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Modern amplitude techniques

Taro V. Brown[a]

[a] *Department of Physics, UC Davis, One Shields Avenue, Davis, CA 95616, USA*

E-mail: taro.brown@nbi.ku.dk

Abstract: Notes on modern amplitude techniques written as part of a research project with Jaroslav Trnka.

Contents

1 Recursion Relations

On-shell recursion is a systematic procedure for relating an amplitude to its values at singular kinematics. In order to probe these kinematic configurations we define a momentum shift, which is a one-parameter deformation of the external momenta engineered to sample various kinematic limit.
A shift of the form

$$p_i \to p_i(z) = p_i + zq_i, \qquad z \in \mathbb{C}. \tag{1.1}$$

Not all momenta have to be shifted and we restrict the shifted momenta to satisfy momentum conservation as well as being on-shell

$$\sum_i p_i(z) = 0, \qquad p_i(z)^2 = 0 \tag{1.2}$$

This implies the following for the shifts q_i

$$\sum_i q_i = 0, \qquad q_i^2 = q_i p_i = 0. \tag{1.3}$$

These conditions preserve the kinematics of the corresponding shifted amplitude

$$A \to A(z) \tag{1.4}$$

We can obtain the original amplitude from the residue

$$A(0) = \oint_{z=0} \mathrm{d}z \frac{A(z)}{z}. \tag{1.5}$$

One can think of the contour integral as a deltafunction in the point $z = 0$.
Using Cauchy's theorem this can be expressed as minus the sum of all the other residues

$$A(0) = -\sum_I \mathrm{Res}_{z=z_I} \left[\frac{A(z)}{z}\right] + B_\infty, \tag{1.6}$$

where B_∞ is a boundary term that vanishes when $A(z) \to 0$ for $z \to \infty$. This will be another condition on what variables we shift.
If we take a subset of momenta $\{p_i\}_{\in I}$ and define the sum over these

$$P_I \equiv \sum_{i \in I} p_i, \tag{1.7}$$

then we can also defined the shifted momenta $P_I(Z)$

$$P_I(z) = \sum_{i \in I} p_i(z) = P_I + zQ_I, \quad \text{with } Q_I = \sum_{i \in I} q_i \tag{1.8}$$

For simplicity we will assume $q_i q_j = 0$ leading to $Q_I^2 = 0$. In this case $P_I(z)^2$ is linear in z

$$P_I(z)^2 = (P_I + zQ_I)^2 = P_I^2 + zP_I Q_I = -\frac{P_I^2}{z_I}(z - z_I), \tag{1.9}$$

where we have defined $z_I \equiv -\frac{P_I^2}{2P_I Q_I}$.
For some reason the amplitude should factorize into a product of two lower point on-shell amplitudes when $z = z_I$ and $P_I^2(z)$ goes on-shell

$$\lim_{z\to z_I} A(z) = A_L(z_I)\frac{1}{P_I^2(z)}A_R(z_I) = -\frac{z_I}{z-z_I}A_L(z_I)\frac{1}{P^2}A_R(z_I) \tag{1.10}$$

Using this to take the residue at $z = z_I$

$$= -\operatorname{Res}_{z=z_I}\left[\frac{A(z)}{z}\right] = \operatorname{Res}_{z=z_I}\left[\frac{z_I}{z(z-z_I)}A_L(z_I)\frac{1}{P_I^2}A_R(z_I)\right] \tag{1.11}$$

The residue is found by multiplying by $(z - z_I)$ and setting $z = z_I$. Summing over all residues we find the amplitude

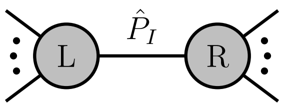
$$A(0) = \sum_I A_L(z_I)\frac{1}{P_I^2}A_R(z_I) + B_\infty \tag{1.12}$$

The boundary contribution B_∞ has no similar general expression in terms of lower-point amplitudes and the simplest way to make it vanish is by requiring

$$A(z) \to 0, \qquad \text{for } z \to \infty \tag{1.13}$$

If this holds then

$$A = \sum_I A_L(z_I)\frac{1}{P_I^2}A_R(z_I) = \sum_{\text{Diagrams } I} \tag{1.14}$$



1.1 BCFW-recursion

A particular recursion technique used often is called BCFW recursion. In four dimensions this can be implemented in the spinor-helicity basis. Denoting the shifted variables by a hat, the shifts that we will employ are

$$|\hat{i}] = |i] + z|j], \quad |\hat{j}] = |j], \quad |\hat{i}\rangle = |i\rangle, \quad |\hat{j}\rangle = |i\rangle - z|j\rangle \tag{1.15}$$

