# A VOLUME FORMULA FOR YANG-MILLS MARGINALS

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Abstract. TODO

## 1. Notations and statement of the main result

1.1. **Conjugacy classes.** Throughout this paper, we fix an integer $n \geq 3$ and denote $\mathcal{H} = (\mathbb{R}/\mathbb{Z})^n/S_n$, where $S_n$ is the symmetric group of degree $n$ acting on $(\mathbb{R}/\mathbb{Z})^n$ by permutations of the coordinates. As a set, we identify $\mathcal{H}$ with $\{\theta = (\theta_1, \ldots, \theta_n) \in [0,1[^n \colon \theta_1 \geq \cdots \geq \theta_n\}$ in the usual way. For our purposes, $\mathcal{H}$ represents the set of conjugacy classes of $\mathrm{U}(n)$, where the conjugacy class of $\theta \in \mathcal{H}$ is

$$\mathcal{O}(\theta) = \left\{U e^{2\pi i\theta} U^{-1} \colon\, U \in \mathrm{U}(n)\right\} \quad \text{with} \quad e^{2\pi i\theta} = \begin{pmatrix} e^{2\pi i\theta_1} & 0 & \ldots & 0 \\ 0 & e^{2\pi i\theta_2} & & \vdots \\ \vdots & & \ddots & \\ 0 & \ldots & & e^{2\pi i\theta_n} \end{pmatrix}.$$

Let us use the notation $\mathcal{H}_{reg} = \{\theta \in \mathcal{H} \colon \theta_1 > \theta_2 > \ldots > \theta_n\}$ which represents the set of regular conjugacy classes of $\mathrm{U}(n)$, namely the ones of maximal dimension in $\mathrm{U}(n)$. Finally, let $\mathcal{H}^0_{reg}$ denote the subset of $\mathcal{H}_{reg}$ corresponding to the regular conjugacy classes of $\mathrm{SU}(n)$. Namely,

$$\mathcal{H}^0_{reg} = \left\{\theta \in \mathcal{H} \,:\, \theta_1 > \theta_2 > \ldots > \theta_n \text{ and } \sum_{i=1}^n \theta_i \in \mathbb{N}\right\}.$$

1.2. **Triangular honeycomb.** Let $S$ be an oriented surface with boundary (possibly empty) endowed with a flat metric $g$ and with its induced volume form $\omega$. Consider, for example, $S = \mathbb{C}$ or $S = \mathbb{C} \setminus \{0\}$ with their usual metrics. Let us denote by $\mathbb{G}$ the set of non-degenerate geodesics of $S$. For $x, y \in S$ and $e$ a geodesic from $x$ to $y$, we then set $\partial e = \{x, y\}$ and denote by $\mathring{e} = e \setminus \{x, y\}$ its interior. Remark that $(S, \mathbb{G})$ is then a graph with an uncountable number of vertices.
Since $S$ is a locally flat and oriented surface, there non-vanishing two form $\omega$ on $S$ associated to its metric. If $e, e' \in \mathcal{E}$ are two geodesics such that $v \in e \cap e'$, we define the angle from $e$ to $e'$ as

$$\widehat{(e, e')} = \arccos(g_v(t, t')),$$

where $t, t' \in T_v S$ are unit tangent vectors of respectively $e$ and $e'$ at $v$ such that either $t = t'$ or $\omega_v(t, t') > 0$.

**Definition 1.1** (Non-degenerated honeycomb). A *honeycomb* $h$ is a union of closed non-degenerate geodesics $\{e\}_{e \in \mathcal{E}}$, where $\mathcal{E} \subset \mathbb{G}$ is finite, together with a color map $c : \mathcal{E} \to \{0, 1, 3\}$ such that :

(1) $\mathring{e} \cap \mathring{e}' \neq \emptyset$ is only possible if, up to a transposition of $e$ and $e'$, $c(e) = 0$, $c'(e') = 1$ and $\widehat{(e, e')} = -2\pi/3$,
(2) if $\partial e \cap \partial e' = \{v\}$, then either $(c(e^1), c(e^2)) \in \{(0,0), (1,1), (0,1), (1,3), (3,0)\}$ and $\widehat{(e, e')} = 2\pi/3$, or $(c(e^1), c(e^2)) = (1, 0)$ and $\widehat{(e, e')} = \pi/3$.

A *structure graph* of a honeycomb $h$ is any finite graph $(V, E)$ with a coloring $c : E \to \{0, 1, 3\}$ with an injection $i : (V, E) \to (S, \mathbb{G})$ such that

$$h = \bigcup_{e \in E} i(e) \quad \text{and} \quad c(e) = c(i(e)), e \in E.$$

In particular, if $h$ is a non-degenerate honeycomb, then $G(h) = (V, \mathcal{E})$ with $V = \{v \in S \mid \exists e \in \mathcal{E}, v \in \partial e\}$ and the coloring $c$ is a structure graph of $h$, called its *canonical structure graph.* Remark from the definition that $\mathcal{E}$ is unambiguously defined from $h$.

By the angle condition, $G = G(h)$ has only vertices of degree less than 3 and the sequence of colors around any trivalent (resp. bivalent) vertex of $G(h)$ belongs to $\{(0,0,0),(1,1,1),(0,3,1)\}$ (resp. is equal to $(0,1)$) in the clockwise order.
The *color number* $c(G)$ of a colored graph $G$ is defined as the number of edges colored 1 and adjacent to a univalent vertex. Then, all structure graphs of a same honeycomb $h$ have the same color number. By abuse of definition, we speak of edges and vertices of a honeycomb to denote edges and vertices of its canonical structure graph.

Remark that in general, one may choose a structure vertices up to degree 4 if we add the crossing of edges from case (1) of Definition 1.1.

Let us denote by $T \coloneqq \{x + ye^{i\pi/3} \mid 0 \leq x, y \leq 1,\ x + y \leq 1\} \subset \mathbb{C}$ the equilateral triangle with vertices $0, 1$ and $e^{i\pi/3}$. To each point $v \in T$ we associate the triple $(v_0, v_1, v_2)$ such that $v = v_1 + v_2 e^{i\pi/3}$ and $v_0 = 1 - v_1 - v_2$. Then, the boundary $\partial T$ can be decomposed as

$$\partial T = \bigsqcup_{i \in \{0,1,2\}} \partial_i T \text{ , where } \partial_i T = \{v \in T \mid v_i = 0\}.$$

**Definition 1.2** (Triangular honeycomb)**.** A *triangular honeycomb* $h$ of size $n$ is a non-degenerated honeycomb $h$ on the surface $T$ endowed with the Euclidian metric such that

(1) $G(h)$ has only univalent and trivalent vertices, and $\bigcup_{e \in h} e \cap \partial T = V_1$, where $V_1$ denotes the set of univalent vertices in $G(h)$,
(2) if $e \in h$, then $e \subset \left\{x + \mathbb{R}e^{2\pi i(\ell(e)+1)/3}\right\}$ for some $\ell(e) \in \{0,1,2\}$ and $x \in T$. The integer $\ell(e)$ is then called the *type* of $e$ and $L(e) = x_{\ell(e)}$ is called the *height* of $e$ which is independent of the choice of $x$,
(3) for $0 \leq i \leq 2$, $\#h \cap \partial_i T = n$ and if $e$ is adjacent to a boundary vertex belonging to $\partial_i T$, then either $c(e) = 0$ and $\ell(e) = i+1$ or $c(e) = 1$ and $\ell(e) = i+2$. Moreover, the color is increasing along each edge : namely, if $e^1$ (resp. $e^2$) meets $\partial_i T$ at $x^1$ (resp. $x^2$) with $x^2_{i+1} > x^1_{i+1}$, then $c(e^2) \geq c(e^1)$.

A triangular honeycomb has always $n^2 + 3n$ vertices and $\frac{3n(n+1)}{2}$ edges, see (**??**). This definition implies that $G(h)$ has only trivalent vertices in $\mathring{T}$. Moreover, the condition on the boundaries yields a natural choice of root $v^0$ of $G$, corresponding to the univalent vertex on $\partial_0 T$ whose coordinate $v^1_1$ is maximal. Then, the cyclic counter-clockwise order on the boundary vertices given by the orientation of $T$ yields an order on the $3n$ boundary vertices $\{v^1, \dots, v^{3n}\}$, with the vertices $\{v^{ni+j}, 1 \leq j \leq n\}$ located on $\partial_i T$.
In the particular case where all edges have the same color, our definition is the original definition of a generic honeycomb from [KT99]. Remark that, besides the coloring, our definition of triangular honeycombs differs slightly from the original one since we impose vertices to be trivalent inside $\mathrm{int}(T)$. The set of honeycombs in their original definition can then be seen as the closure of the ones from the present manuscript (in the case where all colors are the same).

The *boundary*, or the *boundary values*, of a honeycomb $h$ of size $n \geqslant 3$ is the $3n$ tuple

$$\partial h \coloneqq \left((\alpha^0_n \leq \dots \leq \alpha^0_1), (\alpha^1_n \leq \dots \leq \alpha^1_1), (\alpha^2_n \leq \dots \leq \alpha^2_1)\right), \tag{1.1}$$

where $(\alpha^i_n \leq \dots \leq \alpha^i_1)$ is the ordered tuple of the $i+1$-coordinates of boundary points of $h$ on $\partial_i T$. Hence, $\alpha^i_j = v^{ni+j}_{i+1}$ for $0 \leq i \leq 2$ and $1 \leq j \leq n$.

For $\alpha, \beta, \gamma \in \mathcal{H}_{reg}$ let us denote by $\texttt{HONEY}_{d,n}(\alpha,\beta,\gamma)$ the set of triangular honeycombs $h$ having boundary values $\partial h = (\beta, \alpha, \gamma)$, see Figure 1, and such that there are $d$ edges colored 1 meeting one (and thus each) boundary component of $T$. For any colored graph $G$ with an order on the boundary vertices, let us denote by

$$\texttt{HONEY}^G_{n,d}(\alpha,\beta,\gamma)$$

the set of triangular honeycombs of $\texttt{HONEY}_{d,n}(\alpha,\beta,\gamma)$ with canonical graph structure isomorphic to $G$ as colored graph with ordered boundary. Then, if $\texttt{HONEY}^G_{n,d}(\alpha,\beta,\gamma)$ is non-empty, necessarily the graph $G$ has $3d$ univalent vertices adjacent to edges colored 1. We then define the color of $G$ as $c(G) = d$.

Let us denote by $\mathcal{G}_d$ the set of isomorphism classes of colored graphs with ordered boundary appearing in $\{G(h) \mid h \in \mathtt{HONEY}_{n,d}\}$.

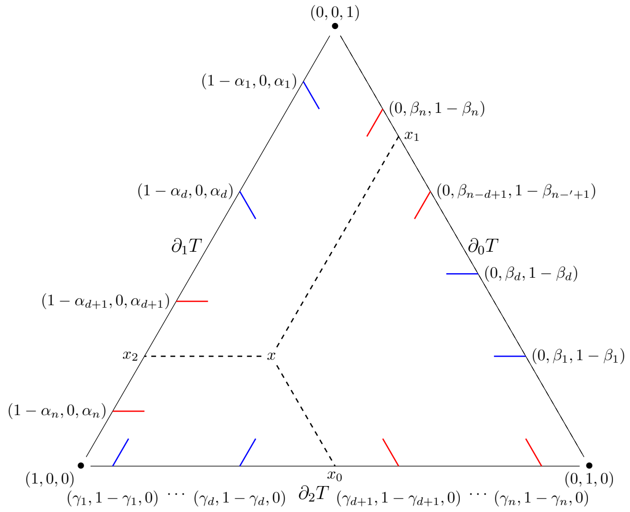



FIGURE 1. Boundaries of a honeycomb in $\mathtt{HONEY}_{n,d}(\alpha, \beta, \gamma)$.

1.3. $(g,p)-$**honeycombs.** Let $g, p \geqslant 0$ be integers and let $S$ be a connected compact orientable surface of genus $g$ with $p$ boundary components $L_1, \ldots, L_p$. Let $\mathcal{M} = (M_1 \ldots, M_{2g+p-2})$ be a decomposition in pants of this surface. We build from $\mathcal{M}$ a surface as follows :

- take $N := 2g + p - 2$ oriented equilateral triangles $(T^1, \ldots, T^{2g+p-2})$, each $T^j$ having three oriented boundaries $\partial_i T^j$ of size 1.
- For each pair of boundary components $\partial_i M^j, \partial_{i'} M^{j'}$ which are identified in the pair of pants decomposition, identify the boundaries $\partial_i T^j$ and $\partial_{i'} T^{j'}$ in an orientation reversing way.

Then, the resulting surface $\mathcal{T}$ is an oriented surface with $p$ boundaries edges $\partial_1 \mathcal{T}, \ldots, \partial_p \mathcal{T}$, and the natural euclidean metric on each equilateral triangle yields a metric on $\mathcal{T}$ which is flat except at the vertices of the triangulation belonging to the interior of $\mathcal{T}$.
Remark that if $h$ is a honeycomb on $S$ and $S' \subset S$ is a closed convex submanifold, then $h \cap S'$ is again a honeycomb.

**Definition 1.3.** A $(g,p)$-*honeycomb* is a honeycomb $h$ on $\mathcal{T}$ such that for each $1 \leq i \leq 2g + p - 2$, $h \cap T_i$ is a triangular honeycomb.

We denote by $\mathtt{HONEY}^{(g,p)}$ the set of $(g,p)$-honeycombs. For $h \in \mathtt{HONEY}^{(g,p)}$ and $1 \leqslant i \leqslant 2g+p-2$, let $G_h^i = (V^i, E^i)$ be the canonical graph structure of the honeycomb $h \cap T_i$. We choose as structure graph of $h$ the graph $\hat{G}[h] = (V, E)$ where

$$V = \bigcup_{1 \leq i \leq 2g+p-2} V_i \quad \text{and} \quad E = \bigcup_{1 \leq i \leq 2g+p-2} E_i \ .$$

In particular, if $h$ is a $(g,p)$ honeycomb and $L_i$ a boundary component, there exists a triangle $T_{j_i}$ and $\ell_i \in \{0,1,2\}$ such that $L_i = \partial_{\ell_i} T^{j_i}$, and we set $\partial_i h = h \cap L_i$. As in the triangular case, the orientation of $S$ yields an orientation on each boundary component. For $h \in \mathtt{HONEY}^{(g,p)}$, order the boundary points so that

$$h \cap \bigcup_{1 \leq i \leq p} L_i = \{v^{(i-1)n+j}, 1 \leq i \leq p, j \leq n\},$$

where $\partial_i h = \{v^{(i-1)n+j}, 1 \leq j \leq n\}$ with $v_{\ell_i+1}^{(i-1)n+j} > v_{\ell_i+1}^{(i-1)n+j'}$ if $1 \leq j < j' \leq n$.
We denote by $\mathtt{HONEY}^{(g,p)}\left(\alpha^1, \ldots, \alpha^p\right)$ the set of $(g,p)$-honeycombs with boundary components $\partial_i h = \alpha^i$.

Beware that the structure graph we are using for $(g,p)$- honeycombs are slightly different from the canonical structure graph introduced after Definition 1.1, since geodesics may cross the boundary $\partial T$ of an equilateral triangle $T$. In this case, the geodesic breaks into two geodesics meeting at a vertex belonging to $\partial T$.

Like in the triangular case, we denote by $\mathtt{HONEY}^G$ the subset of $\mathtt{HONEY}^{(g,p)}$ of honeycombs $h$ with structure graph $\hat{G}$ isomorphic to $G$ as colored graph with ordered boundary vertices. We also denote by $\mathcal{G}^{(p,g)}$ the set of isomorphism classes of colored graphs with ordered boundary appearing in $\mathtt{HONEY}^{(g,p)}$. For $G \in \mathcal{G}^{(g,p)}$, let us set

$$c(G) = \sum_{i=1}^{N} c(G_i).$$

Let us denote by $\mathcal{G}_d^{(g,p)} \subset \mathcal{G}^{(g,p)}$ the subset of graphs $G$ such that $c(G) = d$.

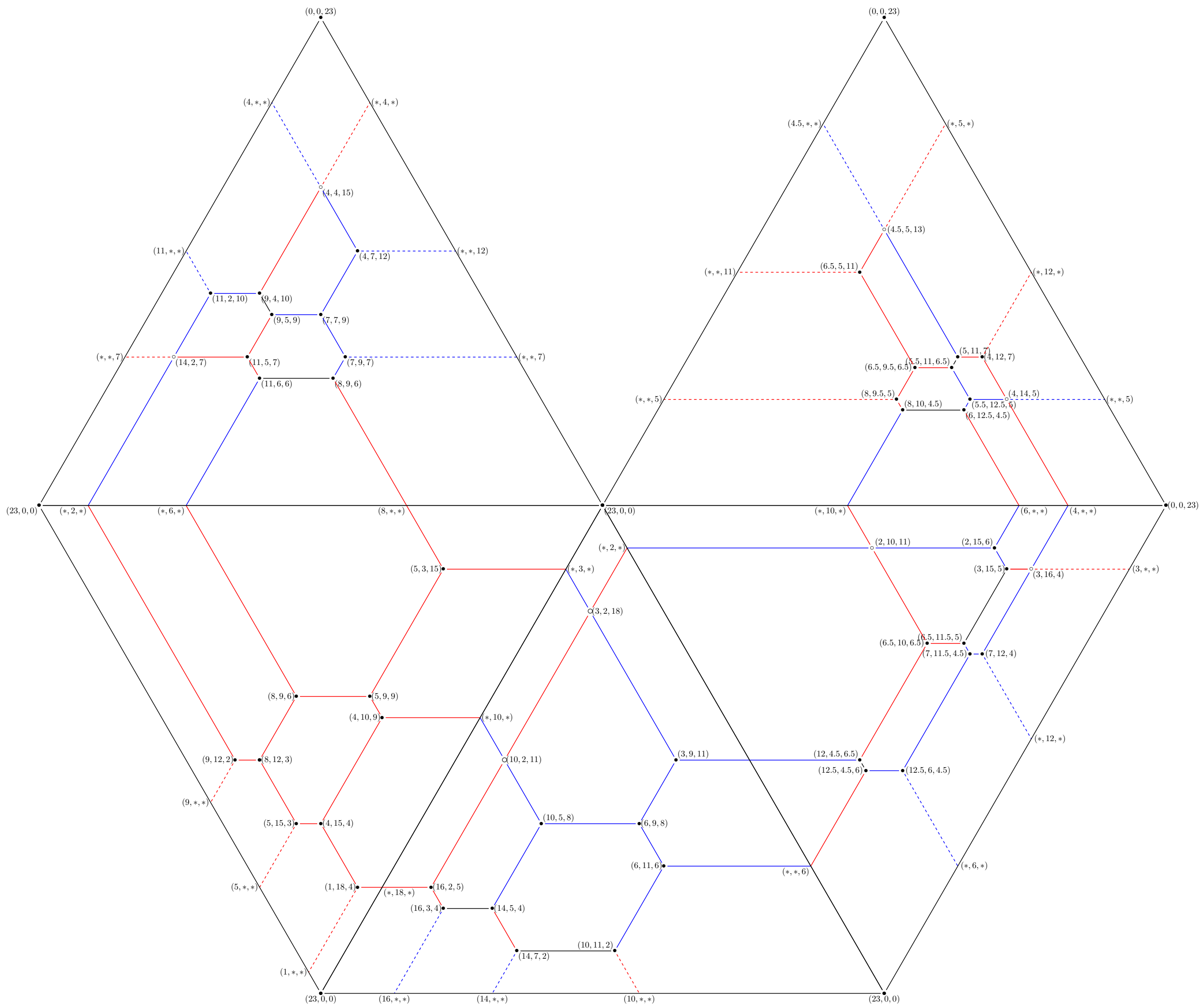



FIGURE 2. A $(2,3)$ honeycomb for a surface with genus 2 and 3 boundaries

For $G = (V, E) \in \mathcal{G}^{(p,g)}$, there is a natural parametrization of $\mathtt{HONEY}^G$ constructed as follows. For $e \in E$ there exists $1 \leq i \leq N$ such that $e \in E_i$. For $h \in \mathtt{HONEY}^G$, set

$$\mathcal{L}[h](e) = L_{T_i}(i(e)),$$

where $L_{|T_i}$ is the height map defined on $T_i$ from Definition 1.2 and $i : E \to \mathbb{G}$ is the injection from Definition 1.1 such that $h \cap T_i = \bigcup_{e \in E_i} i(e)$. Then, introduce the map

$$\mathcal{L} : \mathtt{HONEY}^G \to \mathbb{R}^E$$

sending $h \in \mathtt{HONEY}^G$ to $(\mathcal{L}(e))_{e \in E}$. This map is clearly non-surjective since $\mathcal{L}\left(\mathtt{HONEY}^G\right)$ is a bounded subset of $\left(\mathbb{R}^{3n(n+1)/2}\right)^N$. Moreover, there are several relations among the values of $((\mathcal{L}_{|T_i}(e))_{e \in G_h^i})_{1 \leq i \leq N}$

which implies that $\mathcal{L}$ is an over-parametrization of $\mathtt{HONEY}^G$. However, it will be proven in Proposition 3.3 that $\mathcal{L}$ is injective. In the sequel, we identify $\mathtt{HONEY}^G$ with its image through $\mathcal{L}$.

As a consequence of (??) below, all graphs of $\mathcal{G}^{(g,p)}$ have the same number $n_e = \frac{3Nn(n+1)}{2}$ of edges and $n_v = Nn^2 + \frac{(3N+p)n}{2}$ of vertices. Hence, up to identifying edges and vertices of these graph, one can assume that there is a unique set $E^{(g,p)}$ of edges (resp. set $V^{(g,p)}$ of vertices) common to all graphs of $\mathcal{G}^{(g,p)}$ and that the graph structure of $G \in \mathcal{G}^{(g,p)}$ is encoded in the map $\partial : E^{(g,p)} \to \mathcal{P}(V^{(g,p)})$ which associates to an edge its endpoints. La section "Volume formula" n'a plus raison d'être

1.4. **Volume of flat connections.** Let us denote by $M_{g,n}(\alpha_1, \dots, \alpha_p)$ the moduli space of flat $\mathrm{U}(n)$-valued connections on a compact oriented surface with genus $g$ having $p$ boundary components for which the holonomies around $L_1, \dots, L_p$ respectively belong to $\mathcal{O}_{\alpha_1}, \mathcal{O}_{\alpha_2} \dots, \mathcal{O}_{\alpha_p}$. For $g, p \geq 0$, Set $d\,\mathrm{Vol} \in \Omega^{n_{g,p}}(\mathbb{R}^{E^{(g,p)}})$ for the volume form on $n_{g,p}$-dimensional subspaces induced by the canonical Riemannian metric on $\mathbb{R}^{E^{(g,p)}}$. Hence,

$$d\,\mathrm{Vol} = \sum_{1 \leq i_1 < \dots < i_{n_{g,p}} < \frac{3Nn(n+1)}{2}} dx_{i_1} \wedge \dots \wedge dx_{i_{n_{g,p}}}.$$

In the following statement, we write $\mathrm{Vol}(K)$ for the integration of $d\,\mathrm{Vol}$ on a $n_{g,p}$-dimensional submanifold and we let $\Delta(x) = \prod_{i<j}(x_j - x_i)$ denote the Vandermonde determinant.

