2025 Yokohama National University, Faculty of Science and Engineering, Mathematical Science EP Graduation Research

Magnitude Homology of Graphs and the Magnitude as its Categorification

2264257 Kensho Yachi

https://taro-ken.com

Supervisor : Yuta Nozaki Associate Professor

(January 30th, 2025)

Supervisor's seal	acceptance stamp

Abstract

Sample Abstract

Contents

1 Introduction

Lamport's guide to LaTeX [1]. We denote $\approx$ as isomorphisms and $\cong$ as homeomorphisms.

2 The Magnitude of Graphs

This section introduces the magnitude and the magnitude homology of a graph G, along with fundamental examples and properties. Throughout this paper, a *graph* means a finite undirected graph with no loops or multiple edges. The set of vertices of a graph G is denoted by $V(G)$, and the set of edges of G is denoted by $E(G)$. For vertices $x, y \in V(G)$, the *distance* $d_G(x, y)$ is defined as the length of a shortest edge path between them. If x and y lie in different connected components of G, we set $d_G(x, y) = \infty$.

2.1 The Definition of the Magnitude of Graphs

We begin by defining the magnitude of a graph. This invariant takes values in either the field of rational functions over $\mathbb{Q}$ or the ring of formal power series over $\mathbb{Z}$. Let $\mathbb{Q}(q)$ denote the field of rational functions in a variable q over $\mathbb{Q}$. We also write $\mathbb{Z}[q]$ and $\mathbb{Z}[\![q]\!]$ for the polynomial ring and the ring of formal power series in q over $\mathbb{Z}$, respectively.

Definition 2.1.1. Let G be a graph. We define the *G-matrix* $Z_G = Z_G(q)$ over $\mathbb{Z}[q]$ whose rows and columns are indexed by the vertices of G, and whose (x, y)-entry is given by

$$Z_G(q)(x, y) = q^{d(x,y)} \quad (x, y \in V(G))$$

where by convention $q^\infty = 0$.

G-matrix is the square synmetric matrix.

Proposition 2.1.2. *A の G-matrix is invertible.*

Proof. By definition, the determinant of Z_G has constant term 1, which implies that $\det Z_G \neq 0$. □

Definition 2.1.3. The *magnitude* of a graph G is defined to be the rational function given by

$$\#G(q) = \sum_{x,y \in V(G)} (Z_G(q))^{-1}(x, y).$$

Remark 2.1.4.

$$\#G(q) = \mathrm{sum}(Z_G(q)^{-1}) = \frac{\mathrm{sum}(\mathrm{adj}(Z_G(q)))}{\det(Z_G(q))},$$

where adj is the adjugate matrix and sum is the sum of all entries of a matrix.

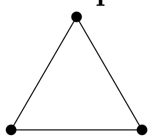
Proposition 2.1.5. $\#G(q)$ *takes values in* $\mathbb{Z}[\![q]\!]$.

Proof. Let $\det Z_G(q) = 1 - qf(q)$ for some $f(q) \in \mathbb{Z}[q]$ by Proposition 2.1.2. Then we have

$$\#G(q) = \frac{\mathrm{sum}(\mathrm{adj}(Z_G))}{\det(Z_G)} = \mathrm{sum}(\mathrm{adj}(Z_G)) \sum_{n=0}^{\infty} q^n f(q)^n.$$

Note that $qf(q)$ has no constant term and then $\sum_{n=0}^{\infty} q^n f(q)^n$ takes values in $\mathbb{Z}[\![q]\!]$. □

Example 2.1.6. Let $G = K_3$ (complete graph with three vertices).



Then, we can calculate the magnitude of K_3 as follows:

$$Z_{K_3}(q) = \begin{pmatrix} 1 & q & q \\ q & 1 & q \\ q & q & 1 \end{pmatrix}, \quad Z_{K_3}(q)^{-1} = \frac{1}{1-3q^2+2q^3} \begin{pmatrix} 1-q^2 & -q+q^2 & -q+q^2 \\ -q+q^2 & 1-q^2 & -q+q^2 \\ -q+q^2 & -q+q^2 & 1-q^2 \end{pmatrix},$$

$\#K_3(q) = \frac{3}{1+2q}$.

Definition 2.1.7. Let G be a graph and $x \in V(G)$. The *weight* of x in G is defined

$$w_G(x)(q) = \sum_{y \in V(G)} (Z_G(q))^{-1}(x, y)$$

The function $w_G : V(G) \to \mathbb{Q}(q)$ is called the *weighting* on G.

The magnitude can be expressed using the weighting as follows:

$$\#G(q) = \sum_{x \in V(G)} w_G(x)$$

Lemma 2.1.8. *For any graph G, the weighting w_G satisfies*

$$\sum_{y \in V(G)} q^{d(x,y)} w_G(y) = 1 \quad (x \in V(G)).$$

Proof. For any vertex $x \in V(G)$, we have

$$\begin{aligned}
\sum_{y \in V(G)} q^{d(x,y)} w_G(y) &= \sum_{y,z \in V(G)} q^{d(x,y)} Z_G^{-1}(y,z) \\
&= \sum_{y,z \in V(G)} Z_G(x,y) Z_G^{-1}(y,z) \\
&= \sum_{z \in V(G)} \sum_{y \in V(G)} Z_G(x,y) Z_G^{-1}(y,z) \\
&= \sum_{z \in V(G)} (Z_G Z_G^{-1})(x,z) \\
&= \sum_{z \in V(G)} I(x,z) \\
&= 1.
\end{aligned}$$

□

This equation is called the *weighting equation.*

Lemma 2.1.9. *Let G be a graph and $\tilde{w}_G : V(G) \to \mathbb{Q}$ be a function satisfying a weighting equation. Then, $\tilde{w}_G = w_G$. Now, w_G is the weighting on G.*

Proof. Let $\boldsymbol{b} = (1, 1, \ldots, 1)^T$ where the length of $\boldsymbol{b}$ is $|V(G)|$ and $\boldsymbol{w}_G = (w_G(x))^T_{x \in V(G)}$. If $\tilde{w}_G$ satisfies the weighting equation, then we have

$$Z_G \tilde{\boldsymbol{w}}_G = \boldsymbol{b}.$$

Since Z_G is invertible by Proposition 2.1.2, we have $\tilde{w}_G = w_G$. □

This lemma shows that the weighting on a graph is unique and we use this frequently to compute the magnitude of graphs.

2.2 Basic Properties and Examples

This subsection presents fundamental properties and examples of magnitude. We focus on vertex-transitive graphs, disjoint unions, and Cartesian products. We also discuss the properties of magnitude within $\mathbb{Z}[\![q]\!]$.

Definition 2.2.1. Let $G = (V(G), E(G)), H = (V(H), E(H))$ be graphs. A *graph homomorphism* from G to H is a map $f : V(G) \to V(H)$ such that if $\{x, y\} \in E(G)$ then $\{f(x), f(y)\} \in E(H)$ or $f(x) = f(y)$.

We can define a *graph automorphism* using the definition above. We denote the group of all graph automorphisms of a graph G by $\mathrm{Aut}(G)$. $\mathrm{Aut}(G)$ includes id_G and for $g, h \in \mathrm{Aut}(G)$ and $x \in V(G)$, $g(h(x)) = (gh)(x)$, which means $\mathrm{Aut}(G)$ acts on $V(G)$.

