4. 解　Jacobi 迭代矩阵 $\boldsymbol{J}$ 的特征方程为

$$\begin{vmatrix}3\lambda & 2 & 1\\ 2 & 3\lambda & -1\\ 1 & 1 & 2\lambda\end{vmatrix} = 0 \Longrightarrow 9\lambda^3 - 4\lambda = 0 \Longrightarrow \lambda_1 = 0,\ \lambda_{2,3} = \pm\frac{2}{3}, \quad (4')$$

因为 $\rho(\boldsymbol{J}) = \dfrac{2}{3} < 1$, 所以Jacobi 迭代收敛. (2′)

Gauss-Seidel 迭代矩阵 $\boldsymbol{G}$ 的特征方程为

$$\begin{vmatrix}3\lambda & 2 & 1\\ 2\lambda & 3\lambda & -1\\ \lambda & \lambda & 2\lambda\end{vmatrix} = 0 \Longrightarrow 9\lambda^3 - 3\lambda^2 - \lambda = 0 \Longrightarrow \lambda_1 = 0,\ \lambda_{2,3} = \frac{1\pm\sqrt{5}}{6}, \quad (4')$$

因为 $\rho(\boldsymbol{G}) = \dfrac{1+\sqrt{5}}{6} < 1$, 所以Gauss-Seidel 迭代收敛. (2′)

5. 解　方法 1: 作一个 2 次多项式 $p(x)$, 满足

$$p(a) = f(a),\ \ p(b) = f(b),\ \ p'(b) = f'(b),$$

则

$$p(x) = f(a) + f[a,b](x-a) + f[a,b,b](x-a)(x-b). \quad (4')$$

记 $R(x) = H(x) - p(x)$, 则有 $R(a) = R(b) = R'(b) = 0$, 因此

$$R(x) = A(x-a)(x-b)^2,$$

其中 A 为常数. 所以

$$H(x) = p(x) + A(x-a)(x-b)^2. \quad (4')$$

求导可得

$$H''(x) = p''(x) + 2A[(x-a) + 2(x-b)] = 2f[a,b,b] + 2A[(x-a) + 2(x-b)],$$

由条件 $H''(a) = f''(a)$ 可得

$$f''(a) = 2f[a,b,b] + 4A(a-b),$$

所以

$$A = \frac{f''(a) - 2f[a,b,b]}{4(a-b)},$$

$$H(x) = f(a) + f[a,b](x-a) + f[a,b,b](x-a)(x-b)$$

$$+\frac{f''(a)-2f[a,b,b]}{4(a-b)}(x-a)(x-b)^2,$$

其中

$$f[a,b]=\frac{f(b)-f(a)}{b-a},\ f[a,b,b]=\frac{(b-a)f'(b)-f(b)+f(a)}{(b-a)^2}. \qquad (4')$$

方法 2: 设 $H'(a)=m$, 则

$$H(x)=f(b)+f[b,b](x-b)+f[b,b,a](x-b)^2+f[b,b,a,a](x-b)^2(x-a). \quad (4')$$

列表求差商:

x_k	$H(x_k)$			
b	$f(b)$	$f'(b)$	$\frac{f[b,a]-f'(b)}{a-b}$	$\frac{m-2f[b,a]+f'(b)}{(a-b)^2}$
b	$f(b)$	$f[b,a]$	$\frac{m-f[b,a]}{a-b}$	
a	$f(a)$	m		
a	$f(a)$			

$(4')$

记 $A=\dfrac{m-2f[b,a]+f'(b)}{(a-b)^2}$, 则

$$H(x)=f(b)+f[b,b](x-b)+f[b,b,a](x-b)^2+A(x-b)^2(x-a),$$

由条件 $H''(a)=f''(a)$ 可得

$$A=\frac{f''(a)-2f[b,b,a]}{4(a-b)},$$

因此

$$\begin{aligned}H(x)=&f(b)+f[b,b](x-b)+f[b,b,a](x-b)^2\\&+\frac{f''(a)-2f[b,b,a]}{4(a-b)}(x-b)^2(x-a).\end{aligned} \qquad (4')$$

6. **解** 记 $f(x)=\ln x$, $p_1(x)=a+bx$, 则当 $x\in(1,2)$ 时, $f''(x)=-\dfrac{1}{x^2}<0$, 因此 $f(x)-p_1(x)$ 在 $[1,2]$ 上恰有 3 个交错偏差点为 $1,x_1,2$, 且满足 $(3')$

$$\begin{cases}f(1)-p_1(1)=-[f(x_1)-p_1(x_1)]=f(2)-p_1(2),\\ f'(x_1)=p_1'(x_1),\end{cases} \qquad (3')$$

即有

$$\begin{cases}0-(a+b)=-[\ln x_1-(a+bx_1)]=\ln 2-(a+2b),\\ \dfrac{1}{x_1}=b,\end{cases} \qquad (3')$$

求得 $a = -\frac{1}{2}(1 + \ln 2 + \ln\ln 2),\ b = \ln 2, x_1 = \frac{1}{\ln 2}$. (3′)

