

习题 3.1

1. (1)

$$f(x) = x^2, x_0 = 1$$

$$\begin{aligned} f'(x_0) &= \lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{(x_0 + \Delta x)^2 - (x_0)^2}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} (2 + \Delta x) \\ &= 2 \end{aligned}$$

(2)

$$f(x) = \frac{1}{x^2}, x_0 = 2$$

$$\begin{aligned} f'(x_0) &= \lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\frac{1}{(x_0 + \Delta x)^2} - \frac{1}{(x_0)^2}}{\Delta x} \\ &= - \lim_{\Delta x \rightarrow 0} \frac{2x_0 + \Delta x}{x_0^2(x_0 + \Delta x)^2} \\ &= -\frac{2}{x_0^3} \\ &= -\frac{1}{4} \end{aligned}$$

(3)

$$f(x) = x(x+1)\dots(x+2020), x_0 = 0$$

$$\begin{aligned} f'(x_0) &= \lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{(x_0 + \Delta x)(x_0 + 1 + \Delta x)\dots(x_0 + 2020 + \Delta x) - x_0(x_0 + 1)\dots(x_0 + 2020)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{x_0[(x_0 + 1 + \Delta x)\dots(x_0 + 2020 + \Delta x) - x_0(x_0 + 1)\dots(x_0 + 2020)]}{\Delta x} \\ &\quad + \lim_{\Delta x \rightarrow 0} (x_0 + 1 + \Delta x) \dots (x_0 + 2020 + \Delta x) \end{aligned}$$

$$\begin{aligned}
 &= \lim_{\Delta x \rightarrow 0} (x_0 + 1 + \Delta x) \dots (x_0 + 2020 + \Delta x) \\
 &= 2020!
 \end{aligned}$$

2. (1)

$$\begin{aligned}
 f'_-(0) &= \lim_{x \rightarrow 0^-} \frac{f(x) - f(0)}{x - 0} \\
 &= \lim_{x \rightarrow 0^-} \frac{f(x)}{x} \\
 &= +\infty
 \end{aligned}$$

$$\begin{aligned}
 f'_+(0) &= \lim_{x \rightarrow 0^+} \frac{f(x) - f(0)}{x - 0} \\
 &= \lim_{x \rightarrow 0^+} \frac{f(x)}{x} \\
 &= +\infty
 \end{aligned}$$

$\therefore f(x)$ 在 $x=0$ 处不可导

(2)

$$\begin{aligned}
 f'_-(0) &= \lim_{x \rightarrow 0^-} \frac{f(x) - f(0)}{x - 0} \\
 &= \lim_{x \rightarrow 0^-} \frac{f(x) - 1}{x} \\
 &= \lim_{x \rightarrow 0^-} x^2 \\
 &= 0
 \end{aligned}$$

$$\begin{aligned}
 f'_+(0) &= \lim_{x \rightarrow 0^+} \frac{f(x) - f(0)}{x - 0} \\
 &= \lim_{x \rightarrow 0^+} \frac{f(x) - 1}{x} \\
 &= \lim_{x \rightarrow 0^+} x \\
 &= 0
 \end{aligned}$$

$$\therefore f'(0) = f'_+(0) = f'_-(0) = 0$$

$\therefore f(x)$ 在 $x=0$ 处可导

3. (1)

$$\because y'|_{x=0} = e^x|_{x=0} = 1$$

$$\therefore k_{\text{切}}=1, k_{\text{法}}=-1$$

$$L_{\text{切}}: y = x+1$$

$$L_{\text{法}}: y = -x+1$$

(2)

设 $P(x_0, \ln x_0)$, 则 $y|_{x=x_0} = \frac{1}{x}$ 令 $\frac{1}{x_0} = \frac{1}{2}$, 解得 $x_0=2$, 即 $P(2, \ln 2)$.

4. 在 $x=1$ 处可导 $\Rightarrow f(x)$ 在 $x=1$ 处连续 $\Rightarrow \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^+} f(x) = f(1)$,

在 $x=1$ 处可导 \Rightarrow 左右导数存在且相等, $\lim_{x \rightarrow 1^-} f'(x) = \lim_{x \rightarrow 1^+} f'(x)$

$$\text{即 } \lim_{x \rightarrow 1^-} 2x = \lim_{x \rightarrow 1^+} a \quad \textcircled{2}$$

解①②得: $a=2, b=-1$

$$\text{5.证明: 左边} = \lim_{h \rightarrow \infty} \frac{f(x_0+h) - f(x_0) + f(x_0) - f(x_0-h)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{f(x_0+h) - f(x_0)}{(x_0+h) - x_0} + \lim_{h \rightarrow 0} \frac{f(x_0) - f(x_0-h)}{x_0 - (x_0-h)}$$

$$= 2f'(x_0)$$

= 右边

6.证明: ①偶函数满足: $f(x) = f(-x)$

两边同时求导: $f'(x) = -f'(-x)$

即偶函数导数为奇函数;

②奇函数满足: $-f(x) = -f(-x)$

两边同时求导: $-f'(x) = -f'(-x)$

$$\Rightarrow f'(x) = f'(-x)$$

即奇函数的导数为偶函数;

③周期函数满足: $f(x) = f(x+T)$

两边同时求导: $f'(x) = f'(x+T)$

即周期函数的导数为周期函数。

$$7. \text{解: } f(x) = \begin{cases} -1, & -1 \leq x < 0 \\ 0, & 0 \leq x < 1 \\ 1, & x = 1 \end{cases}$$

① $f'_{-}(0) = \lim_{\Delta x \rightarrow 0-} \frac{f(0+\Delta x) - f(0)}{\Delta x} = \lim_{\Delta x \rightarrow 0-} \left(-\frac{1}{\Delta x}\right) = +\infty$; (其为函数在 $x=0$ 点的左导数)

$$\text{② } f'_{+}(0) = \lim_{\Delta x \rightarrow 0+} \frac{f(0+\Delta x) - f(0)}{\Delta x} = 0;$$

③ 因 $f'_{+}(0) \neq f'_{-}(0)$, 故 $f'(0)$ 不存在;

④ $\lim_{x \rightarrow 0} f'(x) = \lim_{x \rightarrow 0-} f'(x) = \lim_{x \rightarrow 0+} f'(x) = 0$ (其为在 x 趋向于 0 时函数的导数值)。

8. 解: $|f(0)| \leq 1 - \cos 0 = 0$, 即 $f(0) = 0$

① 如果要证明连续性: $\cos x - 1 \leq f(x) \leq 1 - \cos x$

$$\text{因 } \lim_{x \rightarrow 0-} (\cos 0 - 1) = \lim_{x \rightarrow 0+} (1 - \cos 0) = 0 = f(0)$$

则 $f(x)$ 在 $x=0$ 处连续;

②证明可导性:

$$\begin{aligned}\lim_{x \rightarrow 0-} \frac{-(\cos x - 1) - [-(\cos 0 - 1)]}{x - 0} \\ \leq \lim_{x \rightarrow 0-} \frac{f(x) - f(0)}{x - 0} \leq \lim_{x \rightarrow 0-} \frac{(\cos x - 1) - (\cos 0 - 1)}{x - 0}\end{aligned}$$

$$\text{因 } \lim_{x \rightarrow 0-} \frac{1 - \cos x}{x} = \lim_{x \rightarrow 0-} \frac{\cos x - 1}{x} = 0 \text{ (等价无穷小)}$$

由夹逼定理可得, $f'_+(0) = 0$, 同理可得, $f'_-(0) = 0$

则 $f'(0) = 0$, $f(x)$ 在 $x = 0$ 处可导。

习题 3.2

本颜色字体均为概念或公式

1.

