2001年工程硕士研究生学位考试 "数值分析"试题参考答案

1. ① $\frac{1}{2}(x^2+y^2)=\frac{1}{2}(80.128^2+80.115^2)=6419.454805$

$|e(x)|\le \frac{1}{2}\times10^{-3}$, $|e(y)|\le\frac{1}{2}\times10^{-3}$

$|e(\frac{1}{2}(x^2+y^2))|=|xe(x)+ye(y)|\le x|e(x)|+y|e(y)|$

$\le \frac{1}{2}\times10^{-3}\times(80.128+80.115)=80.1215\times10^{-3}$

$<\frac{1}{2}\times10^{0}$

$\Rightarrow$ $\frac{1}{2}(x^2+y^2)$所得结果至少具有4位有效数.

② $\frac{1}{2}(x^2-y^2)=1.0415795$

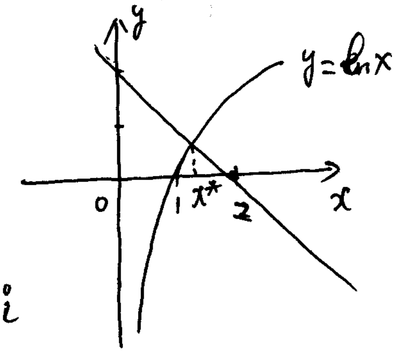
$|e(\frac{1}{2}(x^2-y^2))|=|xe(x)-ye(y)|$

$\le x|e(x)|+y|e(y)|<\frac{1}{2}\times10^{0}$

$\Rightarrow$ $\frac{1}{2}(x^2-y^2)$所得结果至少具有1位有效数.

2. (a) 法一、作图

作 $y=\ln x$ 及 $y=2-x$ 图象



由图象知该方程有唯一实根 $x^*\in[1,2]$.

法二、令 $f(x)=x+\ln x-2$.

由定义域知 $x>0$.

$f'(x)=1+\frac{1}{x}>0$, $x>0\Rightarrow f(x)\nearrow$.

$f(1)=1-2<0$, $f(2)=\ln 2>0$.

∴方程在 $[1,2]$ 中有唯一实根.

(b) $x=2-\ln x$

构造迭代格式

$$\begin{cases}x_{k+1}=2-\ln x_k, & k=0,1,2,\cdots\\ x_0=1.5\end{cases}$$

计算得: $x_1=1.594535$

$x_2=1.533418$

$x_3=1.572501$

$x_4=1.547333$, $x_5=1.563467$

$x_6=1.553094$, $x_7=1.559751$

$x_8=1.555473$, $x_9=1.558220$

$x_{10}=1.556456$, $x_{11}=1.557589$

$x_{12}=1.556861$, $x_{13}=1.557328$

$x_{14}=1.557028$.

$|x_{14}-x_{13}|=0.3\times10^{-3}<\frac{1}{2}\times10^{-3}$

$\therefore x^*\approx1.557028$.

(c) Newton迭代格式

$$\begin{cases}x_{k+1}=x_k-\dfrac{x_k+\ln x_k-2}{1+\frac{1}{x_k}}, & k=0,1,2,\cdots\\ x_0=1.5\end{cases}$$

计算得

$x_1=1.556721$, $x_2=1.557146$

$|x_2-x_1|=0.425\times10^{-3}<\frac{1}{2}\times10^{-3}$.

$\therefore x^*\approx1.557146$

3. $\begin{bmatrix}3&1&-1&4\\4&0&4&8\\12&-3&3&9\end{bmatrix}\xrightarrow{r_1\leftrightarrow r_3}\begin{bmatrix}12&-3&3&9\\4&0&4&8\\3&1&-1&4\end{bmatrix}$

$\xrightarrow[-\frac{1}{4}r_1+r_3]{-\frac{1}{3}r_1+r_2}\begin{bmatrix}12&-3&3&9\\0&1&3&5\\0&\frac{7}{4}&-\frac{7}{4}&\frac{7}{4}\end{bmatrix}\xrightarrow{r_2\leftrightarrow r_3}\begin{bmatrix}12&-3&3&9\\0&\frac{7}{4}&-\frac{7}{4}&\frac{7}{4}\\0&1&3&5\end{bmatrix}$

$\xrightarrow{-\frac{4}{7}r_2+r_3}\begin{bmatrix}12&-3&3&9\\0&\frac{7}{4}&-\frac{7}{4}&\frac{7}{4}\\0&0&4&4\end{bmatrix}$

$\Rightarrow 4x_3=4$, $x_3=1$, $\frac{7}{4}x_2-\frac{7}{4}x_3=\frac{7}{4}$

$x_2=2$, $12x_1-3x_2+3x_3=9$, $x_1=1$.

4. Gauss—Seidel迭代格式为:

$$\begin{cases} x_1^{(k+1)} = (3x_2^{(k)} - 2x_3^{(k)} + 4)/5 \\ x_2^{(k+1)} = (x_1^{(k+1)} + 8x_3^{(k)} - 1) \quad , k=0,1,2,\cdots \\ x_3^{(k+1)} = (-2x_1^{(k+1)} + 3x_2^{(k+1)} - 7)/20 \end{cases}$$

迭代矩阵G的特征方程为:

$$\begin{vmatrix} 5\lambda & -3 & 2 \\ \lambda & -\lambda & 8 \\ 2\lambda & -3\lambda & 20\lambda \end{vmatrix} = 0.$$

$\Rightarrow \lambda(50\lambda^2 - 89\lambda + 24) = 0$

$\lambda_1 = 0$

令 $f(\lambda) = 50\lambda^2 - 89\lambda + 24$

$f(1) = 50 - 89 + 24 < 0$

$f(2) = 46 > 0$

∴至少有一个特征值 $\in (1,2)$

$\Rightarrow \rho(G) > 1$

Gauss-Seidel迭代发散.

5. $L_3(x) = 1 \cdot \frac{(x-2)(x-3)(x-5)}{(0-2)(0-3)(0-5)} + (-3) \cdot \frac{(x-0)(x-3)(x-5)}{(2-0)(2-3)(2-5)}$

$+ (-4) \cdot \frac{(x-0)(x-2)(x-5)}{(3-0)(3-2)(3-5)} + 2 \cdot \frac{(x-0)(x-2)(x-3)}{(5-0)(5-2)(5-3)}$

计算差商

x_k	$f(x_k)$			
0	1	-2	$\frac{1}{3}$	$\frac{1}{5}$
2	-3	-1	$\frac{4}{3}$	
3	-4	3		
5	2			

$N_3(x) = 1 - 2(x-0) + \frac{1}{3}(x-0)(x-2)$

$+ \frac{1}{5}(x-0)(x-2)(x-3).$

6. 设2次拟合多项式是 $p_2(x) = c_0 + c_1 x + c_2 x^2$.

则 $\varphi_0(x) = 1$, $\varphi_1(x) = x$, $\varphi_2(x) = x^2$.

$$\vec{\varphi}_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \vec{\varphi}_1 = \begin{pmatrix} 0 \\ 2 \\ 3 \\ 5 \end{pmatrix}, \vec{\varphi}_2 = \begin{pmatrix} 0 \\ 4 \\ 9 \\ 25 \end{pmatrix}. \vec{y} = \begin{pmatrix} 4 \\ 1 \\ 1 \\ 9 \end{pmatrix}$$

$(\vec{\varphi}_0, \vec{\varphi}_0) = 4$, $(\vec{\varphi}_0, \vec{\varphi}_1) = 10$, $(\vec{\varphi}_0, \vec{\varphi}_2) = 38$

$(\vec{\varphi}_1, \vec{\varphi}_1) = 38$, $(\vec{\varphi}_1, \vec{\varphi}_2) = 160$, $(\vec{\varphi}_2, \vec{\varphi}_2) = 722$

$(\vec{y}, \vec{\varphi}_0) = 15$, $(\vec{y}, \vec{\varphi}_1) = 50$, $(\vec{y}, \vec{\varphi}_2) = 238$

$$\Rightarrow \begin{bmatrix} 4 & 10 & 38 \\ 10 & 38 & 160 \\ 38 & 160 & 722 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 15 \\ 50 \\ 238 \end{bmatrix}$$

$\Rightarrow c_0 = 4.09649$, $c_1 = -3.62198$

$c_2 = 0.91669$.

7. 当 $f(x)=1$ 时, 左 $= \int_0^1 1 dx = 1$, 右 $= \frac{1}{2}(1+1) = 1$

$f(x) = x$ 时, 左 $= \int_0^1 x dx = \frac{1}{2}$, 右 $= \frac{1}{2}(x_0 + x_1)$

$f(x) = x^2$ 时, 左 $= \int_0^1 x^2 dx = \frac{1}{3}$, 右 $= \frac{1}{2}(x_0^2 + x_1^2)$

$$\Rightarrow \begin{cases} \frac{1}{2}(x_0 + x_1) = \frac{1}{2} \\ \frac{1}{2}(x_0^2 + x_1^2) = \frac{1}{3} \end{cases} \Rightarrow \begin{cases} x_0 = \frac{1}{2}(1 - \frac{1}{\sqrt{3}}) \\ x_1 = \frac{1}{2}(1 + \frac{1}{\sqrt{3}}). \end{cases}$$

当 $f(x) = x^3$ 时

左 $= \int_0^1 x^3 dx = \frac{1}{4}$, 右 $= \frac{1}{2}\left[\left[\frac{1}{2}(1 - \frac{1}{\sqrt{3}})\right]^2\right.$

$\left. + \left[\frac{1}{2}(1 + \frac{1}{\sqrt{3}})\right]^2\right] = \frac{1}{4}$.

当 $f(x) = x^4$ 时.

左 $= \int_0^1 x^4 dx = \frac{1}{5}$

右 $= \frac{1}{2}\left\{\left[\frac{1}{2}(1 - \frac{1}{\sqrt{3}})\right]^4 + \left[\frac{1}{2}(1 - \frac{1}{\sqrt{3}})\right]^4\right\} = \frac{7}{36}$

∴代数精度最高为3.

8. (a) $T_n(f)=\sum_{k=0}^{n-1}\frac{h}{2}\left[f(x_k)+f(x_{k+1})\right]$

$$S_n(f)=\sum_{k=0}^{n-1}\frac{h}{6}\left[f(x_k)+4f(x_{k+\frac12})+f(x_{k+1})\right]$$

(b) $T_{2n}(f)=\frac12 T_n(f)+\frac{h}{2}\sum_{k=0}^{n-1}f(x_{k+\frac12})$.

$$\therefore \frac43 T_{2n}(f)-\frac13 T_n(f)$$
$$=\frac23 T_n(f)+\frac23 h\sum_{k=0}^{n-1}f(x_{k+\frac12})-\frac13 T_n(f)$$
$$=\frac13 T_n(f)+\frac23 h\sum_{k=0}^{n-1}f(x_{k+\frac12})$$
$$=\sum_{k=0}^{n-1}\frac{h}{6}\left[f(x_k)+f(x_{k+1})\right]+\frac23 h\sum_{k=0}^{n-1}f(x_{k+\frac12})$$
$$=\sum_{k=0}^{n-1}\frac{h}{6}\left[f(x_k)+4f(x_{k+\frac12})+f(x_{k+1})\right]$$
$$=S_n(f).$$

9. 局部截断误差

$$R_{i+1}=y(x_{i+1})-y(x_i)-\frac{h}{12}\left[5f(x_{i+1},y(x_{i+1}))+8f(x_i,y(x_i))-f(x_{i-1},y(x_{i-1}))\right]$$
$$=y(x_{i+1})-y(x_i)-\frac{5}{12}hy'(x_{i+1})-\frac{8}{12}hy'(x_i)+\frac{1}{12}hy'(x_{i-1})$$
$$=y(x_i)+hy'(x_i)+\frac{h^2}{2!}y''(x_i)+\frac{h^3}{3!}y'''(x_i)+\frac{h^4}{4!}y^{(4)}(x_i)+O(h^5)$$
$$-y(x_i)$$
$$-\frac{5}{12}h\left[y'(x_i)+hy''(x_i)+\frac{h^2}{2!}y'''(x_i)+\frac{h^3}{3!}y^{(4)}(x_i)+O(h^4)\right]$$
$$-\frac{8}{12}hy'(x_i)$$
$$+\frac{1}{12}h\left[y'(x_i)-hy''(x_i)+\frac{h^2}{2!}y'''(x_i)-\frac{h^3}{3!}y^{(4)}(x_i)+O(h^4)\right]$$
$$=-\frac{1}{24}h^4y^{(4)}(x_i)+O(h^5).$$

$\Rightarrow$ 所给公式是3阶的.

