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# Magnitude Homology of Graphs and the Magnitude as its Categorification

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**Abstract**

Sample Abstract

# Contents

<b>1</b>	<b>Introduction</b>	<b>4</b>
<b>2</b>	<b>The Magnitude of Graphs</b>	<b>5</b>
2.1	The Definition of the Magnitude of Graphs . . . . .	5
2.2	Basic Properties and Examples . . . . .	8
2.3	Main Results on the Magnitude of Graphs . . . . .	14
<b>3</b>	<b>The Magnitude Homology of Graphs</b>	<b>19</b>
3.1	The Definition of The Magnitude Homology of Graphs . .	19
3.2	Induced Maps on Magnitude Homology . . . . .	22
3.3	Magnitude Homology of Disjoint Union of Graphs . . . .	24
3.4	Magnitude Homology of Cartesian Products . . . . .	25
3.5	The Mayer-Vietoris Sequence of Magnitude Homology of Graphs . . . . .	27
3.6	Diagonal Graphs and its Property . . . . .	28
<b>4</b>	<b>Motivation: The Magnitude of Enriched Categories</b>	<b>30</b>
4.1	The Magnitude of a Matrix . . . . .	30
4.2	The Definition of Enriched Categories . . . . .	33
4.3	The Magnitude of Enriched Categories . . . . .	35
4.4	The Relation of The Magnitudes of Graphs and Enriched Categories . . . . .	36
<b>Appendix</b>		<b>37</b>
<b>A</b>	<b>The calculation of Graph Automorphisms</b>	<b>37</b>

# 1 Introduction

Lamport's guide to L<sup>A</sup>T<sub>E</sub>X [?]. We denote  $\approx$  as isomorphisms and  $\cong$  as homeomorphisms.

## 2 The Magnitude of Graphs

This section introduces the magnitude and the magnitude homology of a graph  $G$ , along with fundamental examples and properties. Throughout this thesis, a *graph* means a finite undirected graph with no loops or multiple edges. The set of vertices of a graph  $G$  is denoted by  $V(G)$ , and the set of edges of  $G$  is denoted by  $E(G)$ . For vertices  $x, y \in V(G)$ , the *distance*  $d_G(x, y)$  is defined as the length of a shortest path between them, where the length of a path is the number of edges it contains. If  $x$  and  $y$  lie in different connected components of  $G$ , we set  $d_G(x, y) = \infty$ . Now, we say that two vertex  $x, y$  in  $G$  lie in the same connected component if there exists a path between them.

### 2.1 The Definition of the Magnitude of Graphs

We begin by defining the magnitude of a graph. The magnitude is defined to take values in the field of rational functions over  $\mathbb{Q}$ . It can also be interpreted as taking values in the ring of formal power series over  $\mathbb{Z}$ , a property that will be discussed in a later lemma. Let  $\mathbb{Q}(q)$  denote the field of rational functions in a variable  $q$  over  $\mathbb{Q}$ . We also write  $\mathbb{Z}[q]$  and  $\mathbb{Z}[[q]]$  for the polynomial ring and the ring of formal power series in  $q$  over  $\mathbb{Z}$ , respectively.

**Definition 2.1.1.** Let  $G$  be a graph. We define the  *$G$ -matrix*  $Z_G = Z_G(q)$  over  $\mathbb{Z}[q]$  whose rows and columns are indexed by the vertices of  $G$ , and whose  $(x, y)$ -entry is given by

$$Z_G(q)(x, y) = q^{d(x, y)} \quad \text{for } x, y \in V(G),$$

where by convention  $q^\infty = 0$ .

$G$ -matrix is the square symmetric matrix.

**Proposition 2.1.2.** *A  $G$ -matrix is invertible.*

*Proof.* By definition, the determinant of  $Z_G$  has constant term 1, which implies that  $\det Z_G \neq 0$ .  $\square$

**Definition 2.1.3.** The *magnitude* of a graph  $G$  is defined to be the rational function given by

$$\#G(q) = \sum_{x,y \in V(G)} (Z_G(q))^{-1}(x,y).$$

**Remark 2.1.4.**

$$\#G(q) = \text{sum}(Z_G(q)^{-1}) = \frac{\text{sum}(\text{adj}(Z_G(q)))}{\det(Z_G(q))},$$

where  $\text{adj}$  is the adjugate matrix and  $\text{sum}$  is the sum of all entries of a matrix.

**Proposition 2.1.5.**  $\#G(q)$  takes values in  $\mathbb{Z}[[q]]$ .

*Proof.* Let  $\det Z_G(q) = 1 - qf(q)$  for some  $f(q) \in \mathbb{Z}[q]$  by Proposition 2.1.2. Then we have

$$\#G(q) = \frac{\text{sum}(\text{adj}(Z_G))}{\det(Z_G)} = \text{sum}(\text{adj}(Z_G)) \sum_{n=0}^{\infty} q^n f(q)^n.$$

Note that  $qf(q)$  has no constant term and then  $\sum_{n=0}^{\infty} q^n f(q)^n$  takes values in  $\mathbb{Z}[[q]]$ .  $\square$

**Example 2.1.6.** Let  $G = K_3$  (complete graph with three vertices).

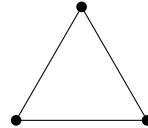


図 1: Complete graph of three vertices

Then, we can calculate the magnitude of  $K_3$  as follows:

$$Z_{K_3}(q) = \begin{pmatrix} 1 & q & q \\ q & 1 & q \\ q & q & 1 \end{pmatrix}, \quad Z_{K_3}(q)^{-1} = \frac{1}{1 - 3q^2 + 2q^3} \begin{pmatrix} 1 - q^2 & -q + q^2 & -q + q^2 \\ -q + q^2 & 1 - q^2 & -q + q^2 \\ -q + q^2 & -q + q^2 & 1 - q^2 \end{pmatrix},$$

$$\#K_3(q) = \frac{3}{1+2q}.$$

**Definition 2.1.7.** Let  $G$  be a graph and  $x \in V(G)$ . The *weight* of  $x$  in  $G$  is defined

$$w_G(x)(q) = \sum_{y \in V(G)} (Z_G(q))^{-1}(x, y)$$

The function  $w_G : V(G) \rightarrow \mathbb{Q}(q)$  is called the *weighting* on  $G$ .

The magnitude can be expressed using the weighting as follows:

$$\#G(q) = \sum_{x \in V(G)} w_G(x)$$

**Lemma 2.1.8.** For any graph  $G$ , the weighting  $w_G$  satisfies

$$\sum_{y \in V(G)} q^{d(x,y)} w_G(y) = 1 \quad \text{for } x \in V(G).$$

*Proof.* For any vertex  $x \in V(G)$ , we have

$$\begin{aligned} \sum_{y \in V(G)} q^{d(x,y)} w_G(y) &= \sum_{y,z \in V(G)} q^{d(x,y)} Z_G^{-1}(y, z) \\ &= \sum_{y,z \in V(G)} Z_G(x, y) Z_G^{-1}(y, z) \\ &= \sum_{z \in V(G)} \sum_{y \in V(G)} Z_G(x, y) Z_G^{-1}(y, z) \\ &= \sum_{z \in V(G)} (Z_G Z_G^{-1})(x, z) \\ &= \sum_{z \in V(G)} I(x, z) \\ &= 1. \end{aligned}$$

□

This equation is called the *weighting equation*.

**Lemma 2.1.9.** Let  $G$  be a graph and let  $\tilde{w}_G : V(G) \rightarrow \mathbb{Q}$  be a function satisfying a weighting equation. Then,  $\tilde{w}_G = w_G$ . Now,  $w_G$  is the weighting on  $G$ .

*Proof.* Let  $\mathbf{b} = (1, 1, \dots, 1)^T$  where the length of  $\mathbf{b}$  is  $|V(G)|$  and  $\mathbf{w}_G = (w_G(x))_{x \in V(G)}^T$ . If  $\tilde{\mathbf{w}}_G$  satisfies the weighting equation, then we have

$$Z_G \tilde{\mathbf{w}}_G = \mathbf{b}.$$

Since  $Z_G$  is invertible by Proposition 2.1.2, we have  $\tilde{\mathbf{w}}_G = \mathbf{w}_G$ .  $\square$

This lemma shows that the weighting on a graph is unique and we use this frequently to compute the magnitude of graphs.

## 2.2 Basic Properties and Examples

This subsection presents fundamental properties and examples of magnitude. We focus on vertex-transitive graphs, disjoint unions, and Cartesian products. We also discuss the properties of magnitude within  $\mathbb{Z}[[q]]$ .

**Definition 2.2.1.** Let  $G = (V(G), E(G))$ ,  $H = (V(H), E(H))$  be graphs. A *graph homomorphism* from  $G$  to  $H$  is a map  $f : V(G) \rightarrow V(H)$  such that if  $\{x, y\} \in E(G)$  then  $\{f(x), f(y)\} \in E(H)$  or  $f(x) = f(y)$ .

We can define a *graph automorphism* using the definition above. We denote the group of all graph automorphisms of a graph  $G$  by  $\text{Aut}(G)$ .  $\text{Aut}(G)$  includes  $\text{id}_G$  and for  $g, h \in \text{Aut}(G)$  and  $x \in V(G)$ ,  $g(h(x)) = (gh)(x)$ , which means  $\text{Aut}(G)$  acts on  $V(G)$ .

