Homework 2

Taro V. Brown

Problem 1

Part a

We will use the map provided in class

$$z = \frac{a}{\pi}(1 + w + e^w) \tag{2.75}$$

which maps a pair of infinite lines in the w plane to a pair of semi-infinite lines in the z plane. Note that the normalization in front makes sure the capacitors go from being at $a, -a$ to $-\pi, \pi$. We will analyze the lines $w = u \pm iv$, since the equipotentials are straight lines. Inserting into the transformation:

$$\begin{aligned} z(u+iv) &= \frac{a}{\pi}(1 + u + iv + e^{u+iv}) \\ &= \frac{a}{\pi}(1 + u + iv + e^u [\cos v + i \sin v]) \\ &= \frac{a}{\pi}([1 + u + e^u \cos v] + i [v + e^u \sin v]) \end{aligned} \tag{2.76}$$

From which we can identify x and y by setting $z = x + iy$, so

$$\begin{aligned} x &= \frac{a}{\pi}[1 + u + e^u \cos v] \\ y &= \frac{a}{\pi}[v + e^u \sin v] \end{aligned} \tag{2.77}$$

The potential in the infinite capacitor is $V = V_0 \frac{v}{\pi}$ between the places. This potential is the imaginary part of the analytic function $\Phi(w) = V_0 \frac{w}{\pi}$. This means that varying u at constant some constant v will give the the equipotential lines and similarly varying v at constant u values give the electric field lines. We now provide a sketch of this as well as the Mathematica code written to do so. We have chosen $a = \pi$ for convenience.

```
Potential =
ParametricPlot[{{1 + u + Exp[u] Cos[Pi / 6],
                 Pi / 6 + Exp[u] Sin[Pi / 6]}, {1 + u + Exp[u],
                 0}, {1 + u + Exp[u] Cos[Pi / 3],
                 Pi / 3 + Exp[u] Sin[Pi / 3]}, {1 + u + Exp[u]
                 Cos[Pi / 2],
                 Pi / 2 + Exp[u] Sin[Pi / 2]}, {1 + u + Exp[u]
```

```
                    Cos[2 Pi / 3],
                    2 Pi / 3 + Exp[u] Sin[2 Pi / 3]}, {1 + u + Exp[u]
                    Cos[5 Pi / 6],
                    5 Pi / 6 + Exp[u] Sin[5 Pi / 6]}, {1 + u + Exp[u]
                    Cos[8 Pi / 9],
                    8 Pi / 9 + Exp[u] Sin[8 Pi / 9]}, {1 + u + Exp[u]
                    Cos[0.97 Pi],
                    0.97 Pi + Exp[u] Sin[0.97 Pi]},
          {1 + u + Exp[u] Cos[-Pi / 6], -Pi / 6 + Exp[u]
                    Sin[-Pi / 6]}, {1 +
                    u + Exp[u],
                    0}, {1 + u + Exp[u] Cos[-Pi / 3], -Pi / 3 +
                    Exp[u] Sin[-Pi / 3]}, {1 + u + Exp[u]
                    Cos[-Pi / 2], -Pi / 2 +
                    Exp[u] Sin[-Pi / 2]}, {1 + u +
                    Exp[u] Cos[-2 Pi / 3], -2 Pi / 3 + Exp[u]
                    Sin[-2 Pi / 3]}, {1 +
                    u + Exp[u] Cos[-5 Pi / 6], -5 Pi / 6 +
                    Exp[u] Sin[-5 Pi / 6]}, {1 + u +
                    Exp[u] Cos[-8 Pi / 9], -8 Pi / 9 + Exp[u]
                    Sin[-8 Pi / 9]}, {1 +
                    u + Exp[u] Cos[-0.97 Pi], -0.97 Pi +
                    Exp[u] Sin[-0.97 Pi]}}, {u, -9, 2.43},
PlotStyle -> {{Gray, Thick}, {Gray, Thick}}];
Plates = ParametricPlot[{{1 + u + Exp[u] Cos[Pi],
                    Pi + Exp[u] Sin[Pi]}, {1 + u + Exp[u]
                    Cos[-Pi], -Pi +
                    Exp[u] Sin[-Pi]}}, {u, -9, 2.43},
PlotStyle -> {{Blue, Thick}, {Blue, Thick}}];
Electric =
ParametricPlot[{{1 + -8. + Exp[-8] Cos[v],
                    v + Exp[-8] Sin[v]}, {1 + -6. + Exp[-6] Cos[v],
                    v + Exp[-6] Sin[v]}, {1 + -4 + Exp[-4] Cos[v],
                    v + Exp[-4] Sin[v]}, {1 + -2 + Exp[-2] Cos[v],
                    v + Exp[-2] Sin[v]}, {1 + -1 + Exp[-1] Cos[v],
                    v + Exp[-1] Sin[v]}, {1 + 0 + Exp[0] Cos[v],
                    v + Exp[0] Sin[v]}, {1 + 1 + Exp[1] Cos[v],
                    v + Exp[1] Sin[v]}, {1 + 2 + Exp[2] Cos[v],
                    v + Exp[2] Sin[v]}, {1 + 1.5 + Exp[1.5] Cos[v],
                    v + Exp[1.5] Sin[v]}}, {v, -\[Pi], Pi},
PlotStyle -> {{Red, Dashed}, {Red, Dashed}}];
```

```
Show[Electric, Plates, Potential]
```