$$(1)(x^n)' = nx^{n-1}$$

$$\Rightarrow y = x^3 - 2x^2 + 3x - 4$$

$$y' = 3x^2 - 4x + 3$$

$$(2)(uv)' = u'v + uv'$$

$$y = (x^2 + 3x + 2)(x^2 - 3x + 2)$$

$$\text{方法} a. y' = (2x + 3)(x^2 - 3x + 2) + (x^2 + 3x - 2)(2x - 3)$$

$$\Rightarrow y' = 4x^3 - 10x$$

$$\text{方法} b. y = (x^2 + 2)^2 - (3x)^2 = x^4 - 5x^2 + 4$$

$$\Rightarrow y' = 4x^3 - 10x$$

$$(3)\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

$$y = \frac{\cos x}{x} + \frac{x}{\cos x}$$

$$\Rightarrow y' = \frac{-\sin x \cdot x - \cos x}{x^2} + \frac{\cos x + x \sin x}{\cos^2 x}$$

$$\Rightarrow y' = -x^{-1} \cdot \sin x - x^{-2} \cos x + \sec x + x \tan x \cdot \sec x$$

$$\text{Tip: } \frac{1}{\sin x} = \csc x, \frac{1}{\cos x} = \sec x$$

$$(4)(\ln x)' = \frac{1}{x} / (\ln|x|)' = \frac{1}{x}$$

$$y = x \ln x$$

$$\Rightarrow y' = \ln x + x \cdot \frac{1}{x}$$

$$\Rightarrow y' = \ln x + 1$$

$$(5)y = x^2 + x^{-2}$$

$$\Rightarrow y' = 2x - 2x^{-3}$$

$$(6)(e^x)' = e^x$$

$$y = e^x \cos x$$

$$\Rightarrow y' = e^x \cos x + e^x(-\sin x)$$

$$\Rightarrow y' = e^x(\cos x - \sin x)$$

$$(7)y = e^x \sin x$$

$$\Rightarrow y' = e^x \sin x + e^x \cos x$$

$$\Rightarrow y' = e^x(\sin x + \cos x)$$

$$(8)y = e^x \ln x$$

$$\Rightarrow y' = e^x \ln x + e^x \cdot \frac{1}{x}$$

$$\Rightarrow y' = e^x \left(\ln x + \frac{1}{x} \right)$$

2.

$$(1) \left(f(g(x)) \right)' = f'(g(x)) \cdot g'(x)$$

$$y = e^{x^2 + \sin x}$$

$$\Rightarrow y' = e^{x^2 + \sin x} \cdot (2x + \cos x) \cdot g'(x)$$

$$(2)y = x \ln(x^2 + e^x)$$

$$\Rightarrow y' = \ln(x^2 + e^x) + x \cdot \frac{1}{x^2 + e^x} \cdot (2x + e^x)$$

$$\Rightarrow y' = \ln(x^2 + e^x) + \frac{2x^2 + xe^x}{x^2 + e^x}$$

$$(3)y = \sin 2x$$

$$\Rightarrow y' = 2 \cos 2x$$

$$(4)y = \cos 2x$$

$$\Rightarrow y' = -2 \sin 2x$$

$$(5)y = \sqrt{x} \arcsin \sqrt{x}$$

$$\text{Tip: } (\arcsin x)' = \frac{1}{\sqrt{1-x^2}};$$

$$(\arccos x)' = -\frac{1}{\sqrt{1-x^2}};$$

$$(\arctan x)' = \frac{1}{1+x^2};$$

$$(\operatorname{arccot} x)' = -\frac{1}{1+x^2}$$

$$\Rightarrow y' = \frac{1}{2\sqrt{x}} \cdot \arcsin \sqrt{x} + \sqrt{x} \cdot \frac{1}{\sqrt{1-x}} \cdot \frac{1}{2\sqrt{x}}$$

$$\Rightarrow y' = \frac{\arcsin \sqrt{x}}{2\sqrt{x}} + \frac{1}{2\sqrt{1-x}} \quad (x \neq 0)$$

$$(6)y = \sqrt{x} \arccos \sqrt{x}$$

$$\Rightarrow y' = \frac{1}{2\sqrt{x}} \cdot \arccos \sqrt{x} + \sqrt{x} \cdot \left(-\frac{1}{\sqrt{1-x}}\right) \cdot \frac{1}{2\sqrt{x}}$$

$$\Rightarrow y' = \frac{\arccos \sqrt{x}}{2\sqrt{x}} - \frac{1}{2\sqrt{1-x}} \quad (x \neq 0)$$

$$(7)y = x^2 \arctan \frac{1}{x}$$

$$\Rightarrow y' = 2x \cdot \arctan \frac{1}{x} + x^2 \frac{1}{1 + \left(\frac{1}{x}\right)^2} \cdot \left(-\frac{1}{x^2}\right)$$

$$\Rightarrow y' = 2x \arctan \frac{1}{x} - \frac{x^2}{x^2 + 1}$$

$$(8)y = x^2 \operatorname{arccot} \frac{1}{x}$$

$$\Rightarrow y' = 2x \cdot \operatorname{arccot} \frac{1}{x} + x^2 \left(-\frac{1}{1 + \left(\frac{1}{x}\right)^2} \right) \cdot \left(\frac{1}{x^2}\right)$$

$$\Rightarrow y' = 2x \operatorname{arccot} \frac{1}{x} - \frac{x^2}{x^2 + 1}$$

$$(9)(\sec x)' = \tan x \cdot \sec x$$

$$(\csc x)' = -\cot x \csc x$$

$$(\tan x)' = \sec^2 x$$

$$(\cot x)' = -\csc^2 x$$

$$y = \sec x^2$$

$$\Rightarrow y' = \tan x^2 - \sec x^2 \cdot (2x)$$

$$\Rightarrow y' = 2x \tan x^2 \sec x^2$$

$$(10)y = \csc \sqrt{x}$$

$$\Rightarrow y' = -\cot \sqrt{x} \csc \sqrt{x} \cdot \frac{1}{2\sqrt{x}}$$

$$\Rightarrow y' = -\frac{\cot \sqrt{x} \csc \cos x}{2\sqrt{x}}$$

$$(11)y = x^{\frac{1}{2}} + x^{-\frac{1}{2}}$$

$$\Rightarrow y' = \frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{2}x^{-\frac{3}{2}}$$

$$\Rightarrow y' = \frac{1}{2} \left(x^{-\frac{1}{2}} - x^{-\frac{3}{2}} \right)$$

$$(12) y = \frac{e^{\sqrt{x}} - e^{-\sqrt{x}}}{e^{\sqrt{x}} + e^{-\sqrt{x}}}$$

$$\Rightarrow y = \frac{e^{\sqrt{x}} + e^{-\sqrt{x}} - 2e^{-\sqrt{x}}}{e^{\sqrt{x}} + e^{-\sqrt{x}}}$$

$$y = 1 - 2 \frac{1}{e^{2\sqrt{x}} + 1}$$

$$\Rightarrow y' = -2 \frac{-e^{2\sqrt{x}} \cdot \frac{1}{\sqrt{x}}}{(e^{2\sqrt{x}} + 1)^2}$$

$$\Rightarrow y' = \frac{2e^{2\sqrt{x}}}{\sqrt{x}(e^{2\sqrt{x}} + 1)^2} \quad (x \neq 0)$$

3.

$$(1) \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

对两边关于 x 求导

$$\Rightarrow \frac{1}{a^2} \cdot 2x + \frac{1}{b^2} \cdot 2y \cdot y' = 0$$

$$\Rightarrow y' = -\frac{b^2 x}{a^2 y} \quad (y \neq 0)$$

$$(2) x^2 + 2xy - y^2 = 2x$$

对两边关于 x 求导

$$\Rightarrow 2x + 2y + 2xy' - 2y \cdot y' = 2$$

$$\Rightarrow (x - y)y' = 1 - x - y$$

$$\Rightarrow y' = \frac{1 - x - y}{x - y} \quad (x \neq y)$$

$$(3) \sqrt{x} + \sqrt{y} = \sqrt{a}$$

对两边关于 x 求导

$$\Rightarrow \frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}} \cdot y' = 0$$

$$\Rightarrow y' = -\frac{\sqrt{xy}}{x} \quad (x > 0, y > 0)$$

$$(4) x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$$

对两边关于 x 求导

$$\Rightarrow \frac{2}{3}x^{-\frac{1}{3}} + \frac{2}{3}y^{-\frac{1}{3}} \cdot y' = 0$$

$$\Rightarrow y' = -\left(\frac{y}{x}\right)^{\frac{1}{3}} \quad (x \neq 0)$$

$$(5) \arctan \frac{y}{x} = \ln \sqrt{x^2 + y^2}$$

对两边关于 x 求导

$$\Rightarrow \frac{1}{1 + \left(\frac{y}{x}\right)^2} \cdot \frac{y'x - y}{x^2} = \frac{1}{\sqrt{x^2 + y^2}} \cdot \frac{2x + 2y \cdot y'}{2\sqrt{x^2 + y^2}}$$

$$\Rightarrow \frac{y'x - y}{x^2 + y^2} = \frac{x + y \cdot y'}{x^2 + y^2}$$

$$\Rightarrow y'x - y = x + y - y'$$

$$\Rightarrow y' = \frac{x + y}{x - y} \quad (x \neq y, x \neq 0)$$

$$(6) x^y = y^x \quad (x > 0, y > 0)$$

$$\Rightarrow y \ln x = x \ln y$$

对两边关于 x 求导

$$\Rightarrow y' \ln x + y \frac{1}{x} = \ln y + x \frac{1}{y} \cdot y'$$

1° 由 $y \ln x = x \ln y$ 变形得

$$\Rightarrow y' \left(\ln x - \frac{x}{y} \right) = y' (\ln y - 1) \cdot \frac{x}{y}$$

$$\ln y - \frac{y}{x} = \frac{y}{x} (\ln x - 1)$$

$$\Rightarrow y' = \frac{y^2 (\ln x - 1)}{x^2 (\ln y - 1)} \quad (x > 0, y > 0)$$

$$2^\circ \Rightarrow y' = \frac{xy \ln y - y^2}{xy \ln x - x^2} \quad (x > 0, y > 0)$$

$$(7) x - y + \xi \sin y = 0 \quad (\xi \text{ 为参数})$$

对两边关于 x 求导

$$\Rightarrow 1 - y' + \xi \cos y \cdot y' = 0$$

$$\Rightarrow y' = \frac{1}{1 - \xi \cos y}$$

4.

