

## 习题 2.1

1. 利用数列极限定义证明下列各式

$$(1) \lim_{n \rightarrow \infty} \frac{n}{n+1} = 1$$

对于  $\forall \varepsilon > 0$ , 要使  $\left| \frac{n}{n+1} - 1 \right| = \frac{1}{n+1} < \varepsilon$

只需要  $\frac{1}{n} < \varepsilon$  即  $n > \frac{1}{\varepsilon}$

取  $N = \left[ \frac{1}{\varepsilon} \right]$ , 则当  $n > N$  时

$$\text{有 } \left| \frac{n}{n+1} - 1 \right| = \frac{1}{n+1} < \frac{1}{n} < \varepsilon$$

由定义知  $\lim_{n \rightarrow \infty} \frac{n}{n+1} = 1$

$$(2) \lim_{n \rightarrow \infty} \left[ 1 + \frac{(-1)^n}{n} \right]^n = 1$$

对于  $\forall \varepsilon > 0$  要使  $\left| 1 + \frac{(-1)^n}{n} - 1 \right| = \left| \frac{(-1)^n}{n} \right| < \varepsilon$

即使  $\frac{1}{n} < \varepsilon$  即  $n > \frac{1}{\varepsilon}$

取  $N = \left[ \frac{1}{\varepsilon} \right] + 1$ , 则当  $n > N$  时

$$\text{有 } \left| 1 + \frac{(-1)^n}{n} - 1 \right| < \varepsilon$$

由定义知  $\lim_{n \rightarrow \infty} \left[ 1 + \frac{(-1)^n}{n} \right]^n = 1$

$$(3) \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n+1}} = 0$$

对于  $\forall \varepsilon > 0$  要使  $\left| \frac{1}{\sqrt{n+1}} - 0 \right| = \frac{1}{\sqrt{n+1}} < \varepsilon$

只需要  $\frac{1}{\sqrt{n}} < \varepsilon$  即  $n > \frac{1}{\varepsilon^2}$

取  $N = \left[ \frac{1}{\varepsilon^2} \right]$ , 则当  $n > N$  时

有  $\left| \frac{1}{\sqrt{n+1}} - 0 \right| = \frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}} < \varepsilon$

由定义知:  $\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n+1}} = 0$

(4)  $\lim_{n \rightarrow \infty} \frac{1}{n^\alpha} = 0$  ( $\alpha$  为正常数)

对于  $\forall \varepsilon > 0$ , 要使  $\left| \frac{1}{n^\alpha} - 0 \right| = \frac{1}{n^\alpha} < \varepsilon$

只需要  $n > \sqrt[\alpha]{\frac{1}{\varepsilon}}$

取  $N = \left[ \sqrt[\alpha]{\frac{1}{\varepsilon}} \right] + 1$ , 则当  $n > N$

有  $\left| \frac{1}{n^\alpha} - 0 \right| < \varepsilon$

由定义知  $\lim_{n \rightarrow \infty} \frac{1}{n^\alpha} = 0$

2、

(1) 若  $\lim_{n \rightarrow \infty} a_n = a$ , 其中  $a \neq 0$ , 则  $\lim_{n \rightarrow \infty} |a_n| = |a|$ ; 问反之是否成立

因为  $\lim_{n \rightarrow \infty} a_n = a$ ,  $\forall \varepsilon > 0, \exists N_0$

当  $n > N_0$  时,  $|a_n - a| < \varepsilon$

则  $\forall \varepsilon > 0$ , 取  $N = N_0$ , 当  $n > N$  时,  $||a_n| - |a|| < |a_n - a| < \varepsilon$

因此  $\lim_{n \rightarrow \infty} |a_n| = |a|$

反之不成立，如： $a_n = (-1)^n$

$$\lim_{n \rightarrow \infty} |a_n| = \lim_{n \rightarrow \infty} |(-1)^n| = 1$$

$$\text{但 } \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} (-1)^n \neq 1$$

(2) 试证明  $\lim_{n \rightarrow \infty} a_n = 0$  当且仅当  $\lim_{n \rightarrow \infty} |a_n| = 0$

$$\because \lim_{n \rightarrow \infty} a_n = 0, \forall \varepsilon > 0, \exists N$$

当  $n > N$  时,  $|a_n - 0| < \varepsilon$

则  $\forall \varepsilon > 0, \exists N$ , 当  $n > N$  时,  $||a_n| - 0| < \varepsilon$

$$\therefore \lim_{n \rightarrow \infty} |a_n| = 0$$

而当  $\lim_{n \rightarrow \infty} |a_n| = 0, \forall \varepsilon > 0, \exists N$

当  $n > N$  时,  $||a_n| - 0| < \varepsilon$

则  $\forall \varepsilon > 0, \exists N$ , 当  $n > N$  时,  $|a_n - 0| < \varepsilon$

$$\therefore \lim_{n \rightarrow \infty} a_n = 0$$

综上所述,  $\lim_{n \rightarrow \infty} a_n = 0$  当且仅当  $\lim_{n \rightarrow \infty} |a_n| = 0$