Which results in the following figure

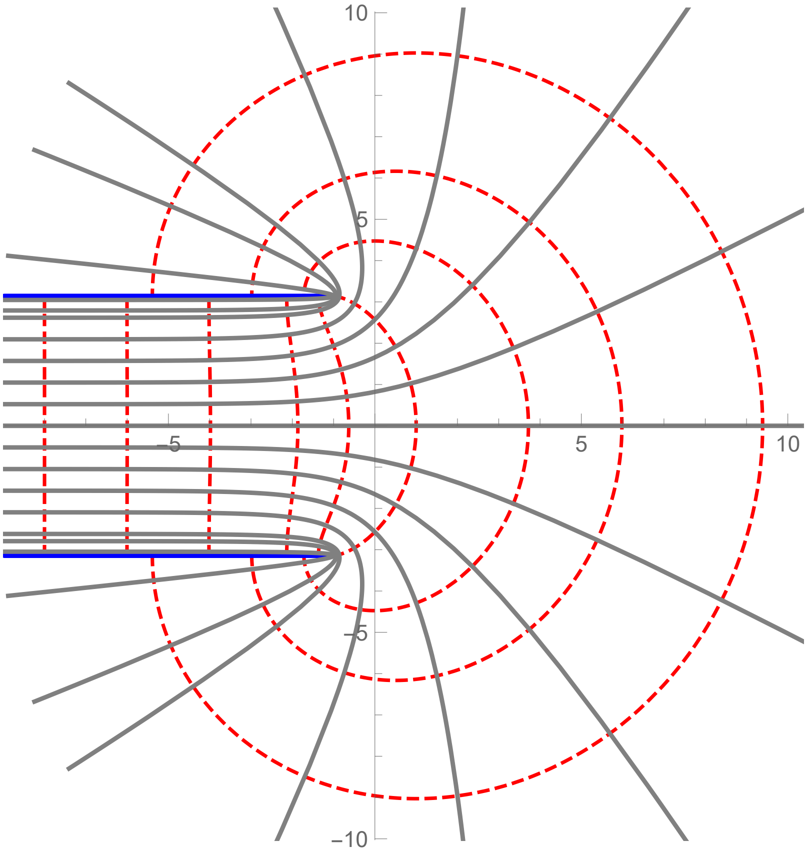

Figure 9. Plot of equipotential lines (grey) and field lines (red, dashed)

