$N < 4$ **On-Shell Diagrams**

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Abstract: Notes on modern amplitude techniques written as part of a research project with Jaroslav Trnka.

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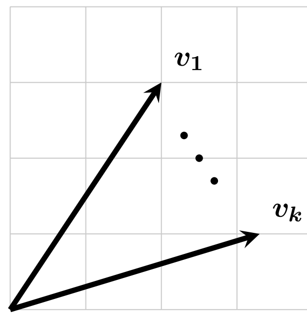


1 Introduction

2 Grassmanian

The Grassmanian $G(k, n)$ is the space of k-planes going through the origin in n dimensions. It can be though of as a generalization of P^{n-1} which is the space of lines going through the origin in n-dimensions since $G(1, n) = P^{n-1}$. One can e.g. take k vectors in n dimensions The span of these vectors give me the k- plane. If we stack them we get

$$\begin{array}{cc} & n \\ k & \begin{bmatrix} V_1 \\ \vdots \\ V_k \end{bmatrix} \end{array} \equiv C_{\alpha a}, \qquad \alpha = 1, \ldots, k \quad a = 1, \ldots, n \tag{2.1}$$

These are in general not unique since there is a GL(k) redundant.

$$C_{\alpha a} \sim L_{\alpha}^{\beta} C_{\beta a} \tag{2.2}$$

The dimensionality of the Grassmanian is

$$\dim G(k, n) = \overbrace{k \times n}^{k \times n \text{ matrix}} \underbrace{-k^2}_{GL(k) \text{ red}} \tag{2.3}$$

The redudancy means that we can gaugefix the matrix using a linear transformation by setting any $k \times k$ blok to the identity. This is equivalent to the rescaling of vectors in projective space to $\begin{pmatrix} 1 & v_2 & v_3 & v_4 \cdots \end{pmatrix}$. Taking e.g. $G(3, 5)$, we have six degrees of freedom:

$$G(3, 5) = \left[\begin{array}{ccc|cc} 1 & 0 & 0 & x_4 & x_5 \\ 0 & 1 & 0 & y_4 & y_5 \\ 0 & 0 & 1 & z_4 & z_5 \end{array}\right] \tag{2.4}$$

The dimensionality of the Grassmanian are symmetric under $n \leftrightarrow k$. This is because there is a bijection between the Grassmania: k and $n - k$ planes in n dimensions, since these planes are orthogonal. In the case above $C^{\perp}$ is a 2-plane in 5 dimensions, so

$$\left[\begin{array}{ccc|cc} & 1\;0\;0 & & x_4 & x_5 \\ & 0\;1\;0 & & y_4 & y_5 \\ & 0\;0\;1 & & z_4 & z_5 \\ \hline -x_4 & -y_4 & -z_4 & 1 & 0 \\ -x_5 & -y_5 & -z_5 & 0 & 1 \end{array}\right] \tag{2.5}$$

With the bottom part just being the negative transpose of the x, y and z coordinates in the upper right corner.

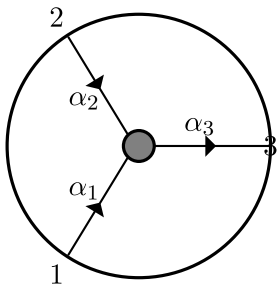
SL(k) invariant are determinants of any k coloumns of the matrix (the minors), labeling these by their indices:

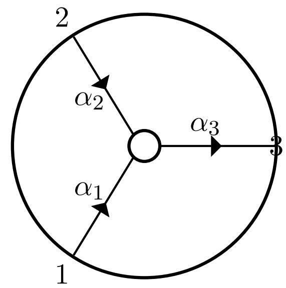
$$\left(a_1\ a_2\ \cdots\ a_k\right) \tag{2.6}$$

3 On-shell diagrams

3.1 Using three point on shell functions

The three-point vertex can be found from the following MHV and $\overline{\text{MHV}}$ diagrams





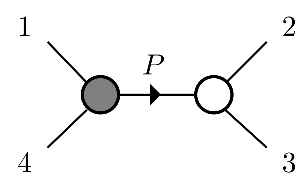
These produce the two following C matrices, respectively

$$\begin{aligned} C &= \begin{pmatrix} 1 & 0 & \alpha_1\alpha_2 \\ 0 & 1 & \alpha_2\alpha_3 \end{pmatrix} \\ C &= \begin{pmatrix} \alpha_1\alpha_3\alpha_2\alpha_3 & 1 \end{pmatrix} \end{aligned} \tag{3.1}$$

Because of momentum conservation and little group invariance, the solution of the delta functions in this case leads to

$$\begin{aligned} A_3^{\text{MHV}}(1,2,3) &= \frac{\delta^8\left(\sum_{i=1}^3 \lambda_i\tilde{\eta}_i\right)\delta^4\left(\sum_{i=1}^3 \lambda_i\tilde{\lambda}_i\right)}{\langle 12\rangle\langle 23\rangle\langle 31\rangle} \\ A_3^{\overline{\text{MHV}}}(1,2,3) &= \frac{\delta^4\left([12]\tilde{\eta}_3 + [23]\tilde{\eta}_1 + [31]\tilde{\eta}_2\right)\delta^4\left(\sum_{i=1}^3 \lambda_i\tilde{\lambda}_i\right)}{[12][23][31]} \end{aligned} \tag{3.2}$$

We start by constructing the simplest possible diagram out of two opposite helicity ($k = 1$ and $k = 2$) amplitudes, see figure below:



To construct the four-point diagram we then glue the two three-point amplitudes together by integrating over the internal degrees of freedom through

$$\prod_I \int \mathrm{d}^4\tilde{\eta}_I \int \frac{d^2\lambda_I d^2\tilde{\lambda}_I}{GL(1)} \tag{3.3}$$

Explicitly we have

$$\int \mathrm{d}\tilde{\eta}_P \int \frac{d^2\lambda_P d^2\tilde{\lambda}_P}{GL(1)} \frac{\delta^8\left(\lambda_1\tilde{\eta}_1 + \lambda_4\tilde{\eta}_4 + \lambda_P\tilde{\eta}_P\right)\delta^4\left(\lambda_1\tilde{\lambda}_1 + \lambda_4\tilde{\lambda}_4 + \lambda_P\tilde{\lambda}_P\right)}{\langle 14\rangle\langle 4P\rangle\langle P1\rangle} \\ \times \frac{\delta^4\left([23]\tilde{\eta}_P + [3P]\tilde{\eta}_2 + [P3]\tilde{\eta}_3\right)\delta^4\left(\lambda_2\tilde{\lambda}_2 + \lambda_3\tilde{\lambda}_3 - \lambda_P\tilde{\lambda}_P\right)}{[23][3P][P2]} \tag{3.4}$$

First we solve the delta-function constraint by projecting along λ_1

$$\begin{aligned} &\lambda_1\tilde{\lambda}_1 + \lambda_4\tilde{\lambda}_4 + \lambda_P\tilde{\lambda}_P = 0 \\ \Rightarrow &\tilde{\lambda}_P = \frac{\langle 41\rangle}{\langle 1P\rangle}\tilde{\lambda}_4 \end{aligned} \tag{3.5}$$

Similarly we use the other delta-function and project using $\tilde{\lambda}_3$

$$\begin{aligned} &\lambda_2\tilde{\lambda}_2 + \lambda_3\tilde{\lambda}_3 - \lambda_P\tilde{\lambda}_P = 0 \\ \Rightarrow &\lambda_P = \frac{[23]}{[P3]}\lambda_2 \end{aligned} \tag{3.6}$$

combining these we obtain

$$\begin{aligned} \tilde{\lambda}_P\lambda_P = \lambda_2\tilde{\lambda}_4\frac{\langle 41\rangle[23]}{\langle 1P\rangle[P3]} &= \lambda_2\tilde{\lambda}_4\frac{[23]}{[43]} \\ &= \lambda_2\tilde{\lambda}_4\frac{\langle 41\rangle}{\langle 12\rangle} \end{aligned} \tag{3.7}$$

where we have used $P = -1 - 4 = 2 + 3$ in the last two equalities. Solving this collapses the momentum conservation delta function as well as giving a Jacobian factor of $\frac{1}{\langle 23\rangle[32]}$

$$\begin{aligned} \lambda_P &= \lambda_2 \\ \tilde{\lambda}_P &= \lambda_4\frac{\langle 41\rangle}{\langle 12\rangle} = \tilde{\lambda}_4\frac{[23]}{[43]} \end{aligned} \tag{3.8}$$

