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Magnitude Homology of Graphs and the Magnitude as its Categorification

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Supervisor's seal	acceptance stamp

Abstract

Sample Abstract

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Lamport's guide to LaTeX [1].

2 The Magnitude of Graphs

In this section, we define the magnitude of a graph G and the magnitude homology of a graph G, give some very basic examples and properties. By a *graph* we mean a finite undirected graph with no loops or multiple edges. The set of vertices of a graph G is denoted by $V(G)$, and the set of edges of G is denoted by $E(G)$. If x and y are vertices of a graph G, then the *distance* $d_G(x, y)$ between x and y is defined to be the length of a shortest edge path from x to y. If x and y lie in different components of G then $d(x, y) = \infty$.

2.1 The definition of the magnitude of Graphs

Here, we difine the magnitude of a graph, which can be expressed as either a rational function over $\mathbb{Q}$ or a formal power series over $\mathbb{Z}$. Write $\mathbb{Z}[q]$ for the polynomial ring over the integers in one variable q and $\mathbb{Z}[[q]]$ for the ring of formal power series over the integers in one variable q.

Definition 2.1.1. Let G be a graph. We define the *G-matrix* $Z_G = Z_G(q)$ over $\mathbb{Z}[q]$ whose rows and columns are indexed by the vertices of G, and whose (x, y)-entry is given by

$$Z_G(q)(x, y) = q^{d(x,y)} \quad (x, y \in V(G))$$

where by convention $q^\infty = 0$.

G-matrix is the square synmetric matrix.

Proposition 2.1.2. *G-matrix is invertible.*

Proof. By definition, the determinant of Z_G has constant term 1, which implies that $\det Z_G \neq 0$. $\square$

Definition 2.1.3. The *magnitude* of a graph G is defined to be the rational function given by

$$\#G(q) = \sum_{x,y \in V(G)} (Z_G(q))^{-1}(x, y)$$

in the rational function field $\mathbb{Q}(q)$.

Remark 2.1.4.

$$\#G(q) = \mathrm{sum}(Z_G(q)^{-1}) = \frac{\mathrm{sum}(\mathrm{adj}(Z_G(q)))}{\det(Z_G(q))}$$

where adj is the adjugate matrix and sum is the sum of all entries of a matrix.

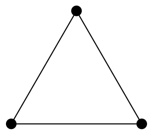
Proposition 2.1.5. *$\#G(q)$ takes values in $\mathbb{Z}[[q]]$.*

Proof. Let $\det Z_G(q) = 1 - qf(q)$ for some $f(q) \in \mathbb{Z}[q]$ by theorem 2.1.2. Then we have

$$\#G(q) = \frac{\mathrm{sum}(\mathrm{adj}(Z_G))}{\det(Z_G)} = \mathrm{sum}(\mathrm{adj}(Z_G)) \sum_{n=0}^{\infty} q^n f(q)^n$$

Note that $qf(q)$ has no constant term and then $\sum_{n=0}^{\infty} q^n f(q)^n$ takes values in $\mathbb{Z}[[q]]$. $\square$

Example 2.1.6. Let $G = K_3$(complete graph with three vertices).



Then, you can calculate the magnitude of K_3 as follows:

$$Z_{K_3}(q) = \begin{pmatrix} 1 & q & q \\ q & 1 & q \\ q & q & 1 \end{pmatrix}, \quad Z_{K_3}(q)^{-1} = \frac{1}{1-3q^2+2q^3} \begin{pmatrix} 1-q^2 & -q+q^2 & -q+q^2 \\ -q+q^2 & 1-q^2 & -q+q^2 \\ -q+q^2 & -q+q^2 & 1-q^2 \end{pmatrix},$$

$\#K_3(q) = \frac{3}{1+2q}$

Definition 2.1.7. Let G be a graph and $x \in V(G)$. The *weight* of x in G is defined

$$w_G(x)(q) = \sum_{y \in V(G)} (Z_G(q))^{-1}(x, y)$$

The function $w_G : V(G) \to \mathbb{Q}(q)$ is called the *weighting* on G.

