习题 3.1

1. (1)

$f(x) = x^2$, $x_0 = 1$

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x_0+\Delta x)^2-(x_0)^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} (2+\Delta x)$$

$$=2$$

(2)

$f(x)=\frac{1}{x^2}$, $x_0 = 2$

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\frac{1}{(x_0+\Delta x)^2}-\frac{1}{(x_0)^2}}{\Delta x}$$

$$=- \lim_{\Delta x \to 0} \frac{2x_0+\Delta x}{{x_0}^2(x_0+\Delta x)^2}$$

$$=-\frac{2}{{x_0}^3}$$

$$=-\frac{1}{4}$$

(3)

$f(x)=x(x+1)\dots(x+2020)$, $x_0 = 0$

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x_0+\Delta x)(x_0+1+\Delta x)\dots(x_0+2020+\Delta x)-x_0(x_0+1)\dots(x_0+2020)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x_0[(x_0+1+\Delta x)\dots(x_0+2020+\Delta x)-x_0(x_0+1)\dots(x_0+2020)]}{\Delta x}$$

$$+\lim_{\Delta x \to 0}(x_0+1+\Delta x)\dots(x_0+2020+\Delta x)$$

$=\lim\limits_{\Delta x \to 0}(x_0+1+\Delta x)\dots(x_0+2020+\Delta x)$

=2020!

2. (1)

$f'_-(0)=\lim\limits_{x \to 0^-}\frac{f(x)-f(0)}{x-0}$

$=\lim\limits_{x \to 0^-}\frac{f(x)}{x}$

$=+\infty$

$f'_+(0)=\lim\limits_{x \to 0^+}\frac{f(x)-f(0)}{x-0}$

$=\lim\limits_{x \to 0^+}\frac{f(x)}{x}$

$=+\infty$

∴f(x)在 x=0 处不可导

(2)

$f'_-(0)=\lim\limits_{x \to 0^-}\frac{f(x)-f(0)}{x-0}$

$=\lim\limits_{x \to 0^-}\frac{f(x)-1}{x}$

$=\lim\limits_{x \to 0^-}x^2$

=0

$f'_+(0)=\lim\limits_{x \to 0^+}\frac{f(x)-f(0)}{x-0}$

$=\lim\limits_{x \to 0^+}\frac{f(x)-1}{x}$

$=\lim\limits_{x \to 0^+}x$

=0

∵ $f'(0)=f'_+(0)=f'_-(0)=0$

∴ f(x)在 x=0 处可导

3. (1)

∵ $y'|_{x=0} = e^{x}|_{x=0} = 1$

∴ $k_{切}=1$，$k_{法}=-1$

$L_{切}$： $y = x+1$

$L_{法}$： $y =-x+1$

（2）

设 $P(x_o,\ln x_o)$，则 $y|_{x=x_o}=\frac{1}{x}$ 令$\frac{1}{x_o}=\frac{1}{2}$，解得 $x_o=2$， 即 $P(2,\ln 2)$.

4. 在 x=1 处可导=> f(x)在 x=1 处连续=> $\lim\limits_{x\to1-} f(x) = \lim\limits_{x\to1+} f(x) = f(1)$,

在 x=1 处可导=>左右导数存在且相等，$\lim\limits_{x\to1-} f'(x) = \lim\limits_{x\to1+} f'(x)$

即 $\lim\limits_{x\to1-} 2x = \lim\limits_{x\to1+} a$ ②

解①②得：$\mathbf{a}= 2$，$\boldsymbol{b} = -1$

5.证明：左边= $\lim\limits_{h\to\infty}\frac{f(x0+h)-f(x0)+f(x0)-f(x0-h)}{h}$

$= \lim\limits_{h\to0}\frac{f(x0+h)-f(x0)}{(x0+h)-x0} + \lim\limits_{h\to0}\frac{f(x0)-f(x0-h)}{x0-(x0-h)}$

$= 2f'(x_0)$

= 右边

6.证明：①偶函数满足：$f(x)= f(-x)$

两边同时求导：$f'(x)=-f'(-x)$

即偶函数导数为奇函数；

②奇函数满足：**$-f(x)=-f(-x)$**

两边同时求导**:** $-f'(x)=-f'(-x)$

$=> f'(x)=f'(-x)$

即奇函数的导数为偶函数；

③周期函数满足：$f(x)=f(x+T)$

两边同时求导：$f'(x)=f'(x+T)$

即周期函数的导数为周期函数。

7.解：$f(x)=\begin{cases} -1,-1\le x<0 \\ 0,0\le x<1 \\ 1,x=1 \end{cases}$

①$f'_{+}(0)=\lim\limits_{\Delta x\to 0-}\frac{f(0+\Delta x)-f(0)}{\Delta x}=\lim\limits_{\Delta x\to 0-}(-\frac{1}{\Delta x})=+\infty$；(其为函数在 x=0 点的左导数)

②$f'_{+}(0)=\lim\limits_{\Delta x\to 0+}\frac{f(0+\Delta x)-f(0)}{\Delta x}=0$；

③因$f'_{+}(0)\neq f_{-}'(0)$,故$f'(0)$不存在；

④$\lim\limits_{x\to 0}f'(x)=\lim\limits_{x\to 0-}f'(x)=\lim\limits_{x\to 0+}f'(x)=0$ (其为在 x 趋向于 0 时函数的导数值)。

8.解：$|f(0)|\le 1-\cos 0=0$,即$f(0)=0$

①如果要证明连续性：$\cos x-1\le f(x)\le 1-\cos x$

因$\lim\limits_{x\to 0-}(cos0-1)=\lim\limits_{x\to 0+}(1-cos0)=0=f(0)$

则$f(x)$在 x=0 处连续；

②证明可导性：

$$\lim_{x\to 0-}\frac{-(\cos x-1)-[-(cos0-1)]}{x-0}$$

$$\leq \lim_{x\to 0-}\frac{f(x)-f(0)}{x-0} \leq \lim_{x\to 0-}\frac{(\cos x-1)-(cos0-1)}{x-0}$$

因$\lim_{x\to 0-}\frac{1-\cos x}{x}=\lim_{x\to 0-}\frac{\cos x-1}{x}$=0(等价无穷小)

由夹逼定理可得，f'_{+}(0)=0,同理可得，f'_{-}(0)=0

则f'(0)=0，f(x)在 x=0 处可导。

习题 3.2

本颜色字体均为概念或公式

1.

(1)$(x^n)' = nx^{n-1}$

$\Longrightarrow y = x^3 - 2x^2 + 3x - 4$

$y' = 3x^2 - 4x + 3$

(2)$(uv)' = u'v + uv'$

$y = (x^2 + 3x + 2)(x^2 - 3x + 2)$

方法$a. y' = (2x+3)(x^2 - 3x + 2) + (x^2 + 3x - 2)(2x - 3)$

$\Rightarrow y' = 4x^3 - 10x$

方法$b. y = (x^2 + 2)^2 - (3x)^2 = x^4 - 5x^2 + 4$

$\Rightarrow y' = 4x^3 - 10x$

(3) $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$

$$y = \frac{\cos x}{x} + \frac{x}{\cos x}$$

$$\Rightarrow y' = \frac{-\sin x \cdot x - \cos x}{x^2} + \frac{\cos x + x \sin x}{\cos^2 x}$$

$\Rightarrow y' = -x^{-1} \cdot \sin x - x^{-2} \cos x + \sec x + x \tan x \cdot \sec x$

Tip： $\frac{1}{\sin x} = \csc x, \frac{1}{\cos x} = \sec x$

(4)$(\ln x)' = \frac{1}{x} / (\ln|x|)' = \frac{1}{x}$

$y = x \ln x$

$$\Rightarrow y' = \ln x + x \cdot \frac{1}{x}$$

$$\Rightarrow y' = \ln x + 1$$

$(5) y = x^2 + x^{-2}$

$$\Rightarrow y' = 2x - 2x^{-3}$$

$(6) (e^x)' = e^x$

$y = e^x \cos x$

$$\Rightarrow y' = e^x \cos x + e^x(-\sin x)$$

$$\Rightarrow y' = e^x(\cos x - \sin x)$$

(7)y=$exsinx$

$$\Rightarrow y' = e^x \sin x + e^x \cos x$$

$$\Rightarrow y' = e^x(\sin x + \cos x)$$

$(8) y = e^x \ln x$

$$\Rightarrow y' = e^x \ln x + e^x \cdot \frac{1}{x}$$

$$\Rightarrow y' = e^x \left(\ln x + \frac{1}{x}\right)$$

2.

