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Magnitude Homology of Graphs and the Magnitude as its Categorification

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Supervisor's seal	acceptance stamp

Abstract

Sample Abstract

Contents

1 Introduction

Lamport's guide to LaTeX [1].

2 The Magnitude of Graphs

In this section, we define the magnitude of a graph G and the magnitude homology of a graph G, give some very basic examples and properties. By a *graph* we mean a finite undirected graph with no loops or multiple edges. The set of vertices of a graph G is denoted by $V(G)$, and the set of edges of G is denoted by $E(G)$. If x and y are vertices of a graph G, then the *distance* $d_G(x, y)$ between x and y is defined to be the length of a shortest edge path from x to y. If x and y lie in different components of G then $d(x, y) = \infty$.

2.1 The definition of the magnitude of Graphs

Here, we difine the magnitude of a graph, which can be expressed as either a rational function over $\mathbb{Q}$ or a formal power series over $\mathbb{Z}$. Write $\mathbb{Z}[q]$ for the polynomial ring over the integers in one variable q and $\mathbb{Z}[[q]]$ for the ring of formal power series over the integers in one variable q.

Definition 2.1.1. Let G be a graph. We define the *G-matrix* $Z_G = Z_G(q)$ over $\mathbb{Z}[q]$ whose rows and columns are indexed by the vertices of G, and whose (x, y)-entry is given by

$$Z_G(q)(x, y) = q^{d(x,y)} \quad (x, y \in V(G))$$

where by convention $q^\infty = 0$.

G-matrix is the square synmetric matrix.

Proposition 2.1.2. *G-matrix is invertible.*

Proof. By definition, the determinant of Z_G has constant term 1, which implies that $\det Z_G \neq 0$. $\square$

Definition 2.1.3. The *magnitude* of a graph G is defined to be the rational function given by

$$\#G(q) = \sum_{x,y \in V(G)} (Z_G(q))^{-1}(x, y)$$

in the rational function field $\mathbb{Q}(q)$.

Remark 2.1.4.

$$\#G(q) = \mathrm{sum}(Z_G(q)^{-1}) = \frac{\mathrm{sum}(\mathrm{adj}(Z_G(q)))}{\det(Z_G(q))}$$

where adj is the adjugate matrix and sum is the sum of all entries of a matrix.

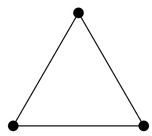
Proposition 2.1.5. *$\#G(q)$ takes values in $\mathbb{Z}[[q]]$.*

Proof. Let $\det Z_G(q) = 1 - qf(q)$ for some $f(q) \in \mathbb{Z}[q]$ by theorem 2.1.2. Then we have

$$\#G(q) = \frac{\mathrm{sum}(\mathrm{adj}(Z_G))}{\det(Z_G)} = \mathrm{sum}(\mathrm{adj}(Z_G)) \sum_{n=0}^{\infty} q^n f(q)^n$$

Note that $qf(q)$ has no constant term and then $\sum_{n=0}^{\infty} q^n f(q)^n$ takes values in $\mathbb{Z}[[q]]$. $\square$

Example 2.1.6. Let $G = K_3$(complete graph with three vertices).



Then, you can calculate the magnitude of K_3 as follows:

$$Z_{K_3}(q) = \begin{pmatrix} 1 & q & q \\ q & 1 & q \\ q & q & 1 \end{pmatrix}, \quad Z_{K_3}(q)^{-1} = \frac{1}{1-3q^2+2q^3} \begin{pmatrix} 1-q^2 & -q+q^2 & -q+q^2 \\ -q+q^2 & 1-q^2 & -q+q^2 \\ -q+q^2 & -q+q^2 & 1-q^2 \end{pmatrix},$$

$\#K_3(q) = \frac{3}{1+2q}$

Definition 2.1.7. Let G be a graph and $x \in V(G)$. The *weight* of x in G is defined

$$w_G(x)(q) = \sum_{y \in V(G)} (Z_G(q))^{-1}(x, y)$$

The function $w_G : V(G) \to \mathbb{Q}(q)$ is called the *weighting* on G.