**Theorem 1.4** (Volume formula for $(g,p)$ partition function)**.** *Let $Z_{g,p,0}(\alpha_1, \dots, \alpha_p)$ be the volume function for the moduli space of flat connection on a compact oriented surface with $p$ boundary components and with genus $g$. Then, for $\alpha_1, \dots, \alpha_p \in \mathcal{H}_{reg}$ such that $\sum_{i=1}^p |\alpha_i|_1 \in \mathbb{N}$, we have*

$$Z_{g,p,0}(\alpha_1, \dots, \alpha_p) = \frac{c_{0,3}^{2g+p-2}}{n^{2g+p-3}} \sum_{G \in \mathcal{G}^{(g,p)}} \frac{\mathrm{Vol}\left[\mathtt{HONEY}^G(\alpha_1, \dots, \alpha_p)\right]}{\sqrt{\#\ \textit{Spanning trees of } G}}, \tag{1.2}$$

*where* $c_{0,3} = \frac{2^{(n+1)[2]}(2\pi)^{(n-1)(n-2)}}{n!}$.

Remark that the sum on the right hand-side is finite, since each set $\mathcal{G}_d^{(g,p)}$ is finite and $d \leq n-1$. As it will appear below, for given $\alpha_1, \dots, \alpha_p \in \mathcal{H}_{reg}$ the volumes appearing in the sum will be non-zero only for $G \in \mathcal{G}_d^{(g,p)}$ where $d = \sum_{i=1}^p \sum_{j=1}^n \alpha_j^i - (p-2)n$.

The formula of Theorem 1.4 also yields a formula for $\mathrm{SU}(n)$-valued connections, since the volume for the $\mathrm{SU}(n)$ case is equal to the one of the $\mathrm{U}(n)$ case for $\alpha_1, \dots, \alpha_p \in \mathcal{H}^0_{reg}$, see (5.2).

**Yang-Mills marginal for disjoint curves.** Theorem 1.4 provides an explicit formula for the marginal Yang–Mills partition function of an oriented surface of genus $g$ with prescribed non-degenerate holonomies (up to conjugation) on a finite set of disjoint loops. This formula is given in Corollary 1.5 below. As it is proven in XXXRefXXX, the partition function only depends on the prescribed conjugacy classes and on the areas of each connected components delimited by the loops.

Let $S$ be a connected compact oriented surface of genus $g \geqslant 0$ together with $p$ disjoints Jordan curves $\Gamma_1, \dots, \Gamma_p$ on $S$. For each $\Gamma_i$, $1 \leq i \leq p$, let $\alpha_i$ be an element of $\mathcal{H}_{reg}$.
We associate to $(S, \Gamma_1, \dots, \Gamma_p)$ a labeled finite tree $T = (V, E)$ such that vertices are labeled by $\mathbb{N} \times \mathbb{R}^+$ and edges are labeled by $\mathcal{H}_{reg}$ as follows :

- the set $V$ of vertices of $T$ is the set of connected components of $S \setminus \bigcup_{i=1}^p \Gamma_i$. Each vertex $v \in V$ is labeled $(A_v, g_v)$ where $A_v$ is the area of the corresponding connected component and $g_v$ is its genus.
- for $v_1, v_2 \in V$, there is an edge $e$ between $v_1$ and $v_2$ for each boundary component. Since loops of $\mathcal{L}$ are non-intersecting, the each boundary component corresponds to a unique loop $\Gamma_j$ of $\mathcal{L}$, and then we label $\alpha_j$ the edge $e$.

In the following, denote by $d_v$ the degree of a vertex $v \in V$.

**Corollary 1.5** (Yang-Mills partition function)**.** *The Yang-Mills partition function associated to the data $(S, \Gamma_1, \dots, \Gamma_p)$ is*

$$\mathrm{YM}(\alpha_1, \dots, \alpha_p) = \int_{\mathcal{H}^{\sum_{i=1}^{\#V} d_v}} \prod_{v \in V} Z_{g,p,0}(\mu_1^v, \dots, \mu_{v_v}^v) \prod_{e_i, v \in e_i} p_{A_v/d_v}(\mu_i^v, \alpha_i) \prod \mathrm{d}\mu_i^v, \tag{1.3}$$

*where* $Z_{0,1,0}(\mu) = \delta_0$*,* $Z_{0,2,0}(\mu,\nu) = \delta_{\mu=\nu}$ *and* $Z_{g,p,0}$ *is given in* (1.2) *otherwise.*

XXXTo do :
- Explain why we remove the crossings : parametrization with spanning tree
- Explain that it is really the volume of a random process.XXX

XXXReference ?XXX

**Organisation of the paper.** In Section **??**, we establish a parametrization of honeycombs which we view as particular instances of *differential structures* introduced in Section **??** on which we define a canonical volume form in Section **??**. The parametrization of honeycombs is then given in Section 3.1. In Section 3, Theorem 3.15, we prove the result of Theorem 1.4 in the case of the three holed-sphere which corresponds to $(g,p) = (0,3)$. In Section 2.2, we introduce the *sieving* and the *contraction* operations on differential structures. Two applications to honeycombs are given in Proposition 4.4 and Proposition 4.5 in Section 4. In Section **??**, apply the previous to construct the volume form on honeycombs, proving Theorem **??**. We then prove Theorem 1.4 for genus zero surfaces in Section 5 using the operations of Section **??** and recursion formulas from [MW99]. The full generality of Theorem 1.4 and Corollary 1.5 are then proved in Section 6.

## 2. Graphs, divergence of flows and volume measures

We will see a honeycomb as graph $G$ endowed with a flow (an antisymmetric function on its edge set). The honeycomb information of $G$, i.e., the precise drawing on the equilateral triangle of size 1, will be obtained once we fix the distance of each edge to the side of the corresponding triangle parallel to it, which are embodied by the map $L$. Is is thus important to give an explicit expression of the volume measure associated to this distances, which is the main objective of this section.

Let us consider a finite graph $G = (V, E)$ and let us denote $\vec{E} = \{(a, b) \in V \times V : \{a, b\} \in E\}$ its oriented edge set. A function $\omega : \vec{E} \to \mathbb{R}$ is called antisymmetric if

$$\omega(a, b) = -\omega(b, a).$$

Let us denote by $\Omega^0(G)$ the vector space of real functions on $V$ endowed with the inner product given by $\langle f_1, f_2\rangle = \sum_{x\in V} f_1(x)f_2(x)$. The space of flows is the vector space $\Omega^1(G)$ of antisymmetric real functions $\omega$ on $\vec{E}$ endowed with the inner product $\langle \omega_1, \omega_2\rangle = \sum_{e\in E} \omega_1(e)\omega_2(e)$, where we are using that, due to the antisymmetry of $\omega_1$ and $\omega_2$, the product $\omega_1(e)\omega_2(e)$ does not depend on the orientating of $e$ as long as we choose the same for both arguments. For any map $\phi : V \to \mathbb{R}$, set

$$\mathcal{F}(\phi) = \left\{ \lambda \in \Omega^1(G) \mid \forall v \in V : \sum_{x\sim v} \lambda(v, x) = \phi(v) \right\}. \tag{2.1}$$

We will recall some properties of the *divergence* operator used in the previous equation and explore the space of solutions to provide a convenient measure on $\mathcal{F}(\phi)$.

2.1. **Canonical measure.** The analogue of the differential of a function is the map $d : \Omega^0(G) \to \Omega^1(G)$,

$$df(a, b) = f(b) - f(a).$$

For $x \in V$ define $\delta_x \in \Omega^0(G)$ that takes the value 1 at $x$ and 0 at other vertices, and for $e \in \vec{E}$ define $\delta_e \in \Omega^1(G)$ that takes the value 1 at $e$, $-1$ at the opposite of $e$ and 0 at other edges. So, we may evaluate $d$ at the basis $\{\delta_x\}_{x\in V}$ of $\Omega^0(G)$ to obtain $d\delta_x = -\sum_{e\in\vec{E}, \underline{e}=x} \delta_e$, where $\underline{(a, b)} = a$. This tells us that we can write $d$ in terms of $\delta_e$ and $\delta_x$ as

$$d = -\sum_{x\in V} \sum_{e\in\vec{E},\underline{e}=x} \delta_e \otimes \delta_x$$

by identifying $\Omega^0(G)$ with its dual. Its adjoint $d^*$ would be given by permuting the terms

$$d^* = -\sum_{x\in V} \sum_{e\in\vec{E},\underline{e}=x} \delta_x \otimes \delta_e$$

or, in a more explicit way, for $\omega \in \Omega^1(G)$,

$$d^*\omega(x) = -\sum_{e\in\vec{E},\underline{e}=x} \omega(e).$$

We define div $= -d^*$ so that (2.1) rewrites as div$\lambda = \phi$, yielding to the reformulation of the corresponding affine space

$$\mathcal{F}(\phi) = \{\lambda \in \Omega^1(G) : \text{div}\lambda = \phi\}.$$

Let us for now assume that $G$ is connected. Remark first that the solution space may be empty since we always have, by antisymmetry, $\sum_{v\in V} \text{div}\lambda(v) = 0$ which is reminiscent of Stokes' theorem. But this is the only restriction as explained in Proposition 2.1. If $\lambda_0 \in \mathcal{F}(\phi)$ we have $\mathcal{F}(\phi) = \lambda_0 + \mathcal{F}(0)$ and $\mathcal{F}(0)$ is precisely Ker(div). So, if $\mathcal{F}(\phi)$ is non-empty, we can calculate

$$\dim(\mathcal{F}(\phi)) = \dim(\text{Ker}(\text{div})) = |E| - \dim(\text{Im}(d)) = |E| - |V| + 1,$$

where we used that $\text{Ker}(d) = \{\text{constant functions}\}$ so that $\dim(\text{Im}(d)) = \dim(\Omega^0(G)) - 1 = |V| - 1$.

**Proposition 2.1.** *If $G$ is connected, the set $\mathcal{F}(\phi)$ is not empty if and only if*

$$\sum_{v\in V} \phi(v) = 0. \tag{2.2}$$

*Moreover, if $S \subset E$ and $\Omega^1(S)$ denotes the space of antisymmetric functions on $\vec{S} \subset \vec{E}$, the composition*

$$\varphi_S : \mathcal{F}(\phi) \xrightarrow{inclusion} \Omega^1(G) \xrightarrow{restriction} \Omega^1(S)$$

*is a bijection if and only if $\mathcal{F}(\phi) \neq \emptyset$ and $(V, E \setminus S)$ is a spanning tree. In case $\varphi_S$ is a bijection, the matrices of $\varphi_S$ and $\varphi_S^{-1}$ in the bases $\{\hat{e}, e \in E\}, \{\hat{e}, e \in S\}$ have integer coefficients.*

*Proof of Proposition 2.1.* We already know that the condition (2.2) is necessary. Let us show it is sufficient. Consider first the case where $G$ is a **tree**. To solve div$\lambda = \phi$ we look for the edges where $\lambda$ is most easily determined by the boundary conditions. Let us denote by $V_1$ the set of leaves of $T$. Then, the divergence condition yields $\lambda(e) = \phi(x)$ if $x \in V_1$ and $e$ is the unique edge starting from $x$.

Next, denote by $V_{>1}$ the vertices of $G$ of degree larger than one. Let us consider edges with an endpoint whose other edges connect only to leaves or, equivalently, the edges in $G \setminus V_1$ adjacent to leaves of $G \setminus V_1$. Take one such leaf $x \in V_{>1}$ of $G \setminus V_1$ and notice that, since $x$ is not a leaf of $G$, the set $V_{1,x}$ of vertices in $V_1$ connected to $x$ is non-empty. Let us reduce our task to finding a solution on $G \setminus V_{1,x}$ as follows. If $e$ denotes the edge of $G \setminus V_1$ starting at $x$, using the previous case of $x$ being a leaf yields that a solution should satisfy

$$\phi(x) = \Big(\sum_{y\in V_{1,x}} \lambda(x,y)\Big) + \lambda(e) = -\Big(\sum_{y\in V_{1,x}} \lambda(y,x)\Big) + \lambda(e) = -\Big(\sum_{y\in V_{1,x}} \phi(y)\Big) + \lambda(e)$$

which determines $\lambda$ at $e$ as linear combinations of different values $\phi$ with integer coefficients. Now, we consider the graph $G \setminus V_{1,x}$, define $\phi(x) = \Big(\phi(x) + \sum_{y\in V_{1,x}} \phi(y)\Big)$ and notice that (2.2) is satisfied for $G \setminus V_{1,x}$. We may proceed by induction until the tree consists solely of leaves in which case the condition (2.2) is precisely the equation div$\lambda = \phi$. By this procedure we have seen that, for a tree, in case there is a solution, it is unique, and its value at $e \in G$ is a linear combination with integer coefficients of the values $\phi(x)$ for $x \in V$.

For a **general graph** $G$ we may take a spanning tree $T$ of $G$ and try to solve the equation for $T$. We may choose the values of $\lambda$ arbitrarily at the edges that are not in $T$. More precisely, if $S$ is the set of edges not belonging to $T$, we may consider any antisymmetric function $\widetilde{\lambda} : \vec{S} \to \mathbb{R}$ and look for a solution $\lambda \in \mathcal{F}(\phi)$ satisfying $\lambda|_{\vec{S}} = \widetilde{\lambda}$. To be able to forget the edges in $S$ we change $\phi$ to $\phi_{new} : V \to \mathbb{R}$ given by

$$\phi_{new}(x) = \phi(x) - \sum_{e\in\vec{S}, \underline{e}=x} \widetilde{\lambda}(e)$$

so that if the divergence of $\lambda$ in $T$ at $x$ is $\phi_{new}(x)$, the divergence of the extension by $\widetilde{\lambda}$ of $\lambda$ in $G$ at $x$ would be $\phi(x)$. Notice that the sum of $\phi_{new}(x)$ for $x \in V$ is the same as the sum of $\phi(x)$. This holds because each edge in $S$ is appears twice in the sum, once with each orientation, and thus the contribution to the sum cancels due to the antisymmetry. Then (2.2) holds for $T$ and $(\phi_{new}; \gamma_{new})$, and we may find a unique solution $\lambda \in \mathcal{F}(\phi_{new})$ on $T$ such that $\lambda(x)$ is a linear combination with integer coefficients of the values of $\phi_{new}$, and thus also of the values of $\phi$ and $\widetilde{\lambda}$. We extend $\lambda$ by $\widetilde{\lambda}$ and remark that it is a solution in $\mathcal{F}(\phi)$.

On the other hand, if $\lambda$ is a solution in $\mathcal{F}(\phi)$ its restriction to the directed edge set of $T$ is a solution in $\mathcal{F}(\phi_{new})$ so that there is only one solution $\lambda$ in $\mathcal{F}(\phi)$ satisfying $\lambda|_{\vec{S}} = \widetilde{\lambda}$. Notice that this already shows that $\varphi_S$ is a bijection when $(V, E \setminus S)$ is a spanning tree of $G$.

It is clear that if $\varphi_S$ is a bijection then $\mathcal{F}(\phi)$ is not empty because $\Omega^1(S)$ would be empty which is impossible. So, for the rest of the proof we may assume that $\mathcal{F}(\phi)$ is not empty. By taking any $\lambda_0 \in \mathcal{F}(\phi)$, using that $\mathcal{F}(\phi) = \mathcal{F}(0)$ and that the map $\mathcal{F}(\phi) \to \Omega^1(S)$ is a composition

$$\mathcal{F}(\phi) \xrightarrow{\text{translation by } -\lambda_0} \mathcal{F}(0) \xrightarrow{\varphi_S} \Omega^1(S) \xrightarrow{\text{translation by } \lambda_0|_{\vec{S}}} \Omega^1(S),$$

we find that it is enough to prove the statements for $\phi = 0$. Now, let us show that if $\varphi_S$ is a bijection then $(V, E \setminus S)$ is a spanning tree. If $(V, E \setminus S)$ had a cycle $(v_1, \dots, v_k, v_{k+1})$ with $v_{k+1} = v_1$ we could define $\lambda(v_i, v_{i+1}) = -\lambda(v_{i+1}, v_i) = 1$ for $i \in \{1, \dots, k\}$ and zero elsewhere. Such $\lambda$ would belong to $\mathcal{F}(0)$ but its image in $\Omega^1(S)$ would be zero so that $\varphi_S$ would not be injective. To show that $(V, E \setminus S)$ is connected we may notice that if it were a disconnected forest we could find $S' \subset S$ such that $(V, E \setminus S')$ is a spanning tree. But this would imply that the values at $S'$ determine the solution and, thus, determine the values at $S \setminus S'$. Then $\varphi_S$ could not be surjective.

$\square$

**Proposition 2.2.** *Suppose that $G$ is connected. If $(V, E \setminus S)$ is a spanning tree, then for all $\phi : V \to \mathbb{R}$ satisfying* (2.2),

$$(\varphi_S)_* \mathcal{L}eb_{\mathcal{F}(\phi)} = \left(\sqrt{\# \text{ Spanning trees of } G}\right) \mathcal{L}eb_{\Omega^1(S)}.$$

*Proof.* Now, assuming that $(V, E \setminus S)$ is a spanning tree, let us look for the constant $C > 0$ such that

$$(\varphi_S)_* \mathcal{L}eb_{\mathcal{F}(0)} = C \mathcal{L}eb_{\Omega^1(S)}$$

Denoting the dimension of $\mathcal{F}(0)$ by $k = |E| - |V| + 1$, we want to study the map $(\varphi_S)_*$ induced on $k$-forms. This is equivalent to looking at the pushforward map on $k$-vectors

$$(\varphi_S)_* : \Lambda^k \mathcal{F}(0) \to \Lambda^k \Omega^1(S),$$

taking the dual and inverting the resulting map. The constant $C > 0$ would be found by taking normalized vectors $w_1 \in \Lambda^k \mathcal{F}(0)$, $w_2 \in \Lambda^k \Omega^1(S)$ and solving

$$(\varphi_S)_* w_1 = \pm C^{-1} w_2$$

or, what is the same, taking $C^{-1} = |\langle (\varphi_S)_* w_1, w_2 \rangle|$. The vector $w_2$ can be explicitly obtained as $\wedge_{e \in S} \delta_e$, where we have chosen an orientation for each edge $e \in E$ and an order to perform the product. We recall that $\delta_e$ takes the value 1 at $e$ with our chosen orientation, $-1$ if we reverse the orientation and 0 at all other edges. To explicitly construct $w_1$ is less obvious since we would be dealing with $\mathrm{Ker}(\mathrm{div})$. We could instead use its orthogonal complement $\mathrm{Im}(d)$. By fixing a leaf $v_0$ of $(V, E \setminus S)$ we may consider $\{d\delta_v\}_{v \in V \setminus \{v_0\}}$, where we recall that $\delta_v$ is the function which is 1 at $v$ and 0 elsewhere. Since $\sum_{v \in V \setminus \{v_0\}} d\delta_v + d\delta_{v_0} = d1 = 0$, this family generates $\mathrm{Im}(d)$. Now, we obtain an element of $\Lambda^k \mathcal{F}(0; 0)$ by taking $*(\wedge_{v \in V \setminus \{v_0\}} d\delta_v)$, where $*$ denotes the Hodge star operator. We have not yet normalized this element nor shown it is non-zero but let us calculate, using that $\wedge_{e \in S} \delta_e = \pm * (\wedge_{e \in E \setminus S} \delta_e)$,

$$\left\langle * (\wedge_{v \in V \setminus \{v_0\}} d\delta_v), \wedge_{e \in S} \delta_e \right\rangle = \pm \left\langle \wedge_{v \in V \setminus \{v_0\}} d\delta_v, \wedge_{e \in E \setminus S} \delta_e \right\rangle = \pm \det \langle d\delta_v, \delta_e \rangle_{v \in V \setminus \{v_0\}, e \in E \setminus S}.$$

This determinant is a sum over all bijections $\sigma : E \setminus S \to V \setminus \{v_0\}$ of the product $(-1)^\sigma \prod_{e \in E \setminus S} \langle \widehat{d\sigma(e)}, \delta_e \rangle$, where the sign $(-1)^\sigma$ is only defined up to an overall sign. Notice that $\langle d\delta_x, \delta_e \rangle = \pm 1$ if and only if $x$ is an endpoint of $e$ and, if not, $\langle d\delta_x, \delta_e \rangle = 0$. So, for a bijection to contribute, the unique edge adjacent to $v_0$ should correspond to the unique other endpoint $v_1$ of this edge. Then, for every other edge adjacent to $v_1$ we do not have a choice but to take the endpoint different from $v_1$. If we continue in this way we get that there is only one bijection that contributes and therefore

$$\left\langle * (\wedge_{v \in V \setminus \{v_0\}} d\delta_v), \wedge_{e \in S} \delta_e \right\rangle = \pm 1.$$

This proves in particular that $*(\wedge_{v \in V \setminus \{v_0\}} d\delta_v)$ is not zero so the explicit formula for $w_1$ as the normalized $*(\wedge_{v \in V \setminus \{v_0\}} d\delta_v)$ works. If $(V, E \setminus S)$ is not a spanning tree of $G$ we already know that $\varphi_S$ is not a bijection so that the previous inner product is zero. Using that $\{\wedge_{e \in S} \delta_e\}_{|S|=k}$ forms an orthonormal basis of $\Omega^1(G)$ we can calculate the norm as the sum of squares of inner products to obtain

$$\| * (\wedge_{v \in V \setminus \{v_0\}} d\delta_v) \|^2 = \# \text{ Spanning trees of } G.$$

This yields that

$$\langle (\varphi_S)_* w_1, w_2\rangle = \pm \frac{1}{\sqrt{\#\text{ Spanning trees of } G}}$$

which implies the final statement of the proposition. ☐

In the following, be set

$$d\,\mathrm{Vol} = \frac{1}{\sqrt{\#\text{ Spanning trees of } G}} \mathcal{L}eb_{\mathcal{F}(\phi)}\ , \tag{2.3}$$

so that $(\phi_S)_* d\,\mathrm{Vol} = \mathcal{L}eb_{\Omega^1(S)}$.