**Definition 2.2.2.** A graph  $G$  is *vertex-transitive* if  $\text{Aut}(G)$  acts transitively on  $V(G)$ . It says that for any vertices  $x$  and  $y$  of  $G$ , there exists an automorphism  $g : G \rightarrow G$  such that  $y = g(x)$ .

**Lemma 2.2.3.** Let  $G$  be a vertex-transitive graph. Then,

$$\#G(q) = \frac{|V(G)|}{\sum_{y \in V(G)} q^{d(x,y)}}$$

for any vertex  $x \in V(G)$ .

*Proof.* Let  $S(x) = \sum_{y \in V(G)} q^{d(x,y)}$  for a vertex  $x \in V(G)$ . We show that  $S(x)$  does not depend on the choice of  $x$ . Take any vertices  $a, b \in V(G)$ .

Since  $G$  is vertex-transitive, there exists  $g \in \text{Aut}(G)$  such that  $b = g(a)$ . Then,

$$\begin{aligned}
S(b) &= \sum_{y \in V(G)} q^{d(b,y)} \\
&= \sum_{y \in V(G)} q^{d(g(a),y)} \\
&= \sum_{y \in V(G)} q^{d(g(a),g(y))} \quad (\text{since } g \text{ is bijective}) \\
&= \sum_{y \in V(G)} q^{d(a,y)} \quad (\text{since } g \text{ is an isomorphism}) \\
&= S(a).
\end{aligned}$$

Thus,  $S(x)$  does not depend on the choice of  $x$ , denoting it by  $S$ . Now, we define a function  $\tilde{w}_G : V(G) \rightarrow \mathbb{Q}(q)$  by  $\tilde{w}_G(x) = \frac{1}{S}$  for any vertex  $x \in V(G)$ . Then  $\tilde{w}_G$  satisfies the weighting equation and by Lemma 2.1.9, we have  $\#G = \frac{|V(G)|}{S}$ .

□

**Example 2.2.4.** (i)  $G = V_n$  (edgeless graph with  $n$  vertices).



図 2: The graph with no edges

Then,  $\text{Aut}(G) \approx \mathfrak{S}_n$  and  $G$  is vertex-transitive.  $S = 1$  and we have  $\#V_n = n$ . Now, the empty graph can be denoted by  $V_0$ .

(ii)  $G = K_n$  (complete graph with  $n$  vertices).

Then,  $\text{Aut}(G) \approx \mathfrak{S}_n$  and  $G$  is vertex-transitive.  $S = 1 + (n - 1)q$  and we have  $\#K_n = \frac{n}{1 + (n - 1)q}$ .

(iii)  $G = C_n$  (cycle graph with  $n$  vertices).

Then,  $\text{Aut}(G) \approx D_{2n}$  and  $G$  is vertex-transitive. If  $n = 2m$ , then  $S = 1 + 2(q + q^2 + \cdots + q^{m-1}) + q^m = \frac{1+q-q^m-q^{m+1}}{1-q}$ . Thus, we have

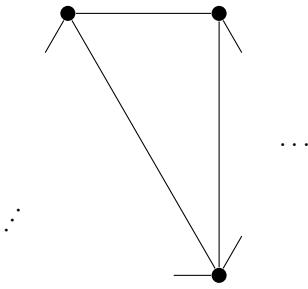


図 3: Complete graph

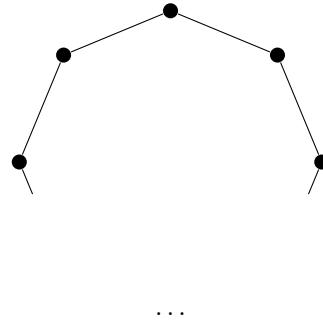


図 4: Cycle graph

$\#C_{2m} = \frac{2m(1-q)}{(1+q)(1-q^m)} = \frac{n(1-q)}{(1+q)(1-q^m)}$ . If  $n = 2m - 1$ , then similarly  
 $\#C_{2m-1} = \frac{n(1-q)}{1+q-2q^m}$ .

(iv)  $G$  is the Petersen graph.

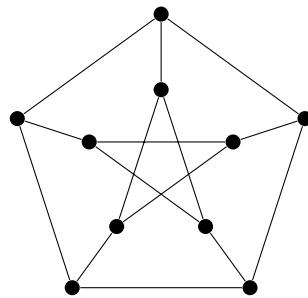


図 5: Petersen graph

Then,  $\text{Aut}(G)$  contains  $D_{10}$  as its subgroup and  $G$  is vertex-transitive.  
 $S = 1 + 3q + 6q^2$  and we have  $\#G = \frac{10}{1+3q+6q^2}$ .

(v)  $G = K_{m,n}$  (complete bipartite graph).

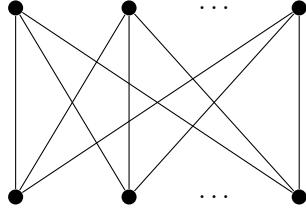


図 6: complete bipartite graph

Then,  $\text{Aut}(G) \approx \mathfrak{S}_m \times \mathfrak{S}_n$  if  $m \neq n$  and  $G$  is not vertex-transitive. We can calculate the magnitude with other methods. Let  $a, b$  be the weight of vertices on each part of  $K_{m,n}$ . Then, the weighting equation is written by two equations as follows:

$$\begin{cases} \{q^0 + (m-1)q^2\}a + nqb = 1. \\ \{q^0 + (n-1)q^2\}b + mq a = 1. \end{cases}$$

We can solve this and we have

$$\#K_{m,n} = ma + nb = \frac{(m+n) - (2mn - m - n)q}{(1+q)(1 - (m-1)(n-1)q^2)}.$$

**Lemma 2.2.5.** *Let  $G$  and  $H$  be graphs. Then,*

$$\#(G \sqcup H) = \#G + \#H,$$

where  $G \sqcup H$  is the disjoint union of  $G$  and  $H$ .

*Proof.*  $Z_{G \sqcup H} = \begin{pmatrix} Z_G & O \\ O & Z_H \end{pmatrix}, Z_{G \sqcup H}^{-1} = \begin{pmatrix} Z_G^{-1} & O \\ O & Z_H^{-1} \end{pmatrix}$ .  
Thus,

$$\#(G \sqcup H) = \text{sum}(Z_{G \sqcup H}^{-1}) = \text{sum}(Z_G^{-1}) + \text{sum}(Z_H^{-1}) = \#G + \#H.$$

□

**Definition 2.2.6.** Let  $G$  and  $H$  be graphs. The *cartesian product*  $G \square H$  of  $G$  and  $H$  is the graph defined as follows;

- $V(G \square H) = V(G) \times V(H)$ .
- $E(G \square H) = \{(x, y), (x', y')\} \mid x = x' \text{ and } \{y, y'\} \in E(H) \text{ or } y = y' \text{ and } \{x, x'\} \in E(G)\}$ .

**Lemma 2.2.7.**  $\#(G \square H) = \#G \cdot \#H$ .

*Proof.* For  $x, x' \in V(G)$  and  $y, y' \in V(H)$ ,  
 $d_{G \square H}((x, y), (x', y')) = d_G(x, x') + d_H(y, y')$ ,  
 $Z_{G \square H}((x, y), (x', y')) = q^{d_{G \square H}((x, y), (x', y'))} = q^{d_G(x, x')} q^{d_H(y, y')} = Z_G(x, x') Z_H(y, y')$ ,  
 $Z_{G \square H} = Z_G \otimes Z_H$  and then  $Z_{G \square H}^{-1} = Z_G^{-1} \otimes Z_H^{-1}$ .  
Thus,  $\#G \square H = \text{sum}(Z_{G \square H}^{-1}) = \text{sum}(Z_G^{-1} \otimes Z_H^{-1}) = \text{sum}(Z_G^{-1}) \cdot \text{sum}(Z_H^{-1}) = \#G \cdot \#H$ .

We used the fact that  $(P \otimes Q)(R \otimes S) = (PR) \otimes (QS)$  for matrices  $P, Q, R$ , and  $S$  such that the products of  $PR$  and  $QS$  are defined.  $\square$

**Example 2.2.8.**  $G = K_2 \square K_3$ .

$$\#K_2 \square K_3 = \#K_2 \cdot \#K_3 = \frac{2}{1+q} \cdot \frac{3}{1+2q} = \frac{6}{(1+q)(1+2q)} = \#K_{3,3}.$$

**Remark 2.2.9.** Here we use the catesian product for graph product, but there are other graph products. However, there is a reason that we use the catesian product. This will be clear in Section 4.

