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# Magnitude Homology of Graphs and the Magnitude as its Categorification

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**Abstract**

Sample Abstract

# Contents

# 1 Introduction

Lamport's guide to LaTeX [1]. We denote $\approx$ as isomorphisms and $\cong$ as homeomorphisms.

# 2 The Magnitude of Graphs

This section introduces the magnitude and the magnitude homology of a graph $G$, along with fundamental examples and properties. Throughout this thesis, a *graph* means a finite undirected graph with no loops or multiple edges. The set of vertices of a graph $G$ is denoted by $V(G)$, and the set of edges of $G$ is denoted by $E(G)$. For vertices $x, y \in V(G)$, the *distance* $d_G(x, y)$ is defined as the length of a shortest path between them, where the length of a path is the number of edges it contains. If $x$ and $y$ lie in different connected components of $G$, we set $d_G(x, y) = \infty$. Now, we say that two vertex $x, y$ in $G$ lie in the same connected component if there exists a path between them.

## 2.1 The Definition of the Magnitude of Graphs

We begin by defining the magnitude of a graph. The magnitude is defined to take values in the field of rational functions over $\mathbb{Q}$. It can also be interpreted as taking values in the ring of formal power series over $\mathbb{Z}$, a property that will be discussed in a later lemma. Let $\mathbb{Q}(q)$ denote the field of rational functions in a variable $q$ over $\mathbb{Q}$. We also write $\mathbb{Z}[q]$ and $\mathbb{Z}[\![q]\!]$ for the polynomial ring and the ring of formal power series in $q$ over $\mathbb{Z}$, respectively.

**Definition 2.1.1.** Let $G$ be a graph. We define the *G-matrix* $Z_G = Z_G(q)$ over $\mathbb{Z}[q]$ whose rows and columns are indexed by the vertices of $G$, and whose $(x, y)$-entry is given by

$$Z_G(q)(x, y) = q^{d(x,y)} \quad (x, y \in V(G)),$$

where by convention $q^\infty = 0$.

$G$-matrix is the square synmetric matrix.

**Proposition 2.1.2.** *A の G-matrix is invertible.*

*Proof.* By definition, the determinant of $Z_G$ has constant term 1, which implies that $\det Z_G \neq 0$. $\square$

**Definition 2.1.3.** The *magnitude* of a graph $G$ is defined to be the rational function given by

$$\#G(q) = \sum_{x,y \in V(G)} (Z_G(q))^{-1}(x,y).$$

**Remark 2.1.4.**

$$\#G(q) = \mathrm{sum}(Z_G(q)^{-1}) = \frac{\mathrm{sum}(\mathrm{adj}(Z_G(q)))}{\det(Z_G(q))},$$

where adj is the adjugate matrix and sum is the sum of all entries of a matrix.

**Proposition 2.1.5.** *$\#G(q)$ takes values in $\mathbb{Z}[\![q]\!]$.*

*Proof.* Let $\det Z_G(q) = 1 - qf(q)$ for some $f(q) \in \mathbb{Z}[q]$ by Proposition 2.1.2. Then we have

$$\#G(q) = \frac{\mathrm{sum}(\mathrm{adj}(Z_G))}{\det(Z_G)} = \mathrm{sum}(\mathrm{adj}(Z_G)) \sum_{n=0}^{\infty} q^n f(q)^n.$$

Note that $qf(q)$ has no constant term and then $\sum_{n=0}^{\infty} q^n f(q)^n$ takes values in $\mathbb{Z}[\![q]\!]$. □

**Example 2.1.6.** Let $G = K_3$ (complete graph with three vertices).

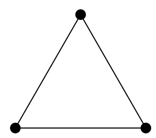

図 1: Complete graph of three vertices

Then, we can calculate the magnitude of $K_3$ as follows:

$$Z_{K_3}(q) = \begin{pmatrix} 1 & q & q \\ q & 1 & q \\ q & q & 1 \end{pmatrix}, \quad Z_{K_3}(q)^{-1} = \frac{1}{1-3q^2+2q^3} \begin{pmatrix} 1-q^2 & -q+q^2 & -q+q^2 \\ -q+q^2 & 1-q^2 & -q+q^2 \\ -q+q^2 & -q+q^2 & 1-q^2 \end{pmatrix},$$

$\#K_3(q) = \frac{3}{1+2q}$.

**Definition 2.1.7.** Let $G$ be a graph and $x \in V(G)$. The *weight* of $x$ in $G$ is defined

$$w_G(x)(q) = \sum_{y \in V(G)} (Z_G(q))^{-1}(x, y)$$

The function $w_G : V(G) \to \mathbb{Q}(q)$ is called the *weighting* on $G$.

The magnitude can be expressed using the weighting as follows:

$$\#G(q) = \sum_{x \in V(G)} w_G(x)$$

**Lemma 2.1.8.** *For any graph $G$, the weighting $w_G$ satisfies*

$$\sum_{y \in V(G)} q^{d(x,y)} w_G(y) = 1 \quad (x \in V(G)).$$

*Proof.* For any vertex $x \in V(G)$, we have

$$\begin{aligned}
\sum_{y \in V(G)} q^{d(x,y)} w_G(y) &= \sum_{y,z \in V(G)} q^{d(x,y)} Z_G^{-1}(y, z) \\
&= \sum_{y,z \in V(G)} Z_G(x, y) Z_G^{-1}(y, z) \\
&= \sum_{z \in V(G)} \sum_{y \in V(G)} Z_G(x, y) Z_G^{-1}(y, z) \\
&= \sum_{z \in V(G)} (Z_G Z_G^{-1})(x, z) \\
&= \sum_{z \in V(G)} I(x, z) \\
&= 1.
\end{aligned}$$

□

This equation is called the *weighting equation.*

**Lemma 2.1.9.** *Let $G$ be a graph and $\tilde{w}_G : V(G) \to \mathbb{Q}$ be a function satisfying a weighting equation. Then, $\tilde{w}_G = w_G$. Now, $w_G$ is the weighting on $G$.*

*Proof.* Let $\boldsymbol{b} = (1, 1, \ldots, 1)^T$ where the length of $\boldsymbol{b}$ is $|V(G)|$ and $\boldsymbol{w}_G = (w_G(x))^T_{x \in V(G)}$. If $\tilde{w}_G$ satisfies the weighting equation, then we have

$$Z_G \tilde{\boldsymbol{w}}_G = \boldsymbol{b}.$$

Since $Z_G$ is invertible by Proposition 2.1.2, we have $\tilde{w}_G = w_G$. □

This lemma shows that the weighting on a graph is unique and we use this frequently to compute the magnitude of graphs.

## 2.2 Basic Properties and Examples

This subsection presents fundamental properties and examples of magnitude. We focus on vertex-transitive graphs, disjoint unions, and Cartesian products. We also discuss the properties of magnitude within $\mathbb{Z}[\![q]\!]$.

**Definition 2.2.1.** Let $G = (V(G), E(G)), H = (V(H), E(H))$ be graphs. A *graph homomorphism* from $G$ to $H$ is a map $f : V(G) \to V(H)$ such that if $\{x, y\} \in E(G)$ then $\{f(x), f(y)\} \in E(H)$ or $f(x) = f(y)$.

We can define a *graph automorphism* using the definition above. We denote the group of all graph automorphisms of a graph $G$ by $\mathrm{Aut}(G)$. $\mathrm{Aut}(G)$ includes $id_G$ and for $g, h \in \mathrm{Aut}(G)$ and $x \in V(G)$, $g(h(x)) = (gh)(x)$, which means $\mathrm{Aut}(G)$ acts on $V(G)$.

**Definition 2.2.2.** A graph $G$ is *vertex-transitive* if $\mathrm{Aut}(G)$ acts transitively on $V(G)$. It says that for any vertices $x$ and $y$ of $G$, there exists an automorphism $g : G \to G$ such that $y = g(x)$.

