An Introduction to Abelian Categories

Y

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Abstract

I and my friends were studying category theory, and learned it such that categories, functors, natural transformations, Yoneda's lemma, and adjoint functors. But we didn't deal with abelian categories. This document is a summary of that. The goal of this document is to solve the following proposition.

Proposition. Let $\mathfrak{C}$ be an abelian category and the sequence $P_\bullet$ below is complex at B.

$$\cdots \to A \xrightarrow{f} B \xrightarrow{g} C \to \cdots$$

Then,

$$P_\bullet \text{ is exact at B} \iff \operatorname{Coker} f = \operatorname{Coker}(\ker g)$$

1 Some Basic Notions of Category Theory

At first, we introduce some definitions. The definition of a category is very abstract and can be interpreted in many ways by different people, but here is the definiton from [1].

Definition 1.1. $\mathfrak{C}$ is a **category** if it consists of a collection of **objects**, denoted by $\mathfrak{Ob}(\mathfrak{C})$ and for each $A, B \in \mathfrak{Ob}(\mathfrak{C})$, a collection of **morphisms** $\mathrm{Mor}(A, B)$ between A and B which satisfies the following conditions:

(i) For each object $A \in \mathfrak{Ob}(\mathfrak{C})$, there exists an identity morphism $1_A \in \mathrm{Mor}(A, A)$ such that for each $f \in \mathrm{Mor}(A, B)$, $f \circ 1_A = f$ and $1_B \circ f = f$.

(ii) For each $A, B, C \in \mathfrak{Ob}(\mathfrak{C})$, there exists a composition of morphisms, that is, for each $f \in \mathrm{Mor}(A, B)$, $g \in \mathrm{Mor}(B, C)$, $g \circ f \in \mathrm{Mor}(A, C)$.

(iii) The composition of morphisms is associative.

Example 1.2. The collection of groups with homomorphisms is a category, denoted by *Grp*.

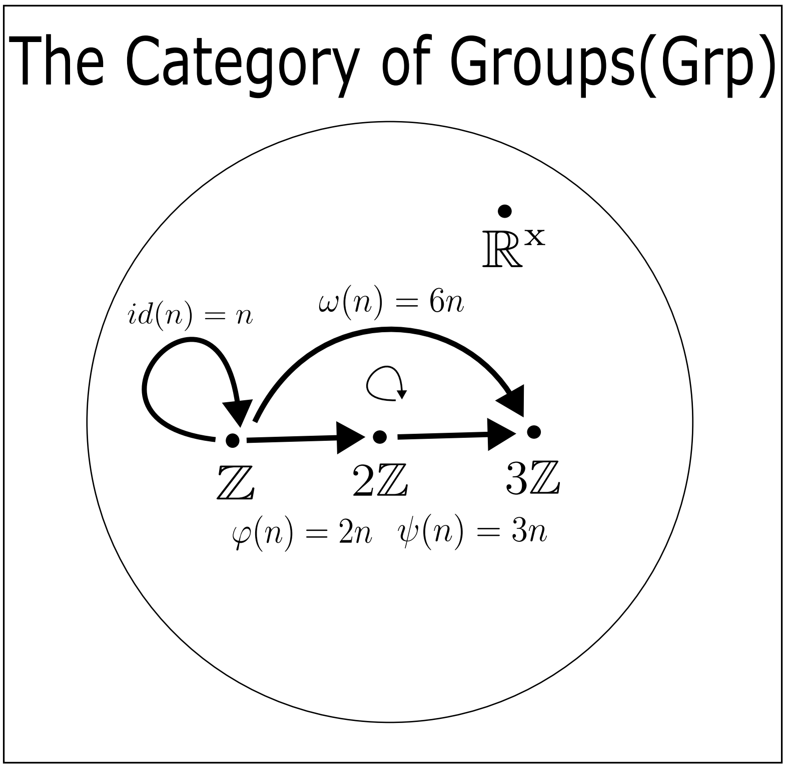

Figure 1: The image of Group

Remark 1.3. One can write down the condition (iii) of the above definition as follows:

$$h \circ (g \circ f) = (h \circ g) \circ f$$

for each $f \in \mathrm{Mor}(A, B)$, $g \in \mathrm{Mor}(B, C)$, $h \in \mathrm{Mor}(C, D)$.

Remark 1.4. $\mathfrak{C}$ is a locally small category if for each $A, B \in \mathfrak{Ob}(\mathfrak{C}), \mathrm{Mor}(A, B)$ is a set.

In this document, a category means a locally small category.

