Riemann Surfaces

An introduction

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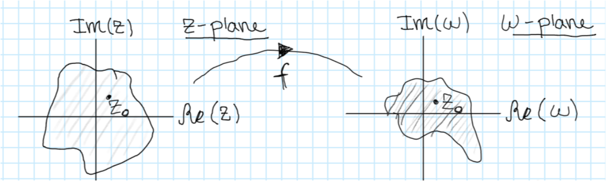
Overview

Recap of complex analysis so far

Functions of one complex variable

Take some open set U in the complex plane and a function f which takes complex variables z and maps them to $\omega = f(z)$ in the open domain V



Recap of complex analysis so far

Functions of one complex variable

Requiring f to be holomorphic in a neighborhood around z_0 put great constraints on our functions, i.e. Cauchy Riemann conditions (among others) with $\omega = u + iv$, $z = x + iy$

$$\partial_x u = \partial_y v, \quad \partial_x v = -\partial_y u \tag{1}$$

Some definitions

Some definitions

- **Isomorphism**: Structure-preserving mapping between two structures of the same type that can be reversed by an inverse mapping.
- **Homeomorphism**: Isomorphism in the category of topological spaces. I.e. they are the mappings that preserve all the topological properties of a given space
- Injective holomorphic map is a holomorphic isomorphism

Given a holomorphic injective map from an open set U to $\mathbb{C}$

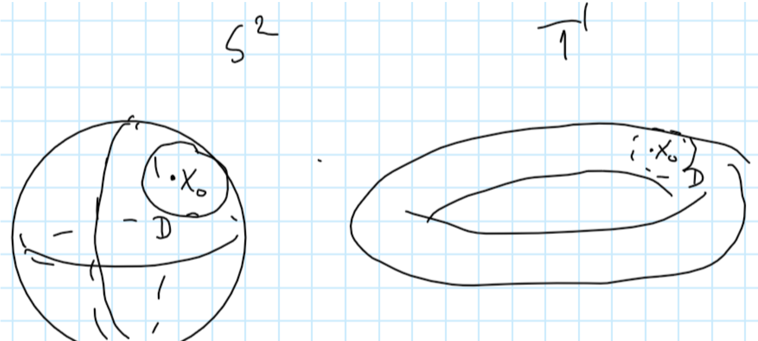
$$f : U \to \mathbb{C} \tag{2}$$

then $f(U)$ is open, and the inverse map is also holomorphic.

Complex analysis on surfaces

Complex analysis on surface

- Take a known surface that we can visualize.
- Pick some point x_0 on the surface on some disc-like domain $\mathcal{D}$.
- Introduce function $f : \mathcal{D} \to \mathbb{C}$ that takes on complex values on $\mathcal{D}$.
- Extend definition of holomorphic function at x_0 so that we can use tools of complex analysis on the surface.



Pick a simple surface

Pick a simple surface

Do this by identifying $\mathcal{D}$ with an open subset, say

$$\Delta = \{z \in \mathbb{C} \mid |z| = 1\} \tag{3}$$

by choosing a homeomorphism

$$\phi : \mathcal{D} \to \Delta \subset \mathbb{C} \tag{4}$$

Hence we now have a map from the unit disc on the surface to the complex plane

$$\begin{tikzcd} \Delta \arrow[rr, bend right, "f \circ \phi^{-1}"'] & \mathcal{D} \arrow[l, "\phi"'] \arrow[r, "f"] & \mathbb{C} \end{tikzcd}$$

and require that $f \circ \phi^{-1}$ is holomorphic at the point.

Holomorphic requirement

Holomorphic requirement

So: we get a function from the disc in the complex plane to the complex numbers.

Here it is it is easy to see when it is holomorphic at x_0.

f is holomorphic on all of $\mathcal{D}$ if $f \circ \phi^{-1}$ is holomorphic on Δ.

Complex coordinate chart

Complex coordinate chart

The pair $(\mathcal{D}, \phi)$ is called a *complex coordinate chart*

It allows us to do complex analysis on the disc

In this example $z = \phi(x)$, $x \in X$ provides us with a new symbol in a continuously isomorphic way.

The resulting function is a function of one complex variable on an open subset on the complex plane, where we can do complex analysis.

Extending to other surfaces

Extending to other surfaces

We could have taken any open set, not just a disc-like neighborhood.

Would have to choose a different homeomorphism to an open set on the complex plane.