The magnitude can be expressed using the weighting as follows:

$$\#G(q) = \sum_{x \in V(G)} w_G(x)$$

Lemma 2.1.8. *For any graph G, the weighting w_G satisfies*

$$\sum_{y \in V(G)} q^{d(x,y)} w_G(y) = 1 \quad (x \in V(G))$$

Proof. For any vertex $x \in V(G)$, we have

$$\begin{aligned}
\sum_{y \in V(G)} q^{d(x,y)} w_G(y) = & \sum_{y,z \in V(G)} q^{d(x,y)} Z_G^{-1}(y,z) \\
& = \sum_{y,z \in V(G)} Z_G(x,y) Z_G^{-1}(y,z) \\
& = \sum_{z \in V(G)} \sum_{y \in V(G)} Z_G(x,y) Z_G^{-1}(y,z) \\
& = \sum_{z \in V(G)} (Z_G Z_G^{-1})(x,z) \\
& = \sum_{z \in V(G)} I(x,z) \\
= 1.
\end{aligned}$$

□

This equation is called the *weighting equation.*

Lemma 2.1.9. *Let G be a graph and $\tilde{w}_G : V(G) \to \mathbb{Q}$ be a function satisfying a weighting equation. Then, $\tilde{w}_G = w_G$. Now, w_G is the weighting on G.*

Proof. Let $\boldsymbol{b} = (1, 1, \ldots, 1)^T$ where the length of $\boldsymbol{b}$ is $|V(G)|$ and $\boldsymbol{w}_G = (w_G(x))^T_{x \in V(G)}$. If $\tilde{w}_G$ satisfies the weighting equation, then we have

$$Z_G \tilde{\boldsymbol{w}}_G = \boldsymbol{b}$$

Since Z_G is invertible by theorem 2.1.2, we have $\tilde{w}_G = w_G$ □

This lemma shows that the weighting on a graph is unique and we use this frequently to compute the magnitude of graphs.

2.2 Basic Properties and Examples

Here we give the most basic facts about magnitude. We focus on transitive graphs, disjoint unions, catesian products, and how the magnitude behaves within $\mathbb{Z}[[q]]$.

Definition 2.2.1. Let $G = (V(G), E(G)), H = (V(H), E(H))$ be a graph. An *graph homomorphism* from G to H is a map $f : V(G) \to V(H)$ such that if $\{x, y\} \in E(G)$ then $\{f(x), f(y)\} \in E(H)$.

We can define a *graph automorphism* using the definition above. We denote the group of all graph automorphisms of a graph G by $\mathrm{Aut}(G)$. $\mathrm{Aut}(G)$ includes id_G and for $g, h \in \mathrm{Aut}(G)$ and $x \in V(G)$, $g(h(x)) = (gh)(x)$, which means $\mathrm{Aut}(G)$ acts on $V(G)$.

Definition 2.2.2. A graph G is *vertex-transitive* if $\mathrm{Aut}(G)$ acts transitively on $V(G)$. It says that for any vertices x and y of G, there exists an automorphism $g : G \to G$ such that $y = g(x)$.

Lemma 2.2.3. *Let G be a vertex-transitive graph. Then,*

$$\#G(q) = \frac{|V(G)|}{\sum_{y \in V(G)} q^{d(x,y)}}$$

for any vertex $x \in V(G)$.

Proof. Let $S(x) = \sum_{y \in V(G)} q^{d(x,y)}$ for a vertex $x \in V(G)$. We show that $S(x)$ does not depend on the choice of x. Take any vertices $a, b \in V(G)$. Since G is vertex-transitive, there exists $g \in \mathrm{Aut}(G)$ such that $b = g(a)$. Then,

$$\begin{aligned}
S(b) &= \sum_{y \in V(G)} q^{d(b,y)} \\
&= \sum_{y \in V(G)} q^{d(g(a),y)} \\
&= \sum_{y \in V(G)} q^{d(g(a),g(y))} \quad \text{(since } g \text{ is bijective)} \\
&= \sum_{y \in V(G)} q^{d(a,y)} \quad \text{(since } g \text{ is an isomorphism)} \\
&= S(a)
\end{aligned}$$

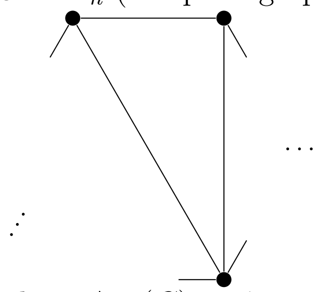
Thus, $S(x)$ does not depend on the choice of x, denoting it by S. Now, we define a function $\tilde{w}_G : V(G) \to \mathbb{Q}(q)$ by $\tilde{w}_G(x) = \frac{1}{S}$ for any vertex $x \in V(G)$. Then $\tilde{w}_G$ satisfies the weighting equation and by theorem 2.1.9, we have $\#G = \frac{|V(G)|}{S}$.