$(1) \left(f\left(g(x)\right)\right)' = f'\left(g(x)\right) \cdot g'(x)$

$y = e^{x^2 + \sin x}$

$$\Rightarrow y' = e^{x^2 + \sin x} \cdot (2x + \cos x)\Big) \cdot g'(x)$$

$(2) y = x \ln(x^2 + e^x)$

$$\Rightarrow y' = \ln(x^2 + e^x) + x \cdot \frac{1}{x^2 + e^x} \cdot (2x + e^x)$$

$$\Rightarrow y' = ln(x^2 + e^x) + \frac{2x^2 + xe^x}{x^2 + e^x}$$

$(3)y = sin\,2x$

$\Rightarrow y' = 2\cos 2x$

$(4)y = cos\,2x$

$\Rightarrow y' = -2\sin 2x$

$(5)y = \sqrt{x}\,arcsin\sqrt{x}$

Tip：$(arcsin\,x)' = \dfrac{1}{\sqrt{1-x^2}}$;

$(arccos\,x)' = -\dfrac{1}{\sqrt{1-x^2}}$;

$(arctan\,x)' = \dfrac{1}{1+x^2}$;

$(arccotx)' = -\dfrac{1}{1+x^2}$

$$\Rightarrow y' = \frac{1}{2\sqrt{x}} \cdot arcsin\sqrt{x} + \sqrt{x} \cdot \frac{1}{\sqrt{1-x}} \cdot \frac{1}{2\sqrt{x}}$$

$$\Rightarrow y' = \frac{arcsin\sqrt{x}}{2\sqrt{x}} + \frac{1}{2\sqrt{1-x}} \quad (x \neq 0)$$

$(6)y = \sqrt{x}\,arccos\sqrt{x}$

$$\Rightarrow y' = \frac{1}{2\sqrt{x}} \cdot arccos\sqrt{x} + \sqrt{x} \cdot \left(-\frac{1}{\sqrt{1-x}}\right) \cdot \frac{1}{2\sqrt{x}}$$

$$\Rightarrow y' = \frac{arccos\sqrt{x}}{2\sqrt{x}} - \frac{1}{2\sqrt{1-x}} \quad (x \neq 0)$$

$(7)y = x^2\,arctan\dfrac{1}{x}$

$$\Rightarrow y' = 2x \cdot arctan\frac{1}{x} + x^2\frac{1}{1+\left(\frac{1}{x}\right)^2} \cdot \left(-\frac{1}{x^2}\right)$$

$$\Rightarrow y' = 2x\,arctan\frac{1}{x} - \frac{x^2}{x^2+1}$$

$$(8)y = x^2 arccot\frac{1}{x}$$

$$\Rightarrow y' = 2x \cdot arccot\frac{1}{x} + x^2\left(-\frac{1}{1+\left(\frac{1}{x}\right)^2}\right) \cdot \left(\frac{1}{x^2}\right)$$

$$\Rightarrow y' = 2xarccot\frac{1}{x} - \frac{x^2}{x^2+1}$$

$$(9)(sec\,x)' = tan\,x \cdot sec\,x$$

$$(csc\,x)' = -\,cot\,x\,csc\,x$$

$$(tan\,x)' = sec^2\,x$$

$$(cot\,x)' = -\,csc^2\,x$$

$$y = sec\,x^2$$

$$\Rightarrow y' = tan\,x^2 - sec\,x^2 \cdot (2x)$$

$$\Rightarrow y' = 2x\,tan\,x^2\,sec\,x^2$$

$$(10)y = csc\,\sqrt{x}$$

$$\Rightarrow y' = -\,cot\,\sqrt{x}\,csc\,\sqrt{x} \cdot \frac{1}{2\sqrt{x}}$$

$$\Rightarrow y' = -\frac{cot\,\sqrt{x}\,csc\,cos\,x}{2\sqrt{x}}$$

$$(11)y = x^{\frac{1}{2}} + x^{-\frac{1}{2}}$$

$$\Rightarrow y' = \frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{2}x^{-\frac{3}{2}}$$

$$\Rightarrow y' = \frac{1}{2}\left(x^{-\frac{1}{2}} - x^{-\frac{3}{2}}\right)$$

$$(12) y = \frac{e^{\sqrt{x}} - e^{-\sqrt{x}}}{e^{\sqrt{x}} + e^{-\sqrt{x}}}$$

$$\Rightarrow y = \frac{e^{\sqrt{x}} + e^{-\sqrt{x}} - 2e^{-\sqrt{x}}}{e^{\sqrt{x}} + e^{-\sqrt{x}}}$$

$$y = 1 - 2\frac{1}{e^{2\sqrt{x}} + 1}$$

$$\Rightarrow y' = -2\frac{-e^{2\sqrt{x}} \cdot \frac{1}{\sqrt{x}}}{\left(e^{2\sqrt{x}} + 1\right)^2}$$

$$\Rightarrow y' = \frac{2e^{2\sqrt{x}}}{\sqrt{x}\left(e^{2\sqrt{x}} + 1\right)^2} \quad (x \neq 0)$$

3.

$$(1) \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

对两边关于x求导

$$\Rightarrow \frac{1}{a^2} \cdot 2x + \frac{1}{b^2} \cdot 2y \cdot y' = 0$$

$$\Rightarrow y' = -\frac{b^2 x}{a^2 y} \quad (y \neq 0)$$

$$(2) x^2 + 2xy - y^2 = 2x$$

对两边关于x求导

$$\Rightarrow 2x + 2y + 2xy' - 2y \cdot y' = 2$$

$$\Rightarrow (x - y)y' = 1 - x - y$$

$$\Rightarrow y' = \frac{1 - x - y}{x - y} \quad (x \neq y)$$

$$(3) \sqrt{x} + \sqrt{y} = \sqrt{a}$$

对两边关于x求导

$$\Rightarrow \frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{y}}\cdot y'=0$$

$$\Rightarrow y'=-\frac{\sqrt{xy}}{x} \quad (x>0, y>0)$$

$(4) x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$

对两边关于x求导

$$\Rightarrow \frac{2}{3}x^{-\frac{1}{3}}+\frac{2}{3}y^{-\frac{1}{3}}-y'=0$$

$$\Rightarrow y'=-\left(\frac{y}{x}\right)^{\frac{1}{3}} \quad (x\neq 0)$$

$(5)\ arctan\frac{y}{x}=ln\sqrt{x^2+y^2}$

对两边关于x求导

$$\Rightarrow \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{y'x-y}{x^2}=\frac{1}{\sqrt{x^2+y^2}}\cdot\frac{2x+2y\cdot y'}{2\sqrt{x^2+y^2}}$$

$$\Rightarrow \frac{y'x-y}{x^2+y^2}=\frac{x+y\cdot y'}{x^2+y^2}$$

$$\Rightarrow y'x-y=x+y-y'$$

$$\Rightarrow y'=\frac{x+y}{x-y} \quad (x\neq y, x\neq 0)$$

$(6) x^y=y^x \quad (x>0, y>0)$

$$\Rightarrow y\,ln\,x=x\,ln\,y$$

对两边关于x求导

$$\Rightarrow y'\,ln\,x+y\frac{1}{x}=ln\,y+x\frac{1}{y}\cdot y'$$

1°由$y\,ln\,x=x\,ln\,y$变形得

$$\Rightarrow y'\left(\ln x-\frac{x}{y}\right)=y'(\ln y-1)\cdot\frac{x}{y}$$

$$\ln y-\frac{y}{x}=\frac{y}{x}(\ln x-1)$$

$$\Rightarrow y'=\frac{y^2(\ln x-1)}{x^2(\ln y-1)}\quad (x>0,y>0)$$

$$2^\circ\Rightarrow y'=\frac{xy\ln y-y^2}{xy\ln x-x^2}\quad (x>0,y>0)$$

$(7)x-y+\xi\sin y=0$ $(\xi$为参数$)$

对两边关于x求导

$$\Rightarrow 1-y'+\xi\cos y\cdot y'=0$$

$$\Rightarrow y'=\frac{1}{1-\xi\cos y}$$

4.

