Homework 9

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Problem 1

Q1. We have computed the path integral for a harmonic oscillator in the position basis. First, what would be corresponding result for the momentum basis propagator? Second, chose instead to work with coherent states, which we recall also obeys a completeness relation, write down the expression for the propagator.

The path integral, or propagator, for the harmonic oscillator was derived in class to be

$$\langle x_f, t_f | x_i, t_i \rangle = \sqrt{\frac{m\omega}{2\pi i \sin\omega T}} \exp\left[\frac{im\omega}{2} \frac{(x_i^2 + x_f^2)\cos\omega T - 2x_i x_f}{\sin\omega T}\right]. \tag{1}$$

Note that we have set $\hbar = 1$. We will now employ the following trick, which Mukund gave at his office-hour:

Since the harmonic oscillator is symmetric in x and p, meaning

$$H = \frac{p^2}{2p_0^2} + \frac{x^2}{2x_0^2} \tag{2}$$

So any result in the x-basis should hold in the p-basis under the exchange

$$\frac{p}{p_0} \Leftrightarrow \frac{x}{x_0} \tag{3}$$

here $p_0 = \sqrt{m}$ and $x_0 = \frac{1}{\sqrt{m\omega}}$, so by inspection we have

$$\begin{aligned}\langle p_f, t_f | p_i, t_i \rangle &= \langle x_f, t_f | x_i, t_i \rangle|_{x\to p,\ \ x_0 \to p_0} \\ &= \sqrt{\frac{1}{2\pi i m\omega \sin\omega T}} \exp\left[\frac{i}{2\omega m} \frac{(p_i^2 + p_f^2)\cos\omega T - 2p_i p_f}{\sin\omega T}\right].\end{aligned} \tag{4}$$

Let us explicitly check this, using that we in the previous homework found that the momentum space propagator is related to the positions space propagator through a double Fourier-transform, i.e.

$$\langle p_f, t_f | p_i, t_i \rangle = \frac{1}{2\pi} \int \mathrm{d}x_f \int \mathrm{d}x_i \, e^{-ip_f x_f} e^{ip_i x_i} \, \langle x_f, t_f | x_i, t_i \rangle \tag{5}$$

Inserting into this, one finds

$$\begin{aligned}\langle p_f, t_f | p_i, t_i \rangle =& \frac{1}{2\pi} \sqrt{\frac{m\omega}{2\pi i \sin\omega T}} \int \mathrm{d}x_f \int \mathrm{d}x_i \, e^{-ip_f x_f} e^{ip_i x_i} \exp\left[\frac{im\omega}{2} \frac{(x_i^2 + x_f^2)\cos\omega T - 2x_i x_f}{\sin\omega T}\right] \\ =& \sqrt{\frac{m\omega}{i8\pi^3 \sin\omega T}} \int \mathrm{d}x_f \int \mathrm{d}x_i \, \exp\left[-a x_i^2 + b(x_f) x_i + c(x_f)\right]\end{aligned} \tag{6}$$

where

$$\begin{aligned} a &\equiv -\frac{i}{2} m\omega \cot T\omega \\ b(x_f) &\equiv ip_i - im\omega x_f \csc T\omega \\ c(x_f) &\equiv \frac{i}{2} m\omega x_f^2 \cot T\omega - ip_f x_f \end{aligned} \tag{7}$$

Performing the Gaussian integration we find

$$\begin{aligned}\langle p_f, t_f | p_i, t_i \rangle =& \frac{1}{\sqrt{4\pi^2 \cos\omega T}} \int \mathrm{d}x_f \, \exp\left[-\frac{i}{2}\left(2p_f x_f - 2p_i x_f \sec\omega T + \left[\frac{p_i^2}{m\omega} + m\omega x_f^2\right] \tan\omega T\right)\right] \\ =& \frac{1}{\sqrt{4\pi^2 \cos\omega T}} \int \mathrm{d}x_f \, \exp\left[-\alpha x_f^2 + \beta x_f + \gamma\right]\end{aligned} \tag{8}$$

where we have collected the constants:

$$
\begin{aligned}
\alpha &\equiv \frac{i}{2} m\omega \tan \omega T \\
\beta &\equiv i p_i \sec \omega T - i p_f \\
\gamma &\equiv \frac{i p_i^2 \tan \omega T}{2m\omega}
\end{aligned} \tag{9}
$$