$$(1) y = x^{\sin x}$$

对两边取对数

$$\Rightarrow \ln y = \sin x \ln x$$

$$\Rightarrow \frac{1}{y} \cdot y' = \cos x \ln x + \sin x \frac{1}{x}$$

$$\Rightarrow y' = y \cos x \ln x + \frac{y \sin x}{x}$$

$$\Rightarrow y' = x^{\sin x} \left(\cos x \ln x + \frac{\sin x}{x} \right)$$

$$(2) y = x^{\ln x}$$

对两边取对数

$$\Rightarrow \ln y = \ln x \cdot \ln x = \ln^2 x$$

$$\Rightarrow \frac{1}{y} \cdot y' = 2 \ln x \cdot \frac{1}{x}$$

$$\Rightarrow y' = \frac{2y \ln x}{x}$$

$$\Rightarrow y' = 2x^{(\ln x)-1} \ln x$$

$$(3)y = \sqrt[3]{\frac{(x-1)(x-2)}{(x-3)(x-4)}}$$

对两边取对数

$$\Rightarrow \ln y = \frac{1}{3} \ln \frac{(x-1)(x-2)}{(x-3)(x-4)}$$

$$= \frac{1}{3} \ln(x-1) + \frac{1}{3} \ln(x-2) - \frac{1}{3} \ln(x-3) - \frac{1}{3} \ln(x-4)$$

$$\Rightarrow \frac{1}{y} \cdot y' = \frac{1}{3} \frac{1}{x-1} + \frac{1}{3} \cdot \frac{1}{x-2} - \frac{1}{3} \cdot \frac{1}{x-3} - \frac{1}{3} \cdot \frac{1}{x-4}$$

$$\Rightarrow y' = \frac{-4x^2 + 20x - 22}{3(x-1)(x-2)(x-3)(x-4)} \cdot y$$

$$\Rightarrow y' = \frac{-4x^2 + 20x - 22}{3(x-1)(x-2)(x-3)(x-4)} \sqrt[3]{\frac{(x-1)(x-2)}{(x-3)(x-4)}}$$

5.

$$y' = \frac{dy}{dx}, y = y(t), x = x(t) \Rightarrow y' = \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$\Rightarrow x = x(t)$ 在 $t \in D$ 时单调

$$(1) \begin{cases} x = 1 - t^2 \\ y = 1 - t^3 \end{cases} \Rightarrow \frac{dx}{dt} = -2t, \quad \frac{dy}{dt} = -3t^2 \Rightarrow y' = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3}{2}t$$

$$(2) \begin{cases} x = \ln(1 + t^2) \\ y = t - \arctan t \end{cases}$$

$$\Rightarrow \frac{dx}{dt} = \frac{2t}{1+t^2}, \frac{dy}{dt} = 1 - \frac{1}{1+t^2} = \frac{t^2}{1+t^2} \Rightarrow y' = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{t}{2}$$

$$(3) \begin{cases} x = a \cos^3 t \\ y = a \sin^3 t \end{cases} \Rightarrow \frac{dx}{dt} = 3a \cos^2 t (-\sin t), \quad \frac{dy}{dt} = 3a \sin^2 t \cdot \cos t$$

$$\Rightarrow y' = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = -\tan t$$

6.

证明: $\because \sqrt{x} + \sqrt{y} = \sqrt{a} (a > 0) \quad (x \geq 0, y \geq 0)$

\therefore 抛物线与 x 轴交点为 $P_1(a, 0)$, 与 y 轴交点为 $P_2(0, a)$

对 $\sqrt{x} + \sqrt{y} = \sqrt{a} (a > 0)$ 两边关于 x 求导

$$\Rightarrow \frac{1}{2\sqrt{x}} + \frac{y'}{2\sqrt{y}} = 0$$

$$\Rightarrow y' = -\frac{\sqrt{xy}}{x} = k_{\text{切}}$$

$$\Rightarrow l_{\text{切}}: y - y_0 = -\frac{\sqrt{x_0 y_0}}{x_0} (x - x_0) \Rightarrow \text{抛物线在 } x_0 \text{ 点处的切线}$$

$$\text{在 } x_0 \text{ 点处 } \sqrt{x_0} + \sqrt{y_0} = \sqrt{a} \Rightarrow x_0 + 2\sqrt{x_0 y_0} + y_0 = a$$

$$\Rightarrow l_{\text{切}} \text{ 与 } x \text{ 轴交点为 } P_1(x_0 + \sqrt{x_0 y_0}, 0), \text{ 与 } y \text{ 轴交点为 } P_2(0, y_0 + \sqrt{x_0 y_0})$$

$$x_0 y_0 = \sqrt{x_0 y_0} (x - x_0), \quad y - y_0 = \sqrt{x_0 y_0}$$

$$\Rightarrow x_0 + \sqrt{x_0 y_0} + y_0 + \sqrt{x_0 y_0} = x_0 + 2\sqrt{x_0 y_0} + y_0 = a$$

故抛物线 $\sqrt{x} + \sqrt{y} = \sqrt{a} (a > 0)$ 上任一点的切线截两个坐标轴的截距之和为 a

7.

$$\text{证明: } \begin{cases} x = a(\cos t + t \sin t) \\ y = a(\sin t + t \cos t) \end{cases}$$

$$\begin{aligned}\frac{dx}{dt} &= a(-\sin t + \sin t + t \cos t) = at \cos t \\ \Rightarrow \frac{dy}{dt} &= a(\cos t - \cos t + t \sin t) = at \sin t\end{aligned}$$

$$\Rightarrow y' = \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{\sin t}{\cos t} = \tan t$$

$$\Rightarrow k_{\text{切}} = -\frac{1}{y'} = -\cot t$$

$$\Rightarrow \text{曲线上点 } x_0 \text{ 的切线为 } y - y_0 = -\cot t_0 (x - x_0)$$

$$l_{\text{法}}: \cos t_0 x + \sin t_0 y - x_0 \cos t_0 - y_0 \sin t_0 = 0$$

$$\Rightarrow d = \frac{|\cos t_0 \cdot x_1 + \sin t_0 \cdot y_1 - x_0 \cos t_0 - y_0 \sin t_0|}{\sqrt{\cos^2 t_0 + \sin^2 t_0}} \quad ((x_1, y_1) \text{ 为原点})$$

$$= a \cos^2 t_0 + at_0 \sin t_0 \cos t_0 + a \sin^2 t_0 - at_0 \sin t_0 \cos t_0$$

$$= a(\cos^2 t_0 + \sin^2 t_0)$$

$$= a$$

8.