$\square$

Example 2.2.4. (i) $G = V_n$ (edgeless graph with n vertices).

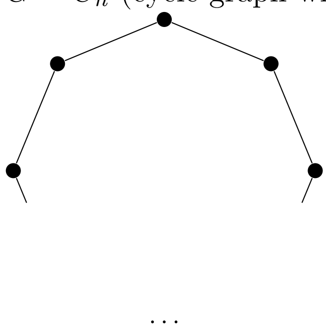
Then, $\mathrm{Aut}(G) \approx \mathfrak{S}_n$ and G is vertex-transitive. $S = 1$ and we have $\#V_n = n$.

(ii) $G = K_n$ (complete graph with n vertices).



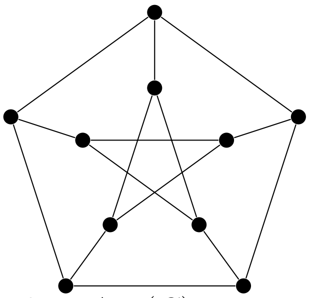
Then, $\mathrm{Aut}(G) \approx \mathfrak{S}_n$ and G is vertex-transitive. $S = 1 + (n-1)q$ and we have $\#K_n = \frac{n}{1+(n-1)q}$.

(iii) $G = C_n$ (cycle graph with n vertices).



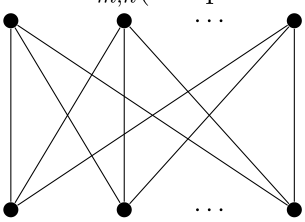
Then, $\mathrm{Aut}(G) \approx D_{2n}$ and G is vertex-transitive. If $n = 2m$(even), then $S = 1 + 2(q + q^2 + \cdots + q^{m-1}) + q^m = \frac{1+q-q^m-q^{m+1}}{1-q}$. Thus, we have $\#C_{2m} = \frac{2m(1-q)}{(1+q)(1-q^m)} = \frac{n(1-q)}{(1+q)(1-q^m)}$. If $n = 2m-1$(odd), then similarly $\#C_{2m-1} = \frac{n(1-q)}{1+q-2q^m}$.

(iv) G is a Petersen graph.



Then, $\mathrm{Aut}(G)$ contains D_{10} as its subgroup and G is vertex-transitive. $S = 1 + 3q + 6q^2$ and we have $\#G = \frac{10}{1+3q+6q^2}$.

(v) $G = K_{m,n}$(complete bipartite graph).



Then, $\mathrm{Aut}(G) \approx \mathfrak{S}_m \times \mathfrak{S}_n$ if $m \neq n$ and G is not vertex-transitive. You can calculate the magnitude with other methods. Let a, b be the weight of vertices on each part of $K_{m,n}$. Then, the weighting equation is written by two equations as follows:

$$\begin{cases} \{q^0 + (m-1)q^2\}a + nqb = 1 \\ \{q^0 + (n-1)q^2\}b + mqa = 1 \end{cases}$$

You can solve this and we have

$$\#K_{m,n} = ma + nb = \frac{(m+n) - (2mn - m - n)q}{(1+q)(1-(m-1)(n-1)q^2)}$$

Lemma 2.2.5. *Let G and H be graphs. Then,*

$$\#(G \sqcup H) = \#G + \#H$$

where $G \sqcup H$ is the disjoint union of G and H.

Proof. $Z_{G \sqcup H} = \begin{pmatrix} Z_G & O \\ O & Z_H \end{pmatrix}$,

$Z_{G \sqcup H}^{-1} = \begin{pmatrix} Z_G^{-1} & O \\ O & Z_H^{-1} \end{pmatrix}$.

Thus,

$$\#(G \sqcup H) = \mathrm{sum}(Z_{G\sqcup H}^{-1}) = \mathrm{sum}(Z_G^{-1}) + \mathrm{sum}(Z_H^{-1}) = \#G + \#H$$

$\square$

Definition 2.2.6. Let G and H be graphs. The *cartesian product* $G\square H$ of G and H is the graph defined as follows;

- $V(G\square H) = V(G) \times V(H)$
- $E(G\square H) = \{\{(x,y),(x',y')\} | x = x' \text{ and } \{y,y'\} \in E(H) \text{ or } y = y' \text{ and } \{x,x'\} \in E(G)\}$.