2.2. **Boundary and sieving of graphs.** For $G$ a finite graph, let us denote by $\partial G$ (resp. $\mathrm{int}(G)$) the set vertices of $G$ of degree 1 (resp. degree larger than 1) and by $\partial E$ the set of edges adjacent to $\partial G$. If $\phi \in \Omega^0(G)$, we denote by $\partial\phi$ the restriction of $\phi$ to $\partial G$. Likewise, we denote by $\partial\lambda$ the restriction of $\lambda \in \Omega^1(G)$ to $\{(a,b), \{a,b\} \in \partial E\}$. By the previous section, for $R \subset V$, $\mathcal{F}(\phi) \neq \emptyset$ if and only if

$$\sum_{v\in R} \phi(v) = -\sum_{v\in V\setminus R} \phi(v)\ .$$

**Definition 2.3** (Sieving of graphs)**.** Let $r \geq 1$. Let $G = (V, E)$ be a finite graph (non necessarily connected) and $W, W' \subset \partial G$ with a bijection $g : W \to W'$. The sieving of $G$ along $(R_1, R_2)$ is the graph $G_{W*W'} = (\tilde V, \tilde E)$ obtained as follows:

- $\tilde V$ is the quotient of $V$ by the equivalence relation generated by $(v, g(v))$ for $v \in W$,
- $\tilde E = \pi(E)$, where $\pi : V \times V \to \tilde V \times \tilde V$ is the quotient map.

This construction informally amounts to merge $v$ and $g(v)$ for $v \in W$ and considering the resulting edge structure inferred by $E$. Since $W, W' \subset \partial G$, $|\tilde E| = |E|$ as long as each connected component of $G$ has a size at least 3 (which will always by the case in the present paper). There is moreover a canonical bijection between $E$ and $\tilde E$. Let us closely look at the behavior of the equation (2.1) with respect to the sieving of graphs.
In the following, if $G = (V, E)$ is a finite graph, $V = S_1 \sqcup \ldots \sqcup S_r$ is a partition of $V$ and $\phi \in \Omega^0(G)$, we write by abuse of notation $(\phi_{|S_1}, \ldots, \phi_{|S_r})$ instead of $\phi$ to detail the decomposition of $\phi$ along this partition. Moreover, we write $\mathcal{F}_G(\phi)$ instead of $\mathcal{F}(\phi)$ to emphasize that the equation (2.1) is considered in the graph $G$.

**Proposition 2.4** (Product formula)**.** *Let $G_1 = (V_1, E_1), G_2 = (V_2, E_2)$ be two connected finite graphs, such that $V_1 \cap V_2 = \emptyset$. Let $W \subset \partial G_1, W' \subset \partial G_2$ and let $g : W \to W'$ be a bijection. Set $\tilde G = (G_1 \cup G_2)_{W*W'}$ and let $\phi \in \Omega^0(\tilde G)$ be such that $\sum_{v\in\tilde V} \phi(v) = 0$. Let $\lambda \in \Omega^1(\tilde G)$ and for $w \in W$, let us denote by $x_w = \lambda(\tilde e)$ where $\tilde e$ is the unique edge of $\tilde G$ starting from $\tilde w$ and ending on $V_1$. Then,*

1. *$\lambda \in \mathcal{F}_{\tilde G}(\phi)$ if and only if*
   (a) *$\sum_{v\in V_1\setminus W} \phi(v) + \sum_{w\in W} x_w = 0$,*
   (b) *$\lambda_{|E_1} \in \mathcal{F}_{G_1}(\phi_1)$ where $\phi_1 = \big(\phi_{|V_1\setminus W}, \phi_W\big)$ and where $\phi_W(w) = x_w$ for $w \in W$,*
   (c) *$\lambda_{|E_2} \in \mathcal{F}_{G_2}(\phi_2)$ where $\phi_2 = \big(\phi_{|V_2\setminus W'}, \phi'_W\big)$ and where $\phi'_W(w') = \phi(w') - \phi_W(g^{-1}(w'))$ for $w' \in W'$.*
2. *For $w_0 \in W$, $K_1 \subset \Omega^1(G_1)$ and $K_2 \subset \Omega^1(G_2)$,*

$$\begin{aligned}\mathrm{Vol}_{\tilde G}((K_1\times K_2)\cap \mathcal{F}_{\tilde G}(\phi)) = \int_{\mathbb{R}^{|W|-1}} &\mathrm{Vol}_{G_1}\left[K_1 \cap \mathcal{F}_{G_1}(\phi_{|V_1}, (x_w)_{w\in W\setminus W_0}, y(x)))\right]\\ &\cdot \mathrm{Vol}_{G_2}\left[K_2 \cap \mathcal{F}_{G_2}(\phi_{|V_2}, (\phi(\tilde w) - x_w)_{w\in W\setminus\{w_0\}}), \phi(\tilde w) - y(w))\right] dx,\end{aligned}$$

*where $y(w)$ is the unique solution to $\sum_{v\in V_1\setminus W} \phi(v) + \sum_{w\in W\setminus\{w_0\}} x_w + y(w) = 0$.*

*Proof.* (1) Recall that $\lambda \in \mathcal{F}_{\tilde G}(\phi)$ if and only if

$$\forall x \in \tilde V : \sum_{e, \bar e = x} \lambda(e) = \phi(x)\ .$$

Let $\lambda \in \mathcal{F}_{\tilde G}(\phi)$ and let $x \in V_1$. If $x \in V_1 \setminus W$, then

$$\sum_{v\in V_1 : \{v,x\}\in E_1} \lambda_{|E_1}(v, x) = \sum_{v\in \tilde V : \{v,x\}\in \tilde E} \mathbb{1}_{\{v,x\}\in E_1} \lambda(v, x) = \sum_{v\in\tilde V : \{v,x\}\in\tilde E} \lambda(v, x) = \phi(x)\ .$$

If $w \in W$, then $\sum_{v\in V_1:\{v,w\}\in E_1} \lambda_{|E_1}(v,w) = x_w$ as required, which gives that $\lambda_{|E_1} \in \mathcal{F}_{G_1}(\phi_{V_1\setminus W}, \phi_W)$. Using the divergence condition for $\phi$, one must have

$$\sum_{v\in V_1} \phi(v) + \sum_{w\in W} x_w = 0 \ .$$

Similarly, $\lambda_{|E_2} \in \mathcal{F}_{G_2}(\phi_{V_2\setminus W'}, (y_w)_{w\in W'})$ with $y_w = \lambda(e)$, where $e$ is the unique edge of $\tilde{E}$ starting from $w$ and ending on $V_2$. By using the divergence condition on $\tilde{w} = \{w, g(w)\}$ for $w \in W$, one has $\phi(\tilde{w}) = x_w + y_{g(w)}$, and thus $y_{g(w)} = \phi(\tilde{w}) - x_w$ .

Reciprocally, one checks that if $\lambda \in \Omega^1(\tilde{G})$ is such that $\lambda_{|E_1} \in \mathcal{F}_{G_1}(\phi_{V_1\setminus W}, (x_w)_{w\in W})$ and $\lambda_{|E_2} \in \mathcal{F}_{G_2}(\phi_{V_2\setminus W'}, (\phi(\tilde{w}) - x_{g^{-1}(w)})_{w\in W})$, then $\lambda \in \mathcal{F}(\phi)$.

(2) Set $s+1 = |W| = |W'|$ and write $W = \{w_0, \dots, w_s\}$. Let $e_i$ be the edge of $E_1$ adjacent to $w_i$. Let $T_i$ be a spanning tree of $G_i$ for $i \in \{1,2\}$. Since each edge $\tilde{w}, w \in W$ is bivalent in $\tilde{G}$, removing the edges $e_j$, $1 \le j \le s$ from $T_1 \cup T_2$ yields a spanning tree $T$ of $\tilde{G}$. By Proposition 2.1 applied to $S = T^c$, the restriction map $\varphi_S$ is a bijection from $\mathcal{F}_{\tilde{G}}(\phi)$ to $\Omega^1(S)$. Moreover, by Proposition 2.2, $(\varphi_S)_*\mathrm{d}\,\mathrm{Vol} = \mathcal{L}eb_{\Omega^1(S)}$. Let us write $S_i = E_i \setminus T_i$ and $R = \{e_1, \dots, e_s\}$. Then,

$$S = T^c = [E_1 \setminus T_1] \cup [E_2 \setminus T_2] \cup R = S_1 \cup S_2 \cup R \ .$$

Let $K_1 \subset \mathbb{R}^{E_1}$ and $K_2 \subset \mathbb{R}^{E_2}$. Then, for $(t_1, t_2, x) \in \mathbb{R}^{S_1} \times \mathbb{R}^{S_2} \times \mathbb{R}^{s-1}$,

$$\lambda = \lambda(t_1, t_2, x) = \varphi_S^{-1}(t_1, t_2, x) \in \mathcal{F}_{\tilde{G}}(\phi) \ .$$

By the previous statement, this is equivalent to the fact that $\lambda_{|E_1} \in \mathcal{F}_{G_1}(\phi_{|V_1\setminus W}, (x_w)_{w\in R}, y)$ where $y$ is the unique solution to $\sum_{v\in V_1\setminus R} \phi(v) + \sum_{w\in R} x_w + y = 0$, and $\lambda_{|E_2} \in \mathcal{F}_{G_2}(\phi_{|V_2\setminus W'}, (\phi(\tilde{w}) - x_w))_{w\in R}, \phi(w_0) - y)$. Let $\varphi^x_{S_1}$ be the projection from $\mathcal{F}_{G_1}(\phi_{|V_1\setminus W}, (x_w)_{w\in R}, y))$ to $\Omega^1(S_1)$ and $\varphi^x_{S_2}$ be the projection from $\mathcal{F}_{G_2}(\phi_{|V_2\setminus W'}, (\phi(\tilde{w}) - x_w))_{w\in R}, \phi(w_0) - y)$ to $\Omega^1(S_2)$. Since $E_i \setminus S_i$ is a spanning tree of $G_i$, each map $\varphi^x_{S_i}$ is bijective. Moreover, since then $\varphi^x_{S_1} \circ \lambda(t_1, t_2, x)$ is well-defined and equal to $t_1$, we have

$$\lambda_{|E_1} = (\varphi^x_{S_1})^{-1}(t_1) \text{ and, likewise, } \lambda_{|E_2} = (\varphi^x_{S_2})^{-1}(t_2) \ .$$

Therefore, $\lambda(t_1, t_2, x) \in K_1 \times K_2 \cap \mathcal{F}(\phi)$ if and only if $(\varphi^x_{S_1})^{-1}(t_1) \in K_1$ and $(\varphi^x_{S_1})^{-1}(t_2) \in K_2$. Hence,

$$\begin{aligned}
Vol(K_1 \times K_2 \cap \mathcal{F}(\phi)) &= \int_{\mathbb{R}^{S_1}\times\mathbb{R}^{S_2}\times\mathbb{R}^s} \mathbf{1}_{\lambda(t_1,t_2,x)\in K_1\times K_2} \mathrm{d}t_1\mathrm{d}t_2\mathrm{d}x \\
&= \int_{\mathbb{R}^s} \left( \int_{\mathbb{R}^{S_1}\times\mathbb{R}^{S_2}} \mathbf{1}_{(\varphi^x_{S_1})^{-1}(t_1)\in K_1} \mathbf{1}_{(\varphi^x_{S_2})^{-1}(t_2)\in K_2} \mathrm{d}t_1\mathrm{d}t_2 \right) \mathrm{d}x \\
&= \int_{\mathbb{R}^s} \left( \int_{\mathbb{R}^{S_1}} \mathbf{1}_{(\varphi^x_{S_1})^{-1}(t_1)\in K_1} \mathrm{d}t_1 \right) \cdot \left( \int_{\mathbb{R}^{S_2}} \mathbf{1}_{(\varphi^x_{S_2})^{-1}(t_2)\in K_2} \mathrm{d}t_2 \right) \mathrm{d}x \\
&= \int_{\mathbb{R}^s} \mathrm{Vol}\left[K_1 \cap \mathcal{F}_{G_1}(\phi_{|V_1\setminus W}, (x_w)_{w\in R}, y)\right] \\
&\qquad \cdot \mathrm{Vol}\left[K_2 \cap \mathcal{F}(\phi_{|V_2\setminus W'}, (\phi(\tilde{w}) - x_w))_{w\in R}, \phi(w_0) - y)\right] \mathrm{d}x \ .
\end{aligned}$$

□

**Proposition 2.5** (Contraction formula)**.** *Let $G = (V, E)$ be a connected finite graph, $W, W' \subset \partial G$ with $W \cap W' = \emptyset$ and $g : W \to W'$ a bijection and set $G_{W*W'} = (\tilde{V}, \tilde{E})$. Then, for $\phi \in \Omega^0(G_{W*W'})$ such that $\sum_{v\in\tilde{V}} \phi(v) = 0$ and $K \subset \Omega^1(\tilde{G})$,*

$$Vol(K \cap \mathcal{F}_{\tilde{G}}(\phi)) = \int_{\mathbb{R}^{|W|-1}} \mathrm{Vol}\left[K \cap \mathcal{F}_G(\phi_{|V\setminus(W\cup W')}, (x_w)_{w\in W}, (\phi(\tilde{w}) - x_{g^{-1}(w')})_{w\in W'})\right] \mathrm{d}x \ .$$

*Proof.* The proof is similar to the one of Proposition 2.4. □

## 3. The three-holed sphere

The goal of this section is to establish Theorem 1.4 in the case where $(g, p) = (0, 3)$, that is, for the three-holed sphere. Section 3.1 gives an injection of honeycombs into flows which is Proposition 3.3. The latter thus gives a parametrization of triangular honeycombs. In Section 3.2, we recall a combinatorial model from [FT24] called *dual hive*. Section 3.3 and Section 3.4 show that there is an linear bijection with integer coefficients between dual hives and triangular honeycombs. We finally prove Theorem 1.4 for the three-holed sphere in Section 3.5.

3.1. **Parametrization of triangular honeycombs.** The goal of this section is to view triangular honeycombs as flows on a graph with prescribed divergence. Proposition 2.2 then yields a volume form on this set of flows. Recall that for $d \geqslant 0$, $\mathcal{G}_d$ denotes the set of isomorphism classes of colored graphs with ordered boundary appearing in $\{G[h],\, h \in \texttt{HONEY}_{n,d}\}$. For a honeycomb $h \in \texttt{HONEY}^G$ and an edge $e \in E$, let us denote by $\ell^h(e)$ the type of $e$, defined in Definition 1.2, (2).

**Lemma 3.1** (Boundary determine type and colors)**.** *Let $G \in \mathcal{G}_d$ and let $h \in \texttt{HONEY}^G$. Then, the type $\ell^h : E \to \{0,1,2\}$ and color $c^h : E \to \{0,1,3\}$ are independent of $h$.*

*Proof.* Remark that such label and color maps are defined for any honeycomb $h \subset T$ such that any geodesic $e \subset h$ is contained in some ray $\{x + \mathbb{R}e^{2(\ell+1)i\pi/3}\}$ for some $\ell \in \{0,1,2\}$. Let us call such a honeycomb *admissible* and let us prove by induction on the number $M$ of inner vertices the following : *for any admissible honeycomb $h$, the induced label and color map on $G[h]$ only depends on the type and color map of the boundary edges and on the order on the boundary vertices.*

If $M = 1$, then all edges of $G[h]$ are boundary edges, and the assertion holds. Let $M > 1$. Suppose that there is $v \in \mathrm{int}(G)$ which is adjacent to two boundary edges $e_1 = \{v, v_1\}, e_2 = \{v, v_2\}$ and one non boundary edge $e$. Then, the type $e$ is uniquely determined by the relation $\{\ell(e), \ell(e_1), \ell(e_2)\} = \{0,1,2\}$ and the value of $\ell(e_1)$ and $\ell(e_2)$. Next, since the cyclic order of the boundary vertices is given, by Definition 1.2 the color $c(e)$ is uniquely determined by $c(e_1)$ and $c(e_2)$. Hence, on $\tilde{G} = (V\backslash\{v_1, v_2\}, E\backslash\{e_1, e_2\}$ the type and color of the boundary edges is known, and by induction, the type and colors of all edges of $\tilde{G}$ only depends on the graph structure and their value on the boundary.

Suppose that all vertices of $G$ are adjacent to at most one boundary edge. Let $(v_1, \dots, v_m)$, $m \geq 1$ be the boundary vertices in the cyclic order. By hypothesis, there exist $(w_1, \dots, w_m)$ such that $\partial G = \{e^i := \{v_i, w_i\}, 1 \leq i \leq m\}$ and $w_i \neq w_j$ when $i \neq j$. We claim that there exists $w_i, w_{i'}$ such that $\{w_i, w_{i'}\} \in E$ and $\{\ell(\{w_i, w_{i'}\}), \ell(e_i), \ell(e_{i'})\} = \{0,1,2\}$. Let $\tilde{G} = (V \setminus V_1, E \setminus \partial G)$ and $\tilde{h} = \bigcup_{e \in E\setminus\partial G} i(e)$. Then, $\tilde{h} \subset \mathring{T}$, and thus there exists a unique connected component $K_0$ in $T \setminus \tilde{h}$ which is adjacent to $\partial T$. Let $L$ be a boundary component of $K_0$. Then, $L$ is a close polygonal line with vertices $\{z_1, \dots, z_p\}$ enumerated in the cyclic order. At each $z_i$, $L$ has an angle $\alpha_i$ so that

- $\alpha_i = \pi/3$ is $z_i$ is the intersection of the interior of two geodesics of $\mathcal{E}$,
- $\alpha_i = 2\pi/3$ if $z_i = w_{j_i}$ with $1 \leq j_i \leq m$,
- $\alpha_i = 4\pi/3$ if $z_i \in \{w_1, \dots, w_n\}$.

Since $L$ is a close polygonal curve, there is at least two consecutive vertices $z_i, z_{i+1}$ such that $\alpha_i = \alpha_{i+1} = 4\pi/3$. Hence, $[w_{j_i}, w_{j_{i+1}}]$ is a geodesic, and thus $\{w_{j_i}, w_{j_{i+1}}\} \in E$. Moreover, the angle from $\{v_{j_i}, w_{j_i}\}$ (resp. $\{v_{j_{i+1}}, w_{j_{i+1}}\}$) to $\{w_{j_i}, w_{j_{i+1}}\}$ is $-2\pi/3$ (resp. $2\pi/3$), so that

$$\big\{\ell(\{w_{j_i}, w_{j_{i+1}}\}),\ \ell(\{v_{j_i}, w_{j_i}\}),\ \ell(\{v_{j_{i+1}}, w_{j_{i+1}}\})\big\} = \{0,1,2\}.$$

Let $w_i, w_{i'}$ be such that $e := \{w_i, w_{i'}\} \in E$, $e^1 := \{w_i, w_{i'}\}$ and $e^2 := \{v_i, w_i\}$ satisfy the angle condition $\{\ell(e^1), \ell(e^2), \ell(e^3)\} = \{0,1,2\}$. Then, $\ell(e^3)$ is determined by $\ell(e^1)$ and $\ell(e^2)$. Let $f^1, f^2$ be the third edge around $w_i$ (resp. $w_{i'}$). Then, $\ell(f^i)$ is determined by $\ell(e^i)$ and $\ell(e^i)$ for $i \in \{1,2\}$. By the color condition from Definition 1.1, $c(e) = 3$ if $c(e^1) \neq c(e^2)$ and otherwise $c(e) = c(e^1) = c(e^2)$. Then, $c(f^1)$ and $c(f^2)$ are uniquely determined by $\{c(e), c(e^2), c(e^2)\}$. Let $\hat{G} = \big(V \setminus \{v_i, v_{i'}\}, E \setminus \{e, e^1, e^2\}\big)$. Then, $\hat{h} = (h \setminus \{e \cup e^1 \cup e^2\}) \cup \{w_i, w_{i'}\}$ is a honeycomb such $G[\hat{h}] = \hat{G}$ having $M - 1$ inner vertices, and such that the type and color of the boundary edges are known. By induction, the type and color of all edges of $\hat{G}$, and thus of $G$ are known. ☐

Let us provide a description of honeycombs with structure graph $G$ in terms of flows. Suppose that $G \in \mathcal{G}_d$. By the condition (1) Definition 1.2, $G$ has only vertices of degree 1 or 3 and thus $v$ has three adjacent edges $e_\ell$, $\ell \in \{0,1,2\}$. Denote by $\mathrm{int}(G)$ the set of vertices of degrees 3 and let $v \in \mathrm{int}(G)$. By (2) of Definition 1.1, the angle between two successive edges at $v$ is $2\pi/3$ and by (2) of Definition 1.2, each edge is oriented along $e^{2(\ell+1)i\pi/3}$ for some $\ell \in \{0,1,2\}$. Hence, there exists a sign $s(v) \in \{-1, +1\}$ such that, up to a relabeling, $e_\ell \subset \big\{v + s(v)e^{2(\ell+1)i\pi/3}\mathbb{R}_{\geq 0}\big\}$. For a univalent vertex $v \in \partial G$ connected to a unique trivalent vertex $v' \in \mathrm{int}(G)$, we set $s(v) = -s(v')$.

Let $v, v' \in V$ be such that $e = \{v, v'\} \in E$. By the previous reasoning, there exist $\ell, \ell'$ such that $e \subset \left\{v + s(v)\mathrm{e}^{2(\ell+1)i\pi/3}\mathbb{R}_{\geq 0}\right\}$ and $e \subset \left\{v' + s(v')\mathrm{e}^{2(\ell'+1)i\pi/3}\mathbb{R}_{\geq 0}\right\}$. Necessarily, $\ell = \ell'$ and $s(v) = -s(v')$, see Figure 3.

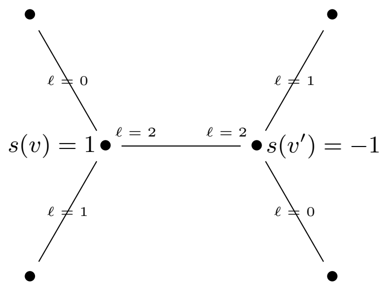



FIGURE 3. Two adjacent vertices in $\mathrm{int}(G)$ for which $\ell = \ell' = 2$.

In particular, the map $s : v \mapsto s(v)$ only depends on the value of $s$ on the boundary $\partial G$. Since $s(v)$ for $v \in \partial G$ is given by the type and thus the color of the unique adjacent edge, see Definition 1.2.(3), we deduce by Lemma 3.1 that $s$ only depends on $G$. In the following definition, recall the definition of the height of an edge in Definition 1.2.

**Definition 3.2** (Flow of a honeycomb)**.** The *flow* of a honeycomb $h \in \mathtt{HONEY}^G_{n,d}$ is the map $\mathcal{L}[h] \in \Omega^1(G)$ which to an oriented edge $(v, v') \in \vec{E}$ associates $\mathcal{L}[h](v, v') = s(v)L(\{v, v'\})$.