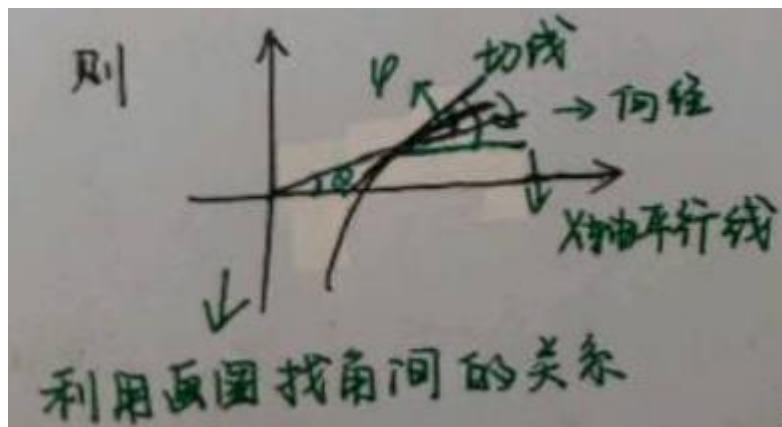
$$(1) \text{ 由题得 } l: r = r(\theta). \begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases}, r = r(\theta) \text{ 关于 } \theta \text{ 可导}$$

$$\frac{dx}{d\theta} = r' \cos \theta - r \sin \theta = r'(\theta) \cos \theta - r(\theta) \sin \theta$$

$$\Rightarrow \frac{dy}{d\theta} = r' \sin \theta + r \cos \theta = r'(\theta) \sin \theta + r(\theta) \cos \theta$$

$$\Rightarrow \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{r'(\theta) \sin \theta + r(\theta) \cos \theta}{r'(\theta) \cos \theta - r(\theta) \sin \theta} = \frac{r'(\theta) \tan \theta + r(\theta)}{r'(\theta) - r(\theta) \tan \theta}$$

(2) 设点 $P(r, \theta)$ 处切线与 x 轴夹角为 α



\Rightarrow 同位角: $\theta = \alpha - \varphi$

即 $\varphi = \alpha - \theta$

$$\begin{aligned} \tan \varphi &= \tan(\alpha - \theta) = \frac{\tan \alpha - \tan \theta}{1 + \tan \alpha \tan \theta} \\ \Rightarrow \tan \alpha &= k_{\text{切}} = \frac{dy}{dx} = \frac{r'(\theta) \tan \theta + r(\theta)}{r'(\theta) - r(\theta) \tan \theta} \end{aligned}$$

上下同乘 $[r'(\theta) - r(\theta) \tan \theta]$

$$\Rightarrow \tan \varphi = \frac{\frac{r'(\theta) \tan \theta + r(\theta)}{r'(\theta) - r(\theta) \tan \theta} - \tan \theta}{1 + \frac{r'(\theta) \tan \theta + r(\theta)}{r'(\theta) - r(\theta) \tan \theta} \tan \theta}$$

$$\Rightarrow \tan \varphi = \frac{r'(\theta) \tan \theta + r(\theta) - r'(\theta) \tan \theta + r(\theta) \tan^2 \theta}{r'(\theta) - r(\theta) \tan \theta + r'(\theta) \tan^2 \theta + r(\theta) \tan \theta}$$

$$\Rightarrow \tan \varphi = \frac{r(\theta)(1 + \tan^2 \theta)}{r'(\theta)(1 + \tan^2 \theta)} = \frac{r(\theta)}{r'(\theta)}$$

9.

解: \because 极坐标曲线 $r = e^\theta$

\therefore 其可用极角 θ 作为参数

表示如下

$$\begin{cases} x = r \cos \theta = e^\theta \cos \theta \\ y = r \sin \theta = e^\theta \sin \theta \end{cases}$$

$$\frac{dy}{d\theta} = e^{\theta} \sin \theta + e^{\theta} \cos \theta$$

$$\frac{dx}{d\theta} = e^{\theta} \cos \theta - e^{\theta} \sin \theta$$

$$\Rightarrow y' = \frac{\sin \theta + \cos \theta}{\cos \theta - \sin \theta}$$

$$\text{点} \left(e^{\frac{\pi}{2}}, \frac{\pi}{2} \right) \text{处的极角为} \frac{\pi}{2}$$

$$\Rightarrow y'_1 = \frac{1+0}{0-1} = -1 \quad y_1 = e^{\frac{\pi}{2}}, x_1 = 0$$

$$\Rightarrow l_{\text{切}} = y - y_1 = y'_1(x - x_1)$$

$$\Rightarrow l_{\text{切}}: y = -x + e^{\frac{\pi}{2}}$$

$$\Rightarrow l_{\text{切}} = x + y - e^{\frac{\pi}{2}} = 0$$

习题 3.3

1. 求函数 $y=x^2$ 在点 $x=1$ 的微分, 其中自变量 x 的增量 Δx 分别如下:

$$\Delta x = 0.1; \Delta x = 0.01; \Delta x = 0.001;$$

解: 由公式可得 $dy|_{\{x=1\}} = f'(1)\Delta x$, 分别代入 Δx 可得:

$$dy=0.2; dy=0.02; dy=0.002$$

2. 求下列函数的微分

$$(1) y = \frac{x^2-1}{x^2+1}$$

$$\begin{aligned} \text{解: } y &= \frac{x^2+1-2}{x^2+1} = 1 - \frac{2}{x^2+1} \quad \therefore y' = \left(0 - \frac{-2 \cdot 2x}{(x^2+1)^2}\right) = \frac{4x}{(x^2+1)^2} \\ &\therefore dy = \frac{4x}{(x^2+1)^2} dx \end{aligned}$$

$$(2) y = \tan x + \sec x$$

$$\begin{aligned} \text{解: } y' &= \frac{\sin x}{\cos x} + \frac{1}{\cos x} \quad \therefore y' = \frac{\sin^2 x + \cos^2 x}{\cos^2 x} + \frac{\sin x}{\cos^2 x} = \sec^2 x + \sec x \tan x \\ &\therefore dy = (\sec^2 x + \sec x \tan x) dx \end{aligned}$$

$$(3) y = \arccos \frac{1}{x}$$

$$\text{解: } y' = \frac{-1}{\sqrt{1-\frac{1}{x^2}}} \cdot \left(\frac{1}{-x^2}\right) = \frac{1}{|x| \cdot \sqrt{x^2-1}}$$

$$\therefore dy = \frac{1}{|x| \cdot \sqrt{x^2-1}} dx \quad \text{且 } (|x|>1)$$

$$(4) y = \arcsin \sqrt{1-x^2}$$

$$\begin{aligned} \text{解: } y' &= \frac{1}{\sqrt{1-(\sqrt{1-x^2})^2}} \cdot \left(\frac{1}{2} \cdot \frac{1}{\sqrt{1-x^2}}\right) \cdot (-2x) \\ &= \frac{1}{\sqrt{x^2}} \cdot \frac{-x}{\sqrt{1-x^2}} \\ &= \frac{-x}{|x| \sqrt{1-x^2}} \\ &\therefore dy = \frac{-x}{|x| \sqrt{1-x^2}} dx \quad \text{且 } (|x|<1) \end{aligned}$$

$$(5) y = \arctan \frac{x^2-1}{x^2+1}$$

$$\begin{aligned}\text{解: } y' &= \frac{1}{1 + \left(\frac{x^2-1}{x^2+1}\right)^2} \cdot \left(\frac{x^2-1}{x^2+1}\right)' \\ &= \frac{1}{2x^4+2} \cdot 4x = \frac{2x}{x^4+1} \\ \therefore dy &= \frac{2x}{x^4+1} dx\end{aligned}$$

$$\text{由(1)知} \left(\frac{x^2-1}{x^2+1}\right)' = \frac{4x}{(x^2+1)^2}$$

$$(6) \quad y = (x^2 + 4x + 1)(x^2 - \sqrt{x})$$

$$\begin{aligned}\text{解: } y' &= (2x+4)(x^2-\sqrt{x}) + (x^2+4x+1)\left(2x-\frac{1}{2\sqrt{x}}\right) \\ &= 2x^3 - 2x^{\frac{3}{2}} + 4x^2 - 4x^{\frac{1}{2}} + 2x^3 - \frac{1}{2}x^{\frac{3}{2}} + 8x^2 - 2x^{\frac{1}{2}} + 2x - 2x^{-\frac{1}{2}} \\ &= 4x^3 + 12x^2 - \frac{5}{2}x^{\frac{3}{2}} + 2x - 6x^{\frac{1}{2}} - \frac{1}{2}x^{-\frac{1}{2}} \\ \therefore dy &= \left(4x^3 + 12x^2 - \frac{5}{2}x^{\frac{3}{2}} + 2x - 6x^{\frac{1}{2}} - \frac{1}{2}x^{-\frac{1}{2}}\right) dx\end{aligned}$$

