Homework 4

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Problem 1

Want to solve ODE

$$L_{zz}\phi(z) = \left[\frac{\mathrm{d}^2}{\mathrm{d}z^2} + \frac{1}{z}\frac{\mathrm{d}}{\mathrm{d}z} + \left(1 - \frac{\nu^2}{z^2}\right)\right]\phi(z) = 0 \tag{7.1}$$

Given the kernel

$$K(z,t) = \left(\frac{z}{2}\right)^{\nu} \exp\left[z - \frac{z^2}{4t}\right] \tag{7.2}$$

such that

$$\phi(z) = \int_C K(z,t)\xi(t)\,\mathrm{d}t \tag{7.3}$$

Applying the different parts of L_{zz}

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}z^2}K =& 2^{-\nu}(\nu-1)\nu e^{z-\frac{z^2}{4t}} z^{\nu-2} + 2^{1-\nu}\nu e^{z-\frac{z^2}{4t}}\left(1-\frac{z}{2t}\right)z^{\nu-1} \\ &+ 2^{-\nu}e^{z-\frac{z^2}{4t}}\left(1-\frac{z}{2t}\right)^2 z^{\nu} - \frac{2^{-\nu-1}e^{z-\frac{z^2}{4t}}z^{\nu}}{t} \\ =& \frac{2^{-\nu-2}e^{z-\frac{z^2}{4t}}z^{\nu-2}\left(4t^2\left((\nu+z)^2-\nu\right) - 2tz^2(2\nu+2z+1)+z^4\right)}{t^2}\end{aligned} \tag{7.4}$$

Similarly

$$\begin{aligned}\frac{1}{z}\frac{\mathrm{d}}{\mathrm{d}z}K =& \frac{2^{-\nu}\nu e^{z-\frac{z^2}{4t}}z^{\nu-1} + 2^{-\nu}e^{z-\frac{z^2}{4t}}\left(1-\frac{z}{2t}\right)z^{\nu}}{z} \\ =& -\frac{2^{-\nu-1}e^{z-\frac{z^2}{4t}}z^{\nu-2}\left(z^2-2t(\nu+z)\right)}{t}\end{aligned} \tag{7.5}$$

and finally

$$\left(1-\frac{\nu^2}{z^2}\right)K = 2^{-\nu}e^{z-\frac{z^2}{4t}}z^{\nu}\left(1-\frac{\nu^2}{z^2}\right) \tag{7.6}$$

Putting it all together we get

$$\oint_C \left[\frac{2^{-\nu-2}e^{t-\frac{z^2}{4t}}z^{\nu+2}}{t^2} + 2^{-\nu}e^{t-\frac{z^2}{4t}}z^{\nu} - \frac{2^{-\nu}e^{t-\frac{z^2}{4t}}z^{\nu}}{t} - \frac{2^{-\nu}\nu e^{t-\frac{z^2}{4t}}z^{\nu}}{t}\right]\xi(t)\,\mathrm{d}t \tag{7.7}$$

which is the same as

$$\int_C \xi(t)\left[\frac{\mathrm{d}}{\mathrm{d}t} - \frac{\nu+1}{t}\right]K(x,t)\,\mathrm{d}t \tag{7.8}$$

Using integration by part to expand this, $L_{zz}\phi(z) = 0$ is satisfied by finding a $\xi(t)$ for which

$$-\left[\frac{\mathrm{d}}{\mathrm{d}t} + \frac{\nu+1}{t}\right]\xi(t) = 0 \tag{7.9}$$

Which leads to $\xi(t) = t^{-\nu-1}$, hence

$$J(z) = \frac{1}{2\pi i}\int_C \mathrm{d}t\, t^{-\nu-1}\left(\frac{z}{2}\right)^{\nu}\exp\left[t - \frac{z^2}{4t}\right] \tag{7.10}$$

Further, the boundary terms have to vanish

$$\left[t^{-\nu-1}\left(\frac{z}{2}\right)^{\nu}\exp\left[t - \frac{z^2}{4t}\right]\right]_{\partial C} = 0 \tag{7.11}$$

This holds for $t = -\infty \pm i\epsilon$ and hence we can use the Hankel contour along the negative real axis, see figure below.

