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Magnitude Homology of Graphs and the Magnitude as its Categorification

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Supervisor's seal	acceptance stamp

Abstract

Sample Abstract

Contents

1 Introduction

Lamport's guide to LaTeX [1].

2 The Magnitude of Graphs

In this section, we define the magnitude of a graph G and the magnitude homology of a graph G, give some very basic examples and properties. By a *graph* we mean a finite undirected graph with no loops or multiple edges. The set of vertices of a graph G is denoted by $V(G)$, and the set of edges of G is denoted by $E(G)$. If x and y are vertices of a graph G, then the *distance* $d_G(x, y)$ between x and y is defined to be the length of a shortest edge path from x to y. If x and y lie in different components of G then $d(x, y) = \infty$.

2.1 The definition of the magnitude of Graphs

Here, we difine the magnitude of a graph, which can be expressed as either a rational function over $\mathbb{Q}$ or a formal power series over $\mathbb{Z}$. Write $\mathbb{Z}[q]$ for the polynomial ring over the integers in one variable q and $\mathbb{Z}[[q]]$ for the ring of formal power series over the integers in one variable q.

Definition 2.1.1. Let G be a graph. We define the *G-matrix* $Z_G = Z_G(q)$ over $\mathbb{Z}[q]$ whose rows and columns are indexed by the vertices of G, and whose (x, y)-entry is given by

$$Z_G(q)(x, y) = q^{d(x,y)} \quad (x, y \in V(G))$$

where by convention $q^\infty = 0$.

G-matrix is the square synmetric matrix.

Proposition 2.1.2. *G-matrix is invertible.*

Proof. By definition, the determinant of Z_G has constant term 1, which implies that $\det Z_G \neq 0$. □

Definition 2.1.3. The *magnitude* of a graph G is defined to be the rational function given by

$$\#G(q) = \sum_{x,y \in V(G)} (Z_G(q))^{-1}(x, y)$$

in the rational function field $\mathbb{Q}(q)$.

Remark 2.1.4.

$$\#G(q) = \mathrm{sum}(Z_G(q)^{-1}) = \frac{\mathrm{sum}(\mathrm{adj}(Z_G(q)))}{\det(Z_G(q))}$$

where adj is the adjugate matrix and sum is the sum of all entries of a matrix.

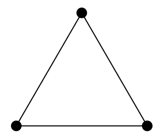
Proposition 2.1.5. *$\#G(q)$ takes values in $\mathbb{Z}[[q]]$.*

Proof. Let $\det Z_G(q) = 1 - qf(q)$ for some $f(q) \in \mathbb{Z}[q]$ by Proposition 2.1.2. Then we have

$$\#G(q) = \frac{\mathrm{sum}(\mathrm{adj}(Z_G))}{\det(Z_G)} = \mathrm{sum}(\mathrm{adj}(Z_G)) \sum_{n=0}^{\infty} q^n f(q)^n$$

Note that $qf(q)$ has no constant term and then $\sum_{n=0}^{\infty} q^n f(q)^n$ takes values in $\mathbb{Z}[[q]]$. $\square$

Example 2.1.6. Let $G = K_3$(complete graph with three vertices).



Then, we can calculate the magnitude of K_3 as follows:

$$Z_{K_3}(q) = \begin{pmatrix} 1 & q & q \\ q & 1 & q \\ q & q & 1 \end{pmatrix}, \quad Z_{K_3}(q)^{-1} = \frac{1}{1-3q^2+2q^3} \begin{pmatrix} 1-q^2 & -q+q^2 & -q+q^2 \\ -q+q^2 & 1-q^2 & -q+q^2 \\ -q+q^2 & -q+q^2 & 1-q^2 \end{pmatrix},$$

$\#K_3(q) = \frac{3}{1+2q}$

Definition 2.1.7. Let G be a graph and $x \in V(G)$. The *weight* of x in G is defined

$$w_G(x)(q) = \sum_{y \in V(G)} (Z_G(q))^{-1}(x, y)$$

The function $w_G : V(G) \to \mathbb{Q}(q)$ is called the *weighting* on G.

The magnitude can be expressed using the weighting as follows:

$$\#G(q) = \sum_{x \in V(G)} w_G(x)$$

Lemma 2.1.8. *For any graph G, the weighting w_G satisfies*

$$\sum_{y \in V(G)} q^{d(x,y)} w_G(y) = 1 \quad (x \in V(G))$$

Proof. For any vertex $x \in V(G)$, we have

$$\begin{aligned}
\sum_{y \in V(G)} q^{d(x,y)} w_G(y) &= \sum_{y,z \in V(G)} q^{d(x,y)} Z_G^{-1}(y,z) \\
&= \sum_{y,z \in V(G)} Z_G(x,y) Z_G^{-1}(y,z) \\
&= \sum_{z \in V(G)} \sum_{y \in V(G)} Z_G(x,y) Z_G^{-1}(y,z) \\
&= \sum_{z \in V(G)} (Z_G Z_G^{-1})(x,z) \\
&= \sum_{z \in V(G)} I(x,z) \\
&= 1.
\end{aligned}$$

□

This equation is called the *weighting equation.*

Lemma 2.1.9. *Let G be a graph and $\tilde{w}_G : V(G) \to \mathbb{Q}$ be a function satisfying a weighting equation. Then, $\tilde{w}_G = w_G$. Now, w_G is the weighting on G.*

Proof. Let $\boldsymbol{b} = (1, 1, \ldots, 1)^T$ where the length of $\boldsymbol{b}$ is $|V(G)|$ and $\boldsymbol{w}_G = (w_G(x))^T_{x \in V(G)}$. If $\tilde{w}_G$ satisfies the weighting equation, then we have

$$Z_G \tilde{\boldsymbol{w}}_G = \boldsymbol{b}$$

Since Z_G is invertible by Proposition 2.1.2, we have $\tilde{w}_G = w_G$ □

This lemma shows that the weighting on a graph is unique and we use this frequently to compute the magnitude of graphs.