3. 求下列复合函数的微分

$$(1) \quad y = \ln \sqrt{1+x^2}$$

$$\begin{aligned}\text{解: } y' &= \frac{1}{\sqrt{1+x^2}} \cdot \frac{1}{2\sqrt{1+x^2}} \cdot 2x = \frac{x}{1+x^2} \\ \therefore dy &= \frac{x}{1+x^2} dx\end{aligned}$$

$$(2) \quad y = \arcsin \frac{1}{x}$$

$$\begin{aligned}\text{解: } y' &= \frac{1}{\sqrt{1-\left(\frac{1}{x}\right)^2}} \cdot \left(-\frac{1}{x^2}\right) \\ &= \sqrt{\frac{x^2}{x^2-1}} \cdot \left(-\frac{1}{x^2}\right) \\ &= \frac{|x|}{\sqrt{x^2-1}} \cdot \frac{1}{-|x|^2} = \frac{1}{|x| \cdot \sqrt{x^2-1}} \\ \therefore dy &= \frac{1}{|x|\sqrt{x^2-1}} dx \left(\frac{1}{\sqrt{x^4-x^2}} dx\right)\end{aligned}$$

$$(3) \quad y = \arctan \sqrt{x}$$

$$\begin{aligned}\text{解: } y' &= \frac{1}{1+(\sqrt{x})^2} \cdot \frac{1}{2\sqrt{x}} = \frac{1}{2(1+x)\sqrt{x}} \\ \therefore dy &= \frac{1}{2(1+x)\sqrt{x}} dx\end{aligned}$$

$$(4) \quad y = e^{\sin x}$$

$$\text{解: } y' = e^{\sin x} \cdot \cos x$$

$$\therefore dy = (e^{\sin x} \cdot \cos x) dx$$

4. 求下列各式的近似值

解: 由微分定义: $\Delta y = f'(x)\Delta x + o(\Delta x)$ ($\Delta x \rightarrow 0$), 即 $f(x + \Delta x) - f(x) \approx f'(x)\Delta x$

$$\therefore f(x + \Delta x) \approx f(x) + f'(x)\Delta x$$

$$(1) \quad \text{令 } \sqrt[3]{1.02} = \sqrt[3]{1 + 0.02} \quad \therefore f(x) = x^{\frac{1}{3}} \quad f'(x) = \frac{1}{3}x^{-\frac{2}{3}} \quad \Delta x = 0.02$$

$$\therefore f(1.02) \approx f(1) + f'(1) \cdot 0.02 = 1 + \frac{1}{3} \times 0.02 = 1.00666 \dots \approx 1.007$$

$$(2) \quad \text{令 } \ln 1.005 = \ln(1 + 0.005) \quad \therefore f(x) = \ln x, \quad f'(x) = \frac{1}{x} \quad \Delta x = 0.005$$

$$\therefore f(1.005) \approx f(1) + f'(1) \cdot 0.005 = 0 + 1 \times 0.005 = 0.005$$

习题 3.4

1. 求下列函数的二阶导数

$$(1) \quad y = x^3 + 2x^2 + 3x + 4$$

$$\text{解: } y' = 3x^2 + 4x + 3$$

$$y'' = 6x + 4$$

$$(2) \quad y = x^4 \ln x$$

$$\text{解: } y' = 4x^3 \ln x + x^3$$

$$y'' = 12x^2 \ln x + 4x^2 + 3x^2$$

$$= 12x^2 \ln x + 7x^2$$

$$(3) \quad y = \frac{x^2}{\sqrt{1+x}}$$

$$\text{解: } y = x^2(1+x)^{-\frac{1}{2}}$$

$$\therefore y' = 2x(1+x)^{-\frac{1}{2}} - \frac{1}{2}x^2(1+x)^{-\frac{3}{2}}$$

$$y'' = 2(1+x)^{-\frac{1}{2}} - \frac{1}{2} \cdot 2x(1+x)^{-\frac{3}{2}} - x(1+x)^{-\frac{3}{2}} + \frac{3}{4}x^2(1+x)^{-\frac{5}{2}}$$

$$= (1+x)^{-\frac{5}{2}} \left[2(1+x)^2 - x(1+x) - x(1+x) + \frac{3}{4}x^2 \right]$$

$$= (1+x)^{-\frac{5}{2}} \left(\frac{3}{4}x^2 + 2x + 2 \right)$$

$$(4) \quad y = \frac{\ln x}{x^2}$$

$$\text{解: } y = x^{-2} \ln x$$

$$\therefore y' = -2x^{-3} \ln x + x^{-3}$$

$$\begin{aligned} y'' &= 6x^{-4} \ln x + (-2)x^{-4} - 3x^{-4} \\ &= (6 \ln x - 5)x^{-4} \end{aligned}$$

$$(5) \quad y = \sin x^2$$

$$\text{解: } y' = \cos x^2 \cdot 2x$$

$$\begin{aligned} y'' &= 2 \cos x^2 + 2x(-\sin x^2 \cdot 2x) \\ &= -4x^2 \sin x^2 + 2 \cos x^2 \end{aligned}$$

$$(6) \quad y = x^3 \cos \sqrt{x}$$

$$\text{解: } y' = 3x^2 \cos \sqrt{x} + x^3(-\sin \sqrt{x}) \frac{1}{2}(x)^{-\frac{1}{2}}$$

$$= 3x^2 \cos \sqrt{x} - \frac{1}{2}x^{\frac{5}{2}} \sin \sqrt{x}$$

$$y'' = 6x \cos \sqrt{x} + 3x^2(-\sin \sqrt{x}) \frac{1}{2}(x)^{-\frac{1}{2}}$$

$$- \left(\frac{5}{4}x^{\frac{3}{2}} \sin \sqrt{x} + \frac{1}{2}x^{\frac{5}{2}} \cos \sqrt{x} \frac{1}{2}(x)^{-\frac{1}{2}} \right)$$

$$= 6x \cos \sqrt{x} - \frac{3}{2}x^{\frac{3}{2}} \sin \sqrt{x} - \frac{5}{4}x^{\frac{3}{2}} \sin \sqrt{x} - \frac{1}{4}x^2 \cos \sqrt{x}$$

$$= \left(6x - \frac{1}{4}x^2 \right) \cos \sqrt{x} - \frac{11}{4}x^{\frac{3}{2}} \sin \sqrt{x}$$

$$(7) \quad y = x^2 e^{3x}$$

$$\text{解: } y' = 2xe^{3x} + x^2 \cdot 3e^{3x}$$

$$\begin{aligned}
 y'' &= 2e^{3x} + 2xe^{3x} \cdot 3 + 2x \cdot 3e^{3x} + 3x^2 \cdot 3e^{3x} \\
 &= e^{3x}(2 + 6x + 6x + 9x^2) \\
 &= (9x^2 + 12x + 2)e^{3x}
 \end{aligned}$$

$$(8) \quad y = e^{-x^2} \arcsin x$$

$$\text{解: } y' = -2xe^{-x^2} \arcsin x + e^{-x^2} \frac{1}{\sqrt{1-x^2}}$$

$$\begin{aligned}
 y'' &= -2xe^{-x^2} \frac{1}{\sqrt{1-x^2}} - 2xe^{-x^2}(-2x) \arcsin x - 2e^{-x^2} \arcsin x \\
 &\quad + e^{-x^2}(-2x)(1-x^2)^{-\frac{1}{2}} +
 \end{aligned}$$

$$\left(-\frac{1}{2}\right)e^{-x^2}(1-x^2)^{-\frac{3}{2}}(-2x)$$

$$\begin{aligned}
 \therefore y'' &= -2x(1-x^2)^{-\frac{1}{2}}e^{-x^2} + 4x^2e^{-x^2} \arcsin x - 2e^{-x^2} \arcsin x \\
 &\quad - 2x(1-x^2)^{-\frac{1}{2}}e^{-x^2} + xe^{-x^2}(1-x^2)^{-\frac{3}{2}} \\
 &= (4x^2 - 2)e^{-x^2} \arcsin x - 4xe^{-x^2}(1-x^2)^{-\frac{1}{2}} \\
 &\quad + xe^{-x^2}(1-x^2)^{-\frac{3}{2}}
 \end{aligned}$$

$$(9) \quad y = x^2 \cos 3x$$

$$\text{解: } y' = 2x \cos 3x + x^2(-\sin 3x) \cdot 3$$

$$\begin{aligned}
 y'' &= 2 \cos 3x + 2x(-\sin 3x) \cdot 3 + 6x(-\sin 3x) - 3x^2 \cos 3x \cdot 3 \\
 &= 2 \cos 3x - 6x \sin 3x - 6x \sin 3x - 9x^2 \cos 3x \\
 &= (2 - 9x^2) \cos 3x - 12x \sin 3x
 \end{aligned}$$

$$(10) \quad y = x^2 \ln x$$

解: $y' = 2x \ln x + x$

$$y'' = 2 \ln x + 2 + 1$$

$$= 2 \ln x + 3$$

2. 求下列函数的 n 阶导数

(1) $y = \ln(x+1)$

解: $y' = \frac{1}{x+1} \quad y'' = -\frac{1}{(x+1)^2}$

$$y''' = \frac{2}{(1+x)^3} \quad y^{(n)} = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}$$

