# $N < 4$ On-Shell Diagrams

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Abstract: Notes on modern amplitude techniques written as part of a research project with Jaroslav Trnka.

## Contents

## 1 Introduction

During the last two decades, significant strides have been made in the study of scattering amplitudes. New modern on-shell techniques have made calculations with a large number of particles and loops feasible. Many of the significant advancements in the field has come by studying $\mathcal{N} = 4$ super Yang-Mills (sYM) and using its planar, or large $N$, limit as a sandbox to explore the underlying geometric structure.

In recent years this geometric structure of scattering amplitudes in $\mathcal{N} = 4$ sYM has been explored using the positive Grassmannian, on-shell diagrams, and the Amplituhedron. A natural progression is to try and extend these discoveries beyond planar $\mathcal{N} = 4$ sYM to explore whether the geometric interpretation is a more general feature of quantum field theories (QFTs). Similarly, considering theories with $\mathcal{N} \neq 4$, such as supergravity ($\mathcal{N} = 8$), will teach us to what extend these geometric pictures apply to other field theories.

The focus of this project has been some of these extensions, with the report structured as follows. In section —— we give a short review of the Grassmaniann, followed by an introduction to on-shell diagrams. Then, in section —– we calculate various examples using the methods described. We extend this approach to multiple loops in section —–, giving examples at four and five points, and then proceed to describe the approach used when considering theories for $\mathcal{N} \neq 4$, explicitly calculating multiple examples at five and six points. Finally we consider the pole structure of the results found.

## 2 Grassmanian

In this section we give a brief review of the Grassmanian, which will serve as a crucial ingredient for our discussion of on-shell diagrams. The Grassmanian, denoted $G(k, n)$, is the space of $k$-planes going through the origin in $n$ dimensions. It can be thought of as a generalization of projective space $P^{n-1}$, which is the space of lines going through the origin in $n$-dimensions, since $G(1, n) = P^{n-1}$. One can e.g. take $k$ vectors in $n$ dimensions, see figure 1

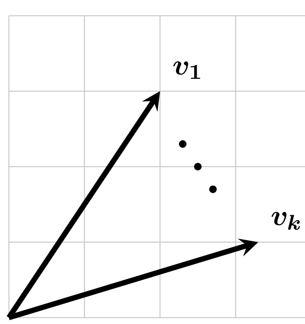



**Figure 1**.

The span of these vectors gives the $k$- plane. If they are stacked, we get a matrix $C_{\alpha a}$

$$\begin{array}{cc} & n \\ k & \begin{bmatrix} V_1 \\ \vdots \\ V_k \end{bmatrix} \end{array} \equiv C_{\alpha a}, \qquad \alpha = 1, \dots, k \quad a = 1, \dots, n \tag{2.1}$$

This matrix are generally not unique since it is GL($k$) redundant, $C_{\alpha a} \sim L_\alpha^\beta C_{\beta a}$. The dimensionality of the Grassmanian is

$$\dim G(k,n) = \overbrace{k \times n}^{k \times n \text{ matrix}} \quad \underbrace{-k^2}_{GL(k) \text{ red}} \tag{2.2}$$

The redudancy means that we can gaugefix the matrix using a linear transformation by setting any $k \times k$ blok to the identity. This is equivalent to the rescaling of a vector in projective space to $\begin{pmatrix} 1 & v_2 & v_3 & v_4 \cdots \end{pmatrix}$. Taking e.g. $G(3,5)$, we have six degrees of freedom:

$$G(3,5) = \left[\begin{array}{ccc|cc} 1 & 0 & 0 & x_4 & x_5 \\ 0 & 1 & 0 & y_4 & y_5 \\ 0 & 0 & 1 & z_4 & z_5 \end{array}\right] \tag{2.3}$$

The dimensionality of the Grassmanian are symmetric under $n \leftrightarrow k$. This is because there is a bijection between the Grassmania: $k$ and $n-k$ planes in $n$ dimensions, since these planes are orthogonal. In the case above $C^\perp$ is a 2-plane in 5 dimensions. We Can illustrate both the matrices like so,

$$\left[\begin{array}{ccc|cc} 1 & 0 & 0 & x_4 & x_5 \\ 0 & 1 & 0 & y_4 & y_5 \\ 0 & 0 & 1 & z_4 & z_5 \\ \hline -x_4 & -y_4 & -z_4 & 1 & 0 \\ -x_5 & -y_5 & -z_5 & 0 & 1 \end{array}\right] \tag{2.4}$$

With the bottom part just being the negative transpose of the $x, y$ and $z$ coordinates in the upper right corner. The SL($k$) invariants are determinants of any $k$ coloumns of the matrix (the minors), labeling these by their indices:

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_k \end{pmatrix} \tag{2.5}$$

We will now proceed to used the properties of the Grassmanian by introducing the concept of on-shell diagrams.

## 3 Introduction to on-shell diagrams

In this section we introduce the notion of on-shell diagrams, and give examples on different ways one can calculate diagrams from these.

One can construct $(k \times n)$ C-matrices out of on-shell diagrams, where $k$ labels the number of negative helicity particles and $n$ is the total number of particles. The procedure to construct a matrix from a diagram is as follows

- Label the edges of the on-shell diagram by $\alpha$'s.
- Starting at one of the incoming particles (say, 2), trace a path to one of the outgoing particles (say, 4). Multiply the $\alpha$'s encountered on the path. This would be entry $C_{2\,4}$ in the C-matrix, i.e.

$$C_{ab} = \sum_{\Gamma(a \to b)} \prod_j \alpha_j \tag{3.1}$$

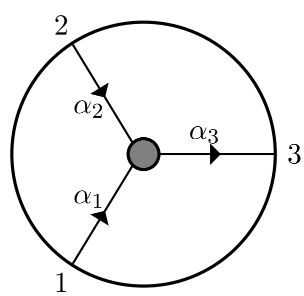
- Going from an incoming particle to another incoming particle, put a 0 in that matrix entry, unless it is the a path starting and ending in the same particle, then put a 1.
- If the path encounters an internal cycle that forms a complete loop, the matrix element is multiplied by the geometric series, labeled by $\delta$, containing the edge-variables encountered in that loop, i.e.

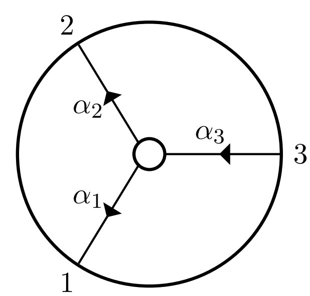
$$\delta = \sum_{n=0}^{\infty} (\alpha_i \alpha_j \alpha_k \cdots)^n = \frac{1}{1 - (\alpha_i \alpha_j \alpha_k \cdots)} \tag{3.2}$$

We will in this section calculate amplitudes through various techniques using the on-shell diagrams.

### 3.1 Three point on-shell form

The easiest example is the two diagrams representing the three-point MHV and $\overline{\text{MHV}}$ amplitudes. Taking e.g the amplitudes to be $A_3^{\text{MHV}}(1^-, 2^-, 3^+)$ and $A_3^{\overline{\text{MHV}}}(1^+, 2^+, 3^-)$, the on-shell diagrams are



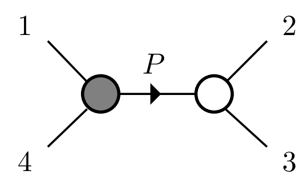
These produce the two following $C$ matrices, respectively

$$C = \begin{pmatrix} 1 & 0 & \alpha_1\alpha_2 \\ 0 & 1 & \alpha_2\alpha_3 \end{pmatrix}, \qquad\qquad C = \begin{pmatrix} \alpha_1\alpha_3 & \alpha_2\alpha_3 & 1 \end{pmatrix} \tag{3.3}$$

Because of momentum conservation and little group invariance, the solution of the delta functions in this case leads to

$$\begin{aligned} A_3^{\mathrm{MHV}}(1,2,3) &= \frac{\delta^8\left(\sum_{i=1}^3 \lambda_i \tilde{\eta}_i\right)\delta^4\left(\sum_{i=1}^3 \lambda_i \tilde{\lambda}_i\right)}{\langle 12\rangle\langle 23\rangle\langle 31\rangle} \\ A_3^{\overline{\mathrm{MHV}}}(1,2,3) &= \frac{\delta^4\left([12]\tilde{\eta}_3 + [23]\tilde{\eta}_1 + [31]\tilde{\eta}_2\right)\delta^4\left(\sum_{i=1}^3 \lambda_i \tilde{\lambda}_i\right)}{[12][23][31]} \end{aligned} \tag{3.4}$$

One can further construct more elaborate diagrams by gluing these three point functions together. The simplest possible diagram out of two opposite helicity ($k = 1$ and $k = 2$) amplitudes, is shown below




To construct the four-point diagram we then glue the two three-point amplitudes together by integrating over the internal degrees of freedom through

$$\prod_I \int \mathrm{d}^4\tilde{\eta}_I \int \frac{d^2\lambda_I d^2\tilde{\lambda}_I}{GL(1)} \tag{3.5}$$

Explicitly we have

$$\begin{aligned} \int \mathrm{d}\tilde{\eta}_P \int \frac{d^2\lambda_P d^2\tilde{\lambda}_P}{GL(1)} &\frac{\delta^8\left(\lambda_1\tilde{\eta}_1 + \lambda_4\tilde{\eta}_4 + \lambda_P\tilde{\eta}_P\right)\delta^4\left(\lambda_1\tilde{\lambda}_1 + \lambda_4\tilde{\lambda}_4 + \lambda_P\tilde{\lambda}_P\right)}{\langle 14\rangle\langle 4P\rangle\langle P1\rangle} \\ \times &\frac{\delta^4\left([23]\tilde{\eta}_P + [3P]\tilde{\eta}_2 + [P3]\tilde{\eta}_3\right)\delta^4\left(\lambda_2\tilde{\lambda}_2 + \lambda_3\tilde{\lambda}_3 - \lambda_P\tilde{\lambda}_P\right)}{[23][3P][P2]} \end{aligned} \tag{3.6}$$

First we solve the delta-function constraint by projecting along $\lambda_1$

$$\begin{aligned} &\lambda_1\tilde{\lambda}_1 + \lambda_4\tilde{\lambda}_4 + \lambda_P\tilde{\lambda}_P = 0 \\ \Rightarrow &\tilde{\lambda}_P = \frac{\langle 41\rangle}{\langle 1P\rangle}\tilde{\lambda}_4 \end{aligned} \tag{3.7}$$

Similarly we use the other delta-function and project using $\tilde{\lambda}_3$

$$\begin{aligned} &\lambda_2\tilde{\lambda}_2 + \lambda_3\tilde{\lambda}_3 - \lambda_P\tilde{\lambda}_P = 0 \\ \Rightarrow &\lambda_P = \frac{[23]}{[P3]}\lambda_2 \end{aligned} \tag{3.8}$$

combining these we obtain

$$\begin{aligned}\tilde{\lambda}_P \lambda_P = \lambda_2 \tilde{\lambda}_4 \frac{\langle 41 \rangle [23]}{\langle 1P \rangle [P3]} &= \lambda_2 \tilde{\lambda}_4 \frac{[23]}{[43]} \\ &= \lambda_2 \tilde{\lambda}_4 \frac{\langle 41 \rangle}{\langle 12 \rangle}\end{aligned} \tag{3.9}$$

where we have used $P = -1 - 4 = 2 + 3$ in the last two equalities. Solving this collapses the momentum conservation delta function as well as giving a Jacobian factor of $\frac{1}{\langle 23 \rangle [32]}$

$$\begin{aligned}\lambda_P &= \lambda_2 \\ \tilde{\lambda}_P &= \lambda_4 \frac{\langle 41 \rangle}{\langle 12 \rangle} = \tilde{\lambda}_4 \frac{[23]}{[43]}\end{aligned} \tag{3.10}$$