1.2 Example of BCFW-recursion

As an example, let us calculate the amplitude $A_5(1_g^+, 2_g^-, 3_g^+, 4_g^-, 5_g^-)$ Since we are dealing with an $\overline{\text{MHV}}$ amplitude we can immediately read of the good shift since the shifts

$$|1] \to |\hat{1}] = |1] + z|5] \tag{1.16}$$
$$|5\rangle \to |\hat{5}\rangle = |5\rangle - z|1\rangle \tag{1.17}$$

will shift the amplitude by

$$A_5(1^+, 2^-, 3^+, 4^-, 5^-) = \frac{[13]^4}{[12][23][34][45][51]} \to \frac{\langle([13] + z[53])^4\rangle}{([12] + z[52])[23][34][45]} \sim z^3$$

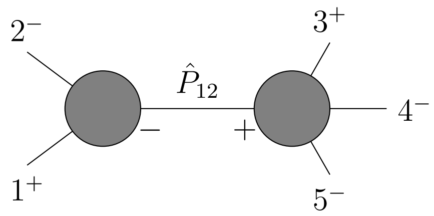
While the shifts

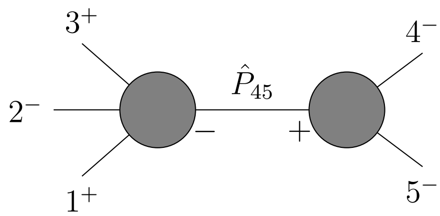
$$|5] \to |\hat{5}] = |5] + z|1] \tag{1.18}$$
$$|1\rangle \to |\hat{1}\rangle = |1\rangle - z|5\rangle \tag{1.19}$$

will shift the amplitude by

$$A_5(1^+,2^-,3^+,4^-,5^-) = \frac{[13]}{[12][23][34][45][51]} \to \frac{[13]^4}{[12][23][34]([45]+z[41])([51]\underbrace{+z[11]}_{0})} \sim \frac{1}{z}$$

Now since we want $A \to 0$ for $z \to \infty$ the *good shift* is the second one, meaning $[5]\ |1\rangle$ which corresponds to a $[-,+\rangle$ shift in Elvangs notation. We could have seen the good shifts by little group scaling of the amplitude since leg one has little group weight 1 and so will the amplitude under a shift will scale like z if we shift the square brackets The corresponding diagrams are





Looking at the first diagram we see that it contains a 3-point MHV amplitude

$$A_3(1^+,2^-,-\hat{P}_{12}^-) = \frac{\left\langle 2\hat{P}_{12}\right\rangle^3}{\langle\hat{1}2\rangle\left\langle\hat{P}_{12}\hat{1}\right\rangle} \tag{1.20}$$

Since we impose that the propagating momentum is on shell we see that

$$0 = \hat{P}_{12} = \langle\hat{1}2\rangle\,[\hat{1}2] = \langle\hat{1}2\rangle\,[12]$$

So the only way to impose on shell conditions is by setting $\langle\hat{1}2\rangle = 0$ similarly one can show that the numerator vanishes and we must have

$$A_3(1^+,2^-,-\hat{P}_{12}^-) = 0$$

which means the first diagram doesn't contribute. For the second diagram we also have a 3-point MHV amplitude, but in this case the shift is in $|5]$ so the sub-diagram isn't zero.

We can then proceed to calculate the second diagram explicitly

$$\begin{aligned} A_5(1^+,2^-,3^+,4^-,5^-) &= A_3(\hat{P}_{45}^+,4^-,\hat{5}^-)\frac{1}{P_{45}^2}A_4(\hat{1},^+,2^-,3^+,-\hat{P}_{45}^-) \\ &= \frac{\langle 4\hat{5}\rangle^3}{\left\langle \hat{P}_{45}4\right\rangle \left\langle \hat{5}\hat{P}_{45}\right\rangle}\frac{1}{\langle 45\rangle\,[45]}\frac{[\hat{1}3]^4}{[\hat{1}2][23][3\hat{P}_{45}][\hat{P}_{45}\hat{1}]} \end{aligned}$$

Since the shift is in $[5,1\rangle$ we can remove the hat on all but the P's:

$$\begin{aligned} A_5(1^+,2^-,3^+,4^-,5^-) &= \frac{\langle 45\rangle^3}{\left\langle \hat{P}_{45}4\right\rangle \left\langle 5\hat{P}_{45}\right\rangle}\frac{1}{\langle 45\rangle\,[45]}\frac{[13]^4}{[12][23][3\hat{P}_{45}][\hat{P}_{45}1]} \\ &= \frac{\langle 45\rangle^3}{\left\langle \hat{P}_{45}4\right\rangle \left\langle 5\hat{P}_{45}\right\rangle}\frac{1}{\langle 45\rangle\,[45]}\frac{[13]^4}{[12][23][3\hat{P}_{45}][\hat{P}_{45}1]} \end{aligned}$$

