

$N < 4$ On-Shell Diagrams

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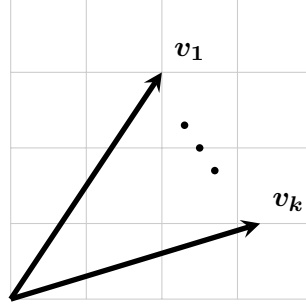
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ABSTRACT: Notes on modern amplitude techniques written as part of a research project with Jaroslav Trnka.

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1 Introduction

2 Grassmanian

The Grassmanian $G(k, n)$ is the space of k -planes going through the origin in n dimensions. It can be thought of as a generalization of P^{n-1} which is the space of lines going through the origin in n -dimensions since $G(1, n) = P^{n-1}$. One can e.g. take k vectors in n dimensions. The span of these vectors give me the k - plane. If we stack them we get

$$k \begin{bmatrix} V_1 \\ \vdots \\ V_k \end{bmatrix} \equiv C_{\alpha a}, \quad \alpha = 1, \dots, k \quad a = 1, \dots, n \quad (2.1)$$

These are in general not unique since there is a $GL(k)$ redundant.

$$C_{\alpha a} \sim L_{\alpha}^{\beta} C_{\beta a} \quad (2.2)$$

The dimensionality of the Grassmanian is

$$\dim G(k, n) = \overbrace{k \times n}^{k \times n \text{ matrix}} \underbrace{-k^2}_{GL(k) \text{ red}} \quad (2.3)$$

The redundancy means that we can gaugefix the matrix using a linear transformation by setting any $k \times k$ block to the identity. This is equivalent to the rescaling of vectors in projective space to $(1 \ v_2 \ v_3 \ v_4 \ \dots)$. Taking e.g. $G(3, 5)$, we have six degrees of freedom:

$$G(3, 5) = \left[\begin{array}{ccc|cc} 1 & 0 & 0 & x_4 & x_5 \\ 0 & 1 & 0 & y_4 & y_5 \\ 0 & 0 & 1 & z_4 & z_5 \end{array} \right] \quad (2.4)$$

The dimensionality of the Grassmanian are symmetric under $n \leftrightarrow k$. This is because there is a bijection between the Grassmanian: k and $n - k$ planes in n dimensions, since these planes

are orthogonal. In the case above C^\perp is a 2-plane in 5 dimensions, so

$$\left[\begin{array}{ccc|cc} 1 & 0 & 0 & x_4 & x_5 \\ 0 & 1 & 0 & y_4 & y_5 \\ 0 & 0 & 1 & z_4 & z_5 \\ \hline -x_4 & -y_4 & -z_4 & 1 & 0 \\ -x_5 & -y_5 & -z_5 & 0 & 1 \end{array} \right] \quad (2.5)$$

With the bottom part just being the negative transpose of the x, y and z coordinates in the upper right corner.

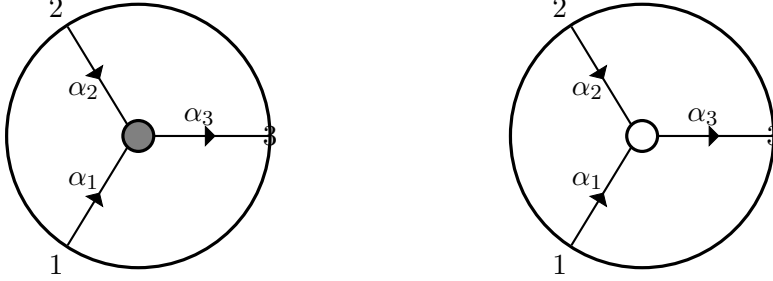
$\text{SL}(k)$ invariant are determinants of any k columns of the matrix (the minors), labeling these by their indices:

$$\left(a_1 \ a_2 \ \cdots \ a_k \right) \quad (2.6)$$

3 On-shell diagrams

3.1 Using three point on shell functions

The three-point vertex can be found from the following MHV and $\overline{\text{MHV}}$ diagrams



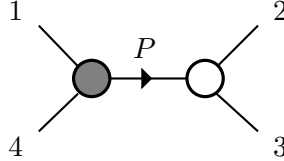
These produce the two following C matrices, respectively

$$\begin{aligned} C &= \begin{pmatrix} 1 & 0 & \alpha_1 \alpha_2 \\ 0 & 1 & \alpha_2 \alpha_3 \end{pmatrix} \\ C &= \begin{pmatrix} \alpha_1 \alpha_3 & \alpha_2 \alpha_3 & 1 \end{pmatrix} \end{aligned} \quad (3.1)$$

Because of momentum conservation and little group invariance, the solution of the delta functions in this case leads to

$$\begin{aligned} A_3^{\text{MHV}}(1, 2, 3) &= \frac{\delta^8 \left(\sum_{i=1}^3 \lambda_i \tilde{\eta}_i \right) \delta^4 \left(\sum_{i=1}^3 \lambda_i \tilde{\lambda}_i \right)}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle} \\ A_3^{\overline{\text{MHV}}}(1, 2, 3) &= \frac{\delta^4 ([12] \tilde{\eta}_3 + [23] \tilde{\eta}_1 + [31] \tilde{\eta}_2) \delta^4 \left(\sum_{i=1}^3 \lambda_i \tilde{\lambda}_i \right)}{[12][23][31]} \end{aligned} \quad (3.2)$$

We start by constructing the simplest possible diagram out of two opposite helicity ($k = 1$ and $k = 2$) amplitudes, see figure below:



To construct the four-point diagram we then glue the two three-point amplitudes together by integrating over the internal degrees of freedom through

$$\prod_I \int d^4 \tilde{\eta}_I \int \frac{d^2 \lambda_I d^2 \tilde{\lambda}_I}{GL(1)} \quad (3.3)$$

Explicitly we have

$$\begin{aligned} \int d\tilde{\eta}_P \int \frac{d^2\lambda_P d^2\tilde{\lambda}_P}{GL(1)} & \frac{\delta^8(\lambda_1\tilde{\eta}_1 + \lambda_4\tilde{\eta}_4 + \lambda_P\tilde{\eta}_P) \delta^4\left(\lambda_1\tilde{\lambda}_1 + \lambda_4\tilde{\lambda}_4 + \lambda_P\tilde{\lambda}_P\right)}{\langle 14 \rangle \langle 4P \rangle \langle P1 \rangle} \\ & \times \frac{\delta^4([23]\tilde{\eta}_P + [3P]\tilde{\eta}_2 + [P3]\tilde{\eta}_3) \delta^4\left(\lambda_2\tilde{\lambda}_2 + \lambda_3\tilde{\lambda}_3 - \lambda_P\tilde{\lambda}_P\right)}{[23][3P][P2]} \end{aligned} \quad (3.4)$$

First we solve the delta-function constraint by projecting along λ_1

$$\begin{aligned} \lambda_1\tilde{\lambda}_1 + \lambda_4\tilde{\lambda}_4 + \lambda_P\tilde{\lambda}_P &= 0 \\ \Rightarrow \tilde{\lambda}_P &= \frac{\langle 41 \rangle}{\langle 1P \rangle} \tilde{\lambda}_4 \end{aligned} \quad (3.5)$$

Similarly we use the other delta-function and project using $\tilde{\lambda}_3$

$$\begin{aligned} \lambda_2\tilde{\lambda}_2 + \lambda_3\tilde{\lambda}_3 - \lambda_P\tilde{\lambda}_P &= 0 \\ \Rightarrow \lambda_P &= \frac{[23]}{[P3]} \lambda_2 \end{aligned} \quad (3.6)$$

combining these we obtain

$$\begin{aligned} \tilde{\lambda}_P \lambda_P &= \lambda_2 \tilde{\lambda}_4 \frac{\langle 41 \rangle [23]}{\langle 1P \rangle [P3]} = \lambda_2 \tilde{\lambda}_4 \frac{[23]}{[43]} \\ &= \lambda_2 \tilde{\lambda}_4 \frac{\langle 41 \rangle}{\langle 12 \rangle} \end{aligned} \quad (3.7)$$

where we have used $P = -1 - 4 = 2 + 3$ in the last two equalities. Solving this collapses the momentum conservation delta function as well as giving a Jacobian factor of $\frac{1}{\langle 23 \rangle [32]}$

$$\begin{aligned} \lambda_P &= \lambda_2 \\ \tilde{\lambda}_P &= \lambda_4 \frac{\langle 41 \rangle}{\langle 12 \rangle} = \tilde{\lambda}_4 \frac{[23]}{[43]} \end{aligned} \quad (3.8)$$

We then use these in one of the grassmann delta-functions

$$\begin{aligned} \tilde{\eta}_P &= \frac{-[3P]\tilde{\eta}_2 - [P2]\tilde{\eta}_3}{[23]} \\ &= -\frac{1}{[23]} \times \frac{[34][23]}{[43]} \times \tilde{\eta}_2 - \frac{1}{[23]} \times \frac{[42]\langle 41 \rangle}{\langle 12 \rangle} \times \tilde{\eta}_3 \\ &= \tilde{\eta}_2 + \frac{\langle 13 \rangle}{\langle 12 \rangle} \times \tilde{\eta}_3 \end{aligned} \quad (3.9)$$

