On an MHV-like vertex expansion for gravity amplitudes

Johannes M. Sørensen

Niels Bohr Institute
University of Copenhagen

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Overview

Content of the thesis

- The spinor-helicity formalism
  - Helicity basis
  - Helicity spinors
  - Little group scaling and locality
- Recursion techniques
  - BCFW recursion
  - CSW expansion
- The KLT relations
- A modified all-line shift
  - Zeros of graviton amplitude
  - Bad/good all-line shifts
  - An MHV-like vertex expansion for gravitons

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Introducing the all-line shift

General shifts considered in this project are deformations

$$p_i \longrightarrow \hat{p}_i = p_i + z r_i,$$

which have to fulfil:

(i) $\sum_{i=1}^{n} r_i = 0$, (Momentum conservation).

(ii) $r_i \cdot r_j = 0$, for all i, j including $i = j$.

(iii) $p_i \cdot r_i = 0$, where there is no sum over i.

Introducing the all-line shift

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(iii) $p_i \cdot r_i = 0$, where there is no sum over i.

From [1] the all-line shift is defined

$$|i] \longrightarrow |\hat{i}] = |i] + z c_i |X] \tag{1}$$

where it is required that $\sum_i c_i |i\rangle = 0$ such that (i) is fulfilled. (ii) and (iii) follows from (1) being an only square spinor shift.

Bad all-line shifts

All-line shift can be chosen specifically such that given an arbitrary square spinor $|Z]$ of unit length it holds for all i that

$$c_i\,|Z] = P_Z\,|i]\,, \text{ where } P_Z \text{ is a projection operator.} \tag{2}$$

Now there are two cases:

$$\left.\begin{array}{l} \textit{1)}\ |Z] \text{ is orthogonal to } |X]. \\ \textit{2)}\ |Z] \text{ is parallel to } |X]. \end{array}\right\} \text{Both shift the amplitude trivially.}$$

$$\hat{P}_I^2(z) = P_I^2 - z\sum_i c_i\,\langle i|\,P_I\,|X] \tag{3}$$

Bad all-line shifts

If *1)* is true, it holds that $c_i = [Xi]$, because of antisymmetry of the spinor product. Therefore

$$P_I^2 = \hat{P}_I^2 + z \sum_{i \in I} c_i \langle i | P_I | X] = \hat{P}_I^2 + z \sum_{i,j \in I} [iX][jX] \langle ij \rangle = \hat{P}_I^2. \quad (4)$$

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If *2)* is true, all spinors are parallel at $z = -1$. Therefore

$$[\hat{i}\hat{j}] = [ij](z+1), \quad (5)$$

from which it follows that every subset of momenta fulfil

$$\hat{P}_I^2 = P_I^2 (z+1). \quad (6)$$

Good all-line shifts: Definition

To avoid bad all-line shift, require further that

$$z_\alpha = \frac{P_\alpha^2}{\sum_{i\in\alpha} c_i \langle i| P_\alpha |X]} \neq \frac{P_\beta^2}{\sum_{i\in\beta} c_i \langle i| P_\beta |X]} = z_\beta \quad \text{for} \quad \alpha \neq \beta. \quad (7)$$

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- Different subsets of momenta P_I are shifted differently.
- Finite number of conditions.
- Conditions are possible to fulfil.

Good all-line shifts: Existence

Note first that $\left[\hat{i}\hat{j}\right] = [ij]\,\frac{z_{ij}-z}{z_{ij}}$ is linear in z. Furthermore

$$\hat{P}_I^2 = \sum_{i,j\in I} \langle ij\rangle \left[\hat{i}\hat{j}\right] = \sum_{i,j\in I} \langle ij\rangle\,[ij]\,\frac{z_{ij}-z}{z_{ij}}. \tag{8}$$

Therefore, if every z_{ij} is chosen to be different, the condition (7) will be fulfilled for generic momenta.

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Process of choosing cs:

(1) Choose first two cs arbitrarily.