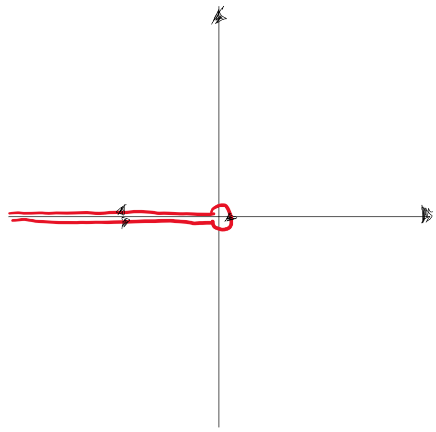

Figure 28. Hankel contour

No information on ν has been given, so assuming it is not an integer, we need a branch cut integral. Setting first $t = uz/2$ and then $t = e^w$ we get a new contour integral,

$$J_\nu(z) = \frac{1}{2\pi i}\int_\gamma \mathrm{d}w\, e^{z\sinh w - vw} \tag{7.12}$$

over the contour shown in the figure below

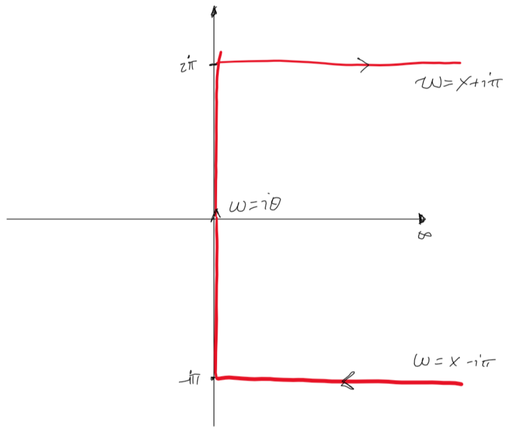

Figure 29. Deformed contour for $J_\nu(z)$

Splitting the contour integral into 2 pieces and setting $w = t + i\pi$ on the flat and $w = i\theta$ on the vertical parts, we get

$$J_\nu(z) = \frac{1}{\pi}\int_0^\pi \mathrm{d}\theta \cos(\nu\theta - z\sin\theta) - \frac{\sin\nu\pi}{\pi}\int_0^\infty \mathrm{d}t e^{-vt - z\sinh t} \tag{7.13}$$

To make sure we have independent solutions of J_ν from $J_{-\nu}$ we then define the Neumann function

$$\begin{aligned} N_\nu(z) \equiv & \frac{J_\nu(z)\cos\nu\pi - J_{-\nu}(z)}{\sin\nu\pi} \\ = & \int_0^\pi \frac{\cot\nu\pi}{\pi}\cos(\nu\theta - z\sin\theta) - \frac{\pi}{\sin\nu\pi}\cos(\nu\theta + z\sin\theta)\mathrm{d}\theta \\ & - \int_0^\infty \frac{\cos\nu\pi}{\pi}e^{-vt - z\sinh t} + \frac{1}{\pi}e^{vt - z\sinh t}\mathrm{d}t \end{aligned} \tag{7.14}$$

We now define the simpler Hankel functions

$$\begin{aligned} H_\nu^{(1)}(z) &= \frac{1}{i\pi}\int_{-\infty}^{\infty + i\pi} e^{z\sinh w - vw}\mathrm{d}w, \qquad |\arg z| < \pi/2 \\ H_\nu^{(2)}(z) &= -\frac{1}{i\pi}\int_{-\infty}^{\infty - i\pi} e^{z\sinh w - vw}\mathrm{d}w, \qquad |\arg z| < \pi/2 \end{aligned} \tag{7.15}$$

Their contour integrals go like

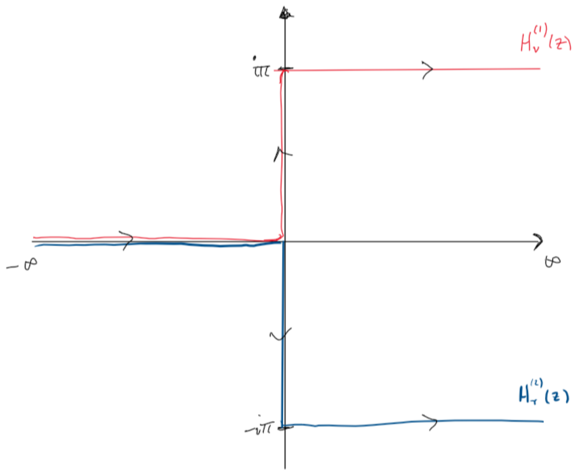

Figure 30. $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$ contours.

