Quantum Error Correction in AdS/CFT

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Abstract: a

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1 Introduction

The AdS/CFT correspondence is a duality relating a $(d+1)$-dimensional bulk theory of gravity in anti-de Sitter spacetime (AdS) to a d-dimensional boundary conformal field theory (CFT). The importance of the duality can not be understated, as the mathematical structure, and especially ulta-violet (UV) behavior, is not fully understood on the gravity side, as opposed to the framework of CFT's, which has been studied for a long time. The duality therefore gives a path towards a theory of quantum gravity, or at least gives hints on certain properties that such a theory should have.

The correspondence provides a dictionary between the CFT and Gravity sides. The focus of this project is on how one relates operators on the boundary to operators in the bulk. This might seem strange at first, since the operators on the boundary live in one spacetime dimension lower than the bulks ones and so one might expect the dictionary to only work one way, but as we will show, one is able to *reconstruct* information in the bulk from data on the boundary. Further, one can reconstruct operators using only part of the boundary information. Here we will focus on Rindler reconstruction, but other subregion dualities exist.

Using Rindler reconstruction, some subtleties and puzzles in the prescription appear. We will describe the problems and then show how treating AdS/CFT as a Quantum Error Correcting Code (QECC), resolves the problems. For this purpose we introduce a toy model: the 3 qutrit model. The model is very coarse grained and we give a brief introduction to how to introduce more degrees of freedom by using tensor networks.

2 Bulk and Rindler Reconstruction

We are going to consider global AdS spacetime with metric

$$\begin{aligned} \mathrm{d}s^2 &= -\left(1-r^2\right)\mathrm{d}t^2 + \left(1-r^2\right)^{-1}\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega_{d-1}^2 \\ &= \omega(\rho)^2\left(-\mathrm{d}t^2 + \mathrm{d}\rho^2 + \sin^2\rho\,\mathrm{d}\Omega_{d-1}^2\right), \end{aligned} \tag{2.1}$$

where we work in natural units, such that the AdS length $\ell_{\mathrm{AdS}} = 1$ and we have defined $r \equiv \tan\rho$ as well as the the conformal factor $\omega(\rho) \equiv \frac{1}{\cos\rho}$ in the second line. This metric is conformally related to the Einstein Static Universe, and its conformal structure can be illustrated on a cylinder. The boundary is located at $r = \infty$ and can be viewed as the cylindrical shell of the global AdS space time.

We are going to denote bulk coordinates by $x \equiv (r, t, \Omega)$ and boundary coordinates by $X \equiv (t, \Omega)$. The AdS/CFT duality provides a map between operators on the boundary $\mathcal{O}(X)$ and operators in the bulk $\phi(x)$. This is illustrated the global AdS

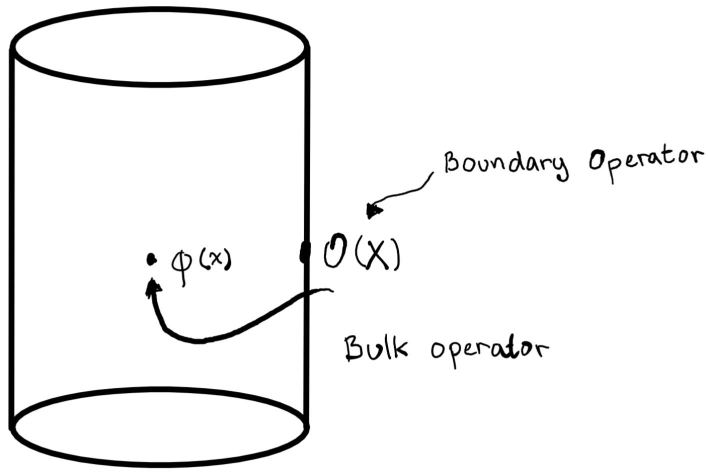

Figure 1. Global AdS as a cylinder with bulk operator $\phi(x)$ and boundary operator $\mathcal{O}(X)$

cylinder on figure 1. If one solves the equations of motion of the bulk field $\phi(x)$ near the boundary and demands that the solution is normalizable, the following extrapolate dictionary relates it to the boundary operator,

$$\lim_{r\to\infty} r^{\Delta}\phi(x) = \mathcal{O}(X), \tag{2.2}$$

where for a generic scalar of mass m, Δ satisfies $\Delta(\Delta - d) = m^2$. This limit seems very natural and one can ask whether it is possible to obtain the bulk solutions if the boundary operator is known. I.e. we are interested in the following

$$\phi(x) = \int \mathrm{d}X K(x;X)\mathcal{O}(X) \tag{2.3}$$

under the boundary condition (2.2). Here $K(x;X)$ is a Greens function, often refered to as the *smearing function*. If one takes a bulk point x, then $K(x;X)$ has support on set of boundary points that are spacelike separated from x, see Figure 2a.