(2) $y = \sin^2(\omega x)$

解: $y = \sin^2(\omega x) = \frac{1 - \cos(2\omega x)}{2} = \frac{1}{2} - \frac{1}{2}\cos(2\omega x)$

$$y^{(n)} = -2^{n-1}\omega^n \cos\left(2\omega x + \frac{n}{2}\pi\right)$$

(3) $y = \frac{1}{x^2-3x+2}$

解: $y = \frac{1}{x^2-3x+2} = \frac{1}{(x-1)(x-2)} = \frac{1}{x-2} - \frac{1}{x-1}$

又由: $\left(\frac{1}{x+1}\right)^{(n)} = (-1)^n \frac{n!}{(x+1)^{n+1}}$

$$\therefore y^{(n)} = (-1)^n \frac{n!}{(x-2)^{n+1}} - (-1)^n \frac{n!}{(x-1)^{n+1}}.$$

$$= (-1)^n n! \left[(x-2)^{-(n+1)} - (x-1)^{-(n+1)} \right]$$

$$(4) \quad y = \cos^2(\omega x)$$

$$\text{解: } y = \cos^2(\omega x) = \frac{1 + \cos(2\omega x)}{2} = \frac{1}{2} + \frac{1}{2}\cos(2\omega x)$$

$$y^{(n)} = 2^{n-1}\omega^n \cos\left(2\omega x + \frac{n}{2}\pi\right)$$

3. 求下列函数的高阶导数

$$(1) \quad y = a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n, \text{ 求 } y^{(n)}, y^{(n+1)};$$

$$\text{解: } y^{(n)} = n! a_0 \quad y^{(n+1)} = 0$$

$$(2) \quad y = x^2(1+x)^3(2+x)^4, \text{ 求 } y^{(9)}, y^{(10)};$$

$$\text{解: } y^{(9)} = 9! \quad y^{(10)} = 0$$

$$(3) \quad y = x^2 e^{2x}, \text{ 求 } y^{(20)};$$

$$\text{解: } \because (x^2)' = 2x \quad (x^2)'' = 2 \quad (x^3)''' = 0$$

由莱布尼茨公式可知: $y^{(n)}$

$$= C_{20}^0 x^2 (e^{2x})^{(20)} + C_{20}^1 2x (e^{2x})^{(19)} + C_{20}^2 2 (e^{2x})^{(18)}$$

$$y^{(n)} = 2^{20} x^2 e^{2x} + 20 \cdot 2^{20} x e^{2x} + \frac{20 \cdot 19}{2} 2^{19} e^{2x}$$

$$= 2^{20} e^{2x} (x^2 + 20x + 95)$$

$$(4) \quad y = x \ln x, \text{ 求 } y^{(5)};$$

$$\text{解: } \because (x)' = 1 \quad (x)'' = 0 \quad (\ln x)^{(n)} = \frac{(-1)^{n-1} (n-1)!}{x^n}$$

$$y^{(5)} = C_5^0 x (\ln x)^{(5)} + C_5^1 (\ln x)^{(4)}$$

$$= x \frac{4!}{x^5} + 5 \left(-\frac{3!}{x^4} \right)$$

$$= 24x^{-4} - 30x^{-4} = -6x^{-4}$$

$$(5) \quad y(n) = (\sqrt{2})^n e^x \sin\left(x + \frac{n}{4}\pi\right)$$

$$\text{当 } n = 1 \text{ 时 } y' = \sqrt{2} \cdot e^x \sin\left(x + \frac{\pi}{4}\pi\right)$$

$$y(k) = (\sqrt{2})^k e^x \sin\left(x + \frac{k}{4}\pi\right)$$

$$y^{(k+1)} = (y^{(k)})' = (\sqrt{2})^k e^x \left[\sin\left(x + \frac{k}{4}\pi\right) + \cos\left(x + \frac{k}{4}\pi\right) \right]$$

$$= (\sqrt{2})^k e^x \cdot \sqrt{2} \sin\left(x + \frac{k}{4}\pi + \frac{\pi}{4}\right)$$

$$= (\sqrt{2})^{k+1} e^x \sin\left(x + \frac{k+1}{4}\pi\right)$$

4. 求下列函数的二阶微分

$$(1) \quad y = \sin x$$

$$\text{解: } y'' = -\sin x$$

$$\therefore d^2y = -\sin x \, dx^2$$

$$(2) \quad y = xe^x$$

$$\text{解: } y' = e^x + xe^x$$

$$y'' = e^x + e^x + xe^x = 2e^x + xe^x$$

$$\therefore d^2y = (2e^x + xe^x)dx^2$$

$$(3) \quad y = x \ln x$$

$$\text{解: } y' = \ln x + 1 \quad y'' = \frac{1}{x}$$

$$\therefore dy^2 = \frac{1}{x} dx^2$$

$$(4) y = x \sin x$$

$$\text{解: } y' = \sin x + x \cos x$$

$$y'' = \cos x + \cos x - x \sin x$$

$$\therefore d^2y = (2 \cos x - x \sin x) dx^2$$

5. 设 x 为中间变量, 求下列函数的二阶微分

$$(1) y = \sin x, x = at + b, \text{ 其中 } a, b \text{ 为常数}$$

$$\text{解: } y = \sin(at + b) \quad y' = \cos(at + b) \cdot a$$

$$y'' = -a^2 \sin(at + b)$$

$$\therefore d^2y = -a^2 \sin(at + b) dt^2$$

$$(2) y = e^x, x = at^2 + bt + c, \text{ 其中 } a, b, c \text{ 为常数}$$

$$\text{解: } y = e^{at^2+bt+c}$$

$$y' = (2at + b)e^{at^2+bt+c}$$

$$y'' = (2a)e^{at^2+bt+c} + (2at + b)^2 e^{at^2+bt+c}$$

$$= (4a^2t^2 + 4abt + b^2 + 2a)e^{at^2+bt+c}$$

$$\therefore d^2y = (4a^2t^2 + 4abt + b^2 + 2a)e^{at^2+bt+c} dx^2$$

第 3 章复习题

$$1. \lim_{h \rightarrow 0} \frac{f(1-h)-f(1)}{2h} = \lim_{h \rightarrow 0} \frac{f(1+h)-f(1)}{-2h} = -\frac{1}{2} \lim_{h \rightarrow 0} \frac{f(1+h)-f(1)}{h} = -\frac{1}{2} f'(1) = -1$$

$$2. f(x)=x-[x], f(0)=0. (\lim_{x \rightarrow 0+} [x] = 0 \therefore \lim_{x \rightarrow 0+} (x - [x]) = \lim_{x \rightarrow 0+} x = 0)$$

$$f'_+(0) = \lim_{x \rightarrow 0+} \frac{f(x) - f(0)}{x} = 1$$

$$f'_-(0) = \lim_{x \rightarrow 0-} \frac{f(x)-f(0)}{x} = \lim_{x \rightarrow 0-} \frac{x-[x]}{x} = \infty$$

$$(\lim_{x \rightarrow 0-} [x] = 1, \therefore \lim_{x \rightarrow 0-} (x - [x]) = \lim_{x \rightarrow 0-} (x + 1) = 1)$$

$$f'_-(0) \neq f'_+(0) \therefore f'(0) \text{ 不存在.}$$

$$x \in (0,1) \text{ 时, } f'(x) = (x - [x])' = 1 \quad x \in (-1,0) \text{ 时, } f'(x) = (x - [x])' = 1$$

$$\therefore \lim_{x \rightarrow 0} f'(x) = 1.$$

$$3. e^y + 6xy + x^2 - 1 = 0, x=0 \text{ 时, } y=0.$$

$$y'e^y + 6y + 6xy' + 2x = 0, y'(0)=0$$

$$y''e^y + (y')^2e^y + 6y' + 6y' + 6xy''y' + 2 = 0$$

$$\therefore y'' + 2 = 0$$

$$\therefore y''(0) = -2.$$

$$4. 2y \sin x + x \ln y = 0.$$

$$\text{两边对 } x \text{ 求导: } 2y' \sin x + 2y \cos x + \ln y + x \frac{y'}{y} = 0.$$

$$y' = -\frac{2y^2 \cos x + y \ln y}{x + 2y \sin x}.$$

再对 x 求导:

$$2y'' \sin x + 2y' \cos x - 2y \sin x + 2y' \cos x + 2 \frac{y'}{y} + \frac{xyy'' - x(y')^2}{y^2} = 0.$$