Extending to other surfaces

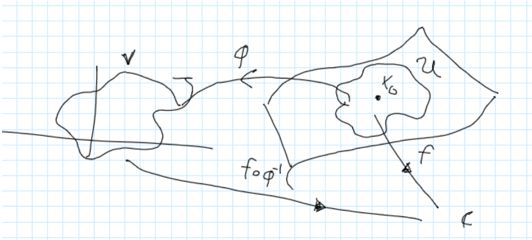
Extending to other surfaces

More generally: a complex coordinate chart is a pair

$$(\mathcal{U}, \phi) \tag{5}$$

where $\mathcal{U}$ is an open subset of X and $\phi : \mathcal{U} \to \mathcal{V}$ is a homemorphism onto an open subset $\mathcal{V}$ of $\mathbb{C}$.

If we we have a function f on $\mathcal{D}$ that takes complex values, f is holomorphic if $f \circ \phi^{-1}$ is holomorphic.



Naïve preliminary definition of Riemann Surface

Riemann surface

A surface X covered by a collection of charts that span of all of X

$$\{(\mathcal{U}_\alpha, \phi_\alpha) \mid \alpha \in I\} \tag{6}$$

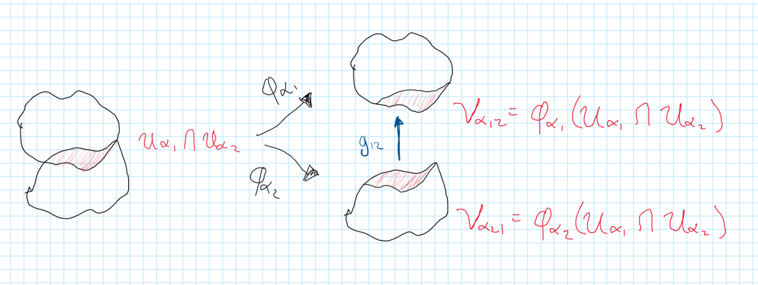
Naïve preliminary definition of Riemann Surface

Problem since charts can in principle intersect

Consider e.g.

$$f : \mathcal{U}_{\alpha_1} \cap \mathcal{U}_{\alpha_2} \to \mathbb{C} \tag{7}$$

so we have two charts $(\mathcal{U}_{\alpha_1}, \phi_{\alpha_1})$ and $(\mathcal{U}_{\alpha_2}, \phi_{\alpha_2})$. To be a Riemann surface, both charts should be holomorphic in the domain.



Naïve preliminary definition of Riemann Surface

Problem since charts can in principle intersect

We then require the transition function g_{12} between $\mathcal{V}_{\alpha_{12}}$ and $\mathcal{V}_{\alpha_{21}}$ to be holomorphic

$$g_{12} = \phi_{\alpha_1}\big|_{\mathcal{U}_{\alpha_1} \cap \mathcal{U}_{\alpha_2}} \circ \phi_{\alpha_2}^{-1}\big|_{\mathcal{U}_{\alpha_1} \cap \mathcal{U}_{\alpha_2}} \tag{8}$$

since this condition leads to (ignoring subscripts)

$$f \circ \phi_{\alpha_1}^{-1} \circ g_{12} = f \circ \phi_{\alpha_2}^{-1} \tag{9}$$

g_{12} is holomorphic isomorphism from the holomorphic requirement (i.e. it has an inverse g_{21})
This implies that $\phi_{\alpha_1}^{-1}$ and $\phi_{\alpha_2}^{-1}$ have to both be holomorphic.
It is exactly what we wanted!

Definition of Riemann Surface

Summary

- We require that transition functions are holomorphic for $\mathcal{U}_{\alpha_1} \cap \mathcal{U}_{\alpha_2} \neq \emptyset$
- This gives us a collection of charts that are *compatible*
- Such a collection that covers all of X is known as a complex *Atlas*
- A surface X is a Riemann surface if there exists such an atlas.

Examples of Riemann Surfaces

Example I: $X = \mathbb{R}^2$

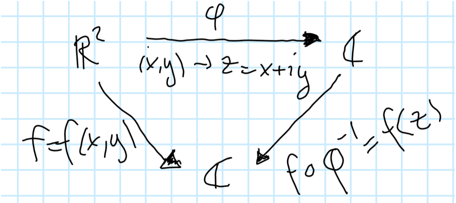
We only need one chart

$$X = \{(\mathbb{R}^2, \phi)\} \tag{10}$$

with

$$\phi : \mathbb{R}^2 \to \mathbb{C} \tag{11}$$

I.e. $(x, y) \to x + iy$. The holomorphic functions on X are the holomorphic functions on $\mathbb{C}$.



