Riemann Surfaces

An introduction

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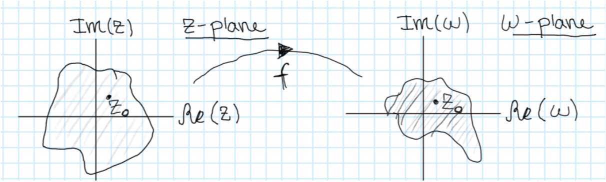
Overview

Recap of complex analysis so far

Functions of one complex variable

Take some open set U in the complex plane and a function f which takes complex variables z and maps them to $\omega = f(z)$ in the open domain V



Recap of complex analysis so far

Functions of one complex variable

Requiring f to be holomorphic in a neighborhood around z_0 put great constraints on our functions, i.e. Cauchy Riemann conditions (among others) with $\omega = u + iv$, $z = x + iy$

$$\partial_x u = \partial_y v, \qquad \partial_x v = -\partial_y u \tag{1}$$

Some definitions

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- **Isomorphism**: Structure-preserving mapping between two structures of the same type that can be reversed by an inverse mapping.
- **Homeomorphism**: Isomorphism in the category of topological spaces. I.e. they are the mappings that preserve all the topological properties of a given space
- Injective holomorphic map is a holomorphic isomorphism

Given a holomorphic injective map from an open set U to $\mathbb{C}$

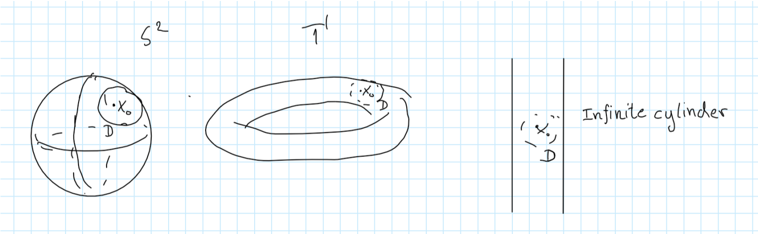
$$f : U \to \mathbb{C} \tag{2}$$

then $f(u)$ is open, and the inverse map is also holomorphic.

Complex analysis on surfaces

Complex analysis on surface

- Take a known surface that we can visualize.
- Pick some point x_0 on the surface on some disc-like domain $\mathcal{D}$.
- Introduce function f that takes on complex values on $\mathcal{D}$.
- Extend definition of holomorphic function at x_0 so that we can use tools of complex analysis on the surface.



Complex analysis on surfaces

Complex analysis on surface

Do this by identifying $\mathcal{D}$ with an open subset, say

$$\Delta = \{z \in \mathbb{C} \;\big|\; |z| = 1\} \tag{3}$$

by choosing a homeomorphism

$$\phi : \mathcal{D} \to \Delta \tag{4}$$

Hence we now have a map from the unit disc on the surface to the complex plane

$$\begin{tikzcd} \Delta & \mathcal{D} \arrow[l, "\phi"'] \arrow[r, "f"] & \mathbb{C} \\ \arrow[from=1-1, to=1-3, bend right, "f \circ \phi^{-1}"'] \end{tikzcd}$$

Complex analysis on surfaces

Complex analysis on surface

Require that $f \circ \phi^{-1}$ is holomorphic in points x ind $\mathcal{D}$.
Just like on slide 5, we can say that f is holomorphic on D if $f \circ \phi^{-1}$ is holomorphic on Δ.

Complex analysis on surfaces

Complex analysis on surface

The pair $(\mathcal{D}, \phi)$ is called a *complex coordinate chart*
It allows us to do complex analysis on the disc, but we could have taken any open set.

Complex analysis on surfaces

Complex analysis on surface

More generally: a complex coorddinate chart is a pair

$$(\mathcal{U}, \phi) \tag{5}$$

where $\mathcal{U}$ is an open subset of X and $\phi : \mathcal{U} \to \mathcal{V}$ is a homemorphism onto an open subset $\mathcal{V}$ of $\mathbb{C}$.

Applications

QFT scattering

Overlap between two asymptotic states

$$\langle f|i\rangle = (2\pi)^D \delta^D\left(\sum_i k_i\right)(\mathbb{1}_{fi} + iT_{fi}), \tag{6}$$

Scattering cross section proportional to $|T_{fi}|^2$.
We refer to T_{fi} as the scattering amplitude and denote it by $\mathcal{A}(...)$ where $(...)$ is the scattering data.

The CHY formalism

Scattering equations and amplitudes

The *scattering equations* are given by

$$\mathcal{S}_i = \sum_{j \neq i} \frac{s_{ij}}{z_i - z_j} = 0, \quad i \in \{1, 2, .., n\}. \tag{7}$$

One can obtain amplitudes of various theories from the formula

$$\mathcal{A}_n(1, ..., n) = \int d\Omega_{\mathsf{CHY}} \mathcal{I}(z_i, k_i, \epsilon_i, ...), \tag{8}$$

with $d\Omega_{\mathsf{CHY}} = \frac{d^n z}{\mathsf{Vol}(\mathsf{SL}(2,\mathbb{C}))} \prod_i' \delta(\mathcal{S}_i)$.

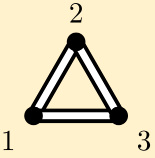
Integration rules

Graphic representation of Möbious invariant integrands

We represent the integrands by four-regular graphs. Every factor of z_{ij}^{-1} is a line between vertices i and j and every factor z_{ij} is a dashed line.

$$
\begin{aligned}
A_3^{\mathsf{YM}}(1,2,3) &= \mathsf{PT}(1,2,3)\mathsf{Pf}'\Psi^{1,3} \\
&= \frac{1}{z_{12}^2 z_{23}^2 z_{13}^2}\{-\epsilon_{12}\epsilon k_{32} + \epsilon_{13}\epsilon k_{23} + \epsilon_{23}\epsilon k_{12}\}
\end{aligned}
$$

$=$



$$\{-\epsilon_{12}\epsilon k_{32} + \epsilon_{13}\epsilon k_{23} + \epsilon_{23}\epsilon k_{12}\} \tag{9}$$

$$= -\epsilon_{12}\epsilon k_{32} - \epsilon_{13}\epsilon k_{21} + \epsilon_{23}\epsilon k_{12},$$

Thank you for your attention.