Modern amplitude techniques

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ABSTRACT: Notes on modern amplitude techniques written as part of a research project with Jaroslav Trnka.

Contents

1   3 point amplitudes bootstrapping

Three particle amplitudes are special since they can be completely determined by their little group scaling. From momentum conservation as well as on-shell massless kinematics we have

$$p_1 + p_1 + p_3 = 0 \quad \Rightarrow \quad \begin{cases} s_{12} = (p_1 + p_2)^2 = \langle 12 \rangle [21] = p_3^3 = 0 \\ s_{13} = (p_1 + p_3)^2 = \langle 13 \rangle [31] = p_2^3 = 0 \\ s_{23} = (p_2 + p_3)^2 = \langle 23 \rangle [32] = p_1^3 = 0 \end{cases} \tag{1.1}$$

For complex momenta we can treat $\rangle p|$ and $[p|$ as independent and so for the above relations to be satisfied we need either

$$[12] = [23] = [13] = 0 \qquad \text{or} \qquad \langle 12 \rangle = \langle 23 \rangle = \langle 13 \rangle = 0. \tag{1.2}$$

Denoting the helicity of the i'th particle by h_i the following result for the 3 point amplitude can be obtained by using the fact that under a little group scaling, the amplitudes transforms with weight according to that particles helicity:

$$A(1^{h_1}, 2^{h_2}, 3^{h_3}) = \begin{cases} \langle 12 \rangle^{h_3 - h_2 - h_1} \langle 23 \rangle^{h_1 - h_2 - h_3} \langle 13 \rangle^{h_2 - h_3 - h_1}, & \sum_i h_i \leq 0 \\ [12]^{h_1 + h_2 - h_3} [23]^{h_2 + h_3 - h_1} [13]^{h_1 + h_3 - h_2}, & \sum_i h_i \geq 0 \end{cases} \tag{1.3}$$

where the two cases arise because of *locality*, which means that only terms that show up in a local Lagrangian such as $AA\partial A$ (from gauge term $\text{Tr}[F^{\mu\nu} F_{\mu\nu}]$) can contribute. The condition is then there to get the correct mass dimension, meaning only amplitudes with dimensions according to local lagrangian interaction terms.

2   Recursion Relations

On-shell recursion is a systematic procedure for relating an amplitude to its values at singular kinematics. In order to probe these kinematic configurations we define a momentum shift, which is a one-parameter deformation of the external momenta engineered to sample various kinematic limit.
A shift of the form

$$p_i \to p_i(z) = p_i + z q_i, \quad z \in \mathbb{C}. \tag{2.1}$$

Not all momenta have to be shifted and we restrict the shifted momenta to satisfy momentum conservation as well as being on-shell

$$\sum_i p_i(z) = 0, \qquad p_i(z)^2 = 0 \tag{2.2}$$

This implies the following for the shifts q_i

$$\sum_i q_i = 0, \qquad q_i^2 = q_i p_i = 0. \tag{2.3}$$

These conditions preserve the kinematics of the corresponding shifted amplitude

$$A \to A(z) \tag{2.4}$$

We can obtain the original amplitude from the residue

$$A(0) = \oint_{z=0} dz \frac{A(z)}{z}. \tag{2.5}$$

One can think of the contour integral as a deltafunction in the point $z = 0$.
Using Cauchy's theorem this can be expressed as minus the sum of all the other residues

$$A(0) = -\sum_I \operatorname{Res}_{z=z_I} \left[\frac{A(z)}{z} \right] + B_\infty, \tag{2.6}$$

where B_∞ is a boundary term that vanishes when $A(z) \to 0$ for $z \to \infty$. This will be another condition on what variables we shift.
If we take a subset of momenta $\{p_i\}_{\in I}$ and define the sum over these

$$P_I \equiv \sum_{i \in I} p_i, \tag{2.7}$$

then we can also defined the shifted momenta $P_I(Z)$

$$P_I(z) = \sum_{i \in I} p_i(z) = P_I + zQ_I, \quad \text{with } Q_I = \sum_{i \in I} q_i \tag{2.8}$$