(2) Choose c_3 such that $z_{i3} \neq z_{12}$, where $i = 1, 2$.

(3) Continue until only three cs are left.

(4) Choose c_{n-2}, c_{n-1} and c_n such that momentum is conserved and (7) is fulfilled.

Zeros of graviton amplitudes

From the KLT relations one gets [9]

$$\mathcal{M}_n^{\mathsf{N}^k\mathsf{MHV}} = \sum_i \prod_{j=1}^{n-3} s_{ij} \mathcal{A}_{n\ i}^{\mathsf{N}^k\mathsf{MHV}} \tilde{\mathcal{A}}_{n\ i}^{\mathsf{N}^k\mathsf{MHV}}. \tag{9}$$

Undesired large z behavior for large n.

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Undesired large z behavior for large n.

At tree level the graviton amplitude is a rational function of helicity spinor products. Therefore

$$\hat{\mathcal{M}}_n^{\mathsf{N}^k\mathsf{MHV}} = \frac{Pol_N(z)}{Pol_M(z)}, \tag{10}$$

where $Pol_N(z)$ and $Pol_M(z)$ have no common roots. Therefore $\hat{\mathcal{M}}_n^{\mathsf{N}^k\mathsf{MHV}}$ has zeros $\{z_1, z_2, ..., z_N\}$.

Zeros of graviton amplitudes

Given the zeros, define the following

$$Z(z) = \prod_i (z - z_i).$$

Then the following holds

$$\frac{\hat{\mathcal{M}}_n^{\mathsf{N}^k\mathsf{MHV}}(z)}{Z(z)} = \mathcal{O}(z^{-M}). \tag{11}$$

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- Fit for recursion given that $M \geq 1$.
- $M \geq 1$ can be seen from the Feynman expansion. $\hat{\mathcal{M}}_n^{\mathsf{N}^k\mathsf{MHV}}$ has a pole whenever $\hat{P}_I^2 = 0$.

Recursion relation

Under complex deformation, the physical amplitude can be reconstructed by Cauchy's theorem

$$\frac{1}{2\pi i}\oint_{\mathbb{C}\text{ at }\infty}\frac{dz}{zZ(z)}\hat{\mathcal{M}}_n(z)=\mathrm{Res}\left(\frac{\hat{\mathcal{M}}_n(z)}{zZ(z)},z=\infty\right). \tag{12}$$

Given that $M\geq 1$ one concludes

$$0=\frac{1}{2\pi i}\oint_{\mathbb{C}\text{ at }\infty}\frac{dz}{zZ(z)}\hat{\mathcal{M}}_n(z)=\frac{\mathcal{M}_n}{Z(0)}+\sum_I\mathrm{Res}\left(\frac{\hat{\mathcal{M}}_n(z)}{zZ(z)},z=z_I\right),$$

The amplitude only has poles from propagators going on-shell, therefore

$$\mathcal{M}_n=\sum_I\frac{Z(0)}{Z(z_I)}\hat{\mathcal{M}}^L_{n_I}\frac{1}{P_I^2}\hat{\mathcal{M}}^R_{\overline{n_I}},\quad n_I+\overline{n_I}=n+2. \tag{13}$$

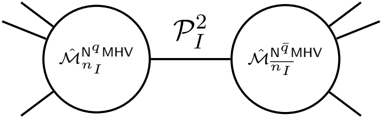
Recursion relation

Under all-line shift it holds that

$$\mathcal{M}_n(1^\pm, 2^+, 3^+, ..., n^+) = 0 \quad \text{and} \quad \hat{\mathcal{M}}_3^{\mathsf{AMHV}} = 0. \tag{14}$$

Therefore it must hold that

$$\mathcal{M}_n^{\mathsf{N}^k\mathsf{MHV}} = \sum_{\text{Diagrams } I}$$



$$, \qquad \bar{q} = k - q - 1,$$

where the propagator is

$$\mathcal{P}_I^2 \longrightarrow \frac{\prod_i \frac{z_i}{z_i - z_I}}{P_I^2}. \tag{15}$$