We then use these in one of the grassmann delta-functions

$$\begin{aligned} \tilde{\eta}_P &= \frac{-[3P]\tilde{\eta}_2 - [P2]\tilde{\eta}_3}{[23]} \\ &= -\frac{1}{[23]} \times \frac{[34][23]}{[43]} \times \tilde{\eta}_2 - \frac{1}{[23]} \times \frac{[42]\langle 41\rangle}{\langle 12\rangle} \times \tilde{\eta}_3 \\ &= \tilde{\eta}_2 + \frac{\langle 13\rangle}{\langle 12\rangle} \times \tilde{\eta}_3 \end{aligned} \tag{3.9}$$

This can be obtained from contracting

$$\lambda_P\tilde{\eta}_P = \lambda_2\tilde{\eta}_2 + \lambda_3\tilde{\eta}_3 \tag{3.10}$$

with λ_1, since $\lambda_P = \lambda_2$. Using this in the other grassmann delta function we get $[23]^4\delta^8(\sum_i \lambda_i\tilde{\eta}_i)$. Finally we take the solutions (3.8) and insert them into the bosonic delta-function

$$\begin{aligned} 0 &= \lambda_1\tilde{\lambda}_1 + \lambda_4\tilde{\lambda}_4 + \lambda_P\tilde{\lambda}_P = \lambda_1\tilde{\lambda}_1 + \tilde{\lambda}_4\left(\lambda_4 + \lambda_2\frac{[23]}{[43]}\right) \\ &= \lambda_1\tilde{\lambda}_1 + \tilde{\lambda}_4\left(\frac{\lambda_4[43] + \lambda_2[23]}{[43]}\right) = \lambda_1\left(\tilde{\lambda}_1 + \tilde{\lambda}_4\frac{[13]}{[34]}\right) \\ &= \lambda_1\left(\frac{\tilde{\lambda}_1[34] + \tilde{\lambda}_4[13]}{[34]}\right) = \lambda_1\tilde{\lambda}_3\frac{[14]}{[34]} \end{aligned} \tag{3.11}$$

Since $\lambda_1 \neq 0$, and $\tilde{\lambda}_3 \neq 0$ this leads to $[14] = 0$ which in turn gives us

$$(p_1 + p_4)^2 = \langle 14\rangle\,[41] = 0 \tag{3.12}$$

Now we only need the kinematic part of the integrand. Including the Jacobians and using $\langle 1P\rangle\,[P3] = \langle 12\rangle\,[23]$ and $\langle 4P\rangle\,[P2] = \langle 43\rangle\,[32]$ we obtain

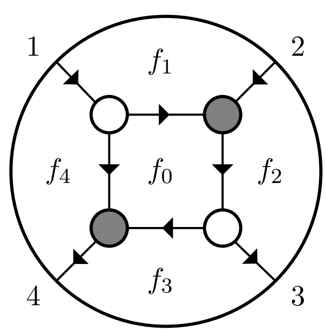
$$\frac{1}{\langle 14\rangle\,\langle 4P\rangle\,\langle P1\rangle} \times \frac{1}{[23][3P][P2]} \times \frac{[23]^4}{\langle 23\rangle\,[23]} = \frac{1}{\langle 12\rangle\,\langle 23\rangle\,\langle 34\rangle\,\langle 41\rangle} \tag{3.13}$$

Such that we in total have the amplitude

$$\frac{\delta^8(\mathcal{Q})\,\delta^4(P)}{\langle 12\rangle\,\langle 23\rangle\,\langle 34\rangle\,\langle 41\rangle}\delta((p_1 + p_4)^2) \tag{3.14}$$

3.2 Four-point directly from C(2,4) matrix

The calculation this C-matrix can be performed either using face or edge variables. We are going to do both for good measure. The four-point diagram with face-variables looks like this



While the

$$C_{ab} = -\sum_{\Gamma(a\to b)}\prod_j(-f_j)\,, \qquad \text{on the right} \tag{3.15}$$

with the constraint

$$\prod_j f_j = -1 \tag{3.16}$$

$$C = \begin{pmatrix} 1 & 0 & f_0 f_3 f_4 & f_4(1-f_0) \\ 0 & 1 & -f_0 f_1 f_3 f_4 & f_0 f_1 f_4 \end{pmatrix} \tag{3.17}$$

Note that f_2 doesn't show up, which means that according to (3.16) we can take the remaining f's as independent.

Positivity (all minors are positive) then demands that

$$f_0 < 0, \qquad f_1 > 0, \qquad f_2 < 0, \qquad f_3 < 0, \tag{3.18}$$

While the perpendicular C-matrix satisfying $C \cdot C^{\perp} = 0$ is easily obtained

$$C^{\perp} = \begin{pmatrix} -f_0 f_3 f_4 & f_0 f_1 f_3 f_4 & 1 & 0 \\ -f_4(1-f_0) & -f_0 f_1 f_4 & 0 & 1 \end{pmatrix} \tag{3.19}$$

we can then find the form through

$$\mathrm{d}\Omega = \frac{\mathrm{d}f_0}{f_0}\frac{\mathrm{d}f_1}{f_1}\frac{\mathrm{d}f_3}{f_3}\frac{\mathrm{d}f_4}{f_4}\delta(C\cdot\tilde{\lambda})\delta(C^{\perp}\cdot\lambda)\delta(C\cdot\tilde{\eta}) \tag{3.20}$$

First let us look at the delta-functions, such that we can specify the face-variables in terms of the spinor products. We start by looking at $\lambda \cdot C^{\perp} = 0$, from which we can two equations

$$C^{\perp}\cdot\lambda = 0 \Rightarrow \begin{cases} -\lambda_1 f_0 f_3 f_4 + \lambda_2 f_0 f_1 f_3 f_4 + \lambda_3 & = 0 \\ -\lambda_1 f_4(1-f_0) - \lambda_2 f_0 f_1 f_4 + \lambda_4 & = 0 \end{cases} \tag{3.21}$$

By multiplying the first equation by $\tilde{\lambda}_2$ one obtains $f_0 f_3 f_4 = -\frac{\langle 23\rangle}{\langle 12\rangle}$. Similarly multiplying the second equation by $\tilde{\lambda}_1$ we get $f_0 f_1 f_4 = \frac{\langle 14\rangle}{\langle 12\rangle}$. Combining these two,

$$f_1 = -\frac{\langle 14\rangle}{\langle 23\rangle} f_3 \tag{3.22}$$

Then multiplying the first equation by $\tilde{\lambda}_1$ we have $f_0 f_1 f_3 f_4 = -\frac{\langle 13\rangle}{\langle 12\rangle}$ together with the previous result, this leads to

$$f_3 = -\frac{\langle 13\rangle}{\langle 14\rangle} \qquad \text{and} \qquad f_1 = \frac{\langle 13\rangle}{\langle 23\rangle} \tag{3.23}$$

The other equations are solved similarly and we obtain

$$\begin{aligned} f_0 &= -\frac{\langle 14\rangle\langle 23\rangle}{\langle 12\rangle\langle 34\rangle} \\ f_4 &= -\frac{\langle 34\rangle}{\langle 13\rangle} \end{aligned} \tag{3.24}$$

Let us now evaluate the two remaining delta-functions. From $C\cdot\tilde{\lambda}$ we get two equations.

$$0 = \tilde{\lambda}_1 + f_0 f_3 f_4 \tilde{\lambda}_3 + f_4(1-f_0)\tilde{\lambda}_4 = \tilde{\lambda}_1 + \frac{\langle 32\rangle}{\langle 12\rangle}\tilde{\lambda}_3 + \frac{\langle 42\rangle}{\langle 12\rangle}\tilde{\lambda}_4 \tag{3.25}$$

and

$$0 = \tilde{\lambda}_2 + \frac{\langle 13\rangle}{\langle 12\rangle}\tilde{\lambda}_3 + \frac{\langle 14\rangle}{\langle 12\rangle}\tilde{\lambda}_4 \tag{3.26}$$

where we have used a Schouten identity for the coefficient of $\tilde{\lambda}_4$

$$\langle 41\rangle\langle 23\rangle + \langle 12\rangle\langle 34\rangle = \langle 13\rangle\langle 24\rangle \tag{3.27}$$

We see that these equations can all be obtained from a momentum conservation delta-function by contracting it with λ_1 and λ_2

$$\delta^4(\lambda_1\tilde{\lambda}_1 + \lambda_2\tilde{\lambda}_2 + \lambda_3\tilde{\lambda}_3 + \lambda_4\tilde{\lambda}_4) \equiv \delta^4(P) \tag{3.28}$$