**Lemma 2.2.3.** *Let $G$ be a vertex-transitive graph. Then,*

$$\#G(q) = \frac{|V(G)|}{\sum_{y \in V(G)} q^{d(x,y)}}$$

*for any vertex $x \in V(G)$.*

*Proof.* Let $S(x) = \sum_{y \in V(G)} q^{d(x,y)}$ for a vertex $x \in V(G)$. We show that $S(x)$ does not depend on the choice of $x$. Take any vertices $a, b \in V(G)$.

Since $G$ is vertex-transitive, there exists $g \in \mathrm{Aut}(G)$ such that $b = g(a)$. Then,

$$\begin{aligned}
S(b) &= \sum_{y \in V(G)} q^{d(b,y)} \\
&= \sum_{y \in V(G)} q^{d(g(a),y)} \\
&= \sum_{y \in V(G)} q^{d(g(a),g(y))} \quad \text{(since } g \text{ is bijective)} \\
&= \sum_{y \in V(G)} q^{d(a,y)} \quad \text{(since } g \text{ is an isomorphism)} \\
&= S(a).
\end{aligned}$$

Thus, $S(x)$ does not depend on the choice of $x$, denoting it by $S$. Now, we define a function $\tilde{w}_G : V(G) \to \mathbb{Q}(q)$ by $\tilde{w}_G(x) = \frac{1}{S}$ for any vertex $x \in V(G)$. Then $\tilde{w}_G$ satisfies the weighting equation and by Lemma 2.1.9, we have $\#G = \frac{|V(G)|}{S}$.

□

**Example 2.2.4.** (i) $G = V_n$ (edgeless graph with $n$ vertices).

図 2: The graph with no edges

Then, $\mathrm{Aut}(G) \approx \mathfrak{S}_n$ and $G$ is vertex-transitive. $S = 1$ and we have $\#V_n = n$.

(ii) $G = K_n$ (complete graph with $n$ vertices).

Then, $\mathrm{Aut}(G) \approx \mathfrak{S}_n$ and $G$ is vertex-transitive. $S = 1 + (n-1)q$ and we have $\#K_n = \frac{n}{1+(n-1)q}$.

(iii) $G = C_n$ (cycle graph with $n$ vertices).

Then, $\mathrm{Aut}(G) \approx D_{2n}$ and $G$ is vertex-transitive. If $n = 2m$, then $S = 1 + 2(q + q^2 + \cdots + q^{m-1}) + q^m = \frac{1+q-q^m-q^{m+1}}{1-q}$. Thus, we have

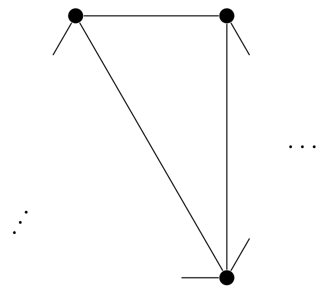

図 3: Complete graph

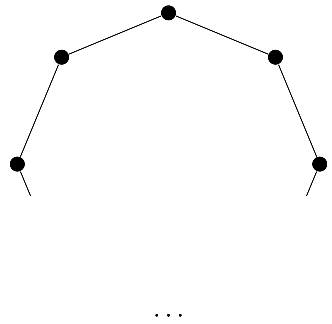

図 4: Cycle graph

$\#C_{2m} = \frac{2m(1-q)}{(1+q)(1-q^m)} = \frac{n(1-q)}{(1+q)(1-q^m)}$. If $n = 2m-1$, then similarly $\#C_{2m-1} = \frac{n(1-q)}{1+q-2q^m}$.

(iv) $G$ is the Petersen graph.

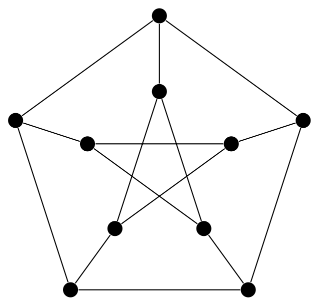

図 5: Petersen graph

Then, $\mathrm{Aut}(G)$ contains $D_{10}$ as its subgroup and $G$ is vertex-transitive. $S = 1 + 3q + 6q^2$ and we have $\#G = \frac{10}{1+3q+6q^2}$.

(v) $G = K_{m,n}$(complete bipartite graph).

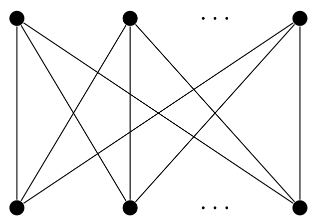

図 6: complete bipartite graph

Then, $\mathrm{Aut}(G) \approx \mathfrak{S}_m \times \mathfrak{S}_n$ if $m \neq n$ and $G$ is not vertex-transitive. We can calculate the magnitude with other methods. Let $a, b$ be the weight of vertices on each part of $K_{m,n}$. Then, the weighting equation is written by two equations as follows:

$$\begin{cases} \{q^0 + (m-1)q^2\}a + nqb = 1. \\ \{q^0 + (n-1)q^2\}b + mqa = 1. \end{cases}$$

We can solve this and we have

$$\#K_{m,n} = ma + nb = \frac{(m+n) - (2mn - m - n)q}{(1+q)(1-(m-1)(n-1)q^2)}.$$

**Lemma 2.2.5.** *Let $G$ and $H$ be graphs. Then,*

$$\#(G \sqcup H) = \#G + \#H,$$

*where $G \sqcup H$ is the disjoint union of $G$ and $H$.*

*Proof.* $Z_{G \sqcup H} = \begin{pmatrix} Z_G & O \\ O & Z_H \end{pmatrix}, Z_{G \sqcup H}^{-1} = \begin{pmatrix} Z_G^{-1} & O \\ O & Z_H^{-1} \end{pmatrix}$.

Thus,

$$\#(G \sqcup H) = \mathrm{sum}(Z_{G \sqcup H}^{-1}) = \mathrm{sum}(Z_G^{-1}) + \mathrm{sum}(Z_H^{-1}) = \#G + \#H.$$

□

**Definition 2.2.6.** Let $G$ and $H$ be graphs. The *cartesian product* $G \square H$ of $G$ and $H$ is the graph defined as follows;

- $V(G\Box H) = V(G) \times V(H)$.
- $E(G\Box H) = \{\{(x,y),(x',y')\} | x = x' \text{ and } \{y,y'\} \in E(H) \text{ or } y = y' \text{ and } \{x,x'\} \in E(G)\}$.

**Lemma 2.2.7.** *$\#(G\Box H) = \#G \cdot \#H$.*

*Proof.* For $x, x' \in V(G)$ and $y, y' \in V(H)$,
$d_{G\Box H}((x,y),(x',y')) = d_G(x,x') + d_H(y,y')$,
$Z_{G\Box H}((x,y),(x',y')) = q^{d_{G\Box H}((x,y),(x',y'))} = q^{d_G(x,x')}q^{d_H(y,y')} = Z_G(x,x')Z_H(y,y')$,
$Z_{G\Box H} = Z_G \otimes Z_H$ and then $Z_{G\Box H}^{-1} = Z_G^{-1} \otimes Z_H^{-1}$.
Thus, $\#G\Box H = \mathrm{sum}(Z_{G\Box H}^{-1}) = \mathrm{sum}(Z_G^{-1} \otimes Z_H^{-1}) = \mathrm{sum}(Z_G^{-1}) \cdot \mathrm{sum}(Z_H^{-1}) = \#G \cdot \#H$.

We used the fact that $(P \otimes Q)(R \otimes S) = (PR) \otimes (QS)$ for matrices $P, Q, R,$ and $S$ such that the products of $PR$ and $QS$ are defined. $\Box$

**Example 2.2.8.** $G = K_2 \Box K_3$.
$\#K_2\Box K_3 = \#K_2 \cdot \#K_3 = \frac{2}{1+q} \cdot \frac{3}{1+2q} = \frac{6}{(1+q)(1+2q)} = \#K_{3,3}$.