$(1)y=x^{\sin x}$

对两边取对数

$$\Rightarrow \ln y=\sin x\ln x$$

$$\Rightarrow \frac{1}{y}\cdot y'=\cos x\ln x+\sin x\frac{1}{x}$$

$$\Rightarrow y'=y\cos x\ln x+\frac{y\sin x}{x}$$

$$\Rightarrow y'=x^{\sin x}\left(\cos x\ln x+\frac{\sin x}{x}\right)$$

$(2)y=x^{\ln x}$

对两边取对数

$$\Rightarrow \ln y=\ln x\cdot\ln x=\ln^2 x$$

$$\Rightarrow \frac{1}{y}\cdot y' = 2\ln x\cdot\frac{1}{x}$$

$$\Rightarrow y' = \frac{2y\ln x}{x}$$

$$\Rightarrow y' = 2x^{(\ln x)-1}\ln x$$

$$(3) y = \sqrt[3]{\frac{(x-1)(x-2)}{(x-3)(x-4)}}$$

对两边取对数

$$\Rightarrow \ln y = \frac{1}{3}\ln\frac{(x-1)(x-2)}{(x-3)(x-4)}$$

$$= \frac{1}{3}\ln(x-1) + \frac{1}{3}\ln(x-2) - \frac{1}{3}\ln(x-3) - \frac{1}{3}\ln(x-4)$$

$$\Rightarrow \frac{1}{y}\cdot y' = \frac{1}{3}\frac{1}{x-1} + \frac{1}{3}\cdot\frac{1}{x-2} - \frac{1}{3}\cdot\frac{1}{x-3} - \frac{1}{3}\cdot\frac{1}{x-4}$$

$$\Rightarrow y' = \frac{-4x^2+20x-22}{3(x-1)(x-2)(x-3)(x-4)}\cdot y$$

$$\Rightarrow y' = \frac{-4x^2+20x-22}{3(x-1)(x-2)(x-2)(x-4)}\sqrt[3]{\frac{(x-1)(x-2)}{(x-3)(x-4)}}$$

5.

$$y' = \frac{dy}{dx}, y = y(t), x = x(t) \Rightarrow y' = \frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx}$$

$\Rightarrow x = x(t)$ 在 $t \in D$ 时单调

$$(1)\begin{cases} x = 1 - t^2 \\ y = 1 - t^3 \end{cases} \Rightarrow \frac{dx}{dt} = -2t, \quad \frac{dy}{dt} = -3t^2 \Rightarrow y' = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3}{2}t$$

$$(2)\begin{cases} x = \ln(1+t^2) \\ y = t - \arctan t \end{cases}$$

$$\Rightarrow \frac{dx}{dt}=\frac{2t}{1+t^2},\frac{dy}{dt}=1-\frac{1}{1+t^2}=\frac{t^2}{1+t^2}\Rightarrow y'=\frac{dy}{dx}=\frac{\frac{dy}{dx}}{\frac{dx}{dt}}=\frac{t}{2}$$

$$(3)\begin{cases}x=a\cos^3 t\\y=a\sin^3 t\end{cases}\Rightarrow \frac{dx}{dt}=3a\cos^2 t\,(-\sin t),\quad \frac{dy}{dt}=3a\sin^2 t\cdot\cos t$$

$$\Rightarrow y'=\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=-\tan t$$

6.

证明：$\because \sqrt{x}+\sqrt{y}=\sqrt{a}(a>0)\quad(x\geq 0,y\geq 0)$

$\therefore$ 抛物线与x轴交点为$P_1(a,0)$，与y轴交点为$P_2(0,a)$

对$\sqrt{x}+\sqrt{y}=\sqrt{a}(a>0)$两边关于$x$求导

$$\Rightarrow \frac{1}{2\sqrt{x}}+\frac{y'}{2\sqrt{y}}=0$$

$$\Rightarrow y'=-\frac{\sqrt{xy}}{x}=k_{切}$$

$\Rightarrow l_{切}:y-y_0=-\dfrac{\sqrt{x_0y_0}}{x_0}(x-x_0)\Rightarrow$ 抛物线在x_0点处的切线

在x_0点处$\sqrt{x_0}+\sqrt{y_0}=\sqrt{a}\Rightarrow x_0+2\sqrt{x_0y_0}+y_0=a$

$\Rightarrow l_{切}$与x轴交点为$P_1\left(x_0+\sqrt{x_0y_0},0\right)$，与$y$轴交点为$P_2\left(0,y_0+\sqrt{x_0y_0}\right)$

$x_0\,y_0=\sqrt{x_0\,y_0}(x-x_0)$，$y-y_0=\sqrt{x_0y_0}$

$$\Rightarrow x_0+\sqrt{x_0y_0}+y_0+\sqrt{x_0y_0}=x_0+2\sqrt{x_0y_0}+y_0=a$$

故抛物线$\sqrt{x}+\sqrt{y}=\sqrt{a}(a>0)$上任一点的切线截两个坐标轴的截距之和为$a$

7.

证明：$\begin{cases}x=a(\cos t+t\sin t)\\y=a(\sin t+t\cos t)\end{cases}$

$$\Rightarrow \begin{array}{l} \dfrac{dx}{dt} = a(-\sin t + \sin t + t\cos t) = at\cos t \\ \dfrac{dy}{dt} = a(\cos t - \cos t + t\sin t) = at\sin t \end{array}$$

$$\Rightarrow y' = \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{\sin t}{\cos t} = \tan t$$

$$\Rightarrow k_{切} = -\frac{1}{y'} = -\cot t$$

$\Rightarrow$ 曲线上点x_0的切线为$y - y_0 = -\cot t_0\,(x - x_0)$

$l_{法}$： $\cos t_0\, x + \sin t_0\, y - x_0\cos t_0 - y_0 \sin t_0 = 0$

$$\Rightarrow d = \frac{|\cos t_0 \cdot x_1 + \sin t_0 \cdot y_1 - x_0\cos t_0 - y_0\sin t_0|}{\sqrt{\cos^2 t_0 + \sin^2 t_0}}\Big((x_1, y_1)为原点\Big)$$

$$= a\cos^2 t_0 + at_0\sin t_0\cos t_0 + a\sin^2 t_0 - at_0\sin t_0\cos t_0$$

$$= a(\cos^2 t_0 + \sin^2 t_0)$$

$$=\ a$$

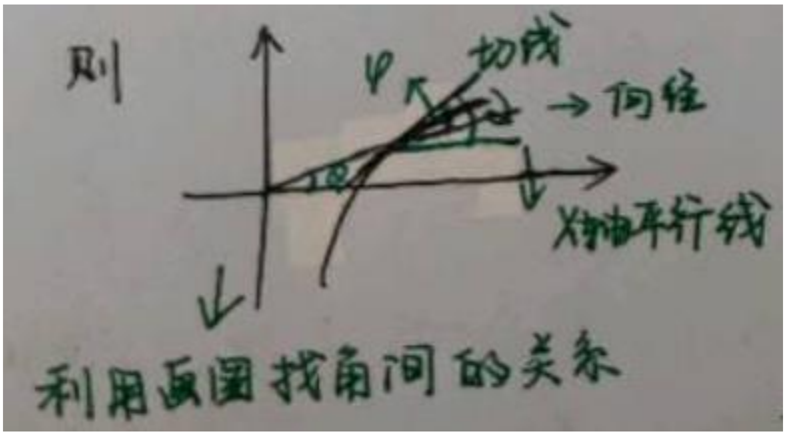
8.