2.2 Basic Properties and Examples

Here we give the most basic facts about magnitude. We focus on transitive graphs, disjoint unions, catesian products, and how the magnitude behaves within $\mathbb{Z}[[q]]$.

Definition 2.2.1. Let $G = (V(G), E(G)), H = (V(H), E(H))$ be a graph. An *graph homomorphism* from G to H is a map $f : V(G) \to V(H)$ such that if $\{x, y\} \in E(G)$ then $\{f(x), f(y)\} \in E(H)$.

We can define a *graph automorphism* using the definition above. We denote the group of all graph automorphisms of a graph G by $\mathrm{Aut}(G)$. $\mathrm{Aut}(G)$ includes id_G and for $g, h \in \mathrm{Aut}(G)$ and $x \in V(G)$, $g(h(x)) = (gh)(x)$, which means $\mathrm{Aut}(G)$ acts on $V(G)$.

Definition 2.2.2. A graph G is *vertex-transitive* if $\mathrm{Aut}(G)$ acts transitively on $V(G)$. It says that for any vertices x and y of G, there exists an automorphism $g : G \to G$ such that $y = g(x)$.

Lemma 2.2.3. *Let G be a vertex-transitive graph. Then,*

$$\#G(q) = \frac{|V(G)|}{\sum_{y \in V(G)} q^{d(x,y)}}$$

for any vertex $x \in V(G)$.

Proof. Let $S(x) = \sum_{y \in V(G)} q^{d(x,y)}$ for a vertex $x \in V(G)$. We show that $S(x)$ does not depend on the choice of x. Take any vertices $a, b \in V(G)$. Since G is vertex-transitive, there exists $g \in \mathrm{Aut}(G)$ such that $b = g(a)$. Then,

$$\begin{aligned}
S(b) &= \sum_{y \in V(G)} q^{d(b,y)} \\
&= \sum_{y \in V(G)} q^{d(g(a),y)} \\
&= \sum_{y \in V(G)} q^{d(g(a),g(y))} \quad \text{(since } g \text{ is bijective)} \\
&= \sum_{y \in V(G)} q^{d(a,y)} \quad \text{(since } g \text{ is an isomorphism)} \\
&= S(a)
\end{aligned}$$

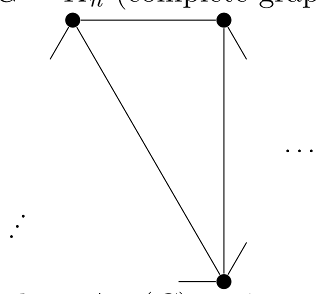
Thus, $S(x)$ does not depend on the choice of x, denoting it by S. Now, we define a function $\tilde{w}_G : V(G) \to \mathbb{Q}(q)$ by $\tilde{w}_G(x) = \frac{1}{S}$ for any vertex $x \in V(G)$. Then $\tilde{w}_G$ satisfies the weighting equation and by Lemma 2.1.9, we have $\#G = \frac{|V(G)|}{S}$.

$\square$

Example 2.2.4. (i) $G = V_n$ (edgeless graph with n vertices).

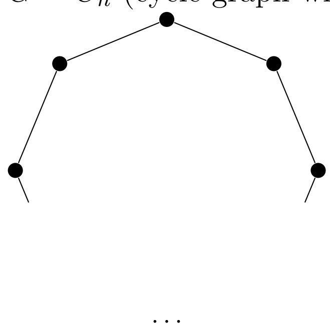
Then, $\mathrm{Aut}(G) \approx \mathfrak{S}_n$ and G is vertex-transitive. $S = 1$ and we have $\#V_n = n$.

(ii) $G = K_n$ (complete graph with n vertices).



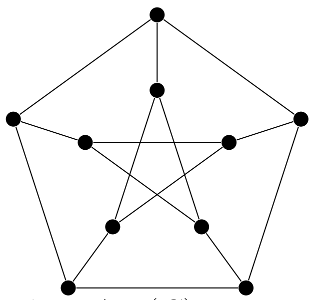
Then, $\mathrm{Aut}(G) \approx \mathfrak{S}_n$ and G is vertex-transitive. $S = 1 + (n-1)q$ and we have $\#K_n = \frac{n}{1+(n-1)q}$.

(iii) $G = C_n$ (cycle graph with n vertices).



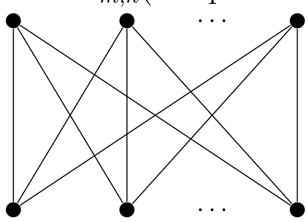
Then, $\mathrm{Aut}(G) \approx D_{2n}$ and G is vertex-transitive. If $n = 2m$(even), then $S = 1 + 2(q + q^2 + \cdots + q^{m-1}) + q^m = \frac{1+q-q^m-q^{m+1}}{1-q}$. Thus, we have $\#C_{2m} = \frac{2m(1-q)}{(1+q)(1-q^m)} = \frac{n(1-q)}{(1+q)(1-q^m)}$. If $n = 2m-1$(odd), then similarly $\#C_{2m-1} = \frac{n(1-q)}{1+q-2q^m}$.

(iv) G is a Petersen graph.



Then, $\mathrm{Aut}(G)$ contains D_{10} as its subgroup and G is vertex-transitive. $S = 1 + 3q + 6q^2$ and we have $\#G = \frac{10}{1+3q+6q^2}$.

(v) $G = K_{m,n}$(complete bipartite graph).