Part b

We will here use the following map, obtained from the conformal dictionary

$$w = \frac{2}{\pi}\,\mathrm{asin}(z) \tag{2.78}$$

This maps two lines to two parallel plates. The inverse, which is the one we are interested in, is

$$z = \sin\left(\frac{2}{\pi}w\right) \tag{2.79}$$

First we write it in terms of its real and complex parts using a trig identity

$$x + iy = z = \sin\left((u+iv)\frac{2}{\pi}\right) = \sin\frac{\pi}{2}u\cosh\frac{\pi}{2}v + i\cos\frac{\pi}{2}u\sinh\frac{\pi}{2}v \tag{2.80}$$

Then to find equipotentials we first identity

$$\begin{aligned} x &= \sin\frac{\pi}{2}u\cosh\frac{\pi}{2}v \\ y &= \cos\frac{\pi}{2}u\sinh\frac{\pi}{2}v \end{aligned} \tag{2.81}$$

Then taking $u = constant$ we get

$$\left(\frac{x}{\sin\frac{\pi}{2}u}\right)^2 - \left(\frac{y}{\cos\frac{\pi}{2}u}\right)^2 = \cosh^2\frac{\pi v}{2} - \sinh^2\frac{\pi v}{2} = 1 \tag{2.82}$$

or a hyperbola. Smiliar holding v constant to find the field lines, we get elipses

$$\left(\frac{x}{\cosh\frac{\pi}{2}v}\right)^2 + \left(\frac{y}{\sinh\frac{\pi}{2}v}\right)^2 = \cos^2\frac{\pi u}{2} + \sin^2\frac{\pi u}{2} = 1 \tag{2.83}$$

Both hyperboles and elipses are conic sections. Sketching these for different values of u and v we get, using the following mathematica code

```
ContourPlot[{x^2/Sin[1]^2 - y^2/Cos[1]^2 == 1,
        x^2/Sin[2]^2 - y^2/Cos[2]^2 == 1,
        x^2/Sin[2.5]^2 - y^2/Cos[2.5]^2 == 1,
        x^2/Sin[3]^2 - y^2/Cos[3]^2 == 1,
        x^2/Cosh[1]^2 + y^2/Sinh[1]^2 == 1,
        x^2/Cosh[2]^2 + y^2/Sinh[2]^2 == 1,
        x^2/Cosh[2.5]^2 + y^2/Sinh[2.5]^2 == 1,
        x^2/Cosh[3]^2 + y^2/Sinh[3]^2 == 1},
        {x, -6.5, 6.5}, {y, -6.5, 6.5}]
```