If $v \in \mathrm{int}(G)$, $v$ corresponds to a point if $T$ and thus its coordinates $(v_1, v_2, v_3)$ satisfies $v_1+v_2+v_3 = 1$. If $e_1, e_2, e_3$ are the three edges adjacent to $v$, we have by Definition 1.2(2), $L(e_1) + L(e_2) + L(e_3) = 1$. Hence, considering now oriented edges yields

$$\forall v \in \mathrm{int}(G) : \sum_{v\sim v'} \mathcal{L}[h](v, v') = s(v) \ . \tag{3.1}$$

Let $(\alpha, \beta, \gamma) \in \mathcal{H}^3_{reg}$ and let $G \in \mathcal{G}_d$. Recall that for a structure graph $G$, its boundary $\partial G = \{\{v, v'\} \mid v \in \partial V\}$ consists of edges having a vertex of degree one. Since by Definition 1.2, $\partial G \subset \partial T$, we may write $\partial G = (e_1, \ldots, e_{3n})$, where for $0 \leqslant l \leqslant 2$ and $1 \leqslant i \leqslant n$, the edge $e_{\ell n+i}$ is adjacent to $v^{\ell n+i}$ on $\partial_\ell T$. Let us denote by $g^{\alpha,\beta,\gamma} \in \mathbb{R}^{\partial G}$ the boundary condition given by

$$g^{\alpha,\beta,\gamma}_{\ell n+i} := g^{\alpha,\beta,\gamma}(e_{\ell n+i}) = \begin{cases} \beta_i & \text{if } c(e_{\ell n+i}) = 0 \\ \beta_i - 1 & \text{if } c(e_{\ell n+i}) = 1 \end{cases}$$

for $\ell = 0$ and $1 \leq i \leq n$ and replacing $\beta_i$ by $\alpha_i$ (resp. $\gamma_i$) when $\ell = 1$ (resp. $\ell = 2$). Remark that for $v \in \partial G$, $s(v) = 1$ if and only if $c(e) = 1$, where $e$ is the unique edge adjacent to $v$, see for example Figure 1. Hence, the boundary condition translates into the condition

$$\mathcal{L}[h](v_{\ell n+i}, v') = g^{\alpha,\beta,\gamma}_{\ell n+i} \ , \tag{3.2}$$

where $v'$ is the unique vertex of $V$ adjacent to $v_{\ell n+i}$. Hence, setting $\phi_{\alpha,\beta,\gamma}(v) = s(v)$ for $v \in \mathrm{int}(G)$ and $\phi_{\alpha,\beta,\gamma}(v_{\ell n+i}) = g^{\alpha,\beta,\gamma}_{\ell n+i}$ for $v^{\ell n+i} \in \partial G$, (3.1) and (3.2) yields that for $h \in \mathtt{HONEY}^G_{n,d}(\alpha, \beta, \gamma)$,

$$\mathcal{L}[h] \in \mathcal{F}_G(\phi_{\alpha,\beta,\gamma}) \ .$$

**Proposition 3.3** (Honeycomb injection)**.** *Let $d \geq 0$ and $G \in \mathcal{G}_d$. The map $\mathcal{L} : h \to \mathcal{L}[h]$ is an injective map from $\mathtt{HONEY}^G$ to $\Omega^1(G)$ and*

$$\mathcal{L}\left(\mathtt{HONEY}^G_{n,d}(\alpha, \beta, \gamma)\right) \subset \mathcal{F}_G(\phi_{\alpha,\beta,\gamma}) \ .$$

*Proof.* The fact that $\mathcal{L}\left(\mathtt{HONEY}^G_{n,d}(\alpha, \beta, \gamma)\right) \subset \mathcal{F}_G(\phi_{\alpha,\beta,\gamma})$ is given by the previous discussion. Let us prove the injectivity of the map. Let $h_1, h_2 \in \mathtt{HONEY}^G$ such that $\mathcal{L}[h_1] = \mathcal{L}[h_2]$. Since $G = (V, E)$ is isomorphic to the canonical graph structure of both $h_1$ and $h_2$, there are two isomorphisms $i_i : G \to (V^i, \mathcal{E}^i)$, $i \in \{1, 2\}$, where $\mathcal{E}^i$ is the set of geodesics associated to $h_i$ by Definition 1.1 and $V^i$ are the endpoints of these geodesics. Since both $h$ and $h'$ are honeycombs on the equilateral triangle, for which there exists a unique geodesic between two points, and $(V^1, \mathcal{E}^1)$ is isomorphic to $(V^2, \mathcal{E}^2)$ so that it suffices to show the equality $V^1 = V^2$.

Let $v \in V$. First, suppose that $v \in V_1$. Since the boundary $\partial G \simeq (v^1, \dots, v^{3n})$ of $G$ is ordered, there exists $1 \leq j \leq 3n$ such that $v = v^j$ and there exists a unique edge $e \in E$ adjacent to $v$. Since $(V^1, \mathcal{E}^1)$ and $(V^2, \mathcal{E}^2)$ are isomorphic to $G$ as colored graph with ordered boundary, $\iota_i(v)$ is the $j$-th vertex of the boundary of $h_i$ and $c(\iota_i(e)) = c(e)$ for $i \in \{1, 2\}$. By Definition 1.2, $\iota_1(v)$ and $\iota_2(v)$ belong to the same boundary $\partial_\ell$ of $T$ and their $(\ell+1)-$coordinates are

$$\begin{aligned}\iota_1(v)_{\ell+1} = \delta_{c(\iota_1(e))=1} + (-1)^{\delta_{c(\iota_1(e))=1}} L(\iota_1(e)) =& \delta_{c(e)=1} + (-1)^{\delta_{c(e)=1}} L(e)\\ =& \delta_{c(\iota_2(e))=1} + (-1)^{\delta_{c(\iota_2(e))=1}} L(\iota_2(e)) = \iota_2(v)_{\ell+1}.\end{aligned}$$

Hence, $\iota_1(v) = \iota_2(v)$.

Suppose that $v \in \mathrm{int}(G)$. By Definition 1.2, $v$ is a trivalent vertex and they are three edges $e^0, e^1, e^2$ adjacent to $v$. Moreover, by Lemma 3.1, the type and color of $e^i$ is given by the graph structure and the type and color of the boundary edges. Suppose without loss of generality that for $1 \leqslant i \leqslant 2$, $\ell(e^i) = i$. Then,

$$\iota_1(v) = (L(e^0), L(e^1), L(e^2)) = \iota_2(v) \ .$$

We deduce that $V^1 = \iota_1(V) = \iota_2(V) = V^2$ and thus $h_1 = h_2$. □

3.2. **Dual hive.** Let us recall the definition of a *dual hive* from [FT24]. For $n \geqslant d \geqslant 0$, let us consider the graph $H_{d,n} = (R_{d,n}, E_{d,n})$ with vertices $R_{d,n}$ and edges $E_{d,n}$. Each vertex $v = r + se^{\pi i/3} \in H_{d,n}$ comes with a coordinate $(v_0, v_1, v_2) = (n+d-r-s, r, s)$. Each edge $e$ of $H_{d,n}$ written $e = (v, v - e^{2\pi i \ell/3})$ with $\ell \in \{0, 1, 2\}$ is labeled $(\ell(e), h(e)) \in \{0, 1, 2\} \times \{0, \dots, n+d\}$ with

$$\ell(e) = \ell \text{ and } h(e) = v_\ell \ . \tag{3.3}$$

Table 1 below shows the different edge types for dual hives and honeycombs, dual to each other.

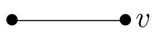
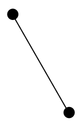
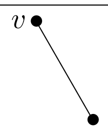
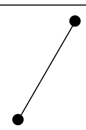
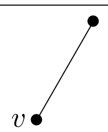
| Type $\ell$ | $e = \big(v, v - e^{2\pi i \ell/3}\big) \in E_{n,d}$ | $e \subset x + \mathrm{e}^{2i\pi(\ell+1)/3} \in \texttt{HONEY}_{n,d}$ |
|---|---|---|
| 0 |  |  |
| 1 |  |  |
| 2 |  |  |

TABLE 1. Edge types in dual hives and honeycombs.

**Definition 3.4** (Color map)**.** A *color map* is a map $C : E_{n,d} \to \{0, 1, 3, m\}$ such that the boundary colors around each triangular face in the clockwise order is either $(0, 0, 0), (1, 1, 1), (1, 0, 3)$ or $(0, 1, m)$ up to a cyclic rotation.

**Definition 3.5** (Non-degenerated dual hive)**.** For $(\alpha, \beta, \gamma) \in \mathcal{H}^3_{reg}$, such that $|\alpha| + |\beta| = |\gamma| + d$, the set of *dual hives*, denoted by $\mathtt{DH}(\alpha, \beta, \gamma)$, is the set of pairs $(C, L)$ such that :

(1) $C : E_{n,d} \to \{0, 1, 3, m\}$ is a color map,
(2) $L : E_{n,d} \to \mathbb{R}_{\geqslant 0}$ is the label map satisfying
  (a) $L(e_1) + L(e_2) + L(e_3) = 1$ for every triangular face of $H_{d,n}$,
  (b) if $e, e'$ are edges of same type on the boundary of a same lozenge $f$,
    (i) $L(e) = L(e')$ if the middle edge of $f$ is colored $m$,
    (ii) $L(e) > L(e')$ if $h(e) > h(e')$ and the middle edge of $f$ is not colored $m$.

(c) The values of $L$ on $\partial E_{n,d}$ are given by $(\alpha,\beta,\gamma)$ so that, sorted in decreasing height of edges, see Figure 4 below.

$$\ell^{(0,1)} = (1-\alpha_d,\dots,1-\alpha_1), \qquad \ell^{(2,2)} = (\alpha_{d+1},\dots,\alpha_n)$$
$$\ell^{(2,0)} = (1-\beta_d,\dots,1-\beta_1), \qquad \ell^{(1,1)} = (\beta_n,\dots,\beta_{d+1})$$
$$\ell^{(1,2)} = (\gamma_n,\dots,\gamma_{n-d+1}), \qquad \ell^{(2,2)} = (1-\gamma_{n-d},\dots,1-\gamma_1).$$

Moreover, for $\ell \in \{0,1,2\}$, the values of the color map $C$ on $\partial^{(\ell,\ell)}$ is set to 0 while equal to 1 on other boundary edges. We call the triple $(\alpha,\beta,\gamma)$ the *boundary* of $L$, or of the dual hive.

Figure 5 shows an example of a dual hive for $d = 1$ and $n = 3$ with boundary

$$(\alpha,\beta,\gamma) = \left(\left(\frac{14}{23},\frac{7}{23},\frac{2}{23}\right),\left(\frac{18}{23},\frac{10}{23},\frac{3}{23}\right),\left(\frac{19}{23},\frac{10}{23},\frac{2}{23}\right)\right). \tag{3.4}$$

Colors red, blue, black and greeen correspond to values $0, 1, 3$ and $m$ of the color map respectively.

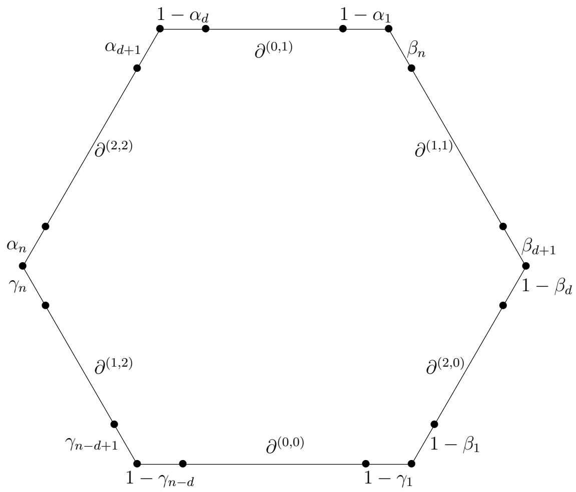



FIGURE 4. Boundary condition in $\mathtt{DH}(\alpha,\beta,\gamma)$.

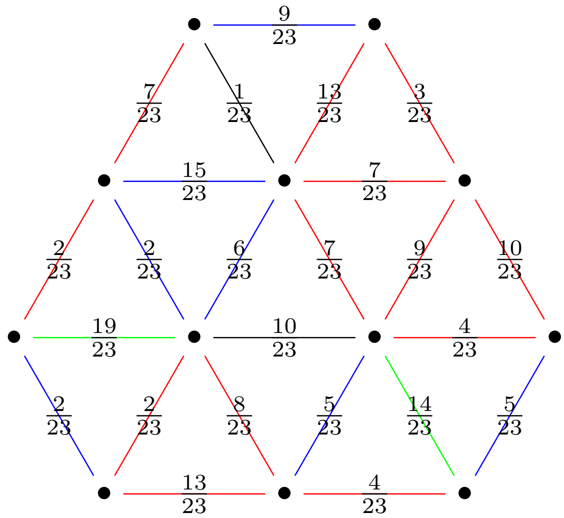



FIGURE 5. A dual hive with boundary condition (3.4).

For a given color map $C$, let us denote by $\mathtt{DH}^C(\alpha,\beta,\gamma)$ the set of dual hives with boundary $(\alpha,\beta,\gamma)$ and color map $C$. Since an element of $\mathtt{DH}^C(\alpha,\beta,\gamma)$ is uniquely defined by its map $L : E_{n,d} \to \mathbb{R}_{\geq 0}$, the set $\mathtt{DH}^C(\alpha,\beta,\gamma)$ can be seen as an affine polytope of $\mathbb{R}^{E_{n,d}}$ written as

$$\mathtt{DH}^C(\alpha,\beta,\gamma) = A^C \cap K_{n,d}\ ,$$

where $K_{n,d}$ is the cone of induced by (2)(b)(ii) and $A^C$ is the affine subspace induced by the equalities coming from (2)(a), (2)(b)(i) and (2)(c).

## 3.3. From dual hive to triangular honeycomb.

**Definition 3.6** ($\Gamma_{d,n}$ graph)**.** Let $n \geqslant d$ be two integers. The *dual graph* $\Gamma_{d,n} = (V^\Gamma, E^\Gamma)$ of $H_{d,n}$ is the following graph :

- there is one vertex $v_f$ for each triangular face $f$ of $H_{d,n}$ and one vertex $v_{\tilde e}$ for each outer edge $\tilde e$ of $H_{d,n}$ ,
- there is an edge $e$ between $v_f$ and $v_{f'}$ (resp. between $v_f$ and $v_{\tilde e}$) if the faces $f$ and $f'$ share an edge $\tilde e$ in $H_{d,n}$ (resp. if $\tilde e$ is a boundary edge of $f$ in $H_{d,n}$).

The map $e \mapsto \tilde e$ yields a bijection from $E^\Gamma$ to $E_{d,n}$ and $e$ is then said *dual* to $\tilde e$. Hence, any color map $C : E_{d,n} \to \{0,1,3,m\}$ yields a color map, also denoted by $C$, from $E^\Gamma$ to $\{0,1,3,m\}$ by setting $C(e) = C(\tilde e)$. Likewise, any edge of $E^\Gamma$ inherits the type $\ell(e) = \ell(\tilde e) \in \{0,1,2\}$, the height $h(e) = h(\tilde e)$ and the label $L(e) = L(\tilde e)$ of its dual edge.

To a dual hive $H = (C, L) \in \mathtt{DH}(\alpha,\beta,\gamma)$, we associate a collection $\mathcal{S}(H)$ of geodesics of $T$:

(1) For each $v \in V^\Gamma$ :
- if $v$ is adjacent to three edges $(e^0, e^1, e^2) \in E_\Gamma^3$ with $e^\ell$ of type $\ell$. We then set $x_v = \big(L(e^0), L(e^1), L(e^2)\big) \in T$,
- if $v \in V_1^\Gamma$ and $v$ is adjacent to an edge $e$ such that $\tilde{e} \in \ell^{(i,i)}$ (resp. in $\ell^{(i+1,i+2)}$), we set $x_v = (L(e)\delta_{i,0}, L(e)\delta_{i,1}, L(e)\delta_{i,2})$ (resp. $x_v = ((1-L(e))\delta_{i,0}, (1-L(e))\delta_{i,1}, (1-L(e))\delta_{i,2})$), where $\delta_{i,j} = 1$ if $i = j$ and 0 otherwise.

(2) Then, we set
$$\mathcal{S}(H) = \left\{[x_v, x_{v'}] \mid e = \{v, v'\} \in E^\Gamma\right\}.$$

For $e = \{v, v'\} \in E^\Gamma$, let us denote its associated geodesic by
$$\Phi(e) = [x_v, x_{v'}] \ . \tag{3.5}$$

As shown below, the collection of geodesics $\mathcal{S}(H)$ is almost the edge set of the canonical structure graph of a honeycomb.

**Lemma 3.7** (Edge geodesics)**.** *Suppose that $\tilde{e} \in E_{n,d}$ is an edge of type $\ell$ adjacent to a face $\tilde{v}$ of $H_{n,d}$ and set $\epsilon = +1$ (resp. $\epsilon = -1$) if this face is a lower (resp. upper.) triangular face. Then, either $c(e) \neq m$ and*
$$\Phi(e) \subset x_v + \epsilon e^{2\pi i(\ell+1)/3}\mathbb{R}_{>0} \ ,$$
*or $c(e) = m$ and*
$$\Phi(e) = \{x_v\} \ .$$

*Proof.* Suppose without loss of generality that $e$ is of type 0. Then, $(x_v)_0 = (x_{v'})_0 = L(e)$, so that $x_0 = L(e)$ for any $x \in \Phi(e) = [x_v, x_{v'}]$. We deduce that $\Phi(e) \subset v + \mathbb{R}e^{2\pi/3}$.
If $e$ is not colored $m$ and is of the form $e = \{v_f, v_{f'}\}$, consider the lozenge of $H_{n,d}$ consisting of faces $f$ and $f'$ whose middle edge is $\tilde{e}$. Denote by $\tilde{f}, \tilde{f}'$ the two edges of type 2 of this lozenge, with the convention that $h(\tilde{f}') > h(\tilde{f})$ and $\tilde{f}$ (resp. $\tilde{f}'$) is a boundary edge of the face dual of $v$ (resp. $v'$). Then, by Condition (2)(ii) of Definition 3.5, $L(\tilde{f}') > L(\tilde{f})$ and thus $(x_{v'})_2 > (x_v)_2$. If $e$ is of the form $e = \{v_f, v_{\tilde{e}}\}$, we have that $x(v_{\tilde{e}})_2 = 0$ if $\tilde{e} \in \partial^{(0,0)}$ or $x(v_{\tilde{e}})_2 = 1 - L(e)$ if $\tilde{e} \in \partial^{(0,1)}$. The two previous cases correspond to $\epsilon = -1$ and $\epsilon = 1$ respectively. In both cases $\epsilon \cdot (x_{v_{\tilde{e}}})_2 > (x_v)_2$. We deduce that $e \subset x_v + \mathbb{R}_{>0}e^{2\pi/3}$. A consequence of this fact is that the angle between two consecutive edges adjacent to an edge $v$ is $2\pi/3$. If $e$ is colored $m$ Condition (2)(b)(i) of Definition 3.5 implies that $x_v = x_{v'}$ and thus $\Phi(e) = \{x_v\}$. □

**Lemma 3.8** (Distinct edges give disjoint geodesics)**.** *If $e, e' \in E^\Gamma$ are distinct, then*
$$int(\Phi(e)) \cap int(\Phi(e')) = \emptyset \ .$$

This lemma is a rephrasing in the continuous case of the statements of Lemma [FT24, p. 5.9] and Lemma [FT24, p. 5.10]. We provide here a proof which is much simpler in its continuous version.

*Proof.* For $\tilde{e} \in E_{n,d}$, denote by $\tilde{e}_i = \tilde{v}_i$ (resp. $\tilde{e}^i$), where $\tilde{v}$ is the upper-triangular face (resp. lower-triangular) which is delimited by $e$. Let $e, e'$ be of same type $\ell$. First, by iterating Condition (2)(ii) of Definition 3.5, $L(e) > L(e')$ if $e_{\ell+1} = e'_{\ell+1}$ and $e_\ell > e'_\ell$. Next, using Condition (2)(ii) of Definition 3.5 and the fact that $C$ is a color map, $L(e) > L(e')$ if $e_{\ell+1} = e'_{\ell+1} - 1$ and $e_\ell = e'_\ell + 1$. Therefore, $L(e) > L(e')$ if $e$ and $e'$ are of same type $\ell$ and $e_{\ell+1} \leq e'_{\ell+1}$, $e_\ell > e'_\ell$. The same reasoning yields that $L(e) \geq L(e')$ if $e$ and $e'$ are of same type $\ell$ and $e_{\ell+1} \leq e'_{\ell+1}$, $e_\ell = e'_\ell$ with equality only if all edges of type $\ell - 1$ between $e$ and $e'$ are colored $m$.
Next suppose that $e = \{v_1, v_2\}$ and $e' = \{v'_1, v'_2\}$ with $e \neq e'$, with $v_1, v'_1$ being dual to an upper-triangular face and $v_2, v'_2$ being dual to a lower-triangular face. If $v_i = v'_j$ for some $i, j \in \{1, 2\}$, then $int(\Phi(e)) \cap int(\Phi(e')) = \emptyset$ by Lemma 3.7.
Otherwise, suppose without loss of generality that $(v_1)_0 < (v'_1)_0$. Since $\sum_{j=0}^2 (v_1)_j = \sum_{j=0}^2 (v'_1)_j = 1$, we can assume without loss of generality that $(v_1)_2 > (v'_2)_2$. Then, by the reasoning above, the edge $e_1^2$ (resp. $e_2^2$) of type 2 adjacent to $v_1$ (resp. $v_2$) satisfy $L(e_1^2) > L(e_2^2)$.
Set $x^i := x_{v_i}$ and $y^i = x_{v'_i}$ for $i = 1, 2$. Since $x_2^1 = L(e_1^2)$ and $y_2^1 = L(e_2^2)$, by the previous reasoning $x_2^1 > y_2^1$. Doing the same with the lower triangular faces $v_2, v'_2$, which must be adjacent respectively to $v_1$ and $v'_1$, yield that $x_2^2 \geq y_2^2$. Hence, the geodesics $\Phi(e) = [x^1, x^2]$ and $\Phi(e') = [y^1, y^2]$ can only meet at $y^2$, and $int(\Phi(e)) \cap int(\Phi(e')) = \emptyset$. □

**Definition 3.9** (Maximal chain, reduced graph)**.** Let $n \geqslant d$ be integers and let $C : E_{n,d} \to \{0,1,3,m\}$ be a color map.