Examples of Riemann Surfaces

Example I: $X = \mathbb{R}^2$

We could also have taken a difference chart $\phi : \mathbb{R}^2 \to \mathbb{C}$ with

$$\begin{aligned} \phi : (x, y) &\mapsto \frac{z}{1+|z|} = \frac{x}{1+\sqrt{x^2+y^2}} + i\frac{y}{1+\sqrt{x^2+y^2}} \\ \phi^{-1} : z &\mapsto \frac{z}{1-|z|} = \left(\frac{x}{1-\sqrt{x^2+y^2}}, \frac{y}{1-\sqrt{x^2+y^2}}\right) \end{aligned} \tag{12}$$

(I.e.) mapping $\mathbb{C}$ onto Δ. This is a homeomorphism. This is not the same Riemann surface!

Examples of Riemann Surfaces

Example I: $X = \mathbb{R}^2$

For f to be holomorphic, $f \circ \phi^{-1}$ has to also be.

$$f \circ \phi^{-1}(z) = f\left(\frac{z}{1-|z|}\right) = f\left(\frac{x}{1-\sqrt{x^2+y^2}}, \frac{y}{1-\sqrt{x^2+y^2}}\right) \tag{13}$$

Not holomorphic for the natural identification $f(x,y) \to x + iy = z$

$$f\left(\frac{z}{1-|z|}\right) = \frac{z}{1-|z|} \tag{14}$$

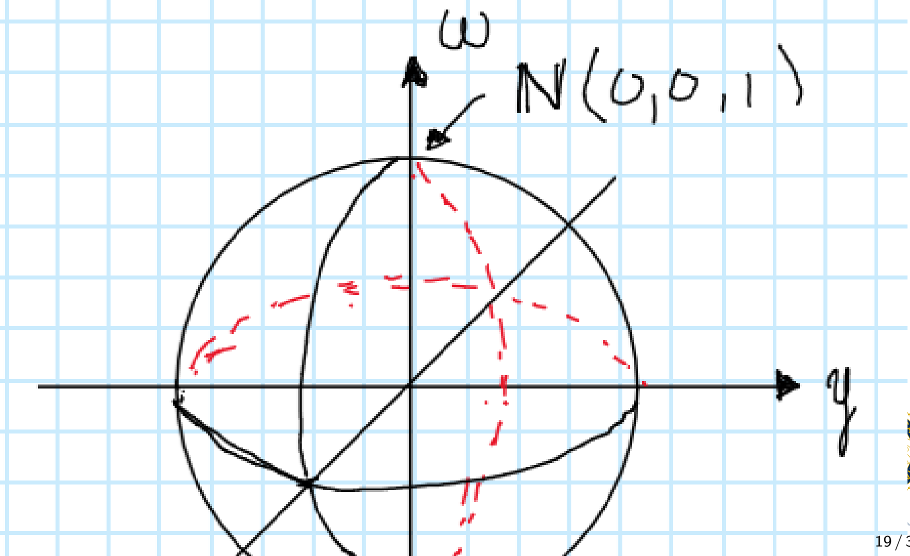
Completely different structure! It is the Riemann surface structure on the unit disc.

Riemann mapping theorem: unit disc is not equal to complex plane.

Examples of Riemann Surfaces

Example: $X = S^2$

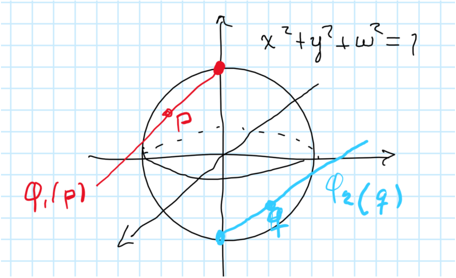
Take the two-dimensional sphere.



Examples of Riemann Surfaces

Example: $X = S^2$

Stereographic projection works in the following way



$\phi_1(p) =$ point of intersection of north pole with xy-plane
$\phi_2(p) =$ point of intersection of south pole with xy-plane

Examples of Riemann Surfaces

Check for compatiblity

$$\begin{aligned}\mathcal{U}_1 \cap \mathcal{U}_2 &= S^2 \backslash \{N, S\} \\ \phi_1(\mathcal{U}_1 \cap \mathcal{U}_2) &= \mathbb{C} \backslash \{0\} \\ \phi_2(\mathcal{U}_1 \cap \mathcal{U}_2) &= \mathbb{C} \backslash \{0\}\end{aligned} \tag{15}$$

The transition function from $\mathbb{C} \backslash \{0\} \to \mathbb{C} \backslash \{0\}$ is $z \to \frac{1}{z}$, which is holomorphic. This Riemann surface structure is the Riemann Sphere

Are there any more atlas?

