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# Magnitude Homology of Graphs and the Magnitude as its Categorification

**2264257 Kensho Yachi**

<https://taro-ken.com>

**Supervisor : Yuta Nozaki Associate  
Professor**

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指導教員印	受理印

## **Abstract**

Sample Abstract

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# 1 Introduction

Lamport's guide to L<sup>A</sup>T<sub>E</sub>X [1].

## 2 The Magnitude of Graphs

In this section, we define the magnitude of a graph  $G$  and the magnitude homology of a graph  $G$ , give some very basic examples and properties. By a *graph* we mean a finite undirected graph with no loops or multiple edges. The set of vertices of a graph  $G$  is denoted by  $V(G)$ , and the set of edges of  $G$  is denoted by  $E(G)$ . If  $x$  and  $y$  are vertices of a graph  $G$ , then the *distance*  $d_G(x, y)$  between  $x$  and  $y$  is defined to be the length of a shortest edge path from  $x$  to  $y$ . If  $x$  and  $y$  lie in different components of  $G$  then  $d(x, y) = \infty$ .

### 2.1 The definition of the magnitude of Graphs

Here, we define the magnitude of a graph, which can be expressed as either a rational function over  $\mathbb{Q}$  or a formal power series over  $\mathbb{Z}$ . Write  $\mathbb{Z}[q]$  for the polynomial ring over the integers in one variable  $q$  and  $\mathbb{Z}[[q]]$  for the ring of formal power series over the integers in one variable  $q$ .

**Definition 2.1.1.** Let  $G$  be a graph. We define the  $G$ -matrix  $Z_G = Z_G(q)$  over  $\mathbb{Z}[q]$  whose rows and columns are indexed by the vertices of  $G$ , and whose  $(x, y)$ -entry is given by

$$Z_G(q)(x, y) = q^{d(x, y)} \quad (x, y \in V(G))$$

where by convention  $q^\infty = 0$ .

$G$ -matrix is the square symmetric matrix.

**Proposition 2.1.2.**  $G$ -matrix is invertible.

*Proof.* By definition, the determinant of  $Z_G$  has constant term 1, which implies that  $\det Z_G \neq 0$ .  $\square$

**Definition 2.1.3.** The *magnitude* of a graph  $G$  is defined to be the rational function given by

$$\#G(q) = \sum_{x, y \in V(G)} (Z_G(q))^{-1}(x, y)$$

in the rational function field  $\mathbb{Q}(q)$ .

**Remark 2.1.4.**

$$\#G(q) = \text{sum}(Z_G(q)^{-1}) = \frac{\text{sum}(\text{adj}(Z_G(q)))}{\det(Z_G(q))}$$

where  $\text{adj}$  is the adjugate matrix and  $\text{sum}$  is the sum of all entries of a matrix.

**Proposition 2.1.5.**  $\#G(q)$  takes values in  $\mathbb{Z}[[q]]$ .

*Proof.* Let  $\det Z_G(q) = 1 - qf(q)$  for some  $f(q) \in \mathbb{Z}[q]$  by theorem 2.1.2. Then we have

$$\#G(q) = \frac{\text{sum}(\text{adj}(Z_G(q)))}{\det(Z_G(q))} = \text{sum}(\text{adj}(Z_G(q))) \sum_{n=0}^{\infty} q^n f(q)^n$$

Note that  $qf(q)$  has no constant term and then  $\sum_{n=0}^{\infty} q^n f(q)^n$  takes values in  $\mathbb{Z}[[q]]$ .  $\square$

**Example 2.1.6.** Let  $G = K_3$  (complete graph with three vertices).

## 2.2 Basic Properties and Examples

Magnitude

## 2.3 The magnitude of union

### **3 The Magnitude Homology of Graphs**

In this section, we define the magnitude homology of a graph  $G$ , give some very

#### **3.1 The Definition of the magnitude homology of graphs**

#### **3.2 Magnitude Homology of Graphs is Categorification of Magnitude of Graphs**

#### **3.3 u**

## 4 Motivation : The Magnitude of Enriched Categories



## References

- [1] Leslie Lamport. *LaTeX: A Document Preparation System*. Addison-Wesley, 2nd edition, 1994.
- [2] Donald E. Knuth. *The TeXbook*. Addison-Wesley, 1984.