Then, $\mathrm{Aut}(G) \approx \mathfrak{S}_m \times \mathfrak{S}_n$ if $m \neq n$ and G is not vertex-transitive. We can calculate the magnitude with other methods. Let a, b be the weight of vertices on each part of $K_{m,n}$. Then, the weighting equation is written by two equations as follows:

$$\begin{cases} \{q^0 + (m-1)q^2\}a + nqb = 1 \\ \{q^0 + (n-1)q^2\}b + mqa = 1 \end{cases}$$

We can solve this and we have

$$\#K_{m,n} = ma + nb = \frac{(m+n) - (2mn - m - n)q}{(1+q)(1-(m-1)(n-1)q^2)}$$

Lemma 2.2.5. *Let G and H be graphs. Then,*

$$\#(G \sqcup H) = \#G + \#H$$

where $G \sqcup H$ is the disjoint union of G and H.

Proof. $Z_{G \sqcup H} = \begin{pmatrix} Z_G & O \\ O & Z_H \end{pmatrix}$,

$Z_{G \sqcup H}^{-1} = \begin{pmatrix} Z_G^{-1} & O \\ O & Z_H^{-1} \end{pmatrix}$.

Thus,

$$\#(G \sqcup H) = \mathrm{sum}(Z_{G \sqcup H}^{-1}) = \mathrm{sum}(Z_G^{-1}) + \mathrm{sum}(Z_H^{-1}) = \#G + \#H$$

$\square$

Definition 2.2.6. Let G and H be graphs. The *cartesian product* $G \square H$ of G and H is the graph defined as follows;

- $V(G \square H) = V(G) \times V(H)$
- $E(G \square H) = \{\{(x, y), (x', y')\} | x = x' \text{ and } \{y, y'\} \in E(H) \text{ or } y = y' \text{ and } \{x, x'\} \in E(G)\}$.

Lemma 2.2.7. $\#G \square H = \#G \cdot \#H$

Proof. For $x, x' \in V(G)$ and $y, y' \in V(H)$,
$d_{G \square H}((x, y), (x', y')) = d_G(x, x') + d_H(y, y')$
$\Rightarrow Z_{G \square H}((x, y), (x', y')) = q^{d_{G \square H}((x,y),(x',y'))} = q^{d_G(x,x')} q^{d_H(y,y')} = Z_G(x, x') Z_H(y, y')$
$\Rightarrow Z_{G \square H} = Z_G \otimes Z_H$ and then $Z_{G \square H}^{-1} = Z_G^{-1} \otimes Z_H^{-1}$
$\Rightarrow \#G \square H = \mathrm{sum}(Z_{G \square H}^{-1}) = \mathrm{sum}(Z_G^{-1} \otimes Z_H^{-1}) = \mathrm{sum}(Z_G^{-1}) \cdot \mathrm{sum}(Z_H^{-1}) = \#G \cdot \#H$

We used the fact that $(P \otimes Q)(R \otimes S) = (PR) \otimes (QS)$ for proper matrices P, Q, R, S. $\square$

Example 2.2.8. See $G = K_2 \square K_3$.
$\#K_2 \square K_3 = \#K_2 \cdot \#K_3 = \frac{2}{1+q} \cdot \frac{3}{1+2q} = \frac{6}{(1+q)(1+2q)} = \#K_{3,3}$.

Remark 2.2.9. Here we use the catesian product for graph product, but there are other graph products such as the tensor product and strong product. However, there is a reason that we use the catesian product. This will be clear in Section 4.

Proposition 2.2.10. *Let G be a graph. Then,*

$$\begin{aligned}\#G(q) &= \sum_{k=0}^{\infty} (-1)^k \sum_{x_0 \neq x_1 \neq \cdots \neq x_k} q^{d(x_0,x_1)+d(x_1,x_2)+\cdots+d(x_{k-1},x_k)} \\ &= \sum_{n=0}^{\infty} c_n q^n\end{aligned}$$

where $c_n = \sum_{k=0}^{n} (-1)^k |\{(x_0, \ldots, x_k) | x_0 \neq x_1 \neq \cdots \neq x_k, d(x_0, x_1) + \cdots + d(x_{k-1}, x_k) = n\}|$

Proof. aaa $\square$

Corollary 2.2.11. *Let G be a graph. $|V(G)| = \#G(0), |E(G)| = -\frac{1}{2}\frac{d}{dq}\#G(q)\Big|_{q=0}$. Here, the derivative is taken in $\mathbb{Z}[[q]]$.*

Proof. From the previous proposition, we have

$$\begin{aligned} c_0 &= \sum_{k=0}^{0}(-1)^k|\{(x_0,\dots,x_k)|x_0 \neq x_1 \neq \cdots \neq x_k, d(x_0,x_1)+\cdots+d(x_{k-1},x_k)=n\}| \\ &= |\{(x_0)|x_0 \in V(G)\}| \\ &= |V(G)| \end{aligned}$$

and

$$\begin{aligned} c_1 &= |\{(x_0)|d(x_0,x_0)=1\}| - |\{(x_0,x_1)|x_0 \neq x_1, d(x_0,x_1)=1\}| \\ &= 0 - 2|E(G)| \\ &= -2|E(G)| \end{aligned}$$

This corollary immidiately follows from these equations. $\square$

Remark 2.2.12. $c_0 \geq 0, c_1 \leq 0,$ and $c_2 \geq 0$. $c_2 = 0$ if and only if

2.3 The main result of magnitude of graphs

This section states the inclusion-exclusion principle of magnitude of graphs under some conditions. First, we see that the magnitude does not satisfy the inclusion-exclusion principle in general. Then, we introduce the sufficient condition for the inclusion-exclusion principle to hold.