Definition 2.2.2. A graph G is *vertex-transitive* if $\mathrm{Aut}(G)$ acts transitively on $V(G)$. It says that for any vertices x and y of G, there exists an automorphism $g : G \to G$ such that $y = g(x)$.

Lemma 2.2.3. *Let G be a vertex-transitive graph. Then,*

$$\#G(q) = \frac{|V(G)|}{\sum_{y \in V(G)} q^{d(x,y)}}$$

for any vertex $x \in V(G)$.

Proof. Let $S(x) = \sum_{y \in V(G)} q^{d(x,y)}$ for a vertex $x \in V(G)$. We show that $S(x)$ does not depend on the choice of x. Take any vertices $a, b \in V(G)$. Since G is vertex-transitive, there exists $g \in \mathrm{Aut}(G)$ such that $b = g(a)$. Then,

$$\begin{aligned} S(b) &= \sum_{y \in V(G)} q^{d(b,y)} \\ &= \sum_{y \in V(G)} q^{d(g(a),y)} \\ &= \sum_{y \in V(G)} q^{d(g(a),g(y))} \quad (g \text{ is bijective}) \\ &= \sum_{y \in V(G)} q^{d(a,y)} \quad (g \text{ is an isomorphism}) \\ &= S(a). \end{aligned}$$

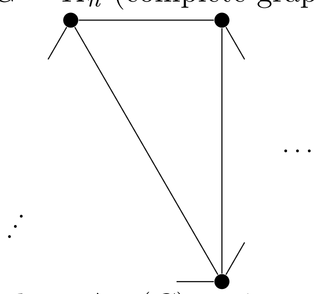
Thus, $S(x)$ does not depend on the choice of x, denoting it by S. Now, we define a function $\tilde{w}_G : V(G) \to \mathbb{Q}(q)$ by $\tilde{w}_G(x) = \frac{1}{S}$ for any vertex $x \in V(G)$. Then $\tilde{w}_G$ satisfies the weighting equation and by Lemma 2.1.9, we have $\#G = \frac{|V(G)|}{S}$.

□

Example 2.2.4. (i) $G = V_n$ (edgeless graph with n vertices).

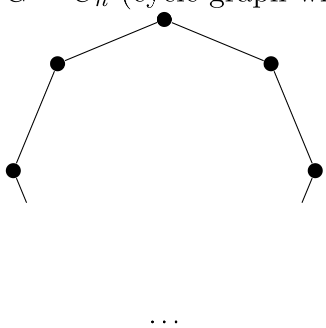
Then, $\mathrm{Aut}(G) \approx \mathfrak{S}_n$ and G is vertex-transitive. $S = 1$ and we have $\#V_n = n$.

(ii) $G = K_n$ (complete graph with n vertices).



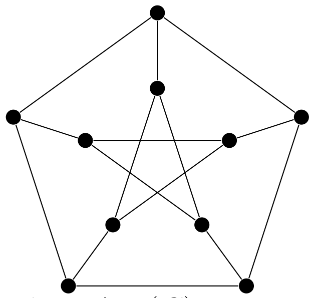
Then, $\mathrm{Aut}(G) \approx \mathfrak{S}_n$ and G is vertex-transitive. $S = 1 + (n-1)q$ and we have $\#K_n = \frac{n}{1+(n-1)q}$.

(iii) $G = C_n$ (cycle graph with n vertices).



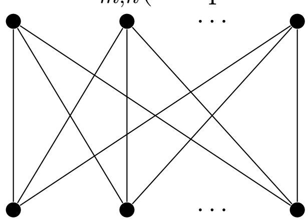
Then, $\mathrm{Aut}(G) \approx D_{2n}$ and G is vertex-transitive. If $n = 2m$, then $S = 1 + 2(q + q^2 + \cdots + q^{m-1}) + q^m = \frac{1+q-q^m-q^{m+1}}{1-q}$. Thus, we have $\#C_{2m} = \frac{2m(1-q)}{(1+q)(1-q^m)} = \frac{n(1-q)}{(1+q)(1-q^m)}$. If $n = 2m-1$, then similarly $\#C_{2m-1} = \frac{n(1-q)}{1+q-2q^m}$.

(iv) G is a Petersen graph.



Then, $\mathrm{Aut}(G)$ contains D_{10} as its subgroup and G is vertex-transitive. $S = 1 + 3q + 6q^2$ and we have $\#G = \frac{10}{1+3q+6q^2}$.

(v) $G = K_{m,n}$(complete bipartite graph).



Then, $\mathrm{Aut}(G) \approx \mathfrak{S}_m \times \mathfrak{S}_n$ if $m \neq n$ and G is not vertex-transitive. We can calculate the magnitude with other methods. Let a, b be the weight of vertices on each part of $K_{m,n}$. Then, the weighting equation is written by two equations as follows:

$$\begin{cases} \{q^0 + (m-1)q^2\}a + nqb = 1 \\ \{q^0 + (n-1)q^2\}b + mqa = 1. \end{cases}$$

We can solve this and we have

$$\#K_{m,n} = ma + nb = \frac{(m+n) - (2mn - m - n)q}{(1+q)(1-(m-1)(n-1)q^2)}.$$

Lemma 2.2.5. *Let G and H be graphs. Then,*

$$\#(G \sqcup H) = \#G + \#H,$$

where $G \sqcup H$ is the disjoint union of G and H.

Proof. $Z_{G \sqcup H} = \begin{pmatrix} Z_G & O \\ O & Z_H \end{pmatrix}$, $Z_{G \sqcup H}^{-1} = \begin{pmatrix} Z_G^{-1} & O \\ O & Z_H^{-1} \end{pmatrix}$.

Thus,

$$\#(G \sqcup H) = \mathrm{sum}(Z_{G \sqcup H}^{-1}) = \mathrm{sum}(Z_G^{-1}) + \mathrm{sum}(Z_H^{-1}) = \#G + \#H.$$

□

Definition 2.2.6. Let G and H be graphs. The *cartesian product* $G\Box H$ of G and H is the graph defined as follows;

- $V(G\Box H) = V(G) \times V(H)$.
- $E(G\Box H) = \{\{(x,y),(x',y')\} | x = x' \text{ and } \{y,y'\} \in E(H) \text{ or } y = y' \text{ and } \{x,x'\} \in E(G)\}$.

Lemma 2.2.7. *$\#G\Box H = \#G \cdot \#H$.*

Proof. For $x, x' \in V(G)$ and $y, y' \in V(H)$,
$d_{G\Box H}((x,y),(x',y')) = d_G(x,x') + d_H(y,y')$,
$Z_{G\Box H}((x,y),(x',y')) = q^{d_{G\Box H}((x,y),(x',y'))} = q^{d_G(x,x')}q^{d_H(y,y')} = Z_G(x,x')Z_H(y,y')$,
$Z_{G\Box H} = Z_G \otimes Z_H$ and then $Z_{G\Box H}^{-1} = Z_G^{-1} \otimes Z_H^{-1}$.
Thus, $\#G\Box H = \mathrm{sum}(Z_{G\Box H}^{-1}) = \mathrm{sum}(Z_G^{-1} \otimes Z_H^{-1}) = \mathrm{sum}(Z_G^{-1}) \cdot \mathrm{sum}(Z_H^{-1}) = \#G \cdot \#H$.