2002年工程硕士学位考试
数值分析试题参考答案

1. 设底面半径为r，高为h，体积为V.

$V=\pi r^2 h$

已知 $|e(r)|\le 0.005$, $|e(h)|\le 0.005$

$$|e(V)| \approx |2\pi r h\, e(r) + \pi r^2 e(h)|$$

$$\le \pi r(2h+r)\times 0.005 = 196.3495$$

$$|e_r(V)| = \left|\frac{e(V)}{V}\right| \le \frac{196.3495}{\pi\times 50^2\times 100} = 2.5\times 10^{-4}$$

2. 设: $\varphi(x)=\sqrt{1+\frac{1}{x}}$, $\varphi'(x) = -\dfrac{\frac{1}{x^2}}{2\sqrt{1+\frac{1}{x}}}$

$$= -\frac{1}{2x^2\sqrt{1+\frac{1}{x}}}$$

当 $x\in[1,2]$ 时, $|\varphi'(x)| \le \frac{1}{2} < 1$

$1<\sqrt{1+\frac{1}{2}} \le \varphi(x) \le \sqrt{2} < 2$

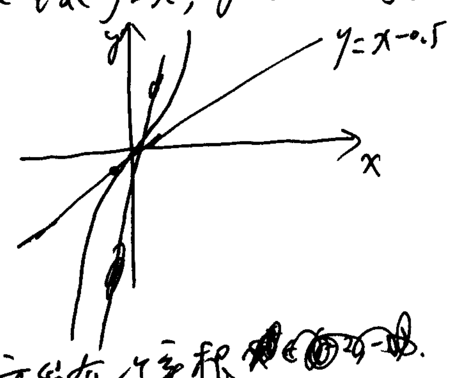
由 Th.2.1 得, 对 $\forall x_0\in[1,2]$ 迭代

$x_{k+1} = \varphi(x_k)$ $k=0,1,\ldots$

收敛.

3. $x^3 = x - 0.5$

作函数 $y=x^3$, $y=x-0.5$ 图象.



方程有一个实根 ~~∈(…)~~.

令 $f(x)=x^3-x+0.5$

$f(-1)=0.5>0$

$f(-2)=-8+2+0.5<0$

$\Rightarrow x^*\in[-1,-2]$

Newton迭代格式:

$$x_{k+1} = x_k - \frac{x_k^3 - x_k + 0.5}{3x_k^2-1}$$

$x_0=-1.5$

$\Rightarrow x_1=-1.2609$, $x_2=-1.19623$

$x_3=-1.1915$ $x_4=-1.191487$

$x_5=-1.191487$

$|x_5-x_4| < \frac{1}{2}\times 10^{-3}$

$x^*\approx -1.191487$

4.
$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 3 & 1 & -1 & 4 \\ 12 & -3 & 3 & 9 \end{bmatrix} \xleftrightarrow{r_1\leftrightarrow r_2} \begin{bmatrix} 12 & -3 & 3 & 9 \\ 3 & 1 & -1 & 4 \\ 1 & 0 & 1 & 2 \end{bmatrix}$$

$$\xrightarrow[-\frac{1}{12}r_1+r_3]{-\frac{1}{4}r_1+r_2} \begin{bmatrix} 12 & -3 & 3 & 9 \\ 0 & \frac{7}{4} & -\frac{7}{4} & \frac{7}{4} \\ 0 & \frac{1}{4} & \frac{3}{4} & \frac{5}{4} \end{bmatrix} \xrightarrow{-\frac{1}{7}r_2+r_3} \begin{bmatrix} 12 & -3 & 3 & 9 \\ 0 & \frac{7}{4} & -\frac{7}{4} & \frac{7}{4} \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$\Rightarrow x_3=1$, $x_2=2$, $x_1=1$

5. (a) Jacobi迭代格式

$$\begin{cases} x_1^{(k+1)} = (3x_2^{(k)} - 2x_3^{(k)} + 4)/15 \\ x_2^{(k+1)} = (x_1^{(k)} + 8x_3^{(k)} - 1) \\ x_3^{(k+1)} = (-2x_1^{(k)} + 3x_2^{(k)} - 7)/20 \end{cases}$$

Gauss-Seidel 迭代格式 $k=0,1,\ldots$

$$\begin{cases} x_1^{(k+1)} = (3x_2^{(k)} - 2x_3^{(k)} + 4)/15 \\ x_2^{(k+1)} = (x_1^{(k+1)} + 8x_3^{(k)} - 1) \\ x_3^{(k+1)} = (-2x_1^{(k+1)} + 3x_2^{(k+1)} - 7)/20 \end{cases}$$

5) Gauss-Seidel迭代矩阵G的特征方程为:

$$\begin{vmatrix} 15\lambda & -3 & 2 \\ \lambda & -\lambda & 8 \\ 2\lambda & -3\lambda & 20\lambda \end{vmatrix} = 0$$

$\Rightarrow \lambda(300\lambda^2 - 418\lambda + 48) = 0$

$\lambda_1 = 0$

$\lambda_{2,3} = \dfrac{418 \pm 342.234}{600}$

$\Rightarrow \rho(G) > 1.$

Gauss-Seidel 迭代发散.

6、 $f(x) - L_n(x) = \dfrac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{i=0}^{n}(x-x_i) \quad \xi \in (3,6),$

$f(x) = \ln x, \quad f^{(n+1)}(x) = (-1)^n \dfrac{n!}{x^{n+1}}$

$\Rightarrow$

$|f(x) - L_n x| = \left| \dfrac{(-1)^n n!}{(n+1)!} \dfrac{1}{\xi^{n+1}} \prod_{i=0}^{n}(x-x_i) \right|$

$\leq \dfrac{1}{(n+1)3^{n+1}} \cdot 3^{n+1} = \dfrac{1}{n+1} \to 0 \ (n \to \infty)$ $x \in [3,6]$.

$\therefore \lim\limits_{n\to\infty} \max\limits_{x\in[3,6]} |f(x) - L_n x| = 0.$

7.

x_k	$f(x_k)$					
1	3	2	-6	11	-25/6	55/36
1	3	-4	5	-3/2	5/12	
2	-1	1	1/2	-1/4		
2	-1	2	0			
4	3	2				
4	3					

$H(x) = 3 + 2(x-1) - 6(x-1)^2 + 11(x-1)^2(x-2) - \dfrac{25}{6}(x-1)^2(x-2)^2 + \dfrac{25}{36}(x-1)^2(x-2)^2(x-4).$

8. 见2001年第8题.

9.(a) 令 $x = \dfrac{a+b}{2} + \dfrac{b-a}{2}t$

$\int_a^b f(x)dx = \dfrac{b-a}{2}\int_{-1}^{1} f\left(\dfrac{a+b}{2} + \dfrac{b-a}{2}t\right)dt$

$\approx \dfrac{b-a}{2}\left[\dfrac{5}{9} f\left(\dfrac{a+b}{2} - \dfrac{b-a}{2}\sqrt{\dfrac{3}{5}}\right) + \dfrac{8}{9} f\left(\dfrac{a+b}{2}\right) + \dfrac{5}{9} f\left(\dfrac{a+b}{2} + \dfrac{b-a}{2}\sqrt{\dfrac{3}{5}}\right)\right].$

(b) $\int_3^6 e^{-x} \approx \dfrac{3}{2}\left[\dfrac{5}{9} e^{-4.5+1.5\sqrt{\frac{3}{5}}} + \dfrac{8}{9} e^{-4.5} + \dfrac{5}{9} e^{-4.5-1.5\sqrt{\frac{3}{5}}}\right]$

$= 0.0472954 1.$

10. 局部截断误差为

$R_{i+1} = y(x_{i+1}) - y(x_i) - h[\alpha f(x_i, y(x_i)) + (1-\alpha) f(x_i+\lambda h, y(x_i)+\lambda h f(x_i, y(x_i)))]$

$= y(x_i) + h y'(x_i) + \dfrac{h^2}{2} y''(x_i) + \dfrac{h^3}{3!} y'''(x_i) + O(h^4)$

$- y(x_i) - \alpha h y'(x_i)$

$- (1-\alpha)h\Big[f(x_i, y(x_i)) + \dfrac{\partial f}{\partial x}\lambda h + \dfrac{\partial f}{\partial y}\lambda h y'(x_i) + \dfrac{1}{2}\Big[\dfrac{\partial^2 f}{\partial x^2}(\lambda h)^2 + 2\dfrac{\partial^2 f}{\partial x \partial y}(\lambda h)^2 y'(x_i) + \dfrac{\partial^2 f}{\partial y^2}(\lambda h)^2 y'(x_i)^2\Big] + O(h^3)\Big]$

$= \left[\dfrac{1}{2} - (1-\alpha)\lambda\right] h^2 y''(x_i) + \left(\dfrac{1}{6} - \dfrac{1}{2}(1-\alpha)\lambda^2\right) y'''(x_i) h^3 + \dfrac{\partial f}{\partial y} y''(x_i) h^3 + O(h^4)$

$\therefore$ 当 $(1-\alpha)\lambda = \dfrac{1}{2}$ 时, $R_{i+1} = O(h^3)$.

公式为2阶.

$$
\begin{aligned}
&= y(x_i) + hy'(x_i) + \frac{h^2}{2}y''(x_i) + O(h^3) - y(x_i) \\
&\quad - h\left[\alpha y'(x_i) + (1-\alpha)f(x_i + \lambda h, y(x_i) + \lambda h y'(x_i))\right] \\
&= hy'(x_i) + \frac{h^2}{2}y''(x_i) + O(h^3) \\
&\quad - h\Big[\alpha y'(x_i) + (1-\alpha)\Big(f(x_i, y(x_i)) + \lambda h\,\frac{\partial f(x_i, y(x_i))}{\partial x} \\
&\quad + \lambda h y'(x_i)\,\frac{\partial f(x_i, y(x_i))}{\partial y} + O(h^2)\Big)\Big] \qquad (5') \\
&= hy'(x_i) + \frac{h^2}{2}y''(x_i) - h\left[y'(x_i) + (1-\alpha)\lambda h y''(x_i)\right] + O(h^3) \\
&= h^2\left(\frac{1}{2} - (1-\alpha)\lambda\right)y''(x_i) + O(h^3) \qquad (2')
\end{aligned}
$$

当$(1-\alpha)\lambda = \dfrac{1}{2}$时，$R_{i+1} = O(h^3)$，所给公式为 2 阶公式. (1′)

2003 年工程硕士研究生学位课程考试

1. 解

$$x_1 \approx 6.1025, \qquad x_2 \approx 80.115$$

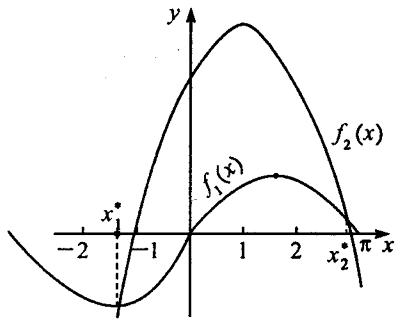
$$|e(x_1)| \leqslant \frac{1}{2}\times 10^{-4}, \qquad |e(x_2)| \leqslant \frac{1}{2}\times 10^{-3} \qquad (1')$$

$$
\begin{aligned}
&e(x_1 x_2) \approx x_2 e(x_1) + x_1 e(x_2) \\
|e(x_1 x_2)| &\approx |x_2 e(x_1) + x_1 e(x_2)| \\
&\leqslant x_2|e(x_1)| + x_1|e(x_2)| \\
&\leqslant 80.115 \times \frac{1}{2}\times 10^{-4} + 6.1025 \times \frac{1}{2}\times 10^{-3} \\
&= (8.015 + 6.1025)\times \frac{1}{2}\times 10^{-3} \\
&= 7.057 \times 10^{-3} \qquad (4')
\end{aligned}
$$

$$
\begin{aligned}
&e_r(x_1 x_2) \approx e_r(x_1) + e_r(x_2) \\
|e_r(x_1 x_2)| &\approx |e_r(x_1) + e_r(x_2)| \\
&\leqslant |e_r(x_1)| + |e_r(x_2)| \\
&\leqslant \frac{\frac{1}{2}\times 10^{-4}}{6.1025} + \frac{\frac{1}{2}\times 10^{-3}}{80.115} \\
&= \left(\frac{1}{6.1025} + \frac{1}{8.0115}\right)\times \frac{1}{2}\times 10^{-4} \\
&= 0.144344 \times 10^{-4} \qquad (4')
\end{aligned}
$$

2. **解** (1) $\sin x = -(x^2 - 2x - 3)$

$$f_1(x) = \sin x,\qquad f_2(x) = -(x^2 - 2x - 3) = -(x+1)(x-3)$$



作 $y = f_1(x)$ 和 $y = f_2(x)$ 的图像知方程 $f(x) = 0$ 有且仅有两根

$$x_1^* \in [-2, -1],\qquad x_2^* \in [2,3] \qquad (3')$$

(2) 原方程可改写为

$$x^2 = 2x + 3 - \sin x$$

当 $x \in [2,3]$ 时,原方程与方程 $x = \sqrt{2x + 3 - \sin x}$ 同解.取迭代格式

$$\begin{cases} x_{k+1} = \sqrt{2x_k + 3 - \sin x_k}, & k = 0,1,2,\cdots \\ x_0 = 2.5 \end{cases} \qquad (2')$$

当 $x \in [-2, -1]$ 时,原方程与方程 $x = -\sqrt{2x + 3 - \sin x}$ 同解.