Once again, performing the Gaussian integration we find

$$
\begin{aligned}
\langle p_f, t_f | p_i, t_i \rangle =& \sqrt{\frac{1}{2\pi i m\omega \sin \omega T}} \exp \left[\frac{i}{2m\omega} \left(p_f^2 \cot \omega T + 2p_i^2 \csc 2\omega T - 2p_i p_f \csc \omega T \right) \right] \\
=& \sqrt{\frac{1}{2\pi i m\omega \sin \omega T}} \exp \left[\frac{i}{2\omega m} \frac{(p_i^2 + p_f^2) \cos \omega T - 2p_i p_f}{\sin \omega T} \right]
\end{aligned} \tag{10}
$$

Just like we found in (4). Taking

$$
U(t) |\zeta\rangle = U(t) e^{\zeta a^\dagger(t)} |0\rangle \tag{11}
$$

Problem 2

Q2. Compute the following correlation functions in the n^{th} excited state of the harmonic oscillator: $\langle \hat{x}(t_1)\,\hat{x}(t_2)\rangle$ and $\langle \hat{x}(t_1)\,\hat{x}(t_2)\,\hat{x}(t_3)\rangle$. You can assume $t_1 > t_2 > t_3$ but do indicate what would change if this temporal ordering was not imposed.

First let us note that

$$\begin{aligned}
\langle n|x|m\rangle &= \frac{x_0}{\sqrt{2}}\,\langle n|(a+a^\dagger)|m\rangle \\
&= \langle n|a|m\rangle + \langle n|a^\dagger|m\rangle \\
&= \sqrt{m}\,\langle n|m-1\rangle + \sqrt{m+1}\,\langle n|m+1\rangle \\
&= \sqrt{m}\delta_{n\,(m-1)} + \sqrt{m+1}\delta_{n\,(m+1)} \\
\Rightarrow \sum_m \langle n|x|m\rangle &= \sqrt{n+1} + \sqrt{n}
\end{aligned} \tag{12}$$

Then computing

$$\begin{aligned}
\langle n_1|x(t_1)x(t_2)|n_1\rangle =& \langle n|e^{iHt_1}xe^{iH(t_2-t_1)}xe^{-iHt_2}|n_1\rangle \\
=& \sum_{n_2} \langle n_1|e^{iHt_1}x\,|n_2\rangle\!\langle n_2|\,e^{iH(t_2-t_1)}xe^{-iHt_2}|n_1\rangle \\
=& \sum_{n_2} e^{iE_{n_1}(t_1-t_2)}e^{iE_{n_2}(t_2-t_1)}\,\langle n_1|x\,|n_2\rangle\!\langle n_2|\,x|n_1\rangle \\
=& \sum_{n_2} e^{iE_{n_1}(t_1-t_2)}e^{iE_{n_2}(t_2-t_1)}\times \\
& \left(\sqrt{n_2}\delta_{n_1\,(n_2-1)} + \sqrt{n_2+1}\delta_{n_1\,(n_2+1)}\right)\left(\sqrt{n_2}\delta_{n_1\,(n_2-1)} + \sqrt{n_2+1}\delta_{n_1\,(n_2+1)}\right) \\
=& x_0 e^{iE_{n_1}(t_1-t_2)}e^{iE_{n_1+1}(t_2-t_1)}(n_1+1) + e^{iE_{n_1}(t_1-t_2)}e^{iE_{n_1-1}(t_2-t_1)}n_1 \\
=& x_0 e^{i\omega(n_1+\frac{1}{2})(t_1-t_2)}e^{i\omega(n_1+\frac{3}{2})(t_2-t_1)}(n_1+1) + x_0 e^{i\omega(n_1+\frac{1}{2})(t_1-t_2)}e^{i\omega(n_1-\frac{1}{2})(t_2-t_1)}n_1 \\
=& x_0 e^{i\omega(n_1+\frac{1}{2})(t_1-t_2)}e^{i\omega(-n_1-\frac{3}{2})(t_1-t_2)}(n_1+1) + x_0 e^{i\omega(n_1+\frac{1}{2})(t_1-t_2)}e^{i\omega(-n_1+\frac{1}{2})(t_1-t_2)}n_1 \\
=& x_0 e^{-i\omega(t_1-t_2)}(n_1+1) + e^{i\omega(t_1-t_2)}n_1 \\
=& x_0 n_1\left(e^{-i\omega(t_1-t_2)} + e^{i\omega(t_1-t_2)}\right) + x_0 e^{-i\omega(t_1-t_2)} \\
=& x_0\left(2n_1\cos\omega(t_1-t_2) + e^{-i\omega(t_1-t_2)}\right)
\end{aligned} \tag{13}$$