For the last delta-function we get the exact same thing except for replacing $\tilde{\lambda}_i \to \tilde{\eta}_i$

$$\delta^8(\lambda_1\tilde{\eta}_1 + \lambda_2\tilde{\eta}_2 + \lambda_3\tilde{\eta}_3 + \lambda_4\tilde{\eta}_4) \equiv \delta^8(\mathcal{Q}) \tag{3.29}$$

Note that we get an extra factor of $\frac{1}{\langle 12\rangle^4}$ from re-writing the delta-functions by projecting along λ_1 and λ_2. Finally we get a Jacobian.

$$J = |J_{ij}| = f_0^2 f_1 f_3 f_4^3 = \frac{\langle 23\rangle\langle 34\rangle\langle 41\rangle}{\langle 12\rangle^2\langle 13\rangle} \tag{3.30}$$

where

$$J_{ij} = \frac{\partial E_i}{\partial f_j} = \begin{pmatrix} f_3 f_3 & 0 & f_0 f_3 & f_0 f_4 \\ f_1 f_3 f_4 & f_0 f_3 f_4 & f_0 f_1 f_4 & f_0 f_1 f_3 \\ f_4 & 0 & 0 & 1 - f_0 \\ f_1 f_4 & f_0 f_4 & 0 & f_0 f_1 \end{pmatrix} \tag{3.31}$$

and

$$E_1 = f_0 f_3 f_4, \quad E_2 = f_0 f_1 f_3 f_4, \quad E_3 = f_4(1 - f_0), \quad E_4 = f_0 f_1 f_3 \tag{3.32}$$

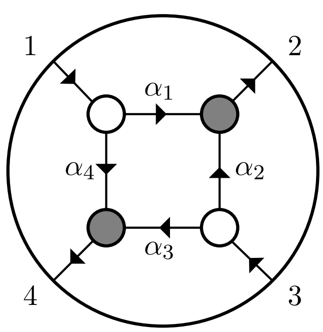
Now using

$$f_0 f_1 f_3 f_4 = \frac{\langle 13\rangle}{\langle 12\rangle} \tag{3.33}$$

We can put it all together to obtain the form

$$\mathrm{d}\Omega = \frac{\langle 12\rangle}{\langle 13\rangle} \times \frac{\langle 12\rangle^2\langle 13\rangle}{\langle 23\rangle\langle 34\rangle\langle 41\rangle} \times \frac{1}{\langle 12\rangle^4} \times \delta^4(P)\delta^8(\mathcal{Q}) = \frac{\delta^8(\mathcal{Q})\,\delta^4(P)}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle} \tag{3.34}$$

For the edge-variable case let us try a different orientation



Here the C-matrix is now giving by

$$C_{ab} = \sum_{\Gamma(a\to b)} \prod_j \alpha_j \tag{3.35}$$

SO we get the following

$$C = \begin{pmatrix} 1 & \alpha_1 & 0 & \alpha_4 \\ 0 & \alpha_2 & 1 & \alpha_3 \end{pmatrix} \tag{3.36}$$

With the inverse

$$C^{\perp} = \begin{pmatrix} -\alpha_1 & 1 & -\alpha_2 & 0 \\ -\alpha_4 & 0 & -\alpha_3 & 1 \end{pmatrix} \tag{3.37}$$

$$C^{\perp} \cdot \lambda = 0 \Rightarrow \begin{cases} -\alpha_1\lambda_1 + \lambda_2 - \alpha_3\lambda_3 & = 0 \\ -\alpha_4\lambda_1 - \alpha_2\lambda_3 + \lambda_4 & = 0 \end{cases} \tag{3.38}$$

turns into

$$\begin{aligned} \langle 21\rangle - \alpha_2 \langle 31\rangle = 0 \Rightarrow \alpha_2 = \frac{\langle 12\rangle}{\langle 13\rangle} \\ \alpha_1 \langle 12\rangle - \alpha_3 \langle 23\rangle = 0 \Rightarrow \alpha_1 = \alpha_2 \frac{\langle 23\rangle}{\langle 12\rangle} = \frac{\langle 23\rangle}{\langle 13\rangle} \end{aligned} \tag{3.39}$$

Similarly we find

$$\alpha_3 = \frac{\langle 14\rangle}{\langle 13\rangle}, \qquad \alpha_4 = \frac{\langle 43\rangle}{\langle 13\rangle} \tag{3.40}$$

For the other delta functions $C \cdot \tilde{\lambda}$ gives us two equations. The first one is

$$\begin{aligned} 0 = \tilde{\lambda}_1 + \alpha_2\tilde{\lambda}_2 + \alpha_4\tilde{\lambda}_4 = \tilde{\lambda}_1 + \frac{\langle 23\rangle}{\langle 13\rangle}\tilde{\lambda}_2 + \frac{\langle 43\rangle}{\langle 13\rangle}\tilde{\lambda}_4 \\ \Rightarrow 0 = \langle 13\rangle \tilde{\lambda}_1 + \langle 23\rangle \tilde{\lambda}_2 + \langle 43\rangle \tilde{\lambda}_4 \end{aligned} \tag{3.41}$$

While the second one is found similarly

$$0 = \langle 21\rangle \tilde{\lambda}_2 + \langle 31\rangle \tilde{\lambda}_2 + \langle 41\rangle \tilde{\lambda}_4 \tag{3.42}$$

We see that the two equations can be obtained from a single momentum conservation equation by contracting with λ_3 and λ_1 respectively. I.e. we have

$$\delta^4(\lambda_1\tilde{\lambda}_1 + \lambda_2\tilde{\lambda}_2 + \lambda_3\tilde{\lambda}_3 + \lambda_4\tilde{\lambda}_4) \equiv \delta^4(P) \tag{3.43}$$

For the last delta-function we get the exact same thing except for replacing $\tilde{\lambda}_i \to \tilde{\eta}_i$

$$\delta^8(\lambda_1\tilde{\eta}_1 + \lambda_2\tilde{\eta}_2 + \lambda_3\tilde{\eta}_3 + \lambda_4\tilde{\eta}_4) \equiv \delta^8(\mathcal{Q}) \tag{3.44}$$

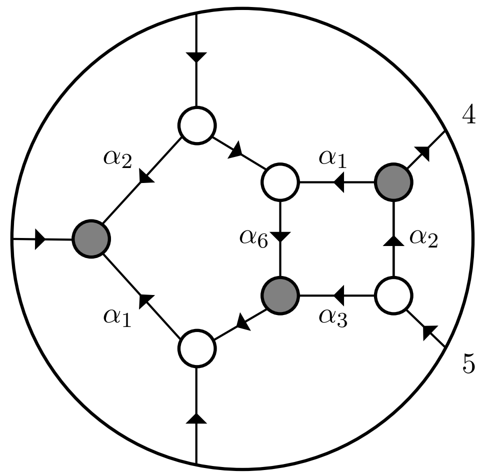
Note that we get an extra factor of $\frac{1}{\langle 13\rangle^4}$ from re-writing the delta-functions in by projecting along λ_1 and λ_3. Finally we have

$$\frac{1}{\alpha_1\alpha_2\alpha_3\alpha_4} = \frac{\langle 13\rangle^4}{\langle 12\rangle \langle 23\rangle \langle 34\rangle \langle 41\rangle} \tag{3.45}$$

We can now calculate the form

$$\mathrm{d}\Omega = \frac{\delta^8(\mathcal{Q})\,\delta^4(P)}{\langle 12\rangle \langle 23\rangle \langle 34\rangle \langle 41\rangle} \tag{3.46}$$

3.3 Five point



Each vertex can fix one edge-variable, however you cannot do it in such a way that all variables in a vertex is fixed. The C matrix is

$$C = \begin{pmatrix} 1 & \alpha_1 + \alpha_2\alpha_6 & \alpha_6 & \alpha_3\alpha_6 & 0 \\ 0 & \alpha_5\alpha_6\alpha_2 & \alpha_5\alpha_6 & \alpha_4 + \alpha_3\alpha_5\alpha_6 & 1 \end{pmatrix} \tag{3.47}$$

with the inverse being

$$C^{\perp} = \begin{pmatrix} -(\alpha_1 + \alpha_2\alpha_6) & 1 & 0 & 0 & -\alpha_5\alpha_6\alpha_2 \\ -\alpha_6 & 0 & 1 & 0 & -\alpha_5\alpha_6 \\ -\alpha_3\alpha_6 & 0 & 0 & 1 & -(\alpha_4 + \alpha_3\alpha_5\alpha_6) \end{pmatrix} \tag{3.48}$$