计算得

$x_1 = 2.7206,\quad x_2 = 2.8342,\quad x_3 = 2.7444,\quad x_4 = 2.8464$

$x_5 = 2.8986,\quad x_6 = 2.9252,\quad x_7 = 2.9387,\quad x_8 = 2.9455$

$x_9 = 2.9489,\quad x_{10} = 2.9506$

$\therefore x_2^* = 2.95$ (3′)

(3) 当原方程与方程 $x = -\sqrt{2x + 3 - \sin x}$ 同解.当 $x \in [-2, -1]$ 时,取迭代格式

$$\begin{cases} x_{k+1} = -\sqrt{3 - \sin x_k + 2x_k}, & k = 0,1,2,\cdots \\ x_0 = -1.5 \end{cases} \qquad (2')$$

令 $x_k = -y_k$,则

$$\begin{cases} y_{k+1} = \sqrt{3 + \sin y_k - 2y_k}, & k = 0,1,2,\cdots \\ y_0 = 1.5 \end{cases}$$

计算得

$y_1 = 0.99875,\quad y_2 = 1.3577,\quad y_3 = 1.1234,\quad y_4 = 1.2864$

$y_5 = 1.1777,\quad y_6 = 1.2523,\quad y_7 = 1.2021,\quad y_8 = 1.2364$

$y_9 = 1.2132,\quad y_{10} = 1.2290,\quad y_{11} = 1.2183,\quad y_{12} = 1.2255$

$y_{13} = 1.2206,\quad y_{14} = 1.2240,\quad y_{15} = 1.2217$

$\therefore x_1^* = -1.22$ (3′)

3. **解**

$$\begin{bmatrix} 3 & 1 & -1 & 4 \\ 1 & 0 & 1 & 2 \\ 12 & -3 & 3 & 9 \end{bmatrix} \xrightarrow{r_3 \leftrightarrow r_1} \begin{bmatrix} 12 & -3 & 3 & 9 \\ 1 & 0 & 1 & 2 \\ 3 & 1 & -1 & 4 \end{bmatrix}$$

$$\xrightarrow[r_3 - \frac{1}{4}r_1]{r_2 - \frac{1}{12}r_1} \begin{bmatrix} 12 & -3 & 3 & 9 \\ 0 & \frac{1}{4} & \frac{3}{4} & \frac{5}{4} \\ 0 & \frac{7}{4} & -\frac{7}{4} & \frac{7}{4} \end{bmatrix} \xrightarrow{r_3 \leftrightarrow r_2} \begin{bmatrix} 12 & -3 & 3 & 9 \\ 0 & \frac{7}{4} & -\frac{7}{4} & \frac{7}{4} \\ 0 & \frac{1}{4} & \frac{3}{4} & \frac{5}{4} \end{bmatrix} \qquad (6')$$

$$\xrightarrow{r_3 - \frac{1}{7}r_2} \begin{bmatrix} 12 & -3 & 3 & 9 \\ 0 & \frac{7}{4} & -\frac{7}{4} & \frac{7}{4} \\ 0 & 0 & 1 & 1 \end{bmatrix} \qquad (4')$$

等价三角方程组为

$$\begin{cases} 12x_1 - 3x_2 + 3x_3 = 9 \\ \frac{7}{4}x_2 - \frac{7}{4}x_3 = \frac{7}{4} \\ x_3 = 1 \end{cases}$$

回代得 $x_3 = 1, x_2 = 2, x_1 = 1$. (3′)

4. **解** (1) Gauss-Seidel 迭代格式为

$$\begin{cases} x_1^{(k+1)} = (15 - 3x_2^{(k)} + x_3^{(k)})/(-18) \\ x_2^{(k+1)} = (6 - 12x_1^{(k+1)} - 3x_3^{(k)})/(-3) \\ x_3^{(k+1)} = (-15 - x_1^{(k+1)} - 4x_2^{(k+1)})/10 \end{cases} \qquad (6')$$

(2) 迭代矩阵 $\boldsymbol{G}$ 的特征方程为

$$\begin{vmatrix} -18\lambda & 3 & -1 \\ 12\lambda & -3\lambda & 3 \\ \lambda & 4\lambda & 10\lambda \end{vmatrix} = 0 \qquad (3')$$

$$\lambda[-18(-30\lambda^2 - 12\lambda) - 12(30\lambda + 4\lambda) + 9 - 3\lambda] = 0$$

解得 $\lambda_1 = 0, \lambda_2 = 0.30678, \lambda_3 = 0.05433$.

$\therefore \rho(\boldsymbol{G}) = 0.30678 < 1$,故 Gauss-Seidel 迭代格式收敛. (3′)(1′)

$$f(x)-N_n(x)=\frac{f^{(n+1)}(\xi)}{(n+1)!}\prod_{i=0}^{n}(x-x_i)$$

$$=\frac{e^{\xi}}{(n+1)!}\prod_{i=0}^{n}(x-x_i),\qquad \xi\in(0,1) \tag{7'}$$

$x\in[0,1]$ 时

$$|f(x)-N_n(x)|\leqslant\frac{e}{(n+1)!} \tag{3'}$$

$$\lim_{n\to\infty}\max_{0\leqslant x\leqslant 1}|f(x)-N_n(x)|\leqslant\lim_{n\to\infty}\frac{e}{(n+1)!}=0 \tag{3'}$$

(1) 由题意知 $f(0)=0, f\left(\frac{\pi}{2}\right)=1.$

$f(x)$ 以 $x_0=0, x_1=\frac{\pi}{2}$ 为节点的 1 次插值多项式为

$$L_1(x)=f(x_0)\frac{x-x_1}{x_0-x_1}+f(x_1)\frac{x-x_0}{x_1-x_0}$$

$$=0\times\frac{x-\frac{\pi}{2}}{0-\frac{\pi}{2}}+1\times\frac{x-0}{\frac{\pi}{2}-0}=\frac{2}{\pi}x=0.63662x \tag{5'}$$

) 记 1 次最佳平方逼近多项式为 $p(x)=c_0+c_1x.$

$$\varphi_0(x)=1,\qquad \varphi_1(x)=x$$

$$(\varphi_0,\varphi_0)=\int_0^{\frac{\pi}{2}}1^2\mathrm{d}x=\frac{\pi}{2},\qquad (\varphi_0,\varphi_1)=\int_0^{\frac{\pi}{2}}x\mathrm{d}x=\frac{1}{2}x^2\Big|_0^{\frac{\pi}{2}}=\frac{1}{8}\pi^2$$

$$(\varphi_1,\varphi_1)=\int_0^{\frac{\pi}{2}}x^2\mathrm{d}x=\frac{\pi^3}{24}$$

$$(\varphi_0,f)=\int_0^{\frac{\pi}{2}}\sin x\mathrm{d}x=1,\qquad (\varphi_1,f)=\int_0^{\frac{\pi}{2}}x\sin x\mathrm{d}x=1$$

正规方程组为

$$\begin{bmatrix}\frac{\pi}{2} & \frac{1}{8}\pi^2\\ \frac{1}{8}\pi^2 & \frac{1}{24}\pi^3\end{bmatrix}\begin{bmatrix}c_0\\ c_1\end{bmatrix}=\begin{bmatrix}1\\ 1\end{bmatrix} \tag{5'}$$

解得 $c_0=\frac{8}{\pi}\left(1-\frac{3}{\pi}\right)=0.11477,\qquad c_1=\frac{96}{\pi^3}\left(1-\frac{1}{4}\pi\right)=0.66444$

$$\therefore p(x)=0.11477+0.66444x \tag{3'}$$

(1) $\qquad I(f)=\int_a^b f(x)\mathrm{d}x$

$$S(f)=\frac{b-a}{6}\left[f(a)+4f\left(\frac{b+a}{2}\right)+f(b)\right] \tag{4'}$$

当 $f(x)=1$ 时

$$S(f)=\frac{b-a}{6}(1+4\times 1+1)=b-a$$

$$I(f)=\int_a^b 1\mathrm{d}x=b-a \tag{1'}$$

$$S(f)=I(f)$$

当 $f(x)=x$ 时

$$S(f)=\frac{b-a}{6}\left(a+4\times\frac{b+a}{2}+b\right)=\frac{1}{2}(b^2-a^2)$$

$$I(f)=\int_a^b x\mathrm{d}x=\frac{1}{2}(b^2-a^2)$$

$$S(f)=I(f)$$

当 $f(x)=x^2$ 时 (1')

$$S(f)=\frac{b-a}{6}\left[a^2+4\times\left(\frac{b+a}{2}\right)^2+b^2\right]$$

$$=\frac{b-a}{6}[a^2+(a+b)^2+b^2]$$

$$=\frac{b-a}{3}(a^2+ab+b^2)=\frac{1}{3}(b^3-a^3)$$

$$I(f)=\int_a^b x^2\mathrm{d}x=\frac{1}{3}(b^3-a^3)$$

$$S(f)=I(f)$$

当 $f(x)=x^3$ 时 (1')

$$S(f)=\frac{b-a}{6}\left[a^3+4\times\left(\frac{a+b}{2}\right)^3+b^3\right]$$

$$=\frac{1}{4}(b^2-a^2)(b^2+a^2)$$

$$I(f)=\int_a^b x^3\mathrm{d}x=\frac{1}{4}(b^4-a^4)$$

$$S(f)=I(f)$$

当 $f(x)=x^4$ 时 (1')

$$S(f)=\frac{b-a}{6}\left[a^4+4\times\left(\frac{a+b}{2}\right)^4+b^4\right]$$

$$I(f)=\int_a^b x^4\mathrm{d}x=\frac{1}{5}(b^5-a^5)$$

$S(f)$ 的 b^5 的系数为$\frac{5}{24}$，而 $I(f)$ 的 b^5 的系数为$\frac{1}{5}$，

$$S(f) \neq I(f) \qquad (1')$$

$\therefore$ Simpson 公式具有 3 次代数精度.