Definition 1.5. Let $\mathfrak{C}$ be a category and A, B be objects of $\mathfrak{C}$. A **product** of A and B is an object $A \times B$ with the morphisms $p_1 \in \mathrm{Mor}(A \times B, A)$, $p_2 \in \mathrm{Mor}(A \times B, B)$ such that for any object W and morphisms $q_1 \in \mathrm{Mor}(W, A)$, $q_2 \in \mathrm{Mor}(W, B)$, there exists a unique morphism $i \in \mathrm{Mor}(W, A \times B)$ such that the diagram in Figure 2 commutes:

$$\begin{tikzcd}
& W \arrow[d, dashed, "i"] \arrow[ddl, bend right, "q_1"'] \arrow[ddr, bend left, "q_2"] & \\
& A \times B \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
A & & B
\end{tikzcd}$$

Figure 2: A commutative diagram of a product

In this figure, two commutative diagrams are written together, that is, $q_1 = p_1 \circ i$, $q_2 = p_2 \circ i$.

Example 1.6. In *Set* (the category of sets with maps), the product of A and B is the direct product in the context of sets $A \times B$ with the projection maps $p_1 \colon A \times B \to A$, $p_2 : A \times B \to B$.

Definition 1.7. $\mathfrak{C}$ is an **additive category** if it is a category and satisfies the following conditions:

(i) For each $A, B \in \mathfrak{Ob}(\mathfrak{C}), \mathrm{Mor}(A, B)$ is an abelian group and its composition distributes over addition.

(ii) $\mathfrak{C}$ has a zero object, denoted 0 such that for each $A \in \mathfrak{Ob}(\mathfrak{C})$, there exists a unique zero morphism $0_{A,0} \in \mathrm{Mor}(A, 0)$ and $0_{0,A} \in \mathrm{Mor}(0, A)$.

(iii) For each $A, B \in \mathfrak{Ob}(\mathfrak{C})$, there exists a product $A \times B$.

Remark 1.8. One can write down the condition (i) of the above definition as follows:

$$h \circ (f + g) = h \circ f + h \circ g$$

for each $f, g \in \mathrm{Mor}(A, B)$, $h \in \mathrm{Mor}(B, C)$. The converse is also.

Definition 1.9. Let $\mathfrak{C}$ be a category, A, B be objects of $\mathfrak{C}$, and $f \in \mathrm{Mor}(A, B)$. f is a **monomorphism** if for any object $W \in \mathfrak{Ob}(\mathfrak{C})$ and morphisms $g, h \in \mathrm{Mor}(W, A)$, $f \circ g = f \circ h$ implies $g = h$.

Definition 1.10. Let $\mathfrak{C}$ be a category, A, B be objects of $\mathfrak{C}$, and $f \in \mathrm{Mor}(A, B)$. f is an **epimorphism** if for any object $W \in \mathfrak{Ob}(\mathfrak{C})$ and morphisms $g, h \in \mathrm{Mor}(B, W)$, $g \circ f = h \circ f$ implies $g = h$.

Remark 1.11. In *Set*, a morphism f is a monomorphism if and only if f is an injective map and a morphism f is an epimorphism if and only if f is a surjective map. This is shown as follows:

Proof. We show only that a monomorphism is an injective map in *Set*.

(i) Let A, B be sets. Assume that $f\colon A \to B$ is a monomorphism and we will show that f is an injective map. Let $m \in \ker f$. Now, let $i\colon \ker f \hookrightarrow A$ be an inclusion and $f(m) = 0 = (f \circ i)(m) = (f \circ 0)(m)$. Thus $f \circ i = f \circ 0$. Since f is a monomorphism, $i = 0$. Therefore, $m = i(m) = 0$ and $\ker f = \{0\}$.

(ii) Let A, B be sets. Assume that $f\colon A \to B$ is an injective map and we will show that f is a monomorphism. Let W be a set and $g, h \in \mathrm{Mor}(W, A)$ be morphisms such that $f \circ g = f \circ h$. Then, for any $w \in W$, $f(g(w)) = f(h(w))$. Since f is an injective map, $g(w) = h(w)$. Therefore, $g = h$.

$\square$

Definition 1.12. Let $\mathfrak{C}$ be an additive category. A **kernel** of a morphism $f \in \mathrm{Mor}(A, B)$ is a pair (K, k) where K is an object of $\mathfrak{C}$ and $k \in \mathrm{Mor}(K, A)$ such that $f \circ k = 0$ and for any object W and morphism $w \in \mathrm{Mor}(W, A)$ such that $f \circ w = 0$, there exists a unique morphism $i \in \mathrm{Mor}(W, K)$ such that the diagram in Figure 3 commutes:

$$\begin{tikzcd}
W \arrow[d, dashed, "i"'] \arrow[dr, "w"] \arrow[drr, bend left, "0"] & & \\
K \arrow[r, "k"'] \arrow[rr, bend right, "0"'] & A \arrow[r, "f"'] & B
\end{tikzcd}$$

Figure 3: A commutative diagram of a kernel

This figure shows three commutative diagrams are written together and says that $f \circ w = 0$ induces the existence of the unique morphism i such that $w = k \circ i$.