This can be obtained from contracting

$$\lambda_P \tilde{\eta}_P = \lambda_2 \tilde{\eta}_2 + \lambda_3 \tilde{\eta}_3 \quad (3.10)$$

with λ_1 , since $\lambda_P = \lambda_2$. Using this in the other grassmann delta function we get $[23]^4 \delta^8(\sum_i \lambda_i \tilde{\eta}_i)$. Finally we take the solutions (3.8) and insert them into the bosonic delta-function

$$\begin{aligned}
0 &= \lambda_1 \tilde{\lambda}_1 + \lambda_4 \tilde{\lambda}_4 + \lambda_P \tilde{\lambda}_P = \lambda_1 \tilde{\lambda}_1 + \tilde{\lambda}_4 \left(\lambda_4 + \lambda_2 \frac{[23]}{[43]} \right) \\
&= \lambda_1 \tilde{\lambda}_1 + \tilde{\lambda}_4 \left(\frac{\lambda_4 [43] + \lambda_2 [23]}{[43]} \right) = \lambda_1 \left(\tilde{\lambda}_1 + \tilde{\lambda}_4 \frac{[13]}{[34]} \right) \\
&= \lambda_1 \left(\frac{\tilde{\lambda}_1 [34] + \tilde{\lambda}_4 [13]}{[34]} \right) = \lambda_1 \tilde{\lambda}_3 \frac{[14]}{[34]}
\end{aligned} \tag{3.11}$$

Since $\lambda_1 \neq 0$, and $\tilde{\lambda}_3 \neq 0$ this leads to $[14] = 0$ which in turn gives us

$$(p_1 + p_4)^2 = \langle 14 \rangle [41] = 0 \tag{3.12}$$

Now we only need the kinematic part of the integrand. Including the Jacobians and using $\langle 1P \rangle [P3] = \langle 12 \rangle [23]$ and $\langle 4P \rangle [P2] = \langle 43 \rangle [32]$ we obtain

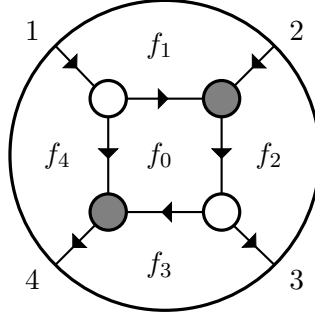
$$\frac{1}{\langle 14 \rangle \langle 4P \rangle \langle P1 \rangle} \times \frac{1}{[23][3P][P2]} \times \frac{[23]^4}{\langle 23 \rangle [23]} = \frac{1}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} \tag{3.13}$$

Such that we in total have the amplitude

$$\frac{\delta^8(Q) \delta^4(P)}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} \delta((p_1 + p_4)^2) \tag{3.14}$$

3.2 Four-point directly from C(2,4) matrix

The calculation this C-matrix can be performed either using face or edge variables. We are going to do both for good measure. The four-point diagram with face-variables looks like this



While the

$$C_{ab} = - \sum_{\Gamma(a \rightarrow b)} \prod_j (-f_j), \quad \text{on the right} \tag{3.15}$$

with the constraint

$$\prod_j f_j = -1 \tag{3.16}$$

$$C = \begin{pmatrix} 1 & 0 & f_0 f_3 f_4 & f_4(1 - f_0) \\ 0 & 1 & -f_0 f_1 f_3 f_4 & f_0 f_1 f_4 \end{pmatrix} \quad (3.17)$$

Note that f_2 doesn't show up, which means that according to (3.16) we can take the remaining f 's as independent.

Positivity (all minors are positive) then demands that

$$f_0 < 0, \quad f_1 > 0, \quad f_2 < 0, \quad f_3 < 0, \quad (3.18)$$

While the perpendicular C-matrix satisfying $C \cdot C^\perp = 0$ is easily obtained

$$C^\perp = \begin{pmatrix} -f_0 f_3 f_4 & f_0 f_1 f_3 f_4 & 1 & 0 \\ -f_4(1 - f_0) & -f_0 f_1 f_4 & 0 & 1 \end{pmatrix} \quad (3.19)$$

we can then find the form through

$$d\Omega = \frac{df_0}{f_0} \frac{df_1}{f_1} \frac{df_3}{f_3} \frac{df_4}{f_4} \delta(C \cdot \tilde{\lambda}) \delta(C^\perp \cdot \lambda) \delta(C \cdot \tilde{\eta}) \quad (3.20)$$

First let us look at the delta-functions, such that we can specify the face-variables in terms of the spinor products. We start by looking at $\lambda \cdot C^\perp = 0$, from which we can two equations

$$C^\perp \cdot \lambda = 0 \Rightarrow \begin{cases} -\lambda_1 f_0 f_3 f_4 + \lambda_2 f_0 f_1 f_3 f_4 + \lambda_3 & = 0 \\ -\lambda_1 f_4(1 - f_0) - \lambda_2 f_0 f_1 f_4 + \lambda_4 & = 0 \end{cases} \quad (3.21)$$

By multiplying the first equation by $\tilde{\lambda}_2$ one obtains $f_0 f_3 f_4 = -\frac{\langle 23 \rangle}{\langle 12 \rangle}$. Similarly multiplying the second equation by $\tilde{\lambda}_1$ we get $f_0 f_1 f_4 = \frac{\langle 14 \rangle}{\langle 12 \rangle}$. Combining these two,

$$f_1 = -\frac{\langle 14 \rangle}{\langle 23 \rangle} f_3 \quad (3.22)$$

Then multiplying the first equation by $\tilde{\lambda}_1$ we have $f_0 f_1 f_3 f_4 = -\frac{\langle 13 \rangle}{\langle 12 \rangle}$ together with the previous result, this leads to

$$f_3 = -\frac{\langle 13 \rangle}{\langle 14 \rangle} \quad \text{and} \quad f_1 = \frac{\langle 13 \rangle}{\langle 23 \rangle} \quad (3.23)$$

The other equations are solved similarly and we obtain

$$\begin{aligned} f_0 &= -\frac{\langle 14 \rangle \langle 23 \rangle}{\langle 12 \rangle \langle 34 \rangle} \\ f_4 &= -\frac{\langle 34 \rangle}{\langle 13 \rangle} \end{aligned} \quad (3.24)$$

Let us now evaluate the two remaining delta-functions. From $C \cdot \tilde{\lambda}$ we get two equations.

$$0 = \tilde{\lambda}_1 + f_0 f_3 f_4 \tilde{\lambda}_3 + f_4(1 - f_0) \tilde{\lambda}_4 = \tilde{\lambda}_1 + \frac{\langle 32 \rangle}{\langle 12 \rangle} \tilde{\lambda}_3 + \frac{\langle 42 \rangle}{\langle 12 \rangle} \tilde{\lambda}_4 \quad (3.25)$$

and

$$0 = \tilde{\lambda}_2 + \frac{\langle 13 \rangle}{\langle 12 \rangle} \tilde{\lambda}_3 + \frac{\langle 14 \rangle}{\langle 12 \rangle} \tilde{\lambda}_4 \quad (3.26)$$

where we have used a Schouten identity for the coefficient of $\tilde{\lambda}_4$

$$\langle 41 \rangle \langle 23 \rangle + \langle 12 \rangle \langle 34 \rangle = \langle 13 \rangle \langle 24 \rangle \quad (3.27)$$

We see that these equations can all be obtained from a momentum conservation delta-function by contracting it with λ_1 and λ_2

$$\delta^4(\lambda_1 \tilde{\lambda}_1 + \lambda_2 \tilde{\lambda}_2 + \lambda_3 \tilde{\lambda}_3 + \lambda_4 \tilde{\lambda}_4) \equiv \delta^4(P) \quad (3.28)$$

For the last delta-function we get the exact same thing except for replacing $\tilde{\lambda}_i \rightarrow \tilde{\eta}_i$

$$\delta^8(\lambda_1 \tilde{\eta}_1 + \lambda_2 \tilde{\eta}_2 + \lambda_3 \tilde{\eta}_3 + \lambda_4 \tilde{\eta}_4) \equiv \delta^8(Q) \quad (3.29)$$

Note that we get an extra factor of $\frac{1}{\langle 12 \rangle^4}$ from re-writing the delta-functions by projecting along λ_1 and λ_2 . Finally we get a Jacobian.

$$J = |J_{ij}| = f_0^2 f_1 f_3 f_4^3 = \frac{\langle 23 \rangle \langle 34 \rangle \langle 41 \rangle}{\langle 12 \rangle^2 \langle 13 \rangle} \quad (3.30)$$

where

$$J_{ij} = \frac{\partial E_i}{\partial f_j} = \begin{pmatrix} f_3 f_3 & 0 & f_0 f_3 & f_0 f_4 \\ f_1 f_3 f_4 & f_0 f_3 f_4 & f_0 f_1 f_4 & f_0 f_1 f_3 \\ f_4 & 0 & 0 & 1 - f_0 \\ f_1 f_4 & f_0 f_4 & 0 & f_0 f_1 \end{pmatrix} \quad (3.31)$$

and

$$E_1 = f_0 f_3 f_4, \quad E_2 = f_0 f_1 f_3 f_4, \quad E_3 = f_4(1 - f_0), \quad E_4 = f_0 f_1 f_3 \quad (3.32)$$

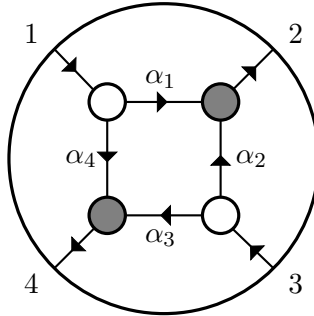
Now using

$$f_0 f_1 f_3 f_4 = \frac{\langle 13 \rangle}{\langle 12 \rangle} \quad (3.33)$$

We can put it all together to obtain the form

$$d\Omega = \frac{\langle 12 \rangle}{\langle 13 \rangle} \times \frac{\langle 12 \rangle^2 \langle 13 \rangle}{\langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} \times \frac{1}{\langle 12 \rangle^4} \times \delta^4(P) \delta^8(Q) = \frac{\delta^8(Q) \delta^4(P)}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} \quad (3.34)$$