### 3. 求下列极限

$$(1) \lim_{n \rightarrow \infty} \frac{3n^5 - 4n^3 + 5n}{n^6 + 4n + 1}$$

$$\text{解: 原式} = \lim_{n \rightarrow \infty} \frac{\frac{3}{n} - \frac{4}{n^3} + \frac{5}{n^5}}{1 + \frac{4}{n^5} + \frac{1}{n^6}} = \frac{3 \times 0 - 4 \times 0 + 5 \times 0}{1 + 4 \times 0 + 0} = 0$$

$$(2) \lim_{n \rightarrow \infty} \frac{n^3 + 3n^2 + 1}{n^3 + 1}$$

$$\text{解: 原式} = \lim_{n \rightarrow \infty} \frac{1 + \frac{3}{n} + \frac{1}{n^3}}{1 + \frac{1}{n^3}} = \frac{1 + 3 \times 0 + 0}{1 + 0} = 1$$

$$(3) \lim_{n \rightarrow \infty} \frac{(-2)^n + 3^n}{(-2)^{n+1} + 3^{n+1}}$$

$$\text{解: 原式} = \lim_{n \rightarrow \infty} \frac{\frac{1}{3} \left( \frac{-2}{3} \right)^n + \frac{1}{3}}{\left( \frac{-2}{3} \right)^{n+1} + 1} = \frac{\frac{1}{3} \times 0 + \frac{1}{3}}{0 + 1} = \frac{1}{3}$$

$$(4) \lim_{n \rightarrow \infty} \frac{1}{n^2} (1 + 2 + \cdots + n)$$

$$\text{解: 原式} = \lim_{n \rightarrow \infty} \frac{\frac{n(n+1)}{2}}{n^2} = \lim_{n \rightarrow \infty} \frac{n^2 + n}{2n^2} = \lim_{n \rightarrow \infty} \frac{1 + \frac{1}{n}}{2} = \frac{1}{2}$$

$$(5) \lim_{n \rightarrow \infty} \left[ \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{(n-1)n} \right]$$

$$\text{解: 原式} = \lim_{n \rightarrow \infty} \left( 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{n-1} - \frac{1}{n} \right)$$

$$= \lim_{n \rightarrow \infty} \left( 1 - \frac{1}{n} \right) = 1$$

$$(6) \lim_{n \rightarrow \infty} \left( \frac{1 + 2 + \cdots + n}{n + 2} - \frac{n}{2} \right)$$

$$\text{解: 原式} = \lim_{n \rightarrow \infty} \left( \frac{\frac{n(n+1)}{2}}{\frac{n}{n+2}} - \frac{n}{2} \right) = \lim_{n \rightarrow \infty} \frac{-n}{2n+4} = -\frac{1}{2}$$

$$(7) \lim_{n \rightarrow \infty} (1 + 2 + 3 + \cdots + K)^{\frac{1}{n}}, \quad (K \text{ 为正整数})$$

$$\text{解: } \because \lim_{n \rightarrow \infty} a^{\frac{1}{n}} = 1$$

$$\therefore \text{原式} = \lim_{n \rightarrow \infty} \left[ \frac{k(k+1)}{2} \right]^{\frac{1}{n}} = 1$$

$$(8) \lim_{n \rightarrow \infty} \left( \sqrt{(n+1)(n+2)} - n \right)$$

$$\begin{aligned} \text{解: 原式} &= \lim_{n \rightarrow \infty} \frac{(n+1)(n+2) - n^2}{\sqrt{(n+1)(n+2)} + n} = \lim_{n \rightarrow \infty} \frac{(1+2)n + 2}{\sqrt{(n+1)(n+2)} + n} \\ &= \lim_{n \rightarrow \infty} \frac{3 + \frac{2}{n}}{\sqrt{\left(1 + \frac{1}{n}\right)\left(1 + \frac{2}{n}\right)} + 1} = \frac{3}{2} \end{aligned}$$

#### 4. 利用单调有界原理求下列数列的极限

$$(1) a_1 = \frac{1}{5}, a_{n+1} = \frac{n}{3n+2} a_n, n = 1, 2, 3, \dots$$

$$\text{解: } \because 0 < \frac{a_{n+1}}{a_n} = \frac{n}{3n+2} < 1$$

$$a_1 = \frac{1}{5}, \text{ 由数学归纳法知: } a_n > 0$$

$$\text{又 } \frac{a_{n+1}}{a_n} < 1, \text{ 则 } \{a_n\} \text{ 是单调递减的且有下界 } 0$$

$\therefore \{a_n\}$  有极限

$$\text{对 } a_{n+1} = \frac{n}{3n+2} a_n \text{ 两边同时取极限}$$

$$\text{则 } \lim_{n \rightarrow \infty} a_{n+1} = \lim_{n \rightarrow \infty} a_n = \frac{1}{3} \lim_{n \rightarrow \infty} a_n \Rightarrow \lim_{n \rightarrow \infty} a_n = 0$$