We then use these in one of the grassmann delta-functions

$$\begin{aligned}\tilde{\eta}_P &= \frac{-[3P]\tilde{\eta}_2 - [P2]\tilde{\eta}_3}{[23]} \\ &= -\frac{1}{[23]} \times \frac{[34][23]}{[43]} \times \tilde{\eta}_2 - \frac{1}{[23]} \times \frac{[42] \langle 41 \rangle}{\langle 12 \rangle} \times \tilde{\eta}_3 \\ &= \tilde{\eta}_2 + \frac{\langle 13 \rangle}{\langle 12 \rangle} \times \tilde{\eta}_3\end{aligned} \tag{3.11}$$

This can be obtained from contracting

$$\lambda_P \tilde{\eta}_P = \lambda_2 \tilde{\eta}_2 + \lambda_3 \tilde{\eta}_3 \tag{3.12}$$

with $\lambda_1$, since $\lambda_P = \lambda_2$. Using this in the other grassmann delta function we get $[23]^4 \delta^8(\sum_i \lambda_i \tilde{\eta}_i)$. Finally we take the solutions (3.10) and insert them into the bosonic delta-function

$$\begin{aligned}0 = \lambda_1 \tilde{\lambda}_1 + \lambda_4 \tilde{\lambda}_4 + \lambda_P \tilde{\lambda}_P &= \lambda_1 \tilde{\lambda}_1 + \tilde{\lambda}_4 \left( \lambda_4 + \lambda_2 \frac{[23]}{[43]} \right) \\ &= \lambda_1 \tilde{\lambda}_1 + \tilde{\lambda}_4 \left( \frac{\lambda_4 [43] + \lambda_2 [23]}{[43]} \right) = \lambda_1 \left( \tilde{\lambda}_1 + \tilde{\lambda}_4 \frac{[13]}{[34]} \right) \\ &= \lambda_1 \left( \frac{\tilde{\lambda}_1 [34] + \tilde{\lambda}_4 [13]}{[34]} \right) = \lambda_1 \tilde{\lambda}_3 \frac{[14]}{[34]}\end{aligned} \tag{3.13}$$

Since $\lambda_1 \neq 0$, and $\tilde{\lambda}_3 \neq 0$ this leads to $[14] = 0$ which in turn gives us

$$(p_1 + p_4)^2 = \langle 14 \rangle [41] = 0 \tag{3.14}$$

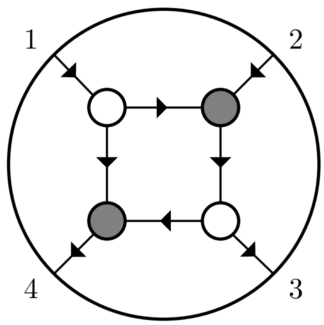
Now we only need the kinematic part of the integrand. Including the Jacobians and using $\langle 1P \rangle [P3] = \langle 12 \rangle [23]$ and $\langle 4P \rangle [P2] = \langle 43 \rangle [32]$ we obtain

$$\frac{1}{\langle 14 \rangle \langle 4P \rangle \langle P1 \rangle} \times \frac{1}{[23][3P][P2]} \times \frac{[23]^4}{\langle 23 \rangle [23]} = \frac{1}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} \tag{3.15}$$

Such that we in total have the form

$$\frac{\delta^8(\mathcal{Q})\,\delta^4(P)}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\delta((p_1+p_4)^2) \tag{3.16}$$

This is not surprising since the diagram we considered is not a Feynman diagram. Rather the first diagram to give the four-point on-shell amplitude arises from gluing 4 three-point functions together




$$\rightarrow \frac{\delta^8(\mathcal{Q})\delta^4(P)}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}$$

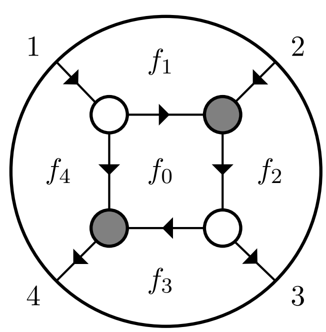
In the next section we will present two methods of getting the amplitude directly from the diagram.

## 3.2 Four-point directly from $G(2,4)$ matrix

The calculation if the C-matrix can be performed either using face or edge variables. We are going to show how to do both for good measure.

### Using face-variables

The four-point diagram with face-variables looks like this




While the

$$C_{ab} = -\sum_{\Gamma(a\to b)}\prod_j(-f_j)\,, \qquad \text{on the right} \tag{3.17}$$

with the constraint

$$\prod_j f_j = -1 \tag{3.18}$$

Using the above, we find the matrix to have the following entries

$$C = \begin{pmatrix} 1 & 0 & f_0 f_3 f_4 & f_4(1-f_0) \\ 0 & 1 & -f_0 f_1 f_3 f_4 & f_0 f_1 f_4 \end{pmatrix} \tag{3.19}$$

Note that $f_2$ doesn't show up, which means that according to (3.18) we can take the remaining $f$'s as independent. Positivity (all minors are positive) then demands that

$$f_0 < 0, \qquad f_1 > 0, \qquad f_2 < 0, \qquad f_3 < 0, \tag{3.20}$$

While the perpendicular C-matrix satisfying $C \cdot C^{\perp} = 0$ is easily obtained

$$C^{\perp} = \begin{pmatrix} -f_0 f_3 f_4 & f_0 f_1 f_3 f_4 & 1 & 0 \\ -f_4(1-f_0) & -f_0 f_1 f_4 & 0 & 1 \end{pmatrix} \tag{3.21}$$

we can then find the form through

$$\mathrm{d}\Omega = \frac{\mathrm{d}f_0}{f_0}\frac{\mathrm{d}f_1}{f_1}\frac{\mathrm{d}f_3}{f_3}\frac{\mathrm{d}f_4}{f_4}\delta(C \cdot \tilde{\lambda})\delta(C^{\perp} \cdot \lambda)\delta(C \cdot \tilde{\eta}) \tag{3.22}$$

First let us look at the delta-functions, such that we can specify the face-variables in terms of the spinor products. We start by looking at $\lambda \cdot C^{\perp} = 0$, from which we can two equations

$$C^{\perp} \cdot \lambda = 0 \Rightarrow \begin{cases} -\lambda_1 f_0 f_3 f_4 + \lambda_2 f_0 f_1 f_3 f_4 + \lambda_3 & = 0 \\ -\lambda_1 f_4(1-f_0) - \lambda_2 f_0 f_1 f_4 + \lambda_4 & = 0 \end{cases} \tag{3.23}$$

By multiplying the first equation by $\tilde{\lambda}_2$ one obtains $f_0 f_3 f_4 = -\frac{\langle 23 \rangle}{\langle 12 \rangle}$. Similarly multiplying the second equation by $\tilde{\lambda}_1$ we get $f_0 f_1 f_4 = \frac{\langle 14 \rangle}{\langle 12 \rangle}$. Combining these two,

$$f_1 = -\frac{\langle 14 \rangle}{\langle 23 \rangle} f_3 \tag{3.24}$$

Then multiplying the first equation by $\tilde{\lambda}_1$ we have $f_0 f_1 f_3 f_4 = -\frac{\langle 13 \rangle}{\langle 12 \rangle}$ together with the previous result, this leads to

$$f_3 = -\frac{\langle 13 \rangle}{\langle 14 \rangle} \qquad \text{and} \qquad f_1 = \frac{\langle 13 \rangle}{\langle 23 \rangle} \tag{3.25}$$

The other equations are solved similarly and we obtain

$$\begin{aligned} f_0 &= -\frac{\langle 14 \rangle \langle 23 \rangle}{\langle 12 \rangle \langle 34 \rangle} \\ f_4 &= -\frac{\langle 34 \rangle}{\langle 13 \rangle} \end{aligned} \tag{3.26}$$

Let us now evaluate the two remaining delta-functions. From $C \cdot \tilde{\lambda}$ we get two equations.

$$0 = \tilde{\lambda}_1 + f_0 f_3 f_4 \tilde{\lambda}_3 + f_4(1-f_0)\tilde{\lambda}_4 = \tilde{\lambda}_1 + \frac{\langle 32 \rangle}{\langle 12 \rangle}\tilde{\lambda}_3 + \frac{\langle 42 \rangle}{\langle 12 \rangle}\tilde{\lambda}_4 \tag{3.27}$$

and

$$0 = \tilde{\lambda}_2 + \frac{\langle 13\rangle}{\langle 12\rangle}\tilde{\lambda}_3 + \frac{\langle 14\rangle}{\langle 12\rangle}\tilde{\lambda}_4 \tag{3.28}$$

where we have used a Schouten identity for the coefficient of $\tilde{\lambda}_4$

$$\langle 41\rangle\langle 23\rangle + \langle 12\rangle\langle 34\rangle = \langle 13\rangle\langle 24\rangle \tag{3.29}$$

We see that these equations can all be obtained from a momentum conservation delta-function by contracting it with $\lambda_1$ and $\lambda_2$

$$\delta^4(\lambda_1\tilde{\lambda}_1 + \lambda_2\tilde{\lambda}_2 + \lambda_3\tilde{\lambda}_3 + \lambda_4\tilde{\lambda}_4) \equiv \delta^4\left(P\right) \tag{3.30}$$

For the last delta-function we get the exact same thing except for replacing $\tilde{\lambda}_i \to \tilde{\eta}_i$

$$\delta^8(\lambda_1\tilde{\eta}_1 + \lambda_2\tilde{\eta}_2 + \lambda_3\tilde{\eta}_3 + \lambda_4\tilde{\eta}_4) \equiv \delta^8\left(\mathcal{Q}\right) \tag{3.31}$$

Note that we get an extra factor of $\frac{1}{\langle 12\rangle^4}$ from re-writing the delta-functions by projecting along $\lambda_1$ and $\lambda_2$. Finally we get a Jacobian.

$$J = |J_{ij}| = f_0^2 f_1 f_3 f_4^3 = \frac{\langle 23\rangle\langle 34\rangle\langle 41\rangle}{\langle 12\rangle^2\langle 13\rangle} \tag{3.32}$$

where

$$J_{ij} = \frac{\partial E_i}{\partial f_j} = \begin{pmatrix} f_3 f_3 & 0 & f_0 f_3 & f_0 f_4 \\ f_1 f_3 f_4 & f_0 f_3 f_4 & f_0 f_1 f_4 & f_0 f_1 f_3 \\ f_4 & 0 & 0 & 1 - f_0 \\ f_1 f_4 & f_0 f_4 & 0 & f_0 f_1 \end{pmatrix} \tag{3.33}$$

and

$$E_1 = f_0 f_3 f_4, \quad E_2 = f_0 f_1 f_3 f_4, \quad E_3 = f_4(1 - f_0), \quad E_4 = f_0 f_1 f_3 \tag{3.34}$$

Now using

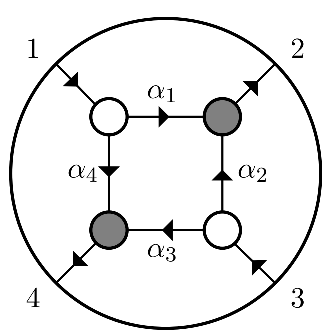
$$f_0 f_1 f_3 f_4 = \frac{\langle 13\rangle}{\langle 12\rangle} \tag{3.35}$$