For the edge-variable case let us try a different orientation



Here the C -matrix is now giving by

$$C_{ab} = \sum_{\Gamma(a \rightarrow b)} \prod_j \alpha_j \quad (3.35)$$

SO we get the following

$$C = \begin{pmatrix} 1 & \alpha_1 & 0 & \alpha_4 \\ 0 & \alpha_2 & 1 & \alpha_3 \end{pmatrix} \quad (3.36)$$

With the inverse

$$C^\perp = \begin{pmatrix} -\alpha_1 & 1 & -\alpha_2 & 0 \\ -\alpha_4 & 0 & -\alpha_3 & 1 \end{pmatrix} \quad (3.37)$$

$$C^\perp \cdot \lambda = 0 \Rightarrow \begin{cases} -\alpha_1 \lambda_1 + \lambda_2 - \alpha_3 \lambda_3 & = 0 \\ -\alpha_4 \lambda_1 - \alpha_2 \lambda_3 + \lambda_4 & = 0 \end{cases} \quad (3.38)$$

turns into

$$\begin{aligned} \langle 21 \rangle - \alpha_2 \langle 31 \rangle &= 0 \Rightarrow \alpha_2 = \frac{\langle 12 \rangle}{\langle 13 \rangle} \\ \alpha_1 \langle 12 \rangle - \alpha_3 \langle 23 \rangle &= 0 \Rightarrow \alpha_1 = \alpha_2 \frac{\langle 23 \rangle}{\langle 12 \rangle} = \frac{\langle 23 \rangle}{\langle 13 \rangle} \end{aligned} \quad (3.39)$$

Similarly we find

$$\alpha_3 = \frac{\langle 14 \rangle}{\langle 13 \rangle}, \quad \alpha_4 = \frac{\langle 43 \rangle}{\langle 13 \rangle} \quad (3.40)$$

For the other delta functions $C \cdot \tilde{\lambda}$ gives us two equations. The first one is

$$\begin{aligned} 0 &= \tilde{\lambda}_1 + \alpha_2 \tilde{\lambda}_2 + \alpha_4 \tilde{\lambda}_4 = \tilde{\lambda}_1 + \frac{\langle 23 \rangle}{\langle 13 \rangle} \tilde{\lambda}_2 + \frac{\langle 43 \rangle}{\langle 13 \rangle} \tilde{\lambda}_4 \\ \Rightarrow 0 &= \langle 13 \rangle \tilde{\lambda}_1 + \langle 23 \rangle \tilde{\lambda}_2 + \langle 43 \rangle \tilde{\lambda}_4 \end{aligned} \quad (3.41)$$

While the second one is found similarly

$$0 = \langle 21 \rangle \tilde{\lambda}_2 + \langle 31 \rangle \tilde{\lambda}_2 + \langle 41 \rangle \tilde{\lambda}_4 \quad (3.42)$$

We see that the two equations can be obtained from a single momentum conservation equation by contracting with λ_3 and λ_1 respectively. I.e. we have

$$\delta^4(\lambda_1 \tilde{\lambda}_1 + \lambda_2 \tilde{\lambda}_2 + \lambda_3 \tilde{\lambda}_3 + \lambda_4 \tilde{\lambda}_4) \equiv \delta^4(P) \quad (3.43)$$

For the last delta-function we get the exact same thing except for replacing $\tilde{\lambda}_i \rightarrow \tilde{\eta}_i$

$$\delta^8(\lambda_1 \tilde{\eta}_1 + \lambda_2 \tilde{\eta}_2 + \lambda_3 \tilde{\eta}_3 + \lambda_4 \tilde{\eta}_4) \equiv \delta^8(\mathcal{Q}) \quad (3.44)$$

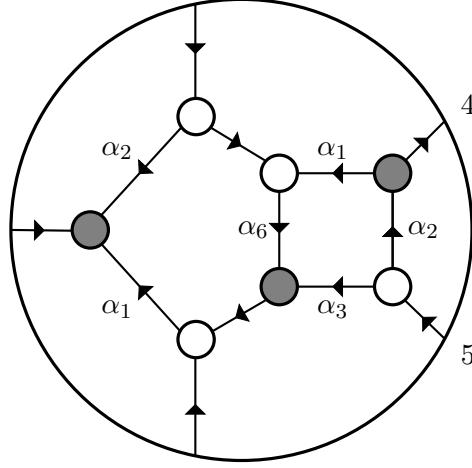
Note that we get an extra factor of $\frac{1}{\langle 13 \rangle^4}$ from re-writing the delta-functions in by projecting along λ_1 and λ_3 . Finally we have

$$\frac{1}{\alpha_1 \alpha_2 \alpha_3 \alpha_4} = \frac{\langle 13 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} \quad (3.45)$$

We can now calculate the form

$$d\Omega = \frac{\delta^8(\mathcal{Q}) \delta^4(P)}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} \quad (3.46)$$

3.3 Five point



Each vertex can fix one edge-variable, however you cannot do it in such a way that all variables in a vertex is fixed. The C matrix is

$$C = \begin{pmatrix} 1 & \alpha_1 + \alpha_2\alpha_6 & \alpha_6 & \alpha_3\alpha_6 & 0 \\ 0 & \alpha_5\alpha_6\alpha_2 & \alpha_5\alpha_6 & \alpha_4 + \alpha_3\alpha_5\alpha_6 & 1 \end{pmatrix} \quad (3.47)$$

with the inverse being

$$C^\perp = \begin{pmatrix} -(\alpha_1 + \alpha_2\alpha_6) & 1 & 0 & 0 & -\alpha_5\alpha_6\alpha_2 \\ -\alpha_6 & 0 & 1 & 0 & -\alpha_5\alpha_6 \\ -\alpha_3\alpha_6 & 0 & 0 & 1 & -(\alpha_4 + \alpha_3\alpha_5\alpha_6) \end{pmatrix} \quad (3.48)$$

The amplitude is found through

$$d\Omega = \frac{d\alpha_1}{\alpha_1} \frac{d\alpha_2}{\alpha_2} \frac{d\alpha_3}{\alpha_3} \frac{d\alpha_4}{\alpha_4} \frac{d\alpha_5}{\alpha_5} \frac{d\alpha_6}{\alpha_6} \delta^{2 \times 2}(C \cdot \tilde{\lambda}) \delta^{2 \times 3}(C^\perp \cdot \lambda) \delta^{4 \times 2}(C \cdot \tilde{\eta}) \quad (3.49)$$

Using the delta-function $\delta^{2 \times 3}(C^\perp \cdot \lambda)$ to solve for the α 's we obtain after contracting with λ_1 , λ_3 , and λ_5

$$\alpha_1 = \frac{\langle 23 \rangle}{\langle 13 \rangle}, \quad \alpha_2 = \frac{\langle 12 \rangle}{\langle 13 \rangle}, \quad \alpha_3 = \frac{\langle 45 \rangle}{\langle 35 \rangle}, \quad \alpha_4 = \frac{\langle 34 \rangle}{\langle 34 \rangle}, \quad \alpha_5 = \frac{\langle 13 \rangle}{\langle 35 \rangle}, \quad \alpha_6 = \frac{\langle 35 \rangle}{\langle 15 \rangle} \quad (3.50)$$

From which we see that we get a Jacobian of $\frac{1}{\langle 15 \rangle^2 \langle 13 \rangle}$. Plugging these α 's back into $(\delta^{2 \times 2} C \cdot \tilde{\lambda})$, we get

$$\begin{aligned} 0 &= \tilde{\lambda}_1 + \tilde{\lambda}_2 \frac{\langle 25 \rangle}{\langle 15 \rangle} + \tilde{\lambda}_3 \frac{\langle 35 \rangle}{\langle 15 \rangle} + \tilde{\lambda}_4 \frac{\langle 45 \rangle}{\langle 15 \rangle} \\ 0 &= \tilde{\lambda}_2 \frac{\langle 12 \rangle}{\langle 15 \rangle} + \tilde{\lambda}_3 \frac{\langle 13 \rangle}{\langle 15 \rangle} + \tilde{\lambda}_4 \frac{\langle 14 \rangle}{\langle 15 \rangle} + \tilde{\lambda}_5 \end{aligned} \quad (3.51)$$

where we have used Schouten identities on the $\tilde{\lambda}_2$ term in the first equation and $\tilde{\lambda}_4$ term in the second equation. We easily see that

$$\delta^{2 \times 2}(C \cdot \lambda) = \langle 15 \rangle^2 \delta^{2 \times 2}(P) \quad (3.52)$$

Then we plug the α 's into the Grassmann delta function. This will of course give a similar result with the exchange of $\tilde{\lambda} \rightarrow \tilde{\eta}$, although with the Jacobian factor now being $\frac{1}{\langle 15 \rangle^4}$. Finally we calculate

$$\prod_i \frac{1}{\alpha_i} = \frac{\langle 13 \rangle \langle 15 \rangle^2 \langle 35 \rangle^2}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 51 \rangle} \quad (3.53)$$

We are now easily able to get the form

$$d\Omega = \frac{\delta^8(\mathcal{Q}) \delta^4(P)}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 51 \rangle} \quad (3.54)$$