We used the fact that $(P \otimes Q)(R \otimes S) = (PR) \otimes (QS)$ for proper matrices $P, Q, R,$ and S. □

Example 2.2.8. $G = K_2 \Box K_3$.
$\#K_2\Box K_3 = \#K_2 \cdot \#K_3 = \frac{2}{1+q} \cdot \frac{3}{1+2q} = \frac{6}{(1+q)(1+2q)} = \#K_{3,3}$.

Remark 2.2.9. Here we use the catesian product for graph product, but there are other graph products. However, there is a reason that we use the catesian product. This will be clear in Section 4.

Proposition 2.2.10. *Let G be a graph. Then,*

$$\begin{aligned}\#G(q) &= \sum_{k=0}^{\infty} (-1)^k \sum_{x_0 \neq x_1 \neq \cdots \neq x_k} q^{d(x_0,x_1)+d(x_1,x_2)+\cdots+d(x_{k-1},x_k)} \\ &= \sum_{n=0}^{\infty} c_n q^n,\end{aligned}$$

where

$$c_n = \sum_{k=0}^{n} (-1)^k |\{(x_0,\ldots,x_k) | x_0 \neq x_1 \neq \cdots \neq x_k, d(x_0,x_1)+\cdots+d(x_{k-1},x_k) = n\}|.$$

Proof. aaa. □

Corollary 2.2.11. *Let G be a graph.* $|V(G)| = \#G(0), |E(G)| = -\frac{1}{2}\frac{d}{dq}\#G(q)\Big|_{q=0}$. *Here, the derivative is taken in* $\mathbb{Z}[\![q]\!]$.

Proof. From the previous proposition, we have

$$\begin{aligned}
c_0 &= \sum_{k=0}^{0}(-1)^k|\{(x_0,\dots,x_k)|x_0 \neq x_1 \neq \cdots \neq x_k, d(x_0,x_1)+\cdots+d(x_{k-1},x_k)=n\}| \\
&= |\{(x_0)|x_0 \in V(G)\}| \\
&= |V(G)|
\end{aligned}$$

and

$$\begin{aligned}
c_1 &= |\{(x_0)|d(x_0,x_0)=1\}| - |\{(x_0,x_1)|x_0 \neq x_1, d(x_0,x_1)=1\}| \\
&= 0 - 2|E(G)| \\
&= -2|E(G)|.
\end{aligned}$$

This corollary immidiately follows from these equations. □

Remark 2.2.12. $c_0 \geq 0, c_1 \leq 0,$ and $c_2 \geq 0$. $c_2 = 0$ if and only if

2.3 Main Results on the Magnitude of Graphs

This subsection states the inclusion-exclusion principle for the magnitude of graphs under specific conditions. We begin by observing that the magnitude does not generally satisfy the inclusion-exclusion principle. We then introduce sufficient conditions for the principle to hold. In this document, we mean $G \cup H$ as a graph $(V(G) \cup V(H), E(G) \cup E(H))$.

Definition 2.3.1. Let R be a ring. A function Φ is an *R-valued graph invariant* if

- $\Phi(G) \in R$ for any graph G,
- If $G \approx H$ as a graph then $\Phi(G) = \Phi(H)$.

Definition 2.3.2. Let Φ be an R-valued graph invariant.
1. Φ is said to be *multiplicative* if

- $\Phi(K_1) = 1$,

- $\Phi(G \square H) = \Phi(G) \cdot \Phi(H)$ for any graphs G and H.

2. Φ is said to satisfy the *inclusion-exclusion principle* if

- $\Phi(\emptyset) = 0$,
- $\Phi(G \cup H) = \Phi(G) + \Phi(H) - \Phi(G \cap H)$ for any graphs G and H.

Lemma 2.3.3. *Let R be a ring containing no nonzero nilpotents and Φ be a multiplicative R-valued graph invariant satisfying the inclusion-exclusion principle. Then, $\Phi(G) = |V(G)|$ for any graph G.*

Proof. aaa $\square$

Corollary 2.3.4. *The magnitude does not satisfy the inclusion-exclusion principle in general.*

Example 2.3.5. labelWillerton

Definition 2.3.6. Let X be a graph and U be a subgraph of X. U is said to be *convex* in X if for any vertices $x, y \in V(U)$, $d_U(x, y) = d_X(x, y)$.

Lemma 2.3.7. *Let X be a graph, G, H be subgraphs of X such that $X = G \cup H$, and $g \in V(G)$ and $h \in V(H)$ such that there is a path $(g = x_0 \to x_1 \to \cdots \to x_n = h)$ in X. Then, there exists a vertex $x_i \in V(G) \cap V(H)$.*

Proof. aaa $\square$

Lemma 2.3.8. *Let X be a graph and G, H be subgraphs of X such that $X = G \cup H$. If $G \cap H$ is convex in X, then G and H are also convex in X.*

Proof. aaa $\square$

Definition 2.3.9. Let X be a graph and U be a subgraph of X such that U is convex in X. We denote $V_U(X) = \{v \in V(X) | d_X(v, u) < \infty \text{ for some } u \in V(U)\}$. Then, we say that X *projects to* U if for any $x \in V_U(X)$, there exists $\pi(x) \in V(U)$ such that for any $u \in V(U)$, $d_X(x, u) = d_X(x, \pi(x)) + d_X(\pi(x), u)$.

Lemma 2.3.10. *If X projects to U, then $\pi(x)$ is uniquely determined for any $x \in V_U(X)$.*

Proof. aaa □

Example 2.3.11. aaa

Lemma 2.3.12. *Let X be a graph and $U \subset X$ be a convex subgraph of X such that X projects to U. Then, for any $u \in V(U)$,*

$$w_U(u) = \sum_{x \in \pi^{-1}(u)} q^{d_X(u,x)} w_X(x).$$

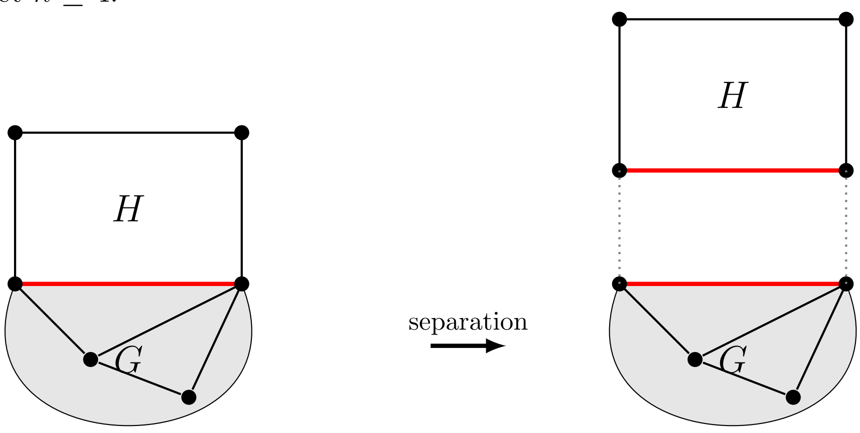
Proof. aaa □

Theorem 2.3.13. *(Main theorem I) Let X be a graph and G, H be subgraphs of X such that $X = G \cup H$. If $G \cap H$ is convex in X and H projects to $G \cap H$, then*

$$\#X = \#G + \#H - \#(G \cap H).$$

Before proving this theorem, we give the example of graphs for which we can apply this theorem.