Definition 2.3.1. Let R be a ring. A function Φ is an *R-valued graph invariant* if

- $\Phi(G) \in R$ for any graph G
- If $G \approx H$ as a graph then $\Phi(G) = \Phi(H)$

Definition 2.3.2. Let Φ be an R-valued graph invariant.
1. Φ is said to be multiplicative if

- $\Phi(K_1) = 1$

- $\Phi(G \square H) = \Phi(G) \cdot \Phi(H)$ for any graphs G and H

2. Φ is said to satisfy the inclusion-exclusion principle if

- $\Phi(\emptyset) = 0$
- $\Phi(G \cup H) = \Phi(G) + \Phi(H) - \Phi(G \cap H)$ for any graphs G and H

Lemma 2.3.3. *Let R be a ring containing no nonzero nilpotents and Φ be a multiplicative R-valued graph invariant satisfying the inclusion-exclusion principle. Then, $\Phi(G) = |V(G)|$ for any graph G.*

Proof. aaa $\square$

Corollary 2.3.4. *The magnitude does not satisfy the inclusion-exclusion principle in general.*

Example 2.3.5. labelWillerton

Definition 2.3.6. Let X be a graph and U be a subgraph of X. U is said to be *convex* in X if for any vertices $x, y \in V(U)$, $d_U(x, y) = d_X(x, y)$.

Lemma 2.3.7. *Let X be a graph and G, H be subgraphs of X such that $X = G \cup H$. In this document, we mean $G \cup H$ as a graph $(V(G) \cup V(H), E(G) \cup E(H))$. Let $g \in V(G)$ and $h \in V(H)$ such that there is a path $(g = x_0 \to x_1 \to \cdots \to x_n = h)$ in X. Then, there exists a vertex $x_i \in V(G) \cap V(H)$.*

Proof. aaa $\square$

Lemma 2.3.8. *Let X be a graph and G, H be subgraphs of X such that $X = G \cup H$. If $G \cap H$ is convex in X, then G and H are also convex in X.*

Proof. aaa $\square$

Definition 2.3.9. Let X be a graph and U be a subgraph of X such that U is convex in X. Write $V_U(X) = \{v \in V(X) | d_X(v, u) < \infty$ for some $u \in V(U)\}$. Then, we say that X projects to U if for any $x \in V_U(X)$, there exists $\pi(x) \in V(U)$ such that for any $u \in V(U)$, $d_X(x, u) = d_X(x, \pi(x)) + d_X(\pi(x), u)$.

Lemma 2.3.10. *If X projects to U, then $\pi(x)$ is uniquely determined for any $x \in V_U(X)$.*

Proof. aaa □

Example 2.3.11. aaa

Lemma 2.3.12. *Let X be a graph and $U \subset X$ be a convex subgraph of X such that X projects to U. Then, for any $u \in V(U)$,*

$$w_U(u) = \sum_{x \in \pi^{-1}(u)} q^{d_X(u,x)} w_X(x)$$

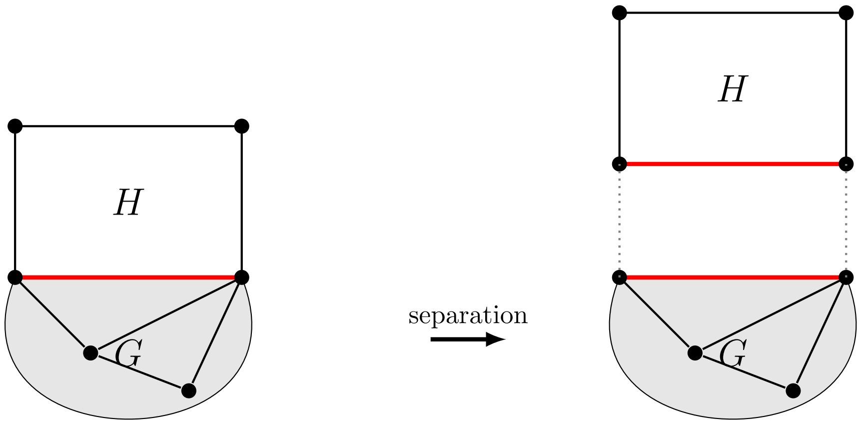
Proof. aaa □

Theorem 2.3.13. *(main theorem I) Let X be a graph and G, H be subgraphs of X such that $X = G \cup H$. If $G \cap H$ is convex in X and H projects to $G \cap H$, then*

$$\#X = \#G + \#H - \#(G \cap H)$$

Before proving this theorem, we give the example of graphs for which we can apply this theorem.

Example 2.3.14. Let G be a graph and consider the graph H formed by identifying one of the edges of a cycle graph C_n with an edge of G. Now, let $n \geq 4$.



$G \cup H$ G, H (separated)

Then, we can apply the main theorem I to $X = G \cup H$ as follows:

$$\#X = \#G + \#C_n - \#K_2$$

Similarly, if G and H are graphs and $G \vee H$ is the graph formed by identifying one vertex of G with one vertex of H, then we have

$$\#(G \vee H) = \#G + \#H - 1$$

Proof. (of main theorem I) aaa □

Example 2.3.15. The three graphs below are divided into a graph C_3, and two graphs C_2, so they all have the same magnitude and can be calculated as follows:

$$\#G = \#C_3 + 2\#C_2 - 2$$

Example 2.3.16. If G is a forest, then we can calculate the magnitude of G as follows:

$$\#G = |V(G)| - 2|E(G)|\frac{q}{1+q}$$

If G is a tree, then

$$\#G = |V(G)| - 2(|V(G)| - 1)\frac{q}{1+q}$$

Furthermore examples.

3 The Magnitude Homology of Graphs

In this section, we define the magnitude homology of a graph G, give some very basic examples and properties, and state the Mayer-Vietoris sequence for the magnitude homology of graphs.