(2) $\quad h = \dfrac{b-a}{n}, \qquad x_i = a + ih, \qquad x_{i+\frac{1}{2}} = \dfrac{1}{2}(x_i + x_{i+1})$

复化 Simpson 公式为

$$S_n(f) = \sum_{i=0}^{n-1} \frac{h}{6}[f(x_i) + 4f(x_{i+\frac{1}{2}}) + f(x_{i+1})] \qquad (4')$$

8. 解　局部截断误差为

$$R_{i+1} = y(x_{i+1}) - y(x_i) - \frac{h}{2}[f(x_i, y(x_i)) + f(x_{i+1}, y(x_i) + hf(x_i, y(x_i)))]$$

$$= y(x_{i+1}) - y(x_i) - \frac{h}{2}[y'(x_i) + f(x_{i+1}, y(x_i) + hy'(x_i))] \quad (3')$$

$$y''(x) = \frac{\partial f(x, y(x))}{\partial x} + y'(x)\frac{\partial f(x, y(x))}{\partial y}$$

$$y'''(x) = \frac{\partial^2 f(x, y(x))}{\partial x^2} + 2y'(x)\frac{\partial^2 f(x, y(x))}{\partial x \partial y} + [y'(x)]^2 \frac{\partial^2 f(x, y(x))}{\partial y^2} + y''(x)\frac{\partial f(x, y(x))}{\partial y}$$

方法 1:

$$R_{i+1} = y(x_i + h) - y(x_i) - \frac{h}{2}[y'(x_i) + f(x_i + h, y(x_i) + hy'(x_i))]$$

$$= y(x_i) + hy'(x_i) + \frac{h^2}{2}y''(x_i) + \frac{h^3}{6}y'''(x_i) + O(h^4) - y(x_i) - \frac{h}{2}y'(x_i) \qquad (2')$$

$$- \frac{h}{2}\Big[f(x_i, y(x_i)) + h\frac{\partial f(x_i, y(x_i))}{\partial x} + hy'(x_i)\frac{\partial f(x_i, y(x_i))}{\partial y} + \frac{1}{2}\Big(h^2\frac{\partial^2 f(x_i, y(x_i))}{\partial x^2} + 2h^2 y'(x_i)\frac{\partial^2 f(x_i, y(x_i))}{\partial x \partial y} + h^2[y'(x_i)]^2\frac{\partial^2 f(x_i, y(x_i))}{\partial y^2}\Big) + O(h^3)\Big] \qquad (3')$$

$$= h^3\Big[\frac{y'''(x_i)}{6} - \frac{1}{4}\Big(\frac{\partial^2 f(x_i, y(x_i))}{\partial x^2} + 2y'(x_i)\frac{\partial^2 f(x_i, y(x_i))}{\partial x \partial y} + [y'(x_i)]^2\frac{\partial^2 f(x_i, y(x_i))}{\partial y^2}\Big)\Big] + O(h^4)$$

$$= h^3\Big[\frac{1}{6}y'''(x_i) - \frac{1}{4}\Big(y'''(x_i) - y''(x_i)\frac{\partial f(x_i, y(x_i))}{\partial y}\Big)\Big] + O(h^4) \qquad (3')$$

$$= \Big[-\frac{1}{12}y'''(x_i) + \frac{1}{4}y''(x_i)\frac{\partial f(x_i, y(x_i))}{\partial y}\Big]h^3 + O(h^4)$$

方法 2:

$$R_{i+1} = y(x_{i+1}) - y(x_i) - \frac{h}{2}[y'(x_i) + f(x_{i+1}, y(x_{i+1}))] + \frac{h}{2}[f(x_{i+1}, y(x_{i+1})) - f(x_{i+1}, y(x_i) + hy'(x_i))]$$

$$= y(x_i + h) - y(x_i) - \frac{h}{2}[y'(x_i) + y'(x_{i+1})] + \frac{h}{2}\frac{\partial f(x_{i+1}, \eta_i)}{\partial y}[y(x_{i+1}) - y(x_i) - hy'(x_i)]$$

$$= y(x_i) + hy'(x_i) + \frac{h^2}{2}y''(x_i) + \frac{h^3}{6}y'''(\xi_i) - y(x_i) - \frac{h}{2}y'(x_i) - \frac{h}{2}\Big[y'(x_i) + hy''(x_i) + \frac{h^2}{2}y'''(\bar{\xi}_i)\Big] - \frac{h}{2}\frac{\partial f(x_{i+1}, \eta_i)}{\partial y}\Big[y(x_i) + hy'(x_i) + \frac{h^2}{2}y''(\tilde{\xi}_i) - y(x_i) - hy'(x_i)\Big]$$

$$= \frac{h^3}{6}y'''(\xi_i) - \frac{h^3}{4}y'''(\bar{\xi}_i) - \frac{h^3}{4}\frac{\partial f(x_{i+1}, \eta_i)}{\partial y}y''(\tilde{\xi}_i)$$

$$= O(h^3)$$

$\therefore$ 所给数值求解公式是 2 阶公式.

1999 年秋季攻读博士学位研究生入学考试

1. 解　$y_n = \displaystyle\int_0^1 \frac{x^n}{4x+1}dx = \frac{1}{4}\int_0^1 \frac{x^{n-1}(4x+1-1)}{4x+1}dx$

$$= \frac{1}{4}\int_0^1 x^{n-1}dx - \frac{1}{4}\int_0^1 \frac{x^{n-1}}{4x+1}dx = \frac{1}{4n} - \frac{1}{4}y_{n-1}, \qquad n = 1,2,3,\cdots$$

$$y_0 = \int_0^1 \frac{1}{4x+1}dx = \frac{1}{4}\ln(4x+1)\Big|_{x=0}^{1}$$

$$= \frac{1}{4}(\ln 5 - \ln 1) = \frac{1}{4}\ln 5$$

按如下递推可计算出 $y_n, n = 1,2,3,\cdots$.

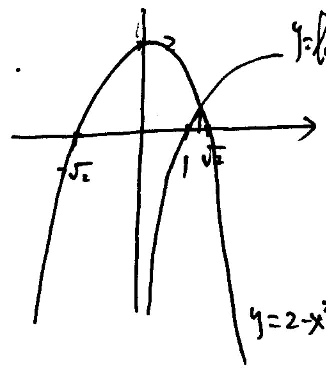
$$\begin{cases} y_n = \dfrac{1}{4n} - \dfrac{1}{4}y_{n-1}, & n = 1,2,3,\cdots \\ y_0 = \dfrac{1}{4}\ln 5 \end{cases}$$

①(5

若 y_0 有一个误差 ε, 则实际计算的值为

2004年工程硕士数值分析试卷

参考答案



1. 1) 法一：作图，$\ln x = 2 - x^2$. 作出 $y=\ln x$ 及 $y=2-x^2$ 的图像.
由图可知方程有唯一实根 $x^* \in [1, \sqrt{2}]$

法二：令 $f(x)=x^2+\ln x-2$ 则 $f(1)=1-2<0$，$f(\sqrt{2})=\frac{1}{2}\ln 2>0$.

$f'(x)=2x+\frac{1}{x}>0 \quad (x>0)$ 则 $f(x)\nearrow$.

∴ 原方程有唯一实根 $x^* \in [1, \sqrt{2}]$.

2) 法一. 构造迭代格式：$\begin{cases} x_{k+1}=\sqrt{2-\ln x_k}, & k=0,1,2\cdots \\ x_0=1.2 \end{cases}$

计算得：$x_1=1.34821$，$x_2=1.30431$，$x_3=1.31694$

$x_4=1.31327$，$x_5=1.31434$，$x_6=1.31403$

$|x_6-x_5|=0.31\times10^{-3}<0.5\times10^{-3} \Rightarrow x^*\approx 1.31403$.

法二. 用Newton迭代法：$\begin{cases} x_{k+1}=x_k-\dfrac{x_k^2+\ln x_k-2}{2x_k+\frac{1}{x_k}} \\ x_0=1.2 \end{cases}$

计算得. $x_1=1.31681$，$x_2=1.31410$，$x_3=1.31410$

$x^*\approx 1.3141$

2. $\begin{pmatrix} 6 & 7 & 9 & 4 \\ -8 & -6 & 3 & -7 \\ 9 & 6 & -15 & -8 \end{pmatrix} \xrightarrow{r_1\leftrightarrow r_3} \begin{pmatrix} 9 & 6 & -15 & -8 \\ -8 & -6 & 3 & -7 \\ 6 & 7 & 9 & 4 \end{pmatrix} \xrightarrow[-\frac{2}{3}r_1+r_3]{\frac{8}{9}r_1+r_2} \begin{pmatrix} 9 & 6 & -15 & -8 \\ 0 & -\frac{2}{3} & -\frac{31}{3} & -\frac{127}{9} \\ 0 & 3 & 19 & \frac{28}{3} \end{pmatrix}$

$\xrightarrow{r_2\leftrightarrow r_3} \begin{pmatrix} 9 & 6 & -15 & -8 \\ 0 & 3 & 19 & \frac{28}{3} \\ 0 & -\frac{2}{3} & -\frac{31}{3} & -\frac{127}{9} \end{pmatrix} \xrightarrow{\frac{2}{9}r_2+r_3} \begin{pmatrix} 9 & 6 & -15 & -8 \\ 0 & 3 & 19 & \frac{28}{3} \\ 0 & 0 & -\frac{55}{9} & -\frac{325}{27} \end{pmatrix} \Rightarrow$

$x_3=\dfrac{\frac{325}{27}}{\frac{55}{9}}=\dfrac{65}{33}=1.96$

$x_2=-9.3637$.

$x_1=8.6364$

3. 1) $\|x\|_1=7$, $\|x\|_2=\sqrt{1+4+16}=\sqrt{21}$, $\|A\|_\infty=19$

$$Ax=\begin{pmatrix}3\\-31\\-8\end{pmatrix} \quad \|Ax\|_\infty=31$$

2) 证明.(法一) 由 $\rho(A)\le\|A\|$, 得 $\rho(A)<1$

$\therefore$ 1不是A的特征值, 即有 $|1\cdot E-A|\ne 0 \Rightarrow$

$E-A$ 可逆, 故 $(E-A)x=b$ 有唯一解.

(法二) 反证. 假设 $(E-A)x=0$ 有非零解, 即

存在 $x_0\in R^n$, $x_0\ne 0$ 使 $(E-A)x_0=0$

$\Rightarrow x_0=Ax_0 \Rightarrow \|x_0\|=\|Ax_0\|\le\|A\|\,\|x_0\|$

由于 $x_0\ne 0\Rightarrow \|x_0\|\ne 0 \Rightarrow 1\le\|A\|$ 矛盾

$\therefore$ 齐次方程 $(E-A)x=0$ 只有零解

即原方程有唯一解.

4. $H''(x)$ 是1次多项式, 由条件 $H''(a)=f''(a)$, $H''(b)=f''(b)$ 得

$$H''(x)=f''(a)+\frac{f''(b)-f''(a)}{b-a}(x-a), \text{(线性插值)}$$

两边积分得:

$$H'(x)=f''(a)(x-a)+\frac{f''(b)-f''(a)}{2(b-a)}(x-a)^2+C_1$$

$$H(x)=\frac{f''(a)}{2}(x-a)^2+\frac{f''(b)-f''(a)}{6(b-a)}(x-a)^3+C_1(x-a)+C_2.$$

由条件 $H(a)=f(a)$ 得 $C_2=f(a)$

由 $H(b)=f(b)$ 得: $\frac{f''(a)}{2}(b-a)^2+\frac{f''(b)-f''(a)}{6}(b-a)^2+C_1(b-a)+f(a)=f(b)$

$$\Rightarrow C_1=\frac{f(b)-f(a)}{b-a}-\frac{(b-a)(f''(b)+2f''(a))}{6}$$

$$\Rightarrow H(x)=f(a)+\left[\frac{f(b)-f(a)}{b-a}-\frac{(b-a)(f''(b)+2f''(a))}{6}\right](x-a)+\frac{f''(a)}{2}(x-a)^2$$

$$+\frac{f''(b)-f''(a)}{6(b-a)}(x-a)^3.$$

5. $\varphi_0(x)=x,\quad \varphi_1(x)=x^2$

$(\varphi_0,\varphi_0)=\int_0^1 x\cdot x\,dx=\frac{1}{3}$, $(\varphi_0,\varphi_1)=\int_0^1 x\cdot x^2\,dx=\frac{1}{4}$

$(\varphi_1,\varphi_1)=\int_0^1 x^2\cdot x^2\,dx=\frac{1}{5}$ $(\varphi_0,f)=\int_0^1 xf(x)\,dx$, $(\varphi_1,f)=\int_0^1 x^2f(x)\,dx$.

$$\Rightarrow \begin{bmatrix}\frac{1}{3} & \frac{1}{4}\\ \frac{1}{4} & \frac{1}{5}\end{bmatrix}\begin{bmatrix}a\\ b\end{bmatrix}=\begin{bmatrix}(f,\varphi_0)\\ (f,\varphi_1)\end{bmatrix} \Rightarrow \begin{cases} a=48(f,\varphi_0)-60(f,\varphi_1)\\ b=80(f,\varphi_1)-60(f,\varphi_0)\end{cases}$$

其中 $(f,\varphi_0)=\int_0^1 x f(x)\,dx$, $(f,\varphi_1)=\int_0^1 x^2 f(x)\,dx$.