MHV-like vertex expansion

It now follows by induction that

$$\mathcal{M}_n^{\mathsf{N}^k\mathsf{MVH}} = \sum_{\substack{\text{MHV diag.}\\ \{\alpha_1,\alpha_2,\ldots,\alpha_k\}}} \prod_{i=1}^{k} k_{\alpha_i} \frac{\hat{\mathcal{M}}^{\mathsf{MHV}}(\alpha_1)\hat{\mathcal{M}}^{\mathsf{MHV}}(\alpha_2)\ldots\hat{\mathcal{M}}^{\mathsf{MHV}}(\alpha_{k+1})}{P_{\alpha_1}^2 P_{\alpha_2}^2 \ldots P_{\alpha_k}^2},$$

where $k_{\alpha_i} = \prod_j \frac{z_j}{z_j - z_{\alpha_i}}$.

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where $k_{\alpha_i} = \prod_j \frac{z_j}{z_j - z_{\alpha_i}}$.

- Gravity analogue to the CSW expansion
- It ties connections to other interesting subjects:
  - Scattering equations [10]
  - Twistor strings and Veronese polynomials [11]

Outlook

Extra relations from the large z fall off $\frac{\hat{\mathcal{M}}(z)}{Z(z)} = \mathcal{O}(z^{-M})$

$$0 = \frac{1}{2\pi i} \oint_{\mathbb{C} \text{ at } \infty} \frac{dz}{z} z^{\omega} \frac{\hat{\mathcal{M}}_n(z)}{Z(z)} = \sum_I \text{Res}\left(z^{\omega-1} \frac{\hat{\mathcal{M}}_n(z)}{Z(z)}, z = z_I \right),$$

where $\omega < M$.

This gives extra relations between residues and zeros of $\hat{\mathcal{M}}_n(z)$.

Interesting theoretical connections

- Scattering equations
- Other representations of the amplitude
- Twistor strings and Veronese polynomials

Outlook

Nummerical work so far

- 5-point NMHV amplitude
  - Zeros found nummerically
  - Process is relatively easy
  - Result agrees with Risager's expansion.

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- 5-point NMHV amplitude
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Future numerical work

- Zeros are to be found analytically
  - Study the explicit form of the zeros
  - Try to design more economic ways of calculating zeros.
- Higher point amplitudes
- Higher k amplitudes (N^kMHV)

Thank you for your attention.

References

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Further reading

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Induction argument

Assume that the following holds true for all $q < k$

$$\mathcal{M}_n^{\mathsf{N}^q\mathsf{MHV}} = \sum_{\substack{\text{MHV diagrams}\\ \{\alpha_1,\alpha_2,\ldots\alpha_q\}}} \prod_{i=1}^{q} k_{\alpha_i} \frac{\hat{\mathcal{M}}^{\mathsf{MHV}}(\alpha_1)\ldots\hat{\mathcal{M}}^{\mathsf{MHV}}(\alpha_{q+1})}{P_{\alpha_1}^2 P_{\alpha_2}^2 \ldots P_{\alpha_q}^2} \quad (16)$$

Now it is clear from the recursion relation that.

$$\mathcal{M}_n^{\mathsf{N}^k\mathsf{MVH}}(I) = \frac{1}{2}\sum_{\alpha} k_\alpha \left.\frac{\hat{\mathcal{M}}^{\mathsf{N}^q\mathsf{MHV}}(\beta)\hat{\mathcal{M}}^{\mathsf{N}^{k-q-1}\mathsf{MHV}}(\gamma)}{P_\alpha^2}\right|_{z=z_\alpha} \quad (17)$$