Let us see what happens when they are combined. We write the sum and split the contour into a part from ∞ to 0 (this is canceled by the sum), then a part from 0 to π with $w = i\theta$ and then lastly a part from 0 to ∞ with $w = x \pm i\pi$ (depending on the which one of the Hankel functions).

Doing this one obtains the following

$$
\begin{aligned}
H_\nu^{(1)}(z) + H_\nu^{(2)}(z) \\
&= \frac{1}{i\pi}\int_0^\infty e^{z\sinh(x+i\pi)-v(x+i\pi)}\mathrm{d}x - \frac{1}{i\pi}\int_0^\infty e^{z\sinh(x-i\pi)-v(x-i\pi)}\mathrm{d}x \\
&\quad - \frac{1}{i\pi}\int_0^\pi e^{z\sinh i\theta-\nu i\theta)} i\mathrm{d}\theta \\
&= \frac{1}{\pi}\int_0^\pi e^{i(\nu\theta - z\sinh\theta)} + e^{-i(\nu\theta-z\sinh\theta)}\mathrm{d}\theta \\
&\quad - \frac{1}{\pi}\int_0^\infty \left(e^{i\nu\pi} - e^{-i\nu\pi}\right) e^{-\nu x - z\sinh\theta x}\mathrm{d}x
\end{aligned}
\tag{7.16}
$$

Converting the exponentials we see that we match the Bessel function just found

$$
\begin{aligned}
H_\nu^{(1)}(z) + H_\nu^{(2)}(z) &= \frac{2}{\pi}\int_0^\pi \mathrm{d}\theta\cos(\nu\theta - z\sin\theta) - \frac{2\sin\nu\pi}{\pi}\int_0^\infty \mathrm{d}t e^{-vt-z\sinh t} \\
&= 2J_\nu(z)
\end{aligned}
\tag{7.17}
$$

Similarly we take take the difference between the two Hankel functions and obtain in the end an expression for $N_\nu(z)$.

$$
\begin{aligned}
&H_\nu^{(1)}(z) - H_\nu^{(2)}(z)\\
&\qquad =\frac{2}{i\pi}\int_{-\infty}^{0} e^{z\sinh x-\nu x}\mathrm{d}x + \frac{1}{i\pi}\int_0^\infty e^{z\sinh(x-i\pi)-\nu(x-i\pi)}\mathrm{d}x\\
&\qquad\quad + \frac{1}{i\pi}\int_0^\infty e^{z\sinh(x+i\pi)-\nu(x+i\pi)}\mathrm{d}x + \frac{1}{i\pi}\int_0^\pi e^{z\sinh i\theta-\nu i\theta} i\mathrm{d}\theta\\
&\qquad\quad + \frac{1}{i\pi}\int_0^{-\pi} e^{z\sinh i\theta-\nu i\theta} i\mathrm{d}\theta\\
&\qquad =-\frac{2}{\pi}\int_0^\infty \left(e^{\nu x} + \cos\nu x e^{-\nu x}\right)e^{-\sinh x}\mathrm{d}x + \frac{1}{\pi}\int_0^\pi \sin(z\sin\theta-\nu\theta)\mathrm{d}\theta\\
&\qquad =2N_\nu(z)
\end{aligned}
\tag{7.18}
$$

where the last line comes from a definition of $N_\nu(z)$ which can be looked up in an integral table, for instance the one provided in the homework statement.

Finally we want to obtain an expression for the asymptotic expansion of $J_\nu(z)$ and $N_\nu(z)$ using the above representations, using the methods of steepest descent. We will look at the Hankel functions and shift to the Hankel contour for the computation of $J_\nu(z)$ and $N_\nu(z)$, which means the Hankel functions are integrated over the following contours

$$
\begin{aligned}
H_\nu^{(1)}(z) &= \frac{1}{i\pi}\int_{i\epsilon}^{-\infty+i\epsilon} \mathrm{d}t\, t^{-\nu-1}e^{-\frac{1}{2}(t-\frac{1}{t})z}\\
H_\nu^{(2)}(z) &= \frac{1}{i\pi}\int_{-\infty-i\epsilon}^{-i\epsilon} \mathrm{d}t\, t^{-\nu-1}e^{-\frac{1}{2}(t-\frac{1}{t})z}
\end{aligned}
\tag{7.19}
$$

The saddlepoint is easily found

$$
\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=t_0} = 0 \Rightarrow t_0 = \pm i \tag{7.20}
$$

For $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$ we have to derform the contour in different ways to go through the saddle point. Picking i for $H_\nu^{(1)}(z)$ and $-i$ for $H_\nu^{(2)}(z)$. We start of by doing $H_\nu^{(1)}(z)$ and translate the result to $H_\nu^{(2}(z)$.