Example 2.3.14. Let G be a graph and consider the graph H formed by identifying one of the edges of a cycle graph C_n with an edge of G. Now, let $n \geq 4$.



$G \cup H$ $\qquad$ G, H (separated)

Then, we can apply Theorem 2.3.13 to $X = G \cup H$ as follows:

$$\#X = \#G + \#C_n - \#K_2.$$

Similarly, if G and H are graphs and $G \vee H$ is the graph formed by identifying one vertex of G with one vertex of H, then we have

$$\#(G \vee H) = \#G + \#H - 1.$$

Proof of Theorem 2.3.13. aaa □

Example 2.3.15. The three graphs below are divided into a graph C_3, and two graphs C_2, so they all have the same magnitude and can be calculated as follows:

$$\#G = \#C_3 + 2 \cdot \#C_2 - 2.$$

Example 2.3.16. If G is a forest, then we can calculate the magnitude of G as follows:

$$\#G = |V(G)| - 2|E(G)|\frac{q}{1+q}.$$

If G is a tree, then

$$\#G = |V(G)| - 2(|V(G)| - 1)\frac{q}{1+q}.$$

Furthermore examples.

3 The Magnitude Homology of Graphs

In this section, we define the magnitude homology of a graph G. We provide fundamental examples and properties, and state the Mayer-Vietoris sequence for magnitude homology.

3.1 The Definition of The Magnitude Homology of Graphs

Definition 3.1.1. Let G be a graph. For positive integers k, the *length* of a tuple $(x_0, \dots, x_k)$ of $V(G)$ is defined to be

$$\begin{aligned} l(x_0, \dots, x_k) &= d(x_0, x_1) + d(x_1, x_2) + \cdots + d(x_{k-1}, x_k) \\ &= \sum_{i=1}^{k} d(x_{i-1}, x_i). \end{aligned}$$

Now, let $l(x_0) = 0$. We say the tuple $(x_0, \dots, x_k)$ is *good* if $x_0 \neq x_1 \neq \cdots \neq x_k$.

Lemma 3.1.2. *(Triangle inequality) If $(x_0, \dots, x_k)$ is a good tuple of $V(G)$, then for any $1 \leq i \leq k-1$,*

$$l(x_0, \dots, x_k) \geq l(x_0, \dots, \hat{x_i}, \dots, x_k).$$

Proof. We obviously have the statement by the triangle inequality of the distance function d. □

Definition 3.1.3. (magnitude chain complex) Let G be a graph. $MC_{*,*}(G)$ is the *magnitude complex* defined as follows:

$$MC_{*,*}(G) = \bigoplus_{l=0}^{\infty} MC_{*,l}(G).$$

For non-negative integers k and l, $MC_{k,l}(G)$ is freely generated by good tuples $(x_0, \dots, x_k)$ of $V(G)$ of length l. The coefficient ring is $\mathbb{Z}$. The differential $\partial : MC_{k,l}(G) \to MC_{k-1,l}(G)$ is defined by

$$\partial = \sum_{i=1}^{k-1} (-1)^{i-1} \partial_i,$$

where

$$\partial_i(x_0,\dots,x_k) = \begin{cases} (x_0,\dots,\hat{x_i},\dots,x_k) & \text{if } l(x_0,\dots,\hat{x_i},\dots,x_k) = l(x_0,\dots,x_k) \\ 0 & \text{otherwise.} \end{cases}$$

Remark 3.1.4. For a good tuple $(x_0,\dots,x_k)$,

$$\partial_i(x_0,\dots,x_k) \neq 0 \iff d(x_{i-1},x_i) + d(x_i,x_{i+1}) = d(x_{i-1},x_{i+1}).$$

Lemma 3.1.5. $\partial \circ \partial = 0$.

Proof. aaa. □

Definition 3.1.6. (magnitude homology) Let G be a graph. The *magnitude homology* $MH_{*,*}(G)$ of G is the homology of the magnitude chain complex $MC_{*,*}(G)$, that is,

$$MH_{k,l}(G) = H_k(MC_{*,l}(G)).$$

Example 3.1.7. (i) $G = V_n$. Then,

$$MC_{k,l}(V_n) = \begin{cases} \mathbb{Z}\{(x) | x \in V(V_n)\} & (k = l = 0) \\ 0 & \text{(otherwise).} \end{cases}$$

$\partial = 0$ inplies that

$$MH_{k,l}(V_n) \approx \begin{cases} \mathbb{Z}^n & (k = l = 0) \\ 0 & \text{(otherwise).} \end{cases}$$

(ii) $G = K_n (n \geq 2)$. Then, $l(x_0,\cdots,x_k) = k$ for any good tuple $(x_0,\cdots,x_k)$ of $V(K_n)$. Thus,

$$MC_{k,l}(K_n) = \begin{cases} \mathbb{Z}\{(x_0,\dots,x_k) | x_0 \neq x_1 \neq \cdots \neq x_k\} & (l = k) \\ 0 & \text{(otherwise).} \end{cases}$$

$\partial = 0$ inplies that

$$MH_{k,l}(K_n) \approx \begin{cases} \mathbb{Z}^{n(n-1)^l} & (l = k) \\ 0 & \text{(otherwise).} \end{cases}$$

(iii) $G = C_5$. Number the vertices of C_5 as shown in the following figure.
ここにナンバリングした C_5 の図を挿入
Let us consider $MH_{2,3}(C_5)$. 続く

Theorem 3.1.8. *Let G be a graph. Then,*

$$\sum_{k,l\geq 0}(-1)^k rank(MH_{k,l}(G))q^l = \#G \text{ in } \mathbb{Z}[[q]].$$

Proof.

$$\begin{aligned}(LHS) &= \sum_{l\geq 0}\chi(MH_{*,l}(G))q^l \\ &= \sum_{l\geq 0}\chi(MC_{*,l}(G))q^l \\ &= \sum_{k,l\geq 0}(-1)^k \mathrm{rank}(MC_{k,l}(G))q^l \\ &= \sum_{k\geq 0}(-1)^k \sum_{x_0\neq\cdots\neq x_k} q^{d(x_0,x_1)+\cdots+d(x_{k-1},x_k)} \\ &= \#G.\end{aligned}$$

The last equation is obtained by Proposition 2.2.10. □

Proposition 3.1.9. *Let G be a graph. Then,*

- $MH_{0,0}(G) \approx \mathbb{Z}^{|V(G)|}$.
- $MH_{1,1}(G) \approx \mathbb{Z}^{2|E(G)|}$.

holds.

Proof.

$$MC_{k,0}(G) = \begin{cases}\mathbb{Z}\{(x)|x\in V(G)\} & (k=0)\\ 0 & (\text{otherwise})\end{cases}$$

and $\partial = 0$ induces the first equation.

$$MC_{k,1}(G) = \begin{cases}\mathbb{Z}\{(x_0,x_1)|x_0\neq x_1\} & (k=1)\\ 0 & (\text{otherwise})\end{cases}$$

and $\partial = 0$ induces the second equation. □

Definition 3.1.10. The diameter d of a graph G is defined by

$$d = \max\{d(x,y) | x, y \in V(G) \text{ and } x, y \text{ lie in the same component of G}\}.$$

If $G = V_n$, then we define $d = 0$. Then, for any graph G, $0 \leq d < \infty$.