The amplitude is found through

$$\mathrm{d}\Omega = \frac{\mathrm{d}\alpha_1}{\alpha_1}\frac{\mathrm{d}\alpha_2}{\alpha_2}\frac{\mathrm{d}\alpha_3}{\alpha_3}\frac{\mathrm{d}\alpha_4}{\alpha_4}\frac{\mathrm{d}\alpha_5}{\alpha_5}\frac{\mathrm{d}\alpha_6}{\alpha_6}\delta^{2\times 2}(C\cdot\tilde{\lambda})\delta^{2\times 3}(C^{\perp}\cdot\lambda)\delta^{4\times 2}(C\cdot\tilde{\eta}) \tag{3.49}$$

Using the delta-function $\delta^{2\times 3}(C^{\perp}\cdot\lambda)$ to solve for the α's we obtain after contracting with λ_1, λ_3, and λ_5

$$\alpha_1 = \frac{\langle 23\rangle}{\langle 13\rangle}, \quad \alpha_2 = \frac{\langle 12\rangle}{\langle 13\rangle}, \quad \alpha_3 = \frac{\langle 45\rangle}{\langle 35\rangle}, \quad \alpha_4 = \frac{\langle 34\rangle}{\langle 34\rangle}, \quad \alpha_5 = \frac{\langle 13\rangle}{\langle 35\rangle}, \quad \alpha_6 = \frac{\langle 35\rangle}{\langle 15\rangle} \tag{3.50}$$

From which we see that we get a Jacobian of $\frac{1}{\langle 15\rangle^2\langle 13\rangle}$. Plugging these α's back into $(\delta^{2\times 2}C\cdot\tilde{\lambda})$, we get

$$\begin{aligned} 0 =& \tilde{\lambda}_1 + \tilde{\lambda}_2\frac{\langle 25\rangle}{\langle 15\rangle} + \tilde{\lambda}_3\frac{\langle 35\rangle}{\langle 15\rangle} + \tilde{\lambda}_4\frac{\langle 45\rangle}{\langle 15\rangle} \\ 0 =& \tilde{\lambda}_2\frac{\langle 12\rangle}{\langle 15\rangle} + \tilde{\lambda}_3\frac{\langle 13\rangle}{\langle 15\rangle} + \tilde{\lambda}_4\frac{\langle 14\rangle}{\langle 15\rangle} + \tilde{\lambda}_5 \end{aligned} \tag{3.51}$$

where we have used Schouten identities on the $\tilde{\lambda}_2$ term in the first equation and $\tilde{\lambda}_4$ term in the second equation. We easily see that

$$\delta^{2\times 2}(C\cdot\lambda) = \langle 15\rangle^2\,\delta^{2\times 2}(P) \tag{3.52}$$

Then we plug the α's into the Grassmann delta function. This will of course give a similar result with the exchange of $\tilde{\lambda}\to\tilde{\eta}$, although with the Jacobian factor now being $\frac{1}{\langle 15\rangle^4}$. Finally we calculate

$$\prod_i \frac{1}{\alpha_i} = \frac{\langle 13\rangle\,\langle 15\rangle^2\,\langle 35\rangle^2}{\langle 12\rangle\,\langle 23\rangle\,\langle 34\rangle\,\langle 45\rangle\,\langle 51\rangle} \tag{3.53}$$

We are now easily able to get the form

$$\mathrm{d}\Omega = \frac{\delta^8\left(\mathcal{Q}\right)\delta^4\left(P\right)}{\langle 12\rangle\,\langle 23\rangle\,\langle 34\rangle\,\langle 45\rangle\,\langle 51\rangle} \tag{3.54}$$

Five-point through loop kinematics

We have the following relations among the helicity variables

$$\begin{gathered}\lambda_1\propto\lambda_\ell\propto\lambda_{\ell-1},\quad \lambda_3\propto\lambda_q\propto\lambda_{q-3},\quad \lambda_{q+4}\propto\lambda_{\ell-1-5}\propto\lambda_{-q+3+2+\ell}\\ \tilde{\lambda}_{\ell-1}\propto\tilde{\lambda}_5\propto\tilde{\lambda}_{\ell-1-5},\quad \tilde{\lambda}_4\propto\tilde{\lambda}_q\propto\tilde{\lambda}_{q+4},\quad \tilde{\lambda}_{q-3-2}\propto\tilde{\lambda}_{q-3-2-\ell}\propto\tilde{\lambda}_\ell\propto\tilde{\lambda}_2\propto\tilde{\lambda}_{q-3}\end{gathered} \tag{3.55}$$

First we write the loop momentum in the box as

$$\begin{aligned}\ell &= \alpha\lambda_1\tilde{\lambda}_2 = -\frac{[15]}{[25]}\lambda_1\tilde{\lambda}_2\\ \ell-1 &= \lambda_1\left(\tilde{\lambda}_1+\alpha\tilde{\lambda}_2\right) = \frac{\lambda_1}{[25]}\left(\tilde{\lambda}_1\,[25]+\tilde{\lambda}_2\,[15]\right) = -\frac{[12]}{[25]}\lambda_1\tilde{\lambda}_5\\ \ell-1-5 &= \frac{\tilde{\lambda}_5}{[25]}\left([21]\,\lambda_1+[25]\,\lambda_5\right) = \frac{\langle\,\cdot\,|1+5|2]}{[25]}\tilde{\lambda}_5\end{aligned} \tag{3.56}$$

where we found $\alpha = -\frac{[15]}{[25]}$ through the on-shell condition

$$0 = (\ell-1-5)^2 = \alpha\,\langle 15\rangle\,[52] + \langle 15\rangle\,[51] \tag{3.57}$$

For the pentagon loop momenta we have

$$\begin{aligned}q &= \beta\lambda_3\tilde{\lambda}_4 = -\frac{[23]}{[24]}\lambda_3\tilde{\lambda}_4\\ q-3 &= \frac{\lambda_3}{[24]}\left(\tilde{\lambda}_4\,[23]+\tilde{\lambda}_3\,[24]\right) = -\frac{[34]}{[24]}\lambda_3\tilde{\lambda}_2\\ q-3-2 &= \frac{\tilde{\lambda}_2}{[24]}\left([34]\,\lambda_3+[24]\,\lambda_2\right) = \frac{\langle\,\cdot\,|1+5|4]}{[24]}\tilde{\lambda}_2\end{aligned} \tag{3.58}$$

Finally we have the momentum shared by the two loops

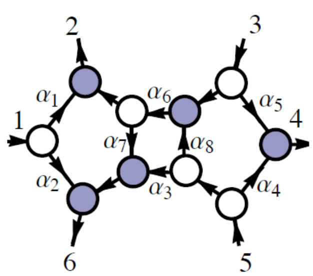
$$q-3-2-\ell = \frac{[45]\,\langle\,\cdot\,|1+5|2]}{[25]\,[24]}\tilde{\lambda}_2 \tag{3.59}$$

From this we have the following subamplitudes

$$
\begin{aligned}
A_4(q+4,5,1,q-3-2) &= \frac{1}{[q+4,5]\,[51]\,[1,q-3-2]\,[q-3-2,q+4]} \\
&= \frac{1}{\frac{[45]}{[24]}\,[51]\,\frac{[12]}{[24]}\,\frac{[24]}{[24]^2}} \\
&= \frac{[24]^3}{[12]\,[45]\,[51]}
\end{aligned}
\tag{3.60}
$$

Six-point NMHV

We first look at the 4+4 diagram



$$
\Leftrightarrow \quad C = \begin{pmatrix} 1 & \alpha_1 & 0 & 0 & 0 & \alpha_2 \\ 0 & \alpha_6 & 1 & \alpha_5 & 0 & \alpha_6\alpha_7 \\ 0 & \alpha_6\alpha_8 & 0 & \alpha_4 & 1 & \alpha_3+\alpha_6\alpha_7\alpha_8 \end{pmatrix}
$$

$$
C^{\perp} = \begin{pmatrix} -\alpha_1 & 1 & -\alpha_6 & 0 & -\alpha_6\alpha_8 & 0 \\ 0 & 0 & -\alpha_5 & 1 & -\alpha_4 & 0 \\ -\alpha_2 & 0 & -\alpha_6\alpha_7 & 0 & -(\alpha_3+\alpha_6\alpha_7\alpha_8) & 1 \end{pmatrix}.
$$

Figure 1.

