Notes on integrands for $\mathcal{N} < 4$

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Loop integrands are rational function and hence may be expanded in arbitrary basis

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In general, one-loop amplitudes can be decomposed in terms of a set of basis functions, I_i, with coefficients, c_i, that are rational in terms of spinor products,

$$A = \sum_i c_i I_i \tag{2.1}$$

For $\mathcal{N} = 4$ amplitudes the set can be taken to contain scalar boxes, I_4. We have here 4 cases, labeled by whether external momenta are massive or mass-less[1]. The cases that we encounter are:

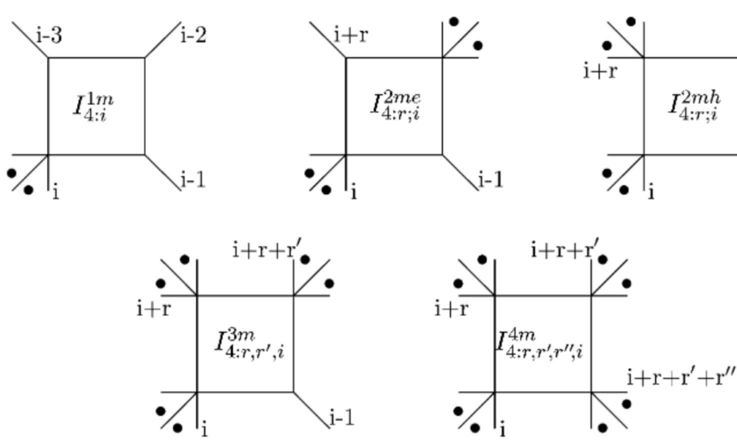
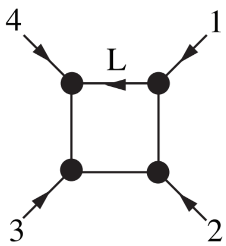


Figure 1.

[1]In reality we are always working with massless particles and the notion of *massive* is taken to mean multiple particles in the external lines such that the summed momenta looks like a massive particle, e.g. $(p_1 + p_2)^2 = "m_{12}^2" \neq 0$

as well as the simplest example the zero mass integral



$$= \int d^4L \frac{N}{L^2(L-p_1)^2(L-p_1-p_2)^2(L-p_1-p_2-p_3)^2}$$

Figure 2.

where $N = (p_1+p_2)^2(p_3+p_4)^2$ is a normalization factor which we will soon see to be convenient.

We can define each momenta as a difference between two spacetime points

$$p_a \equiv x_a - x_{a-1} \tag{2.2}$$

where each variable obeys the symmetry $x_a \to x_a + y$ leaving momentum conservation invariant $\sum p_a \to \sum p_a$.

Further we then also have

$$(x_b - x_a)^2 = (p_a + \cdots + p_{b-1})^2 = s_{a\cdots b-1} \tag{2.3}$$

One can relate null rays and points in an auxiliary space, known as twistor space using something known as the incidence relation:

$$\mu_{\underline{\dot\alpha}} = x_{\underline{\alpha\dot\alpha}}\lambda^{\underline{\alpha}} \tag{2.4}$$

where the twistor then is

$$Z = (\lambda, \mu) \tag{2.5}$$

discussion of twistors

When discussing propagators, it is natural to see quantities of the form

$$p_I^2 = (x_a - x_c)^2 \tag{2.6}$$

with $I = \{a, c\}$. Say that the lines x_a and x_c are related to the twistors Z_A, Z_B and Z_C, Z_D respectively. Then one can write

$$(x_a - x_c)^2 = \frac{\langle Z_A Z_B Z_C Z_D \rangle}{\langle AB \rangle \langle CD \rangle} \tag{2.7}$$

We can now in two steps go to first the dual coordinates x_a and then the twistor coodinates. First take $L = x - x_4$. Then it follows that

$$
\begin{aligned}
(L-p_1) &= ([x-x_4]-[x_1-x_4]) = (x-x_1)\\
(L-p_1-p_2) &= ([x-x_1]-[x_2-x_1]) = (x-x_2)\\
(L-p_1-p_2-p_3) &= ([x-x_2]-[x_3-x_2]) = (x-x_3)\\
(p_1+p_2) &= ([x_1-x_4]+[x_2-x_1]) = (x_2-x_4)\\
(p_2+p_3) &= ([x_2-x_1]+[x_3-x_2]) = (x_3-x_1)\\
\frac{\mathrm{d}L}{\mathrm{d}x} &= 1
\end{aligned}
\tag{2.8}
$$

which leads to

$$\int \mathrm{d}^4x \frac{(x_3-x_1)^2(x_2-x_4)^2}{(x-x_1)^2(x-x_2)^2(x-x_3)^2(x-x_4)^2} \tag{2.9}$$

When translating this integral to twistor space we get a Jacobian

$$\int \mathrm{d}^4x = \int \frac{\mathrm{d}^4Z_A\mathrm{d}^4Z_B}{\mathrm{Vol}(GL(2)) \times \langle AB\rangle^4} \equiv \int_{AB} \frac{1}{\langle AB\rangle^4} \tag{2.10}$$

as we will see, for $\mathcal{N}=4$ at one-loop we only have boxes and the factor $\frac{1}{\langle AB\rangle^4}$ cancels.