We can put it all together to obtain the form

$$\mathrm{d}\Omega = \frac{\langle 12\rangle}{\langle 13\rangle} \times \frac{\langle 12\rangle^2\langle 13\rangle}{\langle 23\rangle\langle 34\rangle\langle 41\rangle} \times \frac{1}{\langle 12\rangle^4} \times \delta^4(P)\delta^8(\mathcal{Q}) = \frac{\delta^8\left(\mathcal{Q}\right)\delta^4\left(P\right)}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle} \tag{3.36}$$

**Using edge-variables**

For the edge-variable case let us try a different orientation



The $C$-matrix and its inverse are

$$C = \begin{pmatrix} 1 & \alpha_1 & 0 & \alpha_4 \\ 0 & \alpha_2 & 1 & \alpha_3 \end{pmatrix}, \qquad C^{\perp} = \begin{pmatrix} -\alpha_1 & 1 & -\alpha_2 & 0 \\ -\alpha_4 & 0 & -\alpha_3 & 1 \end{pmatrix} \tag{3.37}$$

The delta-function constraint on $C^{\perp}$ gives the following equations

$$\delta(C^{\perp} \cdot \lambda) \to \begin{cases} -\alpha_1\lambda_1 + \lambda_2 - \alpha_3\lambda_3 & = 0 \\ -\alpha_4\lambda_1 - \alpha_2\lambda_3 + \lambda_4 & = 0 \end{cases} \tag{3.38}$$

Which after contracting with $\lambda_1$ and $\lambda_2$ turns into

$$\begin{aligned} \langle 21 \rangle - \alpha_2 \langle 31 \rangle = 0 \Rightarrow \alpha_2 = \frac{\langle 12 \rangle}{\langle 13 \rangle} \\ \alpha_1 \langle 12 \rangle - \alpha_3 \langle 23 \rangle = 0 \Rightarrow \alpha_1 = \alpha_2 \frac{\langle 23 \rangle}{\langle 12 \rangle} = \frac{\langle 23 \rangle}{\langle 13 \rangle} \end{aligned} \tag{3.39}$$

Similarly we find

$$\alpha_3 = \frac{\langle 14 \rangle}{\langle 13 \rangle}, \qquad \alpha_4 = \frac{\langle 43 \rangle}{\langle 13 \rangle} \tag{3.40}$$

For the other delta function $\delta(C \cdot \tilde{\lambda})$ we get the two equations. The first one is

$$\begin{aligned} 0 = \tilde{\lambda}_1 + \alpha_2\tilde{\lambda}_2 + \alpha_4\tilde{\lambda}_4 = \tilde{\lambda}_1 + \frac{\langle 23 \rangle}{\langle 13 \rangle}\tilde{\lambda}_2 + \frac{\langle 43 \rangle}{\langle 13 \rangle}\tilde{\lambda}_4 \\ \Rightarrow 0 = \langle 13 \rangle \tilde{\lambda}_1 + \langle 23 \rangle \tilde{\lambda}_2 + \langle 43 \rangle \tilde{\lambda}_4 \end{aligned} \tag{3.41}$$

While the second one is found similarly

$$0 = \langle 21 \rangle \tilde{\lambda}_2 + \langle 31 \rangle \tilde{\lambda}_2 + \langle 41 \rangle \tilde{\lambda}_4 \tag{3.42}$$

We see that the two equations can be obtained from a single momentum conservation equation by contracting with $\lambda_3$ and $\lambda_1$ respectively. I.e. we have

$$\delta^4(\lambda_1\tilde{\lambda}_1 + \lambda_2\tilde{\lambda}_2 + \lambda_3\tilde{\lambda}_3 + \lambda_4\tilde{\lambda}_4) \equiv \delta^4(P) \tag{3.43}$$

For the last delta-function we get the exact same thing except for replacing $\tilde{\lambda}_i \to \tilde{\eta}_i$

$$\delta^8(\lambda_1\tilde{\eta}_1 + \lambda_2\tilde{\eta}_2 + \lambda_3\tilde{\eta}_3 + \lambda_4\tilde{\eta}_4) \equiv \delta^8(\mathcal{Q}) \tag{3.44}$$

Note that we get an extra factor of $\frac{1}{\langle 13 \rangle^4}$ from re-writing the delta-functions in by projecting along $\lambda_1$ and $\lambda_3$. Finally we have

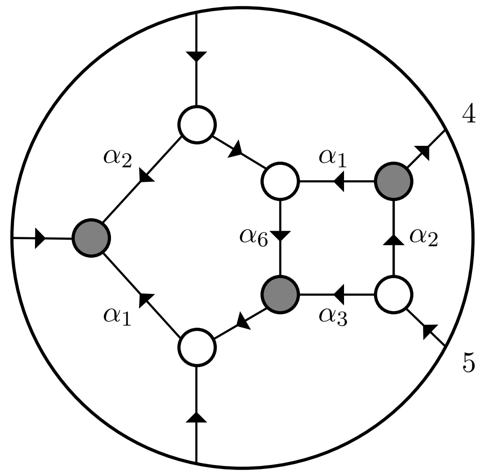
$$\frac{1}{\alpha_1\alpha_2\alpha_3\alpha_4} = \frac{\langle 13 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} \tag{3.45}$$

We can now calculate the form

$$\mathrm{d}\Omega = \frac{\delta^8(\mathcal{Q})\,\delta^4(P)}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} \tag{3.46}$$

# 4 Examples of on-shell diagrams at five and six points

## 4.1 Five point




Each vertex can fix one edge-variable, however you cannot do it in such a way that all variables in a vertex is fixed. The $C$ matrix is

$$C = \begin{pmatrix} 1 & \alpha_1 + \alpha_2\alpha_6 & \alpha_6 & \alpha_3\alpha_6 & 0 \\ 0 & \alpha_5\alpha_6\alpha_2 & \alpha_5\alpha_6 & \alpha_4 + \alpha_3\alpha_5\alpha_6 & 1 \end{pmatrix} \tag{4.1}$$

with the inverse being

$$C^{\perp} = \begin{pmatrix} -(\alpha_1 + \alpha_2\alpha_6) & 1 & 0 & 0 & -\alpha_5\alpha_6\alpha_2 \\ -\alpha_6 & 0 & 1 & 0 & -\alpha_5\alpha_6 \\ -\alpha_3\alpha_6 & 0 & 0 & 1 & -(\alpha_4 + \alpha_3\alpha_5\alpha_6) \end{pmatrix} \tag{4.2}$$

The amplitude is found through

$$\mathrm{d}\Omega = \frac{\mathrm{d}\alpha_1}{\alpha_1}\frac{\mathrm{d}\alpha_2}{\alpha_2}\frac{\mathrm{d}\alpha_3}{\alpha_3}\frac{\mathrm{d}\alpha_4}{\alpha_4}\frac{\mathrm{d}\alpha_5}{\alpha_5}\frac{\mathrm{d}\alpha_6}{\alpha_6}\delta^{2\times 2}(C\cdot\tilde{\lambda})\delta^{2\times 3}(C^{\perp}\cdot\lambda)\delta^{4\times 2}(C\cdot\tilde{\eta}) \tag{4.3}$$

Using the delta-function $\delta^{2\times 3}(C^{\perp}\cdot\lambda)$ to solve for the $\alpha$'s we obtain after contracting with $\lambda_1$, $\lambda_3$, and $\lambda_5$

$$\alpha_1 = \frac{\langle 23\rangle}{\langle 13\rangle}, \quad \alpha_2 = \frac{\langle 12\rangle}{\langle 13\rangle}, \quad \alpha_3 = \frac{\langle 45\rangle}{\langle 35\rangle}, \quad \alpha_4 = \frac{\langle 34\rangle}{\langle 34\rangle}, \quad \alpha_5 = \frac{\langle 13\rangle}{\langle 35\rangle}, \quad \alpha_6 = \frac{\langle 35\rangle}{\langle 15\rangle} \tag{4.4}$$

From which we see that we get a Jacobian of $\frac{1}{\langle 15\rangle^2\langle 13\rangle}$. Plugging these $\alpha$'s back into $(\delta^{2\times 2}C\cdot\tilde{\lambda})$, we get

$$\begin{aligned} 0 =& \tilde{\lambda}_1 + \tilde{\lambda}_2\frac{\langle 25\rangle}{\langle 15\rangle} + \tilde{\lambda}_3\frac{\langle 35\rangle}{\langle 15\rangle} + \tilde{\lambda}_4\frac{\langle 45\rangle}{\langle 15\rangle} \\ 0 =& \tilde{\lambda}_2\frac{\langle 12\rangle}{\langle 15\rangle} + \tilde{\lambda}_3\frac{\langle 13\rangle}{\langle 15\rangle} + \tilde{\lambda}_4\frac{\langle 14\rangle}{\langle 15\rangle} + \tilde{\lambda}_5 \end{aligned} \tag{4.5}$$

where we have used Schouten identities on the $\tilde{\lambda}_2$ term in the first equation and $\tilde{\lambda}_4$ term in the second equation. We easily see that

$$\delta^{2\times 2}(C\cdot\lambda) = \langle 15\rangle^2\,\delta^{2\times 2}(P) \tag{4.6}$$

Then we plug the $\alpha$'s into the Grassmann delta function. This will of course give a similar result with the exchange of $\tilde{\lambda}\to\tilde{\eta}$, although with the Jacobian factor now being $\frac{1}{\langle 15\rangle^4}$. Finally we calculate

$$\prod_i \frac{1}{\alpha_i} = \frac{\langle 13\rangle\,\langle 15\rangle^2\,\langle 35\rangle^2}{\langle 12\rangle\,\langle 23\rangle\,\langle 34\rangle\,\langle 45\rangle\,\langle 51\rangle} \tag{4.7}$$

We are now easily able to get the form

$$\mathrm{d}\Omega = \frac{\delta^8\left(\mathcal{Q}\right)\delta^4\left(P\right)}{\langle 12\rangle\,\langle 23\rangle\,\langle 34\rangle\,\langle 45\rangle\,\langle 51\rangle} \tag{4.8}$$

### 4.2 Five-point through loop kinematics

We have the following relations among the helicity variables

$$\begin{gathered}\lambda_1\propto\lambda_\ell\propto\lambda_{\ell-1},\quad \lambda_3\propto\lambda_q\propto\lambda_{q-3},\quad \lambda_{q+4}\propto\lambda_{\ell-1-5}\propto\lambda_{-q+3+2+\ell}\\ \tilde{\lambda}_{\ell-1}\propto\tilde{\lambda}_5\propto\tilde{\lambda}_{\ell-1-5},\quad \tilde{\lambda}_4\propto\tilde{\lambda}_q\propto\tilde{\lambda}_{q+4},\quad \tilde{\lambda}_{q-3-2}\propto\tilde{\lambda}_{q-3-2-\ell}\propto\tilde{\lambda}_\ell\propto\tilde{\lambda}_2\propto\tilde{\lambda}_{q-3}\end{gathered} \tag{4.9}$$

First we write the loop momentum in the box as

$$\begin{aligned}\ell &= \alpha\lambda_1\tilde{\lambda}_2 = -\frac{[15]}{[25]}\lambda_1\tilde{\lambda}_2\\ \ell-1 &= \lambda_1\left(\tilde{\lambda}_1+\alpha\tilde{\lambda}_2\right) = \frac{\lambda_1}{[25]}\left(\tilde{\lambda}_1\,[25]+\tilde{\lambda}_2\,[15]\right) = -\frac{[12]}{[25]}\lambda_1\tilde{\lambda}_5\\ \ell-1-5 &= \frac{\tilde{\lambda}_5}{[25]}\left([21]\,\lambda_1+[25]\,\lambda_5\right) = \frac{\langle\cdot|1+5|2]}{[25]}\tilde{\lambda}_5\end{aligned} \tag{4.10}$$

where we found $\alpha = -\frac{[15]}{[25]}$ through the on-shell condition

$$0 = (\ell-1-5)^2 = \alpha\,\langle 15\rangle\,[52]+\langle 15\rangle\,[51] \tag{4.11}$$