We then use the following combination of equation from the $C\cdot\tilde{\lambda}$ and $C_{\perp}\cdot\lambda$ delta functions

$$
\begin{aligned}
0 =& -\langle 34\rangle + \alpha_4\langle 35\rangle \\
0 =& -(\alpha_5\langle 35\rangle) + \langle 45\rangle \\
0 =& [12] - \alpha_2\,[26] \\
0 =& [16] + \alpha_1\,[26] \\
0 =& -[23] - \alpha_5\,[24] - \alpha_6\alpha_7\,[26] \\
0 =& \alpha_6\,[26] + [36] + \alpha_5\,[46] \\
0 =& -(\alpha_4\,[24]) - [25] - (\alpha_3+\alpha_6\alpha_7\alpha_8)\,[26] \\
0 =& \alpha_6\alpha_8\,[26] + \alpha_4\,[46] + [56]
\end{aligned}
\tag{3.61}
$$

to obtain

$$
\alpha_1 = -\frac{[16]}{[26]}\,, \quad \alpha_2 = \frac{[12]}{[26]}\,, \quad \alpha_3 = \frac{s_{345}}{\langle 5|Q_{345}|6]}\,, \quad \alpha_4 = \frac{\langle 34\rangle}{\langle 35\rangle}\,, \quad \alpha_5 = \frac{\langle 45\rangle}{\langle 35\rangle}\,,
$$
$$
\alpha_6 = \frac{\langle 5|Q_{345}|6]}{\langle 35\rangle[26]}\,, \alpha_7 = -\frac{\langle 5|Q_{345}|2]}{\langle 5|Q_{345}|6]}\,, \quad \alpha_8 = -\frac{\langle 3|Q_{345}|6]}{\langle 5|Q_{345}|6]}\,.
$$

where $Q_{ijk} = p_i + p_j + p_k$ For the other delta functions we get

$$\begin{aligned}
0 &= \tilde{\eta}_1 - \frac{[16]}{[26]}\tilde{\eta}_2 + \frac{[12]}{[26]}\tilde{\eta}_6 \\
0 &= \frac{\langle 5|Q_{345}|6]}{\langle 35\rangle\,[26]}\tilde{\eta}_2 + \tilde{\eta}_3 + \frac{\langle 45\rangle}{\langle 35\rangle}\tilde{\eta}_4 - \frac{\langle 5|Q_{345}|2]}{\langle 35\rangle\,[26]}\tilde{\eta}_6 \\
0 &= -\frac{\langle 3|Q_{45}|6]}{\langle 35\rangle\,[26]}\tilde{\eta}_2 + \frac{\langle 34\rangle}{\langle 35\rangle}\tilde{\eta}_4 + \tilde{\eta}_5 + \frac{s_{345}\,\langle 35\rangle\,[26] + \langle 3|Q_{345}|6]\langle 5|Q_{345}|2]}{\langle 5|Q_{345}|6]}\tilde{\eta}_5
\end{aligned} \tag{3.62}$$

The Jacobian from the delta functions is

$$J = \frac{1}{[26]^3\,\langle 35\rangle^3\,\langle 5|Q_{345}|6]^2} \tag{3.63}$$

and we get the form

$$\mathrm{d}\Omega = \frac{\delta(\sum P)\delta(\tilde{\eta}_1[26] + \tilde{\eta}_2[61] + \tilde{\eta}_6[12])}{s_{345}\,\langle 34\rangle\,\langle 45\rangle\,[12][16]\langle 3|Q_{345}|6]\langle 5|Q_{345}|2]} \tag{3.64}$$

We later realized that we changed the bcfw bridge scheme in making the next diagram. To match the schemes we permute all labels in this form by 1

$$\mathrm{d}\Omega_1 = \frac{\delta(\sum P)\delta(\tilde{\eta}_5[16] + \tilde{\eta}_6[15] + \tilde{\eta}_1[56])}{s_{234}\,\langle 23\rangle\,\langle 34\rangle\,[61][56]\langle 2|Q_{234}|5]\langle 4|Q_{234}|1]} \tag{3.65}$$

Looking at the 5+3 diagram we use the following C-matrix

$$\begin{aligned}
C &= \begin{pmatrix} \alpha_2 & \alpha_3+\alpha_4 & 1 & 0 & 0 & 0 \\ \alpha_2\alpha_5 & \alpha_3\alpha_5 & 0 & 1 & \alpha_6 & 0 \\ \alpha_8(\alpha_1+\alpha_2\alpha_5) & \alpha_3\alpha_5\alpha_8 & 0 & 0 & \alpha_7 & 1 \end{pmatrix} \\
C_\perp &= \begin{pmatrix} 1 & 0 & -\alpha_2 & -\alpha_2\alpha_5 & 0 & -\alpha_8(\alpha_1+\alpha_2\alpha_5) \\ 0 & 1 & -\alpha_3-\alpha_4 & -\alpha_3\alpha_5 & 0 & -\alpha_3\alpha_5\alpha_8 \\ 0 & 0 & 0 & -\alpha_6 & 1 & -\alpha_7 \end{pmatrix}
\end{aligned} \tag{3.66}$$

From the delta functions $\delta(C\cdot\tilde{\lambda})$ and $\delta(C_\perp\cdot\lambda)$, we use the following equations

$$\begin{aligned}
0 =&\langle 26\rangle - (\alpha_3+\alpha_4)\langle 36\rangle - \alpha_3\alpha_5\langle 46\rangle \\
0 =&-\langle 45\rangle + \alpha_7\langle 46\rangle \\
0 =&-\alpha_6\langle 46\rangle + \langle 56\rangle \\
0 =&-(\alpha_3+\alpha_4)\,[12] - [13] \\
0 =&\alpha_2\,[12] - [23] \\
0 =&\alpha_2\alpha_5\,[12] - [24] - \alpha_6\,[25] \\
0 =&-\alpha_3\alpha_5\alpha_8\,[12] - \alpha_7\,[15] - [16] \\
0 =&(\alpha_1+\alpha_2\alpha_5)\alpha_8\,[12] - \alpha_7\,[25] - [26]
\end{aligned} \tag{3.67}$$

to obtain solutions for the edge-variables

$$\begin{aligned}
&\alpha_1 = \frac{s_{456}}{\langle 4|Q_{456}|1]}\,, \quad \alpha_2 = \frac{[23]}{[12]}\,, \quad \alpha_3 = \frac{[23]\langle 6|Q_{456}|1]}{[12]\langle 6|Q_{456}|2]}\,, \quad \alpha_4 = \frac{\langle 6|Q_{456}|3]}{\langle 6|Q_{456}|2]}\,, \\
&\alpha_5 = \frac{\langle 6|Q_{456}|2]}{\langle 46\rangle[23]}\,, \quad \alpha_6 = \frac{\langle 56\rangle}{\langle 46\rangle}\,, \quad \alpha_7 = \frac{\langle 45\rangle}{\langle 46\rangle}\,, \quad \alpha_8 = \frac{\langle 4|Q_{456}|1]}{\langle 6|Q_{456}|1]}\,.
\end{aligned} \tag{3.68}$$

Here the Jacobian from the delta functions is

$$J = \frac{1}{[12]^3\,\langle 46\rangle^3\,\langle 6|Q_{456}|2]^2} \tag{3.69}$$

and we get the form

$$\mathrm{d}\Omega = \frac{\delta(\sum P)\delta(\tilde{\eta}_1[26] + \tilde{\eta}_2[61] + \tilde{\eta}_3[12])}{s_{456}\,\langle 45\rangle\,\langle 56\rangle\,[12][23]\langle 4|Q_{456}|1]\langle 6|Q_{456}|3]} \tag{3.70}$$

Again we have to permute by 2 to obtain the correct recursion scheme

$$\mathrm{d}\Omega_2 = \frac{\delta(\sum P)\delta(\tilde{\eta}_3[45] + \tilde{\eta}_4[35] + \tilde{\eta}_5[34])}{s_{612}\,\langle 12\rangle\,\langle 16\rangle\,[34][35]\langle 6|Q_{612}|3]\langle 2|Q_{612}|5]} \tag{3.71}$$

The final diagram can be found by permuting by 2 and exchanging square and angle brackets while permuting by 1:

$$\mathrm{d}\Omega_3 = \frac{\delta(\sum P)\delta(\tilde{\eta}_1[23] + \tilde{\eta}_2[13] + \tilde{\eta}_3[12])}{s_{456}\,\langle 45\rangle\,\langle 56\rangle\,[23][12]\langle 4|Q_{456}|1]\langle 6|Q_{456}|3]} \tag{3.72}$$