(1)由题得l：$r = r(\theta).\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$，$r = r(\theta)$关于$\theta$可导

$$\Rightarrow \begin{array}{l} \dfrac{dx}{d\theta} = r'\cos\theta - r\sin\theta = r'(\theta)\cos\theta - r(\theta)\sin\theta \\ \dfrac{dy}{d\theta} = r'\sin\theta + r\cos\theta = r'(\theta)\sin\theta + r(\theta)\cos\theta \end{array}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{r'(\theta)\sin\theta + r(\theta)\cos\theta}{r'(\theta)\cos\theta - r(\theta)\sin\theta} = \frac{r'(\theta)\tan\theta + r(\theta)}{r'(\theta) - r(\theta)\tan\theta}$$

(2)设点$P(r,\theta)$处切线与x轴夹角为α



$\Rightarrow$ 同位角：$\theta = \alpha - \varphi$

即$\varphi = \alpha - \theta$

$$\Rightarrow \begin{matrix} \tan\varphi = \tan(\alpha - \theta) = \dfrac{\tan\alpha - \tan\theta}{1 + \tan\alpha\tan\theta} \\ \tan\alpha = k_{切} = \dfrac{dy}{dx} = \dfrac{r'(\theta)\tan\theta + r(\theta)}{r'(\theta) - r(\theta)\tan\theta} \end{matrix}$$

上下同乘$[r'(\theta) - r(\theta)\tan\theta]$

$$\Rightarrow \tan\varphi = \frac{\dfrac{r'(\theta)\tan\theta + r(\theta)}{r'(\theta) - r(\theta)\tan\theta} - \tan\theta}{1 + \dfrac{r'(\theta)\tan\theta + r(\theta)}{r'(\theta) - r(\theta)\tan\theta}\tan\theta}$$

$$\Rightarrow \tan\varphi = \frac{r'(\theta)\tan\theta + r(\theta) - r'(\theta)\tan\theta + r(\theta)\tan^2\theta}{r'(\theta) - r(\theta)\tan\theta + r'(\theta)\tan^2\theta + r(\theta)\tan\theta}$$

$$\Rightarrow \tan\varphi = \frac{r(\theta)(1 + \tan^2\theta)}{r'(\theta)(1 + \tan^2\theta)} = \frac{r(\theta)}{r'(\theta)}$$

9.

解：$\because$ 极坐标曲线$r = e^\theta$

$\therefore$ 其可用极角θ作为参数

表示如下

$$\begin{cases} x = r\cos\theta = e^\theta\cos\theta \\ y = r\sin\theta = e^\theta\sin\theta \end{cases}$$

$$\frac{dy}{d\theta} = e^{\theta} \sin \theta + e^{\theta} \cos \theta$$

$$\frac{dx}{d\theta} = e^{\theta} \cos \theta - e^{\theta} \sin \theta$$

$$\Rightarrow y' = \frac{\sin \theta + \cos \theta}{\cos \theta - \sin \theta}$$

点$\left(e^{\frac{\pi}{2}}, \frac{\pi}{2}\right)$处的极角为$\frac{\pi}{2}$

$$\Rightarrow y_1' = \frac{1+0}{0-1} = -1 \quad y_1 = e^{\frac{\pi}{2}}, x_1 = 0$$

$$\Rightarrow l_{切} = y - y_1 = y_1'(x - x_1)$$

$$\Rightarrow l_{切}: y = -x + e^{\frac{\pi}{2}}$$

$$\Rightarrow l_{切} = x + y - e^{\frac{\pi}{2}} = 0$$

习题 3.3

1. 求函数 y=x^2在点 x=1 的微分，其中自变量 x 的增量Δx分别如下：

$\Delta x = 0.1$；$\Delta x = 0.01$；$\Delta x = 0.001$；

解：由公式可得$d\,y|_{\{x=1\}} = f^{\prime}(1)\Delta x$，分别代入△x 可得：

dy=0.2；dy=0.02；dy=0.002

2.求下列函数的微分

（1）$y = \frac{x^2-1}{x^2+1}$

$$解：y = \frac{x^2+1-2}{x^2+1} = 1 - \frac{2}{x^2+1} \quad \therefore y^{'} = \left(0 - \frac{-2\cdot 2x}{(x^2+1)^2}\right) = \frac{4x}{(x^2+1)^2}$$

$$\therefore dy = \frac{4x}{(x^2+1)}dx$$

(2) $y = \tan x + \sec x$

$$解：y^{'} = \frac{\sin x}{\cos x} + \frac{1}{\cos x} \quad \therefore y^{'} = \frac{\sin^2 x + \cos^2 x}{\cos^2 x} + \frac{\sin x}{\cos^2 x} = \sec^2 x + \sec x \tan x$$

$$\therefore dy = (\sec^2 x + \sec x \tan x)dx$$

(3) $y = \arccos \frac{1}{x}$

$$解：y^{'} = \frac{-1}{\sqrt{1-\frac{1}{x^2}}} \cdot \left(\frac{1}{-x^2}\right) = \frac{1}{|x|\cdot\sqrt{x^2-1}}$$

$\therefore dy = \frac{1}{|x|\cdot\sqrt{x^2-1}}dx$ 且（$|x|$>1）

（4） $y = \arcsin\sqrt{1-x^2}$

$$解：y^{'} = \frac{1}{\sqrt{1-\left(\sqrt{1-x^2}\right)^2}} \cdot \left(\frac{1}{2}\cdot\frac{1}{\sqrt{1-x^2}}\right)\cdot(-2x)$$

$$= \frac{1}{\sqrt{x^2}}\cdot\frac{-x}{\sqrt{1-x^2}}$$

$$= \frac{-x}{|x|\sqrt{1-x^2}}$$

$\therefore dy = \frac{-x}{|x|\sqrt{1-x^2}}dx$且（$|x|$<1）

(5)$y = \arctan \frac{x^2-1}{x^2+1}$

解：$y' = \dfrac{1}{1+\left(\dfrac{x^2-1}{x^2+1}\right)^2} \cdot \left(\dfrac{x^2-1}{x^2+1}\right)'$　　　　由(1)知$\left(\dfrac{x^2-1}{x^2+1}\right)' = \dfrac{4x}{(x^2+1)^2}$

$$= \frac{1}{2x^4+2} \cdot 4x = \frac{2x}{x^4+1}$$

$$\therefore dy = \frac{2x}{x^4+1}dx$$

（6）$y = (x^2+4x+1)(x^2-\sqrt{x})$

解：$y' = (2x+4)(x^2-\sqrt{x}) + (x^2+4x+1)\left(2x-\dfrac{1}{2\sqrt{x}}\right)$

$$= 2x^3 - 2x^{\frac{3}{2}} + 4x^2 - 4x^{\frac{1}{2}} + 2x^3 - \frac{1}{2}x^{\frac{3}{2}} + 8x^2 - 2x^{\frac{1}{2}} + 2x - 2x^{-\frac{1}{2}}$$

$$= 4x^3 + 12x^2 - \frac{5}{2}x^{\frac{3}{2}} + 2x - 6x^{\frac{1}{2}} - \frac{1}{2}x^{-\frac{1}{2}}$$

$$\therefore dy = \left(4x^3 + 12x^2 - \frac{5}{2}x^{\frac{3}{2}} + 2x - 6x^{\frac{1}{2}} - \frac{1}{2}x^{-\frac{1}{2}}\right)dx$$