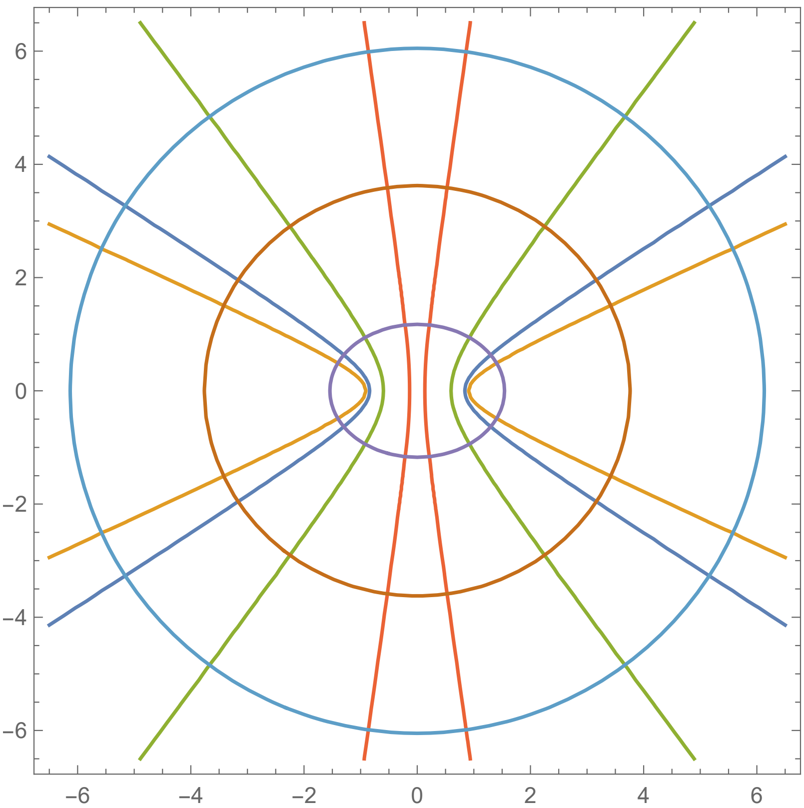

Figure 10. Plot of equipotential lines (parabolas) and field lines (elipses)

Problem 2

The transformation that takes the unit disc to the unit disc is the linear fractional transformation

$$f(z) = e^{i\phi}\frac{z-\alpha}{\bar{\alpha}z-1}, \qquad |\alpha| < 1, \qquad -\pi < \phi < \pi \tag{2.84}$$

The rotation part $e^{i\phi}$ obviously just maps points in the disc to other points in the disc, while the linear fraction part moves the origin from $z = 0$ to $f = \alpha$. We can show that this maps to the unit disc if $|f(z)| < 1$ so that every point in the domain lies within the disc. First we note that

$$\begin{aligned} |z-a|^2 &= (z-a)(\bar{z}-\bar{a}) = |z|^2 + |\alpha|^2 - \alpha\bar{z} - \bar{\alpha}z \\ |\bar{\alpha}z-1|^2 &= (\bar{\alpha}z-1)(\alpha\bar{z}-1) = |z|^2|\alpha|^2 - \alpha\bar{z} - \bar{\alpha}z + 1 \end{aligned} \tag{2.85}$$

If we take the difference of the two we find

$$\begin{aligned} |z-a|^2 - |\bar{\alpha}z-1|^2 &= |z|^2 + |\alpha|^2 - \alpha\bar{z} - \bar{\alpha}z - |z|^2|\alpha|^2 + \alpha\bar{z} + \bar{\alpha}z - 1 \\ &= |z|^2 + |\alpha|^2 - |z|^2|\alpha|^2 - 1 \\ &< 0 \end{aligned} \tag{2.86}$$

where the last inequality only holds since we started within the disc, $|z| < 1$, and since we assumed $|\alpha| < 1$. This implies that $|z-a| < |\bar{\alpha}z-1|$ and hence that

$$|f(z)| = \frac{|z-\alpha|}{|\bar{\alpha}z-1|} < 1 \tag{2.87}$$

which means that every point in the new domain, lies within the unit circle.

Problem 3

Part a

Here we will use the composition law

- If $w = f(z)$ and $\zeta = g(w)$ are analytic then $\zeta = g \circ f(z)$ is a conformal map from $z \to \zeta$ planes

First we will use the following map presented in class, namely

$$w = \frac{z+1}{z-1} \tag{2.88}$$

which maps the half disc $z \in D_+$ to the upper right quadrant (URQ) $w \in 0 < \arg w < \frac{1}{2}\pi$. The map that takes the URQ to the unit disc is found in the conformal dictionary

$$\zeta(w) = \frac{iw^2+1}{iw^2-1} \tag{2.89}$$

Taking the composition we find

$$\zeta(w = \frac{z+1}{z-1}) = \frac{i\left(\frac{z+1}{z-1}\right)^2 + 1}{i\left(\frac{z+1}{z-1}\right)^2 - 1} = \frac{-i\,(1 + 2iz + z^2)}{1 - 2iz + z^2} \tag{2.90}$$