Proposition 3.1.11. *Let G be a graph and d be the diameter of G and assume that $MH_{k,l}(G) \neq 0$ for given non-negative integers k and l. Then,*

- $\frac{l}{d} \leq k \leq l$.
- *If $d > 1$ and $l > 0$, then* $\frac{l}{d} < k \leq l$.

holds.

Proof. Since $MH_{k,l}(G) \neq 0$, there exists a good tuple $(x_0, \ldots, x_k)$ of length l such that $\partial(x_0, \ldots, x_k) = 0$. Thus, $l = l(x_0, \ldots, x_k) = \sum_{i=1}^{k} d(x_{i-1}, x_i) \leq \sum_{i=1}^{k} d = kd$ and $l = \sum_{i=1}^{k} d(x_{i-1}, x_i) \geq k$. This implies that $\frac{l}{d} \leq k \leq l$.

Now, assume that $d > 1$ and $l > 0$ and suppose that $k = \frac{l}{d}$. From the above discussion, we have $d(x_i, x_{i+1}) = d$ for all i. $\partial(x_0, \cdots, x_k) = 0$. For the $(k+1)$-tuple $\partial(x_0, \cdots, x_k)$ is a linear combination of at most $k-1$ distinct terms of k-tuples, so $\partial(x_0, \cdots, x_k) = 0$ implies $d(x_{i-1}, x_i) + d(x_i, x_{i+1}) \neq d(x_{i-1}, x_{i+1})$ for all $1 \leq i \leq k-1$. Since $d(x_0, x_1) = d \geq 2$, there exists a vertex y such that $d(x_0, y) + d(y, x_1) = d(x_0, x_1)$ and $y \neq x_0, x_1$. Then, $(x_0, y, x_1, \cdots, x_k)$ is a good tuple in $MC_{k+1,l}(G)$ and

$$\partial_i(x_0, y, x_1, \cdots, x_k) = \begin{cases} (x_0, x_1, \cdots, x_k) & (i = 1) \\ 0 & (2 \leq i \leq k). \end{cases}$$

. It is obvious for $3 \leq i$ by $d(x_{i-1}, x_i) + d(x_i, x_{i+1}) \neq d(x_{i-1}, x_{i+1})$ and is also true for $i = 2$ since $d(y, x_1) + d(x_1, x_2) = d(y, x_1) + d > d \geq d(y, x_2)$. This implies $MH_{k,l}(G) = 0$ and contradicts the assumption. □

3.2 Induced Maps

Definition 3.2.1. Let G and H be graphs. A map $f : V(G) \to V(H)$ is said to be a *graph map* if for any $\{x, y\} \in E(G)$, either $f(x) = f(y)$ or $\{f(x), f(y)\} \in E(H)$.

Proposition 3.2.2. *$l(f(x_0), \cdots, f(x_k)) \leq l(x_0, \cdots, x_k)$ for any good tuple $(x_0, \cdots, x_k)$ of $V(G)$.*

Proof. For any vertices $x, y \in V(G)$, $d_H(f(x), f(y)) \leq d_G(x, y)$ holds. Indeed, if x, y lie in the same component of G, then there exists a path $(x = x_0 \to x_1 \to \cdots \to x_n = y)$ in G such that $n = d_G(x, y)$. Since f is a graph map, either $f(x_{i-1}) = f(x_i)$ or $\{f(x_{i-1}), f(x_i)\} \in E(H)$ for any $1 \leq i \leq n$. Thus, $(f(x) = f(x_0) \to f(x_1) \to \cdots \to f(x_n) = f(y))$ is a path in H and then $d_H(f(x), f(y)) \leq n = d_G(x, y)$. If x, y do not lie in the same component, then $d_G(x, y) = d_H(f(x), f(y)) = \infty$. Then,

$$\begin{aligned} l(f(x_0), \cdots, f(x_k)) &= \sum_{i=1}^{k} d_H(f(x_{i-1}), f(x_i)) \\ &\leq \sum_{i=1}^{k} d_G(x_{i-1}, x_i) \\ &= l(x_0, \cdots, x_k). \end{aligned}$$

□

Definition 3.2.3. Let G and H be graphs and $f : V(G) \to V(H)$ be a graph map. Then, the *induced map* $f_\# : MC_{*,*}(G) \to MC_{*,*}(H)$ is defined by

$$f_\#(x_0, \cdots, x_k) = \begin{cases} (f(x_0), \cdots, f(x_k)) & l(f(x_0), \cdots, f(x_k)) = l(x_0, \cdots, x_k) \\ 0 & \text{otherwise} \end{cases}$$

for any good tuple $(x_0, \cdots, x_k)$ of $V(G)$.

Proposition 3.2.4. *The induced map $f_\# : MC_{*,*}(G) \to MC_{*,*}(H)$ is a chain map.*

Proof. aaa. □

Definition 3.2.5. (Induced maps in homoplogy) If $f : G \to H$ is a graph map, the *induced map in homology* $f_* : MH_{*,*}(G) \to MH_{*,*}(H)$ is the map induced by the chain map $f_\# : MC_{*,*}(G) \to MC_{*,*}(H)$.

Proposition 3.2.6. *The assignment $G \mapsto MH_{*,*}(G)$ and $f \mapsto f_*$ defines a functor from the category of graphs and graph maps to the category of bigraded abelian groups and bigraded homomorphisms, denoting by* **Graph** $\to$ **BAb**.

Remark 3.2.7. A is called a *bigaded abelian group* if $A = \bigoplus_{k,l\geq 0} A_{k,l}$ where each $A_{k,l}$ is an abelian group. A *bigraded homomorphism* $f : A \to B$ between bigraded abelian groups A and B is a homomorphism such that $f(A_{k,l}) \subset B_{k,l}$ for any $k, l \geq 0$.

Proposition 3.2.8. *Let $f : G \to H$ be a graph map.*

- *$f_* : MH_{0,0}(G) \to MH_{0,0}(H)$ is given by $f_*(x) = f(x)$ for any $x \in V(G)$.*
- *$f_* : MH_{1,1}(G) \to MH_{1,1}(H)$ is given by*

$$f_*(x_0, x_1) = \begin{cases} (f(x_0), f(x_1)) & (\textit{ if }) \\ 0 & (\textit{otherwise}). \end{cases}$$

  for any $(x_0, x_1) \in MH_{1,1}(G)$.

Proof. The first equation is obvious.
For the second equation, we obtain by definition;

$$f_*(x_0, x_1) = \begin{cases} (f(x_0), f(x_1)) & l(f(x_0), f(x_1)) = l(x_0, x_1) = 1 \\ 0 & \text{otherwise} \end{cases}$$

for any $(x_0, x_1) \in MH_{1,1}(G)$. Since f is a graph map, $l(f(x_0), f(x_1)) = 1$ if and only if $f(x_0) \neq f(x_1)$. □

Corollary 3.2.9. *Let $f : G \to H$ be a graph map. f_* is an isomorphism if and only if f is a graph isomorphism.*

Proof. aaa. □

3.3 Disjoint Union

Proposition 3.3.1. *Let G and H be graphs. We define the inclusion graph maps $i : G \to G \bigsqcup H, j : H \to G \bigsqcup H$. Then,*

$$i_* \oplus j_* : MH_{*,*}(G) \oplus MH_{*,*}(H) \to MH_{*,*}(G \bigsqcup H)$$

is an isomorphism for each $k, l \geq 0$.