Also

$$\begin{aligned}
& \langle n_1|x(t_1)x(t_2)x(t_3)|n_1\rangle \\
=& \langle n|e^{iHt_1}xe^{iH(t_2-t_1)}xe^{iH(t_3-t_2)}xe^{-iHt_3}|n_1\rangle \\
=& \sum_{n_2}\sum_{n_3} \langle n_1|e^{iHt_1}x\,|n_2\rangle\!\langle n_2|\,e^{iH(t_2-t_1)}xe^{iH(t_3-t_2)}\,|n_3\rangle\!\langle n_3|\,xe^{-iHt_3}|n_1\rangle \\
=& \sum_{n_2}\sum_{n_3} \underbrace{e^{iE_{n_1}(t_1-t_3)}e^{iE_{n_2}(t_2-t_1)}e^{iE_{n_3}(t_3-t_2)}}_{F(n_1,n_2,n_3)}\,\langle n_1|x\,|n_2\rangle\!\langle n_2|\,x\,|n_3\rangle\!\langle n_3|\,x|n_1\rangle \\
=& \sum_{n_2}\sum_{n_3} F(n_1,n_2,n_3)\left(\sqrt{n_2}\,\delta_{n_1\,(n_2-1)} + \sqrt{n_2+1}\,\delta_{n_1\,(n_2+1)}\right)\,\langle n_2|x|n_3\rangle\,\left(\sqrt{n_3}\,\delta_{n_1\,(n_3-1)} + \sqrt{n_3+1}\,\delta_{n_1\,(n_3+1)}\right) \\
\propto& \langle n_2|x|n_2\rangle \\
=& 0
\end{aligned} \tag{14}$$

Problem 3

Q2. Compute the following correlation functions in the n^{th} excited state of the harmonic oscillator: $\langle \hat{x}(t_1)\,\hat{x}(t_2)\rangle$ and $\langle \hat{x}(t_1)\,\hat{x}(t_2)\,\hat{x}(t_3)\rangle$. You can assume $t_1 > t_2 > t_3$ but do indicate what would change if this temporal ordering was not imposed.