In total we have

$$\begin{aligned}
&\mathrm{d}\Omega_1 + \mathrm{d}\Omega_2 + \mathrm{d}\Omega_3 \\
=&\frac{\delta(\sum P)\delta(\tilde{\eta}_5[16] + \tilde{\eta}_6[15] + \tilde{\eta}_1[56])}{s_{234}\,\langle 23\rangle\,\langle 34\rangle\,[61][56]\langle 2|Q_{234}|5]\langle 4|Q_{234}|1]} + \frac{\delta(\sum P)\delta(\tilde{\eta}_3[45] + \tilde{\eta}_4[35] + \tilde{\eta}_5[34])}{s_{612}\,\langle 12\rangle\,\langle 16\rangle\,[34][35]\langle 6|Q_{612}|3]\langle 2|Q_{612}|5]} \\
&+ \frac{\delta(\sum P)\delta(\tilde{\eta}_1[23] + \tilde{\eta}_2[13] + \tilde{\eta}_3[12])}{s_{456}\,\langle 45\rangle\,\langle 56\rangle\,[23][12]\langle 4|Q_{456}|1]\langle 6|Q_{456}|3]}
\end{aligned} \tag{3.73}$$

Then using the fact that

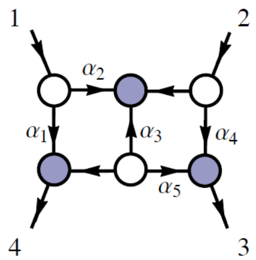
$$\begin{aligned}
&\mathcal{A}^{(0),\mathrm{MHV}} R_{j,j+3,j+5} = \\
&\quad \frac{\delta^{(8)}\left(\sum \lambda_i \eta_i^A\right)}{\langle j\,(j{+}1)\rangle\,\langle (j{+}1)\,(j{+}2)\rangle\,[(j{+}3)\,(j{+}4)]\,[(j{+}4)\,(j{+}5)]} \\
&\quad \times \frac{\delta^{(4)}\left(\eta_{j+3}^A\,[(j{+}4)\,(j{+}5)] + \eta_{j+4}^A\,[(j{+}5)\,(j{+}3)] + \eta_{j+5}^A\,[(j{+}3)\,(j{+}4)]\right)}{\langle j|\,K_{j+1,j+2}\,|(j{+}3)]\,\langle (j{+}2)|\,K_{j+3,j+4}\,|(j{+}5)]\,s_{j,j+1,j+2}}\,.
\end{aligned} \tag{3.74}$$

such that

$$A_6^{\mathrm{NMHV}} = A_6^{\mathrm{MHV}}\,(R_{251} + R_{413} + R_{635}) \tag{3.75}$$

3.4 2 loop four-point

We have the diagram



$$\Leftrightarrow \quad \begin{aligned} C &= \begin{pmatrix} 1 & 0 & \alpha_3\alpha_4\alpha_5 & \alpha_1+\alpha_2\alpha_3 \\ 0 & 1 & \alpha_4+\alpha_3\alpha_5 & \alpha_3 \end{pmatrix} \\ C^{\perp} &= \begin{pmatrix} -\alpha_2\alpha_3\alpha_5 & -(\alpha_4+\alpha_3\alpha_5) & 1 & 0 \\ -(\alpha_1+\alpha_2\alpha_3) & -\alpha_3 & 0 & 1 \end{pmatrix} \end{aligned}$$

Figure 2.

Solving the bosonic delta-functions

$$\begin{aligned} 0 &= \alpha_4\langle 12\rangle + \alpha_3\alpha_5\langle 12\rangle - \langle 13\rangle \\ 0 &= \alpha_2\alpha_3\alpha_5\langle 12\rangle - \langle 23\rangle \\ 0 &= \alpha_3\langle 12\rangle - \langle 14\rangle \\ 0 &= -((\alpha_1+\alpha_2\alpha_3)\langle 12\rangle) - \langle 24\rangle \end{aligned} \tag{3.76}$$

we find

3.5 Calculating $\mathcal{N} < 4$ amplitudes

3.5.1 The measure

The important difference between this and the $\mathcal{N} = 4$ case is that the diagrams are necessarily oriented unlike in the maximally supersymmetric forms where the perfect orientations only played an auxiliary role for constructing the C-matrix. In addition, for perfect orientations with closed internal loops we have to add an extra factor, $\mathcal{J}$, in the measure,

$$\mathrm{d}\Omega = \frac{\mathrm{d}\alpha_1}{\alpha_1}\frac{\mathrm{d}\alpha_2}{\alpha_2}\cdots\frac{\mathrm{d}\alpha_m}{\alpha_m}\mathcal{J}^{\mathcal{N}-4}\cdot\delta(C\cdot Z) \tag{3.77}$$

If there is a collection of closed orbits bounding "faces" f_i, with disjoint pairs (f_i, f_j), disjoint triples (f_i, f_j, f_k) etc., then the Jacobian $\mathcal{J}$ can be expressed as,

$$\mathcal{J} = 1 + \sum_i f_i + \sum_{\substack{\text{disjoint}\\ \text{pairs } i,j}} f_i f_j + \sum_{\substack{\text{disjoint}\\ \text{pairs } i,j,k}} f_i f_j f_k + \cdots \tag{3.78}$$

First we note that each face is defined as clockwise-oriented product of edge-variables, such that a counter clockwise "face" f, gives a Jacobian contribution of f^{-1}. As an example see for instance the following diagram with four internal loops

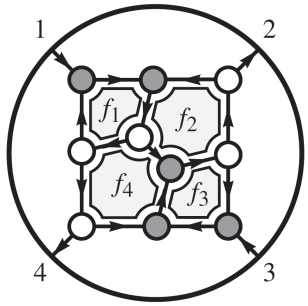


Figure 3.

Here we have three loops containing one phase variable, contributing each f_1, f_2^{-1}, f_3. Further we have the disjoint pair $f_1 f_3$. Lastly we have a close loop containing both f_2 and f_4, i.e. we have in total

$$\mathcal{J} = 1 + f_1 + f_3 + f_2^{-1} + f_2^{-1} f_4^{-1} + f_1 f_3 \tag{3.79}$$

3.5.2 4 pt $\mathcal{N} = 0$

In this case we have two diagrams

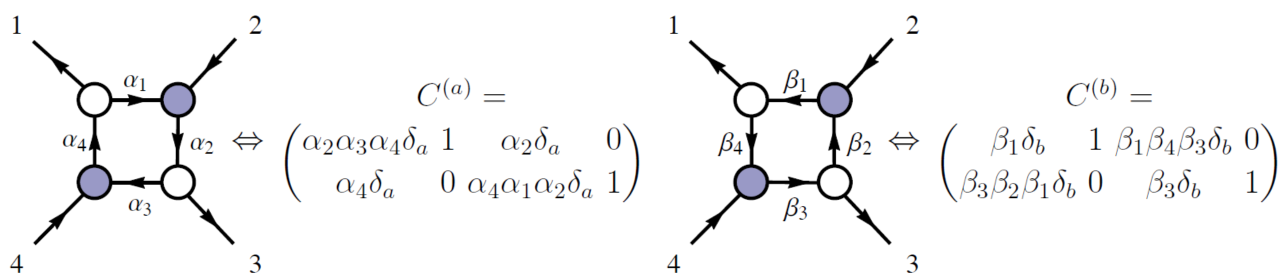


Figure 4.

where the $\delta' s$ are defined by

$$\delta_a = \frac{1}{1 - \prod_{i=1}^{4} \alpha_i} \tag{3.80}$$

Solving the $C_\perp$ delta-function, we get the following equations after contracting with λ_2 and