**Remark 2.2.9.** Here we use the catesian product for graph product, but there are other graph products. However, there is a reason that we use the catesian product. This will be clear in Section 4.

**Proposition 2.2.10.** *Let $G$ be a graph. Then,*

$$\begin{aligned}\#G(q) &= \sum_{k=0}^{\infty} (-1)^k \sum_{x_0 \neq x_1 \neq \cdots \neq x_k} q^{d(x_0,x_1)+d(x_1,x_2)+\cdots+d(x_{k-1},x_k)} \\ &= \sum_{n=0}^{\infty} c_n q^n,\end{aligned}$$

*where*

$$c_n = \sum_{k=0}^{n} (-1)^k |\{(x_0,\ldots,x_k) | x_0 \neq x_1 \neq \cdots \neq x_k, d(x_0,x_1)+\cdots+d(x_{k-1},x_k) = n\}|.$$

*Proof.* aaa. $\Box$

**Corollary 2.2.11.** *Let $G$ be a graph. $|V(G)| = \#G(0), |E(G)| = -\frac{1}{2}\frac{d}{dq}\#G(q)\Big|_{q=0}$. Here, the derivative is taken in $\mathbb{Z}[\![q]\!]$.*

*Proof.* From the previous proposition, we have

$$\begin{aligned}
c_0 = &\sum_{k=0}^{0}(-1)^k|\{(x_0,\dots,x_k)|x_0 \neq x_1 \neq \cdots \neq x_k, d(x_0,x_1)+\cdots+d(x_{k-1},x_k)=n\}| \\
&= |\{(x_0)|x_0 \in V(G)\}| \\
&= |V(G)|
\end{aligned}$$

and

$$\begin{aligned}
c_1 &= |\{(x_0)|d(x_0,x_0)=1\}| - |\{(x_0,x_1)|x_0 \neq x_1, d(x_0,x_1)=1\}| \\
&= 0 - 2|E(G)| \\
&= -2|E(G)|.
\end{aligned}$$

This corollary immidiately follows from these equations. □

**Remark 2.2.12.** $c_0 \geq 0, c_1 \leq 0,$ and $c_2 \geq 0$. $c_2 = 0$ if and only if

## 2.3 Main Results on the Magnitude of Graphs

This subsection states the inclusion-exclusion principle for the magnitude of graphs under specific conditions. We begin by observing that the magnitude does not generally satisfy the inclusion-exclusion principle. We then introduce sufficient conditions for the principle to hold. In this document, we mean $G \cup H$ as a graph $(V(G) \cup V(H), E(G) \cup E(H))$.

**Definition 2.3.1.** Let $R$ be a ring. A function $\Phi$ is an *$R$-valued graph invariant* if

- $\Phi(G) \in R$ for any graph $G$.
- If $G \approx H$ as a graph, then $\Phi(G) = \Phi(H)$.

**Definition 2.3.2.** Let $\Phi$ be an $R$-valued graph invariant.

1. $\Phi$ is said to be *multiplicative* if
   - $\Phi(K_1) = 1$.
   - $\Phi(G \square H) = \Phi(G) \cdot \Phi(H)$ for any graphs $G$ and $H$.
2. $\Phi$ is said to satisfy the *inclusion-exclusion principle* if

- $\Phi(\emptyset) = 0$.
- $\Phi(G \cup H) = \Phi(G) + \Phi(H) - \Phi(G \cap H)$ for any graphs $G$ and $H$.

**Lemma 2.3.3.** *Let $R$ be a ring containing no nonzero nilpotents and $\Phi$ be a multiplicative $R$-valued graph invariant satisfying the inclusion-exclusion principle. Then, $\Phi(G) = |V(G)|$ for any graph $G$.*

*Proof.* aaa □

**Corollary 2.3.4.** *The magnitude does not satisfy the inclusion-exclusion principle in general.*

**Example 2.3.5.**

**Definition 2.3.6.** Let $X$ be a graph and $U$ be a subgraph of $X$. $U$ is said to be *convex* in $X$ if for any vertices $x, y \in V(U)$, $d_U(x, y) = d_X(x, y)$.

**Lemma 2.3.7.** *Let $X$ be a graph, $G, H$ be subgraphs of $X$ such that $X = G \cup H$, and $g \in V(G)$ and $h \in V(H)$ such that there is a path $(g = x_0 \to x_1 \to \cdots \to x_n = h)$ in $X$. Then, there exists a vertex $x_i \in V(G) \cap V(H)$.*

*Proof.* aaa □

**Lemma 2.3.8.** *Let $X$ be a graph and $G, H$ be subgraphs of $X$ such that $X = G \cup H$. If $G \cap H$ is convex in $X$, then $G$ and $H$ are also convex in $X$.*

*Proof.* aaa □

**Definition 2.3.9.** Let $X$ be a graph and $U$ be a subgraph of $X$ such that $U$ is convex in $X$. We denote $V_U(X) = \{v \in V(X) | d_X(v, u) < \infty \text{ for some } u \in V(U)\}$. Then, we say that $X$ *projects to* $U$ if for any $x \in V_U(X)$, there exists $\pi(x) \in V(U)$ such that for any $u \in V(U)$, $d_X(x, u) = d_X(x, \pi(x)) + d_X(\pi(x), u)$.

**Lemma 2.3.10.** *If $X$ projects to $U$, then $\pi(x)$ is uniquely determined for any $x \in V_U(X)$.*

*Proof.* aaa □

**Example 2.3.11.** aaa

**Lemma 2.3.12.** *Let $X$ be a graph and $U \subset X$ be a convex subgraph of $X$ such that $X$ projects to $U$. Then, for any $u \in V(U)$,*

$$w_U(u) = \sum_{x \in \pi^{-1}(u)} q^{d_X(u,x)} w_X(x).$$

*Proof.* aaa $\square$

**Theorem 2.3.13.** *(Main theorem I) Let $X$ be a graph and $G, H$ be subgraphs of $X$ such that $X = G \cup H$. If $G \cap H$ is convex in $X$ and $H$ projects to $G \cap H$, then*

$$\#X = \#G + \#H - \#(G \cap H).$$

Before proving this theorem, we give the example of graphs for which we can apply this theorem.

**Example 2.3.14.** Let $G$ be a graph and consider the graph $H$ formed by identifying one of the edges of a cycle graph $C_n$ with an edge of $G$. Now, let $n \geq 4$.

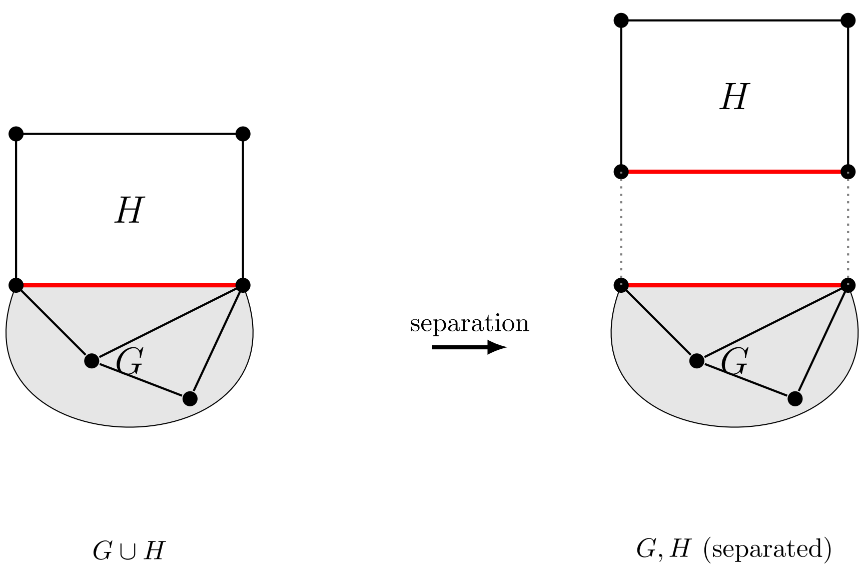


図 7: Cycle graph of four vertices

Then, we can apply Theorem 2.3.13 to $X = G \cup H$ as follows:

$$\#X = \#G + \#C_n - \#K_2.$$

Similarly, if $G$ and $H$ are graphs and $G \vee H$ is the graph formed by identifying one vertex of $G$ with one vertex of $H$, then we have

$$\#(G \vee H) = \#G + \#H - 1.$$

*Proof of Theorem 2.3.13.* aaa □

**Example 2.3.15.** The three graphs below are divided into a graph $C_3$, and two graphs $C_2$, so they all have the same magnitude and can be calculated as follows:

$$\#G = \#C_3 + 2 \cdot \#C_2 - 2.$$

**Example 2.3.16.** If $G$ is a forest, then we can calculate the magnitude of $G$ as follows:

$$\#G = |V(G)| - 2|E(G)|\frac{q}{1+q}.$$

If $G$ is a tree, then

$$\#G = |V(G)| - 2(|V(G)| - 1)\frac{q}{1+q}.$$

Furthermore examples.

# 3 The Magnitude Homology of Graphs

In this section, we define the magnitude homology of a graph $G$. We provide fundamental examples and properties, and state the Mayer-Vietoris sequence for magnitude homology.

## 3.1 The Definition of The Magnitude Homology of Graphs

**Definition 3.1.1.** Let $G$ be a graph. For a positive integer $k$, the *length* of a tuple $(x_0, \dots, x_k)$ of $V(G)$ is defined to be

$$\begin{aligned} l(x_0, \dots, x_k) &= d(x_0, x_1) + d(x_1, x_2) + \cdots + d(x_{k-1}, x_k) \\ &= \sum_{i=1}^{k} d(x_{i-1}, x_i). \end{aligned}$$

Now, let $l(x_0) = 0$. We say the tuple $(x_0, \dots, x_k)$ is *good* if $x_0 \neq x_1 \neq \cdots \neq x_k$.

**Lemma 3.1.2.** *(Triangle inequality) If $(x_0, \dots, x_k)$ is a good tuple of $V(G)$, then for any $1 \leq i \leq k-1$,*

$$l(x_0, \dots, x_k) \geq l(x_0, \dots, \hat{x}_i, \dots, x_k).$$

*Proof.* We obviously have the statement by the triangle inequality of the distance function $d$. $\square$

**Definition 3.1.3.** (*magnitude chain complex*) Let $G$ be a graph. $MC_{*,*}(G)$ is the *magnitude complex* defined as follows:

$$MC_{*,*}(G) = \bigoplus_{l=0}^{\infty} MC_{*,l}(G).$$

For non-negative integers $k$ and $l$, $MC_{k,l}(G)$ is freely generated by good tuples $(x_0, \dots, x_k)$ of $V(G)$ of length $l$ with the ring $\mathbb{Z}$. The differential $\partial : MC_{k,l}(G) \to MC_{k-1,l}(G)$ is defined by

$$\partial = \sum_{i=1}^{k-1} (-1)^{i-1} \partial_i,$$

where

$$\partial_i(x_0,\dots,x_k)=\begin{cases}(x_0,\dots,\hat{x_i},\dots,x_k) & \text{if } l(x_0,\dots,\hat{x_i},\dots,x_k)=l(x_0,\dots,x_k).\\ 0 & \text{otherwise.}\end{cases}$$

**Remark 3.1.4.** For a good tuple $(x_0,\dots,x_k)$,

$$\partial_i(x_0,\dots,x_k)\neq 0 \iff d(x_{i-1},x_i)+d(x_i,x_{i+1})=d(x_{i-1},x_{i+1}).$$

**Lemma 3.1.5.** $\partial\circ\partial=0$.

*Proof.* aaa. □

**Definition 3.1.6.** (*magnitude homology*) Let $G$ be a graph. The *magnitude homology* $MH_{*,*}(G)$ of $G$ is the homology of the magnitude chain complex $MC_{*,*}(G)$, that is,

$$MH_{k,l}(G)=\mathrm{Ker}\partial\cap(MC_{k,l}(G))/\mathrm{Im}\partial\cap(MC_{k,l}(G)).$$

**Example 3.1.7.** (i) $G=V_n$. Then,

$$MC_{k,l}(V_n)=\begin{cases}\mathbb{Z}\{(x)|x\in V(V_n)\} & (k=l=0).\\ 0 & (\text{otherwise}).\end{cases}$$

$\partial=0$ inplies that

$$MH_{k,l}(V_n)\approx\begin{cases}\mathbb{Z}^n & (k=l=0)\\ 0 & (\text{otherwise}).\end{cases}$$

(ii) $G=K_n(n\geq 2)$. Then, $l(x_0,\cdots,x_k)=k$ for any good tuple $(x_0,\cdots,x_k)$ of $V(K_n)$. Thus,

$$MC_{k,l}(K_n)=\begin{cases}\mathbb{Z}\{(x_0,\dots,x_k)|x_0\neq x_1\neq\cdots\neq x_k\} & (l=k).\\ 0 & (\text{otherwise}).\end{cases}$$

$\partial=0$ inplies that

$$MH_{k,l}(K_n)\approx\begin{cases}\mathbb{Z}^{n(n-1)^l} & (l=k).\\ 0 & (\text{otherwise}).\end{cases}$$

(iii) $G = C_5$. Number the vertices of $C_5$ as shown in the following figure.
ここにナンバリングした $C_5$ の図を挿入
Let us consider $MH_{2,3}(C_5)$. 続く

**Theorem 3.1.8.** *Let $G$ be a graph. Then,*

$$\sum_{k,l\geq 0}(-1)^k \mathrm{rank}(MH_{k,l}(G))q^l = \#G \ \mathit{in}\ \mathbb{Z}[\![q]\!].$$

*Proof.*

$$\begin{aligned}
(LHS) &= \sum_{l\geq 0}\chi(MH_{*,l}(G))q^l \\
&= \sum_{l\geq 0}\chi(MC_{*,l}(G))q^l \\
&= \sum_{k,l\geq 0}(-1)^k \mathrm{rank}(MC_{k,l}(G))q^l \\
&= \sum_{k\geq 0}(-1)^k \sum_{x_0\neq\cdots\neq x_k} q^{d(x_0,x_1)+\cdots+d(x_{k-1},x_k)} \\
&= \#G.
\end{aligned}$$

The last equation is obtained by Proposition 2.2.10. □

**Proposition 3.1.9.** *Let $G$ be a graph. Then,*

- $MH_{0,0}(G) \approx \mathbb{Z}^{|V(G)|}$.
- $MH_{1,1}(G) \approx \mathbb{Z}^{2|E(G)|}$.

*holds.*

*Proof.*

$$MC_{k,0}(G) = \begin{cases} \mathbb{Z}\{(x)|x\in V(G)\} & (k=0) \\ 0 & (\text{otherwise}) \end{cases}$$

and $\partial = 0$ induces the first equation.

$$MC_{k,1}(G) = \begin{cases} \mathbb{Z}\{(x_0,x_1)|x_0\neq x_1\} & (k=1) \\ 0 & (\text{otherwise}) \end{cases}$$

and $\partial = 0$ induces the second equation. □

**Definition 3.1.10.** The diameter $d$ of a graph $G$ is defined by

$$d = \max\{d(x,y)|x,y \in V(G) \text{ and } x,y \text{ lie in the same component of } G\}.$$

If $G = V_n$, then we define $d = 0$. Then, for any graph $G$, $0 \leq d < \infty$ holds.