$$(2) a_1 = \sqrt{2}, a_{n+1} = \sqrt{2 + a_n}, n = 1, 2, \dots$$

$$\text{解: } \because a_1 = \sqrt{2} \quad a_{n+1} = \sqrt{2 + a_n}$$

$$\text{由数学归纳法知: } a_{n+1} > a_n$$

$\therefore \{a_n\}$  单调递增

$$\text{则 } a_{n+1} = \sqrt{2 + a_n} > a_n \Rightarrow a_n < 2$$

且  $\{a_n\}$  有界  $\therefore \{a_n\}$  有极限

$$\text{两边取极限, } \lim_{n \rightarrow \infty} a_{n+1} = \lim_{n \rightarrow \infty} \sqrt{2 + a_n}$$

$$\text{则 } \lim_{n \rightarrow \infty} a_n = 2$$

#### 5. 利用夹逼定理求下列极限

$$(1) \lim_{n \rightarrow \infty} (1 + 2^n + 3^n + 4^n)^{\frac{1}{n}}$$

$$\text{解: } 4^n \leq 1 + 2^n + 3^n + 4^n \leq 4 \cdot 4^n$$

$$(4^n)^{\frac{1}{n}} \leq (1 + 2^n + 3^n + 4^n)^{\frac{1}{n}} \leq (4 \cdot 4^n)^{\frac{1}{n}}$$

$$\text{又 } \lim_{n \rightarrow \infty} (4^n)^{\frac{1}{n}} = \lim_{n \rightarrow \infty} (4 \cdot 4^n)^{\frac{1}{n}} = 4$$

$$\text{则由夹逼定理知: } \lim_{n \rightarrow \infty} (1 + 2^n + 3^n + 4^n)^{\frac{1}{n}} = 4$$

$$(2) \lim_{n \rightarrow \infty} [(n+1)^\alpha - n^\alpha], \text{ 其中常数 } \alpha \in (0,1)$$

$$\text{解: } 0 \leq (n+1)^\alpha - n^\alpha = n^\alpha \left[ \left(1 + \frac{1}{n}\right)^\alpha - 1 \right]$$

$$\leq n^\alpha \left[ \left(1 + \frac{1}{n}\right)^1 - 1 \right] = n^\alpha \cdot \frac{1}{n} = \frac{1}{n^{1-\alpha}}$$

$$\text{又 } \lim_{n \rightarrow \infty} \left( \frac{1}{n^{1-\alpha}} \right) = 0$$

$$\therefore \lim_{n \rightarrow \infty} [(n+1)^\alpha - n^\alpha] = 0$$

## 6. 试用子列证明下列数列发散

$$(1) a_n = (-1)^n \frac{n}{n+1}$$

$$\text{证明: } a_{2k-1} = (-1) \cdot \frac{2k-1}{2k} \quad a_{2k} = \frac{2k}{2k+1}$$

$$\because \lim_{k \rightarrow \infty} a_{2k-1} = -1, \quad \lim_{k \rightarrow \infty} a_{2k-1} \neq \lim_{k \rightarrow \infty} a_{2k}, \quad \lim_{k \rightarrow \infty} a_{2k} = 1$$

$\therefore \{a_n\}$  发散

$$(2) a_n = 2 + (-1)^n$$

$$\text{证明: } a_{2k-1} = 2 - 1 = 1 \quad a_{2k} = 2 + 1 = 3$$

$$\because \lim_{k \rightarrow \infty} a_{2k-1} \neq \lim_{k \rightarrow \infty} a_{2k}$$

$\therefore \{a_n\}$  发散

$$(3) \lim_{n \rightarrow \infty} \left( \frac{1}{n} - \frac{2}{n} + \frac{3}{n} - \frac{4}{n} + \cdots + \frac{(-1)^{n-1}n}{n} \right)$$

$$\text{证明: 令 } a_n = \frac{1}{n} - \frac{2}{n} + \frac{3}{n} - \frac{4}{n} + \cdots + \frac{(-1)^{n-1}n}{n}$$

$$\text{则 } a_{2k} = \frac{1 - 2 + 3 - 4 + \cdots - 2k}{2k} = \frac{-k}{2k} = -\frac{1}{2}$$

$$a_{2k+1} = \frac{1 - 2 + 3 - 4 + \cdots + 2k + 1}{2k + 1} = \frac{k + 1}{2k + 1}$$

$$\lim_{k \rightarrow \infty} a_{2k} = -\frac{1}{2} \quad \lim_{k \rightarrow \infty} a_{2k+1} = \lim_{k \rightarrow \infty} \frac{k + 1}{2k + 1} = \frac{1}{2}$$