The magnitude can be expressed using the weighting as follows:

$$\#G(q) = \sum_{x \in V(G)} w_G(x)$$

Lemma 2.1.8. *For any graph G, the weighting w_G satisfies*

$$\sum_{y \in V(G)} q^{d(x,y)} w_G(y) = 1 \quad (x \in V(G))$$

Proof. For any vertex $x \in V(G)$, we have

$$\begin{aligned}
\sum_{y \in V(G)} q^{d(x,y)} w_G(y) &= \sum_{y,z \in V(G)} q^{d(x,y)} Z_G^{-1}(y,z) \\
&= \sum_{y,z \in V(G)} Z_G(x,y) Z_G^{-1}(y,z) \\
&= \sum_{z \in V(G)} \sum_{y \in V(G)} Z_G(x,y) Z_G^{-1}(y,z) \\
&= \sum_{z \in V(G)} (Z_G Z_G^{-1})(x,z) \\
&= \sum_{z \in V(G)} I(x,z) \\
&= 1.
\end{aligned}$$

□

This equation is called the *weighting equation.*

Lemma 2.1.9. *Let G be a graph and $\tilde{w}_G : V(G) \to \mathbb{Q}$ be a function satisfying a weighting equation. Then, $\tilde{w}_G = w_G$. Now, w_G is the weighting on G.*

Proof. Let $\boldsymbol{b} = (1, 1, \ldots, 1)^T$ where the length of $\boldsymbol{b}$ is $|V(G)|$ and $\boldsymbol{w}_G = (w_G(x))^T_{x \in V(G)}$. If $\tilde{w}_G$ satisfies the weighting equation, then we have

$$Z_G \tilde{\boldsymbol{w}}_G = \boldsymbol{b}$$

Since Z_G is invertible by theorem 2.1.2, we have $\tilde{w}_G = w_G$ □

This lemma shows that the weighting on a graph is unique and we use this frequently to compute the magnitude of graphs.

2.2 Basic Properties and Examples

Here we give the most basic facts about magnitude. We focus on transitive graphs, disjoint unions, catesian products, and how the magnitude behaves within $\mathbb{Z}[[q]]$.

Definition 2.2.1. Let $G = (V(G), E(G)), H = (V(H), E(H))$ be a graph. An *graph homomorphism* from G to H is a map $f : V(G) \to V(H)$ such that if $\{x, y\} \in E(G)$ then $\{f(x), f(y)\} \in E(H)$.

We can define a *graph automorphism* using the definition above. We denote the group of all graph automorphisms of a graph G by $\mathrm{Aut}(G)$. $\mathrm{Aut}(G)$ includes id_G and for $g, h \in \mathrm{Aut}(G)$ and $x \in V(G)$, $g(h(x)) = (gh)(x)$, which means $\mathrm{Aut}(G)$ acts on $V(G)$.

Definition 2.2.2. A graph G is *vertex-transitive* if $\mathrm{Aut}(G)$ acts transitively on $V(G)$. It says that for any vertices x and y of G, there exists an automorphism $g : G \to G$ such that $y = g(x)$.

Lemma 2.2.3. *Let G be a vertex-transitive graph. Then,*

$$\#G(q) = \frac{|V(G)|}{\sum_{y \in V(G)} q^{d(x,y)}}$$

for any vertex $x \in V(G)$.

Proof. Let $S(x) = \sum_{y \in V(G)} q^{d(x,y)}$ for a vertex $x \in V(G)$. We show that $S(x)$ does not depend on the choice of x. Take any vertices $a, b \in V(G)$. Since G is vertex-transitive, there exists $g \in \mathrm{Aut}(G)$ such that $b = g(a)$. Then,

$$\begin{aligned}
S(b) &= \sum_{y \in V(G)} q^{d(b,y)} \\
&= \sum_{y \in V(G)} q^{d(g(a),y)} \\
&= \sum_{y \in V(G)} q^{d(g(a),g(y))} \quad \text{(since } g \text{ is bijective)} \\
&= \sum_{y \in V(G)} q^{d(a,y)} \quad \text{(since } g \text{ is an isomorphism)} \\
&= S(a)
\end{aligned}$$

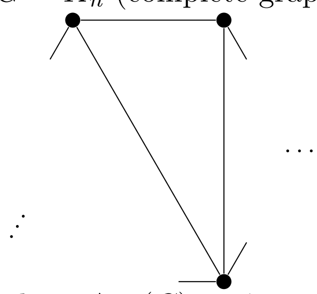
Thus, $S(x)$ does not depend on the choice of x, denoting it by S. Now, we define a function $\tilde{w}_G : V(G) \to \mathbb{Q}(q)$ by $\tilde{w}_G(x) = \frac{1}{S}$ for any vertex $x \in V(G)$. Then $\tilde{w}_G$ satisfies the weighting equation and by theorem 2.1.9, we have $\#G = \frac{|V(G)|}{S}$.