$$y'' = \frac{2y^3 \sin x - 4y'y^2 \cos x - 2yy' + x(y')^2}{xy + 2y^2 \sin x}$$

$$5. (1) y' = [(1 + x^2 + x^4)^{\frac{1}{2}}]' = \frac{1}{2} (2x + 4x^3)(1 + x^2 + x^4)^{-\frac{1}{2}} = x(1 + 2x^2)(1 +$$

$$x^2 + x^4)^{-\frac{1}{2}}$$

$$(2) y = x^{\sin x + 2\cos x}$$

$$\text{两边取对数: } \ln|x|(\sin x + 2\cos x) = \ln|y|$$

$$\text{两边对 } x \text{ 求导: } \frac{y'}{y} = (\cos x - 2\sin x)\ln|x| + \frac{1}{x}(\sin x + 2\cos x)$$

$$y' = x^{\sin x + 2\cos x}[(\cos x - 2\sin x)\ln|x| + \frac{1}{x}(\sin x + 2\cos x)]$$

$$(3) y = (1 + \frac{1}{x})^x$$

$$\text{两边取对数: } \ln|y| = x \ln \left| 1 + \frac{1}{x} \right|$$

$$\text{两边对 } x \text{ 求导: } \frac{y'}{y} = \ln \left| 1 + \frac{1}{x} \right| + \frac{x^2}{1+x} \left(-\frac{1}{x^2} \right)$$

$$y' = (1 + \frac{1}{x})^x \left[\ln \left| 1 + \frac{1}{x} \right| - \frac{1}{1+x} \right]$$

$$(4) y = \sqrt[2]{\frac{\sin^2 x (1 + \cos^2 x)}{1 + \sin^2 x}}$$

$$\text{两边取对数: } \ln y = \frac{1}{2} \ln \frac{\sin^2 x (1 + \cos^2 x)}{1 + \sin^2 x}$$

$$\text{两边对 } x \text{ 求导: } \frac{y'}{y} = \frac{1}{2} \left[\frac{2\sin x \cos x + 2\sin x \cos^3 x - 2\sin^3 x \cos x}{\sin^2 x (1 + \cos^2 x)} - \frac{2\sin x \cos x}{1 + \sin^2 x} \right]$$

$$\text{将 } y = \sqrt[2]{\frac{\sin^2 x (1 + \cos^2 x)}{1 + \sin^2 x}} \text{ 代入上式:}$$

$$\begin{aligned} y' &= \sqrt[2]{\frac{\sin^2 x (1 + \cos^2 x)}{1 + \sin^2 x}} \frac{\cos x (2 + \cos^2 x \sin^2 x + \sin^4 x - \cos^4 x)}{\sin x (1 + \cos^2 x) (1 + \sin^2 x)} \\ &= \frac{\cos x (2 + \cos^2 x \sin^2 x + \sin^4 x - \cos^4 x)}{(1 + \cos^2 x)^{\frac{1}{2}} (1 + \sin^2 x)^{\frac{1}{2}}} \end{aligned}$$

$$6. \text{证明: } \lim_{x \rightarrow 0} \frac{f(x)}{x} = A$$

$$\lim_{x \rightarrow 0} x = 0 \quad \therefore \lim_{x \rightarrow 0} f(x) = 0, \text{ 且 } f(x) \text{ 在 } x=0 \text{ 处连续。}$$

$$\therefore f(0) = \lim_{x \rightarrow 0} f(x) = 0$$

$$f'(0) = \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x} = A$$

7.证明:

$$f(x) = x(x+1)(x+2)\dots(x+n+1)$$

$$f'(x) = (x+1)(x+2)\dots(x+n+1) + x(x+2)\dots(x+n+1) + x(x+1)(x+3)\dots(x+n+1) + x(x+1)(x+2)(x$$

$$+ 4)\dots(x+n+1) + \dots + x(x+1)\dots(x+n)$$

$$f'(-1)=x(x+2)(x+3)\dots(x+n+1)$$

$$=(-1) \times 1 \times 2 \times 3 \dots \times n$$

$$=-n!$$

8.

$$y=\sin^4 x+\cos^4 x$$

$$=(\sin^2 x+\cos^2 x)^2-2\sin^2 x \cos^2 x$$

$$=1-\frac{1}{2}\sin^2 2x=\frac{3}{4}+\frac{1}{4}\cos 4x$$

$$y'=-\sin 4x$$

$$=\cos(4x+\frac{\pi}{2})$$

$$\therefore (\cos \omega x)^{(n)}=\omega^n \cos(\omega x+\frac{n\pi}{2})$$

$$\therefore y^{(n)}=4^{n-1} \cos(4x+\frac{n\pi}{2})$$

9.证明

$$f(x)=(x-a)^n \varphi(x)$$

$\therefore \varphi(x)$ 在点 a 的某领域内有 (n-1) 阶连续导函数

\therefore

$$\therefore f^{(n-1)}(x) = C_{n-1}^0 [(x-a)^n]^{(0)} \varphi^{(n-1)}(x) + C_{n-1}^1 [(x-a)^n]^{(1)} \varphi^{(n-2)}(x) + \dots +$$

$$C_{n-1}^{n-1} [(x-a)^n]^{(n-1)} \varphi^{(0)}(x)$$

$$\therefore f^{(n-1)}(a)=0$$

$$f^{(n)}(a) \lim_{x \rightarrow a} \frac{f^{(n-1)}(x) - f^{(n-1)}(a)}{x-a}, \quad \text{将上式代入}$$

$$\therefore f^{(n)}(a) = \varphi(a)n!$$

10.

(1) $f(x)$ 在 $x=0$ 连续:

$$f(0)=\lim_{x \rightarrow 0} f(x)=0$$

$$\therefore \lim_{x \rightarrow 0} x^m \sin \frac{1}{x}=0$$

$$\therefore \lim_{x \rightarrow 0} x^m=0 \rightarrow m>0$$

(2) $f(x)$ 在 $x=0$ 可导:

在 $m>0$ 前提下, 有 $f'(0)$ 存在

$$f'(0)=\lim_{x \rightarrow 0} \frac{x^m \sin \frac{1}{x}}{x}=\lim_{x \rightarrow 0} x^{m-1} \sin \frac{1}{x}$$

$$\therefore m>1$$

(3) $f'(x)$ 在 $x=0$ 连续:

$$f'(0)=\lim_{x \rightarrow 0} f'(x)$$

由(2)知 $f'(0)$ 若存在则为 0

$$\therefore \lim_{x \rightarrow 0} f'(x)=0=\lim_{x \rightarrow 0} (mx^{m-1} \sin \frac{1}{x} - x^{m-2} \cos \frac{1}{x})$$

$$\begin{cases} m-2>0 \\ m-1>0 \end{cases} \Rightarrow m > 2$$

11. 证明

$$\text{当 } x \neq 0 \text{ 时, } f'(x)=e^{\frac{-1}{x^2}}(\frac{2}{x^3})$$

$$\text{又 } \because f'(0)=\lim_{x \rightarrow 0} \frac{f(x)-f(0)}{x-0}=\lim_{x \rightarrow 0} \frac{1}{xex^2}=0, \lim_{x \rightarrow 0} f'(x)=\lim_{x \rightarrow 0} \frac{2}{x^3ex^2}=0$$

$\therefore f'(0)=0$ 且 $f'(x)$ 在 $x=0$ 连续

$$\therefore f'(x)=\begin{cases} \frac{2}{x^3}e^{\frac{-1}{x^2}}, & x \neq 0 \\ 0, & x=0 \end{cases}, f''(0)=\lim_{x \rightarrow 0} \frac{f'(x)-f'(0)}{x-0}=0$$

$$f''(x)=\begin{cases} (\frac{-6}{x^4} + \frac{4}{x^6})e^{\frac{-1}{x^2}}, & x \neq 0 \\ 0, & x=0 \end{cases} \quad (\text{关于 } x^{-1} \text{ 的六次多项式})$$

$$\text{设 } f^{(n)}(x) = \begin{cases} P_n\left(\frac{1}{x}\right)e^{-\frac{1}{x^2}}, & x \neq 0 \\ 0, & x = 0 \end{cases} \quad (P_n(x^{-1}) \text{ 是关于 } x^{-1} \text{ 的 } 3n \text{ 次多项式})$$

$$\text{则 } f^{(n+1)}(0) = \lim_{x \rightarrow 0} \frac{f^{(n)}(x) - f^{(n)}(0)}{x - 0} = \frac{x^{-1}P_n(x^{-1})}{\frac{1}{x^2}} = 0$$

$$\begin{aligned} f^{(n+1)}(x) &= \left(\frac{2}{x^3} P_n(x^{-1}) - \frac{1}{x^2} P'_n(x^{-1}) \right) e^{-\frac{1}{x^2}} \\ &= P_{n+1}(x^{-1}) e^{-\frac{1}{x^2}} (x \neq 0) \end{aligned}$$

显然 $P_{n+1}(x^{-1})$ 是关于 x 的 $3(n+1)$ 次多项式

$$\therefore f^{(n+1)}(x) = \begin{cases} P_{n+1}\left(\frac{1}{x}\right)e^{-\frac{1}{x^2}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

由数学归纳法可知 $f(x)$ 在 $x=0$ 处 n 阶可导且 $f^{(n)}(0)=0$

12.