**Proposition 2.2.10.** Let  $G$  be a graph. Then,

$$\begin{aligned} \#G(q) &= \sum_{k=0}^{\infty} (-1)^k \sum_{x_0 \neq x_1 \neq \dots \neq x_k} q^{d(x_0, x_1) + d(x_1, x_2) + \dots + d(x_{k-1}, x_k)} \\ &= \sum_{n=0}^{\infty} c_n q^n, \end{aligned}$$

where

$$c_n = \sum_{k=0}^n (-1)^k |\{(x_0, \dots, x_k) \mid x_0 \neq x_1 \neq \dots \neq x_k, d(x_0, x_1) + \dots + d(x_{k-1}, x_k) = n\}|.$$

*Proof.* Let  $\tilde{w}_G : V(G) \rightarrow \mathbb{Z}[q]$  be a map defined by

$$\tilde{w}_G(x) = \sum_{k=0}^{\infty} (-1)^k \sum_{x=x_0 \neq x_1 \neq \dots \neq x_k} q^{d(x_0, x_1) + d(x_1, x_2) + \dots + d(x_{k-1}, x_k)}.$$

Then, for  $x \in V(G)$ ,

$$\begin{aligned}
& \sum_{y \in V(G)} q^{d(x,y)} \tilde{w}_G(y) \\
&= \tilde{w}_G(x) + \sum_{y \in V(G) \setminus \{x\}} q^{d(x,y)} \sum_{k=0}^{\infty} (-1)^k \sum_{y=x_0 \neq x_1 \neq \dots \neq x_k} q^{d(x_0,x_1)+\dots+d(x_{k-1},x_k)} \\
&= \tilde{w}_G(x) + \sum_{k=0}^{\infty} (-1)^k \sum_{x \neq y = x_0 \neq x_1 \neq \dots \neq x_k} q^{d(x,y)+d(x_0,x_1)+\dots+d(x_{k-1},x_k)} \\
&= \tilde{w}_G(x) + \sum_{k=0}^{\infty} (-1)^k \sum_{x \neq x_0 \neq x_1 \neq \dots \neq x_k} q^{d(x,x_0)+d(x_0,x_1)+\dots+d(x_{k-1},x_k)} \\
&= \sum_{k=0}^{\infty} (-1)^k \sum_{x=x_0 \neq \dots \neq x_k} q^{d(x_0,x_1)+\dots+d(x_{k-1},x_k)} \\
&\quad + \sum_{k=0}^{\infty} (-1)^k \sum_{x \neq x_0 \neq \dots \neq x_k} q^{d(x,x_0)+\dots+d(x_{k-1},x_k)} \\
&= 1.
\end{aligned}$$

The last equality holds since the first term at  $k$  is the second term at  $k-1$  in the second term with opposite sign. Now, if  $k$  goes to infinity, the term disappears since  $q^\infty = 0$ . Thus, by Lemma 2.1.9, we have  $\tilde{w}_G = w_G$  and then the first equality of proposition follows. The second equality immediately follows from the first one.  $\square$

**Corollary 2.2.11.** *Let  $G$  be a graph.  $|V(G)| = \#G(0)$ ,  $|E(G)| = -\frac{1}{2} \left. \frac{d}{dq} \#G(q) \right|_{q=0}$ . Here, the derivative is taken in  $\mathbb{Z}[[q]]$ .*

*Proof.* From the previous proposition, we have

$$\begin{aligned}
c_0 &= \sum_{k=0}^0 (-1)^k |\{(x_0, \dots, x_k) \mid x_0 \neq x_1 \neq \dots \neq x_k, d(x_0, x_1) + \dots + d(x_{k-1}, x_k) = n\}| \\
&= |\{(x_0) \mid x_0 \in V(G)\}| \\
&= |V(G)|
\end{aligned}$$

and

$$\begin{aligned}
c_1 &= |\{(x_0) \mid d(x_0, x_0) = 1\}| - |\{(x_0, x_1) \mid x_0 \neq x_1, d(x_0, x_1) = 1\}| \\
&= 0 - 2|E(G)| \\
&= -2|E(G)|.
\end{aligned}$$

This corollary immediately follows from these equations.  $\square$

**Remark 2.2.12.**  $c_0 \geq 0, c_1 \leq 0$ , and  $c_2 \geq 0$  holds. The last inequality follows from

$$c_2 = |\{(x, y, z) \mid d(x, y) = d(y, z) = 1\}| - |\{(x, y) \mid d(x, y) = 2\}|$$

by Proposition 2.2.10. However, in general, the sign of  $c_n$  is not determined. For example, if  $G$  is the Petersen graph, then  $\#G = \frac{10}{1+3q+6q^2} = 10 - 30q + 30q^2 + 90q^3 - 450q^4 + \dots$ . By above equation of  $c_2$ ,  $c_2 = 0$  if and only if  $G$  has no cycles of length 3 and 4.

**Remark 2.2.13.** The magnitude does not have the data of the number of connected components. For example, let  $W$  be the graph obtained by the complete graph  $K_6$  with a triangle of edges removed. By easy calculation,

$$\#W = \frac{6}{1+4q}.$$

Since  $\#K_5 = \frac{5}{1+4q}$ , we have  $\#5W = \frac{30}{1+4q} = \#6K_5$ . Here,  $nG$  means the disjoint union of  $n$  copies of a graph  $G$ .

## 2.3 Main Results on the Magnitude of Graphs

This subsection states the inclusion-exclusion principle for the magnitude of graphs under specific conditions. We begin by observing that the magnitude does not generally satisfy the inclusion-exclusion principle. We then introduce sufficient conditions for the principle to hold. In this document, we mean  $G \cup H$  as a graph  $(V(G) \cup V(H), E(G) \cup E(H))$ .

**Definition 2.3.1.** Let  $R$  be a ring. A function  $\Phi$  is an *R-valued graph invariant* if

- $\Phi(G) \in R$  for any graph  $G$ .
- If  $G \approx H$  as a graph, then  $\Phi(G) = \Phi(H)$ .

**Definition 2.3.2.** Let  $\Phi$  be an *R-valued graph invariant*.

- (1)  $\Phi$  is said to be *multiplicative* if

- $\Phi(V_1) = 1$ .
- $\Phi(G \square H) = \Phi(G) \cdot \Phi(H)$  for any graphs  $G$  and  $H$ .

(2)  $\Phi$  is said to satisfy the *inclusion-exclusion principle* if

- $\Phi(V_0) = 0$ .
- $\Phi(G \cup H) = \Phi(G) + \Phi(H) - \Phi(G \cap H)$  for any graphs  $G$  and  $H$ .

**Lemma 2.3.3.** *Let  $R$  be a ring containing no nonzero nilpotents and let  $\Phi$  be a multiplicative  $R$ -valued graph invariant satisfying the inclusion-exclusion principle. Then,  $\Phi(G) = |V(G)|$  for any graph  $G$ .*

*Proof.* Now,  $\Phi(V_0) = 0$ ,  $\Phi(V_1) = 1$  and by inclusion-exclusion principle, we obtain  $\Phi(V_n) = n$  for any  $n \geq 0$  by induction.

Let  $G$  be any graph and we fix  $e \in E(G)$ . Let  $v_e, w_e$  be the vertices joined by  $e$ . Consider the two subgraphs  $G_e = (V(G), E(G) \setminus \{e\})$  and  $H_e = (\{v_e, w_e\}, \{e\})$  of  $G$ . Then, by inclusion-exclusion principle, we have

$$\begin{aligned}\Phi(G) &= \Phi(G_e) + \Phi(H_e) - \Phi(G_e \cap H_e) \\ &= \Phi(G_e) + \Phi(K_2) - \Phi(V_2) \\ &= \Phi(G_e) + \varepsilon.\end{aligned}$$

here, we denote  $\varepsilon = \Phi(K_2) - \Phi(V_2)$ . Repeating this process for all edges of  $G$ , we have  $\Phi(G) = \Phi(V_{V(G)}) + \varepsilon \cdot |E(G)| = |V(G)| + \varepsilon \cdot |E(G)|$ .

Now, we show that  $\varepsilon = 0$ . Consider the graph  $C_4 = K_2 \square K_2$ . Now,  $\Phi(C_4) = 4 + \varepsilon \cdot 4$  and by multiplicativity, we have

$$\begin{aligned}\Phi(C_4) &= \Phi(K_2) \cdot \Phi(K_2) \\ &= (\Phi(V_2) + \varepsilon)(\Phi(V_2) + \varepsilon) \\ &= 4 + 4\varepsilon + \varepsilon^2.\end{aligned}$$

We obtain  $\varepsilon^2 = 0$  and then  $\varepsilon = 0$  since  $R$  contains no nonzero nilpotents. Thus, we have  $\Phi(G) = |V(G)|$  for any graph  $G$ .  $\square$

The magnitude of graphs is an  $\mathbb{Q}[q]$ -valued graph invariant and is multiplicative and  $\#V_0 = 0$ . Since  $\mathbb{Q}[q]$  has no nonzero nilpotents, we have the following corollary.

**Corollary 2.3.4.** *The magnitude does not satisfy the inclusion-exclusion principle in general.*

**Definition 2.3.5.** Let  $X$  be a graph and let  $U$  be a subgraph of  $X$ .  $U$  is said to be *convex* in  $X$  if for any vertices  $x, y \in V(U)$ ,  $d_U(x, y) = d_X(x, y)$ .

**Lemma 2.3.6.** *Let  $X$  be a graph, let  $G, H$  be subgraphs of  $X$ ,  $g \in V(G)$ , and  $h \in V(H)$  such that  $X = G \cup H$  and there is a path ( $g = x_0 \rightarrow x_1 \rightarrow \dots \rightarrow x_n = h$ ) in  $X$ . Then, there exists a vertex  $x_i \in V(G) \cap V(H)$ .*

*Proof.* aaa □

**Lemma 2.3.7.** *Let  $X$  be a graph and  $G, H$  subgraphs of  $X$  such that  $X = G \cup H$ . If  $G \cap H$  is convex in  $X$ , then  $G$  and  $H$  are also convex in  $X$ .*

*Proof.* aaa □

**Definition 2.3.8.** Let  $X$  be a graph and let  $U$  be a subgraph of  $X$  such that  $U$  is convex in  $X$ . We denote  $V_U(X) = \{v \in V(X) \mid d_X(v, u) < \infty \text{ for some } u \in V(U)\}$ . Then, we say that  $X$  *projects to*  $U$  if for any  $x \in V_U(X)$ , there exists  $\pi(x) \in V(U)$  such that for any  $u \in V(U)$ ,  $d_X(x, u) = d_X(x, \pi(x)) + d_X(\pi(x), u)$ .

