The magnitude of graphs and the magnitude homology as its categorification

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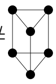
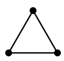
1. 研究背景・動機

The concept of magnitude was introduced by Leinster [2] and it is defined for enriched categories of finite objects such as generalized metric spaces. For example, given a finite graph G, let X_G be the generalized finite metric space consisting of vertices of G equipped with the shortest path metric. Subsequently, Leinster focuses on the magnitude of graphs in [3] using the idea of magnitude of a metric space. Some properties of magnitude are multiplicativity with respect to cartesian product, denoted by $\square$, and an inclusion-exclusion formula for the magnitude of a union under mild hypotheses:

$$\#(G \square H) = \#G \cdot \#H, \quad \#(G \cup H) = \#G + \#H - \#(G \cap H).$$

For example, the magnitude of the Cartesian product of the complete graph on 2 vertices K_2 and that on 3 vertices K_3 satisfies:

\#  = \#  · \# 

Formally, the magnitude of a graph is defined to be a rational function over $\mathbb{Q}$. Since the distance on a graph takes value in integers, it can also be expressed as a formal power series over $\mathbb{Z}$.

Hepworth and Willerton introduced a bigraded homology theory for graphs in [1], which has the magnitude as its graded Euler characteristic, and showed how properties of magnitude proved by Leinster categorify to properties such as a Künneth Theorem and a Mayer-Vietoris Theorem.

Here, we focus on the inclusion-exclusion formula and the relation between the magnitude of graphs and that of enriched categories. We denote the magnitude of a graph G by $\#G$ and the magnitude homology of G by $MH_{*,*}(G)$.

2. 主結果

The first main result establishes the inclusion-exclusion formula, a fundamental property for graph invariants;

$$\#(G \cup H) = \#G + \#H - \#(G \cap H).$$

For this we must impose some hypotheses. Indeed, Leinster [3] shows that there is no nontrivial graph invariant that is fully cardinality-like in the sense of satisfying both multiplication and inclusion-exclusion formula without restriction. However, the hypotheses we impose are mild enough to cover a wide range of examples, including trees, forests, wedge sums, and certain graphs containing a cycle of length at least 4 (for example, see Figure 1).

The second main result confirms that the magnitude defined in the context of enriched categories coincides with that defined in the context of graphs.

3. 意義・証明のアイデア

The significance of this research lies in establishing an effective inclusion-exclusion formula for the magnitude of graphs and clarifying its categorical foundation.

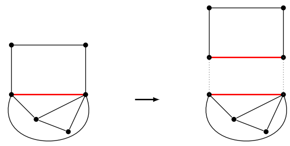


Figure 1: A graph containing a cycle of length 4

For the first main result regarding the inclusion-exclusion formula, the core idea of the proof is to construct the weighting w_X for X linearly as $w_X = w_G + w_H - w_{G\cap H}$. The validity of this construction relies on the metric property induced by the projection $\pi : V(H) \to V(G \cap H)$. Specifically, the projection condition ensures the metric equality $d(g,h) = d(g,\pi(h)) + d(\pi(h),h)$ for any $g \in V(G)$ and $h \in V(H)$. Using this equality, one can verify that the linear combination of weightings satisfies the weighting equation $\sum_{y\in V(X)} q^{d(x,y)} w_X(y) = 1$ for all $x \in V(X)$.

For the second main result connecting graphs to enriched categories, the proof proceeds as follows. A finite graph G is identified with a generalized metric space, which is structurally a $[0,\infty]$-enriched category. We define a map $|\cdot|$ by $|x| = q^x$ for $x \in \mathbb{Z}_{\geq 0} \cup \{\infty\} = Z$. Here, we assume $q^\infty = 0$. Since the distance function d on a graph G takes values in Z, this map satisfies the properties required for G being $[0,\infty]$-enriched category. Under this valuation, the similarity matrix of the category, defined by $\zeta(a,b) = |\mathrm{Hom}(a,b)|$, becomes exactly $q^{d_G(a,b)}$. Thus, the categorical definition of magnitude naturally recovers the graph magnitude defined by weighting vectors.

4. 今後の課題

Based on the results of this thesis, several open problems remain to be explored:

- **Whitney Twist:** Leinster showed that if two graphs differ by a Whitney twist with adjacent gluing points, then their magnitudes are equal. Does this equality extend to an isomorphism of magnitude homology groups?
- **Diagonal Graphs:** Computations suggest that the icosahedral graph has diagonal homology (i.e., $MH_{k,l} = 0$ for $k \neq l$). Is there a general graph-theoretic characterization of diagonal graphs?

参考文献

[1] Richard Hepworth and Simon Willerton. *Categorifying the magnitude of a graph.* Homology Homotopy Appl. 19 (2017), no. 2, 31–60.

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[3] Tom Leinster. *The magnitude of a graph.* Math. Proc. Cambridge Philos. Soc. 166 (2019), no. 2, 247–264.