First we define $\xi \equiv t-i$, then we expand around the saddle point, assuming that z is large

$$
\begin{aligned}
H_\nu^{(1)}(z) &= \frac{1}{i\pi}\int_{i\epsilon}^{-\infty+i\epsilon} \mathrm{d}\xi\,(\xi+i)^{-\nu-1}e^{-\frac{1}{2}(\xi+i-\frac{1}{\xi+i})z}\\
&= \frac{1}{i\pi}\int_{i\epsilon}^{-\infty+i\epsilon} \mathrm{d}\xi\,(\xi+i)^{-\nu-1}e^{iz-i\frac{z}{2}\xi^2+\mathcal{O}(\xi^3)}
\end{aligned}
\tag{7.21}
$$

We then take $\xi = re^{i\theta}$ and expand the factor in the integral $(\xi+i)^{-\nu-1} = i^{-\nu-1} + \mathcal{O}(\xi^1)$, such that

$$
H_\nu^{(1)}(z) \approx \;= \frac{2}{i\pi}\int_0^\infty \mathrm{d}r\, e^{i\theta}(i)^{-\nu-1}e^{iz-i\frac{z}{2}r^2e^{2i\theta}} \tag{7.22}
$$

To be able to perform the saddle point integral we must have $\theta = \frac{3}{4}\pi$, further we write the i's in polar form to get

$$H_\nu^{(1)}(z) \approx \frac{2e^{-\frac{1}{4}\pi i + iz - i\frac{1}{2}\pi\nu}}{\pi} \int_0^\infty \mathrm{d}r\, e^{-\frac{z}{2}r^2} = \sqrt{\frac{2}{\pi z}} e^{-\frac{1}{4}\pi i + iz - i\frac{1}{2}\pi\nu} \tag{7.23}$$

For the Hankel function of the second kind, we had to choose the other maximum. This in turn just changes the sign of the exponentials, since θ now changes and we deduce that

$$\begin{aligned} H_\nu^{(1)}(z) &= \sqrt{\frac{2}{\pi z}} e^{-\frac{1}{4}\pi i + iz - i\frac{1}{2}\pi\nu} + H.C \\ H_\nu^{(2)}(z) &= \sqrt{\frac{2}{\pi z}} e^{+\frac{1}{4}\pi i - iz + i\frac{1}{2}\pi\nu} + H.C \end{aligned} \tag{7.24}$$

with $H.C.$ signefying Higher Order corrections. We then also have

$$\begin{aligned} J_\nu(z) &= \sqrt{\frac{2}{\pi z}} \cos\Big(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\Big) + H.C. \\ N_\nu(z) &= \sqrt{\frac{2}{\pi z}} \sin\Big(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\Big) + H.C. \end{aligned} \tag{7.25}$$

Problem 2

We first start of with the following version of the Riemann-zeta function

$$\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s} \tag{7.26}$$

Then noting that

$$\Gamma(s) = \int_0^\infty e^{-y} y^{s-1} \mathrm{d}y \tag{7.27}$$

We mulitply the numerator and denominator of the RZ function by this and perform a change of variables $y = nu$

$$\begin{aligned} \zeta(s) &= \frac{1}{\Gamma(s)} \sum_{n=1}^\infty \frac{1}{n^s} \int_0^\infty e^{-y} y^{s-1} \mathrm{d}y \\ &= \frac{1}{\Gamma(s)} \sum_{n=1}^\infty \int_0^\infty e^{-nu} u^{s-1} \mathrm{d}u \\ &= \frac{1}{\Gamma(s)} \int_0^\infty \sum_{n=1}^\infty e^{-nu} u^{s-1} \mathrm{d}u \end{aligned} \tag{7.28}$$

where we have changed the order of the limits in the last line. Now note that the following identity holds since $s > 1$

$$\sum_{n=1}^\infty e^{-nu} u^{s-1} = \frac{u^{s-1}}{e^u - 1} \tag{7.29}$$

So that we get the following integral form

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{u^{s-1}}{e^u - 1} \mathrm{d}u \tag{7.30}$$