- A *maximal chain* of $C$ is a path $\gamma = (e_1, \dots, e_{2r+1}) \in (E_{n,d})^{2r+1}$ for some $r \geqslant 0$ such that $c(e_{2i}) = m$, $c(e_{2i+1}) = c(e_1)$ for $1 \leqslant i \leqslant r$, and such that two consecutive edges share a vertex. We write $\gamma = \{x, y\}$ for $x, y \in V^\Gamma$ where $x$ (resp. $y$) is dual to a face $f_x$ (resp $f_y$) in $H_{n,d}$ such that $e_1 \in f_x$ (resp. $e_{2r+1} \in f_y$) and where $f_x$ and $f_y$ do not have any $m$ edges to emphasize that the path goes from $x$ to $y$. Moreover, the color of $\gamma$ is defined as $c(\gamma) = c(e_1)$.

- The *reduced graph of* $C$ is the graph $G^C = (V^C, E^C)$ defined by:
  - $V^C = \mathrm{int}(G)^C \cup V_1^C$, where $\mathrm{int}(G)^C = V^\Gamma \setminus \{u \in V_\Gamma \mid \exists (u,v) \in E_\Gamma, C(\{u,v\}) = m\}$ and $V_1^C = V_1^\Gamma$,
  - $E^C = \{\gamma = \{x,y\} \mid \{x,y\}$ is a maximal chain of $C\}$.

  The boundary vertices of $G^C$ are ordered as the ones of $\Gamma$.

Remark that the definition of the edge set is valid, since any vertex $u \in V^\Gamma$ adjacent to an edge colored $m$ cannot be the endpoint of a maximal chain of $C$. Moreover, by the color condition, a maximal chain with $c(e_1) = 3$ is necessarily of length 1. Note that any edge $e = \{x, y\} \in E_{n,d}$ not adjacent to an edge colored $m$ is a maximal chain (with $r = 0$) and is thus in $E^C$.

The map $C \mapsto C^C$ is injective as we can recover $C$ from $G^C$: it suffices to color the successive edges of a maximal chain $\gamma$ as $c(e_{2i+1}) = c(\gamma)$ and $c(e_{2i}) = m$. In the sequel, we denote by $\partial G^C$ the set of edges adjacent to a univalent vertex of $G^C$. Following Definition 3.5 and Definition 3.9, we introduce a partial order $\leq$ on $E^C$ by completing the relation $e \leq e'$ if there exists an edge $\tilde{e} \in E_{d,n}$ (resp. $\tilde{e}' \in E_{d,n}$) dual to an edge in the equivalence class of $e$ (resp. $e'$) and such that $\tilde{e}, \tilde{e}'$ are of same type, adjacent to the same lozenge and satisfy $h(\tilde{e}') \geq h(\tilde{e})$.

Let $C$ be a color map and let $H \in \mathtt{DH}^C(\alpha, \beta, \gamma)$ be a dual hive. For any $\hat{e} \in E^C$, let us set $\hat{\Phi}[H](\hat{e}) = \bigcup_{e \in \hat{e}} \Phi[H](e)$ and

$$\rho_C(H) = \bigcup_{\hat{e} \in E^C} \hat{\Phi}[H](\hat{e}) = \bigcup_{e \in E} \Phi[H](e)\ .$$

**Lemma 3.10** (Reduced graph geodesics)**.** *The set* $\left\{ \hat{\Phi}[H](\hat{e}) \mid \hat{e} \in E^C \right\}$ *is a set of geodesics of* $T$.

*Proof.* Let $e = \{v_1, v_2\}$ and $e' = \{v_1', v_2'\}$ be edges of $E^\Gamma$ such that $\{v_2, v_1'\} \in E^\Gamma$ and $c(\{v_2, v_1'\}) = m$. Suppose without loss of generality that $v_2$ (resp. $v_1'$) is dual to an upper (resp. lower) triangular face of $H_{d,n}$. Let $\ell \in \{0,1,2\}$ be the type of edges $e$ and $e'$. Then, by Lemma 3.7, $x_{v_2} = x_{v_1'} := x_v$, $\Phi(e) \subset x_{v_2} - \mathbb{R}e^{2(\ell+1)\pi/3}$ (resp. $\Phi(e') \subset x_{v_2} + \mathbb{R}e^{2(\ell+1)\pi/3}$) and $x_v \in \Phi(e) \cap \Phi(e')$, so that $\Phi(e) \cup \Phi(e')$ is a geodesic corresponding to $[x_{v_1}, x_{v_2'}]$. Hence, if $\hat{e} = \{v, v'\}$ is a maximal chain of $C$, $\bigcup_{e \in \hat{e}} \Phi(e)$ is the geodesic $[x_v, x_{v'}]$. $\square$

For $G \in \mathcal{G}_d$, recall that $\phi_{\alpha,\beta,\gamma} \in \Omega^0(V)$ has been defined before Proposition 3.3 in Section 3.1 and that the map $\mathcal{L} : \mathtt{HONEY}^G_{n,d} \to \Omega^1(G)$ has been defined in Definition 3.2.

**Proposition 3.11** (Dual hives as honeycombs)**.** *Let* $C : E_{n,d} \to \{0,1,3\}$ *be a color map. The map* $\rho_C$ *is a injection from* $\mathtt{DH}^C(\alpha,\beta,\gamma)$ *to* $\mathtt{HONEY}^{G^C}_{n,d}(\alpha,\beta,\gamma)$ *such that the map* $\mathcal{L} \circ \rho_C : \mathtt{DH}^C(\alpha,\beta,\gamma) \to \Omega^1(G^C)$ *is the restriction of an affine map with integer coefficients from* $\mathbb{R}^{E_{n,d}}$ *to* $\mathcal{F}_{G^C}(\phi_{\alpha,\beta,\gamma})$.

*Proof.* Let us first prove that for $H \in \mathtt{DH}^C(\alpha,\beta,\gamma)$, $\rho_C(H)$ is a triangular honeycomb. We first check that the two conditions of Definition 1.1 are fullfilled.

(1) Suppose that $\hat{e} \neq \hat{e}'$ and $int(\hat{\Phi}(\hat{e})) \cap int(\hat{\Phi}(\hat{e}')) \neq \emptyset$. Let $x \in int(\hat{\Phi}(\hat{e})) \cap int(\hat{\Phi}(\hat{e}'))$. Since $\hat{\Phi}(\hat{e}) = \bigcup_{e \in \hat{e}} \Phi(e)$, $\hat{\Phi}(\hat{e}') = \bigcup_{e \in \hat{e}'} \Phi(e)$ and, by Lemma 3.8, $int(\Phi(e)) \cap int(\Phi(e')) = \emptyset$ for $e \neq e'$, we have that $x = x_v$ for some $v \in e \cap e'$ with $e \in \hat{e}$, $e' \in \hat{e}'$ not colored $m$. By Lemma 3.7 and up to switching $e$ and $e'$, the angle from $\Phi(e)$ to $\Phi(e')$ is $2\pi/3$. Since $x \in int(\hat{\Phi}(\hat{e}))$, $v$ is adjacent to a third edge colored $m$; since $C$ is a color map, $c(e) = 1$ and $c(e') = 0$.
(2) Suppose that $x \in \partial\hat{\Phi}(\hat{e})) \cap \partial\hat{\Phi}(\hat{e}')$. Then, there exists $e \in \hat{e}$, $e' \in \hat{e}'$, neither of them colored $m$, such that $x \in \partial\Phi(e) \cap \partial\Phi(e')$. Then, Lemma 3.7 and the fact that $C$ is a color map yields the second condition.

Hence, $\rho_C(H)$ is a honeycomb and the canonical structure graph is given by

$$G[\rho_C(H)] = \left(\left\{x_v, v \in V^C\right\}, \left\{\hat{\Phi}[H](\hat{e}), \hat{e} \in G^C\right\}\right),$$

so that $G[\rho_C(H)]$ is isomorphic to $G^C$ as colored graph with ordered boundary. We next turn to the conditions of being a triangular honeycomb.

(1) Let $x$ be a vertex of $G[\rho_C(H)]$. Then, $x$ is the endpoint of a geodesic $\hat{\Phi}(\hat{e}) = \bigcup_{e\in\hat{e}} \Phi(e)$. Hence, $x = x_v$ for some $v \in V^\Gamma$ which is either dual to a triangular face $\tilde{v}$ without edge $m$ on its boundary (for otherwise $x_v \in int(\hat{\Phi}(\hat{e}))$), or is equal to $v_{\tilde{e}}$ for some $\tilde{e} \in E_{d,n}$. In the first case, $x$ is trivalent and, by Lemma 3.7, there are three non-trivial geodesics in $T$ adjacent to $x$, with the angle between two successive geodesics being equal to $2\pi/3$ : this implies that $x \in T \setminus \partial T$. In the second case, $x$ is univalent and belongs to $\partial T$ by construction.

(2) Condition (2) is a direct consequence of Lemma 3.7.

(3) Let $x$ be the $i$-th boundary point of $\rho_C(H)$ on $\partial_1 T$, so that $x_0 = 0$. If $i \leq d$, then $x = x_{v_{\tilde{e}}}$ for the edge $\tilde{e} \in \partial^{(2,0)}$ such that $L(\tilde{e}) = 1-\beta_i$ and $c(e) = 1$. Since $\tilde{e}$ is of type 2, $x_2 = L(\tilde{e}) = 1-\beta_i$, and thus $x_1 = 1-(1-\beta_i) = \beta_i$ . If $d+1 \leq i \leq n$, then $\tilde{e} \in \partial^{(1,1)}$, $L(\tilde{e}) = \beta_i$ and $c(e) = 0$. Moreover, $\tilde{e}$ is of type 1 and thus $x_1 = L(\tilde{e}) = \beta_i$. The cases of other boundaries are similar.

Therefore, $\rho_C(H) \in \mathtt{HONEY}^{G^C}_{n,d}(\alpha,\beta,\gamma)$.

Let us now check that $\mathcal{L} \circ p_C : \mathtt{DH}^C(\alpha,\beta,\gamma) \to \Omega^1(G^C)$ is the restriction of an affine map with integer coefficients. Let $\hat{e} = \{v, v'\} \in E^C$ of type $\ell$ with $s(v) = 1$ and $s(v') = -1$, and suppose without loss of generality that $v_\ell < v'_\ell$. Let $\mathcal{E}(\hat{e}) = \{v, w\}$ be the unique edge of the maximal chain $\hat{e}$ adjacent to $v$. Then, $\mathcal{E}(\hat{e})$ is of type $\ell$ and $\mathcal{L}[\rho_c[H]](\hat{\Phi}[H](\hat{e})) = (x_v)_\ell = L(\mathcal{E}(\hat{e}))$. Hence, $\mathcal{L} \circ \rho_C$ is the restriction of the linear map from $\mathbb{R}^{E_{n,d}}$ to $\Omega^1(G^C)$ mapping $(x(e))_{e\in E_{n,d}}$ to $\sum_{\hat{e}\in E^C} x(\mathcal{E}(\hat{e}))\delta_{\vec{e}}$, where for $e = \{v, v'\} \in E^C$ with $s(v) = 1$ and $s(v') = -1$, $\vec{e} = (v, v')$. Remark that this map has integer coefficients in the canonical bases of both vector spaces.

Finally, since $h \in \mathtt{HONEY}^G_{n,d}(\alpha,\beta,\gamma)$ is uniquely determined by $(L(e))_{e\in G}$ by Proposition 3.3, the injectivity of the map $\mathcal{L} \circ \rho_C$ will be implied by the injectivity of the map $\rho_C$. Suppose that $H_1, H_2 \in \mathtt{DH}^C(\alpha,\beta,\gamma)$ are distinct and denote by $L_1, L_2$ there respective label maps. Then, there exists $e \in E_{n,d}$ such that $L_1(e) \neq L_2(e)$. Denote by $\ell$ the type of $e$ and, up to using Condition (2)(a) of Definition 3.5 on a triangular face next to $e$, assume that $c(e) \neq m$. Let $\hat{e} = \{v, v'\}$ be the maximal chain containing $e$, with the condition that $v_\ell < v'_\ell$. Then, $L_i$ is constant on all edges $e \in \hat{e}$ not colored $m$, so that $L_1(\mathcal{E}(\hat{e})) = L_1(e) \neq L_2(e) = L_2(\mathcal{E}(\hat{e}))$. Hence, $\mathcal{L} \circ \rho_C(H_1) \neq \mathcal{L} \circ \rho_C(H_2)$, and $\rho_C$ is injective. ☐

3.4. **From triangular honeycomb to dual hive.** Let $h \in \mathtt{HONEY}_{d,n}(\alpha,\beta,\gamma)$ be a triangular honeycomb with graph structure $G = (V^G, E^G) \in \mathcal{G}_d$. We construct a graph $\tilde{G} = (\tilde{V}, \tilde{E})$ with color and label $(\tilde{c}, \tilde{L})$ and a map $\tilde{\Phi} : \tilde{E} \to \mathcal{P}(T)$ as follows :

(1) first consider an intermediate augmentation $\hat{G} = \left(\hat{V}, \hat{E}\right)$ of $G$, where $\hat{V}$ consists of vertices $V^G$ of $G$ together with the points which are not locally a one-dimensional variety. For $x, y \in \hat{V}$, we have an edge $\{x, y\} \in \hat{E}$ if and only if $[x, y] \subset e$ for some $e \in h$ and $]x, y[ \cap \hat{V} = \emptyset$. Then, set $\tilde{c}(\{x, y\}) = c(e)$ and $\tilde{L}(\{x, y\}) = L(e) = x_{\ell(e)}$ if $]x, y[ \subset e$. For $\{x, y\} \in \hat{E}$, one sets $\tilde{\Phi}(\{x, y\}) = [x, y] \in \mathcal{P}(T)$.

(2) A vertex $v \in \hat{V}$ is of degree either 1 or 3 if it comes from a vertex of $G$ or of degree 4 if it comes from a non-empty intersection $\iota(e) \cap \iota(e')$ for $e, e' \in E^G$. In the latter case, the four edges $\left\{\{v, x_i^\pm\}, i = 0, 1\right\}$ adjacent to $v$ in $\hat{G}$ are such that $\{v, x_i^\pm\}$ is colored $i$ and of type $\ell - i$ for some $\ell \in \{0, 1, 2\}$. In particular, the angle $\widehat{x_0^\epsilon v x_1^\epsilon} = 2\pi/3$ for $\epsilon \in \{-, +\}$. Replace $v$ by two vertices $v^+, v^-$, add an edge $e$ to $\hat{E}$ with color $m$, type $\ell+1$ and label $1-L(\{v, x_0^\pm\})-L(\{v, x_1^\pm\})$ between $v^+$ and $v^-$. Set $\tilde{\Phi}(e) = \{v\}$. Replace each edge $\{v, x_i^\pm\}$ by $\{v^\pm, x_i^\pm\}$, keeping the same label and color. Repeat the operation successively for each vertex of degree 4.

The resulting augmentation $\tilde{G} = (\tilde{V}, \tilde{E})$ of $G$ has univalent and trivalent vertices, each of the latter being adjacent to one edge of each type $\ell \in \{0, 1, 2\}$. Remark moreover that $\bigcup_{e\in\tilde{E}} e = \bigcup_{e\in E} e := \Lambda$.

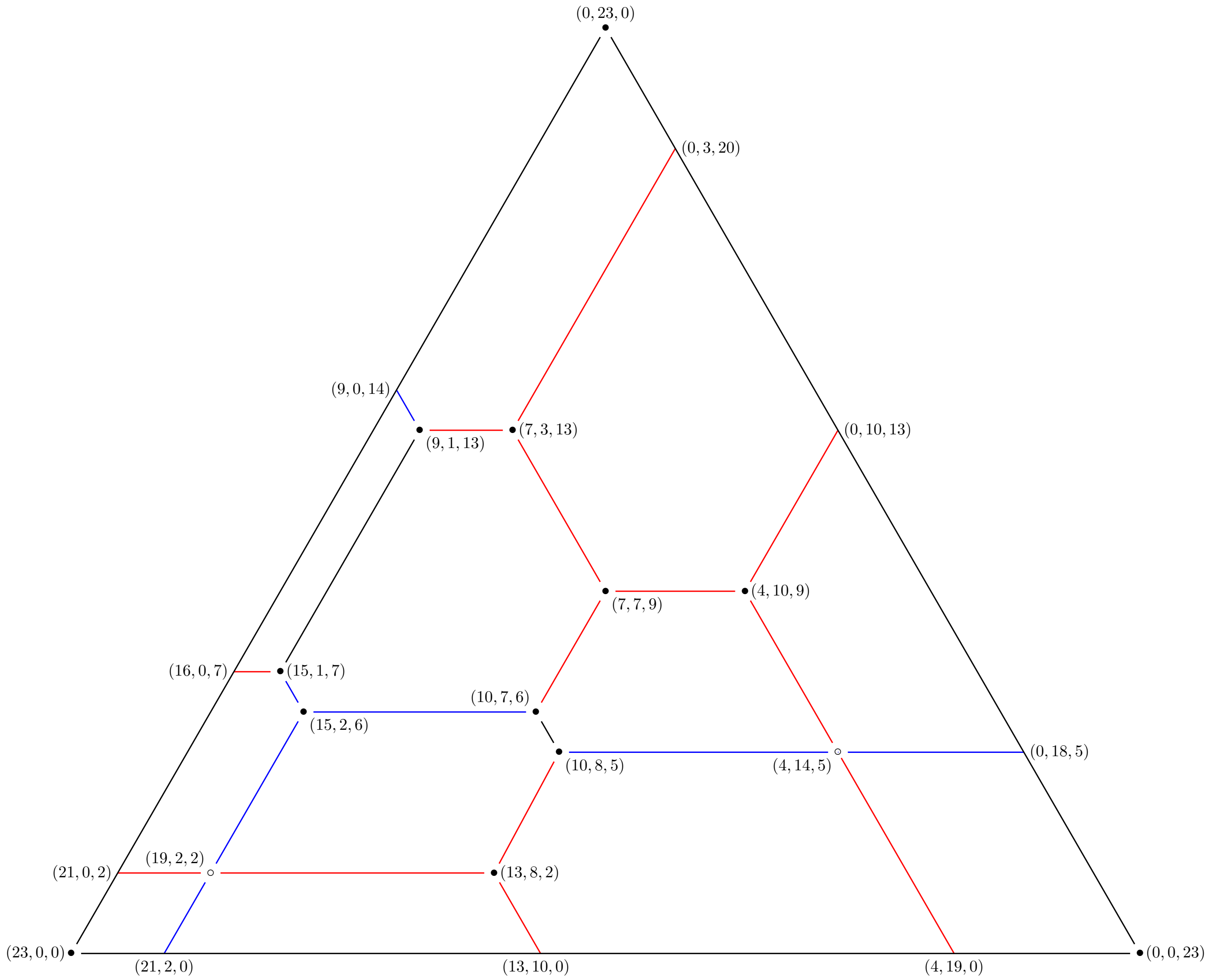



FIGURE 6. The triangular honeycomb corresponding to the dual hive of Figure 5. Coordinates should to be multiplied by 1/23.

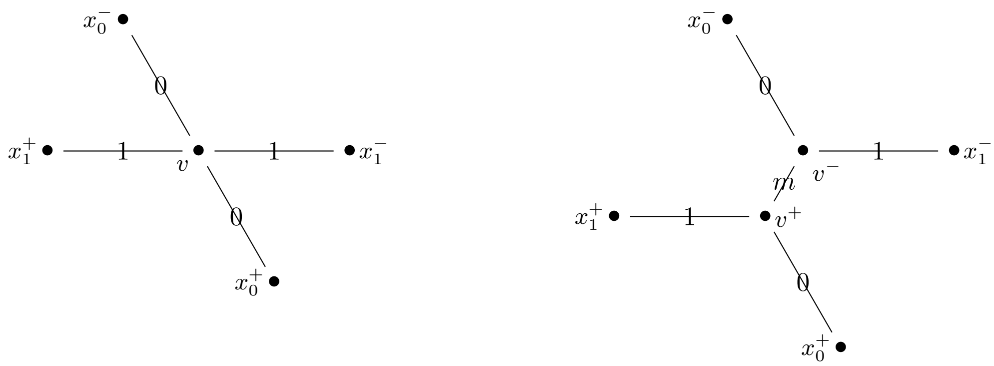



FIGURE 7. The augmentation of a vertex $v$ of degree 4 to two vertices $v^{\pm}$ of degree 3 linked by an edge of type $\ell + 1 = 1$. The configuration is the one on the bottom right of Figure 6.

Hence, a connected region of $\mathbb{C} \setminus \Lambda$ is a polygon with angle either $2\pi/3$ or $\pi/3$. In the latter case, the vertex $v$ of $G$ is a vertex of degree 4 which has been replaced by two vertices of degree 3 and an edge in $\tilde{G}$ as in Figure 7. Hence, any bounded region of $\mathbb{C} \setminus \Lambda$ is bounded by 6 edges of $\tilde{G}$.
Therefore, the dual of $\tilde{G}$ is a graph $\tilde{H}$ with only triangular faces and inner vertices of degree 6. In particular, $\tilde{H}$ is isomorphic to a subgraph of the triangular grid. Let us define the type (resp. color, resp. label) of an edge $e$ of $\tilde{H}$ as the same as the one his dual. In particular, each triangular face is

bordered by three edges $(e^0, e^1, e^2)$, with $e^\ell$ of type $\ell$ and such that $L(e^0)+L(e^1)+L(e^2)=1$. Since the types of the boundary edges is

$$(\underbrace{0,\dots,0}_{n-d},\underbrace{2,\dots,2}_{d},\underbrace{1,\dots,1}_{n-d},\underbrace{0,\dots,0}_{d},\underbrace{2,\dots,2}_{n-d},\underbrace{1,\dots,1}_{d})\ ,$$

$\tilde{H}$ is actually isomorphic to $H_{d,n}$. For $e \in E_{d,n}$, set $C(e) = \tilde{c}(\tilde{e})$ and $L(e) = \tilde{L}(\tilde{e})$

**Lemma 3.12** (Honeycomb to dual hive)**.** *The pair $H = (C, L)$ is a dual hive and $\rho_C(H) = h$.*

*Proof.* It is straightforward to check that the maps $C(e) = \tilde{c}(\tilde{e})$, $L(e) = \tilde{L}(\tilde{e})$ satisfy the properties (1), (2)(a), (2)(b)(i) and (c) of Definition 3.5.
To verify (2)(b)(ii), let $\tilde{e}, \tilde{e}'$ be opposite edges of type $\ell$ of a lozenge of $H_{d,n}$ with middle edge $\tilde{f}$ of type $\ell+1$ not colored $m$ and denote by $e, e', f$ their dual edges in $E^{\tilde{G}}$. Suppose that $h(\tilde{e}) > h(\tilde{e}')$, where $h$ has been defined in (3.3). We want to show that $L(\tilde{e}) = \tilde{L}(e) > L(\tilde{e}') = \tilde{L}(e')$. From the definition above, $\tilde{L}(e) = L(e)$ where for $e \in E^G$, $L(e) = x_\ell$ has been defined in Definition 1.2. We thus need to show that $x_\ell > x'_\ell$. Since $f \in E^{\tilde{G}}$ is of type $\ell+1$, we have that $f \subset x + \mathbb{R}\mathrm{e}^{2i\pi(\ell+2)/3}$ in $T$. The geodesic $f$ cannot be reduced to a point as the honeycomb is non-degenerated and for otherwise condition (2) of Definition 1.1 would not be satisfied. Therefore, the coordinate $\ell$ is strictly decreasing between edges $e$ and $e'$ so that $x_\ell > x'_\ell$. Hence, $H \in \mathtt{DH}^C(\alpha,\beta,\gamma)$.
Remark that a triangular honeycomb $h$ is uniquely determined by its image $\mathcal{S} = \bigcup_{e\in h} e$, since then the elements of $h$ are all the geodesics of $\mathcal{S}$ whose endpoints are univalent or trivalent vertices. Recall that for $e = \{v, v'\} \in E^\Gamma$, $\Phi(e) = [x_v, x_{v'}]$. Hence, to check that $\rho_C(H) = h$, it suffices to show that $\bigcup_{e\in E^\Gamma} \Phi(e) = \bigcup_{e\in h} e$. This is implied by the construction of $\tilde{\Phi}$ at the beginning of the section, since

$$\bigcup_{e\in E^\Gamma} \Phi(e) = \bigcup_{e\in \tilde{E}} \tilde{\Phi}(e) = \bigcup_{\{v,v'\}\in \tilde{E}} [x_v, x_{v'}] = \bigcup_{e\in h} e\ .$$

□

Putting together Lemma 3.11 with Lemma 3.12 yields the following decomposition.