$\therefore \{a_n\}$  发散

7. 试证明：对于数列  $\{a_n\}$ ， $\lim_{n \rightarrow \infty} a_n = a$  的充要条件是  $\{a_n\}$  的

奇子列和偶子列均收敛于  $a$ ，即  $\lim_{k \rightarrow \infty} a_{2k-1} = \lim_{k \rightarrow \infty} a_{2k} = a$

证明：  $\because \lim_{n \rightarrow \infty} a_n = a$

则  $\forall \varepsilon > 0, \exists N > 0, \forall n \geq N$  得  $|a_n - a| < \varepsilon$ , 当  $k > N$  时,

$$n_k \geq K > N$$

则  $|a_{n_k} - a| < \varepsilon$  即  $\lim_{k \rightarrow \infty} a_{n_k} = a$

$$\therefore \lim_{k \rightarrow \infty} a_{2k-1} = \lim_{k \rightarrow \infty} a_{2k} = a$$

又  $\{a_{2k-1}\}$ 、 $\{a_{2k}\}$  包含了  $\{a_n\}$  的所有项

$$\therefore \lim_{n \rightarrow \infty} a_n = a$$

$$\text{则 } \lim_{k \rightarrow \infty} a_{2k-1} = \lim_{k \rightarrow \infty} a_{2k} = a \Leftrightarrow \lim_{n \rightarrow \infty} a_n = a$$

8. 利用柯西收敛准则证明下列数列是收敛的

$$(1) a_n = \frac{\sin 1}{1^2} + \frac{\sin 2}{2^2} + \cdots + \frac{\sin n}{n^2}$$

证明：令  $n > m$

$$\text{则 } |a_n - a_m| = \left| \frac{\sin(m+1)}{(m+1)^2} + \frac{\sin(m+2)}{(m+2)^2} + \cdots + \frac{\sin n}{n^2} \right|$$

$$< \left| \frac{1}{(m+1)^2} + \frac{1}{(m+2)^2} + \cdots + \frac{1}{n^2} \right|$$

$$\begin{aligned}
&< \left| \frac{1}{m(m+1)} + \frac{1}{(m+1)(m+2)} + \cdots + \frac{1}{(n-1)n} \right| \\
&= \left( \frac{1}{m} - \frac{1}{m+1} \right) + \left( \frac{1}{m+1} - \frac{1}{m+2} \right) + \cdots + \left( \frac{1}{n-1} - \frac{1}{n} \right) \\
&= \frac{1}{m} - \frac{1}{n} \\
&< \frac{1}{m}
\end{aligned}$$

又  $\forall \varepsilon < \frac{1}{2}$ , 存在  $N = \left[ \frac{1}{\varepsilon} \right]$ , 当  $m > N$  时,  $\frac{1}{m} < \varepsilon$

即  $N = \left[ \frac{1}{\varepsilon} \right]$ , 当  $n > m > N$  时,  $|a_n - a_m| < \varepsilon$

由柯西收敛准则,  $\{a_n\}$  收敛。

$$(2) a_n = \frac{\cos 1!}{1 \cdot 2} + \frac{\cos 2!}{2 \cdot 3} + \cdots + \frac{\cos n!}{n(n+1)}$$

证明: 令  $m > n$

则

$$\begin{aligned}
|a_m - a_n| &< \left| \frac{1}{(m+1)m} + \cdots + \frac{1}{(n+2)(n+3)} + \frac{1}{(n+1)(n+2)} \right| \\
&= \frac{1}{n+1} - \frac{1}{m+1} < \frac{1}{n+1} < \frac{1}{n}
\end{aligned}$$

$\forall \varepsilon > 0$ , 取  $N = \left[ \frac{1}{\varepsilon} \right]$  当  $n > N$  时,  $|a_m - a_n| < \varepsilon$

由柯西收敛准则,  $\{a_n\}$  收敛。

## 9. 利用柯西收敛准则证明下列数列是发散的

$$(1) a_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$$

证明: 取  $\varepsilon = \frac{1}{4}$ ,  $\forall N \in \mathbb{N}^+$  取  $n = N + 1, m = 2N + 2$



则有  $n, m > N$

$$\text{则 } |a_m - a_n| = \frac{1}{N+2} + \frac{1}{N+3} + \cdots + \frac{1}{2N+2} > \frac{N+1}{2N+2} = \frac{1}{2} > \varepsilon$$

由柯西收敛准则,  $\{a_n\}$  发散

$$(2) a_n = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}}$$

证明: 取  $\varepsilon = \frac{1}{4}$ ,  $\forall N \in \mathbb{N}^+$  取  $n = N+1, m = 2N+2$

则有  $n, m > N$

$$\begin{aligned} \text{则 } |a_m - a_n| &= \frac{1}{\sqrt{N+2}} + \frac{1}{\sqrt{N+3}} + \cdots + \frac{1}{\sqrt{2N+2}} \\ &> \frac{1}{N+2} + \frac{1}{N+3} + \cdots + \frac{1}{2N+2} > \frac{N+1}{2N+2} = \frac{1}{2} > \varepsilon \end{aligned}$$