For the pentagon loop momenta we have

$$\begin{aligned}q &= \beta\lambda_3\tilde{\lambda}_4 = -\frac{[23]}{[24]}\lambda_3\tilde{\lambda}_4\\ q-3 &= \frac{\lambda_3}{[24]}\left(\tilde{\lambda}_4\,[23]+\tilde{\lambda}_3\,[24]\right) = -\frac{[34]}{[24]}\lambda_3\tilde{\lambda}_2\\ q-3-2 &= \frac{\tilde{\lambda}_2}{[24]}\left([34]\,\lambda_3+[24]\,\lambda_2\right) = \frac{\langle\cdot|1+5|4]}{[24]}\tilde{\lambda}_2\end{aligned} \tag{4.12}$$

Finally we have the momentum shared by the two loops

$$q-3-2-\ell = \frac{[45]\,\langle\cdot|1+5|2]}{[25]\,[24]}\tilde{\lambda}_2 \tag{4.13}$$

From this we have the following subamplitudes

$$
\begin{aligned}
A_4(q+4,5,1,q-3-2) &= \frac{1}{[q+4,5]\,[51]\,[1,q-3-2]\,[q-3-2,q+4]} \\
&= \frac{1}{\frac{[45]}{[24]}\,[51]\,\frac{[12]}{[24]}\,\frac{[24]}{[24]^2}} \\
&= \frac{[24]^3}{[12]\,[45]\,[51]}
\end{aligned}
\tag{4.14}
$$

### 4.3 Six-point NMHV

For the NMHV ($k=3$) amplitude at six-points we get three different diagrams. In our case we will look at

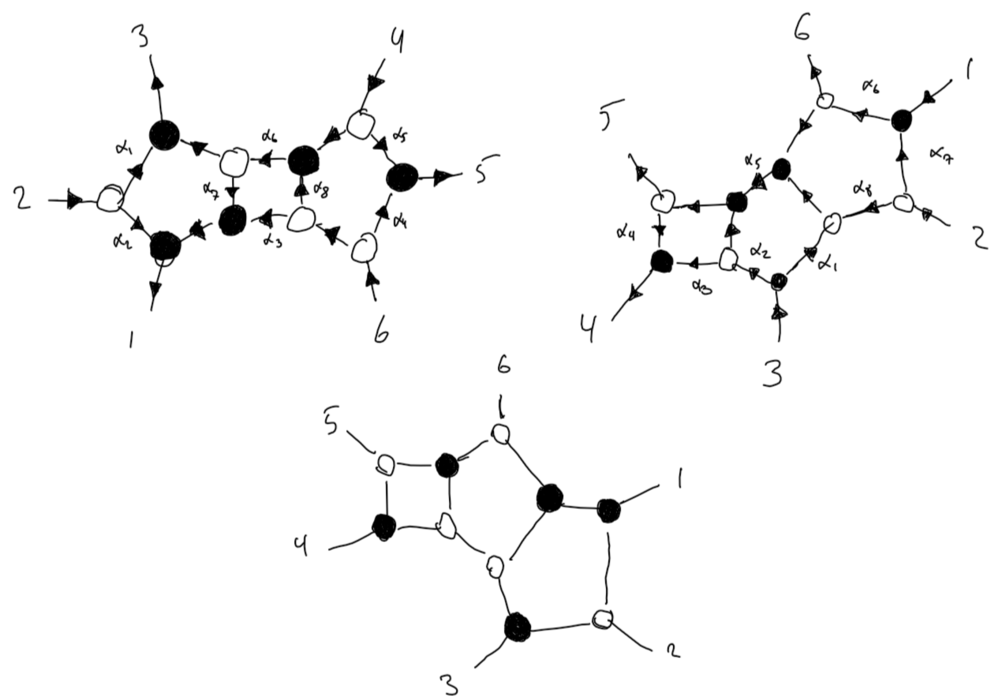


**Figure 2**.

We first look at the 4+4 diagram, which has the following C-matrices

$$
C = \begin{pmatrix} \alpha_2 & 1 & \alpha_1 & 0 & 0 & 0 \\ \alpha_6\alpha_7 & 0 & \alpha_6 & 1 & \alpha_5 & 0 \\ \alpha_3+\alpha_6\alpha_7\alpha_8 & 0 & \alpha_6\alpha_8 & 0 & \alpha_4 & 1 \end{pmatrix}, \quad C^{\perp} = \begin{pmatrix} 1 & -\alpha_2 & 0 & -\alpha_6\alpha_7 & 0 & -\alpha_3-\alpha_6\alpha_7\alpha_8 \\ 0 & -\alpha_1 & 1 & -\alpha_6 & 0 & -\alpha_6\alpha_8 \\ 0 & 0 & 0 & -\alpha_5 & 1 & -\alpha_4 \end{pmatrix} \tag{4.15}
$$

We then use the following combination of equation from the $C \cdot \tilde{\lambda}$ and $C_\perp \cdot \lambda$ delta functions

$$\begin{aligned}
0 =& -\langle 23\rangle + \alpha_4\langle 24\rangle \\
0 =& -(\alpha_5\langle 24\rangle) + \langle 34\rangle \\
0 =& [61] - \alpha_2\,[15] \\
0 =& [65] + \alpha_1\,[15] \\
0 =& -[12] - \alpha_5\,[13] - \alpha_6\alpha_7\,[15] \\
0 =& \alpha_6\,[15] + [25] + \alpha_5\,[35] \\
0 =& -(\alpha_4\,[13]) - [14] - (\alpha_3 + \alpha_6\alpha_7\alpha_8)\,[15] \\
0 =& \alpha_6\alpha_8\,[15] + \alpha_4\,[35] + [45]
\end{aligned} \tag{4.16}$$

to obtain

$$\alpha_1 = -\frac{[16]}{[26]}, \quad \alpha_2 = \frac{[12]}{[26]}, \quad \alpha_3 = \frac{s_{345}}{\langle 5|Q_{345}|6]}, \quad \alpha_4 = \frac{\langle 34\rangle}{\langle 35\rangle}, \quad \alpha_5 = \frac{\langle 45\rangle}{\langle 35\rangle},$$
$$\alpha_6 = \frac{\langle 5|Q_{345}|6]}{\langle 35\rangle[26]}, \alpha_7 = -\frac{\langle 5|Q_{345}|2]}{\langle 5|Q_{345}|6]}, \quad \alpha_8 = -\frac{\langle 3|Q_{345}|6]}{\langle 5|Q_{345}|6]}.$$

where $Q_{ijk} = p_i + p_j + p_k$ For the other delta functions we get

$$\begin{aligned}
0 &= \tilde{\eta}_1 - \frac{[16]}{[26]}\tilde{\eta}_2 + \frac{[12]}{[26]}\tilde{\eta}_6 \\
0 &= \frac{\langle 5|Q_{345}|6]}{\langle 35\rangle\,[26]}\tilde{\eta}_2 + \tilde{\eta}_3 + \frac{\langle 45\rangle}{\langle 35\rangle}\tilde{\eta}_4 - \frac{\langle 5|Q_{345}|2]}{\langle 35\rangle\,[26]}\tilde{\eta}_6 \\
0 &= -\frac{\langle 3|Q_{45}|6]}{\langle 35\rangle\,[26]}\tilde{\eta}_2 + \frac{\langle 34\rangle}{\langle 35\rangle}\tilde{\eta}_4 + \tilde{\eta}_5 + \frac{s_{345}\,\langle 35\rangle\,[26] + \langle 3|Q_{345}|6]\langle 5|Q_{345}|2]}{\langle 5|Q_{345}|6]}\tilde{\eta}_5
\end{aligned} \tag{4.17}$$

The Jacobian from the delta functions is

$$J = \frac{1}{[26]^3\,\langle 35\rangle^3\,\langle 5|Q_{345}|6]^2} \tag{4.18}$$

and we get the form

$$\mathrm{d}\Omega = \frac{\delta(\sum P)\delta(\tilde{\eta}_1[26] + \tilde{\eta}_2[61] + \tilde{\eta}_6[12])}{s_{345}\,\langle 34\rangle\,\langle 45\rangle\,[12][16]\langle 3|Q_{345}|6]\langle 5|Q_{345}|2]} \tag{4.19}$$

We later realized that we changed the bcfw bridge scheme in making the next diagram. To match the schemes we permute all labels in this form by 1

$$\mathrm{d}\Omega_1 = \frac{\delta(\sum P)\delta(\tilde{\eta}_5[16] + \tilde{\eta}_6[15] + \tilde{\eta}_1[56])}{s_{234}\,\langle 23\rangle\,\langle 34\rangle\,[61][56]\langle 2|Q_{234}|5]\langle 4|Q_{234}|1]} \tag{4.20}$$

Looking at the 5+3 diagram we use the following C-matrix

$$\begin{aligned} C &= \begin{pmatrix} \alpha_2 & \alpha_3+\alpha_4 & 1 & 0 & 0 & 0 \\ \alpha_2\alpha_5 & \alpha_3\alpha_5 & 0 & 1 & \alpha_6 & 0 \\ \alpha_8(\alpha_1+\alpha_2\alpha_5) & \alpha_3\alpha_5\alpha_8 & 0 & 0 & \alpha_7 & 1 \end{pmatrix} \\ C_\perp &= \begin{pmatrix} 1 & 0 & -\alpha_2 & -\alpha_2\alpha_5 & 0 & -\alpha_8(\alpha_1+\alpha_2\alpha_5) \\ 0 & 1 & -\alpha_3-\alpha_4 & -\alpha_3\alpha_5 & 0 & -\alpha_3\alpha_5\alpha_8 \\ 0 & 0 & 0 & -\alpha_6 & 1 & -\alpha_7 \end{pmatrix} \end{aligned} \tag{4.21}$$

From the delta functions $\delta(C\cdot\tilde\lambda)$ and $\delta(C_\perp\cdot\lambda)$, we use the following equations

$$\begin{aligned} 0 =& \langle 26\rangle - (\alpha_3+\alpha_4)\langle 36\rangle - \alpha_3\alpha_5\langle 46\rangle \\ 0 =& -\langle 45\rangle + \alpha_7\langle 46\rangle \\ 0 =& -\alpha_6\langle 46\rangle + \langle 56\rangle \\ 0 =& -(\alpha_3+\alpha_4)\,[12] - [13] \\ 0 =& \alpha_2\,[12] - [23] \\ 0 =& \alpha_2\alpha_5\,[12] - [24] - \alpha_6\,[25] \\ 0 =& -\alpha_3\alpha_5\alpha_8\,[12] - \alpha_7\,[15] - [16] \\ 0 =& (\alpha_1+\alpha_2\alpha_5)\alpha_8\,[12] - \alpha_7\,[25] - [26] \end{aligned} \tag{4.22}$$

to obtain solutions for the edge-variables

$$\begin{aligned} &\alpha_1 = \frac{s_{456}}{\langle 4|Q_{456}|1]}\,, \quad \alpha_2 = \frac{[23]}{[12]}\,, \quad \alpha_3 = \frac{[23]\langle 6|Q_{456}|1]}{[12]\langle 6|Q_{456}|2]}\,, \quad \alpha_4 = \frac{\langle 6|Q_{456}|3]}{\langle 6|Q_{456}|2]}\,, \\ &\alpha_5 = \frac{\langle 6|Q_{456}|2]}{\langle 46\rangle[23]}\,, \quad \alpha_6 = \frac{\langle 56\rangle}{\langle 46\rangle}\,, \quad \alpha_7 = \frac{\langle 45\rangle}{\langle 46\rangle}\,, \quad \alpha_8 = \frac{\langle 4|Q_{456}|1]}{\langle 6|Q_{456}|1]}\,. \end{aligned} \tag{4.23}$$