3.求下列复合函数的微分

（1）$y = \ln\sqrt{1+x^2}$

解：$y' = \dfrac{1}{\sqrt{1+x^2}} \cdot \dfrac{1}{2\sqrt{1+x^2}} \cdot 2x = \dfrac{x}{1+x^2}$

$$\therefore dy = \frac{x}{1+x^2}dx$$

（2）$y = \arcsin\frac{1}{x}$

解：$y' = \dfrac{1}{\sqrt{1-\left(\frac{1}{x}\right)^2}} \cdot \left(-\dfrac{1}{x^2}\right)$

$$= \sqrt{\frac{x^2}{x^2-1}} \cdot \left(-\frac{1}{x^2}\right)$$

$$= \frac{|x|}{\sqrt{x^2-1}} \cdot \frac{1}{-|x|^2} = \frac{1}{|x| \cdot \sqrt{x^2-1}}$$

$$\therefore dy = \frac{1}{|x|\sqrt{x^2-1}}dx\left(\frac{1}{\sqrt{x^4-x^2}}dx\right)$$

（3）$y = \arctan\sqrt{x}$

解：$y' = \dfrac{1}{1+(\sqrt{x})^2} \cdot \dfrac{1}{2\sqrt{x}} = \dfrac{1}{2(1+x)\sqrt{x}}$

$$\therefore dy = \frac{1}{2(1+x)\sqrt{x}}dx$$

（4）$y=e^{\sin x}$

解：$y^{'}=e^{\sin x}\cdot\cos x$

$$\therefore dy=\left(e^{\sin x}\cdot\cos x\right)dx$$

4.求下列各式的近似值

解：由微分定义：$\Delta y=f^{'}(x)\Delta x+o\left(\Delta x\right)\quad\left(\Delta x\to 0\right)$, 即 $f\left(x+\Delta x\right)-f(x)\approx f^{'}(x)\Delta x$

$\therefore f\left(x+\Delta x\right)\approx f(x)+f^{'}(x)\Delta x$

（1）令$\sqrt[3]{1.02}=\sqrt[3]{1+0.02}$ $\qquad\therefore f(x)=x^{\frac{1}{3}}\quad f^{'}(x)=\frac{1}{3}x^{-\frac{2}{3}}\quad\Delta x=0.02$

$\therefore f(1.02)\approx f(1)+f^{'}(1)\cdot 0.02=1+\frac{1}{3}\times 0.02=1.00666\ldots\approx 1.007$

（2）令 $\text{In}\ 1.005=\ln(1+0.005)$ $\quad\therefore f(x)=\ln x$，$f'(x)=\frac{1}{x}\quad\Delta x=0.005$

$\therefore f(1.005)\approx f(1)+f^{'}(1)\cdot 0.005=0+1\times 0.005=0.005$

习题 3.4

1. 求下列函数的二阶导数

（1）$y = x^3 + 2x^2 + 3x + 4$

解：$y^{'} = 3x^2 + 4x + 3$

$y^{''} = 6x + 4$

（2）$y = x^4 \ln x$

解：$y^{'} = 4x^3 \ln x + x^3$

$$\begin{aligned} y^{''} &= 12x^2 \ln x + 4x^2 + 3x^2 \\ &= 12x^2 \ln x + 7x^2 \end{aligned}$$

（3）$y = \frac{x^2}{\sqrt{1+x}}$

解：$y = x^2(1+x)^{-\frac{1}{2}}$

$\therefore y^{'} = 2x(1+x)^{-\frac{1}{2}} - \dfrac{1}{2}x^2(1+x)^{-\frac{3}{2}}$

$$\begin{aligned} y^{''} &= 2(1+x)^{-\frac{1}{2}} - \frac{1}{2} \cdot 2x(1+x)^{-\frac{3}{2}} - x(1+x)^{-\frac{3}{2}} + \frac{3}{4}x^2(1+x)^{-\frac{5}{2}} \\ &= (1+x)^{-\frac{5}{2}}\left[2(1+x)^2 - x(1+x) - x(1+x) + \frac{3}{4}x^2\right] \\ &= (1+x)^{-\frac{5}{2}}\left(\frac{3}{4}x^2 + 2x + 2\right) \end{aligned}$$

（4）$y = \frac{\ln x}{x^2}$

解：$y = x^{-2} \ln x$

$\therefore y^{'} = -2x^{-3} \ln x + x^{-3}$

$$\begin{aligned} y^{''} &= 6x^{-4} \ln x + (-2)x^{-4} - 3x^{-4} \\ &= (6 \ln x - 5)x^{-4} \end{aligned}$$

（5）$y = \sin x^2$

解：$y^{'} = \cos x^2 \cdot 2x$

$$\begin{aligned} y^{''} &= 2\cos x^2 + 2x(-\sin x^2 \cdot 2x) \\ &= -4x^2 \sin x^2 + 2\cos x^2 \end{aligned}$$

（6）$y = x^3 \cos \sqrt{x}$

解：$y^{'}=3x^2\cos\sqrt{x}+x^3(-\sin\sqrt{x})\frac{1}{2}(x)^{-\frac{1}{2}}$

$$=3x^2\cos\sqrt{x}-\frac{1}{2}x^{\frac{5}{2}}\sin\sqrt{x}$$

$$y^{''}=6x\cos\sqrt{x}+3x^2(-\sin\sqrt{x})\frac{1}{2}(x)^{-\frac{1}{2}}-\left(\frac{5}{4}x^{\frac{3}{2}}\sin\sqrt{x}+\frac{1}{2}x^{\frac{5}{2}}\cos\sqrt{x}\frac{1}{2}(x)^{-\frac{1}{2}}\right)$$

$$=6x\cos\sqrt{x}-\frac{3}{2}x^{\frac{3}{2}}\sin\sqrt{x}-\frac{5}{4}x^{\frac{3}{2}}\sin\sqrt{x}-\frac{1}{4}x^2\cos\sqrt{x}$$

$$=\left(6x-\frac{1}{4}x^2\right)\cos\sqrt{x}-\frac{11}{4}x^{\frac{3}{2}}\sin\sqrt{x}$$

（7）$y=x^2e^{3x}$

解：$y^{'}=2xe^{3x}+x^2\cdot 3e^{3x}$

$$y^{''}=2e^{3x}+2xe^{3x}\cdot 3+2x\cdot 3e^{3x}+3x^2\cdot 3e^{3x}$$
$$=e^{3x}(2+6x+6x+9x^2)$$
$$=(9x^2+12x+2)e^{3x}$$

（8）$y=e^{-x^2}\arcsin x$

解：$y^{'}=-2xe^{-x^2}\arcsin x+e^{-x^2}\frac{1}{\sqrt{1-x^2}}$

$$y^{''}=-2xe^{-x^2}\frac{1}{\sqrt{1-x^2}}-2xe^{-x^2}(-2x)\arcsin x-2e^{-x^2}\arcsin x+e^{-x^2}(-2x)(1-x^2)^{-\frac{1}{2}}+$$
$$\left(-\frac{1}{2}\right)e^{-x^2}(1-x^2)^{-\frac{3}{2}}(-2x)$$

$$\therefore y^{''}=-2x(1-x^2)^{-\frac{1}{2}}e^{-x^2}+4x^2e^{-x^2}\arcsin x-2e^{-x^2}\arcsin x-2x(1-x^2)^{-\frac{1}{2}}e^{-x^2}$$

$$+xe^{-x^2}(1-x^2)^{-\frac{3}{2}}$$

$$=(4x^2-2)e^{-x^2}\arcsin x-4xe^{-x^2}(1-x^2)^{-\frac{1}{2}}+xe^{-x^2}(1-x^2)^{-\frac{3}{2}}$$

（9）$y=x^2\cos 3x$

解：$y^{'}=2x\cos 3x+x^2(-\sin 3x)\cdot 3$

$$y^{''}=2\cos 3x+2x(-\sin 3x)\cdot 3+6x(-\sin 3x)-3x^2\cos 3x\cdot 3$$
$$=2\cos 3x-6x\sin 3x-6x\sin 3x-9x^2\cos 3x$$
$$=(2-9x^2)\cos 3x-12x\sin 3x$$