Lemma 2.2.7. $\#G\square H = \#G \cdot \#H$

Proof. For $x, x' \in V(G)$ and $y, y' \in V(H)$,
$d_{G\square H}((x,y),(x',y')) = d_G(x,x') + d_H(y,y')$
$\Rightarrow Z_{G\square H}((x,y),(x',y')) = q^{d_{G\square H}((x,y),(x',y'))} = q^{d_G(x,x')}q^{d_H(y,y')} = Z_G(x,x')Z_H(y,y')$
$\Rightarrow Z_{G\square H} = Z_G \otimes Z_H$ and then $Z_{G\square H}^{-1} = Z_G^{-1} \otimes Z_H^{-1}$
$\Rightarrow \#G\square H = \mathrm{sum}(Z_{G\square H}^{-1}) = \mathrm{sum}(Z_G^{-1} \otimes Z_H^{-1}) = \mathrm{sum}(Z_G^{-1}) \cdot \mathrm{sum}(Z_H^{-1}) = \#G \cdot \#H$

We used the fact that $(P \otimes Q)(R \otimes S) = (PR) \otimes (QS)$ for proper matrices P, Q, R, S. $\square$

Example 2.2.8. See $G = K_2\square K_3$.
$\#K_2\square K_3 = \#K_2 \cdot \#K_3 = \frac{2}{1+q} \cdot \frac{3}{1+2q} = \frac{6}{(1+q)(1+2q)} = \#K_{3,3}$.

Remark 2.2.9. Here we use the catesian product for graph product, but there are other graph products such as the tensor product and strong product. However, there is a reason that we use the catesian product. This will be clear in Section 4.

Proposition 2.2.10. *Let G be a graph. Then,*

$$\begin{aligned}\#G(q) &= \sum_{k=0}^{\infty}(-1)^k \sum_{x_0\neq x_1\neq\cdots\neq x_k} q^{d(x_0,x_1)+d(x_1,x_2)+\cdots+d(x_{k-1},x_k)} \\ &= \sum_{n=0}^{\infty} c_n q^n\end{aligned}$$

where $c_n = \sum_{k=0}^{n}(-1)^k|\{(x_0,\ldots,x_k)|x_0 \neq x_1 \neq \cdots \neq x_k, d(x_0,x_1) + \cdots + d(x_{k-1},x_k) = n\}|$

Proof. aaa □

Corollary 2.2.11. *Let G be a graph.* $|V(G)| = \#G(0), |E(G)| = -\frac{1}{2}\frac{d}{dq}\#G(q)\Big|_{q=0}$. *Here, the derivative is taken in* $\mathbb{Z}[[q]]$.

Proof. From the previous proposition, we have

$$\begin{aligned} c_0 &= \sum_{k=0}^{0}(-1)^k|\{(x_0,\ldots,x_k)|x_0 \neq x_1 \neq \cdots \neq x_k, d(x_0,x_1)+\cdots+d(x_{k-1},x_k)=n\}| \\ &= |\{(x_0)|x_0 \in V(G)\}| \\ &= |V(G)| \end{aligned}$$

and

$$\begin{aligned} c_1 &= |\{(x_0)|d(x_0,x_0)=1\}| - |\{(x_0,x_1)|x_0 \neq x_1, d(x_0,x_1)=1\}| \\ &= 0 - 2|E(G)| \\ &= -2|E(G)| \end{aligned}$$

This corollary immidiately follows from these equations. □

Remark 2.2.12. $c_0 \geq 0, c_1 \leq 0,$ and $c_2 \geq 0$. $c_2 = 0$ if and only if

2.3 The main result of magnitude of graphs

This section states the inclusion-exclusion principle of magnitude of graphs under some conditions. First, we see that the magnitude does not satisfy the inclusion-exclusion principle in general. Then, we introduce the sufficient condition for the inclusion-exclusion principle to hold.