**Lemma 2.3.9.** *If  $X$  projects to  $U$ , then  $\pi(x)$  is uniquely determined for any  $x \in V_U(X)$ .*

*Proof.* aaa □

**Example 2.3.10.** aaa

**Lemma 2.3.11.** *Let  $X$  be a graph and let  $U$  be a convex subgraph of  $X$  such that  $X$  projects to  $U$ . Then, for any  $u \in V(U)$ ,*

$$w_U(u) = \sum_{x \in \pi^{-1}(u)} q^{d_X(u, x)} w_X(x).$$

*Proof.* aaa □

**Theorem 2.3.12** (Main Theorem I). *Let  $X$  be a graph and let  $G, H$  be subgraphs of  $X$  such that  $X = G \cup H$ . If  $G \cap H$  is convex in  $X$  and  $H$  projects to  $G \cap H$ , then*

$$\#X = \#G + \#H - \#(G \cap H).$$

Before proving this theorem, we give the example of graphs for which we can apply this theorem.

**Example 2.3.13.** Let  $G$  be a graph and consider the graph  $H$  formed by identifying one of the edges of a cycle graph  $C_n$  with an edge of  $G$ . Now, let  $n \geq 4$ .

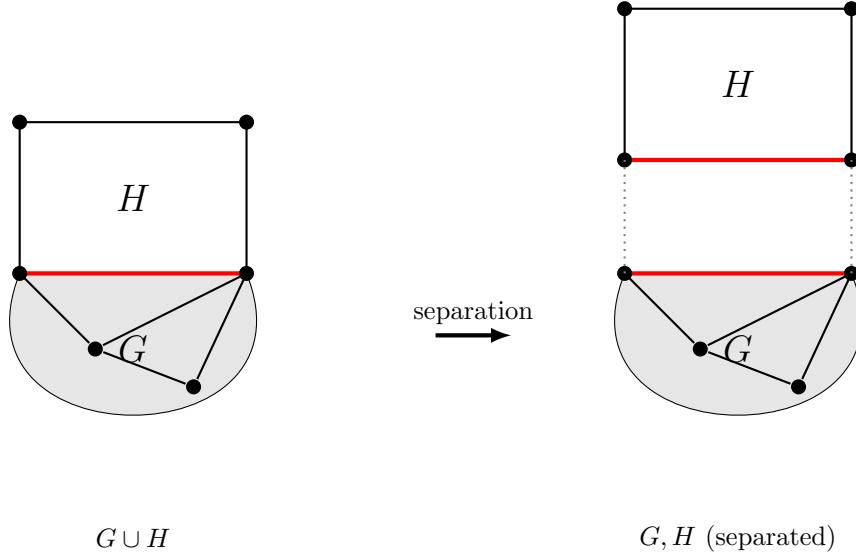


図 7: Cycle graph of four vertices

Then, we can apply Theorem 2.3.12 to  $X = G \cup H$  as follows:

$$\#X = \#G + \#C_4 - \#K_2.$$

Similarly, if  $G$  and  $H$  are graphs and  $G \vee H$  is the graph formed by identifying one vertex of  $G$  with one vertex of  $H$ , then we have

$$\#(G \vee H) = \#G + \#H - 1.$$

*Proof of Theorem 2.3.12. aaa*

□

**Example 2.3.14.** The three graphs below are divided into a graph  $C_3$ , and two graphs  $C_2$ , so they all have the same magnitude and can be calculated as follows:

$$\#G = \#C_3 + 2 \cdot \#C_2 - 2.$$

**Example 2.3.15.** If  $G$  is a forest, then we can calculate the magnitude of  $G$  as follows:

$$\#G = |V(G)| - 2|E(G)| \frac{q}{1+q}.$$

If  $G$  is a tree, then

$$\#G = |V(G)| - 2(|V(G)| - 1) \frac{q}{1+q}.$$

Furthermore examples.

**Remark 2.3.16.** Even though two graphs both have one component and the same magnitude, they may not be isomorphic. We can easily generate such examples using wedge sums of graphs as follows.

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### 3 The Magnitude Homology of Graphs

In this section, we define the magnitude homology of a graph  $G$ . We provide fundamental examples and properties, and state the Mayer-Vietoris sequence for magnitude homology.

#### 3.1 The Definition of The Magnitude Homology of Graphs

**Definition 3.1.1.** Let  $G$  be a graph. For a positive integer  $k$ , the *length* of a tuple  $(x_0, \dots, x_k)$  of  $V(G)$  is defined to be

$$\begin{aligned} l(x_0, \dots, x_k) &= d(x_0, x_1) + d(x_1, x_2) + \cdots + d(x_{k-1}, x_k) \\ &= \sum_{i=1}^k d(x_{i-1}, x_i). \end{aligned}$$

Now, let  $l(x_0) = 0$ . We say a tuple  $(x_0, \dots, x_k)$  is *good* if  $x_0 \neq x_1 \neq \cdots \neq x_k$ .

**Lemma 3.1.2** (Triangle inequality). *If  $(x_0, \dots, x_k)$  is a good tuple of  $V(G)$ , then for any  $1 \leq i \leq k-1$ ,*

$$l(x_0, \dots, x_k) \geq l(x_0, \dots, \hat{x}_i, \dots, x_k).$$

*Proof.* We obviously have the statement by the triangle inequality of the distance function  $d$ .  $\square$

**Definition 3.1.3** (*Magnitude chain complex*). Let  $G$  be a graph.  $MC_{*,*}(G)$  is the *magnitude complex* defined as follows:

$$MC_{*,*}(G) = \bigoplus_{l=0}^{\infty} MC_{*,l}(G).$$

For non-negative integers  $k$  and  $l$ ,  $MC_{k,l}(G)$  is freely generated by good tuples  $(x_0, \dots, x_k)$  of  $V(G)$  of length  $l$  with the ring  $\mathbb{Z}$ . The differential  $\partial : MC_{k,l}(G) \rightarrow MC_{k-1,l}(G)$  is defined by

$$\partial = \sum_{i=1}^{k-1} (-1)^{i-1} \partial_i,$$

where

$$\partial_i(x_0, \dots, x_k) = \begin{cases} (x_0, \dots, \hat{x}_i, \dots, x_k) & \text{if } l(x_0, \dots, \hat{x}_i, \dots, x_k) = l(x_0, \dots, x_k). \\ 0 & \text{otherwise.} \end{cases}$$

**Remark 3.1.4.** For a good tuple  $(x_0, \dots, x_k)$  and  $1 \leq i \leq k - 1$ ,

$$\partial_i(x_0, \dots, x_k) \neq 0 \iff d(x_{i-1}, x_i) + d(x_i, x_{i+1}) = d(x_{i-1}, x_{i+1}).$$

**Lemma 3.1.5.**  $\partial \circ \partial = 0$ .

*Proof.* aaa. □

**Definition 3.1.6.** (*magnitude homology*) Let  $G$  be a graph. The *magnitude homology*  $MH_{*,*}(G)$  of  $G$  is the homology of the magnitude chain complex  $MC_{*,*}(G)$ , that is,

$$MH_{k,l}(G) = \text{Ker}\partial \cap (MC_{k,l}(G)) / \text{Im}\partial \cap (MC_{k,l}(G)).$$

**Example 3.1.7.** (i)  $G = V_n$ . Then,

$$MC_{k,l}(V_n) = \begin{cases} \mathbb{Z}\{(x) \mid x \in V(V_n)\} & \text{if } k = l = 0. \\ 0 & \text{otherwise.} \end{cases}$$

$\partial = 0$  implies that

$$MH_{k,l}(V_n) \approx \begin{cases} \mathbb{Z}^n & \text{if } k = l = 0. \\ 0 & \text{otherwise.} \end{cases}$$

(ii)  $G = K_n (n \geq 2)$ . Then,  $l(x_0, \dots, x_k) = k$  for any good tuple  $(x_0, \dots, x_k)$  of  $V(K_n)$ . Thus,

$$MC_{k,l}(K_n) = \begin{cases} \mathbb{Z}\{(x_0, \dots, x_k) \mid x_0 \neq x_1 \neq \dots \neq x_k\} & \text{if } l = k. \\ 0 & \text{otherwise.} \end{cases}$$

$\partial = 0$  implies that

$$MH_{k,l}(K_n) \approx \begin{cases} \mathbb{Z}^{n(n-1)^l} & \text{if } l = k. \\ 0 & \text{otherwise.} \end{cases}$$

(iii)  $G = C_5$ . Number the vertices of  $C_5$  as shown in the following figure.

ここにナンバリングした  $C_5$  の図を挿入

Let us consider  $MH_{2,3}(C_5)$ . 続く

**Theorem 3.1.8.** *Let  $G$  be a graph. Then,*

$$\sum_{k,l \geq 0} (-1)^k \text{rank}(MH_{k,l}(G)) q^l = \#G \text{ in } \mathbb{Z}[[q]].$$

*Proof.*

$$\begin{aligned} (LHS) &= \sum_{l \geq 0} \chi(MH_{*,l}(G)) q^l \\ &= \sum_{l \geq 0} \chi(MC_{*,l}(G)) q^l \\ &= \sum_{k,l \geq 0} (-1)^k \text{rank}(MC_{k,l}(G)) q^l \\ &= \sum_{k \geq 0} (-1)^k \sum_{x_0 \neq \dots \neq x_k} q^{d(x_0,x_1) + \dots + d(x_{k-1},x_k)} \\ &= \#G. \end{aligned}$$

The last equation is obtained by Proposition 2.2.10.  $\square$

**Proposition 3.1.9.** *Let  $G$  be a graph. Then,*

- $MH_{0,0}(G) \approx \mathbb{Z}^{|V(G)|}$  holds.
- $MH_{1,1}(G) \approx \mathbb{Z}^{2|E(G)|}$  holds.