For simplicity we will assume $q_i q_j = 0$ leading to $Q_I^2 = 0$. In this case $P_I(z)^2$ is linear in z

$$P_I(z)^2 = (P_I + zQ_I)^2 = P_I^2 + zP_I Q_I = -\frac{P_I^2}{z_I}(z - z_I), \tag{2.9}$$

where we have defined $z_I \equiv -\frac{P_I^2}{2P_I Q_I}$.
For some reason the amplitude should factorize into a product of two lower point on-shell amplitudes when $z = z_I$ and $P_I^2(z)$ goes on-shell

$$\lim_{z \to z_I} A(z) = A_L(z_I) \frac{1}{P_I^2(z)} A_R(z_I) = -\frac{z_I}{z - z_I} A_L(z_I) \frac{1}{P^2} A_R(z_I) \tag{2.10}$$

Using this to take the residue at $z = z_I$

$$= -\operatorname{Res}_{z=z_I} \left[\frac{A(z)}{z} \right] = \operatorname{Res}_{z=z_I} \left[\frac{z_I}{z(z - z_I)} A_L(z_I) \frac{1}{P_I^2} A_R(z_I) \right] \tag{2.11}$$

The residue is found by multiplying by $(z - z_I)$ and setting $z = z_I$. Summing over all residues we find the amplitude

$$A(0) = \sum_I A_L(z_I) \frac{1}{P_I^2} A_R(z_I) + B_\infty \tag{2.12}$$

The boundary contribution B_∞ has no similar general expression in terms of lower-point amplitudes and the simplest way to make it vanish is by requiring

$$A(z) \to 0, \qquad \text{for } z \to \infty \tag{2.13}$$

If this holds then

$$A = \sum_I A_L(z_I) \frac{1}{P_I^2} A_R(z_I) = \sum_{\text{Diagrams } I} \vdots \underbrace{L}_{} \overset{\hat{P}_I}{\text{---}} \underbrace{R}_{} \vdots \tag{2.14}$$

2.1 BCFW-recursion

A particular recursion technique used often is called BCFW recursion. In four dimensions this can be implemented in the spinor-helicity basis. Denoting the shifted variables by a hat, the shifts that we will employ are

$$|\hat{i}] = |i] + z|j], \qquad |\hat{j}] = |j], \qquad |\hat{i}\rangle = |i\rangle, \qquad |\hat{j}\rangle = |j\rangle - z|i\rangle \tag{2.15}$$

It can be shown that for Yang-Mills the amplitude vanishes at $z \to \infty$ for the following helicity configurations

$$[i, j\rangle \quad [-, -\rangle \quad [-, +\rangle \quad [+, +\rangle \quad [+, -\rangle \tag{2.16}$$

$$A_n(z) \sim \quad \frac{1}{z} \qquad \frac{1}{z} \qquad \frac{1}{z} \qquad z^3 \tag{2.17}$$

The first 3 types of shifts will be referred to as *good shifts.*

2.2 Example of BCFW-recursion

As an example, let us calculate the amplitude $A_5(1_g^+, 2_g^-, 3_g^+, 4_g^-, 5_g^-)$ Since we are dealing with an $\overline{\text{MHV}}$ amplitude we can immediately read of the good shift since the shifts

$$|1] \to |\hat{1}] = |1] + z|5] \tag{2.18}$$

$$|5\rangle \to |\hat{5}\rangle = |5\rangle - z|1\rangle \tag{2.19}$$

will shift the amplitude by

$$A_5(1^+, 2^-, 3^+, 4^-, 5^-) = \frac{[13]^4}{[12][23][34][45][51]} \to \frac{\langle([13] + z[53])^4\rangle}{([12] + z[52])[23][34][45]} \sim z^3$$

While the shifts

$$|5] \to |\hat{5}] = |5] + z|1] \tag{2.20}$$

$$|1\rangle \to |\hat{1}\rangle = |1\rangle - z|5\rangle \tag{2.21}$$

will shift the amplitude by

$$A_5(1^+, 2^-, 3^+, 4^-, 5^-) = \frac{[13]}{[12][23][34][45][51]} \rightarrow \frac{[13]^4}{[12][23][34]([45] + z[41])([51] \underbrace{+z[11]}_{0})} \sim \frac{1}{z}$$