Definition 2.3.1. Let R be a ring. A function Φ is an *R-valued graph invariant* if

- $\Phi(G) \in R$ for any graph G
- If $G \approx H$ as a graph then $\Phi(G) = \Phi(H)$

Definition 2.3.2. Let Φ be an R-valued graph invariant.
1. Φ is said to be multiplicative if

- $\Phi(K_1) = 1$

- $\Phi(G \square H) = \Phi(G) \cdot \Phi(H)$ for any graphs G and H

2. Φ is said to satisfy the inclusion-exclusion principle if

- $\Phi(\emptyset) = 0$

- $\Phi(G \cup H) = \Phi(G) + \Phi(H) - \Phi(G \cap H)$ for any graphs G and H

Lemma 2.3.3. *Let R be a ring containing no nonzero nilpotents and Φ be a multiplicative R-valued graph invariant satisfying the inclusion-exclusion principle. Then, $\Phi(G) = |V(G)|$ for any graph G.*

Proof. aaa □

Corollary 2.3.4. *The magnitude does not satisfy the inclusion-exclusion principle in general.*

Example 2.3.5. labelWillerton

Definition 2.3.6. Let X be a graph and U be a subgraph of X. U is said to be *convex* in X if for any vertices $x, y \in V(U)$, $d_U(x, y) = d_X(x, y)$.

Lemma 2.3.7. *Let X be a graph and G, H be subgraphs of X such that $X = G \cup H$. In this document, we mean $G \cup H$ as a graph $(V(G) \cup V(H), E(G) \cup E(H))$. Let $g \in V(G)$ and $h \in V(H)$ such that there is a path $(g = x_0 \to x_1 \to \cdots \to x_n = h)$ in X. Then, there exists a vertex $x_i \in V(G) \cap V(H)$.*

Proof. aaa □

Lemma 2.3.8. *Let X be a graph and G, H be subgraphs of X such that $X = G \cup H$. If $G \cap H$ is convex in X, then G and H are also convex in X.*

Proof. aaa □

Definition 2.3.9. Let X be a graph and U be a subgraph of X such that U is convex in X. Write $V_U(X) = \{v \in V(X) | d_X(v, u) < \infty \text{ for some } u \in V(U)\}$. Then, we say that X projects to U if for any $x \in V_U(X)$, there exists $\pi(x) \in V(U)$ such that for any $u \in V(U)$, $d_X(x, u) = d_X(x, \pi(x)) + d_X(\pi(x), u)$.

Lemma 2.3.10. *If X projects to U, then $\pi(x)$ is uniquely determined for any $x \in V_U(X)$.*

Proof. aaa □

Example 2.3.11. aaa

Lemma 2.3.12. *Let X be a graph and $U \subset X$ be a convex subgraph of X such that X projects to U. Then, for any $u \in V(U)$,*

$$w_U(u) = \sum_{x \in \pi^{-1}(u)} q^{d_X(u,x)} w_X(x)$$

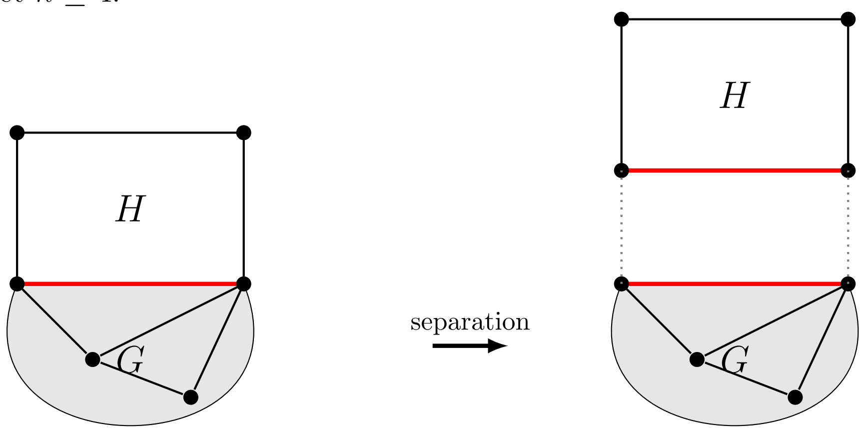
Proof. aaa □

Theorem 2.3.13. *(main theorem I) Let X be a graph and G, H be subgraphs of X such that $X = G \cup H$. If $G \cap H$ is convex in X and H projects to $G \cap H$, then*

$$\#X = \#G + \#H - \#(G \cap H)$$

Before proving this theorem, we give the example of graphs for which we can apply this theorem.

Example 2.3.14. Let G be a graph and consider the graph H formed by identifying one of the edges of a cycle graph C_n with an edge of G. Now, let $n \geq 4$.