We can the rewrite the $\hat{P}$ terms in the following way:

$$\left\langle \hat{P}_{45}4\right\rangle [\hat{P}_{45}1] = -\left\langle 4\hat{P}_{45}\right\rangle [\hat{P}_{45}1] = \langle 4|\hat{P}_{45}|1] = \langle 4|4+\hat{5}|1] = \langle 4|\hat{5}|1] = -\langle 4\hat{5}\rangle\,[\hat{5}1] = -\langle 45\rangle\,[51]$$

$$\begin{aligned} \left\langle 5\hat{P}_{45}\right\rangle [3\hat{P}_{45}] &= -\left\langle 5\hat{P}_{45}\right\rangle [\hat{P}_{45}3] = \langle 5|\hat{P}_{45}|3] = \langle 5|4+\hat{5}|4] = \langle 5|4|3] + \langle 5|\hat{5}|3] \\ &= -\langle 54\rangle\,[43] - \langle 5\hat{5}\rangle\,[\hat{5}3] = -\langle 54\rangle\,[43] = -\langle 45\rangle\,[34] \end{aligned}$$

where we in the first terms have used the fact that $|\hat{5}\rangle = |5\rangle$ and $[\hat{5}1] = [51] + z[11] = [51]$, while in the second term using $\langle 5\hat{5}\rangle = \langle 55\rangle = 0$. Inserting this into the amplitude we get

$$\begin{aligned} A_5(1^+,2^-,3^+,4^-,5^-) &= \frac{[13]^4\,\langle 45\rangle^3}{[12][23][45]\,\langle 45\rangle^3\,[51][34]} \\ &= \frac{[13]^4}{[12][23][34][45][51]} \end{aligned}$$

which is the expected result.

Part b

The soft-limit factorization for tree amplitudes is that for $k_s \to 0$ we can write an n-point amplitude as

$$A_n^{\text{tree}}(1,2,\ldots,a,s^\pm,b,\ldots,n) = \mathcal{S}(a,s^\pm,b)\times A_{n-1}^{\text{tree}}(1,2,\ldots,a,b,\ldots,n) \tag{1.21}$$

where

$$\mathcal{S}(a,s^+,b) = \frac{\langle ab\rangle}{\langle as\rangle\,\langle sb\rangle}, \qquad \mathcal{S}(a,s^-,b) = -\frac{[ab]}{[as][sb]} \tag{1.22}$$

Here this gets us

$$A_5(1^+, 2^-, 3^+, 4^-, 5^-) = -\frac{[41]}{[45][51]} \times \frac{[13]^4}{[12][23][34][41]}$$

which is a valid factorization of the full result.

In the collinear limit for leg 1 and 2 we have the two momenta k_1 and k_2 that become parallel with intermediate momentum k_P. The spinors also have the following relations

$$\begin{aligned} \lambda_a &\simeq \sqrt{z}\lambda_P, \qquad & \lambda_b &\simeq \sqrt{1-z}\lambda_P \\ \tilde{\lambda}_{\dot{a}} &\simeq \sqrt{z}\tilde{\lambda}_P, \qquad & \tilde{\lambda}_{\dot{b}} &\simeq \sqrt{1-z}\tilde{\lambda}_P \end{aligned}$$

taking the amplitude we calculated in part a and shifting it in this limit gives

$$A_5(1^+, 2^-, 3^+, 4^-, 5^-) \to \frac{z^2}{\sqrt{z(1-z)}[12]} \frac{[P3]^4}{[P3][34][45][5P]}$$

which is the result we expected from Dixon:

$$A_n^{\text{tree}}(\ldots, a^{\lambda_a}, b^{\lambda_b}, \ldots) \to \sum_{\lambda_p = \pm} \text{Split}_{-\lambda_P}(a^{\lambda_a}, b^{\lambda_b}; z) A_{n-1}^{\text{tree}}(\ldots, P^{\lambda_P}, \ldots) \tag{1.23}$$

where

$$\text{Split}_{-}(a^+, b^-) = \frac{z^2}{\sqrt{z(1-z)}[ab]} \tag{1.24}$$

References

[1] N. E. J. Bjerrum-Bohr, J. F. Donoghue and P. Vanhove, “On-shell Techniques and Universal Results in Quantum Gravity,” JHEP **02** (2014), 111 doi:10.1007/JHEP02(2014)111 [arXiv:1309.0804 [hep-th]].