Proof. saaa. □

We obtain Lemma 2.2.5 by Proposition 3.3.1 and $\chi(A_* \oplus B_*) = \chi(A_*) + \chi(B_*)$.

3.4 Cartesian Products

Definition 3.4.1. This definition is not true. Fix $l \geq 0$. The *exterior product* is the map

$$\Box : MC_{*,*}(G) \otimes MC_{*,*}(H) \to MC_{*,*}(G \Box H)$$

is defined as follows. Let $\Box$ be the map

$\Box : MC_{k_1,l_1}(G) \times MC_{k_2,l_2}(H) \to MC_{k,l}(G \Box H)$ for $k_1, k_2 \geq 0, k = k_1 + k_2, l = l_1 + l_2$,

which is defined by

$$\Box((x_0, \cdots, x_{k_1}), (y_0, \cdots, y_{k_2})) = \sum_{\sigma} \mathrm{sign}(\sigma)((x_{i_0}, y_{j_0}), (x_{i_1}, y_{j_1}), \cdots, (x_{i_k}, y_{j_k})),$$

where the sum is over all shuffles σ of type (k_1, k_2), that is, all sequences

$$((i_0, j_0), (i_1, j_1), \cdots, (i_k, j_k))$$

such that

$$i_0 = 0, j_0 = 0, 0 \leq i_r \leq k_1, 0 \leq j_r \leq k_2 \text{ for } 0 \leq r \leq k, \text{ and}$$

$$(i_{r+1}, j_{r+1}) = \begin{cases} (i_r + 1, j_r) & \text{or} \\ (i_r, j_r + 1) & \text{for } 0 \leq r < k, \end{cases}$$

and

$$\mathrm{sign}(\sigma) = (-1)^m,$$

where $m = \#\{(i,j) \in \{\{0, \cdots, k_1\} \times \{0, \cdots, k_2\}\} | i = i_r \Rightarrow j < j_r\}$.

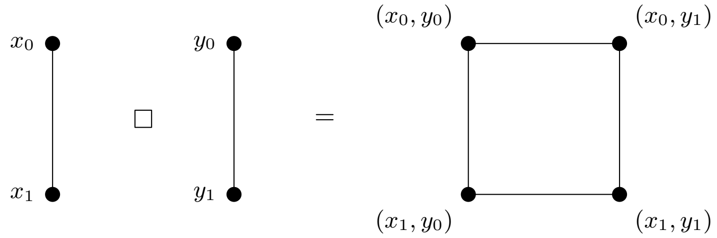
Here, we extend the map $\Box$ bilinearly to the tensor product

$$MC_{k_1,l_1}(G) \otimes MC_{k_2,l_2}(H) \to MC_{k,l}(G \Box H)$$

.

We denote this induced map also by $\Box$ and call it the *exterior product.*

Example 3.4.2. Let $G = C_2 \Box C_2 = C_4$



Consider the exterior product $\Box((x_0, x_1) \otimes (y_0, y_1))$. We have the two shuffles of type $(1, 1)$:

$$((0,0),(1,0),(1,1)),\ ((0,0),(0,1),(1,1)).$$

Thus,

$$\Box((x_0,x_1)\otimes(y_0,y_1)) = -((x_0,y_0),(x_1,y_0),(x_1,y_1)) + ((x_0,y_0),(x_0,y_1),(x_1,y_1)).$$

Remark 3.4.3. As you see in the above example, the number of shuffles is $\binom{k}{k_1}$.

Proposition 3.4.4. *The exterior product $\Box : MC_{*,*}(G) \otimes MC_{*,*}(H) \to MC_{*,*}(G\Box H)$ is a chain map.*

Proof. Let $\mathbf{x} = (\mathbf{x_0}, \cdots, \mathbf{x_{k_1}}), \mathbf{y} = (\mathbf{y_0}, \cdots, \mathbf{y_{k_2}})$. Now, we show that

$$\partial \circ \Box(\mathbf{x} \otimes \mathbf{y}) = \Box((\partial\mathbf{x}) \otimes \mathbf{y}) + (-\mathbf{1})^{\mathbf{k_1}}\Box(\mathbf{x} \otimes (\partial\mathbf{y})) = \Box \circ (\partial \otimes \partial)(\mathbf{x} \otimes \mathbf{y}).$$

Here, we should consider the sequence of tensor products of the magnitude chain complexes defined by

$$\begin{aligned}\partial \otimes \partial : MC_{k_1,l_1}(G) \otimes MC_{k_2,l_2}(H) \to MC_{k_1-1,l_1}(G) \otimes MC_{k_2,l_2}(H)\\ \oplus MC_{k_1,l_1}(G) \otimes MC_{k_2-1,l_2}(H),\\ (\partial \otimes \partial)(\mathbf{x} \otimes \mathbf{y}) = (\partial\mathbf{x}) \otimes \mathbf{y} + (-1)^{k_1}\mathbf{x} \otimes (\partial\mathbf{y}).\end{aligned}$$

Then, we should show only the first equality.
ここに可換図式を挿入. 証明も続く $\Box$

From this proposition, we obtain the induced map in homology, also denoting $\Box$.

Definition 3.4.5. (Tor functor) Let R be a ring and A and B be R-modules. Then, $\mathrm{Tor}(A, B)$ is defined by the derived functor of the tensor product.

Theorem 3.4.6. *Let G and H be graphs.*

$$0 \to MH_{*,*}(G)\otimes MH_{*,*}(H) \xrightarrow{\Box} MH_{*,*}(G\Box H) \to \mathrm{Tor}(MH_{*-1,*}(G), MH_{*,*}(H)) \to 0$$

is a short exact sequence and non-naturally split. In particular, if $MH_{*,*}(G)$ or $MH_{*,*}(H)$ is torsion-free, then the exterior product $\Box$ is an isomorphism.

We don't prove this theorem in this thesis.

Example 3.4.7. $G = C_4 = C_2\Box C_2$.

3.5 The Mayer-Vietoris Sequence

Definition 3.5.1. Let X be a graph and G, H be subgraphs of X.

1. $(X; G, H)$ is said to be a *projecting decomposition* if $X = G \cup H$, $G \cap H$ is convex in X and H projects to $G \cap H$.
   We write $i^G : G \to X$, $i^H : H \to X$, $j^G : G \cap H \to G$, $j^H : G \cap H \to H$ for the inclusion graph maps.

2. Let $(X; G, H), (X'; G', H')$ be projecting decompositions. $f : (X; G, H) \to (X'; G', H')$ is said to be a *decomposition map* if $f : X \to X'$ is a graph map such that $f(V(G)) \subset V(G')$ and $f(V(H)) \subset V(H')$.

3. Let $f : (X; G, H) \to (X'; G', H')$ be a decomposition map. Then, f is said to be a *projecting decomposition map* if $V_{G\cap H}(H) = f^{-1}(V_{G'\cap H'}(H'))$ and $f(\pi(h)) = \pi(f(h))$ for any $h \in V_{G\cap H}(H)$.

