习题 5.3

1.

(1) $\int \mathrm{x}\cos\mathrm{x}\, dx = \int \mathrm{x}\, d\sin\mathrm{x} = x\sin\mathrm{x} - \int \sin\mathrm{x}\,\mathrm{d}\,x$

$= x\sin\mathrm{x} + \cos\mathrm{x} + C$

(2) $\int \ln\mathrm{x}\,\mathrm{d}\,x = \int \mathrm{x}\ln\mathrm{x}\, dx$

$= x\ln\mathrm{x} - \int \mathrm{x}\, d\ln\mathrm{x}$

$= x\ln\mathrm{x} - \int \mathrm{x}\cdot\frac{1}{\mathrm{x}}dx$

$= x\ln\mathrm{x} - x + C$

(3) $\int \mathrm{x}^2\mathrm{e}^{\mathrm{x}}\mathrm{d}\,x = \int \mathrm{x}^2(\mathrm{e}^{\mathrm{x}})'\mathrm{d}\,x = \mathrm{x}^2\mathrm{e}^{\mathrm{x}} - \int \mathrm{e}^{\mathrm{x}}\cdot 2x dx$

$= \mathrm{x}^2\mathrm{e}^{\mathrm{x}} - \left(\mathrm{e}^{\mathrm{x}}\cdot 2\mathrm{x} - \int \mathrm{e}^{\mathrm{x}}\cdot 2\mathrm{dx}\right)$

$= \mathrm{e}^{\mathrm{x}}(\mathrm{x}^2 - 2\mathrm{x} + 2) + C$

(4) $\int \arcsin\mathrm{x}\,\mathrm{d}\,x = \int \mathrm{x}'\arcsin\mathrm{x}\, dx = x\arcsin\mathrm{x} - \int \mathrm{x}\cdot\frac{1}{\sqrt{1-\mathrm{x}^2}}dx$

$= x\arcsin\mathrm{x} + \sqrt{1-\mathrm{x}^2} + C$

(5) $\int \frac{\ln(\ln\mathrm{x})}{\mathrm{x}}dx = \int (\ln\mathrm{x})'\ln(\ln\mathrm{x})\, dx$

$= \ln\mathrm{x}\cdot\ln(\ln\mathrm{x}) - \int \ln\mathrm{x}\cdot\frac{\frac{1}{\mathrm{x}}}{\ln\mathrm{x}}dx$

$= \ln\mathrm{x}\cdot\ln(\ln\mathrm{x}) - \ln\mathrm{x} + C$

(6) $\int e^{2x}\cos x\,dx=\int \frac{1}{2}(e^{2x})'\cos x\,dx$

$=\frac{1}{2}\left(e^{2x}\cos x+\int e^{2x}\sin x\,dx\right)$

$=\frac{1}{2}\left[e^{x}\cos x+\frac{1}{2}\left(e^{2x}\sin x-\int e^{2x}\cos x\,dx\right)\right]$

移项可得 $\int e^{2x}\cos x\,dx=\frac{1}{5}e^{2x}(\sin x+2\cos 2x)+C$

(7) $\int x\sin x\cos x\,dx=\frac{1}{2}\int x\sin 2x dx=-\frac{1}{4}\int x(\cos 2x)'d\,x$

$=-\frac{1}{4}\left(x\cos 2x-\int \cos 2x\,dx\right)$

$=-\frac{1}{4}\left(x\cos 2x-\frac{1}{2}\sin 2x\right)+C$

(8) $\int x f''(x)dx=\int x\left(f'(x)\right)'d\,x$

$=xf'(x)-\int f'(x)d\,x$

$=xf'(x)-f(x)+C$

(9) $\int x\sin^2 x\,dx=\int x\frac{1-\cos 2x}{2}dx$

$=\frac{1}{2}\int x\left(x-\frac{1}{2}\sin 2x\right)'d\,x$

$=\frac{1}{2}\left[x^2-\frac{1}{2}x\sin 2x-\int\left(x-\frac{1}{2}\sin 2x\right)d\,x\right]$

$=\frac{1}{2}\left[x^2-\frac{1}{2}x\sin 2x-\frac{1}{2}x^2+\left(-\frac{1}{4}\cos 2x\right)\right]$