Similarly, we can define a **cokernel** of a morphism $f \in \mathrm{Mor}(A, B)$. I show only Figure 4.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[rr, bend left, "0"] \arrow[drr, bend right, "0"'] & B \arrow[r, "c"] \arrow[dr, "w"'] & C \arrow[d, dashed, "i"] \\
& & W
\end{tikzcd}$$

Figure 4: A commutative diagram of a cokernel

2 Abelian Categories

Definition 2.1. $\mathfrak{C}$ is an **abelian category** if it is an additive category and satisfies the following conditions:

(i) Every map has a kernel and a cokernel.

(ii) Every monomorphism is the kernel of its cokernel.

(iii) Every epimorphism is the cokernel of its kernel.

Example 2.2. *Grp* (the category of groups with homomorphisms) is an abelian category.

Definition 2.3. The **image** of a morphism $f \in \mathrm{Mor}(A, B)$ is the kernel of the cokernel of f, denoted as $\mathrm{Im}(f) = \ker(\mathrm{Coker}(f))$

Example 2.4. In *Grp*, $\mathrm{Im}(f)$ is an image of a homomorphism f.

Definition 2.5. Let $\mathfrak{C}$ be an abelian category and $P_\bullet$ be the following seaquence of morphisms:

$$\cdots \to A \xrightarrow{f} B \xrightarrow{g} C \to \cdots$$

(i) $P_\bullet$ is a **complex** at B if $g \circ f = 0$.

(ii) $P_\bullet$ is **exact** at B if $\mathrm{Im}(f) = \ker(g)$.

(iii) Let $P_\bullet$ is a complex at B. Then a **homology** of B is $H_B :\ = \mathrm{Coker}(i)$ for some monomorphism $i\colon \mathrm{Im}\, f \hookrightarrow \ker g$

Note that the definition of homology is well-defined. Now, I prove this.

Proof. First, we show the existence of a monomorphism $i\colon \mathrm{Im}\, f \hookrightarrow \ker g$. See the diagram below.

$$\begin{tikzcd}
 & & & & \mathrm{Coker} i \arrow[dd, bend right, "\alpha"'] \\
W \arrow[r, shift left, "\varphi"] \arrow[r, shift right, "\varphi'"'] & \mathrm{Im} f \arrow[rr, shift left, "i"] \arrow[rr, shift right, "i'"'] \arrow[dr, "i_f"] & & \mathrm{Ker} g \arrow[ur, "c"] \arrow[dr, "c'"'] \arrow[dl, "k_g"'] & \\
 & & & & \mathrm{Coker} i' \arrow[uu, bend right, "\beta"'] \\
\cdots \arrow[r] & A \arrow[r, "f"] & B \arrow[r, "g"] \arrow[d, "c_f"] & C \arrow[r] \arrow[dl, "h"] & \cdots \\
 & & \mathrm{Coker} f & &
\end{tikzcd}$$

Now, we can obtain c_f, i_f, k_g as $\operatorname{Coker} f, \operatorname{Im} f, \ker g$. Since $g \circ f = 0$, there exists $h\colon C \to \operatorname{Coker} f$ such that $h \circ g = c_f$. Then, $c_f \circ k_g = h \circ g \circ k_g = h \circ 0 = 0$ and there exists $i\colon \operatorname{Im} f \to \ker g$ such that $k_g \circ i = i_f$. This i is a monomorphism. Indeed, let $\varphi, \varphi' \in \operatorname{Hom}(W, \operatorname{Im} f)$ for some object W and assume that $i \circ \varphi = i \circ \varphi'$. Then,

$$k_g \circ i \circ \varphi = k_g \circ i \circ \varphi' \therefore i_f \circ \varphi = i_f \circ \varphi' \therefore \varphi = \varphi'$$

The last equality is due to the uniquness of Ker. Therefore, i is a monomorphism.

Second, assume that there exists another monomorphism $i'\colon \operatorname{Im} f \hookrightarrow \ker g$ and show that $\operatorname{Coker}(i) = \operatorname{Coker}(i')$.

From the properties of the cokernel for i', we get $\alpha\colon \operatorname{Coker} i \to \operatorname{Coker} i'$. Similarly, we get $\beta\colon \operatorname{Coker} i' \to \operatorname{Coker} i$. Then, $\alpha \circ \beta$ is equal to $\mathrm{id}_{\operatorname{Coker} i'}$ from the properties of uniqueness of the cokernel for i'. Similarly, $\beta \circ \alpha$ is equal to $\mathrm{id}_{\operatorname{Coker} i}$. Therefore, $\operatorname{Coker} i = \operatorname{Coker} i'$.

$\square$

Now, let's see the proposition I raised at first.