We are going to focus on constructing operators from boundary subregions. One such subregion is the Rindler wedge, W, defined by metric

$$\mathrm{d}s^2 = -(\rho^2 - 1)\mathrm{d}\tau^2 + (\rho^2 - 1)^{-1}\mathrm{d}\rho^2 + \rho^2 \mathrm{d}H^2_{d-1} \tag{2.4}$$

with $\mathrm{d}H^2_{d-1} = \mathrm{d}\chi^2 + \sinh^2\chi \mathrm{d}\Omega^2_{d-2}$ is the metric on a $d-1$ dimensional hyperbolic ball. The metric has $\rho > 1$ and $-\infty < \tau < \infty$ and its embedding in the global AdS cylinder is shown on Figure 2b. Using similar arguments to the global reconstruction, it can be shown explicitly that a bulk field can be reconstructed from a boundary operator if it lies within the Rindler wedge,

$$\phi(x)|_{x\in W} = \int_{\partial W} \mathrm{d}X K(x;X)\mathcal{O}(X) \tag{2.5}$$

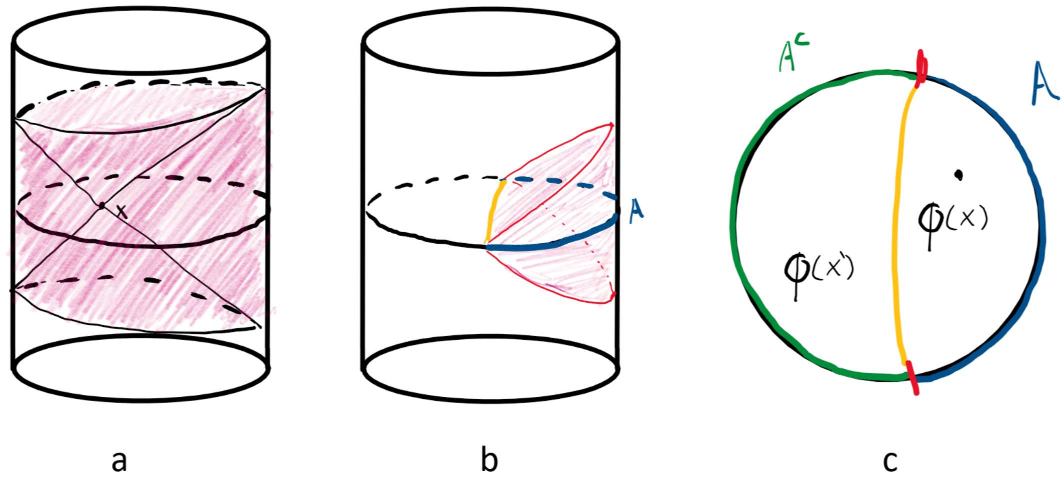

Figure 2.

where $K(x;X)$ now only has support on the boundary region of the wedge. Using this as a jumping off point, we can find a version of (2.3) for any subregion, using the causal wedge, W_C, for any boundary spatial subregion A , defined by

$$W_C[A] = \mathscr{I}^+[D_\partial[A]] \cap \mathscr{I}^-[D_\partial[A]] \tag{2.6}$$

The bulk field can the be reconstructed on the boundary region A as long as it is in the causal wedge, see Figure 2c. Using these definitions we arrive at two puzzles, which we will describe below.