*Proof.*

$$MC_{k,0}(G) = \begin{cases} \mathbb{Z}\{(x) \mid x \in V(G)\} & \text{if } k = 0. \\ 0 & \text{otherwise.} \end{cases}$$

and  $\partial = 0$  induces the first equation.

$$MC_{k,1}(G) = \begin{cases} \mathbb{Z}\{(x_0, x_1) \mid x_0 \neq x_1\} & \text{if } k = 1. \\ 0 & \text{otherwise.} \end{cases}$$

and  $\partial = 0$  induces the second equation.  $\square$

**Definition 3.1.10.** The diameter  $d$  of a graph  $G$  is defined by

$$d = \max\{d(x, y) \mid x, y \in V(G) \text{ and } x, y \text{ lie in the same component of } G\}.$$

If  $G = V_n$ , then we define  $d = 0$ . Then, for any graph  $G$ ,  $0 \leq d < \infty$  holds.

**Proposition 3.1.11.** Let  $G$  be a graph and let  $d$  be the diameter of  $G$  and assume that  $MH_{k,l}(G) \neq 0$  for given non-negative integers  $k$  and  $l$ . Then,

- $\frac{l}{d} \leq k \leq l$  holds.
- If  $d > 1$  and  $l > 0$ , then  $\frac{l}{d} < k \leq l$  holds.

*Proof.* Since  $MH_{k,l}(G) \neq 0$ , there exists a good tuple  $(x_0, \dots, x_k)$  of length  $l$  such that  $\partial(x_0, \dots, x_k) = 0$ . Thus,  $l = l(x_0, \dots, x_k) = \sum_{i=1}^k d(x_{i-1}, x_i) \leq \sum_{i=1}^k d = kd$  and  $l = \sum_{i=1}^k d(x_{i-1}, x_i) \geq k$ . This implies that  $\frac{l}{d} \leq k \leq l$ .

Now, assume that  $d > 1$  and  $l > 0$  and suppose that  $k = \frac{l}{d}$ . From the above discussion, we have  $d(x_i, x_{i+1}) = d$  for all  $i$ .  $\partial(x_0, \dots, x_k) = 0$ . For the  $(k+1)$ -tuple  $\partial(x_0, \dots, x_k)$  is a linear combination of at most  $k-1$  distinct terms of  $k$ -tuples, so  $\partial(x_0, \dots, x_k) = 0$  implies  $d(x_{i-1}, x_i) + d(x_i, x_{i+1}) \neq d(x_{i-1}, x_{i+1})$  for all  $1 \leq i \leq k-1$ . Since  $d(x_0, x_1) = d \geq 2$ , there exists a vertex  $y$  such that  $d(x_0, y) + d(y, x_1) = d(x_0, x_1)$  and  $y \neq x_0, x_1$ . Then,  $(x_0, y, x_1, \dots, x_k)$  is a good tuple in  $MC_{k+1,l}(G)$  and

$$\partial_i(x_0, y, x_1, \dots, x_k) = \begin{cases} (x_0, x_1, \dots, x_k) & \text{for } i = 1. \\ 0 & \text{for } 2 \leq i \leq k. \end{cases}$$

It is obvious for  $3 \leq i$  by  $d(x_{i-1}, x_i) + d(x_i, x_{i+1}) \neq d(x_{i-1}, x_{i+1})$  and is also true for  $i = 2$  since  $d(y, x_1) + d(x_1, x_2) = d(y, x_1) + d > d \geq d(y, x_2)$ . This implies  $MH_{k,l}(G) = 0$  and contradicts the assumption.  $\square$

## 3.2 Induced Maps on Magnitude Homology

**Definition 3.2.1.** Let  $G$  and  $H$  be graphs. A map  $f : V(G) \rightarrow V(H)$  is said to be a *graph map* if for any  $\{x, y\} \in E(G)$ , either  $f(x) = f(y)$  or  $\{f(x), f(y)\} \in E(H)$ .

We note that a graph map is distinguished from a graph homomorphism, which requires that  $\{f(x), f(y)\} \in E(H)$  for any  $\{x, y\} \in E(G)$ .

**Proposition 3.2.2.**  $l(f(x_0), \dots, f(x_k)) \leq l(x_0, \dots, x_k)$  for any good tuple  $(x_0, \dots, x_k)$  of  $V(G)$ .

*Proof.* For any vertices  $x, y \in V(G)$ ,  $d_H(f(x), f(y)) \leq d_G(x, y)$  holds. Indeed, if  $x, y$  lie in the same component of  $G$ , then there exists a path  $(x = x_0 \rightarrow x_1 \rightarrow \dots \rightarrow x_n = y)$  in  $G$  such that  $n = d_G(x, y)$ . Since  $f$  is a graph map, either  $f(x_{i-1}) = f(x_i)$  or  $\{f(x_{i-1}), f(x_i)\} \in E(H)$  for any  $1 \leq i \leq n$ . Thus,  $(f(x) = f(x_0) \rightarrow f(x_1) \rightarrow \dots \rightarrow f(x_n) = f(y))$  is a path in  $H$  and then  $d_H(f(x), f(y)) \leq n = d_G(x, y)$ . If  $x, y$  do not lie in the same component, then  $d_G(x, y) = d_H(f(x), f(y)) = \infty$ . Then,

$$\begin{aligned} l(f(x_0), \dots, f(x_k)) &= \sum_{i=1}^k d_H(f(x_{i-1}), f(x_i)) \\ &\leq \sum_{i=1}^k d_G(x_{i-1}, x_i) \\ &= l(x_0, \dots, x_k). \end{aligned}$$

□

**Definition 3.2.3.** Let  $G$  and  $H$  be graphs and let  $f : V(G) \rightarrow V(H)$  be a graph map. Then, the *induced map*  $f_\# : MC_{*,*}(G) \rightarrow MC_{*,*}(H)$  is defined by

$$f_\#(x_0, \dots, x_k) = \begin{cases} (f(x_0), \dots, f(x_k)) & \text{if } l(f(x_0), \dots, f(x_k)) = l(x_0, \dots, x_k). \\ 0 & \text{otherwise.} \end{cases}$$

for any good tuple  $(x_0, \dots, x_k)$  of  $V(G)$ .

**Proposition 3.2.4.** The induced map  $f_\# : MC_{*,*}(G) \rightarrow MC_{*,*}(H)$  is a chain map.

*Proof.* aaa.

□

**Definition 3.2.5.** (Induced maps in homology) If  $f : G \rightarrow H$  is a graph map, the *induced map in homology*  $f_* : MH_{*,*}(G) \rightarrow MH_{*,*}(H)$  is the map induced by the chain map  $f_\# : MC_{*,*}(G) \rightarrow MC_{*,*}(H)$ .

**Definition 3.2.6.**  $A$  is called a *bigraded abelian group* if  $A = \bigoplus_{k,l \geq 0} A_{k,l}$  where each  $A_{k,l}$  is an abelian group. A *bigraded homomorphism*  $f : A \rightarrow B$  between bigraded abelian groups  $A$  and  $B$  is a homomorphism such that  $f(A_{k,l}) \subset B_{k,l}$  for any  $k, l \geq 0$ .

**Proposition 3.2.7.** *The assignment  $G \mapsto MH_{*,*}(G)$  and  $f \mapsto f_*$  defines a functor from the category of graphs and graph maps to the category of bigraded abelian groups and bigraded homomorphisms, denoted by  $\mathbf{Graph} \rightarrow \mathbf{BAb}$ .*

**Proposition 3.2.8.** *Let  $f : G \rightarrow H$  be a graph map.*

- $f_* : MH_{0,0}(G) \rightarrow MH_{0,0}(H)$  is given by  $f_*(x) = f(x)$  for any  $x \in V(G)$ .

- $f_* : MH_{1,1}(G) \rightarrow MH_{1,1}(H)$  is given by

$$f_*(x_0, x_1) = \begin{cases} (f(x_0), f(x_1)) & \text{if } f(x_0) \neq f(x_1). \\ 0 & \text{otherwise.} \end{cases}$$

for any  $(x_0, x_1) \in MH_{1,1}(G)$ .

*Proof.* The first equation is obvious.

For the second equation, we obtain by definition;

$$f_*(x_0, x_1) = \begin{cases} (f(x_0), f(x_1)) & \text{if } l(f(x_0), f(x_1)) = l(x_0, x_1) = 1. \\ 0 & \text{otherwise.} \end{cases}$$

for any  $(x_0, x_1) \in MH_{1,1}(G)$ . Since  $f$  is a graph map,  $l(f(x_0), f(x_1)) = 1$  if and only if  $f(x_0) \neq f(x_1)$ .  $\square$

**Corollary 3.2.9.** *Let  $f : G \rightarrow H$  be a graph map.  $f_*$  is an isomorphism if and only if  $f$  is a graph isomorphism.*

*Proof.* aaa.  $\square$

### 3.3 Magnitude Homology of Disjoint Union of Graphs

**Proposition 3.3.1.** *Let  $G$  and  $H$  be graphs. We define the inclusion graph maps  $i : G \rightarrow G \sqcup H, j : H \rightarrow G \sqcup H$ . Then,*

$$i_* \oplus j_* : MH_{*,*}(G) \oplus MH_{*,*}(H) \rightarrow MH_{*,*}(G \sqcup H)$$

*is an isomorphism for each  $k, l \geq 0$ .*

*Proof.* saaa. □

We can give another proof of Lemma 2.2.5 by Proposition 3.3.1 and  $\chi(A_* \oplus B_*) = \chi(A_*) + \chi(B_*)$ .