First let us note that

$$
\begin{aligned}
\langle n_1|x(t_1)x(t_2)|n_4\rangle =& \langle n|e^{iHt_1}xe^{iH(t_2-t_1)}xe^{-iHt_2}|n_4\rangle \\
=& \sum_{n_2} \langle n_1|e^{iHt_1}x\,|n_2\rangle\langle n_2|\,e^{iH(t_2-t_1)}xe^{-iHt_2}|n_4\rangle \\
=& \sum_{n_2} e^{i(E_{n_1}-E_{n_2})t_1}e^{i(E_{n_2}-E_{n_4})t_2}\,\langle n_1|x\,|n_2\rangle\langle n_2|\,x|n_4\rangle \\
=& \frac{x_0}{2}\sum_{n_2} e^{i(E_{n_1}-E_{n_2})t_1}e^{i(E_{n_2}-E_{n_4})t_2}\times \\
& \left(\sqrt{n_2}\delta_{n_1\,(n_2-1)} + \sqrt{n_2+1}\delta_{n_1\,(n_2+1)}\right)\left(\sqrt{n_2}\delta_{n_4\,(n_2-1)} + \sqrt{n_2+1}\delta_{n_4\,(n_2+1)}\right) \\
=& \frac{x_0}{2}\sum_{n_2} e^{i(E_{n_1}-E_{n_2})t_1}e^{i(E_{n_2}-E_{n_4})t_2}\times \\
& \Big(\sqrt{n_2}\delta_{(n_1+1)\,n_2}\sqrt{n_2}\delta_{(n_4+1)\,(n_2)} + \sqrt{n_2+1}\delta_{(n_1-1)\,n_2}\sqrt{n_2}\delta_{(n_4+1)\,(n_2)} \\
& \quad + \sqrt{n_2}\delta_{(n_1+1)\,n_2}\sqrt{n_2+1}\delta_{(n_4-1)\,(n_2)} + \sqrt{n_2+1}\delta_{(n_1-1)\,n_2}\sqrt{n_2+1}\delta_{(n_4-1)\,(n_2)}\Big) \\
=& \frac{x_0}{2}e^{i(E_{n_1}-E_{n_1+1})t_1}e^{i(E_{n_4+1}-E_{n_4})t_2}(n_1+1)\delta_{n_1\,n_4} \\
& + \frac{x_0}{2}e^{i(E_{n_1}-E_{n_1-1})t_1}e^{i(E_{n_4+1}-E_{n_4})t_2}\sqrt{(n_1-1)n_1}\delta_{n_1\,(n_4+2)} \\
& + \frac{x_0}{2}e^{i(E_{n_1}-E_{n_1-1})t_1}e^{i(E_{n_4-1}-E_{n_4})t_2}\sqrt{(n+1)(n_1+2)}\delta_{n_1\,n_4-2} \\
& + \frac{x_0}{2}e^{i(E_{n_1}-E_{n_1-1})t_1}e^{i(E_{n_4-1}-E_{n_4})t_2}n_1\delta_{n_1\,n_4} \\
=& \frac{x_0^2}{2}\Big(e^{-i\omega(t_1-t_2)}(n_1+1)\delta_{n_1\,n_4} + e^{i\omega(t_1+t_2)}\sqrt{(n_1-1)n_1}\delta_{n_1\,(n_4+2)} \\
& + e^{-i\omega(t_1+t_2)}\sqrt{(n+1)(n_1+2)}\delta_{n_1\,(n_4-2)} + e^{i\omega(t_1-t_2)}n_1\delta_{n_1\,n_4}\Big)
\end{aligned}
\tag{15}
$$

The diagonal elements here reduce to the same ones we found in problem 2, i.e. we expect this to hold. Also

$$
\begin{aligned}
\langle\zeta|n\rangle =& \langle 0|e^{\zeta^* a}|n\rangle \\
=& e^{\sqrt{n}\zeta^*}\,\langle 0|n-1\rangle \\
=& e^{\sqrt{n}\zeta^*}\,\delta_{n\,1}
\end{aligned}
\tag{16}
$$

$$
\Rightarrow \langle n|\zeta\rangle = (\langle\zeta|n\rangle)^\dagger = e^{\sqrt{n}\zeta}\delta_{n\,1}
$$

Then computing Now taking

$$
\langle\zeta|x(t_1)x(t_2)|\zeta\rangle = \sum_{\substack{n_1\\ n_4}} \langle\zeta|n_1\rangle\langle n_1|x(t_1)x(t_2)|n_4\rangle\langle n_4|\zeta\rangle
\tag{17}
$$

Problem 4

<u>Note:</u> There are many approaches than can be taken in this problem. Since I have solved it before in a previous QM class (see problem 5.4.3 in Shankar), I will do the problem in the same manor I have done previously and not the way it was discussed at Mukunds office hour. I have discussed the answer with classmates and checked the literature, see e.g. [1], and obtain the same result for the position space propagator, although through different means.