7. **解** 当 $f(x) = 1$ 时, 左边 $= \int_0^1 1\,\mathrm{d}x = 1$, 右边 $= \frac{1}{2} + c$;

当 $f(x) = x$ 时, 左边 $= \int_0^1 x\,\mathrm{d}x = \frac{1}{2}$, 右边 $= \frac{1}{2}x_0 + cx_1$;

当 $f(x) = x^2$ 时, 左边 $= \int_0^1 x^2\,\mathrm{d}x = \frac{1}{3}$, 右边 $= \frac{1}{2}x_0^2 + cx_1^2$. (3′)

要使求积公式至少具有 2 次代数精度, 当且仅当

$$\begin{cases} \frac{1}{2} + c = 1, \\ \frac{1}{2}x_0 + cx_1 = \frac{1}{2}, \\ \frac{1}{2}x_0^2 + cx_1^2 = \frac{1}{3}, \end{cases} \quad (3')$$

求得 $c = \frac{1}{2}, x_0 = \frac{1}{2}\left(1 - \frac{1}{\sqrt{3}}\right), x_1 = \frac{1}{2}\left(1 + \frac{1}{\sqrt{3}}\right)$, 所以求积公式为

$$\int_0^1 f(x)\,\mathrm{d}x \approx \frac{1}{2}f\left(\frac{1}{2}\left(1 - \frac{1}{\sqrt{3}}\right)\right) + \frac{1}{2}f\left(\frac{1}{2}\left(1 + \frac{1}{\sqrt{3}}\right)\right). \quad (3')$$

当 $f(x) = x^3$ 时, 左边 $\int_0^1 x^3\,\mathrm{d}x = \frac{1}{4}$,

$$右边 = \frac{1}{2}\left[\frac{1}{2}\left(1 - \frac{1}{\sqrt{3}}\right)\right]^3 + \frac{1}{2}\left[\frac{1}{2}\left(1 + \frac{1}{\sqrt{3}}\right)\right]^3 = \frac{1}{4};$$

当 $f(x) = x^4$ 时, 左边 $\int_0^1 x^4\,\mathrm{d}x = \frac{1}{5}$,

$$右边 = \frac{1}{2}\left[\frac{1}{2}\left(1 - \frac{1}{\sqrt{3}}\right)\right]^4 + \frac{1}{2}\left[\frac{1}{2}\left(1 + \frac{1}{\sqrt{3}}\right)\right]^4 = \frac{7}{36},$$

因为左边 $\neq$ 右边, 所以求积公式的代数精度为 3. (3′)

注 如果最后两步没有验证, 但指出得到的求积公式是两点 Gauss 公式, 代数精度为 3, 同样可得 3 分.

8. **解** 局部截断误差为

$$\begin{aligned} R_{i+1} =& y(x_{i+1}) - 3y(x_i) + 2y(x_{i-1}) - Ahf(x_{i+1}, y(x_{i+1})) \\ & - Bhf(x_{i-1}, y(x_{i-1})) \\ =& y(x_{i+1}) - 3y(x_i) + 2y(x_{i-1}) - Ahy'(x_{i+1}) - Bhy'(x_{i-1}) \end{aligned} \quad (2')$$

$$
\begin{aligned}
&= y(x_i) + hy'(x_i) + \frac{h^2}{2}y''(x_i) + \frac{h^3}{6}y'''(x_i) + O(h^4) - 3y(x_i) \\
&\quad + 2\left[y(x_i) - hy'(x_i) + \frac{h^2}{2}y''(x_i) - \frac{h^3}{6}y'''(x_i) + O(h^4)\right] \\
&\quad - Ah\left[y'(x_i) + hy''(x_i) + \frac{h^2}{2}y'''(x_i) + O(h^3)\right] \\
&\quad - Bh\left[y'(x_i) - hy''(x_i) + \frac{h^2}{2}y'''(x_i) + O(h^3)\right] \\
&= (-1 - A - B)hy'(x_i) + \left(\frac{3}{2} - A + B\right)h^2y''(x_i) \\
&\quad + \left(-\frac{1}{6} - \frac{A}{2} - \frac{B}{2}\right)h^3y'''(x_i) + O(h^4),
\end{aligned} \tag{4'}
$$

要使求解公式至少具有 2 阶精度, 当且仅当

$$
\begin{cases} -1 - A - B = 0, \\ \dfrac{3}{2} - A + B = 0, \end{cases} \tag{2'}
$$

=0

求得 $A = \frac{1}{4}, B = -\frac{5}{4}$, 局部截断误差为

$$
R_{i+1} = \frac{1}{3}y'''(x_i)h^3 + O(h^4),
$$

它是 2 阶公式. (4′)

9. **解** 1) 考虑 (x_i, t_k) 点的方程

$$
\frac{\partial u}{\partial t}(x_i, t_k) - a\frac{\partial^2 u}{\partial x^2}(x_i, t_k) = f(x_i, t_k), \tag{A}
$$