Five-point through loop kinematics

We have the following relations among the helicity variables

$$\begin{aligned} \lambda_1 \propto \lambda_\ell \propto \lambda_{\ell-1}, \quad \lambda_3 \propto \lambda_q \propto \lambda_{q-3}, \quad \lambda_{q+4} \propto \lambda_{\ell-1-5} \propto \lambda_{-q+3+2+\ell} \\ \tilde{\lambda}_{\ell-1} \propto \tilde{\lambda}_5 \propto \tilde{\lambda}_{\ell-1-5}, \quad \tilde{\lambda}_4 \propto \tilde{\lambda}_q \propto \tilde{\lambda}_{q+4}, \quad \tilde{\lambda}_{q-3-2} \propto \tilde{\lambda}_{q-3-2-\ell} \propto \tilde{\lambda}_\ell \propto \tilde{\lambda}_2 \propto \tilde{\lambda}_{q-3} \end{aligned} \quad (3.55)$$

First we write the loop momentum in the box as

$$\begin{aligned} \ell &= \alpha \lambda_1 \tilde{\lambda}_2 = -\frac{[15]}{[25]} \lambda_1 \tilde{\lambda}_2 \\ \ell - 1 &= \lambda_1 \left(\tilde{\lambda}_1 + \alpha \tilde{\lambda}_2 \right) = \frac{\lambda_1}{[25]} \left(\tilde{\lambda}_1 [25] + \tilde{\lambda}_2 [15] \right) = -\frac{[12]}{[25]} \lambda_1 \tilde{\lambda}_5 \\ \ell - 1 - 5 &= \frac{\tilde{\lambda}_5}{[25]} ([21] \lambda_1 + [25] \lambda_5) = \frac{\langle \cdot | 1 + 5 | 2 \rangle}{[25]} \tilde{\lambda}_5 \end{aligned} \quad (3.56)$$

where we found $\alpha = -\frac{[15]}{[25]}$ through the on-shell condition

$$0 = (\ell - 1 - 5)^2 = \alpha \langle 15 \rangle [52] + \langle 15 \rangle [51] \quad (3.57)$$

For the pentagon loop momenta we have

$$\begin{aligned} q &= \beta \lambda_3 \tilde{\lambda}_4 = -\frac{[23]}{[24]} \lambda_3 \tilde{\lambda}_4 \\ q - 3 &= \frac{\lambda_3}{[24]} \left(\tilde{\lambda}_4 [23] + \tilde{\lambda}_3 [24] \right) = -\frac{[34]}{[24]} \lambda_3 \tilde{\lambda}_2 \\ q - 3 - 2 &= \frac{\tilde{\lambda}_2}{[24]} ([34] \lambda_3 + [24] \lambda_2) = \frac{\langle \cdot | 1 + 5 | 4 \rangle}{[24]} \tilde{\lambda}_2 \end{aligned} \quad (3.58)$$

Finally we have the momentum shared by the two loops

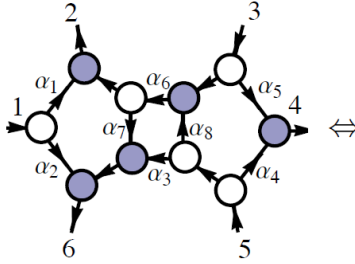
$$q - 3 - 2 - \ell = \frac{[45] \langle \cdot | 1 + 5 | 2 \rangle}{[25] [24]} \tilde{\lambda}_2 \quad (3.59)$$

From this we have the following subamplitudes

$$\begin{aligned}
A_4(q+4, 5, 1, q-3-2) &= \frac{1}{[q+4, 5] [51] [1, q-3-2] [q-3-2, q+4]} \\
&= \frac{1}{\frac{[45]}{[24]} [51] \frac{[12]}{[24]} \frac{[24]}{[24]^2}} \\
&= \frac{[24]^3}{[12] [45] [51]}
\end{aligned} \tag{3.60}$$

Six-point NMHV

We first look at the 4+4 diagram



$$\begin{aligned}
C &= \begin{pmatrix} 1 & \alpha_1 & 0 & 0 & 0 & \alpha_2 \\ 0 & \alpha_6 & 1 & \alpha_5 & 0 & \alpha_6 \alpha_7 \\ 0 & \alpha_6 \alpha_8 & 0 & \alpha_4 & 1 & \alpha_3 + \alpha_6 \alpha_7 \alpha_8 \end{pmatrix} \\
C^\perp &= \begin{pmatrix} -\alpha_1 & 1 & -\alpha_6 & 0 & -\alpha_6 \alpha_8 & 0 \\ 0 & 0 & -\alpha_5 & 1 & -\alpha_4 & 0 \\ -\alpha_2 & 0 & -\alpha_6 \alpha_7 & 0 & -(\alpha_3 + \alpha_6 \alpha_7 \alpha_8) & 1 \end{pmatrix}.
\end{aligned}$$

Figure 1.

We then use the following combination of equation from the $C \cdot \tilde{\lambda}$ and $C_\perp \cdot \lambda$ delta functions

$$\begin{aligned}
0 &= -\langle 34 \rangle + \alpha_4 \langle 35 \rangle \\
0 &= -(\alpha_5 \langle 35 \rangle) + \langle 45 \rangle \\
0 &= [12] - \alpha_2 [26] \\
0 &= [16] + \alpha_1 [26] \\
0 &= -[23] - \alpha_5 [24] - \alpha_6 \alpha_7 [26] \\
0 &= \alpha_6 [26] + [36] + \alpha_5 [46] \\
0 &= -(\alpha_4 [24]) - [25] - (\alpha_3 + \alpha_6 \alpha_7 \alpha_8) [26] \\
0 &= \alpha_6 \alpha_8 [26] + \alpha_4 [46] + [56]
\end{aligned} \tag{3.61}$$

to obtain

$$\begin{aligned}
\alpha_1 &= -\frac{[16]}{[26]}, \quad \alpha_2 = \frac{[12]}{[26]}, \quad \alpha_3 = \frac{s_{345}}{\langle 5|Q_{345}|6 \rangle}, \quad \alpha_4 = \frac{\langle 34 \rangle}{\langle 35 \rangle}, \quad \alpha_5 = \frac{\langle 45 \rangle}{\langle 35 \rangle}, \\
\alpha_6 &= \frac{\langle 5|Q_{345}|6 \rangle}{\langle 35 \rangle [26]}, \quad \alpha_7 = -\frac{\langle 5|Q_{345}|2 \rangle}{\langle 5|Q_{345}|6 \rangle}, \quad \alpha_8 = -\frac{\langle 3|Q_{345}|6 \rangle}{\langle 5|Q_{345}|6 \rangle}.
\end{aligned}$$

where $Q_{ijk} = p_i + p_j + p_k$ For the other delta functions we get

$$\begin{aligned}
0 &= \tilde{\eta}_1 - \frac{[16]}{[26]} \tilde{\eta}_2 + \frac{[12]}{[26]} \tilde{\eta}_6 \\
0 &= \frac{\langle 5|Q_{345}|6\rangle}{\langle 35\rangle [26]} \tilde{\eta}_2 + \tilde{\eta}_3 + \frac{\langle 45\rangle}{\langle 35\rangle} \tilde{\eta}_4 - \frac{\langle 5|Q_{345}|2\rangle}{\langle 35\rangle [26]} \tilde{\eta}_6 \\
0 &= -\frac{\langle 3|Q_{45}|6\rangle}{\langle 35\rangle [26]} \tilde{\eta}_2 + \frac{\langle 34\rangle}{\langle 35\rangle} \tilde{\eta}_4 + \tilde{\eta}_5 + \frac{s_{345} \langle 35\rangle [26] + \langle 3|Q_{345}|6\rangle \langle 5|Q_{345}|2\rangle}{\langle 5|Q_{345}|6\rangle} \tilde{\eta}_5
\end{aligned} \tag{3.62}$$

The Jacobian from the delta functions is

$$J = \frac{1}{[26]^3 \langle 35\rangle^3 \langle 5|Q_{345}|6\rangle^2} \tag{3.63}$$

and we get the form

$$d\Omega = \frac{\delta(\sum P) \delta(\tilde{\eta}_1 [26] + \tilde{\eta}_2 [61] + \tilde{\eta}_6 [12])}{s_{345} \langle 34\rangle \langle 45\rangle [12] [16] \langle 3|Q_{345}|6\rangle \langle 5|Q_{345}|2\rangle} \tag{3.64}$$

We later realized that we changed the bcfw bridge scheme in making the next diagram. To match the schemes we permute all labels in this form by 1

$$d\Omega_1 = \frac{\delta(\sum P) \delta(\tilde{\eta}_5 [16] + \tilde{\eta}_6 [15] + \tilde{\eta}_1 [56])}{s_{234} \langle 23\rangle \langle 34\rangle [61] [56] \langle 2|Q_{234}|5\rangle \langle 4|Q_{234}|1\rangle} \tag{3.65}$$

Looking at the 5+3 diagram we use the following C-matrix

$$\begin{aligned}
C &= \begin{pmatrix} \alpha_2 & \alpha_3 + \alpha_4 & 1 & 0 & 0 & 0 \\ \alpha_2 \alpha_5 & \alpha_3 \alpha_5 & 0 & 1 & \alpha_6 & 0 \\ \alpha_8(\alpha_1 + \alpha_2 \alpha_5) & \alpha_3 \alpha_5 \alpha_8 & 0 & 0 & \alpha_7 & 1 \end{pmatrix} \\
C_\perp &= \begin{pmatrix} 1 & 0 & -\alpha_2 & -\alpha_2 \alpha_5 & 0 & -\alpha_8(\alpha_1 + \alpha_2 \alpha_5) \\ 0 & 1 & -\alpha_3 - \alpha_4 & -\alpha_3 \alpha_5 & 0 & -\alpha_3 \alpha_5 \alpha_8 \\ 0 & 0 & 0 & -\alpha_6 & 1 & -\alpha_7 \end{pmatrix}
\end{aligned} \tag{3.66}$$