由柯西收敛准则,  $\{a_n\}$  发散

## 习题 2.2

### 1. 证明:

(1) 对于  $\forall \varepsilon > 0$ , 要使  $|(2x + 1) - 3| < \varepsilon$

只需  $2|x - 1| < \varepsilon$

即  $|x - 1| < \frac{\varepsilon}{2}$

令  $\delta = \frac{\varepsilon}{2}$ , 则当  $|x - 1| < \delta$  时

恒有  $|(2x + 1) - 3| < \varepsilon$

$\therefore \lim_{x \rightarrow 1} (2x + 1) = 3$

(2) 对于  $\forall \varepsilon > 0$ , 要使  $|(3x + 1) - 7| < \varepsilon$

只需  $3|x - 2| < \varepsilon$

即  $|x - 2| < \frac{\varepsilon}{3}$

令  $\delta = \frac{\varepsilon}{3}$ , 则当  $|x - 2| < \delta$  时

恒有  $|(3x + 1) - 7| < \varepsilon$

$\therefore \lim_{x \rightarrow 2} (3x + 1) = 7$

(3) 对于  $\forall \varepsilon \geq 0$ , 要使  $|\sin x - \sin x_0| < \varepsilon$

即  $2 \left| \sin \frac{x - x_0}{2} \cos \frac{x + x_0}{2} \right| < \varepsilon$

$\Leftrightarrow 2 \left| \sin \frac{x - x_0}{2} \right| < \varepsilon$

$\Leftrightarrow |x - x_0| < \varepsilon$

令  $\delta = \varepsilon$ , 则当  $|x - x_0| < \delta$  时

恒有  $|\sin x - \sin x_0| < \varepsilon$

即  $\lim_{x \rightarrow x_0} \sin x = \sin x_0$

## 2. 证明

$$(1) \lim_{x \rightarrow 0} [x] = 0$$

对于  $\forall \varepsilon > 0$ , 取  $\delta$  为  $\varepsilon$ , 则当  $0 < x < \delta$  时

$$\text{有 } |[x] - 0| = 0 < \delta = \varepsilon$$

$$\therefore \lim_{x \rightarrow 0^+} [x] = 0$$

$$(2) \lim_{x \rightarrow \infty} [x] = -1$$

$\because$  当  $x \in [-1, 0)$  时,  $[x] = -1$

$\therefore$  对于  $\forall \varepsilon > 0$ , 取  $\delta$  为  $\varepsilon$ , 则当  $-\delta < x < 0$  时

$$\text{有 } |[x] + 1| = 0 < \varepsilon =$$

$$\therefore \lim_{x \rightarrow \infty} [x] = -1.$$

$$(3) \lim_{x \rightarrow 0^+} x \operatorname{sgn} x = 0$$

$\because$  当  $x > 0$  时,  $x \operatorname{sgn} x = x$

$\therefore$  对于  $\forall \varepsilon > 0$ , 取  $\delta$  为  $\varepsilon$ , 则当  $0 < x < \delta$  时

$$\text{有 } |x \operatorname{sgn} x - 0| = |x| < \delta = \varepsilon$$

$$\therefore \lim_{x \rightarrow 0^+} x \operatorname{sgn} x = 0$$

$$(4) \lim_{x \rightarrow 0^-} x \operatorname{sgn} x = 0$$

$\because$  当  $x < 0$  时,  $x \operatorname{sgn} x = -x$

$\therefore$  对于  $\forall \varepsilon > 0$ , 取  $\delta$  为  $\varepsilon$ , 则当  $-\delta < x < 0$  时

$$\text{有 } |x \operatorname{sgn} x - 0| = -x < \delta = \varepsilon$$

$$\therefore \lim_{x \rightarrow 0^-} x \operatorname{sgn} x = 0$$

## 3.

(1) 解: 对于  $\forall \varepsilon > 0$ , 取  $X = \frac{1}{\sqrt{\varepsilon}}$ , 则当  $|x| > X$  时,

$$\left| \frac{x^2 + 1}{x^2 + 2} - 1 \right| = \left| \frac{1}{x^2 + 2} \right| = \frac{1}{x^2 + 2} < \frac{1}{X^2} = \varepsilon.$$

$$\therefore \lim_{x \rightarrow \infty} \frac{x^2 + 1}{x^2 + 2} = 1.$$

(2)解：对于  $\forall \varepsilon > 0$ ，取  $X = \frac{1}{\sqrt{\varepsilon}}$ ，则当  $|x| > X$  时，

$$\left| \frac{1}{x^2 + 1} - 0 \right| = \left| \frac{1}{x^2 + 1} \right| = \frac{1}{x^2 + 1} < \frac{1}{X^2} = \varepsilon.$$

$$\therefore \lim_{x \rightarrow \infty} \frac{1}{x^2 + 1} = 0.$$

$$(3) \text{解：} \because \left| \left| \sqrt{x^2 + 1} - x \right| - 0 \right| = \frac{1}{\sqrt{x^2 + 1} + x}$$