This behaves badly for $\operatorname{Re} s = 1$ so we extend it to the complex plane and integrate around the origin by using the Hankel contour from $\infty + i\epsilon$ to $\infty - i\epsilon$ (i.e. pointing to the right.) in the following manor.

$$\zeta(s) = \frac{\Gamma(1-s)}{2\pi i} \int_\gamma \frac{z^{s-1}}{e^z - 1} \mathrm{d}z \tag{7.31}$$

The integral converges for all z and is holomorphic everywhere (i.e. entire). To show that this integral form is correct, consider the integral part along the described Hankel contour.

$$\begin{aligned} I(s) &= \int_\gamma \frac{w^{s-1}}{e^w - 1} \mathrm{d}w \\ &= \int_\infty^\epsilon \frac{e^{(\log u - i\pi)s}}{(e^u - 1)u} \mathrm{d}u + \int_{|w|=\epsilon} \frac{w^s}{(e^w - 1)w} \mathrm{d}w + \int_\epsilon^\infty \frac{e^{(\log u + i\pi)s}}{(e^u - 1)u} \mathrm{d}u \end{aligned} \tag{7.32}$$

The integral has a removable singularity at $z = 0$, so the integral along the origin vanishes as $\epsilon \to 0$. Now taking this limit $\epsilon \to 0$, we find

$$\begin{aligned} I(z) &= \left[e^{i\pi s} - e^{-i\pi s}\right] \int_0^\infty \frac{u^{s-1}}{e^u - 1} \mathrm{d}u \\ &= 2i \sin \pi s \, \Gamma(s) \zeta(s) \\ &= \frac{2\pi i}{\Gamma(1-s)} \zeta(s) \end{aligned} \tag{7.33}$$

where we have used (7.30) and the result obtained in the previous homework

$$\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin \pi z} \tag{7.34}$$

so that

$$\zeta(s) = \frac{\Gamma(1-s)}{2\pi i} I(s) \tag{7.35}$$

Throughout this we have asserted that $\operatorname{Re} s > 1$ for the integral formular of $\zeta(s)$. However, the rhs is analytic everywhere except for $\Gamma(1-s)$ having simple poles at s integer values ≥ 0 and $I(s)$ has zeroes at ≥ 1, so $\zeta(s)$ is analytic everwhere except for a simple pole at $s = 1$ which has residue

$$\begin{aligned} \lim_{s \to 1} (s-1) \frac{\Gamma(1-s)}{2\pi i} I(s) &= -\frac{I(1)}{2\pi i} \\ &= -\frac{1}{2\pi i} \int_{|w|=\epsilon} \frac{w^1}{(e^w - 1)w} \mathrm{d}w \\ &= 1 \end{aligned} \tag{7.36}$$

In conclusion the ζ-function can be analytically continued to a meromorphic function with a simple at $s = 1$ contributing a residue of 1.

Now we use this to calculate $\zeta(-1)$

$$\begin{aligned}
\zeta(-1) &= \frac{\Gamma(2)}{2\pi i}\oint_\gamma \frac{z^{-2}}{e^z - 1}\mathrm{d}z \\
&= \frac{\Gamma(2)}{2\pi i}\oint_\gamma \frac{1}{z^2\left(\left[1 - z + \frac{z^2}{2} - \frac{z^3}{6} + \cdots\right] - 1\right)}\mathrm{d}z \\
&= \frac{\Gamma(2)}{2\pi i}\oint_\gamma \frac{1}{z^2\left(\left[-z + \frac{z^2}{2} - \frac{z^3}{6} + \cdots\right]\right)}\mathrm{d}z \\
&= -\frac{\Gamma(2)}{2\pi i}\oint_\gamma \frac{1}{z^3\left(1 - \left[\frac{z}{2} - \frac{z^2}{6} + \cdots\right]\right)}\mathrm{d}z \\
&= -\frac{\Gamma(2)}{2\pi i}\oint_\gamma \frac{1 + \left[\frac{z}{2} - \frac{z^2}{6} + \cdots\right] + \left[\frac{z}{2} - \frac{z^2}{6} + \cdots\right]^2 + \left[\frac{z}{2} - \frac{z^2}{6} + \cdots\right]^3 + \cdots}{z^3}\mathrm{d}z \\
&= -\frac{\Gamma(2)}{2\pi i}\oint_\gamma \left[\frac{1}{z^3} + \frac{1}{2z^2} + \frac{1}{12z} + \mathcal{O}(z^0)\right]\mathrm{d}z
\end{aligned} \tag{7.37}$$

where we have used the fact that $\Gamma(2) = 1$ and the binomial expansion. We take the contour to be a circle around the residue at $z = 0$. Here only the $\frac{1}{z}$ term contributes, so we get

$$\begin{aligned}
\zeta(-1) &= -\frac{\Gamma(2)}{2\pi i}\oint_\gamma \left[\frac{1}{12z}\right]\mathrm{d}z \\
&= -\frac{1}{12}
\end{aligned} \tag{7.38}$$