The factor $-i$ in front is the same as rotating the disc, which also leaves it invariant so a valid solution is also

$$\zeta(z) = \frac{(1 + 2iz + z^2)}{1 - 2iz + z^2} \tag{2.91}$$

Note that this can also be simplified to

$$\zeta(z) = 1 + \frac{4iz}{1 - 2iz + z^2} \tag{2.92}$$

though the fractional form might be nicer to use. One can further note that this map is not unique and by acting with the transformation found in the previous problem the disc would still be invariant and hence one could use the composition with this, to map the half disc at the origin to the unit disc centered around a different origin:

$$\begin{aligned}\zeta &= e^{i\phi}\frac{\left(1 + \frac{4iz}{1-2iz+z^2}\right) - \alpha}{\bar{\alpha}\left(1 + \frac{4iz}{1-2iz+z^2}\right) - 1} \\ &= e^{i\phi}\frac{(1 + 2iz + z^2) - \alpha(1 - 2iz + z^2)}{-1 + \bar{a} + 2i(1 + \bar{a})z + (-1 + \bar{a})z^2} \\ &= e^{i\phi}\frac{(1 + 2iz + z^2) - \alpha(1 - 2iz + z^2)}{\alpha(1 + 2iz + z^2) - (1 - 2iz + z^2)}, \qquad |\alpha| < 1, \qquad -\pi < \phi < \pi\end{aligned} \tag{2.93}$$

Part b

We will assume the radius of the outer circle to be 1 and inner circle to have the origin at $z = c$ with radius c. Basically we are looking for the following map

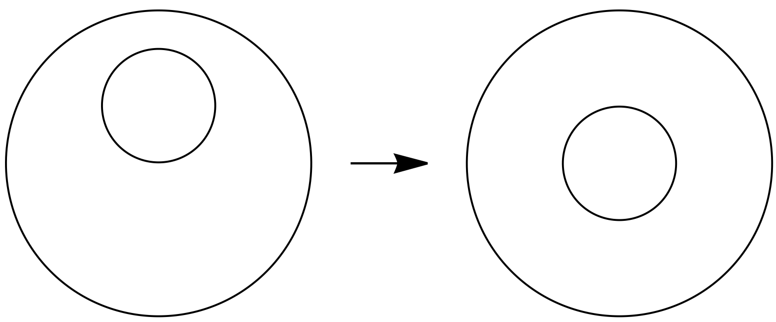

Figure 11. Non-concentric circles to annulus

Using the fractional part of the transformation from problem 2, we have a map that takes circles at origin and map them to circles centered around some other point. This implies that we can use that map here and we just need to find a value of α which takes the inner circle with origin at $z = c$ and radius c to a circle with origin at $z = 0$ and radius r. We will assume $\alpha \in \mathbb{R}$, so $\bar{\alpha} = \alpha$. Mapping the edges of the inner circle to $-r$ and r we find:

$$\begin{aligned} -r = f(z=0) &= \frac{0-\alpha}{\alpha \times 0 - 1} = \alpha \\ r = f(z=2c) &= \frac{2c-\alpha}{2\alpha c - 1} \\ \Rightarrow \quad & 2c - \alpha = -\alpha(2\alpha c - 1) \\ \Rightarrow \quad & 2\alpha^2 c - 2\alpha + 2c = 0 \\ \Rightarrow \quad & \alpha^2 c - \alpha + c = 0 \\ \Rightarrow \quad & \alpha_\pm = \frac{1 \pm \sqrt{1-4c^2}}{2c} \end{aligned} \tag{2.94}$$

The problem states that $0 < c < 1/2$ and since $|\alpha| < 1$ the only valid solution is α_-. Inserting this into the map, we find

$$\begin{aligned} f(z) &= \frac{z - \frac{1-\sqrt{1-4c^2}}{2c}}{\frac{1-\sqrt{1-4c^2}}{2c} z - 1} \\ &= \frac{1 - \sqrt{1-4c^2} - 2cz}{2c + (\sqrt{1-4c^2} - 1)z}, \qquad c < 1/2 \end{aligned} \tag{2.95}$$