**Proposition 3.13** (Color map indexing)**.** *There is a partition*

$$\mathtt{HONEY}_{n,d} = \bigsqcup_{C \text{ color map}} \mathtt{HONEY}^{G^C}_{n,d}\ ,$$

*such that, for each color map $C$, the map $\rho_C$ is a bijection and $(\mathcal{L}\circ\rho_C)^{-1}$ is the restriction of a linear map from $\mathcal{F}_{G^C}(\phi_{\alpha,\beta,\gamma})$ to $\mathbb{R}^{E_{n,d}}$ whose matrix in the canonical bases has integer coefficients.*

*Proof.* By Lemma 3.11, the map $\rho_C : \mathtt{DH}^C(\alpha,\beta,\gamma) \to \mathtt{HONEY}^{G^C}_{n,d}(\alpha,\beta,\gamma)$ is injective.
Let $h \in \mathtt{HONEY}^{G^C}_{n,d}(\alpha,\beta,\gamma)$ be a honeycomb. Then, by Lemma 3.12, there exists $C'$ a color map and $H \in \mathtt{DH}^{C'}(\alpha,\beta,\gamma)$ such that $\rho_{C'}(H) = h$. Hence, $h \in \mathtt{HONEY}^{G^{C'}}_{n,d}(\alpha,\beta,\gamma)$. Since the map $C \mapsto G^C$ is injective, $C = C'$ and $H \in \mathtt{DH}^C(\alpha,\beta,\gamma)$. Therefore, $p_C$ is a bijection and $\mathtt{HONEY}^{G^C}_{n,d}(\alpha,\beta,\gamma) \cap \mathtt{HONEY}^{G^{C'}}_{n,d}(\alpha,\beta,\gamma) = \emptyset$ for $C \neq C'$. Therefore,

$$\mathtt{HONEY}_{n,d} = \bigsqcup_{C \text{ color map}} \mathtt{HONEY}^{G^C}_{n,d}\ .$$

Finally, for a color map $C$, the map $\rho_C^{-1}$ is then obtained as follows : each geodesic $e \in h$ of type $\ell$ corresponds to a maximal chain $\hat{e}$ of type $\ell$ of $\Gamma_{n,d}$ with respect to $C$. Hence, for all edge $f \in \hat{e}$ of type $\ell$, one has $\rho_C^{-1}[h](f) = L(e)$. Then, for any edge $f \in E_{n,d}$ colored $m$, one has $\rho_C^{-1}[h](f) = 1 - \rho_C^{-1}[h](f_1) - \rho_C^{-1}[h](f_2)$, where $f_1$ and $f_2$ are the two other edges of a triangular face bordered by $f$. Hence, the matrix of the map $(\mathcal{L}\circ\rho_C)^{-1}$ has integer coordinates in the canonical bases. □

From the Proposition 3.13, any $G \in \mathcal{G}_d$ is of the form $G = G^C$ for some color map $C$. Since $V^C \subset V^\Gamma$, any vertex $v$ of $G^C$ inherits the coordinates of $\Gamma$ by setting

$$(v_0, v_1, v_2) = (h(e_0), h(e_1), h(e_2))\ ,$$

where $e_\ell$ is the edge of type $\ell$ adjacent to $v$ in $\Gamma_{d,n}$ (even if $e_\ell \notin E^C$). For $e = \{v, w\}, e' = \{v', w'\} \in E^C$ of same type, introduce the cover relation $e < e'$ when, up to a transposition, $\{w, v'\}$ is an edge of $E^C$

of type $\ell+1$ and $v'_{\ell-1} > w_{\ell-1}$. This cover relation translates into a cover relation in $\vec{E}^C$ by saying that $(v,w) < (v',w')$ is and only if $\{v,w\} < \{v',w'\}$ in the former sense.

**Corollary 3.14** (Parametrization of labels)**.** *For a color map $C$ and $G^C \in \mathcal{G}_d$,*

$$\mathcal{L}\left(\mathtt{HONEY}^{G^C}_{n,d}(\alpha,\beta,\gamma)\right) = \mathcal{F}_{G^C}(\phi_{\alpha,\beta,\gamma}) \cap K^C \ ,$$

*where $K^C \subset \Omega^1(G^C)$ is the cone defined as*

$$K^C = \left\{\omega \in \Omega^1(G^C) \mid |\omega(e)| < |\omega(e')| \ \text{ if } e < e'\right\}.$$

*Proof.* Let us define $\Psi : \Omega^1(G) \to \mathbb{R}^{E_{n,d}}$ by

$$\Psi[\omega](e) = \begin{cases} |\omega(\vec{\hat{e}})| & \text{if } e \in \hat{e}, c(e) \neq m \\ 1 - |\omega(\vec{\hat{e_1}})| - |\omega(\vec{\hat{e_2}})| & \text{if } c(e) = m,\ e_1 \in \hat{e}_1, e_2 \in \hat{e}_2,\ (e_1,e_2,e) \text{ triangular face of } H_{n,d} \end{cases}.$$

By Proposition 3.13, we have that $\mathcal{L}\left(\mathtt{HONEY}^{G^C}_{n,d}(\alpha,\beta,\gamma)\right) = \Psi^{-1}\left(\mathtt{DH}^C(\alpha,\beta,\gamma)\right)$.
Then, remark that $\mathtt{DH}^C(\alpha,\beta,\gamma) = \mathcal{H}^C(\alpha,\beta,\gamma) \cap \mathcal{K}_<$, where $\mathcal{H}^C(\alpha,\beta,\gamma) \subset \mathbb{R}^{E_{n,d}}$ is the vector subspace determined by the conditions (2)(a), (2)(b)(i) and (2)(c) of Definition 3.5 and $\mathcal{K}_<$ is the cone given by

$$\mathcal{K}_< = \left\{(H(e))_{e\in E_{n,d}} \mid L(e) < L(e') \text{ if } e,\ e' \text{ satisfy condition (2)(b)(ii) of Definition 3.5}\right\}.$$

Hence,

$$\mathcal{L}\left(\mathtt{HONEY}^{G^C}_{n,d}(\alpha,\beta,\gamma)\right) = \Psi^{-1}\left(\mathcal{H}^C(\alpha,\beta,\gamma) \cap \mathcal{K}_\leq\right) = \Psi^{-1}\left(\mathcal{H}^C(\alpha,\beta,\gamma)\right) \cap \Psi^{-1}\left(\mathcal{K}_<\right).$$

One then checks that $\Psi^{-1}(\mathcal{H}^C(\alpha,\beta,\gamma)) = \mathcal{F}_{G^C}(\alpha,\beta,\gamma)$ and $\Psi^{-1}(\mathcal{K}_<) = K^C$. □

3.5. **Volume of flat connections.** We can now combine the results of [FT24] with the ones of the previous section to prove Theorem 1.4 in the case of the three holed-sphere, that is, for $(g,p) = (0,3)$. Let us denote by $\Sigma_0^3$ the Riemann sphere with three generic marked points removed. The moduli space of flat $\mathrm{SU}(n)$ connections can be described as

$$M_{0,3}(\alpha,\beta,\gamma) = \{(U_1,U_2,U_3) \in \mathcal{O}_\alpha \times \mathcal{O}_\beta \times \mathcal{O}_\gamma \mid U_1U_2U_3 = Id_{\mathrm{SU}(n)}\}/\mathrm{SU}(n)\ ,$$

where $\mathrm{SU}(n)$ acts diagonally by conjugation on each factor. Its volume has been computed in [FT24] using two equivalent models named *toric hives* and *dual hives*. Using the results of this section, we present a reformulation of this results in terms of triangular honeycombs. For $G \in \mathcal{G}_d$, let us set

$$\mathrm{Vol}\left[\mathtt{HONEY}^G_{n,d}(\alpha,\beta,\gamma)\right] := \mathrm{Vol}\left[\mathcal{L}(\mathtt{HONEY}^G_{n,d}(\alpha,\beta,\gamma))\right]\ ,$$

where $\mathcal{L} : \mathtt{HONEY}^G_{n,d}(\alpha,\beta,\gamma) \to \mathcal{F}_G(\phi_{\alpha,\beta,\gamma})$ was defined in Definition 3.2 and Vol is the volume form defined in (2.3).

**Theorem 3.15** (Volume of flat $\mathrm{U}(n)$-connections on the three-holed sphere)**.** *Let $n \geq 3$ and consider the canonical volume form on $\mathrm{U}(n)$. Then, for $\alpha,\beta,\gamma \in \mathcal{H}_{reg}$,*

$$Z_{0,3}(\alpha,\beta,\gamma) := \mathrm{Vol}\left[M_{0,3}(\alpha,\beta,\gamma)\right] \neq 0$$

*only if $\sum_{i=1}^n \alpha_i + \sum_{i=1}^n \beta_i + \sum_{i=1}^n \gamma_i = n+d$ for some $d \in \mathbb{N}$, in which case, if $\tilde{\gamma} = (1-\gamma_n,\dots,1-\gamma_1)$,*

$$Z_{0,3}(\alpha,\beta,\gamma) = \frac{2^{(n+1)[2]}(2\pi)^{(n-1)(n-2)}}{n!\Delta(\alpha)\Delta(\beta)\Delta(\gamma)} \sum_{G\in\mathcal{G}_d} \mathrm{Vol}\left[\mathtt{HONEY}^G_{n,d}(\alpha,\beta,\tilde{\gamma})\right]\ ,$$

*where for $\alpha \in \mathcal{H}_{reg}$, $\Delta(\alpha) = 2^{n(n-1)/2}\prod_{i<j}\sin(\pi(\alpha_i-\alpha_j))$.*

Before proving this theorem, let us recall three results from [FT24].

1. For any pair $(C,C')$ of color maps, there exists a linear isomorphism $Rot[C \to C']$ from $\mathtt{DH}^C(\alpha,\beta,\gamma)$ to $\mathtt{DH}^{C'}(\alpha,\beta,\gamma)$ with integer coefficients on the canonical bases and such that $Rot[C \to C] = Id$ and $Rot[C \to C']^{-1} = Rot[C' \to C]$.
2. There exists a color map $C_0$ and a subset $S \subset E_{n,d}$ such that $p_S : \mathtt{DH}^{C_0}(\alpha,\beta,\gamma) \to \mathbb{R}^S$ which to a label map $L : E_{n,d} \to \mathbb{R}$ associates $(L(e))_{e\in S}$ is an integral isomorphism.

(3) We have the formula

$$Z_{0,3}(\alpha,\beta,\gamma) = \frac{2^{(n+1)[2]}(2\pi)^{(n-1)(n-2)}}{n!\Delta(\alpha)\Delta(\beta)\Delta(\gamma)} \sum_C \mathrm{Vol}_S\left[Rot[C\to C_0]\left(\mathtt{DH}^C(\alpha,\beta,\gamma)\right)\right],$$

where the sum is over color maps $C : E_{n,d} \to \{0,1,3,m\}$ and where $\mathrm{Vol}_S$ is the Lebesgue measure of dimension $|S|$.

*Proof of Theorem 3.15.* Let $C$ be a color map. By Proposition 2.1, there exists a set $R \subset E$ such the restriction map $\varphi_R : \mathcal{F}_{G^C}(\phi_{\alpha,\beta,\gamma})) \to \Omega^1(R)$ is a bijection with integral coefficient in the canonical bases. Denote by $i_R : \Omega^1(R) \to \Omega^1(G)$ the corresponding inverse map, which has thus affine with integer coefficients in the canonical basis and is a bijection from $\Omega^1(R)$ to $\mathcal{F}_{G^C}(\phi_{\alpha,\beta,\gamma})$. Since, by Proposition 3.13 and Corollary 3.14, $\Psi^C : \Omega^1(G^C) \to \mathbb{E}^{E_{n,d}}$ is an affine integral map and a bijection from $\mathcal{F}_{G^C}(\phi_{\alpha,\beta,\gamma})) \cap K^C$ to $\mathtt{DH}^C(\alpha,\beta,\gamma)$ and by (1) above, $Rot[C\to C_0]$ is an integral isomorphism from $\mathtt{DH}^C(\alpha,\beta,\gamma)$ to $\mathtt{DH}^{C_0}(\alpha,\beta,\gamma)$. We deduce that

$$p_S \circ Rot[C\to C_0] \circ \Psi^C \circ i_R : \Omega^1(R) \to \mathbb{R}^S$$

is an integral isomorphism. Likewise, since by (2) above $p_S^{-1} : \mathbb{R}^S \to \mathtt{DH}^{C_0}(\alpha,\beta,\gamma)$ is an integral affine isomorphism and $\mathcal{L}\circ\rho_C$ is an integral affine isomorphism from $\mathtt{DH}^C(\alpha,\beta,\gamma)$ to $\mathcal{F}_{G^C}(\phi_{\alpha,\beta,\gamma})$,

$$F := \left(p_S\circ Rot[C\to C_0]\circ\Psi^C\circ i_R\right)^{-1} = \varphi_R\circ(\mathcal{L}\circ\rho_C)\circ Rot[C_0\to C]\circ p_S^{-1} : \mathbb{R}^S \to \Omega^1(R)$$

is an integral isomorphism. We deduce that its determinant as an isomorphism from $\Omega^1(R)$ to $\mathbb{R}^S$ has modulus one. Hence,

$$\begin{aligned}
\mathrm{Vol}_S\left[Rot[C\to C_0]\left(\mathtt{DH}^C(\alpha,\beta,\gamma)\right)\right] &= \mathrm{Leb}\left[u\in\mathbb{R}^S \mid Rot[C_0\to C]\circ p_S^{-1}(u)\in \mathtt{DH}^C(\alpha,\beta,\gamma)\right]\\
&= \mathrm{Leb}\left[u\in\mathbb{R}^S \mid F(u)\in\varphi_R\left(\mathtt{HONEY}^C_{n,d}(\alpha,\beta,\gamma)\right)\right]\\
&= \mathrm{Leb}\left[z\in\Omega^1(R) \mid z\in\varphi_R\left(\mathtt{HONEY}^C_{n,d}(\alpha,\beta,\gamma)\right)\right]\\
&= \mathrm{Vol}\left[\mathtt{HONEY}^{G^C}_{n,d}(\alpha,\beta,\gamma)\right],
\end{aligned}$$

where we used Proposition 2.2 on the last equality. Hence, by (3),

$$\begin{aligned}
Z_{0,3}(\alpha,\beta,\gamma) &= \frac{2^{(n+1)[2]}(2\pi)^{(n-1)(n-2)}}{n!\Delta(\alpha)\Delta(\beta)\Delta(\gamma)} \sum_{C:E_{n,d}\to\{0,1,3,m\}\text{ color map}} \mathrm{Vol}_S\left[Rot[C\to C_0]\left(\mathtt{DH}^C(\alpha,\beta,\gamma)\right)\right]\\
&= \frac{2^{(n+1)[2]}(2\pi)^{(n-1)(n-2)}}{n!\Delta(\alpha)\Delta(\beta)\Delta(\gamma)} \sum_{C:E_{n,d}\to\{0,1,3,m\}\text{ color map}} \mathrm{Vol}\left[\mathtt{HONEY}^{\Gamma^C}_{n,d}(\alpha,\beta,\gamma)\right]\\
&= \frac{2^{(n+1)[2]}(2\pi)^{(n-1)(n-2)}}{n!\Delta(\alpha)\Delta(\beta)\Delta(\gamma)} \sum_{G\in\mathcal{G}_d} \mathrm{Vol}\left[\mathtt{HONEY}^{G}_{n,d}(\alpha,\beta,\gamma)\right],
\end{aligned}$$

where we used Proposition 3.13 for the last equality. □

## 4. Sieving of honeycombs

Let $\mathcal{T}$ be an oriented surface with boundary obtained by gluing $m$ equilateral triangles $T^1,\dots,T^m$ along their boundaries, such that $p$ edges $L_1,\dots,L_p$ of the equilateral triangles are not glued. Each edge $L_j$ has a natural orientation $\ell_j$ coming from the equilateral it belongs to. Suppose that $h$ is a honeycomb on the surface $\mathcal{T}$ such that each $h\cap\mathcal{T}^i$ is triangular. Then, the structure graph $\hat{G}[h]$ of $h$, as defined after Definition 1.1, has $pn$ univalent vertices, $n$ of them being on each boundary component $L_j$, $1\le j\le p$, and $\frac{(3m-p)n}{2}$ bivalent vertices, $n$ of them being on a the boundary of a common triangle while being in the interior of the surface.

Let us introduce some notations. First, for $1\le j\le p$, let $1\leqslant s_j\leqslant m$ and $0\leqslant \ell_j\leqslant 2$ be such that $L_j = \partial_{\ell_j}T^{s_j}$. For $1\leqslant j\leqslant p$, let us denote by $(v^j_m)_{1\le m\le n}$ the univalent boundary vertices on $L_j$ ranked decreasingly with respect to their $(\ell_j+1)$-coordinate. Let $\mathcal{G}_\mathcal{T}$ be the set of structure graphs appearing in $\mathtt{HONEY}_\mathcal{T}$ rooted at $v^1_1$, and $\mathcal{G}$ the case where $m=1$ of a single triangle as in Section 3. Then, set $N(\mathcal{T}) = \frac{3m-p}{2}$ and denote by $B_1,\dots,B_{N(\mathcal{T})}$ the segments of $\mathcal{T}$ corresponding to boundary of triangles identified together. Each $B_j$ is then adjacent to two triangles $T^{r_j^{(1)}}$ and $T^{r_j^{(2)}}$ and there exists $\ell_j^{(1)},\ell_j^{(2)}$ such that $B_j=\partial_{\ell_j^{(1)}}T^{r_j^{(1)}}=\partial_{\ell_j^{(2)}}T^{r_j^{(2)}}$.

**Lemma 4.1** (Honeycomb decomposition)**.** *There is an injective map $t:\mathcal{G}_{\mathcal{T}}\to\mathcal{G}^m$ such that the map*

$$i:\begin{cases}\mathtt{HONEY}_{\mathcal{T}} &\to\left\{(h^1,\dots,h^m)\in\mathtt{HONEY}_{T^1}\times\dots\times\mathtt{HONEY}_{T^m}\mid h^{r_j^{(1)}}_{|B_j}=h^{r_j^{(2)}}_{|B_j},1\leq j\leq N(\mathcal{T})\right\}\\ h &\mapsto (h\cap T^i)_{1\leq i\leq m}\end{cases}$$

*is bijective and restricts for each $G\in\mathcal{G}_{\mathcal{T}}$ and $(G_1,\dots,G_m)=t(G)$ to a bijection*

$$i:\mathtt{HONEY}^G_{\mathcal{T}}\to\left\{(h^1,\dots,h^m)\in\mathtt{HONEY}^{G_1}_{T^1}\times\dots\times\mathtt{HONEY}^{G_m}_{T^m}\mid h^{r_j^{(1)}}_{|B_j}=h^{r_j^{(2)}}_{|B_j},1\leq j\leq N(\mathcal{T})\right\}.$$

*Proof.* The proof is done by induction on $m$. The result holds for $m=1$. Suppose the result true for $m\geq 1$, and let $\mathcal{T}$ be a surface obtained by pasting $m$ triangles $T^1,\dots,T^m$ along some boundaries and that $L_1,\dots,L_p$ are the boundary which have not been pasted together. Assume without loss of generality that $L_p$ belongs to $T^m$ and let $T^{m+1}$ be another oriented equilateral triangle. Let $\mathcal{T}'$ be the surface obtained by gluing $T^{m+1}$ to $T^m$ along $\partial_0T^{m+1}$ and $L_p$.

Let $h$ be a honeycomb on $\mathcal{T}'$ with structure graph $G\in\mathcal{G}_{\mathcal{T}'}$. The restriction $i(h)=(h\cap T^{m+1},h\cap\mathcal{T})$ yields a pair of honeycomb on $T^{m+1}$ and $\mathcal{T}$ such that $(h\cap\mathcal{T})_{|L_p}=(h\cap T^{m+1})_{|\partial_0T^{m+1}}$. Moreover, the structure graphs $G_1$ of $h\cap T^{m+1}$ and $G_2$ of $h\cap\mathcal{T}$ only depend on $G$. Indeed, by construction of the structure graph $G$, $G_1$ corresponds to all the edge that can be reached from the boundaries $\partial_1T^{m+1}$ and $\partial_2T^{m+1}$ by avoiding bivalent vertices. Hence, the map $i:\mathtt{HONEY}_{\mathcal{T}'}\to\mathtt{HONEY}_{\mathcal{T}^{m+1}}\times\mathtt{HONEY}_{\mathcal{T}}$ and the map

$$\begin{aligned}\tilde{t}:\mathcal{G}_{\mathcal{T}'}&\to\mathcal{G}\times\mathcal{G}_{\mathcal{T}}\\ G&\mapsto(G_1,G_2)\end{aligned}$$

are such that for each $G\in\mathcal{G}_{\mathcal{T}'}$,

$$i\left(\mathtt{HONEY}^G_{\mathcal{T}'}\right)\subset\left\{(h^1,h^2)\in\mathtt{HONEY}^{G_1}_{T^{m+1}}\times\mathtt{HONEY}^{G_2}_{\mathcal{T}}\mid h^1_{|\partial_0T^{m+1}}=h^2_{|L_p}\right\}.$$

If $h,h'\in\mathtt{HONEY}_{\mathcal{T}'}$ are such that $i(h)=i(h')=(h^1,h^2)$, then $h=h^1\cup h^2=h'$ so that $i$ is injective.