λ_4 (which gives a Jacobian of $\langle 24\rangle^2$)

$$\begin{aligned} 0 &= \langle 12\rangle + \frac{\alpha_4\langle 24\rangle}{1-\alpha_1\alpha_2\alpha_3\alpha_4} \\ 0 &= \langle 14\rangle - \frac{\alpha_2\alpha_3\alpha_4\langle 24\rangle}{1-\alpha_1\alpha_2\alpha_3\alpha_4} \\ 0 &= -\langle 23\rangle + \frac{\alpha_1\alpha_2\alpha_4\langle 24\rangle}{1-\alpha_1\alpha_2\alpha_3\alpha_4} \\ 0 &= \langle 34\rangle - \frac{\alpha_2\langle 24\rangle}{1-\alpha_1\alpha_2\alpha_3\alpha_4} \end{aligned} \tag{3.81}$$

from which we obtain

$$\begin{aligned} \alpha_1 &= -\frac{\langle 23\rangle}{\langle 13\rangle}, & \alpha_2 &= \frac{\langle 14\rangle}{\langle 12\rangle}, \\ \alpha_3 &= -\frac{\langle 14\rangle}{\langle 13\rangle}, & \alpha_4 &= -\frac{\langle 13\rangle}{\langle 34\rangle}, \end{aligned} \tag{3.82}$$

along with a Jacobian factor from rewriting the delta functions of $\frac{\langle 24\rangle^2}{\langle 12\rangle^2\langle 34\rangle^2}$. The procedure to find this factor can be found by eg the first line of (3.82), which after insertion of the $\delta_a = \frac{1}{1-\alpha_1\alpha_2\alpha_3\alpha_4}$ reads

$$0 = \langle 12\rangle + \frac{\alpha_4\langle 12\rangle\langle 34\rangle}{\langle 13\rangle} \tag{3.83}$$

Since this is really a delta-function, we can write this as

$$\delta\left(\frac{\langle 34\rangle\langle 12\rangle}{\langle 13\rangle}\left[\frac{\langle 13\rangle}{\langle 34\rangle} + \alpha_4\right]\right) = \frac{\langle 13\rangle}{\langle 34\rangle\langle 12\rangle}\delta\left(\frac{\langle 13\rangle}{\langle 34\rangle} + \alpha_4\right) \tag{3.84}$$

i.e. the Jacobian from this factor is $\frac{\langle 13\rangle}{\langle 34\rangle\langle 12\rangle}$. We can further find an expression for δ_a through this

$$\begin{aligned} 0 &= \langle 12\rangle + \alpha_4\langle 24\rangle\delta_a \\ \Rightarrow \delta_a &= -\frac{1}{\alpha_4}\frac{\langle 12\rangle}{\langle 24\rangle} = \frac{\langle 12\rangle\langle 34\rangle}{\langle 13\rangle\langle 24\rangle} \end{aligned} \tag{3.85}$$

Taking these solutions and inserting into the other delta functions, and multiplying by $\tilde{\lambda}_1$ and $\tilde{\lambda}_3$ we get

$$\begin{aligned} 0 &= [12] + \frac{\langle 34\rangle[13]}{\langle 24\rangle}, & 0 &= [23] + \frac{\langle 14\rangle[13]}{\langle 24\rangle} \\ 0 &= [14] + \frac{\langle 23\rangle[13]}{\langle 24\rangle}, & 0 &= [34] + \frac{\langle 12\rangle[13]}{\langle 24\rangle} \end{aligned} \tag{3.86}$$

Which we can write as

$$\begin{aligned} 0 &= \langle 42\rangle[21] + \langle 43\rangle[31] = \langle 4|P|1], & 0 &= \langle 42\rangle[23] + \langle 41\rangle[13] = \langle 4|P|3] \\ 0 &= \langle 24\rangle[41] + \langle 23\rangle[31] = \langle 2|P|1], & 0 &= \langle 24\rangle[43] + \langle 21\rangle[13] = \langle 2|P|3] \end{aligned} \tag{3.87}$$

Which we can combine into $\delta^4(P)\langle 24\rangle^2$. Finally the Jacobian is

$$\mathcal{J}_a = 1 - \alpha_1\alpha_2\alpha_3\alpha_4 = \delta_a^{-1} = \frac{\langle 13\rangle\langle 24\rangle}{\langle 12\rangle\langle 34\rangle} \tag{3.88}$$

The procedure is the same for the other diagram and here we just summarize the results

$$\beta_1 = \frac{\langle 13\rangle}{\langle 23\rangle}, \qquad \beta_2 = \frac{\langle 21\rangle}{\langle 13\rangle}, \\ \beta_3 = \frac{\langle 13\rangle}{\langle 14\rangle}, \qquad \beta_4 = \frac{\langle 34\rangle}{\langle 13\rangle}, \tag{3.89}$$

$$\mathcal{J}_b = 1 - \beta_1\beta_2\beta_3\beta_4 = \delta_b^{-1} = \frac{\langle 13\rangle\langle 24\rangle}{\langle 14\rangle\langle 23\rangle} \tag{3.90}$$

Combining all these factors with $\prod_i \alpha_i = \frac{\langle 41\rangle\langle 23\rangle}{\langle 12\rangle\langle 34\rangle}$ and $\prod_i \beta_i = \frac{\langle 12\rangle\langle 34\rangle}{\langle 23\rangle\langle 41\rangle}$, as well as using taking the Jacobians to the -4'th power, we have the form

$$\begin{aligned} \mathrm{d}\Omega &= \left(\frac{\langle 24\rangle^4}{\langle 12\rangle^2\langle 34\rangle^2} \frac{\langle 12\rangle\langle 34\rangle}{\langle 41\rangle\langle 23\rangle} \left[\frac{\langle 13\rangle\langle 24\rangle}{\langle 12\rangle\langle 34\rangle}\right]^{-4} + \frac{\langle 24\rangle^4}{\langle 14\rangle^2\langle 23\rangle^2} \frac{\langle 23\rangle\langle 41\rangle}{\langle 12\rangle\langle 34\rangle} \left[\frac{\langle 13\rangle\langle 24\rangle}{\langle 14\rangle\langle 23\rangle}\right]^{-4} \right) \delta(P) \\ &= \frac{\langle 12\rangle^4\langle 34\rangle^4 + \langle 14\rangle^4\langle 23\rangle^4}{\langle 12\rangle\langle 13\rangle^4\langle 23\rangle\langle 34\rangle\langle\rangle} \end{aligned} \tag{3.91}$$

3.5.3 Five-point $\mathcal{N} = 0$

We will once again look at the following diagram

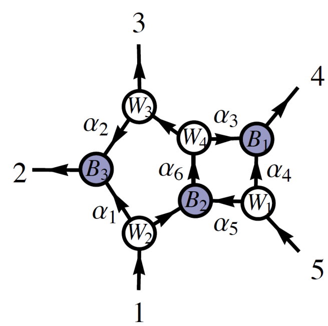


Figure 5.

We already solved this in section ....., here we obtained the following edgevariables after contracting with λ_1, λ_3, and λ_5

$$\alpha_1 = \frac{\langle 23\rangle}{\langle 13\rangle}, \quad \alpha_2 = \frac{\langle 12\rangle}{\langle 13\rangle}, \quad \alpha_3 = \frac{\langle 45\rangle}{\langle 35\rangle}, \quad \alpha_4 = \frac{\langle 34\rangle}{\langle 34\rangle}, \quad \alpha_5 = \frac{\langle 13\rangle}{\langle 35\rangle}, \quad \alpha_6 = \frac{\langle 35\rangle}{\langle 15\rangle} \tag{3.92}$$

and a Jacobian of $\frac{\langle 15\rangle^2}{\langle 35\rangle^2\langle 13\rangle}$. For this diagram we have two negative helicity particles 1 and 5 which is seen from the incoming arrows at those points. The orientation of these external legs do not yield an internal orientation with a closed cycle, and so this is the only diagram we have to take into account and there is no extra Jacobian factor. Using $\prod_i \alpha_i = \frac{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle}{\langle 13\rangle\langle 15\rangle\langle 35\rangle^2}$

The form then yields the amplitude

$$
\begin{aligned}
A_5(1^-,2^+,3^+,4^+,5^-) &= \frac{\langle 15\rangle^2}{\langle 35\rangle^2\,\langle 13\rangle}\frac{\delta(P)}{\prod_i \alpha_i} \\
&= \frac{\langle 15\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}\delta(P)
\end{aligned}
\tag{3.93}
$$

3.5.4 Six-point $\mathcal{N}=0$

Here we start by treating the same diagram as we did in section ...

