习题 5.1

1.C

解析：$\mathrm{F}(x)$仅为I区间内$f(x)$的原函数，非整个区间$f(x)$的原函数，故 C 错误。

2.

（1）$\int f(x)\,dx = C \Rightarrow C' = (\int f(x)\,dx)' = 0 = f(x)$

（2）（3）区间I需连续，并非整个定义域内　　例$f(x) = \frac{1}{\mathrm{x}}$

（3）定义 5.1.1：设函数$f(x)$在某区间I上有定义，如果存在可导函数$\mathrm{F}(x)$，使得对I内每一点x，都有$\mathrm{F}'(x) = f(x)$或$d\mathrm{F}(x) = f(x)\,dx$，则称$\mathrm{F}(x)$为$f(x)$在区间I上的一个原函数。

（4）无穷多个——所有

3.

（1）$\int \left(3x^3 - 5x^2 + \frac{3}{x^2}\right) dx = \int (3x^3)\,dx - \int (5x^2)\,dx + \int \left(\frac{3}{x^2}\right) dx = \frac{3}{4}x^4 - \frac{5}{3}x^3 - \frac{3}{x} + C$

（2）$\int \sqrt{x\sqrt{x\sqrt{x}}}\,dx = \int \sqrt{x\sqrt{x \cdot x^{\frac{1}{2}}}}\,dx = \int \sqrt{x\sqrt{x^{\frac{3}{2}}}}\,dx = \int \sqrt{x \cdot x^{\frac{3}{4}}}\,dx = \int x^{\frac{7}{8}}\,dx = \frac{8}{15}x^{\frac{15}{8}} + C$

（3）$\int (2\tan x + 3\cot x)^2\,dx = \int (4\tan^2 x + 12\tan x \cdot \cot x + 9\cot^2 x)^2\,dx$

$= \int \left[4\left(\frac{1-\cos^2 x}{\cos^2 x}\right) + 12 + 9\left(\frac{1-\sin^2 x}{\sin^2 x}\right)\right] dx = \int \left(4\frac{1}{\cos^2 x} + 9\frac{1}{\sin^2 x} - 1\right) dx$

$= 4\tan x - 9\cot x - x + C$

（5）$\int e^{3x}(3^x - e^{-2x})\,dx = \int e^{3x} \cdot e^{xln3}\,dx - \int e^x\,dx$

$= \int e^{(ln3+3)x}\,dx - \int e^x\,dx = \frac{\mathrm{e}^{(ln3+3)x}}{ln3+3} - e^x + C$

$= \frac{e^x \cdot 3^x}{3+ln3} - e^x + C$

（6）$\int \left(\frac{1}{x} - \frac{3}{\sqrt{1-x^2}}\right) dx = ln|x| - 3\,arc\sin x + C$

（7）$\int \frac{\sqrt{x} - 2\sqrt[3]{x^2} + 1}{4\sqrt{x}}\,dx = \int \frac{\sqrt{x} - 2\sqrt[3]{x^2} + 1}{2}\,d\sqrt{x}$

令$t = \sqrt{x}$

原式$= \int \frac{t}{2}\,dt - \int t^{\frac{4}{3}}\,dt + \frac{1}{2}\int dt$

$= \frac{x}{4} - \frac{3}{7}x^{\frac{7}{6}} + \frac{\sqrt{x}}{2} + C$

（8）$\int \frac{2^{x-1} - 5^{x-1}}{10^x}\,dx = \int \frac{1}{2}\left(\frac{1}{5}\right)^x dx - \int \frac{1}{5}\left(\frac{1}{2}\right)^x dx = \frac{1}{5 \cdot 2^x ln2} - \frac{1}{2 \cdot 5^x ln5} + C$

（9）$\int \frac{(1-x)^2}{x(1+x^2)}\,dx = \int \frac{x^2+1-2x}{x(1+x^2)}\,dx = \int \frac{1}{x}\,dx - \int \frac{2}{x^2+1}\,dx = ln|x| - 2arc\tan x + C$

（10）$\int \frac{x^2}{1+x^2}\,dx = \int \frac{x^2+1-1}{1+x^2}\,dx = \int dx - \int \frac{1}{1+x^2}\,dx = x - arc\tan x + C$

（11）$\int \frac{1+\cos^2 x}{1+\cos 2x} dx = \int \frac{2-\sin^2 x}{2\sin^2 x} dx = -\frac{1}{2}\int dx + \int \frac{1}{\sin^2 x} dx = \cot x - \frac{1}{2}x + C$

4. 解：由题意得 $f'(x) = \frac{2}{\sqrt{1-x^2}}$

$f(x) = \int f'(x)\, dx = 2arc\sin x + C$

因为$f\left(\frac{1}{2}\right) = 0$ 得C=$-\frac{\pi}{3}$

所以$f(x) = 2arc\sin x - \frac{\pi}{3}$

5. 解：由题意得 $x = \int v\, dt = t^3 - t + C\ m$

因为在$x(t)$中，$x(1) = 10m$

所以$C = 10$

所以当$t = 3$时 $x(3) = 34m$

6. 证明：因为$\int f(x)\, dx = \mathrm{F}(x) + C$

所以$\mathrm{F}'(x) = f(x)$ $\mathrm{F}'(ax+b) = af(ax+b)$

对两边积分得 $\mathrm{F}(ax+b) = a\int f(ax+b)\, dx + C$

因为$C \in R$

所以$\frac{1}{a}\mathrm{F}(ax+b) + C = af(ax+b)\, dx$