Now since we want $A \to 0$ for $z \to \infty$ the *good shift* is the second one, meaning $[5]\,|1\rangle$ which corresponds to a $[-, +\rangle$ shift in Elvangs notation. We could have seen the good shifts by little group scaling of the amplitude since leg one has little group weight 1 and so will the amplitude under a shift will scale like z if we shift the square brackets The corresponding diagrams are
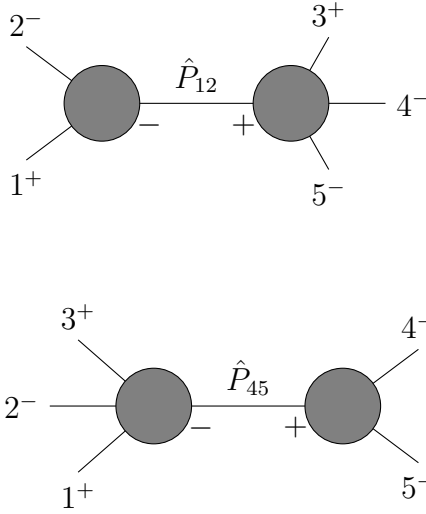
Looking at the first diagram we see that it contains a 3-point MHV amplitude

$$A_3(1^+, 2^-, -\hat{P}_{12}^-) = \frac{\left\langle 2\hat{P}_{12} \right\rangle^3}{\left\langle \hat{1}2 \right\rangle \left\langle \hat{P}_{12}\hat{1} \right\rangle} \tag{2.22}$$

Since we impose that the propagating momentum is on shell we see that

$$0 = \hat{P}_{12} = \left\langle \hat{1}2 \right\rangle [\hat{1}2] = \left\langle \hat{1}2 \right\rangle [12]$$

So the only way to impose on shell conditions is by setting $\left\langle \hat{1}2 \right\rangle = 0$ similarly one can show that the numerator vanishes and we must have

$$A_3(1^+, 2^-, -\hat{P}_{12}^-) = 0$$

which means the first diagram doesn't contribute. For the second diagram we also have a 3-point MHV amplitude, but in this case the shift is in $|5]$ so the sub-diagram isn't zero.

We can then proceed to calculate the second diagram explicitly

$$A_5(1^+, 2^-, 3^+, 4^-, 5^-) = A_3(\hat{P}_{45}^+, 4^-, \hat{5}^-)\frac{1}{P_{45}^2}A_4(\hat{1},^+, 2^-, 3^+, -\hat{P}_{45}^-)$$

$$= \frac{\langle 4\hat{5} \rangle^3}{\langle \hat{P}_{45}4 \rangle \langle \hat{5}\hat{P}_{45} \rangle}\frac{1}{\langle 45 \rangle [45]}\frac{[\hat{1}3]^4}{[\hat{1}2][23][3\hat{P}_{45}][\hat{P}_{45}\hat{1}]}$$

Since the shift is in $[5, 1\rangle$ we can remove the hat on all but the P's:

$$A_5(1^+, 2^-, 3^+, 4^-, 5^-) = \frac{\langle 45 \rangle^3}{\langle \hat{P}_{45}4 \rangle \langle 5\hat{P}_{45} \rangle}\frac{1}{\langle 45 \rangle [45]}\frac{[13]^4}{[12][23][3\hat{P}_{45}][\hat{P}_{45}1]}$$

$$= \frac{\langle 45 \rangle^3}{\langle \hat{P}_{45}4 \rangle \langle 5\hat{P}_{45} \rangle}\frac{1}{\langle 45 \rangle [45]}\frac{[13]^4}{[12][23][3\hat{P}_{45}][\hat{P}_{45}1]}$$

We can the rewrite the $\hat{P}$ terms in the following way:

$$\langle \hat{P}_{45}4 \rangle [\hat{P}_{45}1] = -\langle 4\hat{P}_{45} \rangle [\hat{P}_{45}1] = \langle 4|\hat{P}_{45}|1] = \langle 4|4 + \hat{5}|1] = \langle 4|\hat{5}|1] = -\langle 4\hat{5} \rangle [\hat{5}1] = -\langle 45 \rangle [51]$$

$$\langle 5\hat{P}_{45} \rangle [3\hat{P}_{45}] = -\langle 5\hat{P}_{45} \rangle [\hat{P}_{45}3] = \langle 5|\hat{P}_{45}|3] = \langle 5|4 + \hat{5}|4] = \langle 5|4|3] + \langle 5|\hat{5}|3]$$