Proposition. Let $\mathfrak{C}$ be an abelian category and the sequence $P_\bullet$ below is complex at B.

$$\cdots \to A \xrightarrow{f} B \xrightarrow{g} C \to \cdots$$

Then,

$$P_\bullet \text{ is exact at} B \iff \operatorname{Coker} f = \operatorname{Coker}(\ker g)$$

Here is the proof to this proposition.

Proof. (i) Assume that $P_\bullet$ is exact at B, that is, $\operatorname{Im} f = \ker g \iff \ker(\operatorname{Coker} f) = \ker g$.

$$\begin{tikzcd}
& \ker g \arrow[d, "k_g"] & \\
A \arrow[ur, "\varphi_A"] \arrow[r, "f"] & B \arrow[r, "g"] \arrow[d, "c_f"] \arrow[dr, "\psi"] & C \\
& \operatorname{Coker} f \arrow[r, dashed, "\varphi"] & Z
\end{tikzcd}$$

Now, we obtain c_f, k_g. We want to show $\operatorname{Coker} f = \operatorname{Coker}(\ker g)$. Since $\operatorname{Im} f = \ker(\operatorname{Coker} f) = \ker g$, $c_f \circ k_g = 0$. To prove the unique property, let Z be an object and $\psi\colon B \to Z$ be a morphism such that $\psi \circ k_g = 0$. Since $c_f \circ f = 0$, there exists a morphism $\varphi_A\colon A \to \operatorname{Im} f = \ker g$ such that $k_g \circ \varphi_A = f$. Then, $\psi \circ f = \psi \circ k_g \circ \varphi_A = 0 \circ \varphi_A = 0$ and there exists a morphism $\varphi\colon \operatorname{Coker} f \to Z$ such that $\psi = \varphi \circ c_f$ because of the property of $\operatorname{Coker} f$.

If another $\varphi'\colon \operatorname{Coker} f \to Z$ meets $\psi = \varphi' \circ c_f$. Then, $\varphi = \varphi'$ because of the unique property of $\operatorname{Coker} f$. This shows $\operatorname{Coker} f = \operatorname{Coker}(\ker g)$.

(ii) On the other hand, assume that $\operatorname{Coker} f = \operatorname{Coker}(\ker g)$.

$$
\begin{tikzcd}
 & & C \\
A \arrow[r, "f"] & B \arrow[ur, "g"] \arrow[r, "c_f"] & \operatorname{Coker} f \arrow[u, "\varphi_C"'] \\
 & \operatorname{Ker} g \arrow[u, "k_g"] & Z \arrow[l, dashed, "\varphi"] \arrow[ul, "\psi"']
\end{tikzcd}
$$

Now, we obtain c_f, k_g. We want to show $\operatorname{Im} f = \ker(\operatorname{Coker} f) = \ker g$. Since $\operatorname{Coker} f = \operatorname{Coker}(\ker g)$, $c_f \circ k_g = 0$. To prove the unique property, let Z be an object and $\psi\colon Z \to B$ be a morphism such that $c_f \circ \psi = 0$.

Since $g \circ f = 0$, there exists a morphism $\varphi_C\colon \operatorname{Coker} f \to C$ such that $g = \varphi_C \circ c_f$. Then, $g \circ \psi = \varphi_C \circ c_f \circ \psi = \varphi_C \circ 0 = 0$ and there exists a morphism $\varphi\colon Z \to \ker g$ such that $\psi = k_g \circ \varphi$ because of the property of $\ker g$.

If another $\varphi'\colon Z \to \ker g$ meets $\psi = k_g \circ \varphi'$. Then, $\varphi = \varphi'$ because of the unique property of $\ker g$.

This shows $\ker(\operatorname{Coker} f) = \ker g$.

$\square$

3 Application

Theorem 3.1 (Freyd-Mitchell Embedding Theorem)**.**

Let $\mathfrak{C}$ be an abelian category whose objects form a set. Then there exist a ring A and an exact, fully faithful functor $F\colon \mathfrak{C} \to \mathrm{Mod}_A$.

Now, Mod_A means the category of A-modules with A-linear homomorphisms.

This means that an abelian category is essentially equivalent to the category of modules over a ring. If you want to show some property in an abelian category, you need only show the property in the category of modules over a ring.

References

[1] Vakil, Ravi The rising sea. Foundations of algebraic geometry (to appear). (English) Zbl 07961837 Princeton, NJ: Princeton University Press (ISBN 978-0-691-26866-8/hbk; 978-0-691-26867-5/pbk; 978-0-691-26868-2/ebook). (2025). http://math.stanford.edu/~vakil/216blog/FOAGjul2724public.pdf

[2] Alg-d 壱大整域 圏論 https://alg-d.com/math/kan_extension/ 最終確認日 (2025/03/19)