Here the Jacobian from the delta functions is

$$J = \frac{1}{[12]^3\,\langle 46\rangle^3\,\langle 6|Q_{456}|2]^2} \tag{4.24}$$

and we get the form

$$\mathrm{d}\Omega = \frac{\delta(\sum P)\delta(\tilde\eta_1[26]+\tilde\eta_2[61]+\tilde\eta_3[12])}{s_{456}\,\langle 45\rangle\,\langle 56\rangle\,[12][23]\langle 4|Q_{456}|1]\langle 6|Q_{456}|3]} \tag{4.25}$$

Again we have to permute by 2 to obtain the correct recursion scheme

$$\mathrm{d}\Omega_2 = \frac{\delta(\sum P)\delta(\tilde\eta_3[45]+\tilde\eta_4[35]+\tilde\eta_5[34])}{s_{612}\,\langle 12\rangle\,\langle 16\rangle\,[34][35]\langle 6|Q_{612}|3]\langle 2|Q_{612}|5]} \tag{4.26}$$

The final diagram can be found by permuting by 2 and exchanging square and angle brackets while permuting by 1:

$$\mathrm{d}\Omega_3 = \frac{\delta(\sum P)\delta(\tilde\eta_1[23]+\tilde\eta_2[13]+\tilde\eta_3[12])}{s_{456}\,\langle 45\rangle\,\langle 56\rangle\,[23][12]\langle 4|Q_{456}|1]\langle 6|Q_{456}|3]} \tag{4.27}$$

In total we have

$$
\begin{aligned}
&\mathrm{d}\Omega_1 + \mathrm{d}\Omega_2 + \mathrm{d}\Omega_3 \\
=&\frac{\delta(\sum P)\delta(\tilde\eta_5[16]+\tilde\eta_6[15]+\tilde\eta_1[56])}{s_{234}\,\langle 23\rangle\,\langle 34\rangle\,[61][56]\langle 2|Q_{234}|5]\langle 4|Q_{234}|1]} + \frac{\delta(\sum P)\delta(\tilde\eta_3[45]+\tilde\eta_4[35]+\tilde\eta_5[34])}{s_{612}\,\langle 12\rangle\,\langle 16\rangle\,[34][35]\langle 6|Q_{612}|3]\langle 2|Q_{612}|5]} \\
&+\frac{\delta(\sum P)\delta(\tilde\eta_1[23]+\tilde\eta_2[13]+\tilde\eta_3[12])}{s_{456}\,\langle 45\rangle\,\langle 56\rangle\,[23][12]\langle 4|Q_{456}|1]\langle 6|Q_{456}|3]}
\end{aligned}
\tag{4.28}
$$

Then using the fact that

$$
\begin{aligned}
&\mathcal{A}^{(0),\mathrm{MHV}} R_{j,j+3,j+5} = \\
&\frac{\delta^{(8)}\left(\sum \lambda_i \eta_i^A\right)}{\langle j\,(j{+}1)\rangle\,\langle (j{+}1)\,(j{+}2)\rangle\,[(j{+}3)\,(j{+}4)]\,[(j{+}4)\,(j{+}5)]} \\
&\times\frac{\delta^{(4)}\left(\eta^A_{j+3}\,[(j{+}4)\,(j{+}5)] + \eta^A_{j+4}\,[(j{+}5)\,(j{+}3)] + \eta^A_{j+5}\,[(j{+}3)\,(j{+}4)]\right)}{\langle j|\,K_{j+1,j+2}\,|(j{+}3)]\,\langle (j{+}2)|\,K_{j+3,j+4}\,|(j{+}5)]\,s_{j,j+1,j+2}}\,.
\end{aligned}
\tag{4.29}
$$

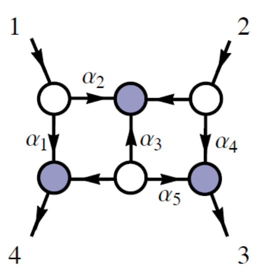
such that

$$
A_6^{\mathrm{NMHV}} = A_6^{\mathrm{MHV}}\left(R_{251} + R_{413} + R_{635}\right) \tag{4.30}
$$

# 5 Examples at higher loops

## 5.1 2 loop four-point

We have the diagram




$$
\Leftrightarrow \quad
\begin{aligned}
C &= \begin{pmatrix} 1 & 0 & \alpha_3\alpha_4\alpha_5 & \alpha_1+\alpha_2\alpha_3 \\ 0 & 1 & \alpha_4+\alpha_3\alpha_5 & \alpha_3 \end{pmatrix} \\
C^{\perp} &= \begin{pmatrix} -\alpha_2\alpha_3\alpha_5 & -(\alpha_4+\alpha_3\alpha_5) & 1 & 0 \\ -(\alpha_1+\alpha_2\alpha_3) & -\alpha_3 & 0 & 1 \end{pmatrix}
\end{aligned}
$$

**Figure 3**.

Solving the bosonic delta-functions

$$
\begin{aligned}
0 &= \alpha_4\langle 12\rangle + \alpha_3\alpha_5\langle 12\rangle - \langle 13\rangle \\
0 &= \alpha_2\alpha_3\alpha_5\langle 12\rangle - \langle 23\rangle \\
0 &= \alpha_3\langle 12\rangle - \langle 14\rangle \\
0 &= -((\alpha_1+\alpha_2\alpha_3)\langle 12\rangle) - \langle 24\rangle
\end{aligned}
\tag{5.1}
$$

we find

# 6 Calculating $\mathcal{N} < 4$ amplitudes

## 6.1 The measure

The important difference between this and the $\mathcal{N} = 4$ case is that the diagrams are necessarily oriented unlike in the maximally supersymmetric forms where the perfect orientations only played an auxiliary role for constructing the C-matrix. In addition, for perfect orientations with closed internal loops we have to add an extra factor, $\mathcal{J}$ , in the measure,

$$\mathrm{d}\Omega = \frac{\mathrm{d}\alpha_1}{\alpha_1}\frac{\mathrm{d}\alpha_2}{\alpha_2}\cdots\frac{\mathrm{d}\alpha_m}{\alpha_m}\mathcal{J}^{\mathcal{N}-4}\cdot\delta(C\cdot Z) \tag{6.1}$$

If there is a collection of closed orbits bounding "faces" $f_i$, with disjoint pairs $(f_i, f_j)$, disjoint triples $(f_i, f_j, f_k)$ etc., then the Jacobian $\mathcal{J}$ can be expressed as,

$$\mathcal{J} = 1 + \sum_i f_i + \sum_{\substack{\text{disjoint}\\ \text{pairs } i,j}} f_i f_j + \sum_{\substack{\text{disjoint}\\ \text{pairs } i,j,k}} f_i f_j f_k + \cdots \tag{6.2}$$

First we note that each face is defined as clockwise-oriented product of edge-variables, such that a counter clockwise "face" $f$, gives a Jacobian contribution of $f^{-1}$. As an example see for instance the following diagram with four internal loops

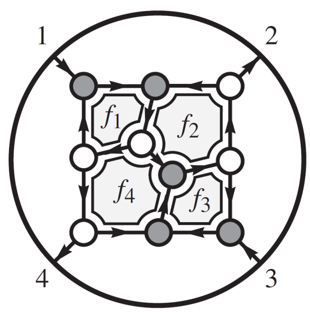



**Figure 4**.

Here we have three loops containing one phase variable, contributing each $f_1, f_2^{-1}, f_3$. Further we have the disjoint pair $f_1 f_3$. Lastly we have a close loop containing both $f_2$ and $f_4$, i.e. we have in total

$$\mathcal{J} = 1 + f_1 + f_3 + f_2^{-1} + f_2^{-1}f_4^{-1} + f_1 f_3 \tag{6.3}$$

### 6.1.1 4 pt $\mathcal{N} = 0$

In this case we have two diagrams

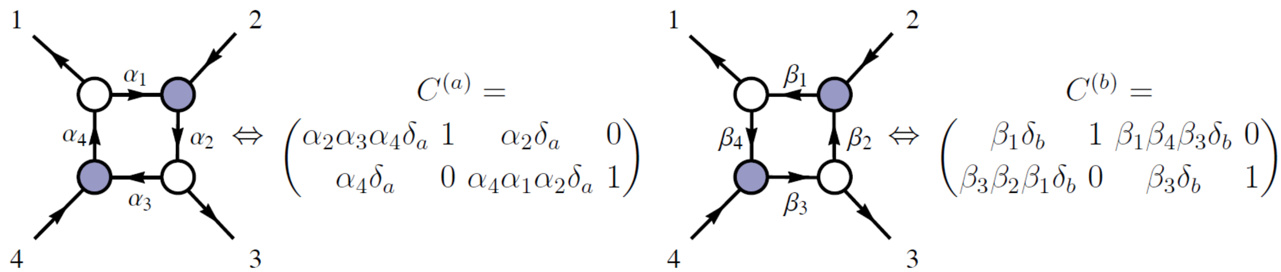



**Figure 5**.

where the $\delta' s$ are defined by

$$\delta_a = \frac{1}{1 - \prod_{i=1}^{4} \alpha_i} \tag{6.4}$$

Solving the $C_\perp$ delta-function, we get the following equations after contracting with $\lambda_2$ and $\lambda_4$ (which gives a Jacobian of $\langle 24\rangle^2$)

$$\begin{aligned} 0 &= \langle 12\rangle + \frac{\alpha_4\langle 24\rangle}{1-\alpha_1\alpha_2\alpha_3\alpha_4} \\ 0 &= \langle 14\rangle - \frac{\alpha_2\alpha_3\alpha_4\langle 24\rangle}{1-\alpha_1\alpha_2\alpha_3\alpha_4} \\ 0 &= -\langle 23\rangle + \frac{\alpha_1\alpha_2\alpha_4\langle 24\rangle}{1-\alpha_1\alpha_2\alpha_3\alpha_4} \\ 0 &= \langle 34\rangle - \frac{\alpha_2\langle 24\rangle}{1-\alpha_1\alpha_2\alpha_3\alpha_4} \end{aligned} \tag{6.5}$$

from which we obtain

$$\begin{aligned} \alpha_1 &= -\frac{\langle 23\rangle}{\langle 13\rangle}, \qquad & \alpha_2 &= \frac{\langle 14\rangle}{\langle 12\rangle}, \\ \alpha_3 &= -\frac{\langle 14\rangle}{\langle 13\rangle}, \qquad & \alpha_4 &= -\frac{\langle 13\rangle}{\langle 34\rangle}, \end{aligned} \tag{6.6}$$

along with a Jacobian factor from rewriting the delta functions of $\frac{\langle 24\rangle^2}{\langle 12\rangle^2\langle 34\rangle^2}$. The procedure to find this factor can be found by eg the first line of (6.6), which after insertion of the $\delta_a = \frac{1}{1-\alpha_1\alpha_2\alpha_3\alpha_4}$ reads

$$0 = \langle 12\rangle + \frac{\alpha_4\langle 12\rangle\langle 34\rangle}{\langle 13\rangle} \tag{6.7}$$