**Proposition 3.1.11.** *Let $G$ be a graph and $d$ be the diameter of $G$ and assume that $MH_{k,l}(G) \neq 0$ for given non-negative integers $k$ and $l$. Then,*

- $\frac{l}{d} \leq k \leq l$.
- *If $d > 1$ and $l > 0$, then* $\frac{l}{d} < k \leq l$.

*holds.*

*Proof.* Since $MH_{k,l}(G) \neq 0$, there exists a good tuple $(x_0, \ldots, x_k)$ of length $l$ such that $\partial(x_0, \ldots, x_k) = 0$. Thus, $l = l(x_0, \ldots, x_k) = \sum_{i=1}^{k} d(x_{i-1}, x_i) \leq \sum_{i=1}^{k} d = kd$ and $l = \sum_{i=1}^{k} d(x_{i-1}, x_i) \geq k$. This implies that $\frac{l}{d} \leq k \leq l$.

Now, assume that $d > 1$ and $l > 0$ and suppose that $k = \frac{l}{d}$. From the above discussion, we have $d(x_i, x_{i+1}) = d$ for all $i$. $\partial(x_0, \cdots, x_k) = 0$. For the $(k+1)$-tuple $\partial(x_0, \cdots, x_k)$ is a linear combination of at most $k-1$ distinct terms of $k$-tuples, so $\partial(x_0, \cdots, x_k) = 0$ implies $d(x_{i-1}, x_i) + d(x_i, x_{i+1}) \neq d(x_{i-1}, x_{i+1})$ for all $1 \leq i \leq k-1$. Since $d(x_0, x_1) = d \geq 2$, there exists a vertex $y$ such that $d(x_0, y) + d(y, x_1) = d(x_0, x_1)$ and $y \neq x_0, x_1$. Then, $(x_0, y, x_1, \cdots, x_k)$ is a good tuple in $MC_{k+1,l}(G)$ and

$$\partial_i(x_0, y, x_1, \cdots, x_k) = \begin{cases} (x_0, x_1, \cdots, x_k) & (i = 1). \\ 0 & (2 \leq i \leq k). \end{cases}$$

It is obvious for $3 \leq i$ by $d(x_{i-1}, x_i) + d(x_i, x_{i+1}) \neq d(x_{i-1}, x_{i+1})$ and is also true for $i = 2$ since $d(y, x_1) + d(x_1, x_2) = d(y, x_1) + d > d \geq d(y, x_2)$. This implies $MH_{k,l}(G) = 0$ and contradicts the assumption. $\square$

## 3.2 Induced Maps

**Definition 3.2.1.** Let $G$ and $H$ be graphs. A map $f : V(G) \to V(H)$ is said to be a *graph map* if for any $\{x, y\} \in E(G)$, either $f(x) = f(y)$ or $\{f(x), f(y)\} \in E(H)$.

**Proposition 3.2.2.** *$l(f(x_0), \cdots, f(x_k)) \leq l(x_0, \cdots, x_k)$ for any good tuple $(x_0, \cdots, x_k)$ of $V(G)$.*

*Proof.* For any vertices $x, y \in V(G)$, $d_H(f(x), f(y)) \leq d_G(x, y)$ holds. Indeed, if $x, y$ lie in the same component of $G$, then there exists a path $(x = x_0 \to x_1 \to \cdots \to x_n = y)$ in $G$ such that $n = d_G(x, y)$. Since $f$ is a graph map, either $f(x_{i-1}) = f(x_i)$ or $\{f(x_{i-1}), f(x_i)\} \in E(H)$ for any $1 \leq i \leq n$. Thus, $(f(x) = f(x_0) \to f(x_1) \to \cdots \to f(x_n) = f(y))$ is a path in $H$ and then $d_H(f(x), f(y)) \leq n = d_G(x, y)$. If $x, y$ do not lie in the same component, then $d_G(x, y) = d_H(f(x), f(y)) = \infty$. Then,

$$\begin{aligned} l(f(x_0), \cdots, f(x_k)) &= \sum_{i=1}^{k} d_H(f(x_{i-1}), f(x_i)) \\ &\leq \sum_{i=1}^{k} d_G(x_{i-1}, x_i) \\ &= l(x_0, \cdots, x_k). \end{aligned}$$

□

**Definition 3.2.3.** Let $G$ and $H$ be graphs and $f : V(G) \to V(H)$ be a graph map. Then, the *induced map* $f_\# : MC_{*,*}(G) \to MC_{*,*}(H)$ is defined by

$$f_\#(x_0, \cdots, x_k) = \begin{cases} (f(x_0), \cdots, f(x_k)) & (l(f(x_0), \cdots, f(x_k)) = l(x_0, \cdots, x_k)) \\ 0 & (\text{otherwise}) \end{cases}$$

for any good tuple $(x_0, \cdots, x_k)$ of $V(G)$.

**Proposition 3.2.4.** *The induced map $f_\# : MC_{*,*}(G) \to MC_{*,*}(H)$ is a chain map.*

*Proof.* aaa. □

**Definition 3.2.5.** (Induced maps in homoplogy) If $f : G \to H$ is a graph map, the *induced map in homology* $f_* : MH_{*,*}(G) \to MH_{*,*}(H)$ is the map induced by the chain map $f_\# : MC_{*,*}(G) \to MC_{*,*}(H)$.

**Remark 3.2.6.** $A$ is called a *bigaded abelian group* if $A = \bigoplus_{k,l \geq 0} A_{k,l}$ where each $A_{k,l}$ is an abelian group. A *bigraded homomorphism* $f : A \to B$ between bigraded abelian groups $A$ and $B$ is a homomorphism such that $f(A_{k,l}) \subset B_{k,l}$ for any $k, l \geq 0$.

**Proposition 3.2.7.** *The assignment* $G \mapsto MH_{*,*}(G)$ *and* $f \mapsto f_*$ *defines a functor from the category of graphs and graph maps to the category of bigraded abelian groups and bigraded homomorphisms, denoting by* **Graph** $\to$ **BAb**.

**Proposition 3.2.8.** *Let* $f : G \to H$ *be a graph map.*

- $f_* : MH_{0,0}(G) \to MH_{0,0}(H)$ *is given by* $f_*(x) = f(x)$ *for any* $x \in V(G)$.
- $f_* : MH_{1,1}(G) \to MH_{1,1}(H)$ *is given by*

$$f_*(x_0, x_1) = \begin{cases} (f(x_0), f(x_1)) & (\text{ if } f(x_0) \neq f(x_1)) \\ 0 & (\text{otherwise}) \end{cases}$$

  *for any* $(x_0, x_1) \in MH_{1,1}(G)$.

*Proof.* The first equation is obvious.
For the second equation, we obtain by definition;

$$f_*(x_0, x_1) = \begin{cases} (f(x_0), f(x_1)) & l(f(x_0), f(x_1)) = l(x_0, x_1) = 1 \\ 0 & \text{otherwise} \end{cases}$$

for any $(x_0, x_1) \in MH_{1,1}(G)$. Since $f$ is a graph map, $l(f(x_0), f(x_1)) = 1$ if and only if $f(x_0) \neq f(x_1)$. □

**Corollary 3.2.9.** *Let* $f : G \to H$ *be a graph map.* $f_*$ *is an isomorphism if and only if* $f$ *is a graph isomorphism.*

*Proof.* aaa. □

## 3.3 Disjoint Union

**Proposition 3.3.1.** *Let* $G$ *and* $H$ *be graphs. We define the inclusion graph maps* $i : G \to G \bigsqcup H, j : H \to G \bigsqcup H$. *Then,*

$$i_* \oplus j_* : MH_{*,*}(G) \oplus MH_{*,*}(H) \to MH_{*,*}(G \bigsqcup H)$$

*is an isomorphism for each* $k, l \geq 0$.