$\frac{\partial u}{\partial t}(x_i, t_k)$ 用向后差商近似, $\frac{\partial^2 u}{\partial x^2}(x_i, t_k)$ 用二阶差商近似, 得

$$
\frac{\partial u}{\partial t}(x_i, t_k) = \frac{1}{\tau}[u(x_i, t_k) - u(x_i, t_{k-1})] + \frac{\tau}{2}\frac{\partial^2 u}{\partial t^2}(x_i, \eta_i^k), \quad \eta_i^k \in (t_{k-1}, t_k),
$$

$$
\frac{\partial^2 u}{\partial x^2}(x_i, t_k) = \frac{1}{h^2}[u(x_{i+1}, t_k) - 2u(x_i, t_k) + u(x_{i-1}, t_k)] - \frac{h^2}{12}\frac{\partial^4 u}{\partial x^4}(\xi_i^k, t_k),
$$

$$
\xi_i^k \in (x_{i-1}, x_{i+1}), \tag{2'}
$$

将上面两式代入方程 (A) 得

$$
\begin{aligned}
&\frac{1}{\tau}[u(x_i, t_k) - u(x_i, t_{k-1})] - \frac{1}{h^2}[u(x_{i+1}, t_k) - 2u(x_i, t_k) + u(x_{i-1}, t_k)] \\
&\quad = f(x_i, t_k) + R_{ik}, \quad 1 \leqslant i \leqslant M - 1, 1 \leqslant k \leqslant N,
\end{aligned} \tag{B}
$$

其中

$$
R_{ik} = -\frac{\tau}{2}\frac{\partial^2 u}{\partial t^2}(x_i, \eta_i^k) - \frac{h^2}{12}\frac{\partial^4 u}{\partial x^4}(\xi_i^k, t_k). \tag{2'}
$$

又由初始条件和边界条件得

$$u(x_i,0)=\varphi(x_i)\quad 1\leqslant i\leqslant M-1,$$

$$u(x_0,t_k)=\alpha(t_k),\quad u(x_M,t_k)=\beta(t_k),\quad 0\leqslant k\leqslant N,$$

在 (B) 中忽略 R_{ik}, 并用 u_i^k 代替 $u(x_i,t_k)$ 得下面的差分格式:

$$\begin{cases}\dfrac{1}{\tau}(u_i^k-u_i^{k-1})-\dfrac{1}{h^2}(u_{i+1}^k-2u_i^k+u_{i-1}^k)=f(x_i,t_k), \\ \qquad\qquad\qquad\qquad 1\leqslant i\leqslant M-1, 1\leqslant k\leqslant N, \\ u_i^0=\varphi(x_i), \qquad\qquad 1\leqslant i\leqslant M-1, \\ u_0^k=\alpha(t_k),\ u_M^k=\beta(t_k),\quad 0\leqslant k\leqslant N.\end{cases} \quad (2'$$

2) 记 $r=\tau/h^2$, 差分方程可以写为下面形式:

$$-ru_{i-1}^k+(1+2r)u_i^k-ru_{i+1}^k=u_i^{k-1}+\tau f(x_i,t_k),$$
$$1\leqslant i\leqslant M-1, 1\leqslant k\leqslant N, \quad (2$$

上式中固定 $k, i=1,2,\cdots,M-1$, 可将其写成矩阵向量形式为

$$\begin{bmatrix}1+2r & -r & & & \\ -r & 1+2r & -r & & \\ & \ddots & \ddots & \ddots & \\ & & -r & 1+2r & -r \\ & & & -r & 1+2r\end{bmatrix}\begin{bmatrix}u_1^k\\ u_2^k\\ \vdots\\ u_{M-2}^k\\ u_{M-1}^k\end{bmatrix}$$

$$=\begin{bmatrix}u_1^{k-1}\\ u_2^{k-1}\\ \vdots\\ u_{M-2}^{k-1}\\ u_{M-1}^{k-1}\end{bmatrix}+\begin{bmatrix}\tau f(x_1,t_k)+r\alpha(t_k)\\ \tau f(x_2,t_k)\\ \vdots\\ \tau f(x_{M-2},t_k)\\ \tau f(x_{M-1},t_k)+r\beta(t_k)\end{bmatrix},\ k=1,2,\cdots,N. \quad ($$

注 本题也可以建立 Crank-Nicolson 格式.

2009 年工程硕士研究生学位课程考试试题

1. 1) 0.004224 2) $x_{k+1}=x_k-\dfrac{3x_k-\mathrm{e}^{x_k}}{3-\mathrm{e}^{x_k}},\quad k=0,1,2,\cdots$ 3) $\sqrt{5}$

4) $(\ln 2)\cdot\dfrac{(x-1)(x-2)}{2}-(\ln 3)\cdot x(x-2)+(\ln 4)\cdot\dfrac{x(x-1)}{2}$

5) $\dfrac{35}{59}, \dfrac{39}{59}$ 6) $f\left(-\dfrac{1}{\sqrt{3}}\right)+f\left(\dfrac{1}{\sqrt{3}}\right)$ (5′