Reciprocally, let $(h^1,h^2)\in\mathtt{HONEY}^{G_1}_{T^{m+1}}\times\mathtt{HONEY}^{G_2}_{\mathcal{T}}$ be such that $h^1_{|\partial_0T^{m+1}}=h^2_{|L_p}$. Suppose without loss of generality that $T^{m+1}=\{-r+se^{2\pi i/3},\,0\leq r,s\leq 1,r+s\leq 1\}$ and that $\partial_0T^{m+1}=\{se^{2\pi i/3}\}=L_p$, so that $T^m=\{re^{\pi i/3}+se^{2\pi i/3},\,0\leq r,s\leq 1,r+s\leq 1\}$. Let $x\in h^1_{\partial_0T^{m+1}}=h^2_{|L_p}$. Then, by Condition (3) of Definition 1.2 (see Figure 8):

- a geodesic $e$ of $h^1$ arriving at $\partial_0T^{m+1}$ at $x$ is either colored $c(e)=0$ and included in $x+\mathbb{R}_{>0}e^{-2\pi i/3}$ or colored $c(e)=1$ and included in $x+\mathbb{R}_{<0}$.
- A geodesic $e'$ of $h^2$ arriving at $L_p$ at $x$ is either colored $c(e)=0$ and included in $x+\mathbb{R}_{<0}e^{-2\pi i/3}$ or colored $c(e)=1$ and included in $x+\mathbb{R}_{>0}$.

In any case, the angle from $e$ to $e'$ is either 0 if $c(e)=c(e')$, in which case $e\cup e'$ is again a geodesic colored $c(e)$, or $2\pi/3$ (resp. $4\pi/3$) if $c(e)=0$ and $c(e')=1$ (resp. $c(e)=1$ and $c(e')=0$). In the latter case, Condition (2) from Definition 1.1 is satisfied. Hence, the set of geodesics

$$h=h_1\cup h_2$$

defines a honeycomb on $\mathcal{T}'$. Moreover, the structure graph $G=(V^G,E^G)$ of $h$ is obtained by setting

$$V^G=(V^{G_1}\cup V^{G_2})/\langle v_i=w_i,1\leq i\leq n\rangle\ ,$$

where $v_1,\dots,v_n$ and $w_1,\dots,w_n$ are the boundary vertices of $G_1$ and $G_2$ corresponding to points on $\partial_0T^{m+1}$ and $L_p$ respectively. The edge set of $G$ is $E^G=E^{G_1}\cup E^{G_2}$. In particular, the map $\tilde{t}:\mathcal{G}_{\mathcal{T}'}\to\mathcal{G}\times\mathcal{G}_{\mathcal{T}}$ is injective, and each restricted map

$$i:\mathtt{HONEY}^G_{\mathcal{T}'}\to\left\{(h^1,h^2)\in\mathtt{HONEY}^{G_1}_{T^{m+1}}\times\mathtt{HONEY}^{G_2}_{\mathcal{T}}\mid h^1_{|\partial_0T^{m+1}}=h^2_{|L_p}\right\}.$$

is bijective. Let $t_m,i_m$ be the maps given by induction on $\mathcal{G}_{\mathcal{T}}$ and $\mathtt{HONEY}_{\mathcal{T}}$. Then, the maps $(t_m\times\mathrm{Id}_{\mathcal{G}})\circ t$ and $(i_m\times\mathrm{Id}_{\mathtt{HONEY}_n})\circ i$ are the bijections of the statement of the lemma. $\square$

Remark from the construction of the reciprocal of $t$ in the latter proof that the injective map $t_{\mathcal{T}} : \mathcal{G}_{\mathcal{T}} \to \mathcal{G}^m$ such that $t(G) = (G_1, \ldots, G_m)$ satisfies

$$G = \left(\bigcup_{i=1}^{m} G_i\right)_{S*S'}$$

in the sense of Definition 2.3, where $S, S' \subset \bigcup_{i=1}^m \partial G_i$ are the vertices which are identified together.

In Section 3.1, we defined for each $G_i = (V_i, E_i) \in \mathcal{G}_{d_i}$ a map $s_i : V_i \to \{-1, 1\}$ such that for each edge $\{v, v'\} \in E_i$, $s_i(v)s_i(v') = -1$. Let us denote by $\mathrm{int}_3(G)$ the set of vertices of degree 3 of $G$. By the construction of $G$ in terms of the graphs $G_i$ and the fact that all vertices of $\mathrm{int}(G_i)$ have degree 3,

$$\mathrm{int}_3(G) = \bigsqcup_{1 \le i \le m} \mathrm{int}(G_i) \ .$$

We can therefore extend the maps $s_i$ defined on each $G_i$ to a map $s : \mathrm{int}_3(G) \to \{-1, 1\}$.

**Definition 4.2** (Label map of a honeycomb)**.** Let $G = (V, E) \in \mathcal{G}_{\mathcal{T}}$ be a structure graph. The *label map* $\mathcal{L} : \mathtt{HONEY}_{\mathcal{T}}^{G} \to \Omega^1(G)$ is defined by setting, for $(v, v') \in \vec{E}$ with $v \in \mathrm{int}_3(G)$,

$$\mathcal{L}[h](v, v') = s(v)L(i(e)) \ ,$$

where $i(e)$ is the injection from Lemma 4.1 and $L$ is the coordinate map defined in Definition 1.2. (2).

Remark that each edge $e \in E$ of $G$ is adjacent to at least one trivalent vertex, and if $e = \{v, v'\}$ with $v, v' \in \mathrm{int}_3(G)$, then necessarily there exists $1 \le i \le m$ such that $v, v' \in V_i$. Therefore $s(v)s(v') = -1$, so that $\mathcal{L}[h] \in \Omega^1(G)$ and $\mathcal{L}$ is a well-defined map from $\mathtt{HONEY}_{\mathcal{T}}^{G}$ to $\Omega^1(G)$.

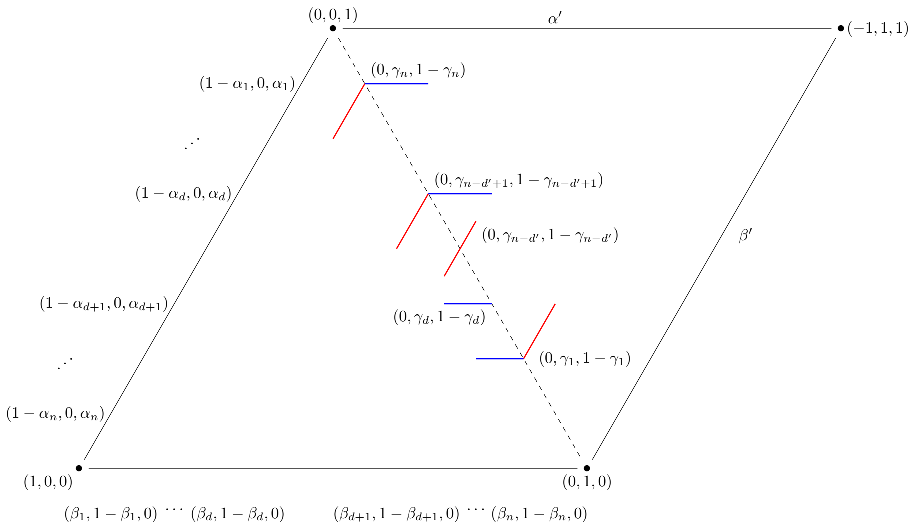



FIGURE 8. Line segments on the common edge of $h$ and the rotation of $h'$ in $\mathbb{R}^3_{\Sigma=1}$. Here, $d < d'$ which can be read from the orientation of line segments.

Let $\gamma^{(1)}, \ldots, \gamma^{(p)} \in \mathcal{H}_{reg}$, and recall that $\mathtt{HONEY}_{\mathcal{T}}^{G}(\gamma^{(1)}, \ldots, \gamma^{(p)})$ denotes the set of honeycombs $h$ on $\mathcal{T}$ having boundary condition

$$(v_s^j)_{\ell_j} = \gamma_s^{(j)}.$$

Let us define $\phi_{\gamma^{(1)},\ldots,\gamma^{(p)}} : V \to \mathbb{R}$ by

$$\phi_{\gamma^{(1)},\ldots,\gamma^{(p)}}(v) = \begin{cases} s(v) & \text{if } \deg v = 3 \\ 1 - c(v,v') - c(v,v'') & \text{if } \deg v = 2, v \sim v', v \sim v'', v' \neq v'' \\ \gamma_s^{(j)} & \text{if } v = v_s^j, v \sim v' \text{ and } c(v,v') = 0 \\ \gamma_s^{(j)} - 1 & \text{if } v = v_s^j, v \sim v' \text{ and } c(v,v') = 1 \end{cases} . \tag{4.1}$$

**Lemma 4.3** (Flow of honeycomb)**.** *For any $G \in \mathcal{G}_{\mathcal{T}}$, $\mathcal{L}$ is injective and for $\gamma^{(1)}, \dots, \gamma^{(p)} \in \mathcal{H}_{reg}$,*

$$\mathcal{L}(\mathtt{HONEY}^G_{\mathcal{T}}(\gamma^{(1)}, \dots, \gamma^{(p)})) = \mathcal{F}_G(\phi_{\gamma^{(1)},\dots,\gamma^{(p)}}) \cap (K_{G_1} \times \cdots \times K_{G_m}) \ ,$$

*where $t(G) = (G_1, \dots, G_m)$ and where $\Omega^1(G)$ is identified with $\prod_{i=1}^m \Omega^1(G_i)$.*

We then write $K_G = K_{G_1} \times \cdots \times K_{G_m}$.

*Proof.* Let $G = (V, E) \in \mathcal{G}_T$. First, by Lemma 4.1, there is a bijection

$$i : \mathtt{HONEY}^G_{\mathcal{T}} \to \left\{ (h^1, \dots, h^m) \in \mathtt{HONEY}^{G_1}_{T^1} \times \dots \times \mathtt{HONEY}^{G_m}_{T^m} \mid h^{r_j^{(1)}}_{|B_j} = h^{r_j^{(2)}}_{|B_j}, 1 \leq j \leq N(\mathcal{T}) \right\}.$$

Next, recall that

$$V = \bigcup_{i=1}^m V_i \,/\, \left\langle \partial_{\ell_j} G^{r_j^{(1)}} = \partial_{\ell'_j} G^{r_j^{(2)}}, 1 \leq j \leq N(\mathcal{T}) \right\rangle$$

and that $E$ is the image of $\bigcup_{i=1}^m E_i$ in this quotient. Since any element of $\bigcup_{i=1}^m E_i$ has at most one endpoint in $\bigcup_{j=1}^{N(\mathcal{T})} \partial_{\ell_j} G^{r_j^{(1)}} \cup \partial_{\ell'_j} G^{r_j^{(2)}}$, there is a canonical identification $\Omega^1(G) = \prod_{j=1}^m \Omega(G^j)$. Moreover, by the definition of $\mathcal{L} : \mathtt{HONEY}^{\mathcal{T}}_G \to \Omega^1(G)$ from Definition 4.2, for $h \in \mathtt{HONEY}^{\mathcal{T}}_G$

$$\mathcal{L}[h] = \prod_{i=1}^m \mathcal{L}_{|T_i}[h_{|T_i}] = \left( \prod_{i=1}^m \mathcal{L}_{|T_i} \right) (i(h))$$

under the previous identification. The injectivity of $i$ together with the injectivity of each map $\mathcal{L}_{|T_i}$ from Proposition 3.3 yield the injectivity of $\mathcal{L}$.

It remains to describe the image $\mathcal{L}\left(\mathtt{HONEY}^G_{\mathcal{T}}(\gamma^{(1)}, \dots, \gamma^{(p)})\right)$. By Lemma 4.1 and the previous reasoning, this amounts to describe the image throught $\prod_{i=1}^m \mathcal{L}_{|T_i}$ of the set

$$\left\{ (h^1, \dots, h^m) \in \mathtt{HONEY}^{G_1}_{T^1} \times \dots \times \mathtt{HONEY}^{G_m}_{T^m} \mid h^{r_j^{(1)}}_{|B_j} = h^{r_j^{(2)}}_{|B_j}, 1 \leq j \leq N(\mathcal{T}) \right\}.$$

Let us first describe how the condition $\partial_{\ell_j} h^{s_j} = \gamma^{(j)}$ for $1 \leq j \leq p$ translates through $\mathcal{L}$. Let $v^j_m \in \partial_{\ell_j} G_{s_j}$ and $x \in \partial_{\ell_j} h^{s_j}$ the point such that $\iota(v^j_m) = x$. Since $h^{s_j} \in \mathtt{HONEY}_{G_{s_j}}$, like in Section 3.1, the condition $x_{\ell_j} = \gamma^{(j)}_m$ is equivalent to the condition

$$\mathcal{L}[h](v^j_m, v) = \gamma^{(j)}_m - c(v^j_m, v) = \phi_{\gamma^{(1)},\dots,\gamma^{(p)}}(v) \ ,$$

where $v$ is the unique element of $V_{s_j}$ such that $v \sim v^j_m$.

Let us now consider the condition $h^{r_j^{(1)}}_{|B_j} = h^{r_j^{(2)}}_{|B_j}$. Let $v \in \partial_{\ell_j^{(1)}} G_{r_j^{(1)}}$ and $v' \in \partial_{\ell_j^{(2)}} G_{r_j^{(2)}}$ such that $v \sim v'$ in $G$, and let $x = \iota_{T^{r_j^{(1)}}}(v), x' = \iota_{T^{r_j^{(2)}}}(v')$ the corresponding image in $h_{|T^{r_j^{(1)}}}$ and $h_{|T^{r_j^{(2)}}}$. Using that the edges of both triangles are identified in the order-reversing way, the condition $h^{r_j^{(1)}}_{|B_j} = h^{r_j^{(2)}}_{|B_j}$ implies that

$$x_{\ell_j^{(1)} - 1} = 1 - x'_{\ell_j^{(2)} - 1} \ .$$

Let $w \in V_{r^{(1)}}$ and $w' \in V_{r^{(2)}}$ such that $v \sim_{G_{r^{(1)}}} w$ and $v' \sim_{G_{r^{(2)}}} w'$. Then, following (3.2), the previous equality is equivalent to $\mathcal{L}[h^{r^{(1)}}](v, w) + c(v, w) = 1 - \mathcal{L}[h^{r^{(1)}}](v, w) - c(v', w')$, which yields

$$\mathcal{L}[h](\bar{v}, w) + \mathcal{L}[h](\bar{v}, w') = 1 - c(v, w) - c(v', w') = \phi_{\gamma^{(1)},\dots,\gamma^{(p)}}(v) \ .$$

□

In order to state the volume formula for $(g, p)$ honeycombs, let us introduce for $r \in [0, 1)$ the notation

$$\mathcal{H}^r_{reg} = \left\{ \gamma \in \mathcal{H}_{reg} \mid \sum_{i=1}^n \gamma_i = r \mod \mathbb{Z} \right\}.$$

This set is a union of affine polytopes of $\mathcal{H}_{reg}$ of dimension $n-1$, and one can check that the volume $d_J u$ on each affine polytope induced by the projection on $\mathbb{R}^J$ is independent of $J$ for any $J \subset \{1, \dots, n\}$ of cardinal $n-1$. We simply denote by $\mathrm{d}u$ this volume form.

**Proposition 4.4.** *Suppose that $\mathcal{T}'$ is obtained by gluing $\mathcal{T}$ and $T^{m+1}$ along the boundaries $L_p$ and $\partial_0 T^{m+1}$. Then, for all $G \in \mathcal{T}'$ and $\gamma^1, \ldots, \gamma^{p+1} \in \mathcal{H}_{reg}$ such that $\sum_{i=1}^{p+1} |\gamma_i| \in \mathbb{N}$,*

$$\sum_{G \in \mathcal{G}_{\mathcal{T}}} \mathrm{Vol}\left[\mathcal{L}\left(\mathtt{HONEY}^{G}_{\mathcal{T}'}(\gamma^1, \ldots, \gamma^{p+1})\right)\right]$$
$$= \int_{\mathcal{H}^{\theta}_{reg}} \sum_{(G_1,G_2) \in \mathcal{G}_{\mathcal{T}} \times \mathcal{G}_{\mathcal{T}}} \mathrm{Vol}\left[\mathcal{L}\left(\mathtt{HONEY}^{G_2}_{\mathcal{T}}(\gamma^1, \ldots, \gamma^{p-1}, u)\right)\right] \mathrm{Vol}\left[\mathcal{L}\left(\mathtt{HONEY}^{G_1}(\tilde{u}, \gamma^p, \gamma^{p+1})\right)\right] \mathrm{d}u,$$

*where $\tilde{u} = (1-u_n, \ldots, 1-u_1)$ and $\theta = -\sum_{i=1}^{p-1} |\gamma^i| \mod \mathbb{Z}$.*

*Proof.* Remark that from (4.1),

$$\sum_{v \in V} \phi_{\gamma^{(1)},\ldots,\gamma^{(p)}}(v) = \sum_{j=1}^{p} \sum_{i=1}^{n} \gamma_s^{(j)} + \sum_{i=1}^{m} \left( \sum_{v \in \mathrm{int}(G_i)} s(v) - \sum_{v \in \partial G_i, v \sim v'} c(v,v') \right).$$

Hence, in order for $\phi_{\gamma^{(1)},\ldots,\gamma^{(p)}}$ to yields a non-empty set $\mathcal{F}_G(\phi_{\gamma^{(1)},\ldots,\gamma^{(p)}})$, by Proposition 2.1 it is necessary that $\sum_{i=1}^{p} |\gamma^{(i)}| \in \mathbb{N}$. Hence, $\mathtt{HONEY}^{G_2}_{\mathcal{T}}(\gamma^1, \ldots, \gamma^{p-1}, u)$ is non-empty if and only if $u \in \mathcal{H}^{\theta}_{reg}$ with $\theta - \sum_{i=1}^{p-1} |\gamma^i| \mod \mathbb{Z}$. Using Proposition 2.4 (2) gives

$$\int_{\mathcal{H}^{\theta}_{reg}} \mathrm{Vol}\left[\mathcal{L}\left(\mathtt{HONEY}^{G_2}_{\mathcal{T}}(\gamma^1, \ldots, \gamma^{p-1}, u)\right)\right] \mathrm{Vol}\left[\mathcal{L}\left(\mathtt{HONEY}^{G_1}(\tilde{u}, \gamma^p, \gamma^{p+1})\right)\right] \mathrm{d}u$$
$$= \mathrm{Vol}\left[(K_{G_1} \times K_{G_2}) \cap \mathcal{F}_{(G_1 \cup G_2)_{S_1 * S_2}}\left(\phi_{\gamma^{(1)},\ldots,\gamma^{(p+1)}}\right)\right].$$

By Lemma 4.1 and Lemma 4.3, the latter volume is non-zero only if $(G_1, G_2) = t(G)$ where $G = (G_1 \cup G_2)_{S_1 * S_2} \in \mathcal{G}_{\mathcal{T}'}$ and where $S_1 \subset \partial G_1$ and $S_2 \subset \partial G_2$ are the vertices identified together on $L_p$ and $\partial_0 T^{m+1}$. In this case, it is equal to

$$\mathbb{1}_{t(G)=(G_1,G_2)} \mathrm{Vol}\left[\mathcal{L}\left(\mathtt{HONEY}^{G}_{\mathcal{T}}\left(\gamma^{(1)}, \ldots, \gamma^{(p+1)}\right)\right)\right].$$

Therefore,

$$\sum_{(G_1,G_2) \in \mathcal{G}_{\mathcal{T}} \times \mathcal{G}_{\mathcal{T}}} \int_{\mathcal{H}^{\theta}_{reg}} \mathrm{Vol}\left[\mathcal{L}\left(\mathtt{HONEY}^{G_2}_{\mathcal{T}}(\gamma^1, \ldots, \gamma^{p-1}, u)\right)\right] \mathrm{Vol}\left[\mathcal{L}\left(\mathtt{HONEY}^{G_1}\left(\tilde{u}, \gamma^p, \gamma^{p+1}\right)\right)\right] \mathrm{d}u$$
$$= \sum_{G \in \mathcal{G}_{\mathcal{T}'}, t(G)=(G_1,G_2)} \int_{\mathcal{H}^{\theta}_{reg}} \mathrm{Vol}\left[\mathcal{L}\left(\mathtt{HONEY}^{G_2}_{\mathcal{T}}\left(\gamma^1, \ldots, \gamma^{p-1}, u\right)\right)\right] \mathrm{Vol}\left[\mathcal{L}\left(\mathtt{HONEY}^{G_1}\left(\tilde{u}, \gamma^p, \gamma^{p+1}\right)\right)\right] \mathrm{d}u$$
$$= \sum_{G \in \mathcal{G}_{\mathcal{T}'}} \mathrm{Vol}\left[\mathcal{L}\left(\mathtt{HONEY}^{G}_{\mathcal{T}}\left(\gamma^{(1)}, \ldots, \gamma^{(p+1)}\right)\right)\right].$$

□

A similar reasoning using Proposition 2.5 yields the following proposition.

**Proposition 4.5.** *Suppose that $\mathcal{T}'$ is obtained by gluing two boundaries of $\mathcal{T}$. Then, for each $G \in \mathcal{G}_{\mathcal{T}'}$, $\mathtt{HONEY}^{G}_{\mathcal{T}'}$ admits a volume form with, for $\gamma^1, \ldots, \gamma^{p-2} \in \mathcal{H}_{reg}$ with $\sum_{i=1}^{p-2} |\gamma^i| \in \mathbb{Z}$,*

$$\sum_{G \in \mathcal{G}_{\mathcal{T}'}} \mathrm{Vol}\left[\mathcal{L}\left(\mathtt{HONEY}^{G}_{\mathcal{T}'}\left(\gamma^{(1)}, \ldots, \gamma^{(p-2)}\right)\right)\right] = \int_{\mathcal{H}_{reg}} \sum_{G \in \mathcal{G}_{\mathcal{T}}} \mathrm{Vol}\left[\mathcal{L}\left(\mathtt{HONEY}^{G}_{\mathcal{T}}\left(\gamma^{(1)}, \ldots, \gamma^{(p-2)}, u, \tilde{u}\right)\right)\right] \mathrm{d}u.$$

## 5. Proof of Theorem 1.4 : volume of flat U($n$)-connections on a compact surface

The goal of this section is the proof of Theorem 1.4, which gives a volume expression for the volume $M_{g,n}(\alpha_1, \ldots, \alpha_p)$ of flat $SU(n)$–connection on surface $\mathcal{M}$ of genus $g$ and $p$ boundary components for $\alpha_1, \ldots, \alpha_p \in \mathcal{H}_{reg}$.