From the delta functions $\delta(C \cdot \tilde{\lambda})$ and $\delta(C_\perp \cdot \lambda)$, we use the following equations

$$\begin{aligned}
0 &= \langle 26 \rangle - (\alpha_3 + \alpha_4) \langle 36 \rangle - \alpha_3 \alpha_5 \langle 46 \rangle \\
0 &= -\langle 45 \rangle + \alpha_7 \langle 46 \rangle \\
0 &= -\alpha_6 \langle 46 \rangle + \langle 56 \rangle \\
0 &= -(\alpha_3 + \alpha_4) [12] - [13] \\
0 &= \alpha_2 [12] - [23] \\
0 &= \alpha_2 \alpha_5 [12] - [24] - \alpha_6 [25] \\
0 &= -\alpha_3 \alpha_5 \alpha_8 [12] - \alpha_7 [15] - [16] \\
0 &= (\alpha_1 + \alpha_2 \alpha_5) \alpha_8 [12] - \alpha_7 [25] - [26]
\end{aligned} \tag{3.67}$$

to obtain solutions for the edge-variables

$$\begin{aligned}\alpha_1 &= \frac{s_{456}}{\langle 4|Q_{456}|1\rangle}, & \alpha_2 &= \frac{[23]}{[12]}, & \alpha_3 &= \frac{[23]\langle 6|Q_{456}|1\rangle}{[12]\langle 6|Q_{456}|2\rangle}, & \alpha_4 &= \frac{\langle 6|Q_{456}|3\rangle}{\langle 6|Q_{456}|2\rangle}, \\ \alpha_5 &= \frac{\langle 6|Q_{456}|2\rangle}{\langle 46\rangle[23]}, & \alpha_6 &= \frac{\langle 56\rangle}{\langle 46\rangle}, & \alpha_7 &= \frac{\langle 45\rangle}{\langle 46\rangle}, & \alpha_8 &= \frac{\langle 4|Q_{456}|1\rangle}{\langle 6|Q_{456}|1\rangle}.\end{aligned}\quad (3.68)$$

Here the Jacobian from the delta functions is

$$J = \frac{1}{[12]^3 \langle 46\rangle^3 \langle 6|Q_{456}|2\rangle^2} \quad (3.69)$$

and we get the form

$$d\Omega = \frac{\delta(\sum P)\delta(\tilde{\eta}_1[26] + \tilde{\eta}_2[61] + \tilde{\eta}_3[12])}{s_{456} \langle 45\rangle \langle 56\rangle [12][23]\langle 4|Q_{456}|1\rangle\langle 6|Q_{456}|3\rangle} \quad (3.70)$$

Again we have to permute by 2 to obtain the correct recursion scheme

$$d\Omega_2 = \frac{\delta(\sum P)\delta(\tilde{\eta}_3[45] + \tilde{\eta}_4[35] + \tilde{\eta}_5[34])}{s_{612} \langle 12\rangle \langle 16\rangle [34][35]\langle 6|Q_{612}|3\rangle\langle 2|Q_{612}|5\rangle} \quad (3.71)$$

The final diagram can be found by permuting by 2 and exchanging square and angle brackets while permuting by 1:

$$d\Omega_3 = \frac{\delta(\sum P)\delta(\tilde{\eta}_1[23] + \tilde{\eta}_2[13] + \tilde{\eta}_3[12])}{s_{456} \langle 45\rangle \langle 56\rangle [23][12]\langle 4|Q_{456}|1\rangle\langle 6|Q_{456}|3\rangle} \quad (3.72)$$

In total we have

$$\begin{aligned}& d\Omega_1 + d\Omega_2 + d\Omega_3 \\ &= \frac{\delta(\sum P)\delta(\tilde{\eta}_5[16] + \tilde{\eta}_6[15] + \tilde{\eta}_1[56])}{s_{234} \langle 23\rangle \langle 34\rangle [61][56]\langle 2|Q_{234}|5\rangle\langle 4|Q_{234}|1\rangle} + \frac{\delta(\sum P)\delta(\tilde{\eta}_3[45] + \tilde{\eta}_4[35] + \tilde{\eta}_5[34])}{s_{612} \langle 12\rangle \langle 16\rangle [34][35]\langle 6|Q_{612}|3\rangle\langle 2|Q_{612}|5\rangle} \\ &+ \frac{\delta(\sum P)\delta(\tilde{\eta}_1[23] + \tilde{\eta}_2[13] + \tilde{\eta}_3[12])}{s_{456} \langle 45\rangle \langle 56\rangle [23][12]\langle 4|Q_{456}|1\rangle\langle 6|Q_{456}|3\rangle}\end{aligned}\quad (3.73)$$

Then using the fact that

$$\begin{aligned}\mathcal{A}^{(0),\text{MHV}} R_{j,j+3,j+5} &= \\ &= \frac{\delta^{(8)}(\sum \lambda_i \eta_i^A)}{\langle j(j+1)\rangle \langle (j+1)(j+2)\rangle [(j+3)(j+4)] [(j+4)(j+5)]} \\ &\times \frac{\delta^{(4)}(\eta_{j+3}^A [(j+4)(j+5)] + \eta_{j+4}^A [(j+5)(j+3)] + \eta_{j+5}^A [(j+3)(j+4)])}{\langle j|K_{j+1,j+2}|(j+3)\rangle \langle (j+2)|K_{j+3,j+4}|(j+5)\rangle s_{j,j+1,j+2}}.\end{aligned}\quad (3.74)$$

such that

$$A_6^{\text{NMHV}} = A_6^{\text{MHV}} (R_{251} + R_{413} + R_{635}) \quad (3.75)$$

3.4 2 loop four-point

We have the diagram

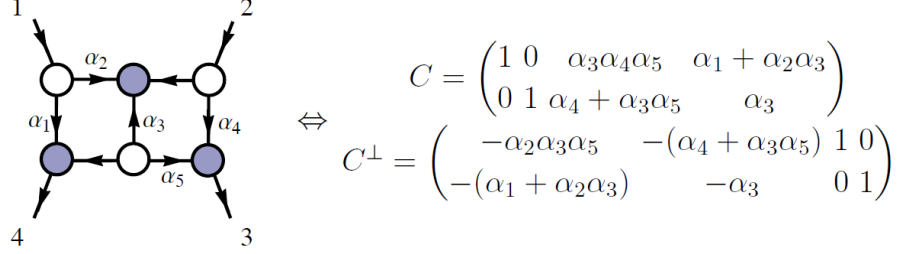


Figure 2.

Solving the bosonic delta-functions

$$\begin{aligned} 0 &= \alpha_4 \langle 12 \rangle + \alpha_3 \alpha_5 \langle 12 \rangle - \langle 13 \rangle \\ 0 &= \alpha_2 \alpha_3 \alpha_5 \langle 12 \rangle - \langle 23 \rangle \\ 0 &= \alpha_3 \langle 12 \rangle - \langle 14 \rangle \\ 0 &= -((\alpha_1 + \alpha_2 \alpha_3) \langle 12 \rangle) - \langle 24 \rangle \end{aligned} \tag{3.76}$$

we find

3.5 Calculating $\mathcal{N} < 4$ amplitudes

3.5.1 The measure

The important difference between this and the $\mathcal{N} = 4$ case is that the diagrams are necessarily oriented unlike in the maximally supersymmetric forms where the perfect orientations only played an auxiliary role for constructing the C-matrix. In addition, for perfect orientations with closed internal loops we have to add an extra factor, \mathcal{J} , in the measure,

$$d\Omega = \frac{d\alpha_1}{\alpha_1} \frac{d\alpha_2}{\alpha_2} \dots \frac{d\alpha_m}{\alpha_m} \mathcal{J}^{\mathcal{N}-4} \cdot \delta(C \cdot Z) \tag{3.77}$$

If there is a collection of closed orbits bounding "faces" f_i , with disjoint pairs (f_i, f_j) , disjoint triples (f_i, f_j, f_k) etc., then the Jacobian \mathcal{J} can be expressed as,

$$\mathcal{J} = 1 + \sum_i f_i + \sum_{\substack{\text{disjoint} \\ \text{pairs } i,j}} f_i f_j + \sum_{\substack{\text{disjoint} \\ \text{pairs } i,j,k}} f_i f_j f_k + \dots \tag{3.78}$$

First we note that each face is defined as clockwise-oriented product of edge-variables, such that a counter clockwise "face" f , gives a Jacobian contribution of f^{-1} . As an example see for instance the following diagram with four internal loops

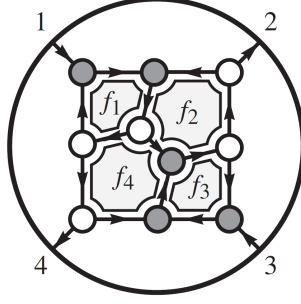


Figure 3.

