

# The magnitude of graphs and the magnitude homology as its categorification

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## 1. 研究背景・動機

The concept of magnitude was introduced by Leinster [2] and it is defined for enriched categories of finite objects such as generalized metric spaces. For example, given a finite graph  $G$ , let  $X_G$  be the generalized finite metric space consisting of vertices of  $G$  equipped with the shortest path metric. Subsequently, Leinster focuses on the magnitude of graphs in [3] using the idea of magnitude of a metric space. Some properties of magnitude are multiplicativity with respect to cartesian product, denoted by  $\square$ , and an inclusion-exclusion formula for the magnitude of a union under mild hypotheses:

$$\#(G \square H) = \#G \cdot \#H, \quad \#(G \cup H) = \#G + \#H - \#(G \cap H).$$

For example, the magnitude of the Cartesian product of the complete graph on 2 vertices  $K_2$  and that on 3 vertices  $K_3$  satisfies:

$$\# \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} = \# \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \cdot \# \begin{array}{c} \bullet \\ / \quad \backslash \\ \bullet \quad \bullet \\ / \quad \backslash \\ \bullet \end{array}$$

Formally, the magnitude of a graph is defined to be a rational function over  $\mathbb{Q}$ . Since the distance on a graph takes value in integers, it can also be expressed as a formal power series over  $\mathbb{Z}$ .

Hepworth and Willerton introduced a bigraded homology theory for graphs in [1], which has the magnitude as its graded Euler characteristic, and showed how properties of magnitude proved by Leinster categorify to properties such as a Künneth Theorem and a Mayer-Vietoris Theorem.

Here, we focus on the inclusion-exclusion formula and the relation between the magnitude of graphs and that of enriched categories. We denote the magnitude of a graph  $G$  by  $\#G$  and the magnitude homology of  $G$  by  $MH_{*,*}(G)$ .

## 2. 主結果

The first main result establishes the inclusion-exclusion formula, a fundamental property for graph invariants;

$$\#(G \cup H) = \#G + \#H - \#(G \cap H).$$

For this we must impose some hypotheses. Indeed, Leinster [3] shows that there is no nontrivial graph invariant that is fully cardinality-like in the sense of satisfying both multiplication and inclusion-exclusion formula without restriction. However, the hypotheses we impose are mild enough to cover a wide range of examples, including trees, forests, wedge sums, and certain graphs containing a cycle of length at least 4 (for example, see Figure 1).

The second main result confirms that the magnitude defined in the context of enriched categories coincides with that defined in the context of graphs.

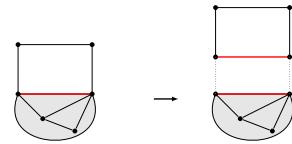


Figure 1: The graph containing a cycle of length 4

## 3. 意義・証明のアイデア

The significance of this research lies in establishing an effective inclusion-exclusion formula for the magnitude of graphs and clarifying its categorical foundation.

For the first main result regarding the inclusion-exclusion formula, the core idea of the proof is to construct the weighting  $w_X$  for  $X$  linearly as  $w_X = w_G + w_H - w_{G \cap H}$ . The validity of this construction relies on the metric property induced by the projection  $\pi : V(H) \rightarrow V(G \cap H)$ . Specifically, the projection condition ensures the metric equality  $d(g, h) = d(g, \pi(h)) + d(\pi(h), h)$  for any  $g \in V(G)$  and  $h \in V(H)$ . Using this equality, one can verify that the linear combination of weightings satisfies the weighting equation  $\sum_{y \in V(X)} q^{d(x,y)} w_X(y) = 1$  for all  $x \in V(X)$ .

For the second main result connecting graphs to enriched categories, the proof proceeds as follows. A finite graph  $G$  is identified with a generalized metric space, which is structurally a  $[0, \infty]$ -enriched category. We define a map  $|\cdot|$  by  $|x| = q^x$ . Since the distance function  $d$  on a graph  $G$  takes values in  $\mathbb{N} \cup \{\infty\}$ , this map satisfies the properties required for a valuation on the enriched category. Under this valuation, the similarity matrix of the category, defined by  $\zeta(a, b) = |\text{Hom}(a, b)|$ , becomes exactly  $q^{d_G(a,b)}$ . Thus, the categorical definition of magnitude naturally recovers the graph magnitude defined by weighting vectors.

## 4. 今後の課題

Based on the results of this thesis, several open problems remain to be explored:

- **Whitney Twist:** Leinster showed that if two graphs differ by a Whitney twist with adjacent gluing points, then their magnitudes are equal. Does this equality extend to an isomorphism of magnitude homology groups?
- **Diagonal Graphs:** Computations suggest that the icosahedral graph has diagonal homology (i.e.,  $MH_{k,l} = 0$  for  $k \neq l$ ). Is there a general graph-theoretic characterization of diagonal graphs?

## 参考文献

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