当  $x \rightarrow \infty$  时，不妨设  $x > 1$ ，有  $\sqrt{x^2 + 1} + x > x$

$$\therefore \left| \left| \sqrt{x^2 + 1} - x \right| - 0 \right| < \frac{1}{x}$$

对于  $\forall \varepsilon > 0$ ，可取  $X = \max \left\{ 1, \frac{1}{\varepsilon} \right\}$

只要  $x > X$  时，就有  $\left| \left| \sqrt{x^2 + 1} - x \right| - 0 \right| < \frac{1}{x} < \frac{1}{X} = \varepsilon$

$$\therefore \lim_{x \rightarrow \infty} (\sqrt{x^2 + 1} - x) = 0$$

$$(4) \because \left| \frac{\sqrt{x+2} - \sqrt{3}}{x-1} \right| = \left| \frac{x+2-3}{(x-1)(\sqrt{x} + \sqrt{3})} \right| = \frac{1}{\sqrt{x+2} + \sqrt{3}} \\ < \frac{1}{\sqrt{x+2}} < \frac{1}{\sqrt{x}}$$

对于  $\forall \varepsilon > 0$ ，可取  $X = \frac{1}{\varepsilon^2}$

只要  $x > X$  时，就有  $\left| \frac{\sqrt{x+2} - \sqrt{3}}{x-1} \right| < \frac{1}{\sqrt{x}} < \frac{1}{\sqrt{X}} = \varepsilon$

$$\therefore \lim_{x \rightarrow \infty} \frac{\sqrt{x+2} - \sqrt{3}}{x-1} = 0$$

4.

$$\text{解: 由题意 } f(x) = \begin{cases} 2, & x > 0 \\ 0, & x < 0 \end{cases}$$

$$\therefore \lim_{x \rightarrow 0^+} f(x) = 2,$$

$$\lim_{x \rightarrow 0^-} f(x) = 0$$

$$\therefore \lim_{x \rightarrow 0^+} f(x) \neq \lim_{x \rightarrow 0^-} f(x)$$

$$\therefore \lim_{x \rightarrow 0} f(x) \text{ 不存在}$$

$$\lim_{x \rightarrow +\infty} f(x) = 2,$$

$$\lim_{x \rightarrow -\infty} f(x) = 0$$

$$\therefore \lim_{x \rightarrow -\infty} f(x) \neq \lim_{x \rightarrow +\infty} f(x)$$

$$\therefore \lim_{x \rightarrow \infty} f(x) \text{ 不存在}$$

5.

$$(1) \text{解: } \frac{x+1}{x^2+2} = \frac{1+1}{1+2} = \frac{2}{3}$$

$$(2) \text{解: } \lim_{x \rightarrow -1} \frac{x^3+1}{x^2+2} = \frac{-1+1}{1+2} = 0$$

$$(3) \text{解: } \lim_{x \rightarrow 2} \frac{x^2-4}{x-2} = \lim_{x \rightarrow 2} (x+2) = 4$$

$$(4) \text{解: } \lim_{x \rightarrow -2} \frac{x^2-4}{x+2} = \lim_{x \rightarrow -2} (x-2) = -4$$

$$(5) \text{解: } \lim_{x \rightarrow \infty} \frac{x^3+x+1}{x^3+2x+1} = \lim_{x \rightarrow \infty} \frac{1+\frac{1}{x^2}+\frac{1}{x^3}}{1+\frac{2}{x^2}+\frac{1}{x^3}} = \frac{\lim_{x \rightarrow \infty} \left(1+\frac{1}{x^2}+\frac{1}{x^3}\right)}{\lim_{x \rightarrow \infty} \left(1+\frac{2}{x^2}+\frac{1}{x^3}\right)} = 1$$

$$(6) \text{解: } \lim_{x \rightarrow \infty} \frac{x^3 + 1}{x^4 + 1} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x} + \frac{1}{x^4}}{1 + \frac{1}{x^4}} = 0$$

$$(7) \text{解: } \lim_{x \rightarrow +\infty} \frac{x+2}{x+1} = 1$$

$$(8) \text{解: } \lim_{x \rightarrow -\infty} \frac{x^2 + 2}{x^2 + 1} = 1$$

$$(9) \text{解: } \lim_{x \rightarrow +\infty} \left( \frac{2x+1}{x+2} \right)^{\frac{\sin x}{x}}$$

$$\because \lim_{x \rightarrow +\infty} \left( \frac{2x+1}{x+2} \right) = 2, \quad \lim_{x \rightarrow \infty} \frac{\sin x}{x} = 0$$