Here we have three loops containing one phase variable, contributing each f_1, f_2^{-1}, f_3 . Further we have the disjoint pair $f_1 f_3$. Lastly we have a close loop containing both f_2 and f_4 , i.e. we have in total

$$\mathcal{J} = 1 + f_1 + f_3 + f_2^{-1} + f_2^{-1} f_4^{-1} + f_1 f_3 \quad (3.79)$$

3.5.2 4 pt $\mathcal{N} = 0$

In this case we have two diagrams

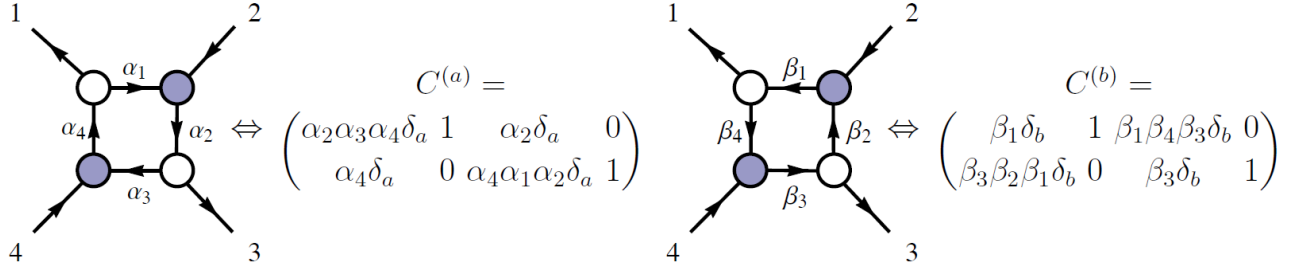


Figure 4.

where the δ 's are defined by

$$\delta_a = \frac{1}{1 - \prod_{i=1}^4 \alpha_i} \quad (3.80)$$

Solving the C_\perp delta-function, we get the following equations after contracting with λ_2 and

λ_4 (which gives a Jacobian of $\langle 24 \rangle^2$)

$$\begin{aligned}
0 &= \langle 12 \rangle + \frac{\alpha_4 \langle 24 \rangle}{1 - \alpha_1 \alpha_2 \alpha_3 \alpha_4} \\
0 &= \langle 14 \rangle - \frac{\alpha_2 \alpha_3 \alpha_4 \langle 24 \rangle}{1 - \alpha_1 \alpha_2 \alpha_3 \alpha_4} \\
0 &= -\langle 23 \rangle + \frac{\alpha_1 \alpha_2 \alpha_4 \langle 24 \rangle}{1 - \alpha_1 \alpha_2 \alpha_3 \alpha_4} \\
0 &= \langle 34 \rangle - \frac{\alpha_2 \langle 24 \rangle}{1 - \alpha_1 \alpha_2 \alpha_3 \alpha_4}
\end{aligned} \tag{3.81}$$

from which we obtain

$$\begin{aligned}
\alpha_1 &= -\frac{\langle 23 \rangle}{\langle 13 \rangle}, & \alpha_2 &= \frac{\langle 14 \rangle}{\langle 12 \rangle}, \\
\alpha_3 &= -\frac{\langle 14 \rangle}{\langle 13 \rangle}, & \alpha_4 &= -\frac{\langle 13 \rangle}{\langle 34 \rangle},
\end{aligned} \tag{3.82}$$

along with a Jacobian factor from rewriting the delta functions of $\frac{\langle 24 \rangle^2}{\langle 12 \rangle^2 \langle 34 \rangle^2}$. The procedure to find this factor can be found by eg the first line of (3.82), which after insertion of the $\delta_a = \frac{1}{1 - \alpha_1 \alpha_2 \alpha_3 \alpha_4}$ reads

$$0 = \langle 12 \rangle + \frac{\alpha_4 \langle 12 \rangle \langle 34 \rangle}{\langle 13 \rangle} \tag{3.83}$$

Since this is really a delta-function, we can write this as

$$\delta \left(\frac{\langle 34 \rangle \langle 12 \rangle}{\langle 13 \rangle} \left[\frac{\langle 13 \rangle}{\langle 34 \rangle} + \alpha_4 \right] \right) = \frac{\langle 13 \rangle}{\langle 34 \rangle \langle 12 \rangle} \delta \left(\frac{\langle 13 \rangle}{\langle 34 \rangle} + \alpha_4 \right) \tag{3.84}$$

i.e. the Jacobian from this factor is $\frac{\langle 13 \rangle}{\langle 34 \rangle \langle 12 \rangle}$. We can further find an expression for δ_a through this

$$\begin{aligned}
0 &= \langle 12 \rangle + \alpha_4 \langle 24 \rangle \delta_a \\
\Rightarrow \delta_a &= -\frac{1}{\alpha_4} \frac{\langle 12 \rangle}{\langle 24 \rangle} = \frac{\langle 12 \rangle \langle 34 \rangle}{\langle 13 \rangle \langle 24 \rangle}
\end{aligned} \tag{3.85}$$

Taking these solutions and inserting into the other delta functions, and multiplying by $\tilde{\lambda}_1$ and $\tilde{\lambda}_3$ we get

$$\begin{aligned}
0 &= [12] + \frac{\langle 34 \rangle [13]}{\langle 24 \rangle}, & 0 &= [23] + \frac{\langle 14 \rangle [13]}{\langle 24 \rangle} \\
0 &= [14] + \frac{\langle 23 \rangle [13]}{\langle 24 \rangle}, & 0 &= [34] + \frac{\langle 12 \rangle [13]}{\langle 24 \rangle}
\end{aligned} \tag{3.86}$$

Which we can write as

$$\begin{aligned}
0 &= \langle 42 \rangle [21] + \langle 43 \rangle [31] = \langle 4 | P | 1 \rangle, & 0 &= \langle 42 \rangle [23] + \langle 41 \rangle [13] = \langle 4 | P | 3 \rangle \\
0 &= \langle 24 \rangle [41] + \langle 23 \rangle [31] = \langle 2 | P | 1 \rangle, & 0 &= \langle 24 \rangle [43] + \langle 21 \rangle [13] = \langle 2 | P | 3 \rangle
\end{aligned} \tag{3.87}$$

Which we can combine into $\delta^4(P) \langle 24 \rangle^2$. Finally the Jacobian is

$$\mathcal{J}_a = 1 - \alpha_1 \alpha_2 \alpha_3 \alpha_4 = \delta_a^{-1} = \frac{\langle 13 \rangle \langle 24 \rangle}{\langle 12 \rangle \langle 34 \rangle} \tag{3.88}$$

The procedure is the same for the other diagram and here we just summarize the results

$$\begin{aligned}\beta_1 &= \frac{\langle 13 \rangle}{\langle 23 \rangle}, & \beta_2 &= \frac{\langle 21 \rangle}{\langle 13 \rangle}, \\ \beta_3 &= \frac{\langle 13 \rangle}{\langle 14 \rangle}, & \beta_4 &= \frac{\langle 34 \rangle}{\langle 13 \rangle},\end{aligned}\tag{3.89}$$

$$\mathcal{J}_b = 1 - \beta_1 \beta_2 \beta_3 \beta_4 = \delta_b^{-1} = \frac{\langle 13 \rangle \langle 24 \rangle}{\langle 14 \rangle \langle 23 \rangle}\tag{3.90}$$

Combining all these factors with $\prod_i \alpha_i = \frac{\langle 41 \rangle \langle 23 \rangle}{\langle 12 \rangle \langle 34 \rangle}$ and $\prod_i \beta_i = \frac{\langle 12 \rangle \langle 34 \rangle}{\langle 23 \rangle \langle 41 \rangle}$, as well as using taking the Jacobians to the -4 'th power, we have the form

$$\begin{aligned}d\Omega &= \left(\frac{\langle 24 \rangle^4}{\langle 12 \rangle^2 \langle 34 \rangle^2} \frac{\langle 12 \rangle \langle 34 \rangle}{\langle 41 \rangle \langle 23 \rangle} \left[\frac{\langle 13 \rangle \langle 24 \rangle}{\langle 12 \rangle \langle 34 \rangle} \right]^{-4} + \frac{\langle 24 \rangle^4}{\langle 14 \rangle^2 \langle 23 \rangle^2} \frac{\langle 23 \rangle \langle 41 \rangle}{\langle 12 \rangle \langle 34 \rangle} \left[\frac{\langle 13 \rangle \langle 24 \rangle}{\langle 14 \rangle \langle 23 \rangle} \right]^{-4} \right) \delta(P) \\ &= \frac{\langle 12 \rangle^4 \langle 34 \rangle^4 + \langle 14 \rangle^4 \langle 23 \rangle^4}{\langle 12 \rangle \langle 13 \rangle^4 \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle^4}\end{aligned}\tag{3.91}$$

3.5.3 Five-point $\mathcal{N} = 0$

We will once again look at the following diagram

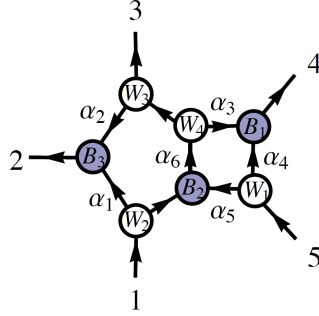


Figure 5.