3.1 The Definition of the magnitude homology of graphs

Definition 3.1.1. Let G be a graph. For positive integers k, the *length* of a tuple $(x_0, \dots, x_k)$ of $V(G)$ is defined to be

$$\begin{aligned} l(x_0, \dots, x_k) &= d(x_0, x_1) + d(x_1, x_2) + \cdots + d(x_{k-1}, x_k) \\ &= \sum_{i=1}^{k} d(x_{i-1}, x_i) \end{aligned}$$

Now, let $l(x_0) = 0$. We say the tuple $(x_0, \dots, x_k)$ is *good* if $x_0 \neq x_1 \neq \cdots \neq x_k$.

Lemma 3.1.2. *(Triangle inequality)*

$$l(x_0, \dots, x_k) \geq l(x_0, \dots, \hat{x}_i, \dots, x_k)$$

Definition 3.1.3. (magnitude chain complex) Let G be a graph. $MC_{*,*}(G)$ is the *magnitude complex* defined as follows:

$$MC_{*,*}(G) = \bigoplus_{l=0}^{\infty} MC_{*,l}(G)$$

For non-negative integers k and l, $MC_{k,l}(G)$ is freely generated by good tuples $(x_0, \dots, x_k)$ of $V(G)$ of length l. The coefficient ring is $\mathbb{Z}$. The differential $\partial : MC_{k,l}(G) \to MC_{k-1,l}(G)$ is defined by

$$\partial = \sum_{i=1}^{k-1} (-1)^{i-1} \partial_i$$

where $\partial_i(x_0, \dots, x_k) = (x_0, \dots, \hat{x}_i, \dots, x_k)$ if $l(x_0, \dots, \hat{x}_i, \dots, x_k) = l(x_0, \dots, x_k)$ and 0 otherwise.

Remark 3.1.4. For a good tuple $(x_0, \ldots, x_k)$,

$$\partial_i(x_0, \ldots, x_k) \neq 0 \iff d(x_{i-1}, x_i) + d(x_i, x_{i+1}) = d(x_{i-1}, x_{i+1})$$

Lemma 3.1.5. $\partial \circ \partial = 0$

Proof. aaa. □

Definition 3.1.6. (magnitude homology) Let G be a graph. The *magnitude homology* $MH_{*,*}(G)$ of G is the homology of the magnitude chain complex $MC_{*,*}(G)$. That is,

$$MH_{k,l}(G) = H_k(MC_{*,l}(G))$$

Example 3.1.7. (i) $G = V_n$. Then,

$$MC_{k,l}(V_n) = \begin{cases} \mathbb{Z}\{(x) | x \in V(V_n)\} & (k = l = 0) \\ 0 & (\text{otherwise}) \end{cases}$$

$\partial = 0$ inplies that

$$MH_{k,l}(V_n) \approx \begin{cases} \mathbb{Z}^n & (k = l = 0) \\ 0 & (\text{otherwise}) \end{cases}$$

(ii) $G = K_n (n \geq 2)$. Then, $l(x_0, \cdots, x_k) = k$ for any good tuple $(x_0, \cdots, x_k)$ of $V(K_n)$. Thus,

$$MC_{k,l}(K_n) = \begin{cases} \mathbb{Z}\{(x_0, \ldots, x_k) | x_0 \neq x_1 \neq \cdots \neq x_k\} & (l = k) \\ 0 & (\text{otherwise}) \end{cases}$$

$\partial = 0$ inplies that

$$MH_{k,l}(K_n) \approx \begin{cases} \mathbb{Z}^{n(n-1)^l} & (l = k) \\ 0 & (\text{otherwise}) \end{cases}$$

(iii) $G = C_5$. Number the vertices of C_5 as shown in the following figure.
ここにナンバリングした C_5 の図を挿入
Let us consider $MH_{2,3}(C_5)$. 続く

Definition 3.1.8. A graph G is said to be *diagonal* if $MH_{k,l}(G) = 0$ for $k \neq l$.

Lemma 3.1.9. *A tree is diagonal.*

Proof. aaa □

Theorem 3.1.10. *Let G be a graph. Then,*

$$\sum_{k,l \geq 0} (-1)^k rank(MH_{k,l}(G)) q^l = \#G \text{ in } \mathbb{Z}[[q]].$$

Proof.

$$\begin{aligned}
(LHS) &= \sum_{l \geq 0} \chi(MH_{*,l}(G)) q^l \\
&= \sum_{l \geq 0} \chi(MC_{*,l}(G)) q^l \\
&= \sum_{k,l \geq 0} (-1)^k \mathrm{rank}(MC_{k,l}(G)) q^l \\
&= \sum_{k \geq 0} (-1)^k \sum_{x_0 \neq \cdots \neq x_k} q^{d(x_0,x_1) + \cdots + d(x_{k-1},x_k)} \\
&= \#G.
\end{aligned}$$

The last equation is obtained by Proposition 2.2.10. □

Proposition 3.1.11. *Let G be a graph.*

- $MH_{0,0}(G) \approx \mathbb{Z}^{|V(G)|}$
- $MH_{1,1}(G) \approx \mathbb{Z}^{2|E(G)|}$

Proof.

$$MC_{k,0}(G) = \begin{cases} \mathbb{Z}\{(x) | x \in V(G)\} & (k = 0) \\ 0 & (\text{otherwise}) \end{cases}$$

and $\partial = 0$ induces the first equation.

$$MC_{k,1}(G) = \begin{cases} \mathbb{Z}\{(x_0, x_1) | x_0 \neq x_1\} & (k = 1) \\ 0 & (\text{otherwise}) \end{cases}$$

and $\partial = 0$ induces the second equation. □

Definition 3.1.12. The diameter d of a graph G is defined by

$$d = \max\{d(x, y) | x, y \in V(G) \text{ and } x, y \text{ lie in the same component of G}\}.$$

If $G = V_n$, then we define $d = 0$. Then, for any graph G, $0 \leq d < \infty$.