*Proof.* saaa. □

We obtain Lemma 2.2.5 by Proposition 3.3.1 and $\chi(A_* \oplus B_*) = \chi(A_*) + \chi(B_*)$.

## 3.4 Cartesian Products

**Definition 3.4.1.** This definition is not true. Fix $l \geq 0$. The *exterior product* is the map

$$\Box : MC_{*,*}(G) \otimes MC_{*,*}(H) \to MC_{*,*}(G\Box H)$$

is defined as follows. Let $\Box$ be the map

$$\Box : MC_{k_1,l_1}(G) \times MC_{k_2,l_2}(H) \to MC_{k,l}(G\Box H) \text{ for } k_1, k_2 \geq 0, k = k_1+k_2, l = l_1+l_2,$$

which is defined by

$$\Box((x_0, \cdots, x_{k_1}), (y_0, \cdots, y_{k_2})) = \sum_{\sigma} \operatorname{sign}(\sigma)((x_{i_0}, y_{j_0}), (x_{i_1}, y_{j_1}), \cdots, (x_{i_k}, y_{j_k})),$$

where the sum is over all shuffles $\sigma$ of type $(k_1, k_2)$, that is, all sequences

$$((i_0, j_0), (i_1, j_1), \cdots, (i_k, j_k))$$

such that

$$i_0 = 0, j_0 = 0, 0 \leq i_r \leq k_1, 0 \leq j_r \leq k_2 \text{ for } 0 \leq r \leq k, \text{ and}$$

$$(i_{r+1}, j_{r+1}) = \begin{cases} (i_r + 1, j_r) & \text{or} \\ (i_r, j_r + 1) & \text{for } 0 \leq r < k, \end{cases}$$

and

$$\operatorname{sign}(\sigma) = (-1)^m,$$

where $m = \#\{(i,j) \in \{\{0, \cdots, k_1\} \times \{0, \cdots, k_2\}\} | i = i_r \Rightarrow j < j_r\}$.

Here, we extend the map $\Box$ bilinearly to the tensor product

$$MC_{k_1,l_1}(G) \otimes MC_{k_2,l_2}(H) \to MC_{k,l}(G\Box H)$$

.

We denote this induced map also by $\Box$ and call it the *exterior product.*

**Example 3.4.2.** Let $G = C_2 \Box C_2 = C_4$

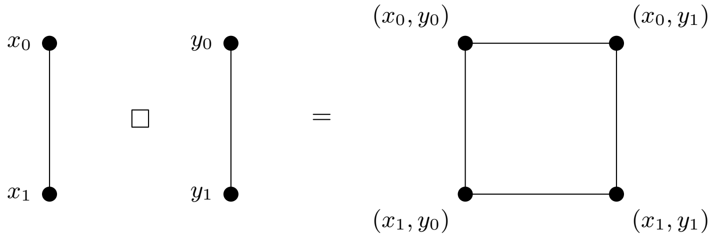


図 8: Square graph

Consider the exterior product $\square((x_0, x_1) \otimes (y_0, y_1))$. We have the two shuffles of type $(1, 1)$:

$$((0,0),(1,0),(1,1)),\ ((0,0),(0,1),(1,1)).$$

Thus,

$$\square((x_0, x_1) \otimes (y_0, y_1)) = -((x_0, y_0), (x_1, y_0), (x_1, y_1)) + ((x_0, y_0), (x_0, y_1), (x_1, y_1)).$$

**Remark 3.4.3.** As you see in the above example, the number of shuffles is $\binom{k}{k_1}$.

**Proposition 3.4.4.** *The exterior product* $\square : MC_{*,*}(G) \otimes MC_{*,*}(H) \to MC_{*,*}(G \square H)$ *is a chain map.*

*Proof.* Let $\mathbf{x} = (\mathbf{x_0}, \cdots, \mathbf{x_{k_1}}), \mathbf{y} = (\mathbf{y_0}, \cdots, \mathbf{y_{k_2}})$. Now, we show that

$$\partial \circ \square(\mathbf{x} \otimes \mathbf{y}) = \square((\partial \mathbf{x}) \otimes \mathbf{y}) + (-\mathbf{1})^{\mathbf{k_1}} \square(\mathbf{x} \otimes (\partial \mathbf{y})) = \square \circ (\partial \otimes \partial)(\mathbf{x} \otimes \mathbf{y}).$$

Here, we should consider the sequence of tensor products of the magnitude chain complexes defined by

$$\begin{aligned} \partial \otimes \partial : MC_{k_1,l_1}(G) \otimes MC_{k_2,l_2}(H) \to (&MC_{k_1-1,l_1}(G) \otimes MC_{k_2,l_2}(H)) \\ &\oplus (MC_{k_1,l_1}(G) \otimes MC_{k_2-1,l_2}(H)), \\ (\partial \otimes \partial)(\mathbf{x} \otimes \mathbf{y}) = (\partial \mathbf{x}) &\otimes \mathbf{y} + (-1)^{k_1} \mathbf{x} \otimes (\partial \mathbf{y}). \end{aligned}$$

Then, we should show only the first equality.
ここに可換図式を挿入. 証明も続く $\square$

From this proposition, we obtain the induced map in homology, also denoting $\square$.

**Definition 3.4.5.** (Tor functor) Let $R$ be a ring and $A$ and $B$ be $R$-modules. Then, $\mathrm{Tor}(A, B)$ is defined by the derived functor of the tensor product.

**Theorem 3.4.6.** *Let $G$ and $H$ be graphs.*

$$0 \to MH_{*,*}(G) \otimes MH_{*,*}(H) \xrightarrow{\square} MH_{*,*}(G \square H) \to \mathrm{Tor}(MH_{*-1,*}(G), MH_{*,*}(H)) \to 0$$

*is a short exact sequence and non-naturally split. In particular, if $MH_{*,*}(G)$ or $MH_{*,*}(H)$ is torsion-free, then the exterior product $\square$ is an isomorphism.*

We don't prove this theorem in this thesis.

**Example 3.4.7.** $G = C_4 = C_2 \square C_2$.

## 3.5 The Mayer-Vietoris Sequence

**Definition 3.5.1.** Let $X$ be a graph and $G, H$ be subgraphs of $X$.

1. $(X; G, H)$ is said to be a *projecting decomposition* if $X = G \cup H$, $G \cap H$ is convex in $X$ and $H$ projects to $G \cap H$.
   We write $i^G : G \to X$, $i^H : H \to X$, $j^G : G \cap H \to G$, $j^H : G \cap H \to H$ for the inclusion graph maps.

2. Let $(X; G, H), (X'; G', H')$ be projecting decompositions. $f : (X; G, H) \to (X'; G', H')$ is said to be a *decomposition map* if $f : X \to X'$ is a graph map such that $f(V(G)) \subset V(G')$ and $f(V(H)) \subset V(H')$.

3. Let $f : (X; G, H) \to (X'; G', H')$ be a decomposition map. Then, $f$ is said to be a *projecting decomposition map* if $V_{G \cap H}(H) = f^{-1}(V_{G' \cap H'}(H'))$ and $f(\pi(h)) = \pi(f(h))$ for any $h \in V_{G \cap H}(H)$.

4. Let $(X; G, H)$ be a projecting decomposition. $MC_{*,*}(G, H)$ denote the subcomplex of $MC_{*,*}(X)$ spanned by good tuples $(x_0, \cdots, x_k)$ whose entries all lie in $G$ or all lie in $H$.