Since this is really a delta-function, we can write this as

$$\delta\left(\frac{\langle 34\rangle\langle 12\rangle}{\langle 13\rangle}\left[\frac{\langle 13\rangle}{\langle 34\rangle} + \alpha_4\right]\right) = \frac{\langle 13\rangle}{\langle 34\rangle\langle 12\rangle}\delta\left(\frac{\langle 13\rangle}{\langle 34\rangle} + \alpha_4\right) \tag{6.8}$$

i.e. the Jacobian from this factor is $\frac{\langle 13\rangle}{\langle 34\rangle\langle 12\rangle}$. We can further find an expression for $\delta_a$ through this

$$\begin{aligned} 0 &= \langle 12\rangle + \alpha_4\langle 24\rangle\delta_a \\ \Rightarrow \delta_a &= -\frac{1}{\alpha_4}\frac{\langle 12\rangle}{\langle 24\rangle} = \frac{\langle 12\rangle\langle 34\rangle}{\langle 13\rangle\langle 24\rangle} \end{aligned} \tag{6.9}$$

Taking these solutions and inserting into the other delta functions, and multiplying by $\tilde{\lambda}_1$ and $\tilde{\lambda}_3$ we get

$$\begin{aligned} 0 &= [12] + \frac{\langle 34\rangle[13]}{\langle 24\rangle}, \qquad & 0 &= [23] + \frac{\langle 14\rangle[13]}{\langle 24\rangle} \\ 0 &= [14] + \frac{\langle 23\rangle[13]}{\langle 24\rangle}, \qquad & 0 &= [34] + \frac{\langle 12\rangle[13]}{\langle 24\rangle} \end{aligned} \tag{6.10}$$

Which we can write as

$$\begin{aligned} 0 &= \langle 42\rangle[21] + \langle 43\rangle[31] = \langle 4|P|1], \qquad & 0 &= \langle 42\rangle[23] + \langle 41\rangle[13] = \langle 4|P|3] \\ 0 &= \langle 24\rangle[41] + \langle 23\rangle[31] = \langle 2|P|1], \qquad & 0 &= \langle 24\rangle[43] + \langle 21\rangle[13] = \langle 2|P|3] \end{aligned} \tag{6.11}$$

Which we can combine into $\delta^4(P)\langle 24\rangle^2$. Finally the Jacobian is

$$\mathcal{J}_a = 1 - \alpha_1\alpha_2\alpha_3\alpha_4 = \delta_a^{-1} = \frac{\langle 13\rangle\langle 24\rangle}{\langle 12\rangle\langle 34\rangle} \tag{6.12}$$

The procedure is the same for the other diagram and here we just summarize the results

$$\begin{aligned} \beta_1 &= \frac{\langle 13\rangle}{\langle 23\rangle}, \qquad & \beta_2 &= \frac{\langle 21\rangle}{\langle 13\rangle}, \\ \beta_3 &= \frac{\langle 13\rangle}{\langle 14\rangle}, \qquad & \beta_4 &= \frac{\langle 34\rangle}{\langle 13\rangle}, \end{aligned} \tag{6.13}$$

$$\mathcal{J}_b = 1 - \beta_1\beta_2\beta_3\beta_4 = \delta_b^{-1} = \frac{\langle 13\rangle\langle 24\rangle}{\langle 14\rangle\langle 23\rangle} \tag{6.14}$$

Combining all these factors with $\prod_i \alpha_i = \frac{\langle 41\rangle\langle 23\rangle}{\langle 12\rangle\langle 34\rangle}$ and $\prod_i \beta_i = \frac{\langle 12\rangle\langle 34\rangle}{\langle 23\rangle\langle 41\rangle}$, as well as using taking the Jacobians to the $-4$'th power, we have the form

$$\begin{aligned} \mathrm{d}\Omega &= \left( \frac{\langle 24\rangle^4}{\langle 12\rangle^2\langle 34\rangle^2}\frac{\langle 12\rangle\langle 34\rangle}{\langle 41\rangle\langle 23\rangle}\left[\frac{\langle 13\rangle\langle 24\rangle}{\langle 12\rangle\langle 34\rangle}\right]^{-4} + \frac{\langle 24\rangle^4}{\langle 14\rangle^2\langle 23\rangle^2}\frac{\langle 23\rangle\langle 41\rangle}{\langle 12\rangle\langle 34\rangle}\left[\frac{\langle 13\rangle\langle 24\rangle}{\langle 14\rangle\langle 23\rangle}\right]^{-4} \right)\delta(P) \\ &= \frac{\langle 12\rangle^4\langle 34\rangle^4 + \langle 14\rangle^4\langle 23\rangle^4}{\langle 12\rangle\langle 13\rangle^4\langle 23\rangle\langle 34\rangle\langle\rangle} \end{aligned} \tag{6.15}$$

### 6.1.2 Five-point $\mathcal{N} = 0$, no internal cycle

For five and six points we will look at two different type of diagrams. First we analyze on without an internal cycle, i.e. has $\mathcal{J} = 0$. As will be evident, this produces the standard Yang-Mills amplitude. We then follow this by evaluating diagrams with internal cycles, which we can use to get forms for $\mathcal{N} \neq 4$. We will once again look at the following diagram

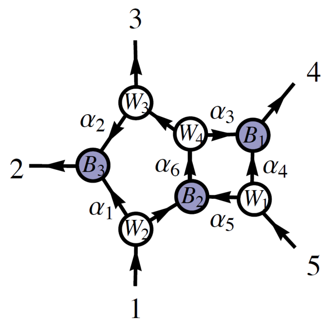



**Figure 6**.

We already solved this in section ——, here we obtainedthe following edgevariables after contracting with $\lambda_1$, $\lambda_3$, and $\lambda_5$

$$\alpha_1 = \frac{\langle 23\rangle}{\langle 13\rangle}, \quad \alpha_2 = \frac{\langle 12\rangle}{\langle 13\rangle}, \quad \alpha_3 = \frac{\langle 45\rangle}{\langle 35\rangle}, \quad \alpha_4 = \frac{\langle 34\rangle}{\langle 34\rangle}, \quad \alpha_5 = \frac{\langle 13\rangle}{\langle 35\rangle}, \quad \alpha_6 = \frac{\langle 35\rangle}{\langle 15\rangle} \tag{6.16}$$

and a Jacobian from solving the delta functions of $\frac{\langle 15\rangle^2}{\langle 35\rangle^2\langle 13\rangle}$. For this diagram we have two negative helicity particles 1 and 5 which is seen from the incoming arrows at those points. The orientation of these external legs do not yield an internal orientation with a closed cycle, and so this is the only diagram we have to take into account and there is no extra Jacobian factor. Using $\prod_i \alpha_i = \frac{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle}{\langle 13\rangle\langle 15\rangle\langle 35\rangle^2}$ The form then yields the amplitude

$$\begin{aligned} d\Omega_5(1^-, 2^+, 3^+, 4^+, 5^-) &= \frac{\langle 15\rangle^2}{\langle 35\rangle^2\langle 13\rangle}\frac{\delta(P)}{\prod_i \alpha_i} \\ &= \frac{\langle 15\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}\delta(P) \end{aligned} \tag{6.17}$$

### 6.2 Five point with internal cycles

We will now consider a diagram containing internal cycles. For the fixed helicity configuration of $(1^+, 2^-, 3^+, 4^+, 5^-)$ the diagrams below contribute

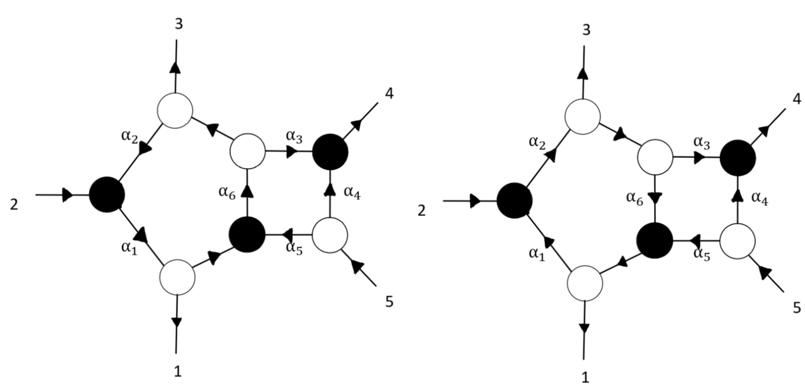



**Figure 7**.

Denoting the left diagram by "$a$" and the right one by "$b$", the $C$-matrices are

$$C_a = \begin{pmatrix} \alpha_1\delta_a & 1 & \alpha_1\alpha_6\delta_a & \alpha_1\alpha_3\alpha_6\delta_a & 0 \\ \alpha_1\alpha_2\alpha_5\alpha_6\delta_a & 0 & \alpha_5\alpha_6\delta_a & \alpha_3\alpha_5\alpha_6\delta_a+\alpha_4 & 1 \end{pmatrix}, \quad C_a^{\perp} = \begin{pmatrix} 1 & -\alpha_1\delta_a & 0 & 0 & -\alpha_1\alpha_2\alpha_5\alpha_6\delta_a \\ 0 & -\alpha_1\alpha_6\delta_a & 1 & 0 & -\alpha_5\alpha_6\delta_a \\ 0 & -\alpha_1\alpha_3\alpha_6\delta_a & 0 & 1 & -\alpha_3\alpha_5\alpha_6\delta_a-\alpha_4 \end{pmatrix}$$
$$C_b = \begin{pmatrix} \beta_2\beta_6\delta_b & 1 & \beta_2\delta_b & \beta_2\beta_3\delta_b & 0 \\ \beta_5\delta_b & 0 & \beta_1\beta_2\beta_5\delta_b & \beta_1\beta_2\beta_3\beta_5\delta_b+\beta_4 & 1 \end{pmatrix}, \quad C_b^{\perp} = \begin{pmatrix} 1 & -\beta_2\beta_6\delta_b & 0 & 0 & -\beta_5\delta_b \\ 0 & -\beta_2\delta_b & 1 & 0 & -\beta_1\beta_2\beta_5\delta_b \\ 0 & -\beta_2\beta_3\delta_b & 0 & 1 & -\beta_1\beta_2\beta_3\beta_5\delta_b-\beta_4 \end{pmatrix} \tag{6.18}$$

Solving the delta-function constraints we are led to the following solutions for the edge variables

$$\begin{aligned} &\alpha_1 = \frac{\langle 13\rangle}{\langle 23\rangle}, \quad \alpha_2 = -\frac{\langle 12\rangle}{\langle 13\rangle}, \quad \alpha_3 = \frac{\langle 45\rangle}{\langle 35\rangle}, \quad \alpha_4 = \frac{\langle 34\rangle}{\langle 35\rangle}, \quad \alpha_5 = \frac{\langle 13\rangle}{\langle 35\rangle}, \quad \alpha_6 = \frac{\langle 35\rangle}{\langle 15\rangle} \\ &\beta_1 = -\frac{\langle 23\rangle}{\langle 13\rangle}, \quad \beta_2 = -\frac{\langle 13\rangle}{\langle 12\rangle}, \quad \beta_3 = \frac{\langle 45\rangle}{\langle 35\rangle}, \quad \beta_4 = \frac{\langle 34\rangle}{\langle 35\rangle}, \quad \beta_5 = -\frac{\langle 13\rangle}{\langle 35\rangle}, \quad \beta_6 = \frac{\langle 15\rangle}{\langle 35\rangle} \end{aligned} \tag{6.19}$$

along with a Jacobian from solving the delta functions of $\frac{\langle 13\rangle\langle 25\rangle^4}{\langle 15\rangle^2\langle 23\rangle^2\langle 35\rangle^2}$ and $\frac{\langle 13\rangle\langle 25\rangle^4}{\langle 12\rangle^2\langle 35\rangle^2}$ respectively. The Jacobian from the internal loops are

$$J_a = \delta_a^{-1} = \frac{\langle 13\rangle\langle 25\rangle}{\langle 15\rangle\langle 23\rangle}, \qquad J_b = \delta_a^{-1} = \frac{\langle 13\rangle\langle 25\rangle}{\langle 12\rangle\langle 35\rangle} \tag{6.20}$$

The $\mathcal{N}=0$ form found by adding the diagrams is then

$$\begin{aligned} \mathrm{d}\Omega &= \left( \frac{\langle 13\rangle\langle 25\rangle^4}{\langle 15\rangle^2\langle 23\rangle^2\langle 35\rangle^2} J_a^{-4} \prod_{i=1}^{5} \frac{1}{\alpha_i} + \frac{\langle 13\rangle\langle 25\rangle^4}{\langle 12\rangle^2\langle 35\rangle^2} J_b^{-4} \prod_{i=1}^{5} \frac{1}{\beta_i} \right) \delta(P) \\ &= \frac{\langle 25\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle} \delta(P) \left( J_a^{-4} + J_b^{-4} \right) \\ &= \frac{\langle 15\rangle^4\langle 23\rangle^4 + \langle 12\rangle^4\langle 35\rangle^4}{\langle 13\rangle^4\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle} \delta(P) \end{aligned} \tag{6.21}$$

### 6.2.1 Six-point $\mathcal{N}=0$

Here we start by treating the same diagram as we did in section ...