6. $S_n=\frac{4}{3}T_{2n}-\frac{1}{3}T_n$.

$T_1=\frac{1}{2}[e^0+e^1]=1.85914$, $T_1=\frac{1}{2}T_0+\frac{1}{2}e^{0.5}=1.753931$

$T_4=\frac{1}{2}T_2+\frac{1}{4}[e^{0.25}+e^{0.75}]=1.727222$

$S_1=\frac{4}{3}T_2-\frac{1}{3}T_1=1.718861$, $S_2=\frac{4}{3}T_4-\frac{1}{3}T_2=1.718319$

$\frac{1}{15}|S_2-S_1|=0.361\times10^{-4}<\frac{1}{2}\times10^{-4}$

$\therefore$ $\int_0^1 e^x\,dx\approx 1.718319$.

7. 1) 当 $f(x)=1$, 左 $=\int_{-1}^1 1\,dx=2$, 右 $=1+1=2$

$f(x)=x$, 左 $=\int_{-1}^1 x\,dx=0$ 右 $=-\frac{1}{\sqrt3}+\frac{1}{\sqrt3}=0$

$f(x)=x^2$, 左 $=\int_{-1}^1 x^2\,dx=\frac{2}{3}$, 右 $=(-\frac{1}{\sqrt3})^2+(\frac{1}{\sqrt3})^2=\frac{2}{3}$

$f(x)=x^3$, 左 $=\int_{-1}^1 x^3\,dx=0$, 右 $=(-\frac{1}{\sqrt3})^3+(\frac{1}{\sqrt3})^3=0$.

$f(x)=x^4$, 左 $=\int_{-1}^1 x^4\,dx=\frac{2}{5}$, 右 $=(\frac{-1}{\sqrt3})^4+(\frac{1}{\sqrt3})^4=\frac{2}{9}$, 左 $\neq$ 右.

$\therefore$ 该求积公式代数精度为3.

8. ① (*)的局部截断误差是

$$R^{(1)}_{i+1} = y(x_{i+1}) - y(x_i) - \frac{h}{2}\left[f(x_i, y(x_i)) + f(x_{i+1}, y(x_i) + hf(x_i, y(x_i)))\right]$$

$$= y(x_{i+1}) - y(x_i) - \frac{h}{2}y'(x_i) - \frac{h}{2}f(x_{i+1}, y(x_i) + hy'(x_i))$$

$$= y(x_i) + hy'(x_i) + \frac{h^2}{2}y''(x_i) + \frac{h^3}{3!}y'''(x_i) + o(h^4)$$

$$- y(x_i) - \frac{h}{2}y'(x_i)$$

$$- \frac{h}{2}\Big[f(x_i, y(x_i)) + \frac{\partial f}{\partial x}h + \frac{\partial f}{\partial y}hy'(x_i) + \frac{1}{2}\Big(\frac{\partial^2 f}{\partial x^2}h^2 + 2\frac{\partial^2 f}{\partial x \partial y}h^2 y'(x_i) + \frac{\partial^2 f}{\partial y^2}h^2 (y'(x_i))^2\Big)$$

$$+ o(h^3)\Big]$$

$$= \frac{h}{2}y'(x_i) + \frac{h^2}{2}y''(x_i) + \frac{h^3}{3!}y'''(x_i) + o(h^4)$$

$$- \frac{h}{2}\Big[y'(x_i) + hy''(x_i) + \frac{h^2}{2}\Big(y'''(x_i) - \frac{\partial f(x_i, y(x_i))}{\partial y}y''(x_i)\Big) + o(h^3)\Big]$$

$$= \Big[-\frac{1}{12}y'''(x_i) + \frac{1}{4}\frac{\partial f(x_i, y(x_i))}{\partial y}y''(x_i)\Big]h^3 + o(h^4).$$

∴公式(*)是2阶公式.

② 公式(**)的局部截断误差为:

$$R^{(2)}_{i+1} = y(x_{i+1}) - y(x_i) - \frac{h}{2}\left[3f(x_i, y(x_i)) - f(x_{i-1}, y(x_{i-1}))\right]$$

$$= y(x_{i+1}) - y(x_i) - \frac{3h}{2}y'(x_i) + \frac{h}{2}y'(x_{i-1})$$

$$= y(x_i) + hy'(x_i) + \frac{h^2}{2}y''(x_i) + \frac{h^3}{3!}y'''(x_i) + o(h^4)$$

$$- y(x_i) - \frac{3h}{2}y'(x_i)$$

$$+ \frac{h}{2}\Big[y'(x_i) - hy''(x_i) + \frac{h^2}{2}y'''(x_i) + o(h^3)\Big]$$

$$= \frac{5}{12}h^3 y'''(x_i) + o(h^4).$$

∴公式(**)是2阶公式

注: 公式(**)有错, 应改为:

$$y_{i+1} = y_i + \frac{h}{2}\left[3f(x_i, y_i) - f(x_{i-1}, y_{i-1})\right].$$

2005年工程硕士数值分析试卷答案

1. (1) $|e(x)|\le\frac{1}{2}\times10^{-4}$, $|e(y)|\le\frac{1}{2}\times10^{-6}$

$$|e(x+y)|=|e(x)+e(y)|\le|e(x)|+|e(y)|=\frac{1}{2}\times10^{-4}+\frac{1}{2}\times10^{-6}=0.505\times10^{-4}$$
$$<0.5\times10^{-3}$$

$x+y=1.4684+0.047154=1.515554$, $\Rightarrow$ $x+y$ 具有4位有效数

$x^2y=1.4684^2\times0.047154=0.101673386$

$$|e(x^2y)|\approx|2xye(x)+x^2e(y)|\le2xy|e(x)|+x^2|e(y)|$$
$$=2\times1.4684\times0.047154\times\frac{1}{2}\times10^{-4}+1.4684^2\times\frac{1}{2}\times10^{-6}$$
$$=0.9076\times10^{-5}<0.5\times10^{-4}$$ $\Rightarrow$ x^2y 具有4位有效数.

(2). Newton迭代格式为:

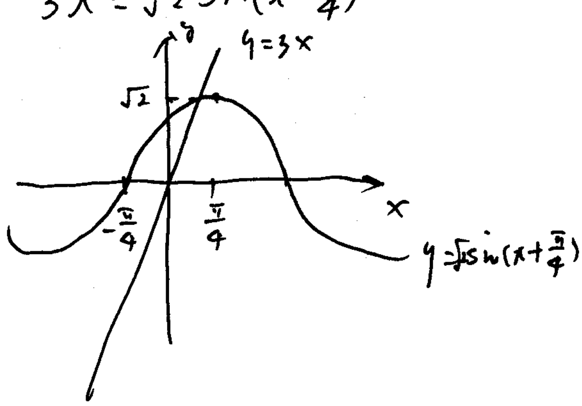
$$x_{k+1}=x_k-\frac{5x_k^4+6x_k^3-4x_k^2-3x_k+1}{20x_k^3+18x_k^2-8x_k-3}=\frac{15x_k^4+12x_k^3-4x_k^2-1}{20x_k^3+18x_k^2-8x_k-3}$$

$$=\frac{((15x_k+12)x_k-4)x_k^2-1}{((20x_k+18)x_k-8)x_k-3},\quad k=0,1,2,\cdots$$

2. (1) 法一. $3x-\sqrt{2}(\sin x\cdot\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}\cos x)=0$ 即 $3x=\sqrt{2}\sin(x+\frac{\pi}{4})$

作图. $y=3x$ 及 $y=\sqrt{2}\sin(x+\frac{\pi}{4})$.

由图可知方程有唯一实根 $x^*\in(0,\frac{\pi}{4})$.



法二. 令 $f(x)=3x-\sin x-\cos x$

则 $f(0)=-1<0$

$f(1)=3-\sin1-\cos1>0$

而 $f'(x)=3-\cos x+\sin x>0$ $x\in(-\infty,+\infty)$.

即 $f(x)\uparrow$. $\therefore$ $f(x)=0$ 有唯一实根 $x^*\in(0,1)$.

(2) 构造迭代格式

$$\begin{cases}x_{k+1}=\frac{1}{3}(\sin x_k+\cos x_k),\ k=0,1,2\cdots\\ x_0=0.5\end{cases}$$

计算得. $x_1=0.452336$, $x_2=0.445499$, $x_3=0.444435$

$x_4=0.444267$, $x_5=0.444241$

$|x_5-x_4|=0.26\times10^{-4}<\frac{1}{2}\times10^{-4}$

$\Rightarrow\ x^*\approx0.444241$

3). 迭代函数 $\varphi(x)=\frac{1}{3}(\sin x+\cos x)$.

① 当 $x\in[0,1]$ 时, $\varphi(x)\le\frac{2}{3}<1$

$\varphi(x)>0$

$\Rightarrow\ \varphi(x)\in[0,1]$.

② $|\varphi'(x)|=\frac{1}{3}|\cos x-\sin x|\le\frac{2}{3}<1$, $\forall x\in[0,1]$.

$\Rightarrow$ 由Th.2.1, 对 $\forall x_0\in[0,1]$, 迭代格式

$x_{k+1}=\frac{1}{3}(\sin x_k+\cos x_k)$, $k=0,1,2,\cdots$

收敛.

3. ~~解~~

$$\begin{bmatrix}\frac15&\frac14&\frac13&\frac{8}{15}\\ \frac14&\frac13&\frac12&\frac34\\ \frac13&\frac12&1&\frac43\end{bmatrix}\xrightarrow{r_1\leftrightarrow r_3}\begin{bmatrix}\frac13&\frac12&1&\frac43\\ \frac14&\frac13&\frac12&\frac34\\ \frac15&\frac14&\frac13&\frac{8}{15}\end{bmatrix}\xrightarrow[-\frac35 r_1+r_3]{-\frac34 r_1+r_2}\begin{bmatrix}\frac13&\frac12&1&\frac43\\ 0&-\frac{1}{24}&-\frac14&-\frac14\\ 0&-\frac{1}{20}&-\frac{4}{15}&-\frac{4}{15}\end{bmatrix}$$

$$\xrightarrow{r_2\leftrightarrow r_3}\begin{bmatrix}\frac13&\frac12&1&\frac43\\ 0&-\frac{1}{20}&-\frac{4}{15}&-\frac{4}{15}\\ 0&-\frac{1}{24}&-\frac14&-\frac14\end{bmatrix}\xrightarrow{-\frac56 r_2+r_3}\begin{bmatrix}\frac13&\frac12&1&\frac43\\ 0&-\frac{1}{20}&-\frac{4}{15}&-\frac{4}{15}\\ 0&0&-\frac{1}{12}&-\frac{1}{12}\end{bmatrix}\Rightarrow\ \begin{matrix}x_3=1,\\ x_2=0\\ x_1=1.\end{matrix}$$

4. ~~解~~ Gauss-Seidel 迭代格式:

$$\begin{cases}x_1^{(k+1)}=(x_2^{(k)}+2)/4\\ x_2^{(k+1)}=(x_1^{(k+1)}-x_3^{(k)}+6)/2\\ x_3^{(k+1)}=(-x_2^{(k+1)}+2)/4\end{cases}$$

$k=0,1,2\cdots$

迭代矩阵 G 的特征方程为:

$$\begin{vmatrix}4\lambda&-1&0\\ -\lambda&2\lambda&1\\ 0&\lambda&4\lambda\end{vmatrix}=0$$

$\Rightarrow\ \lambda^2(4\lambda-1)=0$

$\lambda_1=\lambda_2=0$, $\lambda_3=\frac14$.

$\Rightarrow\ \rho(G)<1$. 迭代收敛.

5. 作 $S(x)$ 的二次插值多项式 $N_2(x)$：

$$N_2(x)=S(0.2)+S[0.2,0.4](x-0.2)+S[0.2,0.4,0.6](x-0.2)(x-0.4)$$

计算差商

x_k	$S(x_k)$		
0.2	0.19956	0.983	−0.057875
0.4	0.39616	0.95985	
0.6	0.58813		

故 $N_2(x)=0.19956+0.983(x-0.2)-0.057875(x-0.2)(x-0.4)$

令 $N_2(x)=0.44$ 时可得方程：

$$0.19956+0.983(x-0.2)-0.057875(x-0.2)(x-0.4)=0.44$$

可解得 $x=0.44525$.