Now identifying x with Z_A and Z_B and using the association

$$x_a \leftrightarrow (Z_a, Z_{a+1}), \tag{2.11}$$

we get, using (2.7),

$$
\begin{aligned}
(x-x_1)^2 &= \frac{\langle AB12\rangle}{\langle AB\rangle\langle 12\rangle}, \quad (x-x_2)^2 = \frac{\langle AB23\rangle}{\langle AB\rangle\langle 23\rangle},\\
(x-x_3)^2 &= \frac{\langle AB34\rangle}{\langle AB\rangle\langle 34\rangle}, \quad (x-x_4)^2 = \frac{\langle AB41\rangle}{\langle AB\rangle\langle 41\rangle},\\
(x_3-x_1)^2 &= \frac{\langle 3412\rangle}{\langle 34\rangle\langle 12\rangle}, \quad (x_2-x_4)^2 = \frac{\langle 2341\rangle}{\langle 23\rangle\langle 41\rangle},
\end{aligned}
\tag{2.12}
$$

which gives

$$\int_{AB} \frac{\langle 1234\rangle^2}{\langle AB12\rangle\langle AB23\rangle\langle AB34\rangle\langle AB41\rangle} \tag{2.13}$$

Let us compute the residue over Z_A for $\langle AB12\rangle = 0$. Schematically the four-brackets contract using the four term Levi-Cevita symbol, so the residue on $\langle AB12\rangle = 0$ us found by setting B equal to either $1, 2$, or A in the remaining brackets depending on which gives a non-zero result, i.e.,

$$\int_A \frac{\langle 1234\rangle^2}{\langle A123\rangle\langle A234\rangle\langle A241\rangle} \tag{2.14}$$

Then taking the residue on $\langle A123\rangle = 0$ we obtain

$$\frac{\langle 1234\rangle^2}{\langle 1234\rangle\langle 3241\rangle} = -1 \tag{2.15}$$

where we have used the anti-symmetric symbol in the last line. I.e. we have a unit leading singularity when we set the maximum (2 in this case) amount of propagators on-shell.

Let us look at the 4-mass box. We are going to ignore the normalization factors in the numerator since these only depend on external kinematics. Note that while we can collect the external legs in each corner to emulate a massive particle, the internal propagators are all still massless.

$I_{\text{Box}} =$

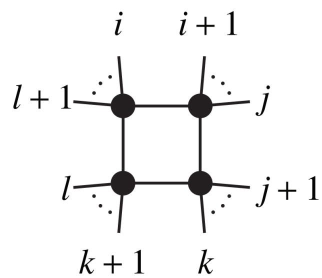


Figure 3.

Defining $P_i \equiv p_i + \cdots + p_{l+1}$ etc, as well as $L = x - x_l$ the formula is

$$\begin{aligned} I &= \int \mathrm{d}^4L \, \frac{1}{L^2(L-P_i)^2(L-P_i-P_j)^2(L-P_i-P_j-P_k)^2} \\ &= \int \mathrm{d}^4x \, \frac{1}{(x-x_i)^2(x-x_j)^2(x-x_k)^2(x-x_l)^2} \\ &= \int_{AB} \frac{1}{\langle AB\rangle^4} \times \frac{\langle AB\rangle^4\langle i\,i+1\rangle\langle j\,j+1\rangle\langle k\,k+1\rangle\langle l\,l+1\rangle}{\langle ABi\,i+1\rangle\langle ABj\,j+1\rangle\langle ABk\,k+1\rangle\langle ABl\,l+1\rangle} \\ &= \int_{AB} \frac{\langle i\,i+1\rangle\langle j\,j+1\rangle\langle k\,k+1\rangle\langle l\,l+1\rangle}{\langle AB\,i\,i+1\rangle\langle AB\,j\,j+1\rangle\langle AB\,k\,k+1\rangle\langle AB\,l\,l+1\rangle} \end{aligned} \tag{2.16}$$

It turns out that the box expansion reproduces all one-loop amplitudes post- integration, it does not match the full structure of the actual loop integrand. Here one has to also include *pentagons*