（10）$y=x^2\ln x$

解：$y^{'}=2x\ln x+x$

$$y^{''}=2\ln x+2+1$$

$= 2\ln x + 3$

2. 求下列函数的 n 阶导数

（1）$y = \ln(x+1)$

解：$y^{'} = \dfrac{1}{x+1}$ $y^{''} = -\dfrac{1}{(x+1)^2}$ $y^{'''} = \dfrac{2}{(1+x)^3}$ $y^{(n)} = \dfrac{(-1)^{n-1}(n-1)!}{(1+x)^n}$

（2）$y = \sin^2(\omega x)$

解：$y = \sin^2(\omega x) = \dfrac{1-\cos(2\omega x)}{2} = \dfrac{1}{2} - \dfrac{1}{2}\cos(2\omega x)$

$y^{(n)} = -2^{n-1}w^n \cos\left(2wx + \dfrac{n}{2}\pi\right)$

（3） $y = \frac{1}{x^2-3x+2}$

解：$y = \dfrac{1}{x^2-3x+2} = \dfrac{1}{(x-1)(x-2)} = \dfrac{1}{x-2} - \dfrac{1}{x-1}$

又由：$\left(\dfrac{1}{x+1}\right)^{(n)} = (-1)^n \dfrac{n!}{(x+1)^{n+1}}$

$\therefore y^{(n)} = (-1)^n \dfrac{n!}{(x-2)^{n+1}} - (-1)^n \dfrac{n!}{(x-1)^{n+1}}.$

$= (-1)^n n! \left[(x-2)^{-(n+1)} - (x-1)^{-(n+1)}\right]$

（4）$y = \cos^2(\omega x)$

解：$y = \cos^2(\omega x) = \dfrac{1+\cos(2\omega x)}{2} = \dfrac{1}{2} + \dfrac{1}{2}\cos(2\omega x)$

$y^{(n)} = 2^{n-1}\omega^n \cos\left(2\omega x + \dfrac{n}{2}\pi\right)$

3. 求下列函数的高阶导数

（1）$y = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$, 求$y^{(n)}, y^{(n+1)}$;

解： $y^{(n)} = n!\, a_0$ $y^{(n+1)} = 0$

（2）$y = x^2(1+x)^3(2+x)^4$, 求 $y^{(9)}, y^{(10)}$;

解： $y^{(9)} = 9!$ $y^{(10)} = 0$

（3）$y = x^2e^{2x}$, 求 $y^{(20)}$;

解： $\because (x^2)^{'} = 2x$ $(x^2)^{''} = 2$ $(x^3)^{'''} = 0$

由莱布尼茨公式可知：$y^{(n)} = C_{20}^0 x^2(e^{2x})^{(20)} + C_{20}^1 2x(e^{2x})^{(19)} + C_{20}^2 2(e^{2x})^{(18)}$

$$y^{(n)}=2^{20}x^2e^{2x}+20\cdot 2^{20}xe^{2x}+\frac{20\text{x}19}{2}2^{19}e^{2x}$$

$$=2^{20}e^{2x}(x^2+20x+95)$$

（4）$y=x\ln x$，求$y^{(5)}$；

解：$\because (x)^{'}=1\quad (x)^{''}=0\quad (\ln x)^{(n)}=\dfrac{(-1)^{n-1}(n-1)!}{x^n}$

$$y^{(5)}=C_5^0x(\ln x)^{(5)}+C_5^1(\ln x)^{(4)}$$

$$=x\frac{4!}{x^5}+5\left(-\frac{3!}{x^4}\right)$$

$$=24x^{-4}-30x^{-4}=-6x^{-4}$$

（5）$y=e^x\sin x$，求$y^{(n)}$；

$$y^{'}=e^x\sin x+e^x\cos x=e^x(\sin x+\cos x)=\sqrt{2}e^x\sin\left(x+\frac{\pi}{4}\right)$$

$$y^{''}=e^x\sin x+e^x\cos x+e^x\cos x-e^x\sin x=2e^x\cos x=2e^x\sin\left(x+\frac{\pi}{2}\right)$$

$$y^{'''}=2e^x\cos x-2e^x\sin x=2\sqrt{2}e^x\sin\left(x+\frac{3}{4}\pi\right)$$

$$\therefore y^{(n)}=2^{\frac{n}{2}}e^x\sin\left(x+\frac{n}{4}\pi\right)$$

4. 求下列函数的二阶微分

（1）$y=\sin x$

解：$y^{''}=-\sin x$

$$\therefore d^2y=-\sin x\,dx^2$$

（2）$y=xe^x$

解：$y^{'}=e^x+xe^x$

$$y^{''}=e^x+e^x+xe^x=2e^x+xe^x$$

$$\therefore d^2y=(2e^x+xe^x)dx^2$$

（3）$y=x\ln x$

解：$y^{'}=\ln x+1\quad y^{''}=\dfrac{1}{x}$

$$\therefore dy^2=\frac{1}{x}dx^2$$

（4）$y=x\sin x$

解：$y^{'}=\sin x+x\cos x$

$$y^{''}=\cos x+\cos x-x\sin x$$

$$\therefore d^2y=(2\cos x-x\sin x)dx^2$$

5. 设 x 为中间变量，求下列函数的二阶微分

（1）$y=\sin x\,, x=at+b$， 其中 a，b 为常数

解：$y=\sin(at+b)$ $y^{'}=\cos(at+b)\cdot a$

$y^{''}=-a^2\sin(at+b)$
$\therefore d^2y=-a^2\sin(at+b)\,dt^2$

（2）$y=e^x, x=at^2+bt+c$，其中 a、b、c 为常数

解：$y=e^{at^2+bt+c}$

$y^{'}=(2at+b)e^{at^2+bt+c}$
$y^{''}=(2a)e^{at^2+bt+c}+(2at+b)^2e^{at^2+bt+c}=(4a^2t^2+4abt+b^2+2a)e^{at^2+bt+c}$
$\therefore d^2y=(4a^2t^2+4abt+b^2+2a)e^{at^2+bt+c}dx^2$

第 3 章复习题

1. $\lim\limits_{h\to 0}\frac{f(1-h)-f(1)}{2h}=\lim\limits_{h\to 0}\frac{f(1+h)-f(1)}{-2h}=-\frac{1}{2}\lim\limits_{h\to 0}\frac{f(1+h)-f(1)}{h}=-\frac{1}{2}f'(1)=-1$

2. f(x)=x-[x], f(0)=0.($\lim\limits_{x\to 0+}[x]=0\ \therefore \lim\limits_{x\to 0+}(x-[x])=\lim\limits_{x\to 0+}x=0$)

   $f'_+(0)=\lim\limits_{x\to 0^+}\frac{f(x)-f(0)}{x}=1$

   $f'_-(0)=\lim\limits_{x\to 0^-}\frac{f(x)-f(0)}{x}=\lim\limits_{x\to 0^-}\frac{x-[x]}{x}=\infty$

   ($\lim\limits_{x\to 0^-}[x]=1,\quad \therefore \lim\limits_{x\to 0^-}(x-[x])=\lim\limits_{x\to 0^-}(x+1)=1$)

   $f'_-(0)\neq f'_+(0)\quad \therefore f'(0)$不存在.

   x$\in(0,1)$时，$f'(x)=(x-[x])'=1$　x$\in(-1,0)$时，$f'(x)=(x-[x])'=1$

   $\therefore \lim\limits_{x\to 0}f'(x)=1$.

3. $e^y+6xy+x^2-1=0$,x=0 时，y=0.

   $y'e^y+6y+6xy'+2x=0$, y'(0)=0

   $y^{"}e^y+(y')^2e^y+6y'+6y'+6xy"y'+2=0$
   $\therefore y"+2=0$

   $\therefore y"(0)=-2$.

4. 2y$\sin x+xlny=0$.

