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Magnitude Homology of Graphs and the Magnitude as its Categorification

2264257 Kensho Yachi

https://taro-ken.com

Supervisor : Yuta Nozaki Associate Professor

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Supervisor's seal	acceptance stamp

Abstract

Sample Abstract

Contents

1 Introduction

Lamport's guide to LaTeX [1].

2 The Magnitude of Graphs

In this section, we define the magnitude of a graph G and the magnitude homology of a graph G, give some very basic examples and properties. By a *graph* we mean a finite undirected graph with no loops or multiple edges. The set of vertices of a graph G is denoted by $V(G)$, and the set of edges of G is denoted by $E(G)$. If x and y are vertices of a graph G, then the *distance* $d_G(x, y)$ between x and y is defined to be the length of a shortest edge path from x to y. If x and y lie in different components of G then $d(x, y) = \infty$.

2.1 The definition of the magnitude of Graphs

Here, we difine the magnitude of a graph, which can be expressed as either a rational function over $\mathbb{Q}$ or a formal power series over $\mathbb{Z}$. Write $\mathbb{Z}[q]$ for the polynomial ring over the integers in one variable q and $\mathbb{Z}[[q]]$ for the ring of formal power series over the integers in one variable q.

Definition 2.1.1. Let G be a graph. We define the G*-matrix* $Z_G = Z_G(q)$ over $\mathbb{Z}[q]$ whose rows and columns are indexed by the vertices of G, and whose (x, y)-entry is given by

$$Z_G(q)(x, y) = q^{d(x,y)} \quad (x, y \in V(G))$$

where by convention $q^\infty = 0$.

G-matrix is the square synmetric matrix.

Proposition 2.1.2. *G-matrix is invertible.*

Proof. By definition, the determinant of Z_G has constant term 1, which implies that $\det Z_G \neq 0$. $\square$

Definition 2.1.3. The *magnitude* of a graph G is defined to be the rational function given by

$$\#G(q) = \sum_{x,y \in V(G)} (Z_G(q))^{-1}(x, y)$$

in the rational function field $\mathbb{Q}(q)$.

Remark 2.1.4.

$$\#G(q) = \mathrm{sum}(Z_G(q)^{-1}) = \frac{\mathrm{sum}(\mathrm{adj}(Z_G(q)))}{\det(Z_G(q))}$$

where adj is the adjugate matrix and sum is the sum of all entries of a matrix.

Proposition 2.1.5. *$\#G(q)$ takes values in $\mathbb{Z}[[q]]$.*

Proof. Let $\det Z_G(q) = 1 - qf(q)$ for some $f(q) \in \mathbb{Z}[q]$ by theorem 2.1.2. Then we have

$$\#G(q) = \frac{\mathrm{sum}(\mathrm{adj}(Z_G(q)))}{\det(Z_G(q))} = \mathrm{sum}(\mathrm{adj}(Z_G(q))) \sum_{n=0}^{\infty} q^n f(q)^n$$

Note that $qf(q)$ has no constant term and then $\sum_{n=0}^{\infty} q^n f(q)^n$ takes values in $\mathbb{Z}[[q]]$. □

Example 2.1.6. Let $G = K_3$(complete graph with three vertices).

2.2 Basic Properties and Examples

Magnitude

2.3 The magnitude of union

3 The Magnitude Homology of Graphs

In this section, we define the magnitude homology of a graph G, give some very

3.1 The Definition of the magnitude homology of graphs

3.2 Magnitude Homology of Graphs is Categorification of Magnitude of Graphs

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4 Motivation : The Magnitude of Enriched Categories

References

[1] Leslie Lamport. *LaTeX: A Document Preparation System*. Addison-Wesley, 2nd edition, 1994.

[2] Donald E. Knuth. *The TeXbook*. Addison-Wesley, 1984.