( $\sin x$  是有界变量,  $\frac{1}{x}$  是无穷小量, 无穷小量与有界变量的乘积是无穷小量)

$$\therefore \lim_{x \rightarrow +\infty} \left( \frac{2x+1}{x+2} \right)^{\frac{\sin x}{x}} = 2^0 = 1$$

$$(10) \lim_{x \rightarrow a} \frac{\sqrt[3]{x} - \sqrt[3]{a}}{\sqrt[3]{x-a}} = \lim_{x \rightarrow a} \frac{\frac{x-a}{x^{\frac{2}{3}} + (ax)^{\frac{1}{3}} + a^{\frac{2}{3}}}}{(x-a)^{\frac{1}{3}}} = \lim_{x \rightarrow a} \frac{(x-a)^{\frac{2}{3}}}{x^{\frac{2}{3}} + (ax)^{\frac{1}{3}} + a^{\frac{2}{3}}} = 0$$

$$x^3 - y^3 = (x-y)(x^2 + xy + y^2)$$

$$\Rightarrow x - y = \frac{x^3 - y^3}{x^2 + xy + y^2}$$

$$\therefore x^{\frac{1}{3}} - y^{\frac{1}{3}} = \frac{x - y}{x^{\frac{2}{3}} + (xy)^{\frac{1}{3}} + y^{\frac{2}{3}}}$$

6.

$$\text{解: } \lim_{x \rightarrow 1^-} \left( \frac{x+5}{x^2+1} + 5 \right) = \frac{1+5}{1+1} + 5 = 8$$

$$\lim_{x \rightarrow 1^+} \left( 6 + \frac{x^2 - 1}{x - 1} \right) = \lim_{x \rightarrow 1^+} (6 + x + 1) = 7 + 1 = 8$$

$$\therefore \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^+} f(x) = 8$$

$$\therefore \lim_{x \rightarrow 1} f(x) = 8$$

7.

(1)要证  $\lim_{x \rightarrow 0} \frac{x+1}{x} = \infty$

即证  $\lim_{x \rightarrow 0} \left(1 + \frac{1}{x}\right) = \infty$

只需证  $\lim_{x \rightarrow 0} \frac{1}{x} = \infty$

只需证  $\lim_{x \rightarrow 0} x = 0$

对于  $\forall \varepsilon > 0$ , 取  $\delta$  为  $\varepsilon$ , 则当  $0 < |x - 0| < \delta$  时

有  $|x - 0| = |x| < \delta = \varepsilon$

$\therefore \lim_{x \rightarrow 0} x = 0$ , 即  $\lim_{x \rightarrow 0} \frac{x+1}{x} = \infty$

(2)要证  $\lim_{x \rightarrow 0^+} e^{\frac{1}{x}} = +\infty$

即证  $\lim_{x \rightarrow 0^+} e^{-\frac{1}{x}} = 0$

对于  $\forall \varepsilon > 0$ , 取  $\delta = -\frac{1}{\ln \varepsilon}$ ,  $0 < |x - 0| < \delta$

有  $\left| e^{-\frac{1}{x}} - 0 \right| = e^{-\frac{1}{x}} < \delta = \varepsilon$

$\therefore \lim_{x \rightarrow 0^+} e^{-\frac{1}{x}} = 0$ , 即  $\lim_{x \rightarrow +\infty} e^{\frac{1}{x}} = +\infty$

(3)要证  $\lim_{x \rightarrow \infty} x^2 = +\infty$

即证  $\lim_{x \rightarrow \infty} \frac{1}{x^2} = 0$

对于  $\forall \varepsilon > 0$ , 取  $X = \frac{1}{\sqrt{\varepsilon}}$ , 则当  $|x| > X$  时,

有  $\left| \frac{1}{x^2} - 0 \right| = \frac{1}{x^2} < \frac{1}{X^2} = \varepsilon$

$\therefore \lim_{x \rightarrow \infty} \frac{1}{x^2} = 0$ ,  $\lim_{x \rightarrow \infty} x^2 = +\infty$

(4)要证  $\lim_{x \rightarrow -\infty} x^3 = -\infty$

即证  $\lim_{x \rightarrow -\infty} \frac{1}{x^3} = 0$

对于  $\forall \varepsilon > 0$ , 取  $X = \frac{1}{\sqrt[3]{\varepsilon}}$ , 则当  $|x| > X$  时,

$$\text{有 } \left| \frac{1}{x^3} - 0 \right| = \left| \frac{1}{x^3} \right| < \frac{1}{X^3} = \varepsilon$$

$$\therefore \lim_{x \rightarrow -\infty} \frac{1}{x^3} = 0, \text{ 即 } \lim_{x \rightarrow -\infty} x^3 = -\infty$$

8.