$G \cup H$ $\qquad$ G, H (separated)

Then, we can apply the main theorem I to $X = G \cup H$ as follows:

$$\#X = \#G + \#C_n - \#K_2$$

Similarly, if G and H are graphs and $G \vee H$ is the graph formed by identifying one vertex of G with one vertex of H, then we have

$$\#(G \vee H) = \#G + \#H - 1$$

Proof. (of main theorem I) aaa □

Example 2.3.15. The three graphs below are divided into a graph C_3, and two graphs C_2, so they all have the same magnitude and can be calculated as follows:

$$\#G = \#C_3 + 2\#C_2 - 2$$

Example 2.3.16. If G is a forest, then we can calculate the magnitude of G as follows:

$$\#G = |V(G)| - 2|E(G)|\frac{q}{1+q}$$

If G is a tree, then

$$\#G = |V(G)| - 2(|V(G)| - 1)\frac{q}{1+q}$$

Furthermore examples.

3 The Magnitude Homology of Graphs

In this section, we define the magnitude homology of a graph G, give some very

3.1 The Definition of the magnitude homology of graphs

Definition 3.1.1. Let G be a graph. For positive integers k, the *length* of a tuple $(x_0, \dots, x_k)$ of $V(G)$ is defined to be

$$\begin{aligned} l(x_0, \dots, x_k) &= d(x_0, x_1) + d(x_1, x_2) + \cdots + d(x_{k-1}, x_k) \\ &= \sum_{i=1}^{k} d(x_{i-1}, x_i) \end{aligned}$$

Now, let $l(x_0) = 0$.

Lemma 3.1.2. *(Triangle inequality)*

$$l(x_0, \dots, x_k) \geq l(x_0, \dots, \hat{x}_i, \dots, x_k)$$

Definition 3.1.3. (magnitude chain complex) Let G be a graph. $MC_{*,*}(G)$ is the *magnitude complex* defined as follows:

$$MC_{*,*}(G) = \bigoplus_{l=0} MC_{*,l}(G)$$

For non-negative integers k and l, $MC_{k,l}(G)$ is freely generated by tuples $(x_0, \dots, x_k)$ of $V(G)$ satisfying $x_0 \neq x_1 \neq \cdots \neq x_k$ and $l(x_0, \dots, x_k) = l$. The coefficient ring is $\mathbb{Z}$. The differential $\partial : MC_{k,l}(G) \to MC_{k-1,l}(G)$ is defined by

$$\partial = \sum_{i=1}^{k-1} (-1)^{i-1} \partial_i$$

where $\partial_i(x_0, \dots, x_k) = (x_0, \dots, \hat{x}_i, \dots, x_k)$ if $l(x_0, \dots, \hat{x}_i, \dots, x_k) = l(x_0, \dots, x_k)$ and 0 otherwise.

Remark 3.1.4.

$$\partial_i(x_0, \dots, x_k) \neq 0 \iff d(x_{i-1}, x_i) + d(x_i, x_{i+1}) = d(x_{i-1}, x_{i+1})$$

Lemma 3.1.5. $\partial \circ \partial = 0$

Proof. aaa. □

3.2 Magnitude Homology of Graphs is Categorification of Magnitude of Graphs

3.3 u

4 Motivation : The Magnitude of Enriched Categories

References

[1] Leslie Lamport. *LaTeX: A Document Preparation System*. Addison-Wesley, 2nd edition, 1994.

[2] Donald E. Knuth. *The TeXbook*. Addison-Wesley, 1984.

Proposition .0.1. *Let $G = K_{n,n}$. Then, $Aut(G) \approx (\mathfrak{S}_n \times \mathfrak{S}_n) \rtimes_s \mathbb{Z}_2$, where $s : \mathbb{Z}_2 \to Aut(G); 0 \mapsto id_G, 1 \mapsto \tau$ and τ is the automorphism which interchanges the two parts of $K_{n,n}$.*

Proof. Now,

$$0 \to \mathfrak{S}_n \times \mathfrak{S}_n \xrightarrow{incl} \mathrm{Aut}(G) \xrightarrow{s} \mathbb{Z}_2 \to 0$$

is exact and this sequence splits. Thus, we have $\mathrm{Aut}(G) \approx (\mathfrak{S}_n \times \mathfrak{S}_n) \rtimes_s \mathbb{Z}_2$. □