   两边对 x 求导：$2y'sinx+2ycosx+lny+x\frac{y\prime}{y}=0$.

   $y'=-\frac{2y^2cosx+ylny}{x+2ysinx}$.

   再对 x 求导：

   $2y"sinx+2y'cosx-2ysinx+2y'cosx+2\frac{y\prime}{y}+\frac{xyy"-x(y')^2}{y^2}=0$.

   $y"=\frac{2y^3sinx-4y'y^2cosx-2yy'+x(y\prime)^2}{xy+2y^2sinx}$

5. (1)　$y'=[(1+x^2+x^4)^{\frac{1}{2}}]'=\frac{1}{2}(2x+4x^3)(1+x^2+x^4)^{-\frac{1}{2}}=x(1+2x^2)(1+$

$x^2+x^4)^{-\frac{1}{2}}$

(2) $y=x^{sinx+2cosx}$

两边取对数：$ln|x|(sinx+2cosx)=ln|y|$

两边对 X 求导：$\frac{y\prime}{y}=(cosx-2sinx)ln|x|+\frac{1}{x}(sinx+2cosx)$

$$y'=x^{sinx+2cosx}[(cosx-2sinx)ln|x|+\frac{1}{x}(sinx+2cosx)]$$

(3) $y=(1+\frac{1}{x})^{x}$

两边取对数: $ln|y|=xln\left|1+\frac{1}{x}\right|$

两边对 X 求导：$\frac{y\prime}{y}=ln\left|1+\frac{1}{x}\right|+\frac{x^2}{1+x}(-\frac{1}{x^2})$

$$y'=(1+\frac{1}{x})^{x}[ln\left|1+\frac{1}{x}\right|-\frac{1}{1+x}]$$

(4) $y=\sqrt[2]{\frac{sin^2x(1+cos^2x)}{1+sin^2x}}$

两边取对数: $lny=\frac{1}{2}ln\frac{sin^2x(1+cos^2x)}{1+sin^2x}$

两边对 X 求导：$\frac{y\prime}{y}=\frac{1}{2}[\frac{2sinxcosx+2sinxcos^3x-2sin^3xcosx}{sin^2x(1+cos^2x)}-\frac{2sinxcosx}{1+sin^2x}]$

将$y=\sqrt[2]{\frac{sin^2x(1+cos^2x)}{1+sin^2x}}$代入上式：

$$y'=\sqrt[2]{\frac{sin^2x(1+cos^2x)}{1+sin^2x}}\frac{cosx(2+cos^2xsin^2x+sin^4x-cos^4x)}{sinx(1+cos^2x)(1+sin^2x)}$$

$$=\frac{cosx(2+cos^2xsin^2x+sin^4x-cos^4x)}{(1+cos^2x)^{\frac{1}{2}}(1+sin^2x)^{\frac{1}{2}}}$$

6.证明：$\lim_{x\to0}\frac{f(x)}{x}=A$

$\lim_{X\to0}x=0$　　$\therefore\lim_{x\to0}f(x)=0$，　且$f(x)$在 x=0 处连续。

$\therefore f(0)=\lim_{x\to0}f(x)=0$

$f'(0)=\lim_{x\to0}\frac{f(x)-f(0)}{x}=A$

7.证明：

f(x)=x(x+1)(x+2)...(x+n+1)

f '(x)=(x+1)(x+2)...(x+n+1)+x(x+2)...(x+n+1)+x(x+1)(x+3)...(x+n+1)+x(x+1)(x+2)(x+4)...(x+n+1)+...+x(x+1)...(x+n)

$f'(-1)=x(x+2)(x+3)\ldots(x+n+1)$

$=(-1)\times1\times2\times3\ldots\times n$

$=-n!$

8.

$y=\sin^4 x+\cos^4 x$

$=(\sin^2 x+\cos^2 x)^2-2\sin^2 x\cos^2 x$

$=1-\frac{1}{2}\sin^2 2x=\frac{3}{4}+\frac{1}{4}\cos 4x$

$y'=-\sin 4x$

$=\cos(4x+\frac{\pi}{2})$

$\because (\cos\omega x)^{(n)}=\omega^n\cos(\omega x+\frac{n\pi}{2})$

$\therefore y^{(n)}=4^{n-1}\cos(4x+\frac{n\pi}{2})$

9.证明

$f(x)=(x-a)^n\varphi(x)$

$\because\varphi(x)$在点 a 的某领域内有（n-1）阶连续导函数

$\therefore$

$\therefore\ f^{(n-1)}(x)=C_{n-1}^{0}[(x-a)^n]^{(0)}\varphi^{(n-1)}(x)+C_{n-1}^{1}[(x-a)^n]^{(1)}\varphi^{(n-2)}(x)+\ldots+C_{n-1}^{n-1}[(x-a)^n]^{(n-1)}\varphi^{(0)}(x)$

$\therefore f^{(n-1)}(a)=0$

$f^{(n)}(a)\lim\limits_{x\to a}\frac{f^{(n-1)}(x)-f^{(n-1)}(a)}{x-a}$，　将上式带入

$\therefore\ f^{(n)}(a)=\varphi(a)n!$

10.

(1) $f(x)$在 x=0 连续：

$f(0)=\lim\limits_{x\to 0} f(x)=0$

$\therefore \lim\limits_{x\to 0} x^m \sin\frac{1}{x}=0$

$\therefore \lim\limits_{x\to 0} x^m=0 \to m>0$

(2) $f(x)$在 x=0 可导:

在 m>0 前提下，有$f'(0)$存在

$f'(0)=\lim\limits_{x\to 0}\frac{x^m \sin\frac{1}{x}}{x}=\lim\limits_{x\to 0} x^{m-1}\sin\frac{1}{x}$

$\therefore m>1$

(3) $f'(x)$在 x=0 连续:

$f'(0)=\lim\limits_{x\to 0} f'(x)$

由(2)知$f'(0)$若存在则为 0

$\therefore \lim\limits_{x\to 0} f'(x)=0=\lim\limits_{x\to 0}(mx^{m-1}\sin\frac{1}{x}-x^{m-2}\cos\frac{1}{x})$

$$\begin{cases} m-2>0 \\ m-1>0 \end{cases} \Longrightarrow m > 2$$

11.证明

当 x≠0 时，$f'(x)=e^{\frac{-1}{x^2}}(\frac{2}{x^3})$

又$\because f'(0)=\lim\limits_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim\limits_{x\to 0}\frac{1}{xe^{\frac{1}{x^2}}}=0$，$\lim\limits_{x\to 0} f'(x)=\lim\limits_{x\to 0}\frac{2}{x^3 e^{\frac{1}{x^2}}}=0$

$\therefore f'(0)=0$且$f'(x)$在 x=0 连续

$\therefore f'(x)=\begin{cases} \frac{2}{x^3}e^{\frac{-1}{x^2}}, x\neq 0 \\ 0, x=0 \end{cases}$，$f''(0)=\lim\limits_{x\to 0}\frac{f'(x)-f'(0)}{x-0}=0$

$f''(x)=\begin{cases} (\frac{-6}{x^4}+\frac{4}{x^6})e^{\frac{-1}{x^2}}, x\neq 0 \\ 0, x=0 \end{cases}$ (关于x^{-1}的六次多项式)

设$f^{(n)}$(x)= $\begin{cases} P_n(\frac{1}{x})e^{-\frac{1}{x^2}}, x\neq 0 \\ 0, x=0 \end{cases}$ (P_n（x^{-1}）是关于x^{-1}的3n 次多项式)

则$f^{(n+1)}$(0)=$\lim\limits_{x\to 0}\frac{f^{(n)}(\mathrm{x})-f^{(n)}(0)}{x-0}$=$\frac{x^{-1}P_n(x^{-1})}{e^{\frac{1}{x^2}}}$=0

$f^{(n+1)}$(x)=($\frac{2}{x^3}P_n$（x^{-1}）$-\frac{1}{x^2}P'_n$（x^{-1}）)$e^{\frac{-1}{x^2}}$

=$P_{n+1}(x^{-1})e^{\frac{-1}{x^2}}$(x≠0)

显然$P_{n+1}(x^{-1})$是关于 x 的 3(n+1)次多项式

∴$f^{(n+1)}$(x)= $\begin{cases} P_{n+1}(\frac{1}{x})e^{\frac{-1}{x^2}}, x\neq 0 \\ 0, x=0 \end{cases}$

由数学归纳法可知f(x)在 x=0 处 n 阶可导且$f^{(n)}$(0)=0

12.