Problem 3

Part a

We have the partition function

$$Z(\beta) = \prod_{n=1}^{\infty} \frac{1}{(1 - e^{-\beta n})^{a_n}} \tag{7.39}$$

taking the logarithm of this gives us

$$\begin{aligned}
\log[Z(\beta)] &= \log\left[\prod_{n=1}^{\infty} \frac{1}{(1 - e^{-\beta n})^{a_n}}\right] \\
&= -\sum_{n=1}^{\infty} a_n \log\left[1 - e^{-\beta n}\right]
\end{aligned} \tag{7.40}$$

Then using the fact that

$$\log(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} x^k \tag{7.41}$$

we get

$$\log[Z(\beta)] = \sum_{k=1}^{\infty} \frac{1}{k} \sum_{n=1}^{\infty} a_n e^{-\beta k n} \tag{7.42}$$

We are now going to use the Cahen-Mellin integral (see e.g. Mellin transformation on wikipedia),

$$e^{-z} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(s) z^{-s} \, \mathrm{d}s \tag{7.43}$$

with $c > 0$, we obtain:

$$\log[Z(\beta)] = \frac{1}{2\pi i} \sum_{k=1}^{\infty} \frac{1}{k} \sum_{n=1}^{\infty} a_n \int_{1+a-i\infty}^{1+a+i\infty} \Gamma(s)(\beta k n)^{-s} \, \mathrm{d}s \tag{7.44}$$

For the integral to converge we take the contour to the right of all the poles coming from the Dirichlet series and zeta function. This in turn gives the condition $1+a > k$. Since the integral now converges, we can switch the order of limits to obtain

$$\begin{aligned} \log[Z(\beta)] &= \frac{1}{2\pi i} \int_{1+a-i\infty}^{1+a+i\infty} \sum_{k=1}^{\infty} \frac{1}{k} \sum_{n=1}^{\infty} a_n \Gamma(s)(\beta k n)^{-s} \, \mathrm{d}s \\ &= \frac{1}{2\pi i} \int_{1+a-i\infty}^{1+a+i\infty} \frac{\Gamma(s)}{\beta^s} \sum_{k=1}^{\infty} \frac{1}{k^{s+1}} \sum_{n=1}^{\infty} \frac{a_n}{n^s} \, \mathrm{d}s \\ &= \frac{1}{2\pi i} \int_{1+a-i\infty}^{1+a+i\infty} \frac{\Gamma(s)}{\beta^s} \sum_{k=1}^{\infty} \frac{1}{k^{s+1}} D(s) \, \mathrm{d}s \end{aligned} \tag{7.45}$$

where we have defined the Dirichlet series $D(s) \equiv \sum_{n=1}^{\infty} \frac{a_n}{n^s}$. Finally note that the sum over $1/k$ produces the Riemann-Zeta function leaving us with

$$\log[Z(\beta)] = \frac{1}{2\pi i} \int_{1+a-i\infty}^{1+a+i\infty} \frac{\Gamma(s)\zeta(s+1)}{\beta^s} D(s) \, \mathrm{d}s \tag{7.46}$$

Part b

We repeat the steps from problem 2 applied to

$$D(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \tag{7.47}$$

Multiplying and dividing by $\Gamma(s)$ and defining $y = nu$

$$
\begin{aligned}
D(s) &= \frac{1}{\Gamma(s)} \sum_{n=1}^{\infty} \frac{a_n}{n^s} \int_0^{\infty} e^{-y} y^{s-1} \mathrm{d}y \\
&= \frac{1}{\Gamma(s)} \sum_{n=1}^{\infty} a_n \int_0^{\infty} e^{-nu} u^{s-1} \mathrm{d}u \\
&= \frac{1}{\Gamma(s)} \int_0^{\infty} \sum_{n=1}^{\infty} a_n e^{-nu} u^{s-1} \mathrm{d}u
\end{aligned} \tag{7.48}
$$