Proposition 3.1.13. *Let G be a graph and d be the diameter of G and assume that $MH_{k,l}(G) \neq 0$ for given non-negative integers k and l. Then,*

- $\frac{l}{d} \leq k \leq l$.
- *If $d > 1$ and $l > 0$, then* $\frac{l}{d} < k \leq l$.

Proof. Since $MH_{k,l}(G) \neq 0$, there exists a good tuple $(x_0, \dots, x_k)$ of length l such that $\partial(x_0, \dots, x_k) = 0$. Thus, $l = l(x_0, \dots, x_k) = \sum_{i=1}^{k} d(x_{i-1}, x_i) \leq \sum_{i=1}^{k} d = kd$ and $l = \sum_{i=1}^{k} d(x_{i-1}, x_i) \geq k$. This implies that $\frac{l}{d} \leq k \leq l$.

Now, assume that $d > 1$ and $l > 0$ and suppose that $k = \frac{l}{d}$. From the above discussion, we have $d(x_i, x_{i+1}) = d$ for all i. $\partial(x_0, \cdots, x_k) = 0$. For the $(k+1)$-tuple $\partial(x_0, \cdots, x_k)$ is a linear combination of at most $k-1$ distinct terms of k-tuples, so $\partial(x_0, \cdots, x_k) = 0$ implies $d(x_{i-1}, x_i) + d(x_i, x_{i+1}) \neq d(x_{i-1}, x_{i+1})$ for all $1 \leq i \leq k-1$. Since $d(x_0, x_1) = d \geq 2$, there exists a vertex y such that $d(x_0, y) + d(y, x_1) = d(x_0, x_1)$ and $y \neq x_0, x_1$. Then, $(x_0, y, x_1, \cdots, x_k)$ is a good tuple in $MC_{k+1,l}(G)$ and

$$\partial_i(x_0, y, x_1, \cdots, x_k) = \begin{cases} (x_0, x_1, \cdots, x_k) & (i = 1) \\ 0 & (2 \leq i \leq k) \end{cases}$$

. It is obvious for $3 \leq i$ by $d(x_{i-1}, x_i) + d(x_i, x_{i+1}) \neq d(x_{i-1}, x_{i+1})$ and is also true for $i = 2$ since $d(y, x_1) + d(x_1, x_2) = d(y, x_1) + d > d \geq d(y, x_2)$. This implies $MH_{k,l}(G) = 0$ and contradicts the assumption. □

3.2 Induced maps

Definition 3.2.1. Let G and H be graphs. A map $f : V(G) \to V(H)$ is said to be a *graph map* if for any $\{x, y\} \in E(G)$, either $f(x) = f(y)$ or $\{f(x), f(y)\} \in E(H)$.

Proposition 3.2.2. *$l(f(x_0), \cdots, f(x_k)) \leq l(x_0, \cdots, x_k)$ for any good tuple $(x_0, \cdots, x_k)$ of $V(G)$.*

Proof. For any vertices $x, y \in V(G)$, $d_H(f(x), f(y)) \leq d_G(x, y)$ holds. Indeed, if x, y lie in the same component of G, then there exists a path $(x = x_0 \to x_1 \to \cdots \to x_n = y)$ in G such that $n = d_G(x, y)$. Since f is a graph map, either $f(x_{i-1}) = f(x_i)$ or $\{f(x_{i-1}), f(x_i)\} \in E(H)$ for any $1 \leq i \leq n$. Thus, $(f(x) = f(x_0) \to f(x_1) \to \cdots \to f(x_n) = f(y))$ is a path in H and then $d_H(f(x), f(y)) \leq n = d_G(x, y)$. If x, y do not lie in the same component, then $d_G(x, y) = d_H(f(x), f(y)) = \infty$. Then,

$$\begin{aligned} l(f(x_0), \cdots, f(x_k)) &= \sum_{i=1}^{k} d_H(f(x_{i-1}), f(x_i)) \\ &\leq \sum_{i=1}^{k} d_G(x_{i-1}, x_i) \\ &= l(x_0, \cdots, x_k). \end{aligned}$$

□

Definition 3.2.3. Let G and H be graphs and $f : V(G) \to V(H)$ be a graph map. Then, the *induced map* $f_\# : MC_{*,*}(G) \to MC_{*,*}(H)$ is defined by

$$f_\#(x_0, \cdots, x_k) = \begin{cases} (f(x_0), \cdots, f(x_k)) & l(f(x_0), \cdots, f(x_k)) = l(x_0, \cdots, x_k) \\ 0 & \text{otherwise} \end{cases}$$

for any good tuple $(x_0, \cdots, x_k)$ of $V(G)$.

Proposition 3.2.4. *The induced map $f_\# : MC_{*,*}(G) \to MC_{*,*}(H)$ is a chain map.*

Proof. aaa. □

Definition 3.2.5. (Induced maps in homoplogy) If $f : G \to H$ is a graph map, the *induced map in homology* $f_* : MH_{*,*}(G) \to MH_{*,*}(H)$ is the map induced by the chain map $f_\# : MC_{*,*}(G) \to MC_{*,*}(H)$.

Proposition 3.2.6. *The assignment $G \mapsto MH_{*,*}(G)$ and $f \mapsto f_*$ defines a functor from the category of graphs and graph maps to the category of bigraded abelian groups and bigraded homomorphisms.*

Remark 3.2.7.

Proposition 3.2.8. *Let $f : G \to H$ be a graph map.*

3.3 u

4 Motivation : The Magnitude of Enriched Categories

In this section, I explain the motivation of why we study the magnitude of graphs in a larger context. We use the notion of enriched categories to define the magnitude.