$=\frac{1}{4}x^2-\frac{x}{4}\sin 2x-\frac{1}{8}\cos 2x+C$

$$(10) \int x(\arctan x)^2 \mathrm{d}\, x$$

$$= \int \frac{1}{2}(x^2)'(\arctan x)^2 \mathrm{d}\, x$$

$$= \frac{1}{2}[x^2 \arctan x)^2 - \int 2 \arctan x \left(1 - \frac{1}{1+x^2}\right) dx\Big]$$

$$= \frac{1}{2}x^2(\arctan x)^2 - \int x' \arctan x\, dx + \frac{1}{2}[[(\arctan x)^2]' dx$$

$$= \frac{1}{2}x^2(\arctan x)^2 - \left(x \arctan x - \int \frac{x}{1+x^2} dx\right) + \frac{1}{2}(\arctan x)^2$$

$$= \frac{1+x^2}{2}(\arctan x)^2 - x \arctan x + \sqrt{1+x^2} + C$$

$$(11) \int \ln\left(x + \sqrt{1+x^2}\right) \mathrm{d}\, x$$

$$= \int x' \ln\left(x + \sqrt{1+x^2}\right) dx$$

$$= x \ln\left(x + \sqrt{1+x^2}\right) - \int \frac{1 + \frac{2x}{2\sqrt{1+x^2}}}{x + \sqrt{1+x^2}} \cdot x dx$$

$$= x \ln\left(x + \sqrt{1+x^2}\right) - \int \frac{x}{\sqrt{1+1^2}} dx$$

$$= x \ln\left(x + \sqrt{1+x^2}\right) - \sqrt{1+x^2} + C$$

$$(12) \int \frac{x \cos x}{\sin^3 x} dx = \int -x \cdot \frac{1}{2}\left(\frac{1}{\sin^2 x}\right)' dx$$

$$= -\left(\frac{x}{\sin^2 x} - \int \frac{1}{\sin^2 x} dx\right)$$

$$= -\frac{x}{2\sin^2 x} - \frac{1}{2}\cot x + C$$

$$(13) \int \sec^5 x\, \mathrm{d}\, x = \int (\tan x)' \sec^3 x\, dx$$

$$= \tan x \sec^3 x - 3\int \tan x \sec^4 \sin x\, dx$$

得①$4\int \sec^5 x\, d\, x = \tan x \sec^3 x + 3\int \sec^3 x\, d\, x$

由$\displaystyle\int \sec^3 x\, d\, x = \tan x \sec x - \int \tan x \frac{\sin x}{\cos^2 x} dx$

得②$2\int \sec^3 x\, d\, x = \tan x \sec x + \int \frac{1}{\cos x} dx$

$= \tan x \sec x + \ln|\sec x + \tan x| + C$

②代入①得

$$\int \sec^5 x\, d\, x = \frac{1}{4}\tan x \sec^3 x + \frac{3}{8}\tan x \sec x + \frac{3}{8}\ln|\sec x + \tan x| + C$$

(14) $\displaystyle\int \frac{x^2 \arctan x}{1+x^2} dx$

$$= \int \left(1 - \frac{1}{1+x^2}\right) \arctan x\, dx$$

$$= (x - \arctan x) \arctan x - \int (x - \arctan x) \frac{1}{1+x^2} dx$$

$$= (x - \arctan x) \arctan x - \frac{1}{2}\ln(1+x^2) + \int \frac{\arctan x}{1+x^2} dx$$

由$\displaystyle\int \frac{\arctan x}{1+x^2} dx = (\arctan x)^2 - \int \frac{\arctan x}{1+x^2} dx$

得$\displaystyle\int \frac{\arctan x}{1+x^2} = \frac{1}{2}(\arctan x)^2 + C$

$$\int \frac{x^2 \arctan x}{1+x^2} dx = x \arctan x - \frac{1}{2}\ln(1+x^2) - \frac{1}{2}(\arctan x)^2 + C$$

2.对于正整数 n≥2,建立$I_n = \int \sin^n x\, d\, x$的递推公式

$$I_n = \int \sin^{n-1} x \cdot \sin x\, dx = -\int \sin^{n-1} x\, (\cos x)'\, d\, x$$

$$= -\sin^{n-1} x \cos x + \int \cos x \,(n-1) \sin^{n-2} x \cos x \, dx$$

$$= -\sin^{n-1} x \cos x + \int (1 - \sin^2 x)(n-1) \sin^{n-2} x \, dx$$

$$= -\sin^{n-1} x \cos x + (n-1)\left(\int \sin^{n-2} x \, \mathrm{d}\, x - \int \sin^{n} x \, \mathrm{d}\, x\right)$$

整理可得$I_n = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} I_{n-2}$