Performing the sum using the binomial coeffecients

$$
D(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} u^{s-1} \left(\frac{1}{(1-e^{-u})^k} - 1 \right) \mathrm{d}u \tag{7.49}
$$

Then using the result from problem 2 we split this up in to two integrals. The first one over the hankel contour.

$$
D(s) = \frac{\Gamma(1-s)}{2\pi i} \int_\gamma z^{s-1} \frac{1}{(1-e^{-z})^{-k}} \mathrm{d}z - \frac{1}{\Gamma(s)} \int_0^{\infty} u^{s-1} \mathrm{d}u \tag{7.50}
$$

Now consider the integral

$$
I(s) = \int_\gamma z^{s-1} \mathrm{d}z \tag{7.51}
$$

This can be has a removable singularity when s gets close to 1 and so using the hankel contour, the contribution from the circle vanishes and we get the contributions from the flat parts, similar to problem 2

$$
I(s) = 2\pi i \sin(\pi s) \int_0^{\infty} u^{s-1} \mathrm{d}u \tag{7.52}
$$

Inserting this into $D(s)$ we can collect the the contour integral, using again

$$
\Gamma(1-z) = \frac{\pi}{\Gamma(z) \sin \pi z} \tag{7.53}
$$

so

$$
D(s) = -\frac{\Gamma(1-s)}{2\pi i} \int_\gamma (-z)^{s-1} \left(\frac{1}{(1-e^{z})^k} - 1 \right) \mathrm{d}z \tag{7.54}
$$

This has simple poles at $s = 1, 2, 3 \cdots, k$. The residues at these points are the following (note that one has to then evaluate the z integral)

$$
\begin{aligned}
&\operatorname{Res}\left[D(s)\right]_{s=1} = 1 - \left(e^{-z} - 1\right)^{-k} \\
&\operatorname{Res}\left[D(s)\right]_{s=2} = z \left(\left(e^{-z} - 1\right)^{-k} - 1 \right) \\
&\operatorname{Res}\left[D(s)\right]_{s=3} z \left(\left(e^{-z} - 1\right)^{-k} - 1 \right)
\end{aligned} \tag{7.55}
$$

$$
\operatorname{Res} \frac{1}{6} z^3 \left(\left(e^{-z} - 1\right)^{-k} - 1 \right)
$$

From this we deduce that the residues at s are given by

$$A_n = \int_\gamma \frac{(-1)^{n+1}\left(e^{-z}-1\right)^{-k}\left(\left(e^{-z}-1\right)^{k}-1\right)z^{n-1}}{\Gamma(n)}\mathrm{d}z \tag{7.56}$$

Part c

Since our current contour for the partition function is located to the right of the poles, we now move our line of integration from $\mathrm{Re}(s) = 1 + a$ to $\mathrm{Re}(s) = -\alpha$ to pick up the contributions from the residues. On this contour we have first order poles at $s = 1, 2, 3, \ldots, k$ and a second order pole at the origin. To find the contribution from the origin, we expand

$$\begin{aligned}\frac{\Gamma(s)\zeta(s+1)D(s)}{\beta^s} &= (1 - s\log\beta + \ldots)(s^{-1} - \gamma + \ldots)(s^{-1} - \gamma + \ldots)(D(0) + D'(0)s + \ldots) \\ &= \frac{D(0)}{s^2} + \frac{1}{s}(D'(0) - D(0)\log\beta) + H.C.\end{aligned} \tag{7.57}$$

While the residues from $s = j\ldots, k$ is given by the sum

$$\sum_{j=1}^{k}\frac{\Gamma(j)\zeta(j+1)A_j}{\beta^j} \tag{7.58}$$

Where A_j is the residue from $D(j)$, so that we in total can express our partition function as

$$\log Z(\beta) = \sum_{j=1}^{k}\frac{\Gamma(j)\zeta(j+1)A_j}{\beta^j} + D'(0) - D(0)\log\beta + H.C \tag{7.59}$$

Or

$$Z(\beta) = \exp\left[\sum_{j=1}^{k}\frac{\Gamma(j)\zeta(j+1)A_j}{\beta^j} + D'(0) - D(0)\log\beta\right] + H.C. \tag{7.60}$$

Part d

Density of states is given by

$$d(n) = \frac{1}{2\pi i}\int_{b-i\pi}^{b+i\pi}\mathrm{d}\beta\, Z(\beta)e^{n\beta} \tag{7.61}$$