6. $I(f)-T(f)=\dfrac{-(b-a)^3}{12}f''(\xi),\quad \xi\in[a,b]$

$I(f)-Q(f)=\dfrac{(b-a)^3}{24}f''(\eta),\quad \eta\in[a,b]$.

当 $b-a$ 很小时，$I(f)-Q(f)\approx-\frac{1}{2}(I(f)-T(f))$

即：$3I(f)\approx 2Q(f)+T(f)$

$\Rightarrow\quad I(f)\approx\frac{2}{3}Q(f)+\frac{1}{3}T(f)$

∴ 构造求积公式

$$R(f)=\frac{2}{3}Q(f)+\frac{1}{3}T(f)=\frac{b-a}{6}[f(a)+f(b)]+\frac{2(b-a)}{3}f\left(\frac{a+b}{2}\right)$$

$$=\frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]$$

∴ $R(f)$ 就是 simpson 公式，显然其精度比 $T(f)$ 和 $Q(f)$ 高

$$I(f)-R(f)=-\frac{(b-a)}{180}\left(\frac{b-a}{2}\right)^4f^{(4)}(\xi),\quad \xi\in[a,b].$$

7. 局部截断误差为:

$$R_{i+1}=y(x_{i+1})-A(y(x_i)+y(x_{i-1}))-h[Bf(x_i,y(x_i))+Cf(x_{i-1},y_{i-1})]$$

$$=y(x_{i+1})-Ay(x_i)-Ay(x_{i-1})-hBy'(x_i)-Chy'(x_{i-1})$$

$$=y(x_i)+hy'(x_i)+\frac{h^2}{2}y''(x_i)+\frac{h^3}{3!}y'''(x_i)+O(h^4)$$

$$-Ay(x_i)$$

$$-A\left[y(x_i)-hy'(x_i)+\frac{h^2}{2}y''(x_i)-\frac{h^3}{3!}y'''(x_i)+O(h^4)\right]$$

$$-Bhy'(x_i)$$

$$-Ch\left[y'(x_i)-hy''(x_i)+\frac{h^2}{2}y'''(x_i)+O(h^3)\right]$$

$$=(1-2A)y(x_i)+(1+A-B-C)hy'(x_i)+\left(\frac{1}{2}-\frac{A}{2}+C\right)h^2y''(x_i)$$

$$+\left(\frac{1}{6}+\frac{A}{6}-\frac{C}{2}\right)h^3y'''(x_i)+O(h^4)$$

要使所求公式精度尽量高，则令

$$\begin{cases}1-2A=0\\1+A-B-C=0\\\frac{1}{2}-\frac{A}{2}+C=0\end{cases}\quad \text{求得}\quad \begin{array}{l}A=\frac{1}{2}\\B=\frac{7}{4}\\C=-\frac{1}{4}\end{array}$$

此时 $R_{i+1}=\frac{9}{24}h^3y'''(x_i)+O(h^4)$

∴ 阶数为2阶.

2005年10月参考答案

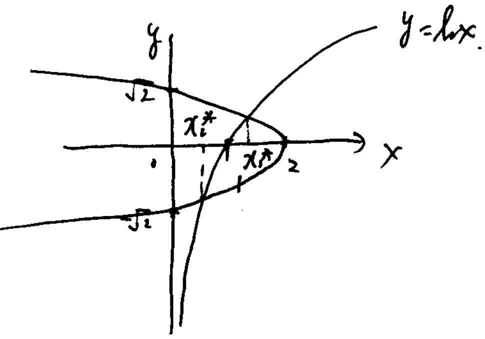
1\. 1) $\|x\|_\infty = 2$, $\|A\|_\infty = 17$, $\|Ax\|_\infty \le \|A\|_\infty \|x\|_\infty = 34$

2) $\text{cond}(A)_2 = \sqrt{\dfrac{\lambda_{max}(A^TA)}{\lambda_{min}(A^TA)}}$, $A^TA = \begin{bmatrix} 4 & 8 \\ 6 & 9 \end{bmatrix}\begin{bmatrix} 4 & 6 \\ 8 & 9 \end{bmatrix} = \begin{bmatrix} 80 & 96 \\ 96 & 117 \end{bmatrix}$

$|\lambda I - A^TA| = \begin{vmatrix} \lambda-80 & -96 \\ -96 & \lambda-117 \end{vmatrix} = 0 \Rightarrow \lambda^2 - 197\lambda + 44 = 0$. $\lambda_{1,2} = \dfrac{197 \pm \sqrt{197^2 - 4\times 44}}{2} = \dfrac{197 \pm \sqrt{38633}}{2}$

$\therefore \text{cond}(A)_2 = \sqrt{\dfrac{197+\sqrt{38633}}{197-\sqrt{38633}}} = \dfrac{197+\sqrt{38633}}{197^2-38633} = \dfrac{197+\sqrt{38633}}{176} = 2.2361$

2\. 1) 法一: $\ln x = \pm\sqrt{2-x}$, 作函数 $y=\ln x$ 及 $y=\pm\sqrt{2-x}$ 的图象



由图象知方程有两个实根 $x_1^* \in [1,2]$, $x_2^* \in [0,1]$.

(也可以直接作函数 $y=\ln^2 x$ 及 $y=2-x$ 的图象).

法二: 令 $f(x) = x + \ln^2 x - 2$, $f'(x) = 1 + \dfrac{2\ln x}{x}$.

令 $f'(x) = 1 + \dfrac{2\ln x}{x} = 0$. 得 $x_0 \in (0,1)$ 使 $f'(x_0) = 1 + \dfrac{2\ln x_0}{x_0} = 0$.

$f''(x_0) = \dfrac{2}{x_0^2} - \dfrac{2\ln x_0}{x_0^2} = 2\dfrac{1}{x_0^2}(1-\ln x_0) > 0$, $x_0 \in (0,1)$.

$\therefore$ $f(x)$ 在 x_0 处取极小值. 极小值为 $f(x_0) = x_0^2 + \ln^2 x_0 - 2$

$= x_0^2 + \dfrac{1}{4}x_0^2 - 2 < 0$.

当 $x \to 0^+$ 时 $f(x) \to +\infty$.
$x \to +\infty$ 时 $f(x) \to +\infty$. $\therefore$ $f(x)=0$ 在 $(0,+\infty)$ 上有两个实根.

$f(1) = 1-2 < 0$, $f(2) = 2 + \ln^2 2 - 2 > 0$, $f(0.5) = 1.98 > 0$.

$\therefore$ 两个根 $x_1^* \in [1,2]$, $x_2^* \in [0.5,1]$.

2) Newton 格式 $x_{k+1} = x_k - \dfrac{x_k + \ln^2 x_k - 2}{1 + \dfrac{2\ln x_k}{x_k}} = \dfrac{2\ln x_k - \ln^2 x_k + 2}{1 + \dfrac{2\ln x_k}{x_k}}$, $k = 0,1,\cdots$

取 $x_0 = 1.5$, $x_1 = 1.7178$, $x_2 = 1.7113$, $x_3 = 1.7113$. $x_1^* \approx 1.711$.

取 $x_0 = 0.3$, $x_1 = 0.26436$, $x_2 = 0.26816$, $x_3 = 0.26821$ $|x_3 - x_2| = 0.5\times 10^{-4}$.

$x_2^* \approx 0.26820$.

3. $\begin{bmatrix} 3 & 1 & -1 & 4 \\ 3 & 0 & 3 & 6 \\ 12 & -3 & 3 & 9 \end{bmatrix} \xrightarrow{r_1 \leftrightarrow r_3} \begin{bmatrix} 12 & -3 & 3 & 9 \\ 3 & 0 & 3 & 6 \\ 3 & 1 & -1 & 4 \end{bmatrix} \xrightarrow[-\frac{1}{4}r_1+r_3]{-\frac{1}{4}r_1+r_2} \begin{bmatrix} 12 & -3 & 3 & 9 \\ 0 & \frac{3}{4} & \frac{9}{4} & \frac{15}{4} \\ 0 & \frac{7}{4} & -\frac{7}{4} & \frac{7}{4} \end{bmatrix} \xrightarrow{r_2 \leftrightarrow r_3} \begin{bmatrix} 12 & -3 & 3 & 9 \\ 0 & \frac{7}{4} & -\frac{7}{4} & \frac{7}{4} \\ 0 & \frac{3}{4} & \frac{9}{4} & \frac{15}{4} \end{bmatrix}$

$\xrightarrow{-\frac{3}{7}r_2+r_3} \begin{bmatrix} 12 & -3 & 3 & 9 \\ 0 & \frac{7}{4} & -\frac{7}{4} & \frac{7}{4} \\ 0 & 0 & 3 & 3 \end{bmatrix} \Rightarrow$ $3x_3=3,\ x_3=1,\quad \frac{7}{4}x_2-\frac{7}{4}=\frac{7}{4},\ x_2=2.$

$12x_1-3\times 2+3\times 1=9 \quad x_1=1.$ 解：$(1,2,1)$.

4. 1) Gauss—seidel 迭代格式

$$\begin{cases} x_1^{(k+1)} = (x_2^{(k)}+2)/4 \\ x_2^{(k+1)} = (x_1^{(k+1)}-x_3^{(k)}+6)/\alpha, \quad k=0,1,2,\cdots \\ x_3^{(k+1)} = (-x_2^{(k+1)}+2)/4 \end{cases}$$

2) 特征方程：

$$\begin{vmatrix} 4\lambda & -1 & 0 \\ -\lambda & \alpha\lambda & 1 \\ 0 & \lambda & 4\lambda \end{vmatrix} = 0 \Rightarrow$$

$8\lambda^2(2\alpha\lambda-1)=0,\quad \lambda_{1,2}=0,\ \lambda_3=\frac{1}{2\alpha}.$

$\therefore \rho(G)=\frac{1}{2|\alpha|}.$

要使 Gauss—seidel 迭代收敛，则 $\frac{1}{2|\alpha|}<1$ 即 $|\alpha|>\frac{1}{2}$.

3) 要使 Gauss—seidel ~~收敛~~ 收敛越快，则要求 $\rho(G)$ 越小，

$\therefore$ 当 $|\alpha|$ 越大时，Gauss—seidel 迭代越快。

5. 由题意知，曲线 $p(x)$ 经过 B、C 两点，且在 B 点与 AB 相切，在 C 点与 CD 相切。

AB 斜率 $=0$，CD 斜率 $=1 \Rightarrow$

$p(1)=0,\ p(3)=0$

$p'(1)=0,\ p'(3)=1.$

$p(x)=f(1)+f[1,1](x-1)+f[1,1,3](x-1)^2+f[1,1,3,3](x-1)^2(x-3).$

x_k	$f(x_k)$			
1	0	0	0	$\frac{1}{4}$
1	0	0	$\frac{1}{2}$	
3	0	1		
3	0			

$\Rightarrow p(x)=\frac{1}{4}(x-1)^2(x-3).$

6. 设 $p_2(x)=c_0+c_1x+c_2x^2$, 则 $\varphi_0=1,\ \varphi_1=x,\ \varphi_2=x^2$.

$(\varphi_0,\varphi_0)=\int_0^\pi 1\cdot 1dx=\pi$, $(\varphi_0,\varphi_1)=\int_0^\pi xdx=\frac{1}{2}\pi^2$, $(\varphi_0,\varphi_2)=\int_0^1 x^2dx=\frac{1}{3}\pi^3$;

$(\varphi_1,\varphi_1)=\int_0^\pi x^2dx=\frac{1}{3}\pi^3$, $(\varphi_1,\varphi_2)=\int_0^\pi x^3dx=\frac{1}{4}\pi^4$, $(\varphi_2,\varphi_2)=\int_0^1 x^4dx=\frac{1}{5}\pi^5$.