Problem 4

Part a

In this problem we will be using the map from the previous problem. On could do the problem for a general c, but we note that (requiring, $c < 1/2$) the following values of c simplify the transformation

```
In[238]:= Simplify[
Table[-((-1 + Sqrt[1 - 4 b^2] + 2 b z)/(
2 b + (-1 + Sqrt[1 - 4 b^2]) z)), {b, {1/3, 1/4, 1/5, 2/5}}]]

Out[238]= {-((-3 + Sqrt[5] + 2 z)/(
          2 + (-3 + Sqrt[5]) z)), -((-2 + Sqrt[3] + z)/(
          1 + (-2 + Sqrt[3]) z)), -((-5 + Sqrt[21] + 2 z)/(
          2 + (-5 + Sqrt[21]) z)), (-1 + 2 z)/(-2 + z)}
```

The simplest transformation happens at $c = \frac{2}{5}$ meaning the center of the small circle is at $z = \frac{2}{5}$ and the radius is $r = \frac{2}{5}$. In this case we get the simple map

$$f(z) = \frac{2z-1}{z-2} \tag{2.96}$$

The potential doesn't change in the region

$$\Delta\phi = 0 \quad \text{in the region } |z| < 1 \;\; \& \;\; |z - \frac{2}{5}| > \frac{2}{5} \tag{2.97}$$

while it must satisfy the following boundary conditions

$$\begin{aligned} \phi &= \phi_a \qquad \text{at } |z| = 1 \\ \phi &= \phi_b \qquad \text{at } |z - \frac{5}{2}| = \frac{2}{5} \end{aligned} \tag{2.98}$$

Using the map the boundary conditions on the coaxial cable become

$$\begin{aligned} \Phi &= \phi_a \qquad \text{at} \qquad |f| = \frac{1}{2} \\ \Phi &= \phi_b \qquad \text{at} \qquad |f| = 1 \end{aligned} \tag{2.99}$$

In the coaxial domain this solves the Laplace equation in cylindrical coordinates which for radial symmetric setups is

$$\Phi(r) = A_0 \log|r| + B_0 \tag{2.100}$$

Matching at the boundary

$$\begin{aligned} \Phi(1) &= A_0 \log|1| + B_0 = B_0 = \phi_b \\ \Phi(1/2) &= A_0 \log|1/2| + \phi_b = \phi_a \Rightarrow A_0 = (\phi_a - \phi_b)\log 2 \end{aligned} \tag{2.101}$$

So we get

$$\begin{aligned} \Phi(f) &= (\phi_a - \phi_b)\log 2 \log|f| + \phi_b \\ \phi(z) &= (\phi_a - \phi_b)\log 2 \log|\frac{2z-1}{z-2}| + \phi_b \end{aligned} \tag{2.102}$$

To write this in terms of x and y coordinates we insert $z = x + iy$ and write the term in the logarithm in terms of its real and complex parts

$$\begin{aligned} \phi(x,y) &= (\phi_a - \phi_b)\log 2 \log|\frac{2(x+iy)-1}{(x+iy)z-2}| + \phi_b \\ &= (\phi_a - \phi_b)\log 2 \log|\frac{2x^2-5x-2y^2+2}{x^2-4x+y^2+4} + i\frac{4xy-5y}{x^2-4x+y^2+4}| + \phi_b \\ &= (\phi_a - \phi_b)\log 2 \log\sqrt{\frac{4x^2-4x+4y^2+1}{x^2-4x+y^2+4}} + \phi_b \\ &= \frac{1}{2}(\phi_a - \phi_b)\log 2 \log\frac{4x^2-4x+4y^2+1}{x^2-4x+y^2+4} + \phi_b \end{aligned} \tag{2.103}$$