**Theorem 3.5.2.** *Let $(X; G, H)$ be a projecting decomposition. Then, the inclusion map*

$$MC_{*,l}(G, H) \hookrightarrow MC_{*,l}(X)$$

*is a quasi-isomorphism for any $l \geq 0$.*

*Proof.* aaa. □

**Theorem 3.5.3.** *(the main theorem II) Let $(X; G, H)$ be a projecting decomposition. Then,*

$$0 \to MH_{*,*}(G \cap H) \xrightarrow{(j_*^G, -j_*^H)} MH_{*,*}(G) \oplus MH_{*,*}(H) \xrightarrow{i_*^G \oplus i_*^H} MH_{*,*}(X) \to 0$$

*is a split short exact sequence.*

*Proof.* aaa. □

**Corollary 3.5.4.** *Let $(X; G, H)$ be a projecting decomposition. Then,*

$$\#X = \#G + \#H - \#(G \cap H)$$

*in $\mathbb{Z}[[q]]$.*

*Proof.* By Theorem 3.5.3,

$$\chi(MH_{*,l}(G \cap H)) - \chi((MH_{*,l}(G)) \oplus \chi(MH_{*,l}(H))) + \chi(MH_{*,l}(X)) = 0.$$
$$\Rightarrow \chi(MH_{*,l}(X)) = \chi(MH_{*,l}(G)) + \chi(MH_{*,l}(H)) - \chi(MH_{*,l}(G \cap H)).$$

For each $l \geq 0$, multiplying by $q^l$ and summing over all $l \geq 0$, we have

$$\sum_{l \geq 0} \chi(MH_{*,l}(X))q^l = \sum_{l \geq 0} \chi(MH_{*,l}(G))q^l + \sum_{l \geq 0} \chi(MH_{*,l}(H))q^l - \sum_{l \geq 0} \chi(MH_{*,l}(G \cap H))q^l.$$

By Theorem 3.1.8, we obtain the desired equation. □

**Corollary 3.5.5.** *Let $T$ be a tree.*

## 3.6 Diagonal Graphs

**Definition 3.6.1.** A graph $G$ is said to be *diagonal* if $MH_{k,l}(G) = 0$ for $k \neq l$.

**Lemma 3.6.2.** *Any trees are diagonal.*

*Proof.* aaa □

**Proposition 3.6.3.** *For a diagonal graph, the magnitude completely determines the magnitude homology ranks.*

*Proof.* Obvious by Theorem 3.1.8. □

# 4 Motivation: The Magnitude of Enriched Categories

In this section, we explain the motivation for studying the magnitude of graphs in a broader context. We employ the notion of enriched categories to define the magnitude.

## 4.1 The Magnitude of a Matrix

**Definition 4.1.1.** Let $k$ be a set and $+, \cdot$ be a binary operation on $k$, and $0_k, 1_k$ be elements of $k$. Then, $(k, +, \cdot, 0_k, 1_k)$ is called a *rig* if the following conditions hold:

- $(k, +, 0_k)$ is a commutative monoid.
- $(k, \cdot, 1_k)$ is a monoid.
- multiplication distributes over addition.

Now, we mean a rig as a commutative rig with the operation $\cdot$.

**Example 4.1.2.** $(\mathbb{Z}_{\geq 0}, +, \cdot, 0, 1)$ is a rig.

**Definition 4.1.3.** Let $k$ be a rig and $I, J$ be finite sets. A $I \times J$-matrix is a function $\zeta : I \times J \to k$.

**Remark 4.1.4.** Let $k$ be a rig, and $I, J,$ and $L$ be finite sets.

1. If $\zeta_1$ is an $I \times J$-matrix and $\zeta_2$ is a $J \times L$-matrix, then the product $\zeta_1\zeta_2$ is defined as follows:
$$(\zeta_1\zeta_2)(i,l) = \sum_{j \in J} \zeta_1(i,j) \cdot \zeta_2(j,l) \quad (i \in I, l \in L)$$
2. $\delta : I \times I \to k$ is called the *identity matrix* if $\delta(i,j) = 1_k$ when $i = j$ and $\delta(i,j) = 0_k$ when $i \neq j$.
3. Let $\zeta : I \times J \to k$ be a matrix. We define $\zeta^* : J \times I \to k$ by $\zeta^*(j,i) = \zeta(i,j)$.

4. Let $\zeta$ be an $I \times I$-matrix. If there exists an $I \times I$-matrix $\zeta^{-1}$ such that $\zeta\zeta^{-1} = \delta$ and $\zeta^{-1}\zeta = \delta$, then $\zeta$ is said to be *invertible* and $\zeta^{-1}$ is called the *inverse* of $\zeta$.

5. $w : I \to k$ is called a *vector*. $w$ can be thought of as an element of $k^I$. If $\zeta$ is an $I \times J$-matrix, $v$ is a $I$-vector, and $w$ is a $J$-vector, then the product $\zeta w : I \to k$ and $v\zeta : J \to k$ are defined by

$$(\zeta w)(i) = \sum_{j \in J} \zeta(i,j) \cdot w(j) \quad (i \in I)$$

$$(v\zeta)(j) = \sum_{i \in I} v(i) \cdot \zeta(i,j) \quad (j \in J)$$

Now, $\zeta w$ is a $I$-vector and $v\zeta$ is a $J$-vector.

6. If $w, v$ are $I$-vectors, then the *inner product* $vw$ is defined by

$$vw = \sum_{i \in I} v(i) \cdot w(i)$$

7. A vector $u_I : I \to k$ is defined by $u_I(i) = 1_k$ for any $i \in I$.

**Definition 4.1.5.** Let $\zeta$ be an $I \times I$-matrix over a rig $k$.

- A *weighting* on $\zeta$ is a vector $w : J \to k$ such that $\zeta w = u_I$. $w(j)$ is called the *weight* of $j \in J$.

- A *coweighting* on $\zeta$ is a vector $v : I \to k$ such that $v\zeta = u_I^*$. $v(i)$ is called the *coweight* of $i \in I$.

**Example 4.1.6.** Let $G$ be a graph. Then, $Z_G(q)$ is a $V(G) \times V(G)$-matrix over the rig $\mathbb{Q}[\![q]\!]$ and the weighting on $Z_G(q)$ is the weighting on $G$ defined in Section 2.1.

**Lemma 4.1.7.** *Let $\zeta$ be an $I \times I$-matrix over a rig $k$. If $\zeta$ has a weighting $w$ and a coweighting $v$, then*

$$\sum_{i \in I} v(i) = \sum_{j \in J} w(j)$$

*Proof.*

$$\begin{aligned}
\sum_{i\in I} v(i) &= vu_I \\
&= v(\zeta w) \\
&= (v\zeta)w \\
&= u_J w \\
&= \sum_{j\in J} w(j)
\end{aligned}$$

□

From this lemma, the sum of the weighting or coweighting on $\zeta$ is unique if they exist. Therefore, we can define the magnitude of $\zeta$ as follows:

**Definition 4.1.8.** Let $\zeta$ be an $I \times J$-matrix over a rig $k$. If $\zeta$ has a weighting and a coweighting, then the *magnitude* of $\zeta$ is defined to be

$$\#\zeta = \sum_{i\in I} v(i) = \sum_{j\in J} w(j),$$

where $w$ is the weighting on $\zeta$ and $v$ is the coweighting on $\zeta$.

**Lemma 4.1.9.** *Let $\zeta$ be an $I \times I$-matrix over a rig $k$.*

1. *If $\zeta$ is invertible, then $\zeta$ has the magnitude.*
2. *If $k$ is a field and $\zeta$ has the magnitude, then $\zeta$ is invertible.*

*Proof.* (1) If $\zeta$ is invertible, then $w = \zeta^{-1}u_I$ and $v = u_I\zeta^{-1}$ obviously satisfy the definition of weighting and coweighting respectively. Thus $\zeta$ has the magnitude by Lemma 4.1.7.