$(f,\varphi_0)=\int_0^\pi \sin x dx=2$, $(f,\varphi_1)=\int_0^\pi x\sin xdx=\pi$, $(f,\varphi_2)=\int_0^1 x^2\sin xdx=\pi^2+4$

$$\begin{bmatrix}\pi & \frac{1}{2}\pi^2 & \frac{1}{3}\pi^3\\ \frac{1}{2}\pi^2 & \frac{1}{3}\pi^3 & \frac{1}{4}\pi^4\\ \frac{1}{3}\pi^3 & \frac{1}{4}\pi^4 & \frac{1}{5}\pi^5\end{bmatrix}\begin{bmatrix}c_0\\ c_1\\ c_2\end{bmatrix}=\begin{bmatrix}2\\ \pi\\ \pi^2+4\end{bmatrix}.$$

$$\begin{cases}C_0=12(\pi^2+10)/\pi^3.\\ C_1=-60(\pi^2+12)/\pi^4\\ C_2=\dfrac{60(\pi^2+12)}{\pi^5}\end{cases}$$

$C_0\approx 7.6899$, $c_1\approx -13.47086$, $c_2=4.287908$.

7. $S_1(f)=\frac{1}{6}[e^0+4e^{0.5}+e^1]=1.71886$

$S_2(f)=\frac{1}{12}[e^0+4e^{0.25}+2e^{0.5}+4e^{0.75}+e^1]=1.71839$

$\frac{1}{15}|S_2-S_1|=0.0000314<0.5\times10^{-4}$

$\therefore\ I(f)\approx 1.71839.$

$<\cancel{0.000\ldots}$

8. 局部截断误差

$R_{i+1}=y(x_{i+1})-\alpha y(x_{i-1})-h[\beta_0f(x_{i+1},y(x_{i+1}))+\beta_1f(x_i,y(x_i))+\beta_2f(x_{i-1},y(x_{i-1}))]$ $<\cancel{0.0005}$ $=\cancel{\frac{1}{2}\times10^{-3}}$

$=y(x_{i+1})-\alpha y(x_{i-1})-\beta_0hy'(x_{i+1})-\beta_1hy'(x_i)-\beta_2hy'(x_{i-1})$

Taylor展开

$=y(x_i)+hy'(x_i)+\frac{h^2}{2}y''(x_i)+\frac{h^3}{3!}y'''(x_i)+\frac{h^4}{4!}y^{(4)}(x_i)+\frac{h^5}{5!}y^{(5)}(x_i)+O(h^6)$

$-\alpha[y(x_i)-hy'(x_i)+\frac{h^2}{2}y''(x_i)-\frac{h^3}{3!}y'''(x_i)+\frac{h^4}{4!}y^{(4)}(x_i)-\frac{h^5}{5!}y^{(5)}(x_i)+O(h^6)]$

$-\beta_0h[y'(x_i)+hy''(x_i)+\frac{h^2}{2}y'''(x_i)+\frac{h^3}{3!}y^{(4)}(x_i)+\frac{h^4}{4!}y^{(5)}(x_i)+O(h^5)]$

$-\beta_1hy'(x_i)$

$-\beta_2h[y'(x_i)-hy''(x_i)+\frac{h^2}{2}y'''(x_i)-\frac{h^3}{3!}y^{(4)}(x_i)+\frac{h^4}{4!}y^{(5)}(x_i)+O(h^5)]$

$=(1-\alpha)y(x_i)+(1+\alpha-\beta_0-\beta_1-\beta_2)hy'(x_i)+(\frac{1}{2}-\frac{\alpha}{2}-\beta_0+\beta_2)h^2y''(x_i)$

$+(\frac{1}{6}+\frac{\alpha}{6}-\frac{\beta_0}{2}-\frac{\beta_2}{2})h^3y'''(x_i)+(\frac{1}{24}-\frac{\alpha}{24}-\frac{\beta_0}{6}+\frac{\beta_2}{6})h^4y^{(4)}(x_i)$

$+(\frac{1}{120}+\frac{\alpha}{120}-\frac{\beta_0}{24}-\frac{\beta_2}{24})h^5y^{(5)}(x_i)+O(h^6).$

要使其精度尽可能高, 则 $\begin{cases}1-\alpha=0\\ 1+\alpha-\beta_0-\beta_1-\beta_2=0\\ \frac{1}{2}-\frac{\alpha}{2}-\beta_0+\beta_2=0\\ \frac{1}{6}+\frac{\alpha}{6}-\frac{\beta_0}{2}-\frac{\beta_2}{2}=0\end{cases}$ 解得: $\begin{cases}\alpha=1\\ \beta_0=\frac{1}{3}\\ \beta_1=\frac{4}{3}\\ \beta_2=\frac{1}{3}\end{cases}$

此时 $R_{i+1}=-\frac{1}{90}h^5y^{(5)}(x_i)$.

$\therefore$ 该公式是4阶公式.

2006年10月工程硕士学位考试参考答案

1. (1) $|e(x_1)| \le \frac{1}{2}\times 10^{-4}$, $|e(x_2)| \le \frac{1}{2}\times 10^{-3}$

$$
\begin{aligned}
|e(x_1x_2)| &\approx |x_1 e(x_2) + x_2 e(x_1)| \\
&\le x_1|e(x_2)| + x_2|e(x_1)| \\
&\le 0.80587\times 10^{-3} < \frac{1}{2}\times 10^{-2}
\end{aligned}
$$

$x_1\cdot x_2 = 649.4045545$

故 x_1x_2 具有5位有效数字.

(2)

	3	4	-2	3	1
$x=2$		6	20	36	78
	3	10	18	39	79

$p(2)=79$

2. (1) 迭代函数 $\varphi(x)=\sqrt{2+x}$

① 当 $x\in[0,4]$ 时, $0<\sqrt{2}\le\varphi(x)\le\sqrt{6}<4$

② $|\varphi'(x)| = \frac{1}{2\sqrt{2+x}} \le \frac{1}{2\sqrt{2}} < 1$, $\forall x\in[0,4]$

∴ 迭代对 $\forall x_0\in[0,4]$ 均收敛于同一个数.

(2) 对 $\forall x_0\in[0,\infty)$, 不妨设 $x_0>2$.

当 $x\in[0,x_0]$ 时, $0<\sqrt{2}<\varphi(x)\le\sqrt{2+x_0}<x_0$

$|\varphi'(x)| = \frac{1}{2\sqrt{2+x}} \le \frac{1}{2\sqrt{2}} < 1$, $x\in[0,x_0]$

因此对 $\forall x_0\in[0,\infty)$, 迭代均收敛.

3.
$$
\begin{bmatrix} 4 & 8 & 8 & 3 \\ 10 & -4 & 6 & 20 \\ 4 & 18 & 6 & -3 \end{bmatrix}
\xrightarrow{r_1\leftrightarrow r_2}
\begin{bmatrix} 10 & -4 & 6 & 20 \\ 4 & 8 & 8 & 3 \\ 4 & 18 & 6 & -3 \end{bmatrix}
$$

$$
\xrightarrow[-\frac{2}{5}r_1+r_3]{-\frac{2}{5}r_1+r_2}
\begin{bmatrix} 10 & -4 & 6 & 20 \\ 0 & \frac{48}{5} & \frac{28}{5} & -5 \\ 0 & \frac{98}{5} & \frac{18}{5} & -11 \end{bmatrix}
$$

$$
\xrightarrow{r_2\leftrightarrow r_3}
\begin{bmatrix} 10 & -4 & 6 & 20 \\ 0 & \frac{98}{5} & \frac{18}{5} & -11 \\ 0 & \frac{48}{5} & \frac{28}{5} & -5 \end{bmatrix}
$$

$$
\xrightarrow{-\frac{48}{98}r_2+r_3}
\begin{bmatrix} 10 & -4 & 6 & 20 \\ 0 & \frac{98}{5} & \frac{18}{5} & -11 \\ 0 & 0 & \frac{188}{49} & \frac{19}{49} \end{bmatrix}
$$

$\Rightarrow$ $x_1 = 1.7074$

$x_2 = -0.5798$

$x_3 = 0.1011$.

4. (1) Gauss-Seidel迭代格式

$$
\begin{cases}
x_1^{(k+1)} = (-4x_2^{(k)} - 3x_3^{(k)} + 31)/2 \\
x_2^{(k+1)} = (x_1^{(k+1)} + 6x_3^{(k)} + 1)/5 \\
x_3^{(k+1)} = (-4x_1^{(k+1)} - 7x_2^{(k+1)} - 6)/3,
\end{cases}
\quad k=0,1,\cdots
$$

(2) 特征方程

$$
\begin{vmatrix} 2\lambda & 4 & +3 \\ \lambda & -5\lambda & 6 \\ 4\lambda & 7\lambda & 3\lambda \end{vmatrix} = 0
$$

$\Rightarrow \lambda(10\lambda^2+5\lambda-32)=0$

$\lambda_1=0$, $\lambda_{2,3} = \frac{-5\pm\sqrt{25+40\times 32}}{20} = \frac{-5\pm 36.1247}{20}$

$\Rightarrow \rho(G)>1$, 迭代发散.

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5. $H(x)=f(a)+f[a,\frac{a+b}{2}](x-a)+f[a,\frac{a+b}{2},\frac{a+b}{2}](x-a)(x-\frac{a+b}{2})$

$\quad+f[a,\frac{a+b}{2},\frac{a+b}{2},b](x-a)(x-\frac{a+b}{2})^2$

$f[a,\frac{a+b}{2}]=\dfrac{f(\frac{a+b}{2})-f(a)}{(b-a)/2}$, $\quad f[\frac{a+b}{2},\frac{a+b}{2}]=f'(\frac{a+b}{2})$

$f[\frac{a+b}{2},b]=\dfrac{f(b)-f(\frac{a+b}{2})}{(b-a)/2}$

$f[a,\frac{a+b}{2},\frac{a+b}{2}]=\dfrac{\frac{b-a}{2}f'(\frac{a+b}{2})-f(\frac{a+b}{2})+f(a)}{(\frac{b-a}{2})^2}$

$f[\frac{a+b}{2},\frac{a+b}{2},b]=\dfrac{f(b)-f(\frac{a+b}{2})-\frac{b-a}{2}f'(\frac{a+b}{2})}{(\frac{b-a}{2})^2}$

$f[a,\frac{a+b}{2},\frac{a+b}{2},b]=\dfrac{f(b)-f(a)-(b-a)f'(\frac{a+b}{2})}{\frac{(b-a)^3}{4}}$

$\therefore H(x)=f(a)+\frac{2}{b-a}(f(\frac{a+b}{2})-f(a))(x-a)$

$\quad+\frac{4}{(b-a)^2}\left[\frac{b-a}{2}f'(\frac{a+b}{2})-f(\frac{a+b}{2})+f(a)\right](x-a)(x-\frac{a+b}{2})$

$\quad+\frac{4}{(b-a)^3}\left[f(b)-f(a)-(b-a)f'(\frac{a+b}{2})\right](x-a)(x-\frac{a+b}{2})^2.$

6. 即求 $f(x)$ 在 $[0,1]$ 上的最佳平方逼近1次多项式 $a+bx$.

记 $\varphi_0(x)=1$, $\varphi_1(x)=x$.

$(\varphi_0,\varphi_0)=\int_0^1 1^2dx=1$, $\quad(\varphi_0,\varphi_1)=\int_0^1 x\,dx=\frac{1}{2}$, $\quad(\varphi_1,\varphi_1)=\int_0^1 x^2dx=\frac{1}{3}$

$(\varphi_0,f)=\int_0^1 f(x)dx$, $\quad(\varphi_1,f)=\int_0^1 xf(x)dx$

正规方程为:
$$\begin{bmatrix}1 & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{3}\end{bmatrix}\begin{bmatrix}a\\ b\end{bmatrix}=\begin{bmatrix}\int_0^1 f(x)dx\\ \int_0^1 xf(x)dx\end{bmatrix}$$

求得 $a=2\int_0^1(2-3x)f(x)dx$, $\quad b=6\int_0^1(2x-1)f(x)dx$.

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7. (1) $\int_{-1}^{1} f(x)dx \approx f(-\frac{1}{\sqrt{3}}) + f(\frac{1}{\sqrt{3}})$.