We already solved this in section, here we obtained the following edgevariables after contracting with λ_1 , λ_3 , and λ_5

$$\alpha_1 = \frac{\langle 23 \rangle}{\langle 13 \rangle}, \quad \alpha_2 = \frac{\langle 12 \rangle}{\langle 13 \rangle}, \quad \alpha_3 = \frac{\langle 45 \rangle}{\langle 35 \rangle}, \quad \alpha_4 = \frac{\langle 34 \rangle}{\langle 34 \rangle}, \quad \alpha_5 = \frac{\langle 13 \rangle}{\langle 35 \rangle}, \quad \alpha_6 = \frac{\langle 35 \rangle}{\langle 15 \rangle}\tag{3.92}$$

and a Jacobian of $\frac{\langle 15 \rangle^2}{\langle 35 \rangle^2 \langle 13 \rangle}$. For this diagram we have two negative helicity particles 1 and 5 which is seen from the incoming arrows at those points. The orientation of these external legs do not yield an internal orientation with a closed cycle, and so this is the only diagram we have to take into account and there is no extra Jacobian factor. Using $\prod_i \alpha_i = \frac{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle}{\langle 13 \rangle \langle 15 \rangle \langle 35 \rangle^2}$

The form then yields the amplitude

$$\begin{aligned}
A_5(1^-, 2^+, 3^+, 4^+, 5^-) &= \frac{\langle 15 \rangle^2}{\langle 35 \rangle^2 \langle 13 \rangle} \frac{\delta(P)}{\prod_i \alpha_i} \\
&= \frac{\langle 15 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 51 \rangle} \delta(P)
\end{aligned} \tag{3.93}$$

3.5.4 Six-point $\mathcal{N} = 0$

Here we start by treating the same diagram as we did in section ...

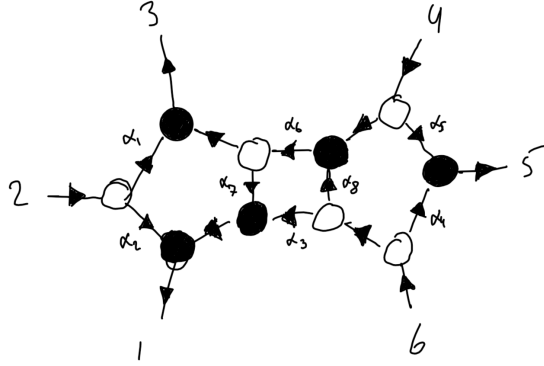


Figure 6.

where we obtained

$$\begin{aligned}
\alpha_1 &= -\frac{[65]}{[15]}, \quad \alpha_2 = \frac{[61]}{[15]}, \quad \alpha_3 = \frac{s_{234}}{\langle 4|Q_{234}|5]}, \quad \alpha_4 = \frac{\langle 23 \rangle}{\langle 24 \rangle}, \quad \alpha_5 = \frac{\langle 34 \rangle}{\langle 24 \rangle}, \\
\alpha_6 &= \frac{\langle 4|Q_{234}|5]}{\langle 24 \rangle [15]}, \quad \alpha_7 = -\frac{\langle 4|Q_{234}|1]}{\langle 4|Q_{234}|5]}, \quad \alpha_8 = -\frac{\langle 2|Q_{234}|5]}{\langle 4|Q_{234}|5]}.
\end{aligned}$$

The Jacobian from the delta functions is

$$J = \frac{[15] \langle 24 \rangle}{\langle 4|Q_{234}|5]^2} \tag{3.94}$$

such that the form is

$$d\Omega_{4+4} = \frac{\langle 24 \rangle^4 [15]^4}{s_{234} \langle 23 \rangle \langle 34 \rangle \langle 2|Q_{234}|5] \langle 4|Q_{234}|1] [61] [56]} \tag{3.95}$$

For the 5+3 diagram we once again use the results from the previous section. The diagram is

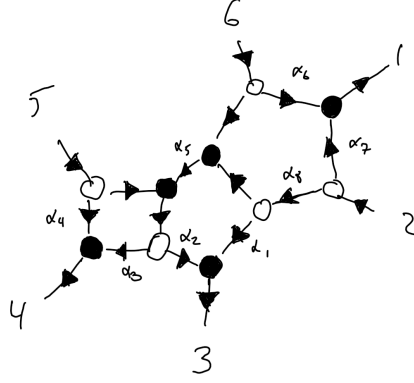


Figure 7.

With the following edgevariables

$$\begin{aligned} \alpha_1 &= \frac{s_{612}}{\langle 6|Q_{612}|3\rangle}, & \alpha_2 &= \frac{[45]}{[34]}, & \alpha_3 &= \frac{[45]\langle 2|Q_{612}|3\rangle}{[34]\langle 2|Q_{612}|4\rangle}, & \alpha_4 &= \frac{\langle 2|Q_{612}|5\rangle}{\langle 2|Q_{612}|4\rangle}, \\ \alpha_5 &= \frac{\langle 2|Q_{612}|4\rangle}{\langle 62\rangle[45]}, & \alpha_6 &= \frac{\langle 12\rangle}{\langle 62\rangle}, & \alpha_7 &= \frac{\langle 61\rangle}{\langle 62\rangle}, & \alpha_8 &= \frac{\langle 6|Q_{612}|3\rangle}{\langle 2|Q_{612}|3\rangle}. \end{aligned} \quad (3.96)$$

Here the Jacobian from the delta functions is

$$J = \frac{[35]^4 \langle 62\rangle}{[34]^3 \langle 2|Q_{612}|4\rangle \langle 6|Q_{612}|3\rangle} \quad (3.97)$$

The form is

$$d\Omega_{5+3} = \frac{\langle 26\rangle^4 [35]^4}{s_{612} \langle 12\rangle \langle 16\rangle \langle 6|Q_{612}|3\rangle \langle 2|Q_{612}|5\rangle [34][45]} \quad (3.98)$$

3.5.5 Jacobian from solving delta-function

The following property holds for the δ -functions

$$\delta(kx) = \frac{1}{k} \delta(x) \quad (3.99)$$

which means that for instance when multiplying a deltafunction by a spinor one has

$$\delta(\lambda_i) = \lambda_j \delta(\lambda_j \lambda_i) \quad (3.100)$$

Or, using spinor helicity bracket notation

$$\delta(\lambda_i) \delta(\lambda_j) = \langle kl \rangle \delta(\lambda_k \lambda_i) \delta(\lambda_l \lambda_j) \quad (3.101)$$

3.6 One loop five-point in $\mathcal{N} = 0$

One might also consider diagrams with an extra internal parameter. At five-point an example of this is shown below

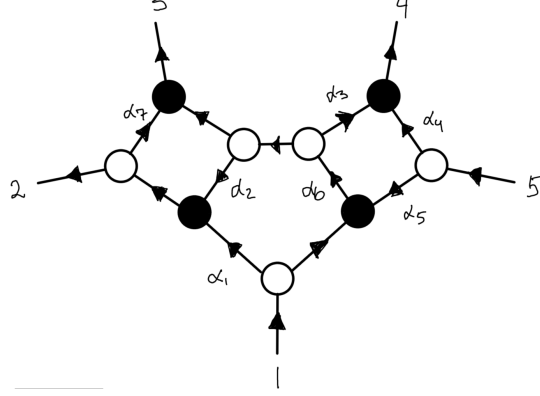


Figure 8.

Such a diagram can be thought of as a loop-level diagram, since the on-shell condition imposed on the three loop momenta leaves us with $3 \times 4 - 11 = 1$ unfixed parameter. The C-matrices are

$$C = \begin{pmatrix} 1 & \alpha_1 + \alpha_2\alpha_6 & \alpha_2\alpha_7\alpha_6 + \alpha_6 + \alpha_1\alpha_7 & \alpha_3\alpha_6 & 0 \\ 0 & \alpha_2\alpha_5\alpha_6 & \alpha_5\alpha_6(\alpha_2\alpha_7 + 1) & \alpha_4 + \alpha_3\alpha_5\alpha_6 & 1 \end{pmatrix}$$

$$C_\perp = \begin{pmatrix} -\alpha_1 - \alpha_2\alpha_6 & 1 & 0 & 0 & -\alpha_2\alpha_5\alpha_6 \\ -\alpha_6 - (\alpha_1 + \alpha_2\alpha_6)\alpha_7 & 0 & 1 & 0 & -\alpha_5\alpha_6(\alpha_2\alpha_7 + 1) \\ -\alpha_3\alpha_6 & 0 & 0 & 1 & -\alpha_4 - \alpha_3\alpha_5\alpha_6 \end{pmatrix} \quad (3.102)$$

$$\delta^6(C_\perp \cdot \lambda) \rightarrow \begin{cases} 0 & = -\langle 12 \rangle + \alpha_2\alpha_5\alpha_6\langle 15 \rangle \\ 0 & = -((\alpha_1 + \alpha_2\alpha_6)\langle 15 \rangle) + \langle 25 \rangle \\ 0 & = -\langle 13 \rangle + \alpha_5\alpha_6(1 + \alpha_2\alpha_7)\langle 15 \rangle \\ 0 & = -((\alpha_6 + \alpha_1\alpha_7 + \alpha_2\alpha_6\alpha_7)\langle 15 \rangle) + \langle 35 \rangle \\ 0 & = -\langle 14 \rangle + (\alpha_4 + \alpha_3\alpha_5\alpha_6)\langle 15 \rangle \\ 0 & = -(\alpha_3\alpha_6\langle 15 \rangle) + \langle 45 \rangle \end{cases} \quad (3.103)$$

Solving the delta-functions we find

$$\begin{aligned} \alpha_1 &= -\frac{\langle 23 \rangle}{\alpha_7\langle 12 \rangle - \langle 13 \rangle}, & \alpha_2 &= -\frac{\langle 12 \rangle}{\alpha_7\langle 12 \rangle - \langle 13 \rangle}, & \alpha_3 &= -\frac{\langle 45 \rangle}{\alpha_7\langle 25 \rangle - \langle 35 \rangle} \\ \alpha_4 &= -\frac{\alpha_7\langle 24 \rangle - \langle 34 \rangle}{\alpha_7\langle 25 \rangle - \langle 35 \rangle}, & \alpha_5 &= \frac{\alpha_7\langle 12 \rangle - \langle 13 \rangle}{\alpha_7\langle 25 \rangle - \langle 35 \rangle}, & \alpha_6 &= -\frac{\alpha_7\langle 25 \rangle - \langle 35 \rangle}{\langle 15 \rangle} \end{aligned} \quad (3.104)$$