### 3.4 Magnitude Homology of Cartesian Products

**Definition 3.4.1.** This definition is not true. Fix  $l \geq 0$ . The *exterior product* is the map

$$\square : MC_{*,*}(G) \otimes MC_{*,*}(H) \rightarrow MC_{*,*}(G \square H)$$

is defined as follows. Let  $\square$  be the map

$$\square : MC_{k_1, l_1}(G) \times MC_{k_2, l_2}(H) \rightarrow MC_{k, l}(G \square H) \text{ for } k_1, k_2 \geq 0, k = k_1 + k_2, l = l_1 + l_2,$$

which is defined by

$$\square((x_0, \dots, x_{k_1}), (y_0, \dots, y_{k_2})) = \sum_{\sigma} \text{sign}(\sigma)((x_{i_0}, y_{j_0}), (x_{i_1}, y_{j_1}), \dots, (x_{i_k}, y_{j_k})),$$

where the sum is over all shuffles  $\sigma$  of type  $(k_1, k_2)$ , that is, all sequences

$$((i_0, j_0), (i_1, j_1), \dots, (i_k, j_k))$$

such that

$$i_0 = 0, j_0 = 0, 0 \leq i_r \leq k_1, 0 \leq j_r \leq k_2 \text{ for } 0 \leq r \leq k, \text{ and}$$

$$(i_{r+1}, j_{r+1}) = \begin{cases} (i_r + 1, j_r) & \text{or} \\ (i_r, j_r + 1) & \text{for } 0 \leq r < k, \end{cases}$$

and

$$\text{sign}(\sigma) = (-1)^m,$$

where  $m = \#\{(i, j) \in \{\{0, \dots, k_1\} \times \{0, \dots, k_2\}\} \mid i = i_r \Rightarrow j < j_r\}$ .

Here, we extend the map  $\square$  bilinearly to the tensor product

$$MC_{k_1, l_1}(G) \otimes MC_{k_2, l_2}(H) \rightarrow MC_{k, l}(G \square H)$$

We denote this induced map also by  $\square$  and call it the *exterior product*.

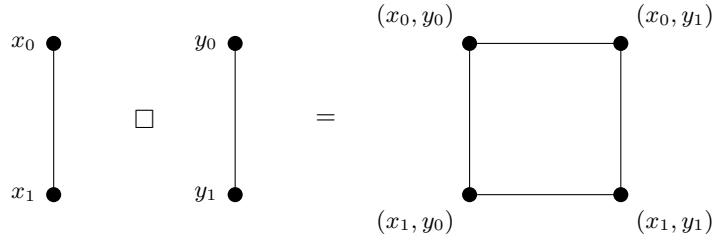


図 8: Square graph

**Example 3.4.2.** Let  $G = C_2 \square C_2 = C_4$

Consider the exterior product  $\square((x_0, x_1) \otimes (y_0, y_1))$ . We have the two shuffles of type  $(1, 1)$ :

$$((0, 0), (1, 0), (1, 1)), ((0, 0), (0, 1), (1, 1)).$$

Thus,

$$\square((x_0, x_1) \otimes (y_0, y_1)) = -((x_0, y_0), (x_1, y_0), (x_1, y_1)) + ((x_0, y_0), (x_0, y_1), (x_1, y_1)).$$

**Remark 3.4.3.** As you see in the above example, the number of shuffles is  $\binom{k}{k_1}$ .

**Proposition 3.4.4.** *The exterior product  $\square : MC_{*,*}(G) \otimes MC_{*,*}(H) \rightarrow MC_{*,*}(G \square H)$  is a chain map.*

*Proof.* Let  $\mathbf{x} = (\mathbf{x}_0, \dots, \mathbf{x}_{k_1})$ ,  $\mathbf{y} = (\mathbf{y}_0, \dots, \mathbf{y}_{k_2})$ . Now, we show that

$$\partial \circ \square(\mathbf{x} \otimes \mathbf{y}) = \square((\partial \mathbf{x}) \otimes \mathbf{y}) + (-1)^{k_1} \square(\mathbf{x} \otimes (\partial \mathbf{y})) = \square \circ (\partial \otimes \partial)(\mathbf{x} \otimes \mathbf{y}).$$

Here, we should consider the sequence of tensor products of the magnitude chain complexes defined by

$$\begin{aligned} \partial \otimes \partial : MC_{k_1, l_1}(G) \otimes MC_{k_2, l_2}(H) &\rightarrow (MC_{k_1-1, l_1}(G) \otimes MC_{k_2, l_2}(H)) \\ &\quad \oplus (MC_{k_1, l_1}(G) \otimes MC_{k_2-1, l_2}(H)), \\ (\partial \otimes \partial)(\mathbf{x} \otimes \mathbf{y}) &= (\partial \mathbf{x}) \otimes \mathbf{y} + (-1)^{k_1} \mathbf{x} \otimes (\partial \mathbf{y}). \end{aligned}$$

Then, we should show only the first equality.

ここに可換図式を挿入. 証明も続く

□

From this proposition, we obtain the induced map in homology, also denoting  $\square$ .

**Definition 3.4.5.** (Tor functor) Let  $R$  be a ring and let  $A, B$  be  $R$ -modules. Then,  $\text{Tor}(A, B)$  is defined by the derived functor of the tensor product.

**Theorem 3.4.6.** *Let  $G$  and  $H$  be graphs.*

$$0 \rightarrow MH_{*,*}(G) \otimes MH_{*,*}(H) \xrightarrow{\square} MH_{*,*}(G \square H) \rightarrow \text{Tor}(MH_{*-1,*}(G), MH_{*,*}(H)) \rightarrow 0$$

*is a short exact sequence and non-naturally split. In particular, if  $MH_{*,*}(G)$  or  $MH_{*,*}(H)$  is torsion-free, then the exterior product  $\square$  is an isomorphism.*

We don't prove this theorem in this thesis.

**Example 3.4.7.**  $G = C_4 = C_2 \square C_2$ .

### 3.5 The Mayer-Vietoris Sequence of Magnitude Homology of Graphs

**Definition 3.5.1.** Let  $X$  be a graph and let  $G, H$  be subgraphs of  $X$ .

- (1)  $(X; G, H)$  is said to be a *projecting decomposition* if  $X = G \cup H$ ,  $G \cap H$  is convex in  $X$  and  $H$  projects to  $G \cap H$ .  
We write  $i^G : G \rightarrow X$ ,  $i^H : H \rightarrow X$ ,  $j^G : G \cap H \rightarrow G$ ,  $j^H : G \cap H \rightarrow H$  for the inclusion graph maps.
- (2) Let  $(X; G, H), (X'; G', H')$  be projecting decompositions.  $f : (X; G, H) \rightarrow (X'; G', H')$  is said to be a *decomposition map* if  $f : X \rightarrow X'$  is a graph map such that  $f(V(G)) \subset V(G')$  and  $f(V(H)) \subset V(H')$ .
- (3) Let  $f : (X; G, H) \rightarrow (X'; G', H')$  be a decomposition map. Then,  $f$  is said to be a *projecting decomposition map* if  $V_{G \cap H}(H) = f^{-1}(V_{G' \cap H'}(H'))$  and  $f(\pi(h)) = \pi(f(h))$  for any  $h \in V_{G \cap H}(H)$ .
- (4) Let  $(X; G, H)$  be a projecting decomposition.  $MC_{*,*}(G, H)$  denote the subcomplex of  $MC_{*,*}(X)$  spanned by good tuples  $(x_0, \dots, x_k)$  whose entries all lie in  $G$  or all lie in  $H$ .

**Theorem 3.5.2.** Let  $(X; G, H)$  be a projecting decomposition. Then, the inclusion map

$$MC_{*,l}(G, H) \hookrightarrow MC_{*,l}(X)$$

is a quasi-isomorphism for any  $l \geq 0$ .

*Proof.* aaa. □

**Theorem 3.5.3. (the main theorem II)** Let  $(X; G, H)$  be a projecting decomposition. Then,

$$0 \rightarrow MH_{*,*}(G \cap H) \xrightarrow{(j_*^G, -j_*^H)} MH_{*,*}(G) \oplus MH_{*,*}(H) \xrightarrow{i_*^G \oplus i_*^H} MH_{*,*}(X) \rightarrow 0$$

is a split short exact sequence.

*Proof.* aaa. □

**Corollary 3.5.4.** Let  $(X; G, H)$  be a projecting decomposition. Then,

$$\#X = \#G + \#H - \#(G \cap H)$$

in  $\mathbb{Z}[[q]]$ .