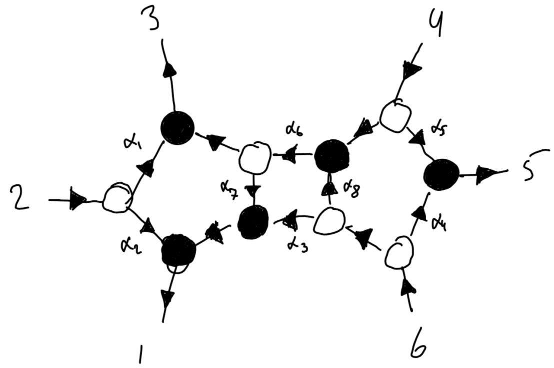



**Figure 8**.

where we obtained

$$\alpha_1 = -\frac{[65]}{[15]}\,, \quad \alpha_2 = \frac{[61]}{[15]}\,, \quad \alpha_3 = \frac{s_{234}}{\langle 4|Q_{234}|5]}\,, \quad \alpha_4 = \frac{\langle 23\rangle}{\langle 24\rangle}\,, \quad \alpha_5 = \frac{\langle 34\rangle}{\langle 24\rangle}\,,$$
$$\alpha_6 = \frac{\langle 4|Q_{234}|5]}{\langle 24\rangle[15]}\,, \alpha_7 = -\frac{\langle 4|Q_{234}|1]}{\langle 4|Q_{234}|5]}\,, \quad \alpha_8 = -\frac{\langle 2|Q_{234}|5]}{\langle 4|Q_{234}|5]}\,.$$

The Jacobian from the delta functions is

$$J = \frac{[15]\,\langle 24\rangle}{\langle 4|Q_{234}|5]^2} \tag{6.22}$$

such that the form is

$$\mathrm{d}\Omega_{4+4} = \frac{\langle 24\rangle^4[15]^4}{s_{234}\langle 23\rangle\langle 34\rangle\langle 2|Q_{234}|5]\langle 4|Q_{234}|1][61][56]} \tag{6.23}$$

For the 5+3 diagram we once again use the results from the previous section. The diagram is

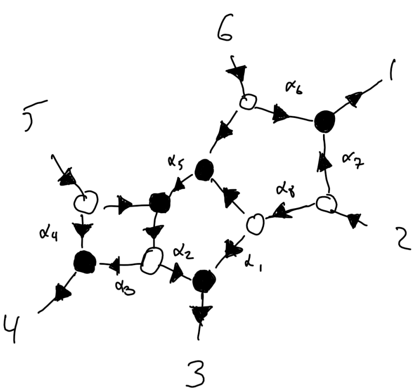



**Figure 9**.

With the following edgevariables

$$
\begin{aligned}
\alpha_1 &= \frac{s_{612}}{\langle 6|Q_{612}|3]}\,, \quad \alpha_2 = \frac{[45]}{[34]}\,, \quad \alpha_3 = \frac{[45]\langle 2|Q_{612}|3]}{[34]\langle 2|Q_{612}|4]}\,, \quad \alpha_4 = \frac{\langle 2|Q_{612}|5]}{\langle 2|Q_{612}|4]}\,, \\
\alpha_5 &= \frac{\langle 2|Q_{612}|4]}{\langle 62\rangle[45]}\,, \quad \alpha_6 = \frac{\langle 12\rangle}{\langle 62\rangle}\,, \quad \alpha_7 = \frac{\langle 61\rangle}{\langle 62\rangle}\,, \quad \alpha_8 = \frac{\langle 6|Q_{612}|3]}{\langle 2|Q_{612}|3]}\,.
\end{aligned} \tag{6.24}
$$

Here the Jacobian from the delta functions is

$$
J = \frac{[35]^4\,\langle 62\rangle}{[34]^3\langle 2|Q_{612}|4]\langle 6|Q_{612}|3]} \tag{6.25}
$$

The form is

$$
\mathrm{d}\Omega_{5+3} = \frac{\langle 26\rangle^4[35]^4}{s_{612}\langle 12\rangle\langle 16\rangle\langle 6|Q_{612}|3]\langle 2|Q_{612}|5][34][45]} \tag{6.26}
$$

### 6.3 Six point NMHV with internal cycles

We now consider a six point diagram with an internal cycle. Fixing the helicity configuration to $(1^+, 2^-, 3^+, 4^-, 5^+, 6^-)$ and considering the 5+3 diagrams yields

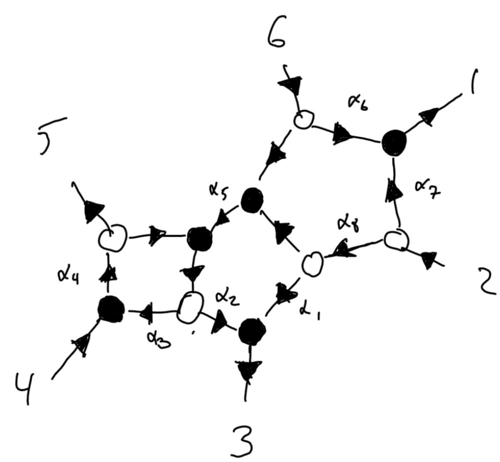



**Figure 10**.

Where another diagram with the internal cycle is reversed also contributes to the form. The edgevariables are found to be

$$
\begin{aligned}
\alpha_1 &= \frac{s_{612}}{\langle 6|Q_{612}|3]}\,, \quad \alpha_2 = \frac{[45]}{[34]}\,, \quad \alpha_3 = \frac{[45]\langle 2|Q_{612}|3]}{[34]\langle 2|Q_{612}|4]}\,, \quad \alpha_4 = -\frac{\langle 2|Q_{612}|4]}{\langle 2|Q_{612}|5]}\,, \\
\alpha_5 &= \frac{\langle 2|Q_{612}|4]}{\langle 62\rangle[45]}\,, \quad \alpha_6 = \frac{\langle 12\rangle}{\langle 62\rangle}\,, \quad \alpha_7 = \frac{\langle 61\rangle}{\langle 62\rangle}\,, \quad \alpha_8 = -\frac{\langle 6|Q_{612}|3]}{\langle 2|Q_{612}|3]}\,.
\end{aligned} \tag{6.27}
$$

The Jacobian from solving the delta functions is

$$
\frac{\langle 26\rangle[35]^4\langle 2|Q_{612}|4]}{[34]^2\langle 2|Q_{612}|5]^2\langle 2|Q_{612}|3]} \tag{6.28}
$$

### 6.4 One loop five-point in $\mathcal{N} = 0$

One might also consider diagrams with an extra internal parameter. At five-point an example of this is shown below

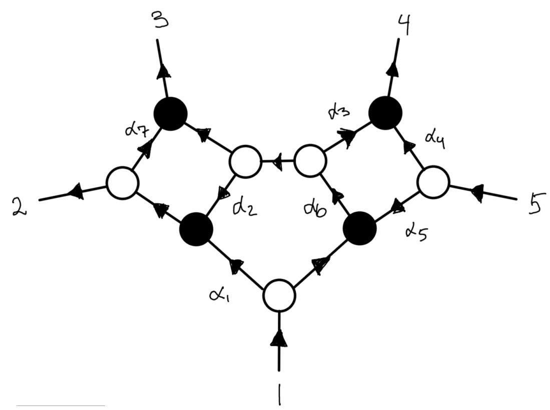



**Figure 11**.

Such a diagram can be thought of as a loop-level diagram, since the on-shell condition imposed on the three loop momenta leaves us with $3 \times 4 - 11 = 1$ unfixed parameter. The C-matrices are

$$
\begin{aligned}
C &= \begin{pmatrix} 1 & \alpha_1 + \alpha_2\alpha_6 & \alpha_2\alpha_7\alpha_6 + \alpha_6 + \alpha_1\alpha_7 & \alpha_3\alpha_6 & 0 \\ 0 & \alpha_2\alpha_5\alpha_6 & \alpha_5\alpha_6\,(\alpha_2\alpha_7 + 1) & \alpha_4 + \alpha_3\alpha_5\alpha_6 & 1 \end{pmatrix} \\
C_\perp &= \begin{pmatrix} -\alpha_1 - \alpha_2\alpha_6 & 1 & 0 & 0 & -\alpha_2\alpha_5\alpha_6 \\ -\alpha_6 - (\alpha_1 + \alpha_2\alpha_6)\,\alpha_7 & 0 & 1 & 0 & -\alpha_5\alpha_6\,(\alpha_2\alpha_7 + 1) \\ -\alpha_3\alpha_6 & 0 & 0 & 1 & -\alpha_4 - \alpha_3\alpha_5\alpha_6 \end{pmatrix}
\end{aligned}
\tag{6.29}
$$

$$
\delta(C_\perp \cdot \lambda) \to \begin{cases} 0 &= -\langle 12\rangle + \alpha_2\alpha_5\alpha_6\langle 15\rangle \\ 0 &= -((\alpha_1 + \alpha_2\alpha_6)\langle 15\rangle) + \langle 25\rangle \\ 0 &= -\langle 13\rangle + \alpha_5\alpha_6(1 + \alpha_2\alpha_7)\langle 15\rangle \\ 0 &= -((\alpha_6 + \alpha_1\alpha_7 + \alpha_2\alpha_6\alpha_7)\langle 15\rangle) + \langle 35\rangle \\ 0 &= -\langle 14\rangle + (\alpha_4 + \alpha_3\alpha_5\alpha_6)\langle 15\rangle \\ 0 &= -(\alpha_3\alpha_6\langle 15\rangle) + \langle 45\rangle \end{cases}
\tag{6.30}
$$