Such that

$$\begin{aligned}
& \delta^4(C_\perp \cdot \lambda) \delta^6(C_\perp \cdot \lambda) \\
&= \frac{\langle 15 \rangle^2 (\alpha_7 \langle 12 \rangle - \langle 13 \rangle)}{\langle 13 \rangle^2 (\alpha_7 \langle 25 \rangle - \langle 35 \rangle)^2} \delta^4(P) \delta \left(\alpha_1 + \frac{\langle 23 \rangle}{\alpha_7 \langle 12 \rangle - \langle 13 \rangle} \right) \delta \left(\alpha_2 + \frac{\langle 12 \rangle}{\alpha_7 \langle 12 \rangle - \langle 13 \rangle} \right) \\
& \delta \left(\alpha_3 + \frac{\langle 45 \rangle}{\alpha_7 \langle 25 \rangle - \langle 35 \rangle} \right) \delta \left(\alpha_4 + \frac{\alpha_7 \langle 24 \rangle - \langle 34 \rangle}{\alpha_7 \langle 25 \rangle - \langle 35 \rangle} \right) \delta \left(\alpha_5 - \frac{\alpha_7 \langle 12 \rangle - \langle 13 \rangle}{\alpha_7 \langle 25 \rangle - \langle 35 \rangle} \right) \delta \left(\alpha_6 + \frac{\alpha_7 \langle 25 \rangle - \langle 35 \rangle}{\langle 15 \rangle} \right)
\end{aligned} \tag{3.105}$$

So that we get the form

$$d\Omega = \frac{(\alpha_7 \langle 12 \rangle - \langle 13 \rangle)^2 \langle 15 \rangle^3}{\alpha_7 \langle 12 \rangle \langle 13 \rangle^2 \langle 23 \rangle \langle 45 \rangle (\alpha_7 \langle 24 \rangle - \langle 34 \rangle)} \tag{3.106}$$

taking the residue of the form at $\alpha_7 = 0$ one recovers the standard YM Parke-Taylor tree-level amplitude

$$d\Omega = \frac{\langle 15 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 51 \rangle} \tag{3.107}$$

Further, we note that the form has one other pole at $\alpha_7 = \frac{\langle 34 \rangle}{\langle 24 \rangle}$.

3.7 One loop five-point in any \mathcal{N}

We will work on the following diagram.

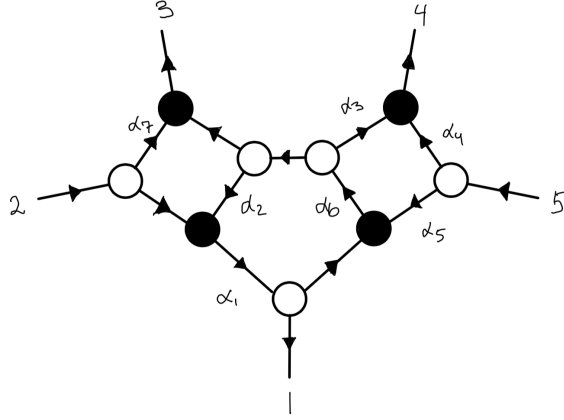


Figure 9.

The C-matrices obtained are

$$\begin{aligned}
C &= \begin{pmatrix} \alpha_1 \delta_a & 1 & \alpha_1 \alpha_6 \delta_a + \alpha_7 & \alpha_1 \alpha_3 \alpha_6 \delta_a & 0 \\ \alpha_1 \alpha_2 \alpha_5 \alpha_6 \delta_a & 0 & \alpha_5 \alpha_6 \delta_a & \alpha_3 \alpha_5 \alpha_6 \delta_a + \alpha_4 & 1 \end{pmatrix} \\
C_\perp &= \begin{pmatrix} 1 & -\alpha_1 \delta_a & 0 & 0 & -\alpha_1 \alpha_2 \alpha_5 \alpha_6 \delta_a \\ 0 & -\alpha_1 \alpha_6 \delta_a - \alpha_7 & 1 & 0 & -\alpha_5 \alpha_6 \delta_a \\ 0 & -\alpha_1 \alpha_3 \alpha_6 \delta_a & 0 & 1 & -\alpha_3 \alpha_5 \alpha_6 \delta_a - \alpha_4 \end{pmatrix},
\end{aligned} \tag{3.108}$$

where $\delta_a = \frac{1}{1-\alpha_1\alpha_2\alpha_6}$. The first delta-function constraint gives us

$$\delta^6(C_\perp \cdot \lambda) \rightarrow \begin{cases} 0 &= \langle 12 \rangle + \alpha_1\alpha_2\alpha_5\alpha_6\delta_a\langle 25 \rangle \\ 0 &= \langle 15 \rangle - \alpha_1\delta_a\langle 25 \rangle \\ 0 &= -\langle 23 \rangle + \alpha_5\alpha_6\delta_a\langle 25 \rangle \\ 0 &= (-\alpha_7 - \alpha_1\alpha_6\delta_a)\langle 25 \rangle + \langle 35 \rangle \\ 0 &= -\langle 24 \rangle - (-\alpha_4 - \alpha_3\alpha_5\alpha_6\delta_a)\langle 25 \rangle \\ 0 &= -(\alpha_1\alpha_3\alpha_6\delta_a\langle 25 \rangle) + \langle 45 \rangle \end{cases} \quad (3.109)$$

Solving the delta-functions we find

$$\begin{aligned} \alpha_1 &= -\frac{\alpha_7\langle 12 \rangle - \langle 13 \rangle}{\langle 23 \rangle}, & \alpha_2 &= -\frac{\langle 12 \rangle}{\alpha_7\langle 12 \rangle - \langle 13 \rangle}, & \alpha_3 &= -\frac{\langle 45 \rangle}{\alpha_7\langle 25 \rangle - \langle 35 \rangle} \\ \alpha_4 &= -\frac{\alpha_7\langle 24 \rangle - \langle 34 \rangle}{\alpha_7\langle 25 \rangle - \langle 35 \rangle}, & \alpha_5 &= \frac{\alpha_7\langle 12 \rangle - \langle 13 \rangle}{\alpha_7\langle 25 \rangle - \langle 35 \rangle}, & \alpha_6 &= -\frac{\alpha_7\langle 25 \rangle - \langle 35 \rangle}{\langle 15 \rangle} \\ \delta_a &= -\frac{\langle 15 \rangle\langle 23 \rangle}{\langle 25 \rangle(\alpha_7\langle 12 \rangle - \langle 13 \rangle)} \end{aligned} \quad (3.110)$$

$$\begin{aligned} &\delta^4(C_\perp \cdot \lambda)\delta^6(C_\perp \cdot \lambda) \\ &= \frac{\langle 15 \rangle^4 (\alpha_7\langle 12 \rangle - \langle 13 \rangle)}{\langle 15 \rangle^2 \langle 23 \rangle^2 (-\alpha_7\langle 25 \rangle + \langle 35 \rangle)^2} \delta^4(P) \delta\left(\alpha_1 + \frac{\alpha_7\langle 12 \rangle - \langle 13 \rangle}{\langle 23 \rangle}\right) \delta\left(\alpha_2 + \frac{\langle 12 \rangle}{\alpha_7\langle 12 \rangle - \langle 13 \rangle}\right) \\ &\quad \delta\left(\alpha_3 + \frac{\langle 45 \rangle}{\alpha_7\langle 25 \rangle - \langle 35 \rangle}\right) \delta\left(\alpha_4 + \frac{\alpha_7\langle 24 \rangle - \langle 34 \rangle}{\alpha_7\langle 25 \rangle - \langle 35 \rangle}\right) \delta\left(\alpha_5 - \frac{\alpha_7\langle 12 \rangle - \langle 13 \rangle}{\alpha_7\langle 25 \rangle - \langle 35 \rangle}\right) \delta\left(\alpha_6 + \frac{\alpha_7\langle 25 \rangle - \langle 35 \rangle}{\langle 15 \rangle}\right) \end{aligned} \quad (3.111)$$

So that we get the form

$$d\Omega = \frac{\langle 25 \rangle^4}{\alpha_7\langle 12 \rangle\langle 51 \rangle\langle 23 \rangle\langle 45 \rangle(\alpha_7\langle 24 \rangle - \langle 34 \rangle)} \quad (3.112)$$

taking the residue of the form at $\alpha_7 = 0$ one recovers the standard YM Parke-Taylor tree-level amplitude

$$d\Omega = \frac{\langle 15 \rangle^4}{\langle 12 \rangle\langle 23 \rangle\langle 34 \rangle\langle 45 \rangle\langle 51 \rangle} \quad (3.113)$$

Further, the $\mathcal{N} = 0$ Jacobian is

$$J_a = \delta_a^{-1} = \frac{\langle 25 \rangle (\alpha_7\langle 12 \rangle - \langle 13 \rangle)}{\langle 15 \rangle\langle 23 \rangle} \quad (3.114)$$

References

- [1] N. E. J. Bjerrum-Bohr, J. F. Donoghue and P. Vanhove, “On-shell Techniques and Universal Results in Quantum Gravity,” JHEP **02** (2014), 111 doi:10.1007/JHEP02(2014)111 [arXiv:1309.0804 [hep-th]].