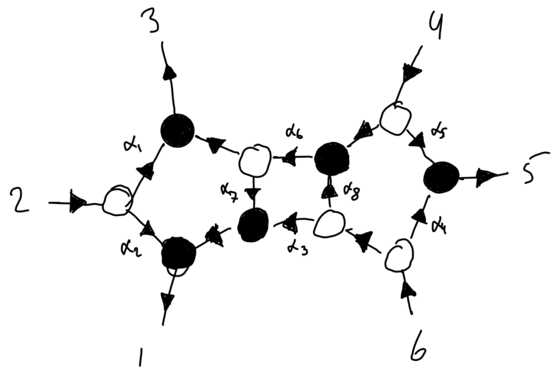


Figure 6.

where we obtained

$$
\alpha_1 = -\frac{[65]}{[15]}\,, \quad \alpha_2 = \frac{[61]}{[15]}\,, \quad \alpha_3 = \frac{s_{234}}{\langle 4|Q_{234}|5]}\,, \quad \alpha_4 = \frac{\langle 23\rangle}{\langle 24\rangle}\,, \quad \alpha_5 = \frac{\langle 34\rangle}{\langle 24\rangle}\,,
$$
$$
\alpha_6 = \frac{\langle 4|Q_{234}|5]}{\langle 24\rangle[15]}\,, \alpha_7 = -\frac{\langle 4|Q_{234}|1]}{\langle 4|Q_{234}|5]}\,, \quad \alpha_8 = -\frac{\langle 2|Q_{234}|5]}{\langle 4|Q_{234}|5]}\,.
$$

The Jacobian from the delta functions is

$$
J = \frac{[15]\,\langle 24\rangle}{\langle 4|Q_{234}|5]^2}
\tag{3.94}
$$

such that the form is

$$
\mathrm{d}\Omega_{4+4} = \frac{\langle 24\rangle^4[15]^4}{s_{234}\langle 23\rangle\langle 34\rangle\langle 2|Q_{234}|5]\langle 4|Q_{234}|1][61][56]}
\tag{3.95}
$$

For the 5+3 diagram we once again use the results from the previous section. The diagram is

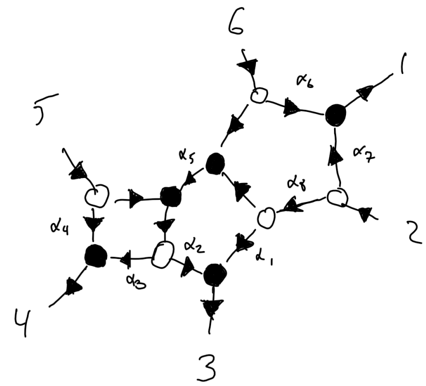


Figure 7.

With the following edgevariables

$$\begin{aligned} \alpha_1 &= \frac{s_{612}}{\langle 6|Q_{612}|3]}, \quad \alpha_2 = \frac{[45]}{[34]}, \quad \alpha_3 = \frac{[45]\langle 2|Q_{612}|3]}{[34]\langle 2|Q_{612}|4]}, \quad \alpha_4 = \frac{\langle 2|Q_{612}|5]}{\langle 2|Q_{612}|4]}, \\ \alpha_5 &= \frac{\langle 2|Q_{612}|4]}{\langle 62\rangle[45]}, \quad \alpha_6 = \frac{\langle 12\rangle}{\langle 62\rangle}, \quad \alpha_7 = \frac{\langle 61\rangle}{\langle 62\rangle}, \quad \alpha_8 = \frac{\langle 6|Q_{612}|3]}{\langle 2|Q_{612}|3]}. \end{aligned} \tag{3.96}$$

Here the Jacobian from the delta functions is

$$J = \frac{[35]^4\,\langle 62\rangle}{[34]^3\langle 2|Q_{612}|4]\langle 6|Q_{612}|3]} \tag{3.97}$$

The form is

$$\mathrm{d}\Omega_{5+3} = \frac{\langle 26\rangle^4[35]^4}{s_{612}\langle 12\rangle\langle 16\rangle\langle 6|Q_{612}|3]\langle 2|Q_{612}|5][34][45]} \tag{3.98}$$

3.5.5 Jacobian from solving delta-function

The following property holds for the δ-functions

$$\delta(kx) = \frac{1}{k}\delta(x) \tag{3.99}$$

which means that for instance when multiplying a deltafunction by a spinor one has

$$\delta(\lambda_i) = \lambda_j\delta(\lambda_j\lambda_i) \tag{3.100}$$

Or, using spinor helicity bracket notation

$$\delta(\lambda_i)\delta(\lambda_j) = \langle kl\rangle\delta(\lambda_k\lambda_i)\delta(\lambda_l\lambda_j) \tag{3.101}$$

3.6 One loop five-point in $\mathcal{N} = 0$

One might also consider diagrams with an extra internal parameter. At five-point an example of this is shown below

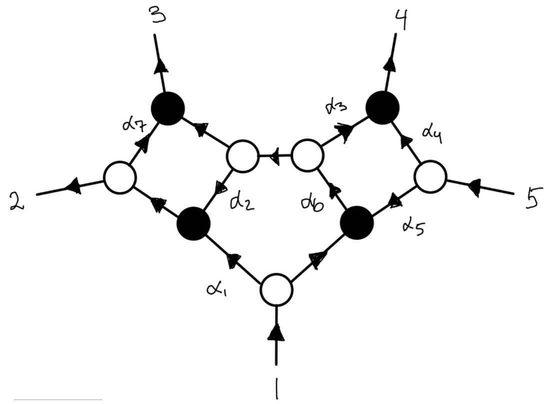


Figure 8.

Such a diagram can be thought of as a "one-loop" diagram, since the on-shell condition imposed on the three loop momenta leaves us with $3 \times 4 - 11 = 1$ unfixed parameter. The C-matrices are

$$
\begin{aligned}
C &= \begin{pmatrix} 1 & \alpha_1 + \alpha_2\alpha_6 & \alpha_2\alpha_7\alpha_6 + \alpha_6 + \alpha_1\alpha_7 & \alpha_3\alpha_6 & 0 \\ 0 & \alpha_2\alpha_5\alpha_6 & \alpha_5\alpha_6\left(\alpha_2\alpha_7 + 1\right) & \alpha_4 + \alpha_3\alpha_5\alpha_6 & 1 \end{pmatrix} \\
C_\perp &= \begin{pmatrix} -\alpha_1 - \alpha_2\alpha_6 & 1 & 0 & 0 & -\alpha_2\alpha_5\alpha_6 \\ -\alpha_6 - \left(\alpha_1 + \alpha_2\alpha_6\right)\alpha_7 & 0 & 1 & 0 & -\alpha_5\alpha_6\left(\alpha_2\alpha_7 + 1\right) \\ -\alpha_3\alpha_6 & 0 & 0 & 1 & -\alpha_4 - \alpha_3\alpha_5\alpha_6 \end{pmatrix}
\end{aligned}
\tag{3.102}
$$

$$
\begin{aligned}
-\langle 12\rangle + \alpha_2\alpha_5\alpha_6\langle 15\rangle \\
-((\alpha_1 + \alpha_2\alpha_6)\langle 15\rangle) + \langle 25\rangle \\
-\langle 13\rangle + \alpha_5\alpha_6(1 + \alpha_2\alpha_7)\langle 15\rangle \\
-((\alpha_6 + \alpha_1\alpha_7 + \alpha_2\alpha_6\alpha_7)\langle 15\rangle) + \langle 35\rangle \\
-\langle 14\rangle + (\alpha_4 + \alpha_3\alpha_5\alpha_6)\langle 15\rangle \\
-(\alpha_3\alpha_6\langle 15\rangle) + \langle 45\rangle \\
[12] - \left(\alpha_6 + \alpha_1\alpha_7 + \alpha_2\alpha_6\alpha_7\right)[23] - \alpha_3\alpha_6\,[24] \\
[13] + \left(\alpha_1 + \alpha_2\alpha_6\right)[23] - \alpha_3\alpha_6\,[34] \\
[14] + \left(\alpha_1 + \alpha_2\alpha_6\right)[24] + \left(\alpha_6 + \alpha_1\alpha_7 + \alpha_2\alpha_6\alpha_7\right)[34] \\
-(\alpha_5\alpha_6(1 + \alpha_2\alpha_7)\,[23]) - \left(\alpha_4 + \alpha_3\alpha_5\alpha_6\right)[24] - [25] \\
\alpha_2\alpha_5\alpha_6\,[23] - \left(\alpha_4 + \alpha_3\alpha_5\alpha_6\right)[34] - [35] \\
\alpha_2\alpha_5\alpha_6\,[24] + \alpha_5\alpha_6(1 + \alpha_2\alpha_7)\,[34] - [45]
\end{aligned}
\tag{3.103}
$$

References

[1] =N. E. J. Bjerrum-Bohr, J. F. Donoghue and P. Vanhove, “On-shell Techniques and Universal Results in Quantum Gravity,” JHEP **02** (2014), 111 doi:10.1007/JHEP02(2014)111 [arXiv:1309.0804 [hep-th]].