$\square$

Example 2.2.4. (i) $G = V_n$ (edgeless graph with n vertices).

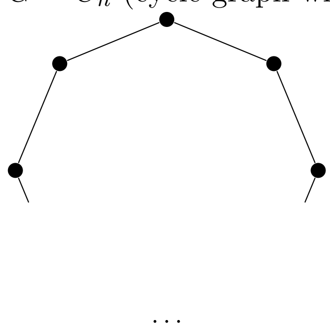
Then, $\mathrm{Aut}(G) \approx \mathfrak{S}_n$ and G is vertex-transitive. $S = 1$ and we have $\#V_n = n$.

(ii) $G = K_n$ (complete graph with n vertices).



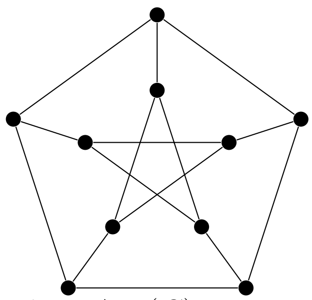
Then, $\mathrm{Aut}(G) \approx \mathfrak{S}_n$ and G is vertex-transitive. $S = 1 + (n-1)q$ and we have $\#K_n = \frac{n}{1+(n-1)q}$.

(iii) $G = C_n$ (cycle graph with n vertices).



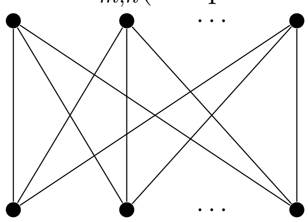
Then, $\mathrm{Aut}(G) \approx D_{2n}$ and G is vertex-transitive. If $n = 2m$(even), then $S = 1 + 2(q + q^2 + \cdots + q^{m-1}) + q^m = \frac{1+q-q^m-q^{m+1}}{1-q}$. Thus, we have $\#C_{2m} = \frac{2m(1-q)}{(1+q)(1-q^m)} = \frac{n(1-q)}{(1+q)(1-q^m)}$. If $n = 2m-1$(odd), then similarly $\#C_{2m-1} = \frac{n(1-q)}{1+q-2q^m}$.

(iv) G is a Petersen graph.



Then, $\mathrm{Aut}(G)$ contains D_{10} as its subgroup and G is vertex-transitive. $S = 1 + 3q + 6q^2$ and we have $\#G = \frac{10}{1+3q+6q^2}$.

(v) $G = K_{m,n}$(complete bipartite graph).



Then, $\mathrm{Aut}(G) \approx \mathfrak{S}_m \times \mathfrak{S}_n$ if $m \neq n$ and G is not vertex-transitive. You can calculate the magnitude as follows:

2.3 The magnitude of union

3 The Magnitude Homology of Graphs

In this section, we define the magnitude homology of a graph G, give some very

3.1 The Definition of the magnitude homology of graphs

3.2 Magnitude Homology of Graphs is Categorification of Magnitude of Graphs

3.3 u

4 Motivation : The Magnitude of Enriched Categories

References

[1] Leslie Lamport. *LaTeX: A Document Preparation System*. Addison-Wesley, 2nd edition, 1994.

[2] Donald E. Knuth. *The TeXbook*. Addison-Wesley, 1984.

Proposition .0.1. *Let $G = K_{n,n}$. Then, $Aut(G) \approx (\mathfrak{S}_n \times \mathfrak{S}_n) \rtimes_s \mathbb{Z}_2$, where $s : \mathbb{Z}_2 \to Aut(G); 0 \mapsto id_G, 1 \mapsto \tau$ and τ is the automorphism which interchanges the two parts of $K_{n,n}$.*

Proof. Now,

$$0 \to \mathfrak{S}_n \times \mathfrak{S}_n \xrightarrow{incl} \mathrm{Aut}(G) \xrightarrow{s} \mathbb{Z}_2 \to 0$$

is exact and this sequence splits. Thus, we have $\mathrm{Aut}(G) \approx (\mathfrak{S}_n \times \mathfrak{S}_n) \rtimes_s \mathbb{Z}_2$. □