*Proof.* By Theorem 3.5.3,

$$\begin{aligned} \chi(MH_{*,l}(G \cap H)) - \chi((MH_{*,l}(G)) \oplus \chi(MH_{*,l}(H))) + \chi(MH_{*,l}(X)) &= 0 \text{ holds and then} \\ \chi(MH_{*,l}(X)) &= \chi(MH_{*,l}(G)) + \chi(MH_{*,l}(H)) - \chi(MH_{*,l}(G \cap H)). \end{aligned}$$

For each  $l \geq 0$ , multiplying by  $q^l$  and summing over all  $l \geq 0$ , we have

$$\sum_{l \geq 0} \chi(MH_{*,l}(X))q^l = \sum_{l \geq 0} \chi(MH_{*,l}(G))q^l + \sum_{l \geq 0} \chi(MH_{*,l}(H))q^l - \sum_{l \geq 0} \chi(MH_{*,l}(G \cap H))q^l.$$

By Theorem 3.1.8, we obtain the desired equation. □

**Corollary 3.5.5.** Let  $T$  be a tree.

## 3.6 Diagonal Graphs and its Property

**Definition 3.6.1.** A graph  $G$  is said to be *diagonal* if  $MH_{k,l}(G) = 0$  for  $k \neq l$ .

**Lemma 3.6.2.** *Any trees are diagonal.*

*Proof.* aaa

□

**Proposition 3.6.3.** *For a diagonal graph, the magnitude completely determines the magnitude homology ranks.*

*Proof.* Obvious by Theorem 3.1.8.

□

## 4 Motivation: The Magnitude of Enriched Categories

In this section, we explain the motivation for studying the magnitude of graphs in a broader context. We employ the notion of enriched categories to define the magnitude.

### 4.1 The Magnitude of a Matrix

**Definition 4.1.1.** Let  $k$  be a set and let  $+, \cdot$  be binary operations on  $k$ , and let  $0_k, 1_k$  be elements of  $k$ . Then,  $(k, +, \cdot, 0_k, 1_k)$  is called a *rig* if the following conditions hold:

- (1)  $(k, +, 0_k)$  is a commutative monoid.
- (2)  $(k, \cdot, 1_k)$  is a monoid.
- (3) multiplication distributes over addition, that is, for any  $a, b, c \in k$ ,

$$a \cdot (b + c) = a \cdot b + a \cdot c, \quad (a + b) \cdot c = a \cdot c + b \cdot c.$$

Now, we mean a rig as a commutative rig with the operation  $\cdot$ .

**Example 4.1.2.**  $(\mathbb{Z}_{\geq 0}, +, \cdot, 0, 1)$  is a rig.

**Definition 4.1.3.** Let  $k$  be a rig and let  $I, J$  be finite sets. An  $I \times J$ -matrix is a function  $\zeta : I \times J \rightarrow k$ .

**Definition 4.1.4.** Let  $k$  be a rig, and let  $I, J$ , and  $L$  be finite sets.

- (1) If  $\zeta_1$  is an  $I \times J$ -matrix and  $\zeta_2$  is a  $J \times L$ -matrix, then the product  $\zeta_1 \zeta_2$  is defined as follows:

$$(\zeta_1 \zeta_2)(i, l) = \sum_{j \in J} \zeta_1(i, j) \cdot \zeta_2(j, l) \quad (i \in I, l \in L)$$

- (2)  $\delta : I \times I \rightarrow k$  is called the *identity matrix* if  $\delta(i, j) = 1_k$  when  $i = j$  and  $\delta(i, j) = 0_k$  when  $i \neq j$ .

- (3) Let  $\zeta : I \times J \rightarrow k$  be a matrix. We define  $\zeta^* : J \times I \rightarrow k$  by  $\zeta^*(j, i) = \zeta(i, j)$ .
- (4) Let  $\zeta$  be an  $I \times I$ -matrix. If there exists an  $I \times I$ -matrix  $\zeta^{-1}$  such that  $\zeta\zeta^{-1} = \delta$  and  $\zeta^{-1}\zeta = \delta$ , then  $\zeta$  is said to be *invertible* and  $\zeta^{-1}$  is called the *inverse* of  $\zeta$ .
- (5)  $w : I \rightarrow k$  is called a *vector*. If  $\zeta$  is an  $I \times J$ -matrix,  $v$  is a  $I$ -vector, and  $w$  is a  $J$ -vector, then the products  $\zeta w : I \rightarrow k$  and  $v\zeta : J \rightarrow k$  are defined by

$$(\zeta w)(i) = \sum_{j \in J} \zeta(i, j) \cdot w(j) \quad (i \in I)$$

$$(v\zeta)(j) = \sum_{i \in I} v(i) \cdot \zeta(i, j) \quad (j \in J)$$

Now,  $\zeta w$  is an  $I$ -vector and  $v\zeta$  is a  $J$ -vector.

- (6) If  $w, v$  are  $I$ -vectors, then the *inner product*  $vw$  is defined by

$$vw = \sum_{i \in I} v(i) \cdot w(i)$$

- (7) Let  $u_I$  be the  $I$ -vector defined by  $u(i) = 1_k$  for any  $i \in I$ .

**Definition 4.1.5.** Let  $\zeta$  be an  $I \times I$ -matrix over a rig  $k$ .

- A *weighting* on  $\zeta$  is a vector  $w : J \rightarrow k$  such that  $\zeta w = u_I$ .  $w(j)$  is called the *weight* of  $j \in J$ .
- A *coweighting* on  $\zeta$  is a vector  $v : I \rightarrow k$  such that  $v\zeta = u_I^*$ .  $v(i)$  is called the *coweight* of  $i \in I$ .

**Example 4.1.6.** Let  $G$  be a graph. Then,  $Z_G(q)$  is a  $V(G) \times V(G)$ -matrix over the rig  $\mathbb{Q}[[q]]$  and the weighting on  $Z_G(q)$  is the weighting on  $G$  defined in Section 2.1.

**Lemma 4.1.7.** Let  $\zeta$  be an  $I \times I$ -matrix over a rig  $k$ . If  $\zeta$  has a weighting  $w$  and a coweighting  $v$ , then

$$\sum_{i \in I} v(i) = \sum_{j \in J} w(j)$$

*Proof.*

$$\begin{aligned}
\sum_{i \in I} v(i) &= vu_I \\
&= v(\zeta w) \\
&= (v\zeta)w \\
&= u_J w \\
&= \sum_{j \in J} w(j)
\end{aligned}$$

□

From this lemma, the sum of the weighting or coweighting on  $\zeta$  is unique if they exist. Therefore, we can define the magnitude of  $\zeta$  as follows:

**Definition 4.1.8.** Let  $\zeta$  be an  $I \times J$ -matrix over a rig  $k$ . If  $\zeta$  has a weighting and a coweighting, then the *magnitude* of  $\zeta$  is defined to be

$$\#\zeta = \sum_{i \in I} v(i) = \sum_{j \in J} w(j),$$

where  $w$  is the weighting on  $\zeta$  and  $v$  is the coweighting on  $\zeta$ .

**Lemma 4.1.9.** Let  $\zeta$  be an  $I \times I$ -matrix over a rig  $k$ .

(1) If  $\zeta$  is invertible, then  $\zeta$  has the magnitude.

(2) If  $k$  is a field and  $\zeta$  has the magnitude, then  $\zeta$  is invertible.

*Proof.* (1) If  $\zeta$  is invertible, then  $w = \zeta^{-1}u_I$  and  $v = u_I\zeta^{-1}$  obviously satisfy the definition of weighting and coweighting respectively. Thus  $\zeta$  has the magnitude by Lemma 4.1.7.

(2) If  $k$  is a field and  $\zeta$  has the magnitude, then there exist a weighting  $w$  and a coweighting  $v$  on  $\zeta$ . Let  $\zeta x$  be a zero-map for some  $x : I \rightarrow k$ . Then,

$$0 = v(\zeta x) = (v\zeta)x = u_I x = \sum_{i \in I} x(i)$$

ここからやり直し

□

**Lemma 4.1.10.** Let  $\zeta$  be an invertible  $I \times I$ -matrix over a rig  $k$ . Then,  $\zeta$  has the unique weighting  $w$  of  $\zeta$ , given by  $w(j) = \sum_{i \in I} \zeta^{-1}(j, i)$  for  $j \in I$ , and the unique coweighting  $v$  of  $\zeta$ , given by  $v(i) = \sum_{j \in I} \zeta^{-1}(j, i)$  for  $i \in I$ . Then,

$$\#\zeta = \sum_{i,j \in I} \zeta^{-1}(j, i)$$

*Proof.* We should check the uniqueness and it holds from the invertibility of  $\zeta$ .  $\square$

## 4.2 The Definition of Enriched Categories

In this document, we only treat the locally small categories, which means that for any objects  $a, b$  of a category  $\mathcal{C}$ , the hom-set  $\text{Hom}_{\mathcal{C}}(a, b)$  is a set.

**Definition 4.2.1.** A category  $\mathcal{C}$  is called a *monoid* if  $\mathcal{C}$  has only one object  $*$  and  $V = \text{Hom}_{\mathcal{C}}(*, *)$  is a monoid with the composition of morphisms as the binary operation and the identity morphism  $id_*$  as the identity element. We denote the operation of  $V$  by  $\otimes$ .