By combining these one obtains

$$\mathcal{M}_n^{\mathsf{N}^k\mathsf{MVH}} = \frac{1}{2}\sum_{\alpha}\sum_{\substack{\mathsf{MHV}\\ \{\beta_i,\gamma_j\}}} \prod_{i=1}^{p}\prod_{j=1}^{k-q-1} k_\alpha k_{\beta_i} k_{\gamma_j} \frac{\hat{\mathcal{M}}^{\mathsf{MHV}}(\beta_1)\ldots\hat{\mathcal{M}}^{\mathsf{MHV}}(\beta_{q+1})\hat{\mathcal{M}}^{\mathsf{MHV}}(\gamma_1)\ldots\hat{\mathcal{M}}^{\mathsf{MHV}}(\gamma_{k-q})}{\hat{P}_{\beta_1}^2(z_\alpha)\ldots\hat{P}_{\beta_q}^2(z_\alpha)P_\alpha^2\hat{P}_{\gamma_1}^2(z_\alpha)\ldots\hat{P}_{\gamma_{k-q-1}}^2(z_\alpha)} \quad (18)$$

Induction continued

By reindexing in the following way

$$\{\alpha_1, \alpha_2, ..., \alpha_k\} \equiv \{\alpha, \beta_1, \beta_2, ..., \beta_q, \gamma_1, \gamma_2, ..., \gamma_{k-q-1}\}$$

one obtains

$$\mathcal{M}_n^{\mathrm{N}^k\mathrm{MVH}} = \sum_{\substack{\text{MHV diag.}\\ \{\alpha_1,\alpha_2,\ldots,\alpha_k\}}} \prod_{i=1}^{k} k_{\alpha_i} \sum_{B=1}^{k} \frac{\hat{\mathcal{M}}^{\mathrm{MHV}}(\alpha_1)\hat{\mathcal{M}}^{\mathrm{MHV}}(\alpha_2)\ldots\hat{\mathcal{M}}^{\mathrm{MHV}}(\alpha_{k+1})}{\hat{P}^2_{\alpha_1}(z_{\alpha_B})\ldots\hat{P}^2_{\alpha_{B-1}}(z_{\alpha_B})P^2_{\alpha_B}\hat{P}^2_{\alpha_{B+1}}(z_{\alpha_B})\ldots\hat{P}^2_{\alpha_k}(z_{\alpha_B})}. \tag{19}$$

Now it is known that

$$\sum_{B=1}^{k} \frac{1}{\hat{P}^2_{\alpha_1}(z_{\alpha_B})\ldots\hat{P}^2_{\alpha_{B-1}}(z_{\alpha_B})P^2_{\alpha_B}\hat{P}^2_{\alpha_{B+1}}(z_{\alpha_B})\ldots\hat{P}^2_{\alpha_k}(z_{\alpha_B})} = \frac{1}{P^2_{\alpha_1}P^2_{\alpha_2}\ldots P^2_{\alpha_k}}$$

which follows from the integral

$$\oint_{\mathbb{C}\text{ at }\infty} \frac{dz}{z} \frac{1}{\hat{P}^2_{\alpha_1}(z)\ldots\hat{P}^2_{\alpha_{B-1}}(z)\hat{P}^2_{\alpha_B}(z)\hat{P}^2_{\alpha_{B+1}}(z)\ldots\hat{P}^2_{\alpha_k}(z)} = 0$$

Extra slides: Existence of good all-line shift

Given the set of all square spinor shifts

$$|i] \longrightarrow |\hat{i}] = |i] + z c_i |X]$$

Define the parametric space of coordinates $\{c_i\}_{i\in\{1,2,\ldots,n\}} = \mathbb{C}^n$.

(i) reduces this space to a hyperplane, $\mathscr{H}$, of dimension $n-2$.

Every condition such as

$$z_\alpha = \frac{P_\alpha^2}{\sum_{i\in\alpha} c_i \langle i| P_\alpha |X]} = \frac{P_\beta^2}{\sum_{i\in\beta} c_i \langle i| P_\beta |X]} = z_\beta \quad \text{for} \quad \alpha \neq \beta, \quad (20)$$

defines a hyperplane of dimension $n-1$.

Intersection is hyperplane of dimension $n-3$.
The union of all these intersections can not span $\mathscr{H}$