(1) 证明

∵φ(x)在 x=a 连续

∴f(x)在 x=a 连续

$\lim\limits_{x\to a+}\frac{f(\mathrm{x})-f(\mathrm{a})}{x-a}$= $\lim\limits_{x\to a+}$ φ(a)=φ(a)

$\lim\limits_{x\to a-}\frac{f(\mathrm{x})-f(\mathrm{a})}{x-a}$= $\lim\limits_{x\to a-}$ φ(a)=φ(a)

∴f(x)在 x=a 可导，且$f'(\mathrm{a}) = \varphi(\mathrm{a})$

(2)

g'_+(a)= $\lim\limits_{x\to a+}\frac{|x-a|\varphi(\mathrm{x})}{x-a}$=φ(a)

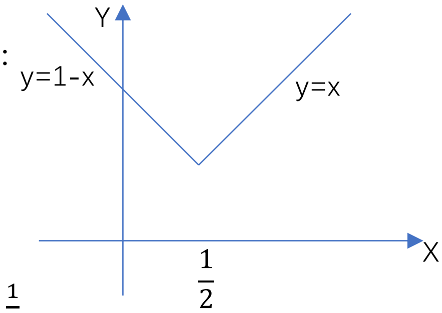
g'_-(a)= $\lim\limits_{x\to a-}\frac{|x-a|\varphi(\mathrm{x})}{x-a}$=−φ(a)

要使 g(x)在 x=a 可导

则g'_+(a)=g'_-(a)

即φ(a)=0

13. 解：



因为$f(x)$为多项式函数

所以$f(x)$可导

由于 $f(x) \geq x$ 所以 $f\left(\frac{1}{2}\right) \geq \frac{1}{2}$

假设 $f\left(\frac{1}{2}\right) = \frac{1}{2}$ 所以 $f\left(\frac{1}{2}\right)$为较小值

由费马定理，$f'\left(\frac{1}{2}\right) = 0$

当$x > \frac{1}{2}$时，$f(x) \geq x$ 则$f'_+\left(\frac{1}{2}\right) \geq 1$，与$f'\left(\frac{1}{2}\right) = 0$相矛盾

所以$f\left(\frac{1}{2}\right) \neq \frac{1}{2}$ 故$f\left(\frac{1}{2}\right) > \frac{1}{2}$

14. 解：$\lim\limits_{x\to\infty}\left(\frac{f\left(\frac{1}{x}\right)}{f(0)}\right)^x = \lim\limits_{x\to\infty}\left(1+\frac{f\left(\frac{1}{x}\right)-f(0)}{f(0)}\right)^x = \lim\limits_{x\to\infty}\left(1+\frac{f\left(\frac{1}{x}\right)-f(0)}{f(0)}\right)^{\frac{f\left(\frac{1}{x}\right)-f(0)}{f(0)}\cdot x\cdot\frac{f(0)}{f\left(\frac{1}{x}\right)-f(0)}} =$

$e^{\lim\limits_{x\to\infty} x\cdot\frac{f\left(\frac{1}{x}\right)-f(0)}{f(0)}}$ 令$x = \frac{1}{t}$ $x \to \infty$，$t \to 0$

$$\therefore \lim_{x\to\infty} x\cdot\frac{f\left(\frac{1}{x}\right)-f(0)}{f(0)} = \lim_{t\to 0}\frac{f(t)-f(0)}{tf(0)} = \frac{f'(0)}{f(0)}$$

$\therefore$ 原式 $= e^{\frac{f'(0)}{f(0)}}$

15. 解：$f(x) = -x^3 + x$ $f(x+1) = -x^3 - 3x^2 - 2x = af(x)$ $x \in [-1,0)$

$f(x)$中$x \in [0,1)$ $f(x+1)$中$x \in [-1,0)$

$$\therefore f(x) = \begin{cases} -x^3 + x, & x \in [0,1) \\ \frac{1}{a}(-x^2 - 3x - 2), & x \in [-1,0) \end{cases}$$

因为$f(0) = 0$ $\lim\limits_{x\to 0} f(x) = f(0)$ $\therefore f(x)$在$x = 0$处连续

$$f'_-(0) = \lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-}\frac{1}{a}(-x^2 - 3x - 2) = \frac{-2}{a}$$

$$f'_+(0) = \lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+}(-x^2+1) = 1$$

又因为$f(x)$在$x=0$处可导 $\therefore f'_-(0)=f'_+(0)$ $\therefore a=-2, f'(0)=1$

16. 解：因为$f'(x)=f^2(x), f(0)=2$ $\therefore f'(0)=f^2(0)=4$

$$f''(0)=\left(f^2(0)\right)'=2f(0)f'(0)=2f^3(0)=2\times 2^3$$

$$f'''(0)=6f^4(0)=6\times 2^4$$

设$f^{(n)}(0)=n!\ 2^{n+1}=n!\,f^{n+1}(0)$

则 $f^{(n+1)}(0)=[n!\,f^{n+1}(0)]'=(n+1)!\,f^n(0)\cdot f^2(0)=(n+1)!\,f^{n+2}(0)$

$\therefore$ 由数学归纳法可知$f^{(n)}(0)=n!\ 2^{n+1}$

17. 解：因为$f(xy)=f(x)+f(y)$ $\therefore f(x)=f(x)+f(1)$ $\therefore f(1)=0$

又因为$f(1)=f(x)+f\left(\frac{1}{x}\right)=0$ $\therefore f(x)=-f\left(\frac{1}{x}\right)$

$$f'(x_0)=\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}=\lim_{h\to 0}\frac{f(x_0+h)+f\left(\frac{1}{x_0}\right)}{h}$$

$$=\lim_{h\to 0}\frac{f\left(1+\frac{h}{x_0}\right)}{h}$$

由洛必达法则可知 $f'(x_0)=\lim_{h\to 0}f'\left(1+\frac{h}{x_0}\right)\cdot\frac{1}{x_0}=\frac{f'(1)}{x_0}=\frac{a}{x_0}$

$\therefore f'(x)=\frac{a}{x}$， $x\in\left(0,\ +\infty\right)$

18. 解：充分性：若$f(x)$在$x=a$处可导且$f'(a)=0$，$f(a)=0$，则$|f(x)|$在$x=a$处可导

因为 $f(a)=0$ $f'(a)=0$ $\therefore \lim_{x\to a}\frac{f(x)}{x-a}=0$，$\therefore |f(a)|=0$

$$|\,f'_+(a)|=\lim_{x\to a^+}\frac{|f(x)|-|f(a)|}{x-a}=\lim_{x\to a^+}\frac{|f(x)|}{x-a}=0$$

$$|\,f'_-(a)|=\lim_{x\to a^-}\frac{|f(x)|-|f(a)|}{x-a}=\lim_{x\to a^-}\frac{|f(x)|}{x-a}=0$$

$\therefore |f(x)|$在$x=a$处可导且$|f(a)|'=0$

必要性：若$|f(x)|$在$x=a$处可导且$f(a)=0$，则$f'(a)=0$

因为$|f(x)|$在$x=a$处可导　　$\lim\limits_{x\to a^+}\frac{|f(x)|}{x-a}=-\lim\limits_{x\to a^-}\frac{|f(x)|}{x-a}$

$\therefore \lim\limits_{x\to a}\dfrac{|f(x)|}{x-a}=0$，$\therefore f'(a)=0$