Radial Commutivity Puzzle

Using the reconstruction procedure just described on a t=0 timeslice with support on a region A that covers almost the whole boundary region, as shown on Figure 3a, one finds that the operator dual to ϕ on the boundary, will necessarily commute with an operator $\mathcal{O}$ that lies in the complement region A^C.

$$[\phi(x), \mathcal{O}(X')] = 0 \tag{2.7}$$

Similarly one can pick a different operator $\mathcal{O}'$ on the boundary and by using a new region A', see Figure 3b, we once again find that ϕ commutes with $\mathcal{O}'$. We can in fact do this with all operators and so must conclude that ϕ commutes with *all* operators on the boundary. This however contradicts a well-known quantum field theory theorem known as the *time-slice axiom*, which states that any operator which commutes with all local operators at a fixes time most be trivial. Since ϕ obviously is not trivial this is quite puzzling.

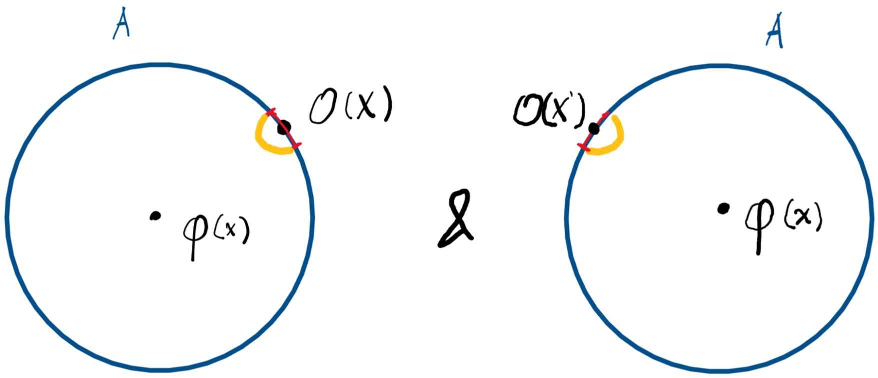

Figure 3.

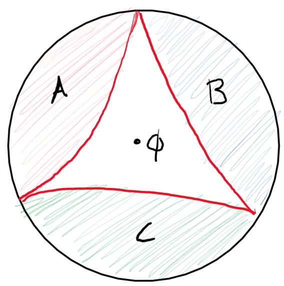

Figure 4.

ABC Puzzle

The ABC-puzzle is best illustrated through Figure 4. Using the reconstruction procedure ϕ can not be reconstructed on regions A, B, or C, it can however be reconstructed from any combination of two regions. So there exists at least 3 representation of ϕ: ϕ_{AB}, ϕ_{BC}, and ϕ_{CA}. The can not be the same operator on the CFT however, and so we get yet another puzzle.

Both puzzles can however be resolved by interpreting the correspondence as an quantum error correcting code, which we will describe next.

3 Quantum Error Correction - The Qutrit Model

Error correction is the procedure off protecting information from being lost. The type of loss can be manifold, but if one is talking about quantum information it could e.g. be decoherence.

In classical computers an easy way to protect information is through redundancy - simply sending your information multiple times. In quantum mechanics however, we are not able to procure our quantum bits (or, as we will see below, qutrits) in the same state due to the no-cloning theorem. Instead we have to come up with a new way to protect information. For this purpose we are going to define the 3-qutrit model. The qutrit is a 3 dimensional extension of the qubit (i.e. Bell state) and it has the basis $\{|0\rangle, |1\rangle, |2\rangle\}$.

$$|\psi\rangle = \sum_{k=0}^{2} c_k |k\rangle \tag{3.1}$$

$$|\tilde{\psi}\rangle = \sum_{k=0}^{2} c_k |\tilde{k}\rangle \tag{3.2}$$

$$\boxed{\begin{aligned} |\tilde{0}\rangle &= \frac{1}{\sqrt{3}} (|000\rangle + |111\rangle + |222\rangle) \\ |\tilde{1}\rangle &= \frac{1}{\sqrt{3}} (|012\rangle + |120\rangle + |201\rangle) \\ |\tilde{2}\rangle &= \frac{1}{\sqrt{3}} (|021\rangle + |102\rangle + |210\rangle) \end{aligned} \qquad (3.3)}$$

$$\rho_{\text{single qutrit}} = \frac{1}{\sqrt{3}} (|0\rangle\langle 0| + |1\rangle\langle 1| + |2\rangle\langle 2|) \tag{3.4}$$

$$\begin{array}{lll} |00\rangle \rightarrow |00\rangle & |11\rangle \rightarrow |01\rangle & |22\rangle \rightarrow |02\rangle \\ |01\rangle \rightarrow |12\rangle & |12\rangle \rightarrow |10\rangle & |20\rangle \rightarrow |11\rangle \\ |02\rangle \rightarrow |21\rangle & |10\rangle \rightarrow |22\rangle & |21\rangle \rightarrow |20\rangle \end{array} . \tag{3.5}$$