$$= -\langle 54 \rangle [43] - \langle 5\hat{5} \rangle [\hat{5}3] = -\langle 54 \rangle [43] = -\langle 45 \rangle [34]$$

where we in the first terms have used the fact that $|\hat{5}\rangle = |5\rangle$ and $[\hat{5}1] = [51] + z[11] = [51]$, while in the second term using $\langle 5\hat{5} \rangle = \langle 55 \rangle = 0$. Inserting this into the amplitude we get

$$A_5(1^+, 2^-, 3^+, 4^-, 5^-) = \frac{[13]^4 \langle 45 \rangle^3}{[12][23][45] \langle 45 \rangle^3 [51][34]}$$

$$= \frac{[13]^4}{[12][23][34][45][51]}$$

which is the expected result.

2.3   Soft limit factorization

The soft-limit factorization for tree amplitudes is that for $k_s \to 0$ we can write an n-point amplitude as

$$A_n^{\text{tree}}(1, 2, \ldots, a, s^{\pm}, b, \ldots, n) = \mathcal{S}(a, s^{\pm}, b) \times A_{n-1}^{\text{tree}}(1, 2, \ldots, a, b, \ldots, n) \qquad (2.23)$$

where

$$\mathcal{S}(a, s^+, b) = \frac{\langle ab \rangle}{\langle as \rangle \langle sb \rangle}, \qquad \mathcal{S}(a, s^-, b) = -\frac{[ab]}{[as][sb]} \qquad (2.24)$$

Here this gets us

$$A_5(1^+, 2^-, 3^+, 4^-, 5^-) = -\frac{[41]}{[45][51]} \times \frac{[13]^4}{[12][23][34][41]}$$

which is a valid factorization of the full result.

2.4 Co-linear limit

In the co-linear limit for leg 1 and 2 we have the two momenta k_1 and k_2 that become parallel with intermediate momentum k_P. The spinors also have the following relations

$$\lambda_a \simeq \sqrt{z}\lambda_P, \qquad \lambda_b \simeq \sqrt{1-z}\lambda_P$$
$$\tilde{\lambda}_{\dot{a}} \simeq \sqrt{z}\tilde{\lambda}_P, \qquad \tilde{\lambda}_{\dot{b}} \simeq \sqrt{1-z}\tilde{\lambda}_P$$

taking the amplitude we calculated in part a and shifting it in this limit gives

$$A_5(1^+, 2^-, 3^+, 4^-, 5^-) \rightarrow \frac{z^2}{\sqrt{z(1-z)}[12]} \frac{[P3]^4}{[P3][34][45][5P]}$$

which is the result we expected from Dixon:

$$A_n^{\text{tree}}(\ldots, a^{\lambda_a}, b^{\lambda_b}, \ldots) \rightarrow \sum_{\lambda_p = \pm} \text{Split}_{-\lambda_P}(a^{\lambda_a}, b^{\lambda_b}; z) A_{n-1}^{\text{tree}}(\ldots, P^{\lambda_P}, \ldots) \qquad (2.25)$$

where

$$\text{Split}_-(a^+, b^-) = \frac{z^2}{\sqrt{z(1-z)}[ab]} \qquad (2.26)$$

3 Unitarity

3.1 Loops in general

Feynman rules require momentum conservation at each vertex of a Feynman diagram. At tree-level, this fixes all momenta of the internal lines in terms of the external momenta. At loop-level, momentum conservation leaves one momentum undetermined per loop and one must integrate over all such unfixed momenta. Thus in D-dimensions, one has a D-dimensional loop-integral for each loop.

4 Unitarity

Take a general analytic function of some variable x

$$f(x) = \beta(x) + i\alpha(x) \qquad (4.1)$$

Defining the discontinuity of this function as $\text{Disc}[f(x)] = if(x + i\epsilon) - if(x - i\epsilon)$ we find as we let $\epsilon \rightarrow 0$

$$
\begin{aligned}
if(x + i\epsilon) - if(x - i\epsilon) &= i\beta(x + i\epsilon) - \alpha(x + i\epsilon) - i\beta(x - i\epsilon) - \alpha(x - i\epsilon) \\
&= -2\left(\alpha(x) + i\beta(i\epsilon)\right) + \mathcal{O}(\epsilon^2) \\
&= -2\alpha(x) + \mathcal{O}(\epsilon) \\
&= -2\,\text{Im}[f(x)]
\end{aligned}
\qquad (4.2)
$$