Examples of Riemann Surfaces

Uniformization Theorem

Every simply connected Riemann surface is conformally equivalent to one of three Riemann surfaces: the open unit disk Δ, the complex plane $\mathbb{C}$, or the Riemann sphere $\mathbb{C} \cup \infty$.

Simply connected: I.e. an object which consists of one piece and does not have any holes that pass all the way through it.

Examples of Riemann Surfaces

Automorphisms

Final note before we look at applications.
Automorphisms of the Riemann sphere are Möbius transformations

$$z \to \frac{az+b}{cz+d}, \qquad ad - bc = 1 \tag{16}$$

Applications

QFT scattering

Overlap between two asymptotic states

$$\langle f|i\rangle = (2\pi)^D \delta^D\left(\sum_i k_i\right)(\mathbb{1}_{fi} + iT_{fi}), \tag{17}$$

Scattering cross section proportional to $|T_{fi}|^2$.
We refer to T_{fi} as the scattering amplitude and denote it by $\mathcal{A}(\ldots)$ where $(\ldots)$ is the scattering data.

The Scattering Equations

Scattering equations and amplitudes

The *scattering equations* live on the Riemann sphere through z_i and are given by

$$\mathcal{S}_i = \sum_{j\neq i} \frac{s_{ij}}{z_i - z_j} = 0, \quad i \in \{1, 2, .., n\}. \tag{18}$$

One can obtain amplitudes of various theories from the formula

$$\mathcal{A}_n(1, ..., n) = \int d\Omega_{\mathsf{CHY}} \mathcal{I}(z_i, k_i, \epsilon_i, ...), \tag{19}$$

with $d\Omega_{\mathsf{CHY}} = \frac{d^n z}{\mathsf{Vol}(\mathsf{SL}(2,\mathbb{C}))} \prod_i' \delta(\mathcal{S}_i)$.

Complex analysis tools

Scattering equations and amplitudes

Since the space is the Riemann sphere, one can use tools of complex analysis. Remember

$$f(a) = \frac{1}{2\pi i} \oint \mathrm{d}z \, \frac{f(z)}{z-a} \tag{20}$$

Reformulate the delta-functions $\rightarrow$ use complex analysis.

From the global residue theorem one can obtain diagrammatic rules to calculate amplitudes.

Integration rules

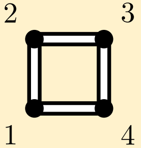
Graphic representation of Möbius invariant integrands

We represent the integrands by four-regular graphs. Every factor of z_{ij}^{-1} is a line between vertices i and j and every factor z_{ij} is a dashed line.

$$A_n^{\varphi^3}(1,2,3,\ldots,n) = \int \mathrm{d}\Omega_{\mathsf{CHY}} \frac{1}{z_{12}^2 z_{23}^2 \cdots z_{n1}^2}. \tag{21}$$

The integrand is

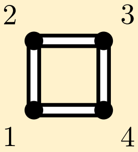
$$\mathcal{I}(z) = \frac{1}{z_{12}^2 z_{23}^2 z_{34}^2 z_{41}^2} \rightarrow \tag{22}$$



Integration rules

Graphic representation of Möbius invariant integrands

Using simple derived rules



$$\rightarrow -\frac{1}{s_{12}} - \frac{1}{s_{14}}, \tag{23}$$

We get the final amplitude to be

$$A_4^{\varphi^3}(1,2,3,4) = -\frac{1}{s_{12}} - \frac{1}{s_{14}}. \tag{24}$$

It only took 1 diagram!

Summary

- Started with a real surface X on which we wanted to do complex analysis
- This was achieved by using complex charts $\{(\mathcal{U}_i, \phi_i) \big| i \in I\}$ such that $\mathcal{U}_i$ covers all of X
- We checked that f was holomorphic by using $f \circ \phi^{-1}$
- Compatible charts (atlas) were introduced to make sure the functions were holomorphic even if charts overlapped.
- We analyzed different Riemann structures and introduced the uniformization theorem.
- Finally an application in scattering amplitudes was reviewed.

Thank you for your attention.