4.1 the magnitude of a matrix

Definition 4.1.1. Let k be a set and $+, \cdot$ be a binary operation on k, and $0_k, 1_k$ be elements of k. Then, $(k, +, \cdot, 0_k, 1_k)$ is called a *rig* if the following conditions hold:

- $(k, +, 0_k)$ is a commutative monoid
- $(k, \cdot, 1_k)$ is a monoid
- multiplication distributes over addition

Now, we mean a rig as a commutative rig with the operation $\cdot$.

Example 4.1.2. $(\mathbb{Z}_{\geq 0}, +, \cdot, 0, 1)$ is a rig.

Definition 4.1.3. Let k be a rig and I, J be finite sets. A $I \times J$-matrix is a function $\zeta : I \times J \to k$.

Remark 4.1.4. Let k be a rig, and I, J, L be finite sets.

1. If ζ_1 is an $I \times J$-matrix and ζ_2 is a $J \times L$-matrix, then the product $\zeta_1\zeta_2$ is defined as follows:

$$(\zeta_1\zeta_2)(i, l) = \sum_{j \in J} \zeta_1(i, j) \cdot \zeta_2(j, l) \quad (i \in I, l \in L)$$

2. $\delta : I \times I \to k$ is called the *identity matrix* if $\delta(i, j) = 1_k$ when $i = j$ and $\delta(i, j) = 0_k$ when $i \neq j$.

3. Let $\zeta : I \times J \to k$ be a matrix. We define $\zeta^* : J \times I \to k$ by $\zeta^*(j, i) = \zeta(i, j)$.

4. Let ζ be an $I \times I$-matrix. If there exists an $I \times I$-matrix ζ^{-1} such that $\zeta\zeta^{-1} = \delta$ and $\zeta^{-1}\zeta = \delta$, then ζ is said to be *invertible* and ζ^{-1} is called the *inverse* of ζ.

5. $w : I \to k$ is called a *vector*. w can be thought of as an element of k^I. If ζ is an $I \times J$-matrix, v is a I-vector, and w is a J-vector, then the product $\zeta w : I \to k$ and $v\zeta : J \to k$ are defined by

$$(\zeta w)(i) = \sum_{j \in J} \zeta(i,j) \cdot w(j) \quad (i \in I)$$

$$(v\zeta)(j) = \sum_{i \in I} v(i) \cdot \zeta(i,j) \quad (j \in J)$$

   Now, ζw is a I-vector and $v\zeta$ is a J-vector.

6. If w, v are I-vectors, then the *inner product* vw is defined by

$$vw = \sum_{i \in I} v(i) \cdot w(i)$$

7. A vector $u_I : I \to k$ is defined by $u_I(i) = 1_k$ for any $i \in I$.

Definition 4.1.5. Let ζ be an $I \times I$-matrix over a rig k.

- A *weighting* on ζ is a vector $w : J \to k$ such that $\zeta w = u_I$. $w(j)$ is called the *weight* of $j \in J$.

- A *coweighting* on ζ is a vector $v : I \to k$ such that $v\zeta = u_I^*$. $v(i)$ is called the *coweight* of $i \in I$.

Example 4.1.6. Let G be a graph. Then, $Z_G(q)$ is a $V(G) \times V(G)$-matrix over the rig $\mathbb{Q}[[q]]$ and the weighting on $Z_G(q)$ is the weighting on G defined in Section 2.1.

Lemma 4.1.7. *Let ζ be an $I \times I$-matrix over a rig k. If ζ has a weighting w and a coweighting v, then*

$$\sum_{i \in I} v(i) = \sum_{j \in J} w(j)$$

Proof.

$$\begin{aligned}\sum_{i\in I} v(i) &= vu_I \\ &= v(\zeta w) \\ &= (v\zeta)w \\ &= u_J w \\ &= \sum_{j\in J} w(j)\end{aligned}$$

□

From this lemma, the sum of the weighting or coweighting on ζ is unique if they exist. Therefore, we can define the magnitude of ζ as follows:

Definition 4.1.8. Let ζ be an $I \times J$-matrix over a rig k. If ζ has a weighting and a coweighting, then the *magnitude* of ζ is defined to be

$$\#\zeta = \sum_{i\in I} v(i) = \sum_{j\in J} w(j)$$

where w is the weighting on ζ and v is the coweighting on ζ.

Lemma 4.1.9. *Let ζ be an $I \times I$-matrix over a rig k.*

1. *If ζ is invertible, then ζ has the magnitude.*

2. *If k is a field and ζ has the magnitude, then ζ is invertible.*

Proof. (1) If ζ is invertible, then $w = \zeta^{-1}u_I$ and $v = u_I\zeta^{-1}$ obviously satisfy the definition of weighting and coweighting respectively. Thus ζ has the magnitude by Lemma 4.1.7.

(2) If k is a field and ζ has the magnitude, then there exist a weighting w and a coweighting v on ζ. Let ζx be a zero-map for some $x : I \to k$. Then,

$$0 = v(\zeta x) = (v\zeta)x = u_I x = \sum_{i\in I} x(i)$$

ここからやり直し □

Lemma 4.1.10. *Let ζ be an invertible $I \times I$-matrix over a rig k. Then, ζ has the unique weighting w of ζ, given by $w(j) = \sum_{i \in I} \zeta^{-1}(j,i)$ for $j \in I$, and the unique coweighting v of ζ, given by $v(i) = \sum_{j \in I} \zeta^{-1}(j,i)$ for $i \in I$. Then,*

$$\#\zeta = \sum_{i,j \in I} \zeta^{-1}(j,i)$$

Proof. We should check the uniqueness and it holds from the invertibility of ζ. □

4.2 The definition of enriched categories

In this document, we only treat the locally small categories, which means that for any objects a, b of a category $\mathcal{C}$, the hom-set $\hom_{\mathcal{C}}(a,b)$ is a set.