5.1. **Parametrizations of conjugacy classes and volume form.** Let us consider the standard parametrization of conjugacy classes in $\mathrm{U}(n)$ given by

$$\mathcal{A} = \{t_1 \geq \ldots \geq t_n, \sum_{i=1}^{n} t_i = 0, t_1 - t_n \leq 1\}.$$

The set $\mathcal{A}$ is called an alcove of type $A_{n-1}$. Remark that $\mathcal{A}$ is a polytope of dimension $n-1$ in $\mathbb{R}^n$, and for any $R \subset \{1, \ldots, n\}$ of cardinal $n-1$, the projection $p_R : \mathcal{A} \to \mathbb{R}^R$ yields a non-zero volume form $p_R^* d\ell_{\mathbb{R}^R}$ on $\mathcal{A}$. This volume form is again independent of $R$ and denoted by $dt$ in the sequel. Choosing for example $R = \{1, \ldots n-1\}$, we have

$$\begin{aligned} Vol(\mathcal{A}) &= \int_{\mathbb{R}^{n-1}} \mathbf{1}_{t_1 \geq \ldots \geq t_{n-1}, t_1 + \sum_{i=1}^{n-1} t_i \leq 1} \prod_{i=1}^{n-1} dt_i \\ &= \int_{\mathbb{R}^{n-1}} \mathbf{1}_{1 \geq u_1 \geq \ldots \geq u_{n-1} \geq 0} \frac{1}{n} \prod_{i=1}^{n-1} du_i = \frac{1}{(n-1)!}, \end{aligned}$$

where we did the change of variable $\phi : (t_i)_{1 \leq i \leq n-1} \mapsto (t_i + \sum_{i=1}^{n-1} t_i)_{1 \leq i \leq n}$ with $Jac(\phi(t)) = n$.

Since $\mathcal{H}^0$ and $\mathcal{A}$ are both parametrizing conjugacy classes of $SU(n)$, there is a natural bijection $\phi : \mathcal{A} \to \mathcal{H}^0$ whose value on $(t_1 \geq \ldots \geq t_i \geq 0 > t_{i+1} \geq \cdots \geq t_n)$ is

$$\phi(t_1, \ldots, t_n) = (1 - t_{i+1}, \ldots, 1 - t_n, t_1, \ldots, t_i). \tag{5.1}$$

One has $Jac(\phi(t)) = 1$ for all $t \in \mathcal{A}$, and thus $\phi$ is volume preserving. One then checks that $\phi^* d\theta = dt$.

5.1.1. *Contraction formula on moduli spaces of flat connections.* Let $\widehat{\mathcal{M}}$ be a (possibly disconnected) oriented surface with $(p+2)$ boundary components $L_1, \ldots, L_p, L_{p+1}, L_{p+2}$. Let $\mathcal{M}$ be the oriented surface obtained by gluing $L_{p+1}$ and $L_{p+2}$ in an orientation reversing way and suppose that $\mathcal{M}$ is connected. Then, the following formula holds for the corresponding volume of flat $SU(n)$-connection.

**Theorem 5.1** ([MW99, Prop. 5.4])**.** *Suppose that $\alpha_1, \ldots, \alpha_p \in \mathcal{H}^0_{reg}$. If $M(\mathcal{M}, \alpha_1, \ldots, \alpha_p)$ contains at least one connection whose stabilizer is $Z(SU(n))$, then*

$$\mathrm{Vol}(M(\mathcal{M}, \alpha_1, \ldots, \alpha_p)) = \frac{1}{k} \int_{\mathcal{A}} \mathrm{Vol}(M(\widehat{\mathcal{M}}, \alpha_1, \ldots, \alpha_p, \phi(t), \phi(-t))) \tilde{d}t,$$

*where $k = 1$ if $\widehat{\mathcal{M}}$ is connected and $\#Z(SU(n)) = n$ otherwise, and $\tilde{d}$ is the unique volume form on $\{(t_1, \ldots, t_n), \sum_{i=1}^{n} t_i = 0\}$ whose volume $\widetilde{\mathrm{Vol}}$ satisfies*

$$\widetilde{\mathrm{Vol}}\left(\left\{(t_1, \ldots, t_n), \sum_{i=1}^{n} t_i = 0, \max_{1 \leq i,j \leq 1} (t_i - t_j) \leq 1\right\}\right) = 1.$$

Let us first remark that considering $U(n)$-valued connection instead of $SU(n)-$valued connection does not change the volume. Indeed, suppose that $\mathcal{M}$ corresponds to a surface of genus $g$ with $p$ boundary components. Then, for any $\alpha_1, \ldots, \alpha_p \in \mathcal{H}_{reg}$,

$$\begin{aligned} &M_{U(n)}(\mathcal{M}, \alpha_1, \ldots, \alpha_p) \\ &\simeq \left\{((U_i)_{1 \leq i \leq 2g}, C_1, \ldots, C_p) \in U(n)^{2g} \times \mathcal{O}_{\alpha_1} \times \cdots \times \mathcal{O}_{\alpha_p} \middle| \prod_{i=1}^{g} [U_{2i-1}, U_{2i}] = \prod_{i=1}^{p} C_i \right\} / U(n), \end{aligned}$$

where $U(n)$ acts diagonally by conjugation. Since $\det \prod_{i=1}^{g} [U_{2i-1}, U_{2i}] = 1$, the latter set is non-empty only if $\sum_{i=1}^{p} |\alpha_i| \in \mathbb{N}$.

Next, remark that any conjugacy class of $SU(n)$ is also a conjugacy class of $U(n)$ and there is a natural action of $\mathbb{R}$ on $\mathcal{H}_{reg}$ given by

$$t \cdot (\theta_1 > \ldots > \theta_n) = std(\theta_i + t \mod \mathbb{Z}),$$

where $std(x_1, \ldots, x_n)$ denotes the standardization $(x_{i_1} > x_{i_2} > \ldots > x_{i_n})$. For $\alpha \in \mathcal{H}_{reg}$, set $t(\alpha) = |\alpha| \mod \mathbb{Z}$ and $\hat{\alpha} = (-t_\alpha / n) \cdot \alpha$, so that $\hat{\alpha} \in \mathcal{H}^0_{reg}$.

Since $Z(U(n))$ acts trivially by conjugation, when $\alpha_1, \ldots \alpha_p \in \mathcal{H}_{reg}$ we have

$$\begin{aligned}
&M_{U(n)}(\mathcal{M}, \alpha_1, \ldots, \alpha_p) \\
&\simeq \left\{ ((U_i)_{1\leq i\leq 2g}, C_1, \ldots, C_p) \in U(n)^{2g} \times \mathcal{O}_{\alpha_1} \times \cdots \times \mathcal{O}_{\alpha_p} \middle| \prod_{i=1}^{g} [U_{2i-1}, U_{2i}] = \prod_{i=1}^{p} C_i \right\} / SU(n) \\
&\simeq \mathbb{T}^{2g} \times \left\{ ((U_i)_{1\leq i\leq 2g}, C_1, \ldots, C_p) \in SU(n)^{2g} \times \mathcal{O}_{\hat{\alpha}_1} \times \cdots \times \mathcal{O}_{\hat{\alpha}_p} \middle| \prod_{i=1}^{g} [U_{2i-1}, U_{2i}] = \prod_{i=1}^{p} C_i \right\} / SU(n) \\
&\simeq \mathbb{T}^{2g} \times M_{\mathrm{SU}(n)}(\mathcal{M}, \hat{\alpha}_1, \ldots, \hat{\alpha}_p),
\end{aligned}$$

so that with the convention that $\mathrm{Vol}(\mathbb{T}) = 1$,

$$\mathrm{Vol}(M_{U(n)}(\mathcal{M}, \alpha_1, \ldots, \alpha_p)) = \mathrm{Vol}(M_{SU(n)}(\mathcal{M}, \hat{\alpha}_1, \ldots, \hat{\alpha}_p)). \tag{5.2}$$

We deduce then from this and the previous theorem the following proposition.

**Proposition 5.2.** *Suppose that $\alpha_1, \ldots, \alpha_p \in \mathcal{H}^0_{reg}$. If either $p \geq 3$, $p = 1$ and $g(\mathcal{M}) = 1$ or $g(\mathcal{M}) \geq 2$, then, if $\widehat{\mathcal{M}}$ is disconnected,*

$$\mathrm{Vol}(M(\mathcal{M}, \alpha_1, \ldots, \alpha_p)) = \frac{1}{n} \int_{\mathcal{H}^0_{reg}} \mathrm{Vol}(M(\widehat{\mathcal{M}}, \alpha_1, \ldots, \alpha_p, \theta, \tilde{\theta}))d\theta,$$

*and, if $\widehat{\mathcal{M}}$ is connected,*

$$\mathrm{Vol}(M(\mathcal{M}, \alpha_1, \ldots, \alpha_p)) = \int_{\mathcal{H}_{reg}} \mathrm{Vol}(M(\widehat{\mathcal{M}}, \alpha_1, \ldots, \alpha_p, \theta, \tilde{\theta}))d\theta,$$

*Proof.* First, by [BL, Thm. 5.20], $M_{g,n}(\alpha_1, \ldots, \alpha_p)$ contains at least one element for which the stabiliser under the diagonal action of $SU(n)$ is $Z(SU(n))$ if $\alpha_1, \ldots, \alpha_p \in \mathcal{H}^0_{reg}$ and either $p \geq 3$, $p = 1$ and $g(\mathcal{M}) = 1$ or $g(\mathcal{M}) \geq 2$.

Remark that the volume form $dt$ on $\mathcal{A}$ introduced in the previous subsection is such that

$$\mathrm{Vol}\left( \left\{ (t_1, \ldots, t_n) \in \mathbb{R}^n, \sum_{i=1}^{n} t_i = 0, \max_{1\leq i,j\leq n} t_i - t_j < 1 \right\} \right) = 1,$$

that $\tilde{d} = d$. By (5.1), $\phi(-t) = \widetilde{\phi(t)}$ and $Jac(\phi(t)) = 1$ for all $t \in \mathcal{A}$. Hence, doing the change of variable $\theta = \phi(t)$ yields

$$\mathrm{Vol}(M(\mathcal{M}, \alpha_1, \ldots, \alpha_p)) = \frac{1}{k} \int_{\mathcal{H}^0_{reg}} \mathrm{Vol}(M(\widehat{\mathcal{M}}, \alpha_1, \ldots, \alpha_p, \theta, \tilde{\theta}))d\theta,$$

where $k = 1$ if $\widehat{\mathcal{M}}$ is connected and $k = n$ otherwise.

In remains to replace the integration on $\mathcal{H}^0$ by the integration on $\mathcal{H}$ in the case where $\widehat{\mathcal{M}}$ is connected. For all $t \in [0, 1]$, $\alpha_i \in \mathcal{H}^0_{reg}$, $1 \leq i \leq p$, and $\theta \in \mathcal{H}^0_{reg}$, by (5.2)

$$\mathrm{Vol}(M_{U(n)}(\widehat{\mathcal{M}}, \alpha_1, \ldots, \alpha_p, t\cdot\theta, \widetilde{t\cdot\theta})) = \mathrm{Vol}(M_{SU(n)}(\widehat{\mathcal{M}}, \alpha_1, \ldots, \alpha_p, \theta, \tilde{\theta}))$$

for any $t \in [0, 1/n[$. Therefore, if $\hat{\mathcal{M}}$ is connected,

$$\begin{aligned}
\frac{1}{k} \int_{\mathcal{H}^0_{reg}} \mathrm{Vol}(M(\widehat{\mathcal{M}}, \alpha_1, \ldots, \alpha_p, \theta, \tilde{\theta}))d\theta &= n \int_0^{1/n} \left( \int_{\mathcal{H}^0_{reg}} \mathrm{Vol}(M_{U(n)}(\widehat{\mathcal{M}}, \alpha_1, \ldots, \alpha_p, t\cdot\theta, \widetilde{t\cdot\theta}))d\theta \right) dt \\
&= \int_{\mathcal{H}_{reg}} \mathrm{Vol}(M_{U(n)}(\widehat{\mathcal{M}}, \alpha_1, \ldots, \alpha_p, u, \tilde{u}))du,
\end{aligned}$$

where we use that the change of variable $(u_1, \ldots, u_n) = (\theta_1 + t, \ldots, \theta_{n-1} + t, -\sum_{i=1}^{n-1} \theta_i + t) =: \phi(\theta_1, \ldots, \theta_{n-1}, t)$ yields $Jac(\phi) = n$. □

The proof of Theorem 1.4 is then a deduction of the previous results and the previous construction on differential structures.

*Proof of Theorem 1.4.* The proof is done by recursion $N = 3g + p$, where $N \geq 3$. If $N = 3$, the result is given by Theorem 3.15. Suppose $N > 3$ and let $S$ be a surface of genus $g$ with $p$ points removed, where $3g + p = N$. Let $\mathcal{T}$ be a surface constructed in Section **??**. Then $\mathcal{T}$ is obtained either by gluing two edges of a connected surface $\mathcal{T}'$ or by gluing one edge of a connected surface $\mathcal{T}'$ to the edge of an equilateral triangle $T$.

In the first case $\mathcal{T}'$ is a flat surface associated to a surface $\mathcal{M}'$ with genus $g-1$ and $p+2$ points removed. Let $\alpha_1, \dots, \alpha_p \in \mathcal{H}_{reg}$ with $\sum_{i=1}^{p} |\alpha_i| \in \mathbb{N}$. Then, by applying (5.2) and Theorem 5.2, we have

$$\begin{aligned} Z_{g,p}(\alpha_1, \dots, \alpha_p) = Z_{g,p}(\hat{\alpha}_1, \dots, \hat{\alpha}_p) &= \int_{\mathcal{H}_{reg}} \mathrm{Vol}(M(\mathcal{M}', \hat{\alpha}_1, \dots, \hat{\alpha}_p, \theta, \tilde{\theta}))d\theta. \\ &= \int_{\mathcal{H}_{reg}} \mathrm{Vol}(M(\mathcal{M}', \alpha_1, \dots, \alpha_p, \theta, \tilde{\theta}))d\theta. \end{aligned}$$

Since $3(g-1) + p + 2 < N$, by induction

$$\mathrm{Vol}(M(\mathcal{M}', \alpha_1, \dots, \alpha_p, \theta, \tilde{\theta})) = c_{g-1,p+2} \sum_{G \in \mathcal{G}^{(g,p)}} \mathrm{Vol}\left[\mathtt{HONEY}^G(\alpha_1, \dots, \alpha_p, \theta, \tilde{\theta})\right],$$

and thus by Proposition 4.5,

$$\begin{aligned} Z_{g,p}(\alpha_1, \dots, \alpha_p) =& c_{g-1,p+2} \int_{\mathcal{H}_{reg}} \sum_{G \in \mathcal{G}^{(g-1,p+2)}} \mathrm{Vol}\left[\mathtt{HONEY}^G(\alpha_1, \dots, \alpha_p, \theta, \tilde{\theta})\right] d\theta \\ =& c_{g-1,p+2} \sum_{G \in \mathcal{G}^{(g,p)}} \mathrm{Vol}\left[\mathtt{HONEY}^G(\alpha_1, \dots, \alpha_p)\right]. \end{aligned}$$

In the second case, $\mathcal{T}'$ is obtained by gluing a a surface $\mathcal{M}'$ with genus $g$ and $p-1$ points removed and a triangle $T$. Let $\widehat{M} = \mathcal{M}' \cup T$ be the corresponding disconnected surface. Then, by (5.2) and Proposition 5.2,

$$\begin{aligned} Z_{g,p}(\alpha_1, \dots, \alpha_p) =& Z_{g,p}(\hat{\alpha}_1, \dots, \hat{\alpha}_p) \\ =& \frac{1}{n} \int_{\mathcal{H}^0_{reg}} \mathrm{Vol}(M(\mathcal{M}', \hat{\alpha}_1, \dots, \hat{\alpha}_{p-2}, \theta)) \mathrm{Vol}(M(T, -\theta, \hat{\alpha}_{p-1}, \hat{\alpha}_p))d\theta. \end{aligned}$$

Set $s = -\sum_{i=1}^{p-2} |\alpha|_i$. By (5.2),

$$\mathrm{Vol}(M(\mathcal{M}', \hat{\alpha}_1, \dots, \hat{\alpha}_{p-2}, \theta)) = \mathrm{Vol}(M(\mathcal{M}', \alpha_1, \dots, \alpha_{p-2}, \theta + s))$$

and

$$\mathrm{Vol}(M(T, \theta, \hat{\alpha}_{p-1}, \hat{\alpha}_p)) = \mathrm{Vol}(M(T, -\theta - s, \alpha_{p-1}, \alpha_p)).$$

Since the map $\theta \mapsto \theta - s$ is volume preserving,

$$\begin{aligned} &\int_{\mathcal{H}^0_{reg}} \mathrm{Vol}(M(\mathcal{M}', \hat{\alpha}_1, \dots, \hat{\alpha}_{p-2}, \theta)) \mathrm{Vol}(M(T, -\theta, \hat{\alpha}_{p-1}, \hat{\alpha}_p))d\theta \\ &\qquad = \int_{\mathcal{H}^s_{reg}} \mathrm{Vol}(M(\mathcal{M}', \alpha_1, \dots, \alpha_{p-2}, \theta)) \mathrm{Vol}(M(T, -\theta, \alpha_{p-1}, \alpha_p))d\theta. \end{aligned}$$

By induction,

$$\mathrm{Vol}(M(\mathcal{M}', \alpha_1, \dots, \alpha_{p-2}, \theta, )) = c_{g,p-1} \sum_{G \in \mathcal{G}^{(g,p-1)}} \mathrm{Vol}\left[\mathtt{HONEY}^G(\alpha_1, \dots, \alpha_{p-2}, \theta)\right],$$

and

$$\mathrm{Vol}(M(T, \theta, \alpha_{p-1}, \alpha_p)) = c_{0,3} \sum_{G \in \mathcal{G}^{(g,p-1)}} \mathrm{Vol}\left[\mathtt{HONEY}^G(\theta, \dots, \alpha_{p-1}, \alpha_p)\right],$$

and thus, by Proposition 4.4,

$$\begin{aligned}
&Z_{g,p}(\alpha_1,\dots,\alpha_p)\\
=&c_{g,p-1}c_{0,3}\frac{1}{n}\int_{\mathcal{H}^s_{reg}}\sum_{G_1\in\mathcal{G}^{(g,p-1)},G_2\in\mathcal{G}^{(0,3)}}\mathrm{Vol}\left[\mathtt{HONEY}^{G_1}(\alpha_1,\dots,\alpha_{p-2},\theta)\right]\mathrm{Vol}\left[\mathtt{HONEY}^{G_2}(\tilde\theta,\alpha_{p-1},\alpha_p)\right]d\theta\\
=&c_{g,p}\sum_{G\in\mathcal{G}^{(g,p)}}\mathrm{Vol}\left[\mathtt{HONEY}^{G}(\alpha_1,\dots,\alpha_p)\right],
\end{aligned}$$

with $c_{g,p}=\frac{c_{g,p-1}c_{0,3}}{n}$. □

## 6. Yang-Mills partition function on compact oriented surfaces

The goal of this section is to give an explicit volume formula for the marginal Yang–Mills partition function of an oriented surface of genus $g$ with prescribed non-degenerated holonomies (up to conjugation) on a finite set of disjoint loops. As it is proven in XXXRefXXX, the corresponding partition function then only depends on the prescribed conjugacy classes and on the areas of each connected components delimites by the loops.

**Definition 6.1.** A disjoint loops configuration $\mathcal{L}=(S,\Gamma_1,\dots,\Gamma_p)$ is the data of a compact oriented surface together with $p$ disjoints Jordan curves $\Gamma_1,\dots,\Gamma_p$ on $S$ and for each $\Gamma_i$, $1\le i\le p$ an element $\alpha_i\in\mathcal{H}_{reg}$.

A skeleton is the data of a labeled finite tree $T=(V,E)$ such vertices are labeled by $\mathbb{N}\times\mathbb{R}^+$ and edges are labeled by $\mathcal{H}_{reg}$.

To each disjoint loops configuration $\mathcal{L}$, one associates a skeleton $T(\mathcal{L})$ as follows :

- the set $V$ of vertices of $T(\mathcal{L})$ is the set of connected components of $S\setminus\bigcup_{i=1}^p\Gamma_i$. Each vertex $v\in V$ is labeled $(A_v,g_v)$ where $A_v$ is the area of the corresponding connected component and $g_v$ is its genus.
- for $v_1,v_2\in V$, there is an edge $e$ between $v_1$ and $v_2$ for each boundary component. Since loops of $\mathcal{L}$ are non-intersecting, the each boundary component corresponds to a unique loop $\Gamma_j$ of $\mathcal{L}$, and then we label $\alpha_j$ the edge $e$.

**Definition 6.2** (Fat $(g,p)$-toric honeycomb)**.** Let

In this section, for $T\geqslant 0$ and $x\in\mathrm{SU}(n)$, we denote by $p_T(x)$ the heat kernel on $\mathrm{SU}(n)$.

**Lemma 6.3** (Partition function of a cylinder)**.** *Let $T>0$, $\alpha\in\mathcal{H}$ and let $x\in\mathcal{O}_\alpha$. Then,*

$$Z_{0,2,T}(1,x)=p_T(x).$$

*Proof.* This is a consequence of Proposition 4.2.4 and (5.3) in [Lév03]. □

**Theorem 6.4** (Volume formula for Yang-Mills partition function)**.** *Let $p,g\geqslant 0$ be integers, $(\alpha_1,\dots,\alpha_p)\in\mathcal{H}^p$ and $T>0$. Assume that loops associated to $\alpha_1,\dots,\alpha_p$ enclose respective areas $t_1,\dots,t_p$. Then, the Yang–Mills partition function is given by*

$$Z_{g,p,T}(\alpha_1,\dots,\alpha_p)=\int Z_{g,p,0}(\alpha_1,\dots,\alpha_{p-1},u)\,Z_{0,2,T}(u^{-1},\alpha_p)\prod_{\ell=1}^p p_{t_\ell}(\alpha_\ell)\mathrm{d}u$$

*Proof.* Recall that □

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