(1) 证明

$\because \varphi(x)$ 在 $x=a$ 连续

$\therefore f(x)$ 在 $x=a$ 连续

$$\lim_{x \rightarrow a+} \frac{f(x) - f(a)}{x - a} = \lim_{x \rightarrow a+} \varphi(a) = \varphi(a)$$

$$\lim_{x \rightarrow a-} \frac{f(x) - f(a)}{x - a} = \lim_{x \rightarrow a-} \varphi(a) = \varphi(a)$$

$\therefore f(x)$ 在 $x=a$ 可导, 且 $f'(a) = \varphi(a)$

(2)

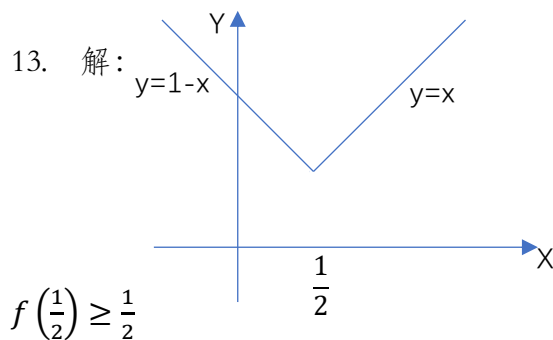
$$g'_+(a) = \lim_{x \rightarrow a+} \frac{|x-a|\varphi(x)}{x-a} = \varphi(a)$$

$$g'_-(a) = \lim_{x \rightarrow a-} \frac{|x-a|\varphi(x)}{x-a} = -\varphi(a)$$

要使 $g(x)$ 在 $x=a$ 可导

$$\text{则 } g'_+(a) = g'_-(a)$$

$$\text{即 } \varphi(a) = 0$$



因为 $f(x)$ 为多项式函数

所以 $f(x)$ 可导

由于 $f(x) \geq x$ 所以

假设 $f\left(\frac{1}{2}\right) = \frac{1}{2}$ 所以

$f\left(\frac{1}{2}\right)$ 为较小值

由费马定理, $f'\left(\frac{1}{2}\right) = 0$

当 $x > \frac{1}{2}$ 时, $f(x) \geq x$ 则 $f'_+\left(\frac{1}{2}\right) \geq 1$, 与 $f'\left(\frac{1}{2}\right) = 0$ 相矛盾

所以 $f\left(\frac{1}{2}\right) \neq \frac{1}{2}$ 故 $f\left(\frac{1}{2}\right) > \frac{1}{2}$

14. 解: $\lim_{x \rightarrow \infty} \left(\frac{f\left(\frac{1}{x}\right)}{f(0)} \right)^x = \lim_{x \rightarrow \infty} \left(1 + \frac{f\left(\frac{1}{x}\right) - f(0)}{f(0)} \right)^x = \lim_{x \rightarrow \infty} \left(1 + \right.$

$$\left. \frac{f\left(\frac{1}{x}\right) - f(0)}{f(0)} \right)^x =$$

$$e^{\lim_{x \rightarrow \infty} x \cdot \frac{f\left(\frac{1}{x}\right) - f(0)}{f(0)}} \quad \text{令 } x = \frac{1}{t} \quad x \rightarrow \infty, t \rightarrow 0$$

$$\therefore \lim_{x \rightarrow \infty} x \cdot \frac{f\left(\frac{1}{x}\right) - f(0)}{f(0)} = \lim_{t \rightarrow 0} \frac{f(t) - f(0)}{tf(0)} = \frac{f'(0)}{f(0)}$$

$$\therefore \text{原式} = e^{\frac{f'(0)}{f(0)}}$$

15. 解: $f(x) = -x^3 + x$ $f(x+1) = -x^3 - 3x^2 - 2x = af(x)$ $x \in [-1, 0)$

$f(x)$ 中 $x \in [0, 1)$ $f(x+1)$ 中 $x \in [-1, 0)$

$$\therefore f(x) = \begin{cases} -x^3 + x, & x \in [0, 1) \\ \frac{1}{a}(-x^2 - 3x - 2), & x \in [-1, 0) \end{cases}$$

因为 $f(0) = 0$ $\lim_{x \rightarrow 0} f(x) = f(0)$ $\therefore f(x)$ 在 $x = 0$ 处连续

$$f'_-(0) = \lim_{x \rightarrow 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0^-} \frac{1}{a}(-x^2 - 3x - 2) = \frac{-2}{a}$$

$$f'_+(0) = \lim_{x \rightarrow 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0^+} (-x^2 + 1) = 1$$

又因为 $f(x)$ 在 $x = 0$ 处可导 $\therefore f'_-(0) = f'_+(0) \therefore a = -2, f'(0) = 1$

16. 解: 因为 $f'(x) = f^2(x), f(0) = 2 \therefore f'(0) = f^2(0) = 4$

$$f''(0) = (f^2(0))' = 2f(0)f'(0) = 2f^3(0) = 2 \times 2^3$$

$$f'''(0) = 6f^4(0) = 6 \times 2^4$$

$$\text{设 } f^{(n)}(0) = n! \cdot 2^{n+1} = n! f^{n+1}(0)$$

则 $f^{(n+1)}(0) = [n! f^{n+1}(0)]' = (n+1)! f^n(0) \cdot f^2(0) = (n+1)! f^{n+2}(0)$

\therefore 由数学归纳法可知 $f^{(n)}(0) = n! \cdot 2^{n+1}$

17. 解: 因为 $f(xy) = f(x) + f(y) \therefore f(x) = f(x) + f(1) \therefore f(1) = 0$

$$\text{又因为 } f(1) = f(x) + f\left(\frac{1}{x}\right) = 0 \therefore f(x) = -f\left(\frac{1}{x}\right)$$

$$\begin{aligned} f'(x_0) &= \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h} = \lim_{h \rightarrow 0} \frac{f(x_0 + h) + f\left(\frac{1}{x_0}\right)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f\left(1 + \frac{h}{x_0}\right)}{h} \end{aligned}$$

$$\text{由洛必达法则可知 } f'(x_0) = \lim_{h \rightarrow 0} f'\left(1 + \frac{h}{x_0}\right) \cdot \frac{1}{x_0} = \frac{f'(1)}{x_0} = \frac{a}{x_0}$$

$$\therefore f'(x) = \frac{a}{x}, x \in (0, +\infty)$$

18. 解: 充分性: 若 $f(x)$ 在 $x = a$ 处可导且 $f'(a) = 0, f(a) = 0$, 则 $|f(x)|$ 在 $x = a$ 处可导

$$\text{因为 } f(a) = 0 \quad f'(a) = 0 \quad \therefore \lim_{x \rightarrow a} \frac{f(x)}{x-a} = 0, \therefore |f(a)| = 0$$

$$|f'_+(a)| = \lim_{x \rightarrow a^+} \frac{|f(x)| - |f(a)|}{x - a} = \lim_{x \rightarrow a^+} \frac{|f(x)|}{x - a} = 0$$

$$|f'_-(a)| = \lim_{x \rightarrow a^-} \frac{|f(x)| - |f(a)|}{x - a} = \lim_{x \rightarrow a^-} \frac{|f(x)|}{x - a} = 0$$

$$\therefore |f(x)| \text{ 在 } x = a \text{ 处可导且 } |f(a)|' = 0$$

必要性: 若 $|f(x)|$ 在 $x = a$ 处可导且 $f(a) = 0$, 则 $f'(a) = 0$

因为 $|f(x)|$ 在 $x = a$ 处可导 $\lim_{x \rightarrow a^+} \frac{|f(x)|}{x-a} = - \lim_{x \rightarrow a^-} \frac{|f(x)|}{x-a}$

$$\therefore \lim_{x \rightarrow a} \frac{|f(x)|}{x-a} = 0, \quad \therefore f'(a) = 0$$