The scattering matrix is unitary

$$S = \mathbb{1} + iT \tag{4.3}$$

Taking

$$
\begin{aligned}
S^\dagger S &= (\mathbb{1} - iT^\dagger)(\mathbb{1} - iT) = 1 \\
&= \mathbb{1} - i(T - T^\dagger) - T^\dagger T = 1 \\
\Rightarrow T^\dagger T &= i(T^\dagger - T) = i\left[(\operatorname{Re}[T] - i\operatorname{Im}[T]) - (\operatorname{Re}[T] + i\operatorname{Im}[T])\right] \\
&= 2\operatorname{Im}[T] = -\operatorname{Disc}[iT]
\end{aligned}
\tag{4.4}
$$

Expanding this order by order in perturbation theory we have for instance at four and five point

$$
\begin{aligned}
T_4 &= g^2 T_4^{(0)} + g^4 T_4^{(1)} + g^6 T_4^{(2)} \\
T_5 &= g^3 T_5^{(0)} + g^5 T_5^{(1)} + g^7 T_5^{(2)}
\end{aligned}
\tag{4.5}
$$

with $T_n^{(L)}$ being the n-point gluon amplitude at L-loop. By inserting these into the equation for the discontinuity we find first to order g^2

$$\operatorname{Disc}[T_4^{(0)}] = 0 \tag{4.6}$$

Since we cant construct the amplitude from the product of two other amplitudes. This simply states that tree level amplitudes have no branch cuts. At order g^4 we have

$$\operatorname{Disc}[T_4^{(1)}] = T_4^{(0)\,\dagger} T_4^{(0)} \tag{4.7}$$

This is equivalent to cutting to lines in the one loop diagram and obtaining two four point tree-level diagrams with an on-shell propagator in between, which momentum obviously must depend on the loop momentum ℓ.

Using the optical theorem the Discontinuity of a function is related directly to the scattering cross-section.

5   Generalized unitarity

When we take the momenta in the scattering amplitudes to be complex and use the unitarity method, we get *generalized unitarity*. The inclusion of complex momenta opens up the possibility of cutting more than two lines. e.g. the three-point amplitudes are only non-zero for complex momenta, so for the four-point one-loop amplitude we could cut four lines and end up with a product of four three-point amplitudes.

In general we will in fact only be able to cut up to four lines since we need momentum conservation at the vertices as well as having the cut momentum being on-shell. This imposes one new condition for every cut, and since ℓ^μ has four components we need 4 equations.

6 Exterior derivatives and forms

An n-form $F_{(n)}$ is a completely antisymmetric tensor

$$F_{\mu_1\,\mu_2\,\mu_3\cdots\mu_n} = -F_{\mu_1\,\mu_3\,\mu_2\cdots\mu_n} = F_{\mu_3\,\mu_1\,\mu_2\cdots\mu_n} \tag{6.1}$$

On this n-form, we can define an exterior derivative, which is an $(n+1)$ form:

$$\mathrm{d}F_{\mu_1\,\mu_2\,\mu_3\cdots\mu_n} = (n+1)\partial_{[\mu_1}F_{\mu_2\,\mu_3\cdots\mu_n]}, \tag{6.2}$$

where the square brackets just mean that we antisymmetrize in the indices, e.g.

$$\begin{aligned}
\mathrm{d}F_{\mu\nu} =&\frac{(1+1)}{2!}\left(\partial_\mu F_\nu - \partial_\nu F_\mu\right)\\
=&\partial_\mu F_\nu - \partial_\nu F_\mu,\\
\mathrm{d}F_{\mu\nu\rho} =&\frac{(2+1)}{3!}\left(\partial_\mu F_{\nu\rho} - \partial_\mu F_{\rho\nu} - \partial_\nu F_{\mu\rho} + \partial_\nu F_{\rho\mu} - \partial_\rho F_{\nu\mu} + \partial_\rho F_{\mu\nu}\right)\\
=&\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu}
\end{aligned} \tag{6.3}$$

Furthermore we have the property

$$\mathrm{d}^2 F = 0 \tag{6.4}$$

And the following nomenclature

- *Exact form*: if we can write $\qquad F = \mathrm{d}G$

- *Closed form*: if $\qquad\qquad \mathrm{d}F = 0$

- $\Rightarrow$ All exact forms are closed.