(2) If $k$ is a field and $\zeta$ has the magnitude, then there exist a weighting $w$ and a coweighting $v$ on $\zeta$. Let $\zeta x$ be a zero-map for some $x : I \to k$. Then,

$$0 = v(\zeta x) = (v\zeta)x = u_I x = \sum_{i\in I} x(i)$$

ここからやり直し □

**Lemma 4.1.10.** *Let $\zeta$ be an invertible $I \times I$-matrix over a rig $k$. Then, $\zeta$ has the unique weighting $w$ of $\zeta$, given by $w(j) = \sum_{i \in I} \zeta^{-1}(j, i)$ for $j \in I$, and the unique coweighting $v$ of $\zeta$, given by $v(i) = \sum_{j \in I} \zeta^{-1}(j, i)$ for $i \in I$. Then,*

$$\#\zeta = \sum_{i,j \in I} \zeta^{-1}(j, i)$$

*Proof.* We should check the uniqueness and it holds from the invertibility of $\zeta$. $\square$

## 4.2 The Definition of Enriched Categories

In this document, we only treat the locally small categories, which means that for any objects $a, b$ of a category $\mathcal{C}$, the hom-set $\mathrm{Hom}_{\mathcal{C}}(a, b)$ is a set.

**Definition 4.2.1.** A category $\mathcal{C}$ is called a *monoid* if $\mathcal{C}$ has only one object $*$ and $V = \mathrm{Hom}_{\mathcal{C}}(*, *)$ is a monoid with the composition of morphisms as the binary operation and the identity morphism $id_*$ as the identity element. We denote the operation of $V$ by $\otimes$. ここに可換図式を挿入

**Definition 4.2.2.** A pair $(\mathcal{V}, \otimes, I)$ is called a *monoidal category* if it satisfies the following conditions:

1. $\mathcal{V}$ is a category.
2. $\otimes : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ is a functor.
3. $I$ is an object of $\mathcal{V}$.
4. There exist the natural isomorphism $\alpha : \otimes \circ (\otimes \times id_{\mathcal{V}}) \Rightarrow \otimes \circ (id_{\mathcal{V}} \times \otimes)$ given by $\alpha_{uvw} : (u \otimes v) \otimes w \xrightarrow{\sim} u \otimes (v \otimes w)$.
5. There exist the natural isomorphism $\lambda : I \otimes - \Rightarrow id_{\mathcal{V}}$ given by $\lambda_u : I \otimes u \xrightarrow{\sim} u$.
6. There exist the natural isomorphism $\rho : - \otimes I \Rightarrow id_{\mathcal{V}}$ given by $\rho_u : u \otimes I \xrightarrow{\sim} u$.
7. The following diagram commutes for any objects $u, v, w, x$ of $\mathcal{V}$: ここに可換図式を挿入

8. The following diagrams commute for any objects $u, v$ of $\mathcal{V}$: ここに可換図式を挿入

**Example 4.2.3.** (i) $(\mathbf{Set}, \times, \{*\})$ is a monoidal category. ここに説明を挿入

(ii) $(\mathbf{Vect}_K, \otimes_K, K)$ is a monoidal category, where $K$ is a field. ここに説明を挿入

(iii) $([0, \infty], +, 0)$ is a monoidal category. ここに説明を挿入

(iv) $(\mathbf{2}, \otimes, t)$ is a monoidal category, where $\mathbf{2}$ is the category defined by $Ob(\mathbf{2}) = \{t, f\}$ and the morphism sets are defined by

| $\hom_{\mathbf{2}}$ | $t$ | $f$ |
|---|---|---|
| $t$ | $\{id_t\}$ | $\emptyset$ |
| $f$ | $\{*\}$ | $\{id_f\}$ |

and the operation $\otimes$ is defined by the following table:

| $\otimes$ | $t$ | $f$ |
|---|---|---|
| $t$ | $t$ | $f$ |
| $f$ | $f$ | $f$ |

ここに説明を挿入

Then, $\mathbf{2}$ is a monoidal subcategory of $[0, \infty]$ by the embedding $t \mapsto 0, f \mapsto \infty$ and of $Set$ by the embedding $t \mapsto \{*\}, f \mapsto \emptyset$.

**Definition 4.2.4.** An *enriched category* $\mathcal{A}$ in a monoidal category $(\mathcal{V}, \otimes, I)$ is defined as follows:

1. For any objects $a, b$ of $\mathcal{A}$, $\mathrm{Hom}_{\mathcal{A}}(a, b)$ is an object of $\mathcal{V}$.

2. For any objects $a, b, c$ of $\mathcal{A}$, there exists a morphism $m_{abc} : \mathrm{Hom}_{\mathcal{A}}(b, c) \otimes \mathrm{Hom}_{\mathcal{A}}(a, b) \to \mathrm{Hom}_{\mathcal{A}}(a, c)$ in $\mathcal{V}$, which defines the composition of morphisms.

3. For any object $a$ of $\mathcal{A}$, there exists a morphism $j_a : I \to \mathrm{Hom}_{\mathcal{A}}(a, a)$ in $\mathcal{V}$, which defines the identity morphism of $a$. あと３つの可換図式を挿入

Then, $\mathcal{A}$ is called a *$\mathcal{V}$-category*.

**Definition 4.2.5.** Let $\mathcal{A}, \mathcal{A}'$ be $\mathcal{V}$-categories. A functor $F : \mathcal{A} \to \mathcal{A}'$ is called a *$\mathcal{V}$-functor* if it satisfies the following conditions. We denote $F_{ab} : \mathrm{Mor}_{\mathcal{A}}(a, b) \to \mathrm{Mor}_{\mathcal{A}'}(F(a), F(b))$ as the morphism function.

1. The following diagram commutes for any objects $a, b, c$ of $\mathcal{A}$: ここに可換図式を挿入

2. The following diagram commutes for any object $a$ of $\mathcal{A}$: ここに可換図式を挿入

**Remark 4.2.6.** The family of all $\mathcal{V}$-categories and $\mathcal{V}$-functors form a category, which is denoted by *$\mathcal{V}$-Cat*.

**Example 4.2.7.** aaa

## 4.3 The Magnitude of Enriched Categories

**Definition 4.3.1.** Let $(\mathcal{V}, \otimes, I)$ be a monoidal category and $k$ be a rig. We define a monoid homomorphism

$$|\cdot| : (\mathcal{O}\mathrm{b}(\mathcal{V})/\approx, \otimes, I) \to (k, \cdot, 1_k)$$

such that $|I| = 1_k$ and $|u \otimes v| = |u| \cdot |v|$ for any objects $u, v$ of $\mathcal{V}$.

**Example 4.3.2.** aaa.

**Definition 4.3.3.** Let $\mathcal{A}$ be a $\mathcal{V}$-category.

## 4.4 The Relation of The Magnitudes of Graphs and Enriched Categories

ここが私が一番説明したい部分です.

# Appendix

## A The calculation of Graph Automorphisms

**Proposition A.0.1.** *Let* $G = K_{n,n}$. *Then,* $\mathrm{Aut}(G) \approx (\mathfrak{S}_n \times \mathfrak{S}_n) \rtimes_s \mathbb{Z}_2$, *where* $s : \mathbb{Z}_2 \to \mathrm{Aut}(G); 0 \mapsto id_G, 1 \mapsto \tau$ *and* $\tau$ *is the automorphism which interchanges the two parts of* $K_{n,n}$.

*Proof.* Now,

$$0 \to \mathfrak{S}_n \times \mathfrak{S}_n \xrightarrow{incl} \mathrm{Aut}(G) \xrightarrow{s} \mathbb{Z}_2 \to 0$$

is exact and this sequence splits. Thus, we have $\mathrm{Aut}(G) \approx (\mathfrak{S}_n \times \mathfrak{S}_n) \rtimes_s \mathbb{Z}_2$. □

## References


[1] Leslie Lamport. *LaTeX: A Document Preparation System*. Addison-Wesley, 2nd edition, 1994.

[2] Donald E. Knuth. *The TeXbook*. Addison-Wesley, 1984.