(2) 令 $x = \frac{1}{2} + \frac{1}{2}t \Rightarrow$

$$\int_0^1 e^{-x}dx = \frac{1}{2}\int_{-1}^{1} e^{-\frac{1}{2}(1+t)}dt \approx \frac{1}{2}\left[e^{-\frac{1}{2}(1-\frac{1}{\sqrt{3}})} + e^{-\frac{1}{2}(1+\frac{1}{\sqrt{3}})}\right]$$

$$\approx 0.6320$$

8. 局部截断误差为

$$R_{i+1} = y(x_{i+1}) - y(x_i) - \frac{h}{3}f(x_i, y(x_i)) - \frac{2}{3}hf(x_i+\alpha h, y(x_i)+\alpha h f(x_i, y(x_i)))$$

$$= y(x_{i+1}) - y(x_i) - \frac{h}{3}y'(x_i) - \frac{2}{3}hf(x_i+\alpha h, y(x_i)+\alpha h y'(x_i))$$

$$= y(x_i) + hy'(x_i) + \frac{h^2}{2}y''(x_i) + \frac{h^3}{6}y'''(x_i) + O(h^4)$$

$$-y(x_i) - \frac{h}{3}y'(x_i)$$

$$-\frac{2}{3}h\Big[f(x_i, y(x_i)) + \frac{\partial f}{\partial x}(x_i, y(x_i))\cdot \alpha h + \frac{\partial f}{\partial y}(x_i, y(x_i))\cdot \alpha h y'(x_i)$$

$$+\frac{1}{2}\Big(\frac{\partial^2 f}{\partial x^2}(\alpha h)^2 + 2\frac{\partial^2 f}{\partial x \partial y}(\alpha h)^2 y'(x_i) + \frac{\partial^2 f}{\partial y^2}(\alpha h)^2 (y'(x_i))^2\Big) + O(h^3)\Big]$$

$$= \frac{2}{3}hy'(x_i) + \frac{h^2}{2}y''(x_i) + \frac{h^3}{6}y'''(x_i) + O(h^4)$$

$$-\frac{2}{3}h\Big[y'(x_i) + \alpha h y''(x_i) + \frac{1}{2}(\alpha h)^2\Big(y'''(x_i) - \frac{\partial f}{\partial y}(x_i, y(x_i))y''(x_i)\Big) + O(h^3)\Big]$$

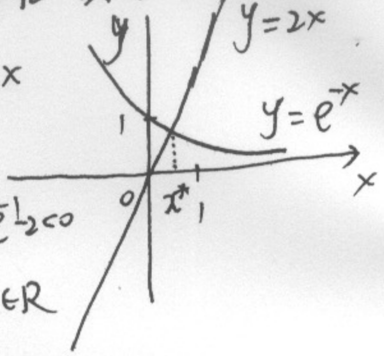
$$= (\frac{1}{2} - \frac{2\alpha}{3})h^2 y''(x_i) + \Big((\frac{1}{6} - \frac{1}{3}\alpha^2)y'''(x_i) + \frac{\alpha^2}{3}\frac{\partial f}{\partial y}(x_i, y(x_i))y''(x_i)\Big)h^3 + O(h^4)$$

要使公式成为二阶公式，则 $\frac{1}{2} - \frac{2\alpha}{3} = 0 \Rightarrow \alpha = \frac{3}{4}$.

$$R_{i+1} = \Big[-\frac{1}{48}y'''(x_i) + \frac{3}{16}\frac{\partial f}{\partial y}(x_i, y(x_i))y''(x_i)\Big]h^3 + O(h^4).$$

2007年10月数值分析学位考试参考答案.

1. (1) (法一) 作函数 $y=e^{-x}$, $y=2x$ 图像, 从图上可知方程有唯一实根 $x^*\in[0,1]$.



(法二). 令 $f(x)=e^{-x}-2x$

$f(x)$为连续函数

又 $f(0)=1>0$, $f(1)=e^{-1}-2<0$

$f'(x)=-e^{-x}-2<0,\ x\in R$

∴方程 $f(x)=0$ 有唯一实根 $x^*\in[0,1]$.

(2) 构造迭代格式

$$\begin{cases} x_{k+1}=\frac{1}{2}e^{-x_k}, & k=0,1,2,\cdots \\ x_0=0.5 \end{cases}$$

计算得 $x_1=0.3033$, $x_2=0.3692$, $x_3=0.3456$

$x_4=0.3539$, $x_5=0.3510$, $x_6=0.3520$

$x_7=0.3516$,

$|x_7-x_6|=0.0004<0.0005=\frac{1}{2}\times10^{-3}$

∴ $x^*\approx0.3516$

(3) 迭代函数 $\varphi(x)=\frac{1}{2}e^{-x}$

① $|\varphi'(x)|=\frac{1}{2}e^{-x}\le\frac{1}{2}$, $x\in[0,1]$

② 当 $x\in[0,1]$ 时,

$0<\varphi(x)\le\frac{1}{2}$

∴由Th2.1上述迭代是收敛的.

2. $$\begin{bmatrix}1&2&1&0\\2&2&3&3\\-1&-3&0&2\end{bmatrix}\xrightarrow{r_1\leftrightarrow r_2}\begin{bmatrix}2&2&3&3\\1&2&1&0\\-1&-3&0&2\end{bmatrix}$$

$$\xrightarrow[\frac{1}{2}r_1+r_3]{-\frac{1}{2}r_1+r_2}\begin{bmatrix}2&2&3&3\\0&1&-\frac{1}{2}&-\frac{3}{2}\\0&-2&\frac{3}{2}&\frac{7}{2}\end{bmatrix}\xrightarrow{r_2\leftrightarrow r_3}\begin{bmatrix}2&2&3&3\\0&-2&\frac{3}{2}&\frac{7}{2}\\0&1&-\frac{1}{2}&-\frac{3}{2}\end{bmatrix}$$

$$\xrightarrow{\frac{1}{2}r_2+r_3}\begin{bmatrix}2&2&3&3\\0&-2&\frac{3}{2}&\frac{7}{2}\\0&0&\frac{1}{4}&\frac{1}{4}\end{bmatrix}\Rightarrow\begin{cases}x_1=1\\x_2=-1\\x_3=1\end{cases}$$

3. (1) 将方程改写为下面形式

$$\begin{pmatrix}1&2&1\\-1&-3&0\\2&2&3\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}3\\2\\0\end{pmatrix}$$

则Gauss-Seidel迭代格式为:

$$\begin{cases}x_1^{(k+1)}=-2x_2^{(k)}-x_3^{(k)}\\ x_2^{(k+1)}=(x_1^{(k+1)}+2)/(-3), \quad k=0,1,2,\cdots\\ x_3^{(k+1)}=(-2x_1^{(k+1)}-2x_2^{(k+1)})/3\end{cases}$$

(2) 迭代矩阵的特征方程为

$$\begin{vmatrix}\lambda&2&1\\-\lambda&-3\lambda&0\\2\lambda&2\lambda&3\lambda\end{vmatrix}=0\Rightarrow 9\lambda^3-10\lambda^2=0$$

$\lambda_1=\lambda_2=0$, $\lambda_3=\frac{10}{9}$

$\rho(G)>0$, 迭代发散.

4. (1) $f(x_0)=-11$, $f(x_1)=-1$, $f(x_2)=34$, $f(x_3)=-10$

$L_3(x)=f(x_0)l_0(x)+f(x_1)l_1(x)+f(x_2)l_2(x)+f(x_3)l_3(x)$

$$=-11\frac{(x-3)(x+2)x}{(1-3)(1+2)\cdot1}-\frac{(x-1)(x+2)x}{(3-1)(3+2)\cdot3}$$

$$+34\frac{(x-1)(x-3)x}{(-2-1)(-2-3)(-2)}-10\frac{(x-1)(x-3)(x+2)}{(-1)(-3)(2)}$$

$$=\frac{11}{6}x(x-3)(x+2)-\frac{1}{30}x(x-1)(x+2)$$

$$-\frac{17}{15}x(x-1)(x-3)-\frac{5}{3}(x-1)(x-3)(x+2)$$

(2) $N_3(x)=f(1)+f[1,3](x-1)$

$+f[1,3,-2](x-1)(x-3)$

$+f[1,3,-2,0](x-1)(x-3)(x+2)$

列表求差商.

x_k	$f(x_k)$			
1	-11	5	4	-1
3	-1	-7	5	
-2	34	-22		
0	-10			

$$\Rightarrow N_3(x) = -11 + 5(x-1) + 4(x-1)(x-3) - (x-1)(x-3)(x+2)$$

(3) $R_n(x) = f(x) - N_3(x) = f(x) - L_3(x)$

$$= \frac{f^{(4)}(\xi)}{4!}(x-1)(x-3)(x+2)x$$

$$= x(x-1)(x-3)(x+2).$$

5. 记 $A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ -1 & 1 \end{pmatrix}$, $b = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$, 则

$$A^TA = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 1 & 1 \end{bmatrix}\begin{pmatrix} 1 & 1 \\ 2 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 2 \\ 2 & 3 \end{pmatrix}$$

$$A^Tb = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 1 & 1 \end{bmatrix}\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 6 & 2 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \qquad \begin{cases} x_1 = -\frac{3}{14} \\ x_2 = \frac{8}{7} \end{cases}$$

6. (1) $S(f) = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$

(2) 作 $f(x)$ 的 2 次插值多项式 $L_2(x)$, 满足

$L_2(a) = f(a)$, $L_2\left(\frac{a+b}{2}\right) = f\left(\frac{a+b}{2}\right)$, $L_2(b) = f(b)$

则 $$L_2(x) = f(a)\frac{(x-\frac{a+b}{2})(x-b)}{(a-\frac{a+b}{2})(a-b)} + f\left(\frac{a+b}{2}\right)\frac{(x-a)(x-b)}{(\frac{a+b}{2}-a)(\frac{a+b}{2}-b)} + f(b)\frac{(x-a)(x-\frac{a+b}{2})}{(b-a)(b-\frac{a+b}{2})}$$

$$\int_a^b f(x)dx \approx \int_a^b L_2(x)dx$$

$$= \frac{2f(a)}{(b-a)^2}\int_a^b \left(x - \frac{a+b}{2}\right)(x-b)dx - \frac{4}{(b-a)^2}f\left(\frac{a+b}{2}\right)\int_a^b (x-a)(x-b)dx + \frac{2f(b)}{(b-a)^2}\int_a^b (x-a)\left(x-\frac{a+b}{2}\right)dx$$

$$= \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] = S(f)$$

(3) $T_n(f) = \sum_{k=0}^{n-1}\frac{h}{2}\left[f(x_k) + f(x_{k+1})\right]$

$S_n(f) = \sum_{k=0}^{n-1}\frac{h}{6}\left[f(x_k) + 4f(x_{k+\frac{1}{2}}) + f(x_{k+1})\right]$

$S_n(f) = \frac{4}{3}T_{2n}(f) - \frac{1}{3}T_n(f).$

7. 局部截断误差为

$$R_{i+1} = y(x_{i+1}) - y(x_i) - \frac{h}{24}\left[55f(x_i, y(x_i)) - 59f(x_{i-1}, y(x_{i-1})) + 37f(x_{i-2}, y(x_{i-2})) - 9f(x_{i-3}, y(x_{i-3}))\right]$$

$$= y(x_{i+1}) - y(x_i) - \frac{h}{24}\left[55y'(x_i) - 59y'(x_{i-1}) + 37y'(x_{i-2}) - 9y'(x_{i-3})\right]$$

$$= y(x_i) + hy'(x_i) + \frac{h^2}{2}y''(x_i) + \frac{h^3}{6}y'''(x_i) + \frac{h^4}{24}y^{(4)}(x_i) + \frac{h^5}{120}y^{(5)}(x_i) + O(h^6) - y(x_i) - \frac{55}{24}hy'(x_i)$$

$$+ \frac{59h}{24}\left[y'(x_i) - hy''(x_i) + \frac{h^2}{2}y'''(x_i) - \frac{h^3}{6}y^{(4)}(x_i) + \frac{h^4}{24}y^{(5)}(x_i)\right]$$

$$- \frac{37}{24}h\left[y'(x_i) - 2hy''(x_i) + \frac{4h^2}{2}y'''(x_i) - \frac{8h^3}{6}y^{(4)}(x_i) + \frac{16h^4}{24}y^{(5)}(x_i)\right]$$

$$+ \frac{9h}{24}\left[y'(x_i) - 3hy''(x_i) + \frac{9h^2}{2}y'''(x_i) - \frac{27h^3}{6}y^{(4)}(x_i) + \frac{81h^4}{24}y^{(5)}(x_i)\right] + O(h^6)$$

$$= \frac{251}{720}h^5y^{(5)}(x_i) + O(h^6)$$

∴ 该公式是 4 阶公式, $R_{i+1} = \frac{251}{720}h^5y^{(5)}(x_i) + O(h^6)$