We also define the wedge product of forms. Given two forms $F_{(n)}$ and $G_{(m)}$, we can define the $n+m$ form $F \wedge G$

$$(F \wedge G)_{\mu_1\cdots\mu_{n+m}} = \frac{(n+m)!}{n!m!}F_{[\mu_1\cdots\mu_n}G_{\mu_{n+1}\cdots\mu_{n+m}]} \tag{6.5}$$

For instance

$$\begin{aligned}
F_\mu \wedge G_\nu =&\frac{(1+1)!}{1!1!}\left(F_\mu G_\nu - F_\nu G_\mu\right)\\
=&2[F_\mu, G_\nu]
\end{aligned} \tag{6.6}$$

Lastly we can define the *Hodge dual*, as the $(D-n)$-form

$$^*F^{\mu_{n+1}\,\mu_{n+2}\cdots\mu_D} = \frac{1}{n!}F_{\mu_1\,\mu_2\cdots\mu_n}\epsilon^{\mu_1\,\mu_2\cdots\mu_n\,\mu_{n+1}\,\mu_{n+2}\cdots\mu_D} \tag{6.7}$$

Using this notation we can write the source-less Maxwell equations as:

$$\begin{aligned}
\mathrm{d}F &= 0\\
\mathrm{d}^*F &= 0
\end{aligned} \tag{6.8}$$

7 Projective spaces

We will start by considering $\mathbb{P}^2$. Any geometric questions that do not involve distance are best thought of projectively. All lines that intersect the origin in $\mathbb{R}^3$ can be though of as points crossing a plane
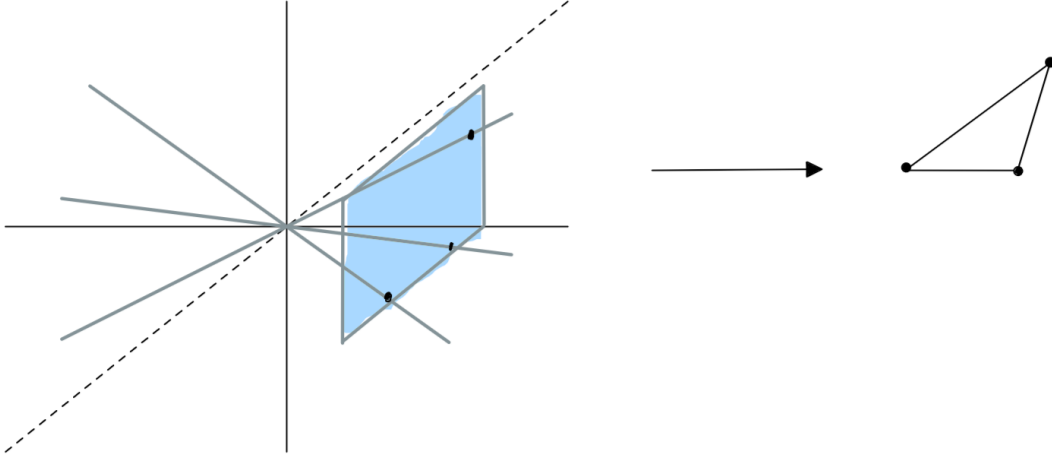
If we denote the points by $Z_I = \begin{pmatrix} Z_0 \\ Z_1 \\ Z_2 \end{pmatrix}$ these can always be rescaled by some factor t while still preserving the geometry of the points in the plane. This means that we really have two degrees of freedom per point (which should not surprise since we have a plane), so that $Z_I = \begin{pmatrix} 1 \\ x_1 \\ x_2 \end{pmatrix}$. The space has an SL(3) = Translations + $\underbrace{\text{Rotations}}_{\text{SL}(2)}$ symmetry, meaning all symmetries that map straight lines to straight lines.

The only invariant tensor is ϵ_{IJK}.