Example

$$\begin{aligned} U_{12}|\tilde{1}\rangle &= \frac{1}{\sqrt{3}} (|122\rangle + |100\rangle + |111\rangle) \\ &= |1\rangle_1 \otimes |\chi\rangle_{23} \end{aligned} \tag{3.6}$$

in general

$$U_{12}|\tilde{\psi}\rangle = |\psi\rangle_1 \otimes |\chi\rangle_{23} \tag{3.7}$$

$$O\,|k\rangle = O_{jk}\,|j\rangle \tag{3.8}$$

$$\tilde{O}|\tilde{k}\rangle = O_{jk}|\tilde{j}\rangle \tag{3.9}$$

$$\tilde{O}_{12} \equiv U_{12}^{\dagger} O_1 U_{12} \tag{3.10}$$

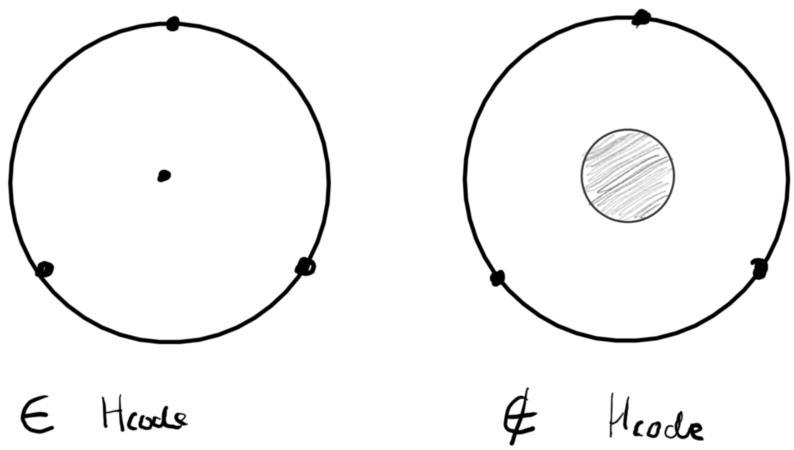

Figure 6.

4 AdS/CFT as a QECC

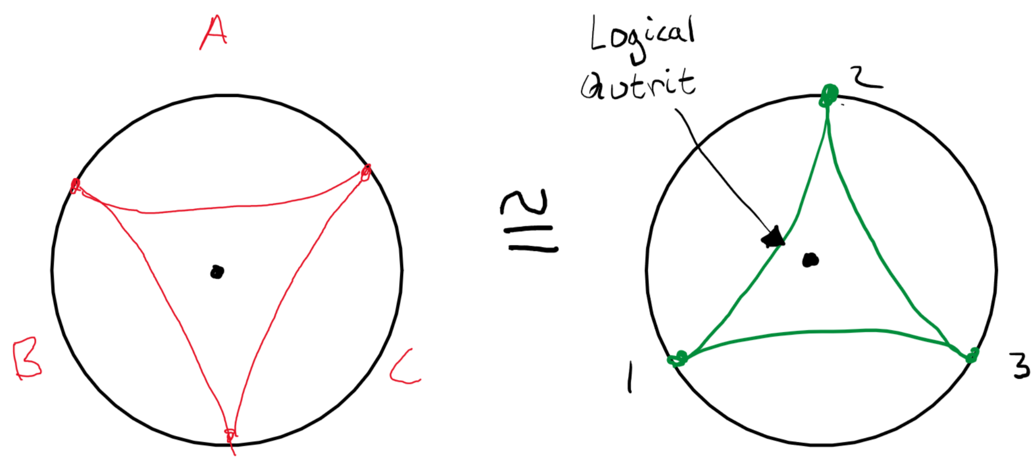

Figure 5.

$$\langle\tilde{\psi}|\,[\phi(x),\mathcal{O}(X)]\,|\tilde{\psi}\rangle = 0, \qquad \forall \quad |\tilde{\psi}\rangle \in H_{\mathrm{code}} \tag{4.1}$$

5 Tensor Models

6 Outlook

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