Part b

We saw in class (and partially in the last HW), that the (normalized to unity) fluid streamline function for uniform flow can be written as $\Phi(z) = z$. To get the fluid flowing around a corner, which in this case is the lower left quadrant, we want a map which takes values from the quadrant to UHP. One such map is $f(z) = (iz)^{\frac{2}{3}}$, we note however that this is not necesarilly unique, since e.g. the maps z^{2n}, $n = 1, 2, 3, \ldots$ do the same job. Using this map for our streamline we simply get

$$\Phi(f) = \Phi((iz)^{\frac{2}{3}}) = (iz)^{\frac{2}{3}} \tag{2.104}$$

Or, using the inverse map

$$\phi = -iz^{\frac{3}{2}} \tag{2.105}$$

using this mapping one can write it in terms of real and complex parts as usual

$$\begin{aligned} x &= -\left(u^2+v^2\right)^{3/4} \cos\left(\frac{3}{2}\arg[iu - v]\right) \\ y &= \left(u^2+v^2\right)^{3/4} \sin\left(\frac{3}{2}\arg[iu - v]\right) \end{aligned} \tag{2.106}$$

Finally the streamlines can be sketched holding u constant and varying v using the following code (see next page for the figure)

```
Pot2 = ParametricPlot[{{-(0.5^2 + v^2)^(3/4) Cos[
                3/2 Arg[I 0.5 - v]], (0.5^2 + v^2)^(3/4)
                Sin[3/2 Arg[I 0.5 - v]]}, {-(1^2 + v^2)^(3/4) Cos[
                3/2 Arg[I 1 - v]], (1^2 + v^2)^(3/4)
                Sin[3/2 Arg[I 1 - v]]}, {-(2^2 + v^2)^(3/4) Cos[
                3/2 Arg[I 2 - v]], (2^2 + v^2)^(3/4)
                Sin[3/2 Arg[I 2 - v]]}, {-(3^2 + v^2)^(3/4) Cos[
                3/2 Arg[I 3 - v]], (3^2 + v^2)^(3/4)
                Sin[3/2 Arg[I 3 - v]]}, {-(4^2 + v^2)^(3/4) Cos[
                3/2 Arg[I 4 - v]], (4^2 + v^2)^(3/4)
                Sin[3/2 Arg[I 4 - v]]},
                {-(5^2 + v^2)^(3/4) Cos[3/2 Arg[I 5 - v]],
                     (5^2 + v^2)^(3/4)
                Sin[3/2 Arg[I 5 - v]]}}, {v, -10, 10},
 PlotStyle -> {{Red, Thick}, {Red, Thick}}];
Show[Pot2]
```

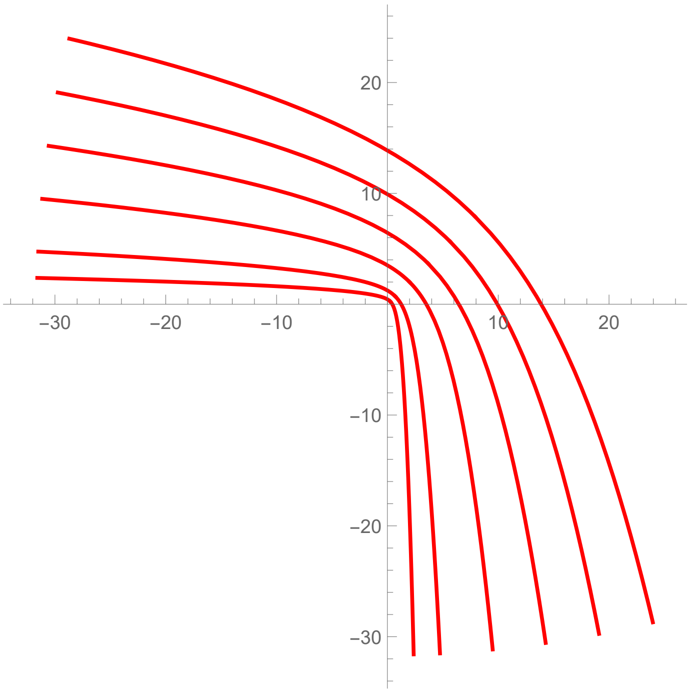

Figure 12. Streamlines around corner