Definition 4.2.1. A category $\mathcal{C}$ is called a *monoid* if $\mathcal{C}$ has only one object $*$ and $V = Hom_{\mathcal{C}}(*,*)$ is a monoid with the composition of morphisms as the binary operation and the identity morphism id_* as the identity element. We denote the operation of V by $\otimes$. ここに可換図式を挿入

Definition 4.2.2. A pair $(\mathcal{V}, \otimes, I)$ is called a *monoidal category* if it satisfies the following conditions:

1. $\mathcal{V}$ is a category.
2. $\otimes : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ is a functor.
3. I is an object of $\mathcal{V}$.
4. There exist the natural isomorphism $\alpha : \otimes \circ (\otimes \times id_{\mathcal{V}}) \Rightarrow \otimes \circ (id_{\mathcal{V}} \times \otimes)$ given by $\alpha_{uvw} : (u \otimes v) \otimes w \xrightarrow{\sim} u \otimes (v \otimes w)$
5. There exist the natural isomorphism $\lambda : I \otimes - \Rightarrow id_{\mathcal{V}}$ given by $\lambda_u : I \otimes u \xrightarrow{\sim} u$
6. There exist the natural isomorphism $\rho : - \otimes I \Rightarrow id_{\mathcal{V}}$ given by $\rho_u : u \otimes I \xrightarrow{\sim} u$
7. The following diagram commutes for any objects u, v, w, x of $\mathcal{V}$: ここに可換図式を挿入

8. The following diagrams commute for any objects u, v of $\mathcal{V}$: ここに可換図式を挿入

Example 4.2.3. (i) $(\mathit{Set}, \times, \{*\})$ is a monoidal category. ここに説明を挿入

(ii) $(\mathit{Vect}_K, \otimes_K, K)$ is a monoidal category, where K is a field. ここに説明を挿入

(iii) $([0, \infty], +, 0)$ is a monoidal category. ここに説明を挿入

(iv) $(\mathbf{2}, \otimes, t)$ is a monoidal category, where $\mathbf{2}$ is the category defined by $Ob(\mathbf{2}) = \{t, f\}$ and $\hom_{\mathbf{2}}(x, y) = \{*\}$ if $x \leq y$ and $\emptyset$ otherwise, and the operation $\otimes$ is defined by the following table:

$\otimes$	t	f
t	t	f
f	f	f

ここに説明を挿入

Then, $\mathbf{2}$ is a monoidal subcategory of $[0, \infty]$ by the embedding $t \mapsto 0, f \mapsto \infty$ and of Set by the embedding $t \mapsto \{*\}, f \mapsto \emptyset$.

Definition 4.2.4. An *enriched category* $\mathcal{A}$ in a monoidal category $(\mathcal{V}, \otimes, I)$ is defined as follows:

1. For any objects a, b of $\mathcal{A}$, $\hom_{\mathcal{A}}(a, b)$ is an object of $\mathcal{V}$.

2. For any objects a, b, c of $\mathcal{A}$, there exists a morphism $m_{abc} : \hom_{\mathcal{A}}(b, c) \otimes \hom_{\mathcal{A}}(a, b) \to \hom_{\mathcal{A}}(a, c)$ in $\mathcal{V}$, which defines the composition of morphisms.

3. For any object a of $\mathcal{A}$, there exists a morphism $j_a : I \to \hom_{\mathcal{A}}(a, a)$ in $\mathcal{V}$, which defines the identity morphism of a. あと３つの可換図式を挿入

Then, $\mathcal{A}$ is called a $\mathcal{V}$*-category.*

Definition 4.2.5. Let $\mathcal{A}, \mathcal{A}'$ be $\mathcal{V}$-categories. A functor $F : \mathcal{A} \to \mathcal{A}'$ is called a $\mathcal{V}$*-functor* if it satisfies the following conditions. We denote $F_{ab} : \mathrm{Mor}_{\mathcal{A}}(a, b) \to \mathrm{Mor}_{\mathcal{A}'}(F(a), F(b))$ as the morphism function.

1. The following diagram commutes for any objects a, b, c of $\mathcal{A}$: ここに可換図式を挿入

2. The following diagram commutes for any object a of $\mathcal{A}$: ここに可換図式を挿入

Remark 4.2.6. The family of all $\mathcal{V}$-categories and $\mathcal{V}$-functors form a category, which is denoted by $\mathcal{V}$-*Cat*.

Example 4.2.7. aaa

4.3 The magnitude of enriched categories

Definition 4.3.1. Let $(\mathcal{V}, \otimes, I)$ be a monoidal category and k be a rig.

References

[1] Leslie Lamport. *LaTeX: A Document Preparation System*. Addison-Wesley, 2nd edition, 1994.

[2] Donald E. Knuth. *The TeXbook*. Addison-Wesley, 1984.

Proposition .0.2. *Let $G = K_{n,n}$. Then, $Aut(G) \approx (\mathfrak{S}_n \times \mathfrak{S}_n) \rtimes_s \mathbb{Z}_2$, where $s : \mathbb{Z}_2 \to Aut(G); 0 \mapsto id_G, 1 \mapsto \tau$ and τ is the automorphism which interchanges the two parts of $K_{n,n}$.*

Proof. Now,

$$0 \to \mathfrak{S}_n \times \mathfrak{S}_n \xrightarrow{incl} \mathrm{Aut}(G) \xrightarrow{s} \mathbb{Z}_2 \to 0$$

is exact and this sequence splits. Thus, we have $\mathrm{Aut}(G) \approx (\mathfrak{S}_n \times \mathfrak{S}_n) \rtimes_s \mathbb{Z}_2$. $\square$