We want to derive an asymptotic expression for this as $n \to \infty$. We will take

$$S(\beta) = \beta n + \log Z(\beta) \tag{7.62}$$

and look at it as $\beta \to 0$ such that we can approximate

$$S(\beta) = n\beta + \sum_{j=1}^{k}\frac{\Gamma(j)\zeta(j+1)A_j}{j\beta^j} \tag{7.63}$$

The saddle point is given by

$$
\begin{aligned}
0 = S'(\beta_n) &= \sum_{j=1}^{k} \frac{\Gamma(j)\zeta(j+1)A_j}{\beta_n^{j+1}} - n \\
\Rightarrow n &= \sum_{j=1}^{k} \frac{\Gamma(j)\zeta(j+1)A_j}{\beta_n^{j+1}}
\end{aligned}
\tag{7.64}
$$

From this we argue that since β_n and n are reciprocal , taking $\beta \to 0$ amounts to $n \to \infty$. We look at the two cases $k = 1$ and $k = 2$ for which we find

$$
\begin{aligned}
k = 1: \qquad \beta_n &= \sqrt{\frac{\Gamma(1)\zeta(2)A_1}{n}} \\
k = 2: \qquad n &= \frac{\Gamma(1)\zeta(2)A_1}{\beta_n^2} + \frac{\Gamma(2)\zeta(3)A_2}{\beta_n^3}
\end{aligned}
\tag{7.65}
$$

Further

$$
S''(\beta_n) = \sum_{j=1}^{k} \frac{(j+1)\Gamma(j)\zeta(j+1)A_j}{\beta^{j+2}}
\tag{7.66}
$$

So for the two cases we are studying

$$
\begin{aligned}
k = 1: \qquad S''(\beta_n) &= \frac{2\Gamma(1)\zeta(2)A_1}{\beta_n^3} \\
k = 2: \qquad S''(\beta_n) &= \frac{2\Gamma(1)\zeta(2)A_1}{\beta_n^3} + \frac{3\Gamma(2)\zeta(3)A_2}{\beta_n^4}
\end{aligned}
\tag{7.67}
$$

These are both positive so when using the steepest descent methods we must integrate along an imaginary path. We then switch variables in our integral $\beta \to iy$ and expand $S(\beta)$ around y to get an integral for the density of states in terms of the saddle point (we extend the limits of the integration to be able to perform the gaussian integral)

$$
d(n) \approx \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}y \, \frac{1}{\sqrt{S''(\beta_n)}} \exp\bigl[S(\beta_n) + D'(0) - D(0)\log\beta - 1/2 S'' y^2\bigr]
\tag{7.68}
$$

Since we take $\beta \to 0$ then S'' in the exponential can be expanded and we truncate to lowest order, after which we perform the gaussian integral:

$$
d(n) \approx \frac{1}{\sqrt{2\pi S''(\beta_n)}} \exp[S(\beta_n) + D'(0) - D(0)\log\beta_n]
\tag{7.69}
$$

For the $k = 1$ solution we insert β_n to get

$$
\begin{aligned}
S(\beta) &= n^{1/2} \frac{\Gamma(1)\zeta(2)A_1}{(\Gamma(1)\zeta(2)A_1)^{1/2}} = n^{1/2}(\Gamma(1)\zeta(2)A_1)^{1/2} \\
S''(\beta_n) &= \frac{2n^{3/2}\Gamma(1)\zeta(2)A_1}{(\Gamma(1)\zeta(2)A_1)^{3/2}} = \frac{2n^{3/2}}{(\Gamma(1)\zeta(2)A_1)^{1/2}}
\end{aligned}
\tag{7.70}
$$

While we for $D(s=0)$ get by performing the binomial sum in mathematica

$$\begin{aligned} D(0) &= -1 \\ D'(0) &= -1 \end{aligned} \tag{7.71}$$

hence we can put together (neglecting the factor of $1/e$) and calling the constant $(\Gamma(1)\zeta(2)A_1)^{1/2} \equiv \kappa$

$$d(n) \approx \sqrt{\frac{\kappa}{4\pi n^{3/2}}} \beta_n e^{(\kappa\sqrt{n})} = \sqrt{\frac{\kappa^2}{4\pi n^{1/2}}} e^{\kappa\sqrt{n}} \tag{7.72}$$

Similarly one could solve the condition posed for $k=2$ (7.65) and then insert this into $S(\beta_n)$ to obtain the asymptotic density of states.