4. Let $(X; G, H)$ be a projecting decomposition. $MC_{*,*}(G, H)$ denote the subcomplex of $MC_{*,*}(X)$ spanned by good tuples $(x_0, \cdots, x_k)$ whose entries all lie in G or all lie in H.

Theorem 3.5.2. *Let $(X; G, H)$ be a projecting decomposition. Then, the inclusion map*

$$MC_{*,l}(G, H) \hookrightarrow MC_{*,l}(X)$$

is a quasi-isomorphism for any $l \geq 0$.

Proof. aaa. □

Theorem 3.5.3. *(the main theorem II) Let $(X; G, H)$ be a projecting decomposition. Then,*

$$0 \to MH_{*,*}(G\cap H) \xrightarrow{(j^G_*, -j^H_*)} MH_{*,*}(G) \oplus MH_{*,*}(H) \xrightarrow{i^G_* \oplus i^H_*} MH_{*,*}(X) \to 0$$

is a split short exact sequence.

Proof. aaa. □

Corollary 3.5.4. *Let $(X; G, H)$ be a projecting decomposition. Then,*

$$\#X = \#G + \#H - \#(G \cap H)$$

in $\mathbb{Z}[[q]]$.

Proof. By Theorem 3.5.3,

$$\chi(MH_{*,l}(G \cap H)) - \chi((MH_{*,l}(G)) \oplus \chi(MH_{*,l}(H))) + \chi(MH_{*,l}(X)) = 0.$$
$$\Rightarrow \chi(MH_{*,l}(X)) = \chi(MH_{*,l}(G)) + \chi(MH_{*,l}(H)) - \chi(MH_{*,l}(G \cap H)).$$

For each $l \geq 0$, multiplying by q^l and summing over all $l \geq 0$, we have

$$\sum_{l\geq 0} \chi(MH_{*,l}(X))q^l = \sum_{l\geq 0} \chi(MH_{*,l}(G))q^l + \sum_{l\geq 0} \chi(MH_{*,l}(H))q^l - \sum_{l\geq 0} \chi(MH_{*,l}(G \cap H))q^l.$$

By Theorem 3.1.8, we obtain the desired equation. □

Corollary 3.5.5. *Let T be a tree.*

3.6 Diagonal Graphs

Definition 3.6.1. A graph G is said to be *diagonal* if $MH_{k,l}(G) = 0$ for $k \neq l$.

Lemma 3.6.2. *A tree is diagonal.*

Proof. aaa □

Proposition 3.6.3. *For a diagonal graph, the magnitude completely determines the magnitude homology ranks.*

Proof. Obvious by Theorem 3.1.8. □

4 Motivation: The Magnitude of Enriched Categories

In this section, we explain the motivation for studying the magnitude of graphs in a broader context. We employ the notion of enriched categories to define the magnitude.

4.1 The Magnitude of a Matrix

Definition 4.1.1. Let k be a set and $+, \cdot$ be a binary operation on k, and $0_k, 1_k$ be elements of k. Then, $(k, +, \cdot, 0_k, 1_k)$ is called a *rig* if the following conditions hold:

- $(k, +, 0_k)$ is a commutative monoid.
- $(k, \cdot, 1_k)$ is a monoid.
- multiplication distributes over addition.

Now, we mean a rig as a commutative rig with the operation $\cdot$.

Example 4.1.2. $(\mathbb{Z}_{\geq 0}, +, \cdot, 0, 1)$ is a rig.

Definition 4.1.3. Let k be a rig and I, J be finite sets. A $I \times J$-matrix is a function $\zeta : I \times J \to k$.

Remark 4.1.4. Let k be a rig, and I, J, L be finite sets.

1. If ζ_1 is an $I \times J$-matrix and ζ_2 is a $J \times L$-matrix, then the product $\zeta_1\zeta_2$ is defined as follows:

$$(\zeta_1\zeta_2)(i, l) = \sum_{j \in J} \zeta_1(i, j) \cdot \zeta_2(j, l) \quad (i \in I, l \in L)$$

2. $\delta : I \times I \to k$ is called the *identity matrix* if $\delta(i, j) = 1_k$ when $i = j$ and $\delta(i, j) = 0_k$ when $i \neq j$.

3. Let $\zeta : I \times J \to k$ be a matrix. We define $\zeta^* : J \times I \to k$ by $\zeta^*(j, i) = \zeta(i, j)$.

4. Let ζ be an $I \times I$-matrix. If there exists an $I \times I$-matrix ζ^{-1} such that $\zeta\zeta^{-1} = \delta$ and $\zeta^{-1}\zeta = \delta$, then ζ is said to be *invertible* and ζ^{-1} is called the *inverse* of ζ.

5. $w : I \to k$ is called a *vector*. w can be thought of as an element of k^I. If ζ is an $I \times J$-matrix, v is a I-vector, and w is a J-vector, then the product $\zeta w : I \to k$ and $v\zeta : J \to k$ are defined by

$$(\zeta w)(i) = \sum_{j \in J} \zeta(i, j) \cdot w(j) \quad (i \in I)$$

$$(v\zeta)(j) = \sum_{i \in I} v(i) \cdot \zeta(i, j) \quad (j \in J)$$

Now, ζw is a I-vector and $v\zeta$ is a J-vector.

6. If w, v are I-vectors, then the *inner product* vw is defined by

$$vw = \sum_{i \in I} v(i) \cdot w(i)$$

7. A vector $u_I : I \to k$ is defined by $u_I(i) = 1_k$ for any $i \in I$.

Definition 4.1.5. Let ζ be an $I \times I$-matrix over a rig k.

- A *weighting* on ζ is a vector $w : J \to k$ such that $\zeta w = u_I$. $w(j)$ is called the *weight* of $j \in J$.

- A *coweighting* on ζ is a vector $v : I \to k$ such that $v\zeta = u_I^*$. $v(i)$ is called the *coweight* of $i \in I$.

Example 4.1.6. Let G be a graph. Then, $Z_G(q)$ is a $V(G) \times V(G)$-matrix over the rig $\mathbb{Q}[[q]]$ and the weighting on $Z_G(q)$ is the weighting on G defined in Section 2.1.

Lemma 4.1.7. *Let ζ be an $I \times I$-matrix over a rig k. If ζ has a weighting w and a coweighting v, then*

$$\sum_{i \in I} v(i) = \sum_{j \in J} w(j)$$

Proof.

$$\begin{aligned}\sum_{i\in I} v(i) &= vu_I \\ &= v(\zeta w) \\ &= (v\zeta)w \\ &= u_J w \\ &= \sum_{j\in J} w(j)\end{aligned}$$

□

From this lemma, the sum of the weighting or coweighting on ζ is unique if they exist. Therefore, we can define the magnitude of ζ as follows:

Definition 4.1.8. Let ζ be an $I \times J$-matrix over a rig k. If ζ has a weighting and a coweighting, then the *magnitude* of ζ is defined to be

$$\#\zeta = \sum_{i\in I} v(i) = \sum_{j\in J} w(j)$$

where w is the weighting on ζ and v is the coweighting on ζ.

Lemma 4.1.9. *Let ζ be an $I \times I$-matrix over a rig k.*

1. *If ζ is invertible, then ζ has the magnitude.*

2. *If k is a field and ζ has the magnitude, then ζ is invertible.*

Proof. (1) If ζ is invertible, then $w = \zeta^{-1}u_I$ and $v = u_I\zeta^{-1}$ obviously satisfy the definition of weighting and coweighting respectively. Thus ζ has the magnitude by Lemma 4.1.7.