Example: straight line A straight line obeys the following equation

$$ax + by + c = 0 \tag{7.1}$$

This can be written using the Z_I variables

$$Z_I = \begin{pmatrix} 1 \\ x \\ y \end{pmatrix}, \qquad W^I = \begin{pmatrix} c \\ a \\ b \end{pmatrix}, \qquad Z_I W^I = 0 \tag{7.2}$$

where W is a line. Take e.g. $Z^1 = \{1, x^1, y^2\} = \{1, 1, 0\}$ and $Z^2 = \{1, x^2, y^2\} = \{1, 10, 10\}$. We check that this gives the expected result

```
In[403]:= ClearAll["Global`*"]

(*Below we define two points from which we then find the line*)

In[404]:= Z1 = {1, x1, y1};
Z2 = {1, x2, y2};
a = (y2 - y1)/(x2 - x1);
W = Z1 . LeviCivitaTensor[3] . Z2;
s = Collect[Solve[W . {1, x, y} == 0, y], x][[1]][[1]];
Equal @@ s;
y = (x (y1 - y2))/(x1 - x2) + (-x2 y1 + x1 y2)/(x1 - x2);

In[411]:= (*Testing against the usual formula*)
x1 = 1; x2 = 10; y1 = 0; y2 = 10;
yy = a x + (y2 - a x2);
yy == y

Out[413]= True
```

Similarly the point where two lines cross is given by

$$Z_I = \epsilon_{IJK} W_1^J W_2^K \tag{7.3}$$

8 Polytopes

Let us define the five-bracket

$$[i, j, k, l, m] \equiv \frac{\delta^4(\chi_{iA} \langle jklm \rangle + \text{ cyclic })}{\langle ijkl \rangle \langle jklm \rangle \langle klmi \rangle \langle lmij \rangle \langle mijk \rangle} \tag{8.1}$$

with $\langle ijkl \rangle \equiv \epsilon_{IJKL} Z_i^I Z_j^J Z_k^K Z_l^L$ and the Z_i^I's are the bosonic component of the momentum supertwistors $Z_i^I = (|i\rangle, [\mu_i|)$. One can write the five-bracket completely in terms of bosonic parts by introducing a five vector $\mathsf{Z}_i^{\mathcal{I}}$,

$$\mathsf{Z}_i^{\mathcal{I}} = \begin{pmatrix} Z_i^I \\ \chi_i \cdot \psi \end{pmatrix}, \qquad \mathcal{I} = 1, \dots, 5 \tag{8.2}$$

with the ψ being an auxiliary Grassmann valued field. The five-bracket can then be written only in terms of the five-vectors by integrating out the fermionic auxilary field.

$$[i, j, k, l, m] = \frac{1}{4!} \int d^4\psi \frac{\langle ijklm \rangle^4}{\langle 0ijkl \rangle \langle 0jklm \rangle \langle 0klmi \rangle \langle 0lmij \rangle \langle 0mijkl \rangle} \tag{8.3}$$

where we have introduced the auxiliary reference spinor

$$
\mathsf{Z}_0^{\mathcal{I}} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}
\tag{8.4}
$$

Focusing on the integrand, it is invariant under $\mathsf{Z}_i^{\mathcal{I}} \to t_i \mathsf{Z}_i^{\mathcal{I}}$ and hence appear projectively which means they can be thought of as points in $\mathbb{CP}^4$ with the reference vector introduced being the only thing that breaks projective invariance. This is similar to the map between the dual momentum coordinates and the momentum twistors

$$
y_{ij}^2 = \frac{\langle i-1, i, j-1, j \rangle}{\langle I_0, i-1, i \rangle \langle I_0, j-1, j \rangle}, \qquad I_0^{IJ} = \begin{pmatrix} 0 & 0 \\ 0 & \epsilon_{\dot{a}\dot{b}} \end{pmatrix}
\tag{8.5}
$$

where the I_0 is known as infinity twistor that is inserted to break SL(4) conformal invariance and gives a definition of distance. The fact that the reference spinor appears five times in equation (8.1) is analogous to how I_0 appears twice to give the distance between i and j, and can be regarded as giving the volume of a simplices.

8.1 Defintions

Simplex: Generalization of the notion of a triangle or tetrahedron to arbitrary dimensions

References

[1] N. E. J. Bjerrum-Bohr, J. F. Donoghue and P. Vanhove, "On-shell Techniques and Universal Results in Quantum Gravity," JHEP **02** (2014), 111 doi:10.1007/JHEP02(2014)111 [arXiv:1309.0804 [hep-th]].