Solving the delta-functions we find

$$
\begin{aligned}
\alpha_1 &= -\frac{\langle 23\rangle}{\alpha_7\langle 12\rangle - \langle 13\rangle}, & \alpha_2 &= -\frac{\langle 12\rangle}{\alpha_7\langle 12\rangle - \langle 13\rangle}, & \alpha_3 &= -\frac{\langle 45\rangle}{\alpha_7\langle 25\rangle - \langle 35\rangle} \\
\alpha_4 &= -\frac{\alpha_7\langle 24\rangle - \langle 34\rangle}{\alpha_7\langle 25\rangle - \langle 35\rangle}, & \alpha_5 &= \frac{\alpha_7\langle 12\rangle - \langle 13\rangle}{\alpha_7\langle 25\rangle - \langle 35\rangle}, & \alpha_6 &= -\frac{\alpha_7\langle 25\rangle - \langle 35\rangle}{\langle 15\rangle}
\end{aligned}
\tag{6.31}
$$

Such that

$$
\begin{aligned}
&\delta(C_\perp \cdot \lambda)\delta(C \cdot \tilde{\lambda}) \\
=&\frac{\langle 15\rangle^3}{\left(\alpha_7\langle 12\rangle - \langle 13\rangle\right)\left(-\alpha_7\langle 25\rangle + \langle 35\rangle\right)^2}\delta^4(P)\delta\left(\alpha_1 + \frac{\langle 23\rangle}{\alpha_7\langle 12\rangle - \langle 13\rangle}\right)\delta\left(\alpha_2 + \frac{\langle 12\rangle}{\alpha_7\langle 12\rangle - \langle 13\rangle}\right) \\
&\delta\left(\alpha_3 + \frac{\langle 45\rangle}{\alpha_7\langle 25\rangle - \langle 35\rangle}\right)\delta\left(\alpha_4 + \frac{\alpha_7\langle 24\rangle - \langle 34\rangle}{\alpha_7\langle 25\rangle - \langle 35\rangle}\right)\delta\left(\alpha_5 - \frac{\alpha_7\langle 12\rangle - \langle 13\rangle}{\alpha_7\langle 25\rangle - \langle 35\rangle}\right)\delta\left(\alpha_6 + \frac{\alpha_7\langle 25\rangle - \langle 35\rangle}{\langle 15\rangle}\right)
\end{aligned}
\tag{6.32}
$$

So that we get the form

$$
\mathrm{d}\Omega = \frac{\langle 15\rangle^3}{\alpha_7\langle 12\rangle\langle 23\rangle\langle 45\rangle\left(\alpha_7\langle 24\rangle - \langle 34\rangle\right)} \tag{6.33}
$$

taking the residue of the form at $\alpha_7 = 0$ one recovers the standard YM Parke-Taylor tree-level amplitude

$$
\mathrm{d}\Omega = \frac{\langle 15\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle} \tag{6.34}
$$

Further, we note that the form has one other pole at $\alpha_7 = \frac{\langle 34\rangle}{\langle 24\rangle}$.

### 6.5 One loop five-point in any $\mathcal{N}$

To get the form for e.g. $\mathcal{N} = 1$, one needs to consider diagrams that contain internal closed cycles. Here we will consider the following

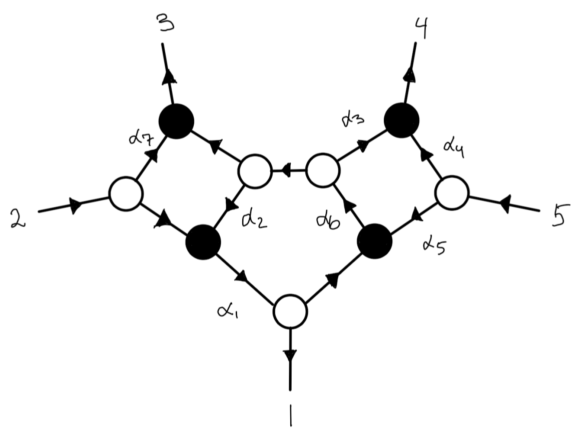



**Figure 12**.

The C-matrices obtained are

$$
\begin{aligned}
C &= \begin{pmatrix} \alpha_1\delta_a & 1 & \alpha_1\alpha_6\delta_a + \alpha_7 & \alpha_1\alpha_3\alpha_6\delta_a & 0 \\ \alpha_1\alpha_2\alpha_5\alpha_6\delta_a & 0 & \alpha_5\alpha_6\delta_a & \alpha_3\alpha_5\alpha_6\delta_a + \alpha_4 & 1 \end{pmatrix} \\
C_\perp &= \begin{pmatrix} 1 & -\alpha_1\delta_a & 0 & 0 & -\alpha_1\alpha_2\alpha_5\alpha_6\delta_a \\ 0 & -\alpha_1\alpha_6\delta_a - \alpha_7 & 1 & 0 & -\alpha_5\alpha_6\delta_a \\ 0 & -\alpha_1\alpha_3\alpha_6\delta_a & 0 & 1 & -\alpha_3\alpha_5\alpha_6\delta_a - \alpha_4 \end{pmatrix},
\end{aligned}
\tag{6.35}
$$

where $\delta_a = \frac{1}{1-\alpha_1\alpha_2\alpha_6}$. The first delta-function constraint gives us

$$\delta^6(C_\perp \cdot \lambda) \to \begin{cases} 0 & = \langle 12\rangle + \alpha_1\alpha_2\alpha_5\alpha_6\delta_a\langle 25\rangle \\ 0 & = \langle 15\rangle - \alpha_1\delta_a\langle 25\rangle \\ 0 & = -\langle 23\rangle + \alpha_5\alpha_6\delta_a\langle 25\rangle \\ 0 & = (-\alpha_7 - \alpha_1\alpha_6\delta_a)\langle 25\rangle + \langle 35\rangle \\ 0 & = -\langle 24\rangle - (-\alpha_4 - \alpha_3\alpha_5\alpha_6\delta_a)\langle 25\rangle \\ 0 & = -(\alpha_1\alpha_3\alpha_6\delta_a\langle 25\rangle) + \langle 45\rangle \end{cases} \tag{6.36}$$

Solving the delta-functions we find

$$\begin{aligned} &\alpha_1 = -\frac{\alpha_7\langle 12\rangle - \langle 13\rangle}{\langle 23\rangle}, \qquad \alpha_2 = -\frac{\langle 12\rangle}{\alpha_7\langle 12\rangle - \langle 13\rangle}, \qquad \alpha_3 = -\frac{\langle 45\rangle}{\alpha_7\langle 25\rangle - \langle 35\rangle} \\ &\alpha_4 = -\frac{\alpha_7\langle 24\rangle - \langle 34\rangle}{\alpha_7\langle 25\rangle - \langle 35\rangle}, \qquad \alpha_5 = \frac{\alpha_7\langle 12\rangle - \langle 13\rangle}{\alpha_7\langle 25\rangle - \langle 35\rangle}, \qquad \alpha_6 = -\frac{\alpha_7\langle 25\rangle - \langle 35\rangle}{\langle 15\rangle} \\ &\delta_a = -\frac{\langle 15\rangle\langle 23\rangle}{\langle 25\rangle\left(\alpha_7\langle 12\rangle - \langle 13\rangle\right)} \end{aligned} \tag{6.37}$$

$$\begin{aligned} &\delta^4(C_\perp \cdot \lambda)\delta^6(C_\perp \cdot \lambda) \\ =&\frac{\langle 15\rangle^4\left(\alpha_7\langle 12\rangle - \langle 13\rangle\right)}{\langle 15\rangle^2\langle 23\rangle^2\left(-\alpha_7\langle 25\rangle + \langle 35\rangle\right)^2}\delta^4(P)\delta\left(\alpha_1 + \frac{\alpha_7\langle 12\rangle - \langle 13\rangle}{\langle 23\rangle}\right)\delta\left(\alpha_2 + \frac{\langle 12\rangle}{\alpha_7\langle 12\rangle - \langle 13\rangle}\right) \\ &\delta\left(\alpha_3 + \frac{\langle 45\rangle}{\alpha_7\langle 25\rangle - \langle 35\rangle}\right)\delta\left(\alpha_4 + \frac{\alpha_7\langle 24\rangle - \langle 34\rangle}{\alpha_7\langle 25\rangle - \langle 35\rangle}\right)\delta\left(\alpha_5 - \frac{\alpha_7\langle 12\rangle - \langle 13\rangle}{\alpha_7\langle 25\rangle - \langle 35\rangle}\right)\delta\left(\alpha_6 + \frac{\alpha_7\langle 25\rangle - \langle 35\rangle}{\langle 15\rangle}\right) \end{aligned} \tag{6.38}$$

So that we get the form

$$\mathrm{d}\Omega = \frac{\langle 25\rangle^4}{\alpha_7\langle 12\rangle\langle 51\rangle\langle 23\rangle\langle 45\rangle\left(\alpha_7\langle 24\rangle - \langle 34\rangle\right)} \tag{6.39}$$

taking the residue of the form at $\alpha_7 = 0$ one recovers the standard YM Parke-Taylor tree-level amplitude

$$\mathrm{d}\Omega = \frac{\langle 15\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle} \tag{6.40}$$

Further, the $\mathcal{N} = 0$ Jacobian is

$$J_a = \delta_a^{-1} = \frac{\langle 25\rangle\left(\alpha_7\langle 12\rangle - \langle 13\rangle\right)}{\langle 15\rangle\langle 23\rangle} \tag{6.41}$$

### 6.6 5 point from stitching subamplitudes

Once again we look at the following diagram

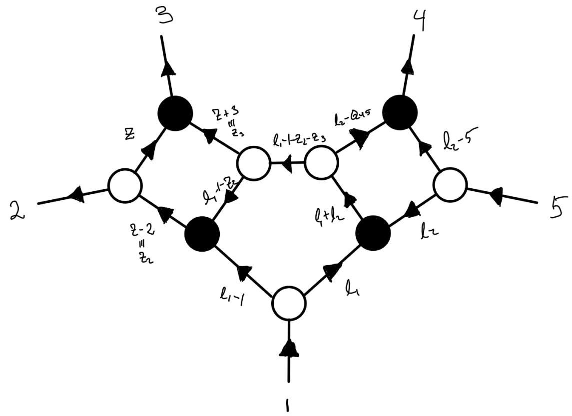



**Figure 13**.

Where we have assigned 3 different loop-momenta, $\ell_1$, $\ell_2$, and $z$. The momenta is found to be

$$z = \alpha\lambda_2\tilde{\lambda}_3, \qquad z_2 = \lambda_2\left(\alpha\tilde{\lambda}_3 - \tilde{\lambda}_2\right), \qquad z_3 = \tilde{\lambda}_3\left(\alpha\lambda_2 + \lambda_3\right) \tag{6.42}$$

giving the full $\mathcal{N} = 4$ form

$$\mathrm{d}\Omega_5^\ell = \frac{\delta(P)\delta(Q)}{\alpha\langle 12\rangle\langle 23\rangle\langle 45\rangle\langle 51\rangle\left(\alpha\langle 24\rangle + \langle 34\rangle\right)} \tag{6.43}$$

## A Jacobian from solving delta-function

The following property holds for the $\delta$-functions

$$\delta(kx) = \frac{1}{k}\delta(x) \tag{A.1}$$

which means that for instance when multiplying a deltafunction by a spinor one has

$$\delta(\lambda_i) = \lambda_j\delta(\lambda_j\lambda_i) \tag{A.2}$$

Or, using spinor helicity bracket notation

$$\delta(\lambda_i)\delta(\lambda_j) = \langle kl\rangle\delta(\lambda_k\lambda_i)\delta(\lambda_l\lambda_j) \tag{A.3}$$

## References

[1] N. E. J. Bjerrum-Bohr, J. F. Donoghue and P. Vanhove, "On-shell Techniques and Universal Results in Quantum Gravity," JHEP **02** (2014), 111 doi:10.1007/JHEP02(2014)111 [arXiv:1309.0804 [hep-th]].