(2) If k is a field and ζ has the magnitude, then there exist a weighting w and a coweighting v on ζ. Let ζx be a zero-map for some $x : I \to k$. Then,

$$0 = v(\zeta x) = (v\zeta)x = u_I x = \sum_{i\in I} x(i)$$

ここからやり直し □

Lemma 4.1.10. *Let ζ be an invertible $I \times I$-matrix over a rig k. Then, ζ has the unique weighting w of ζ, given by $w(j) = \sum_{i \in I} \zeta^{-1}(j,i)$ for $j \in I$, and the unique coweighting v of ζ, given by $v(i) = \sum_{j \in I} \zeta^{-1}(j,i)$ for $i \in I$. Then,*

$$\#\zeta = \sum_{i,j \in I} \zeta^{-1}(j,i)$$

Proof. We should check the uniqueness and it holds from the invertibility of ζ. $\square$

4.2 The Definition of Enriched Categories

In this document, we only treat the locally small categories, which means that for any objects a, b of a category $\mathcal{C}$, the hom-set $\hom_{\mathcal{C}}(a,b)$ is a set.

Definition 4.2.1. A category $\mathcal{C}$ is called a *monoid* if $\mathcal{C}$ has only one object $*$ and $V = Hom_{\mathcal{C}}(*,*)$ is a monoid with the composition of morphisms as the binary operation and the identity morphism id_* as the identity element. We denote the operation of V by $\otimes$. ここに可換図式を挿入

Definition 4.2.2. A pair $(\mathcal{V}, \otimes, I)$ is called a *monoidal category* if it satisfies the following conditions:

1. $\mathcal{V}$ is a category.
2. $\otimes : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ is a functor.
3. I is an object of $\mathcal{V}$.
4. There exist the natural isomorphism $\alpha : \otimes \circ (\otimes \times id_{\mathcal{V}}) \Rightarrow \otimes \circ (id_{\mathcal{V}} \times \otimes)$ given by $\alpha_{uvw} : (u \otimes v) \otimes w \xrightarrow{\sim} u \otimes (v \otimes w)$.
5. There exist the natural isomorphism $\lambda : I \otimes - \Rightarrow id_{\mathcal{V}}$ given by $\lambda_u : I \otimes u \xrightarrow{\sim} u$.
6. There exist the natural isomorphism $\rho : - \otimes I \Rightarrow id_{\mathcal{V}}$ given by $\rho_u : u \otimes I \xrightarrow{\sim} u$.
7. The following diagram commutes for any objects u, v, w, x of $\mathcal{V}$: ここに可換図式を挿入

8. The following diagrams commute for any objects u, v of $\mathcal{V}$: ここに可換図式を挿入

Example 4.2.3. (i) $(Set, \times, \{*\})$ is a monoidal category. ここに説明を挿入

(ii) $(Vect_K, \otimes_K, K)$ is a monoidal category, where K is a field. ここに説明を挿入

(iii) $([0,\infty], +, 0)$ is a monoidal category. ここに説明を挿入

(iv) $(\mathbf{2}, \otimes, t)$ is a monoidal category, where $\mathbf{2}$ is the category defined by $Ob(\mathbf{2}) = \{t, f\}$ and the morphism sets are defined by

$\hom_{\mathbf{2}}$	t	f
t	$\{id_t\}$	$\emptyset$
f	$\{*\}$	$\{id_f\}$

and the operation $\otimes$ is defined by the following table:

$\otimes$	t	f
t	t	f
f	f	f

ここに説明を挿入

Then, $\mathbf{2}$ is a monoidal subcategory of $[0,\infty]$ by the embedding $t \mapsto 0, f \mapsto \infty$ and of Set by the embedding $t \mapsto \{*\}, f \mapsto \emptyset$.

Definition 4.2.4. An *enriched category* $\mathcal{A}$ in a monoidal category $(\mathcal{V}, \otimes, I)$ is defined as follows:

1. For any objects a, b of $\mathcal{A}$, $\hom_{\mathcal{A}}(a,b)$ is an object of $\mathcal{V}$.

2. For any objects a, b, c of $\mathcal{A}$, there exists a morphism $m_{abc} : \hom_{\mathcal{A}}(b,c) \otimes \hom_{\mathcal{A}}(a,b) \to \hom_{\mathcal{A}}(a,c)$ in $\mathcal{V}$, which defines the composition of morphisms.

3. For any object a of $\mathcal{A}$, there exists a morphism $j_a : I \to \hom_{\mathcal{A}}(a,a)$ in $\mathcal{V}$, which defines the identity morphism of a. あと3つの可換図式を挿入

Then, $\mathcal{A}$ is called a $\mathcal{V}$*-category*.

Definition 4.2.5. Let $\mathcal{A}, \mathcal{A}'$ be $\mathcal{V}$-categories. A functor $F : \mathcal{A} \to \mathcal{A}'$ is called a $\mathcal{V}$*-functor* if it satisfies the following conditions. We denote $F_{ab} : \mathrm{Mor}_{\mathcal{A}}(a,b) \to \mathrm{Mor}_{\mathcal{A}'}(F(a), F(b))$ as the morphism function.

1. The following diagram commutes for any objects a, b, c of $\mathcal{A}$: ここに可換図式を挿入

2. The following diagram commutes for any object a of $\mathcal{A}$: ここに可換図式を挿入

Remark 4.2.6. The family of all $\mathcal{V}$-categories and $\mathcal{V}$-functors form a category, which is denoted by $\mathcal{V}$-*Cat*.

Example 4.2.7. aaa

4.3 The Magnitude of Enriched Categories

Definition 4.3.1. Let $(\mathcal{V}, \otimes, I)$ be a monoidal category and k be a rig. We define a monoid homomorphism

$$|\cdot| : (Ob(\mathcal{V})/\approx, \otimes, I) \to (k, \cdot, 1_k)$$

such that $|I| = 1_k$ and $|u \otimes v| = |u| \cdot |v|$ for any objects u, v of $\mathcal{V}$.

Example 4.3.2. aaa.

Definition 4.3.3. Let $\mathcal{A}$ be a $\mathcal{V}$-category.

4.4 The Relation of The Magnitudes of Graphs and Enriched Categories

ここが私が一番説明したい部分です.

Appendix

A The calculation of Graph Automorphisms

Proposition A.0.1. *Let $G = K_{n,n}$. Then, $Aut(G) \approx (\mathfrak{S}_n \times \mathfrak{S}_n) \rtimes_s \mathbb{Z}_2$, where $s : \mathbb{Z}_2 \to Aut(G); 0 \mapsto id_G, 1 \mapsto \tau$ and τ is the automorphism which interchanges the two parts of $K_{n,n}$.*

Proof. Now,

$$0 \to \mathfrak{S}_n \times \mathfrak{S}_n \xrightarrow{incl} \mathrm{Aut}(G) \xrightarrow{s} \mathbb{Z}_2 \to 0$$

is exact and this sequence splits. Thus, we have $\mathrm{Aut}(G) \approx (\mathfrak{S}_n \times \mathfrak{S}_n) \rtimes_s \mathbb{Z}_2$. □

References

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