ここに可換図式を挿入

**Definition 4.2.2.** A pair  $(\mathcal{V}, \otimes, I)$  is called a *monoidal category* if it satisfies the following conditions:

- (1)  $\mathcal{V}$  is a category.
- (2)  $\otimes : \mathcal{V} \times \mathcal{V} \rightarrow \mathcal{V}$  is a functor.
- (3)  $I$  is an object of  $\mathcal{V}$ .
- (4) There exist the natural isomorphism  $\alpha : \otimes \circ (\otimes \times id_{\mathcal{V}}) \Rightarrow \otimes \circ (id_{\mathcal{V}} \times \otimes)$  given by  $\alpha_{uvw} : (u \otimes v) \otimes w \xrightarrow{\sim} u \otimes (v \otimes w)$ .
- (5) There exist the natural isomorphism  $\lambda : I \otimes - \Rightarrow id_{\mathcal{V}}$  given by  $\lambda_u : I \otimes u \xrightarrow{\sim} u$ .
- (6) There exist the natural isomorphism  $\rho : - \otimes I \Rightarrow id_{\mathcal{V}}$  given by  $\rho_u : u \otimes I \xrightarrow{\sim} u$ .

- (7) The following diagram commutes for any objects  $u, v, w, x$  of  $\mathcal{V}$ : ここに可換図式を挿入
- (8) The following diagrams commute for any objects  $u, v$  of  $\mathcal{V}$ : ここに可換図式を挿入

We also call  $\mathcal{V}$  a monoidal category, omitting  $\otimes$  and  $I$ .

**Example 4.2.3.** (i)  $(\mathbf{Set}, \times, \{\ast\})$  is a monoidal category. ここに説明を挿入

(ii)  $(\mathbf{Vect}_K, \otimes_K, K)$  is a monoidal category, where  $K$  is a field. ここに説明を挿入

(iii)  $([0, \infty], +, 0)$  is a monoidal category. ここに説明を挿入

(iv)  $(\mathbf{2}, \otimes, t)$  is a monoidal category, where  $\mathbf{2}$  is the category defined by  $Ob(\mathbf{2}) = \{t, f\}$  and the morphism sets are defined by

hom <sub>2</sub>	$t$	$f$
$t$	$\{id_t\}$	$\emptyset$
$f$	$\{\ast\}$	$\{id_f\}$

and the operation  $\otimes$  is defined by the following table:

$\otimes$	$t$	$f$
$t$	$t$	$f$
$f$	$f$	$f$

ここに説明を挿入

Then,  $\mathbf{2}$  is a monoidal subcategory of  $[0, \infty]$  by the embedding  $t \mapsto 0, f \mapsto \infty$  and of  $Set$  by the embedding  $t \mapsto \{\ast\}, f \mapsto \emptyset$ .

**Definition 4.2.4.** An *enriched category*  $\mathcal{A}$  in a monoidal category  $(\mathcal{V}, \otimes, I)$  is defined as follows:

- (1) For any objects  $a, b$  of  $\mathcal{A}$ ,  $\text{Hom}_{\mathcal{A}}(a, b)$  is an object of  $\mathcal{V}$ .
- (2) For any objects  $a, b, c$  of  $\mathcal{A}$ , there exists a morphism  $m_{abc} : \text{Hom}_{\mathcal{A}}(b, c) \otimes \text{Hom}_{\mathcal{A}}(a, b) \rightarrow \text{Hom}_{\mathcal{A}}(a, c)$  in  $\mathcal{V}$ , which defines the composition of morphisms.

- (3) For any object  $a$  of  $\mathcal{A}$ , there exists a morphism  $j_a : I \rightarrow \text{Hom}_{\mathcal{A}}(a, a)$  in  $\mathcal{V}$ , which defines the identity morphism of  $a$ . あと 3つの可換図式を挿入

Then,  $\mathcal{A}$  is called a  $\mathcal{V}$ -category.

**Definition 4.2.5.** Let  $\mathcal{A}, \mathcal{A}'$  be  $\mathcal{V}$ -categories. A functor  $F : \mathcal{A} \rightarrow \mathcal{A}'$  is called a  $\mathcal{V}$ -functor if it satisfies the following conditions. We denote  $F_{ab} : \text{Mor}_{\mathcal{A}}(a, b) \rightarrow \text{Mor}_{\mathcal{A}'}(F(a), F(b))$  as the morphism function.

- (1) The following diagram commutes for any objects  $a, b, c$  of  $\mathcal{A}$ : ここに可換図式を挿入
- (2) The following diagram commutes for any object  $a$  of  $\mathcal{A}$ : ここに可換図式を挿入

**Remark 4.2.6.** The family of all  $\mathcal{V}$ -categories and  $\mathcal{V}$ -functors form a category, which is denoted by  $\mathcal{V}\text{-Cat}$ .

**Example 4.2.7.** aaa

### 4.3 The Magnitude of Enriched Categories

**Definition 4.3.1.** Let  $\mathcal{C}$  be a category. The *isomorphism classes* of objects of  $\mathcal{C}$  is defined by the quotient classes  $\mathcal{O}\text{b}(\mathcal{C})/\approx$ , where  $a \approx b$  if and only if there exists an isomorphism  $f : a \rightarrow b$  in  $\mathcal{C}$ .

In this section, we consider only monoidal categories  $(\mathcal{V}, \otimes, I)$  satisfying that  $\mathcal{V}/\approx$  forms a set.

**Lemma 4.3.2.** Let  $(\mathcal{V}, \otimes, I)$  be a monoidal category. Then,  $(\mathcal{O}\text{b}(\mathcal{V})/\approx, \otimes, I)$  is a monoid.

*Proof.* aaa. □

**Definition 4.3.3.** Let  $(\mathcal{V}, \otimes, I)$  be a monoidal category and let  $k$  be a rig. We define a monoid homomorphism

$$|\cdot| : (\mathcal{O}\text{b}(\mathcal{V})/\approx, \otimes, I) \rightarrow (k, \cdot, 1_k)$$

such that  $|I| = 1_k$  and  $|u \otimes v| = |u| \cdot |v|$  for any objects  $u, v$  of  $\mathcal{V}$ .

**Example 4.3.4.** aaa.

**Definition 4.3.5.** Let  $\mathcal{V}$  be a monoidal category and let  $\mathcal{A}$  be a  $\mathcal{V}$ -category which has finitely many objects.

- (1)  $\mathcal{O}\mathrm{b}(\mathcal{A}) \times \mathcal{O}\mathrm{b}(\mathcal{A})$ -matrix  $\zeta_{\mathcal{A}}$  is defined by

$$\zeta_{\mathcal{A}}(a, b) = |\mathrm{Hom}_{\mathcal{A}}(a, b)| \quad (a, b \in \mathcal{O}\mathrm{b}(\mathcal{A}))$$

This  $\zeta_{\mathcal{A}}$  is called the *similarity matrix* of  $\mathcal{A}$ .

- (2) A weighting on  $\mathcal{A}$  is defined to be a weighting on  $\zeta_{\mathcal{A}}$ , similarly to a coweighting on  $\mathcal{A}$ .
- (3)  $\mathcal{A}$  is said to *have the magnitude* if  $\zeta_{\mathcal{A}}$  has a weighting and a coweighting, denoting it by  $\#\mathcal{A} = \#\zeta_{\mathcal{A}}$ .
- (4)  $\mathcal{A}$  which has the magnitude is said to *have Mobius inversion* if  $\zeta_{\mathcal{A}}$  is invertible, denoting its inverse by  $\mu_{\mathcal{A}} = \zeta_{\mathcal{A}}^{-1}$ .

**Example 4.3.6.** aaa.

## 4.4 The Relation of The Magnitudes of Graphs and Enriched Categories

**Theorem 4.4.1.** *The magnitude of a graph  $G$  defined in Definition 2.1.3 coincides with the magnitude of the  $[0, \infty]$ -category corresponding to  $G$ .*

*Proof.* We show that a graph  $G$  is  $[0, \infty]$ -category and its magnitude coincides with the magnitude of  $G$  defined in Definition 2.1.3.

First, as we see in examples, generalized metric spaces are  $[0, \infty]$ -categories, which shows that  $G$  is  $[0, \infty]$ -category with the metric  $d_G$ .

Second, let a monoid homomorphism  $|\cdot| : [0, \infty] \rightarrow \mathbb{Q}(q)$  defined by  $|x| = q^x$ . Then, the similarity matrix  $\zeta_G$  of the  $[0, \infty]$ -category corresponding to  $G$  is given by

$$\zeta_G(a, b) = |\mathrm{Hom}_G(a, b)| = |d_G(a, b)| = q^{d_G(a, b)}.$$

This coincides with the  $Z_G(q)$  defined in Definition 2.1.3.

Therefore, the magnitude of the  $[0, \infty]$ -category corresponding to  $G$  coincides with the magnitude of  $G$  defined in Definition 2.1.3.  $\square$

## Appendix

### A The calculation of Graph Automorphisms

**Proposition A.0.1.** *Let  $G = K_{n,n}$ . Then,  $\text{Aut}(G) \approx (\mathfrak{S}_n \times \mathfrak{S}_n) \rtimes_s \mathbb{Z}_2$ , where  $s : \mathbb{Z}_2 \rightarrow \text{Aut}(G)$ ;  $0 \mapsto id_G$ ,  $1 \mapsto \tau$  and  $\tau$  is the automorphism which interchanges the two parts of  $K_{n,n}$ .*

*Proof.* Now,

$$0 \rightarrow \mathfrak{S}_n \times \mathfrak{S}_n \xrightarrow{\text{incl}} \text{Aut}(G) \xrightarrow{s} \mathbb{Z}_2 \rightarrow 0$$

is exact and this sequence splits. Thus, we have  $\text{Aut}(G) \approx (\mathfrak{S}_n \times \mathfrak{S}_n) \rtimes_s \mathbb{Z}_2$ .  $\square$

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