We have the following Hamiltonian

$$H = T + fx \tag{18}$$

And we have to find the propagator given by

$$K(x,t|x',t=0) = \left\langle x \middle| e^{-\frac{i}{\hbar}H(t-t')} \middle| x' \right\rangle \tag{19}$$

Since our Hamiltonians dependence on x is simpler than it's dependence on p it would instead be easier to calculate the propagator in the momentum bases and then transfer to position space afterwards:

$$K(x,t|x',0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}p \int_{-\infty}^{\infty} \mathrm{d}p'\, K(p,t|p',0) e^{\frac{i}{\hbar}px} e^{\frac{i}{\hbar}p'x'}$$

Where we have set $\hbar = 1$. We first have to obtain the initial states. We first write the Schrödinger equation for stationary states:

$$H\left|\psi\right\rangle = E\left|\psi\right\rangle \tag{20}$$

in the momentum basis where we have the operator $\hat{x} = i\frac{\partial}{\partial p}$ this is just:

$$\left(\frac{p^2}{2m} + if\frac{\partial}{\partial p}\right)\psi(p) = E\psi(p) \tag{21}$$

Rearranging the terms we get:

$$\frac{\partial\psi(p)}{\partial p} = \underbrace{\left[\left(E - \frac{p^2}{2m}\right)\frac{i}{\hbar f}\right]}_{=\text{function of } p}\psi(p)$$

The solution to this is just the exponential function:

$$\psi(p) = C\exp\left[\frac{i}{\hbar f}\left(E - \frac{p^2}{6m}\right)p\right]$$

The constant C can be found from the normalization condition

$$\langle\psi_E(p)|\psi_{E'}(p)\rangle = \delta(E-E') \tag{22}$$

We have

$$|C|^2\int_{-\infty}^{\infty}\mathrm{d}p\,\exp\left[\frac{i}{\hbar f}\left(E-E'\right)p\right] = |C|^2 2\pi\hbar f\delta(E-E') \Rightarrow |C|^2 = \frac{1}{2\pi\hbar f}$$

Where we have used the following property of the dirac delta function:

$$\delta(x-x') = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(x-x')p}\,\mathrm{d}p$$

[1] Barry R. Holstein, American Journal of Physics 65, 414 (1997)

We can now find the propagator in the momentum basis

$$K(p,t|p',0)=\frac{1}{2\pi f}\int_{-\infty}^{\infty}\mathrm{d}E\exp\left[\frac{i}{f}\left(Ep-\frac{p^3}{6m}\right)\right]\exp\left[-\frac{i}{f}\left(Ep'-\frac{p'^3}{6m}\right)\right]\exp\left[-iE(t-t')\right]$$

$$K(p,t|p',0)=\frac{1}{2\pi f}\int_{-\infty}^{\infty}\mathrm{d}E\exp\left[\frac{i}{f}E\left(p-p'-f(t-t')\right)\right]\exp\left[\frac{i}{f}\frac{p'^3-p^3}{6m}\right]$$

Again using the fact that this just integrates out to the Dirac Delta function we get:

$$K(p,t|p',0)=\delta(p-p'-f(t-t'))\exp\left[\frac{i}{f}\frac{p'^3-p^3}{6m}\right]$$

We can now obtain the propagator in position space

$$K(x,t|x',0)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}p\int_{-\infty}^{\infty}\mathrm{d}p'\,\delta(p-p'-f(t-t'))\exp\left[\frac{i}{f}\frac{p'^3-p^3}{6m}\right]\exp[ipx]\exp\left[ip'x'\right]$$

Integrating over the deltafunction gives

$$K(x,t|x',0)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}p\;\exp\left[\frac{i}{f}\frac{(p-f(t-t'))^3-p^3}{6m}\right]\exp\left[\frac{i}{\hbar}px\right]\exp\left[\frac{i}{\hbar}(p-f(t-t'))x'\right]$$

plugging this integral into Mathematica and simplifying we get:

$$K(x,t|x',0)=\sqrt{\frac{m}{2\pi i(t-t')}}\exp\left[i\left(\frac{m(x-x')^2}{2(t-t')}+\frac{1}{2}f(t-t')(x-x')-\frac{1}{24m}f^2(t-t')^2\right)\right]$$