解: ① 当  $m = n$  时

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{a_m + a_{m-1} + \frac{1}{x} + \cdots + a_1 \frac{1}{x^{m-1}} + a_0 \frac{1}{x^m}}{b_n + b_{n-1} - \frac{1}{x} + \cdots + b_1 \frac{1}{x^{n-1}} + b_0 \frac{1}{x^n}} = \frac{a_m}{b_n}$$

② 当  $m < n$  时

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{a_m \frac{1}{x^{n-m}} + a_{m-1} \frac{1}{x^{n-m+1}} + \cdots + a_0 \frac{1}{x^n}}{b_n + b_{n-1} \frac{1}{x} + \cdots + b_0 \frac{1}{x^n}} = 0$$

③ 当  $m > n$  时

$$\text{令 } g(x) = a_m x^m + a_{m-1} x^{m-1} + \cdots + a_1 x + a_0$$

$$h(x) = b_n x^n + b_{n-1} x^{n-1} + \cdots + b_1 x + b_0$$

$$\text{由 ② 得 } \lim_{x \rightarrow \infty} \frac{h(x)}{g(x)} = 0$$

$$\therefore \lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{g(x)}{h(x)} = \infty.$$



## 习题 2.3

1.

$\forall \delta > 0$ , 当  $0 < |x - x_0| < \delta$  时

$$\lim_{x \rightarrow x_0} g(x) = A \Rightarrow |g(x) - A| < \varepsilon \Rightarrow A - \varepsilon < g(x) < A + \varepsilon$$

$$\text{同理: } \lim_{x \rightarrow x_0} h(x) = A \Rightarrow A - \varepsilon < h(x) < A + \varepsilon$$

$$\because g(x) \leq f(x) \leq h(x)$$

$$\therefore A - \varepsilon < g(x) \leq f(x) \leq h(x) < A + \varepsilon$$

$$\Rightarrow A - \varepsilon < f(x) < A + \varepsilon \Rightarrow |f(x) - A| < \varepsilon \Rightarrow \lim_{x \rightarrow x_0} f(x) = A$$

$$2. (1) \lim_{x \rightarrow \infty} \frac{[x]}{x}$$

$$x - 1 < [x] \leq x \Rightarrow \frac{x - 1}{x} < \frac{[x]}{x} \leq 1$$

$$\because \lim_{x \rightarrow \infty} \frac{x - 1}{x} = 1, \lim_{x \rightarrow \infty} 1 = 1 \quad \therefore \lim_{x \rightarrow \infty} \frac{[x]}{x} = 1$$

$$(2) \lim_{x \rightarrow +\infty} \sqrt{1 + \frac{1}{x^\alpha}} \quad (\alpha > 0)$$

$$\sqrt{1} < \sqrt{1 + \frac{1}{x^\alpha}} < 1 + \frac{1}{x^\alpha}$$

$$\because \lim_{x \rightarrow +\infty} \sqrt{1} = 1 \quad \lim_{x \rightarrow +\infty} 1 + \frac{1}{x^\alpha} = 1$$

$$\therefore \lim_{x \rightarrow +\infty} \sqrt{1 + \frac{1}{x^\alpha}} = 1$$

$$\frac{1}{x} - 1 < \left[ \frac{1}{x} \right] \leq \frac{1}{x} \Rightarrow 1 - x < x \left[ \frac{1}{x} \right] \leq 1$$

$$\because \lim_{x \rightarrow 0} 1 - x = 1 \quad \lim_{x \rightarrow 0} 1 = 1 \quad \therefore \lim_{x \rightarrow 0} x \left[ \frac{1}{x} \right] = 1$$

3.

$$(1) \lim_{x \rightarrow 0} \sin \frac{1}{x}$$

$$\text{取 } x_1 = \frac{1}{2n\pi}, \quad x_2 = \frac{1}{2n\pi + \frac{\pi}{2}} \quad n \in N^*$$

$$\text{易知 } \lim_{n \rightarrow \infty} x'_n = \lim_{n \rightarrow \infty} x_2 = 0, \quad x''_n \neq x_2 \neq 0$$

$$\therefore \lim_{n \rightarrow \infty} \sin \frac{1}{x'_n} = \lim_{n \rightarrow \infty} \sin 2n\pi = 0$$

$$\lim_{n \rightarrow \infty} \sin \frac{1}{x''_n} = \lim_{n \rightarrow \infty} \sin \left( 2n\pi + \frac{\pi}{2} \right) = 1$$

$$\therefore \lim_{x \rightarrow 0} \sin \frac{1}{x} \text{ 不存在}$$

$$(2) \lim_{x \rightarrow 0} \cos \frac{1}{x}$$

证明过程同(1)

4.

$$(1) \lim_{x \rightarrow 0} \frac{\sin \alpha x}{\sin \beta x} (\beta \neq 0)$$

$$= \lim_{x \rightarrow 0} \frac{\sin \alpha x}{\alpha x} \cdot \frac{\beta x}{\sin \beta x} \cdot \frac{\alpha}{\beta}$$

$$= \lim_{x \rightarrow 0} \frac{\sin \alpha x}{\alpha x} \cdot \lim_{x \rightarrow 0} \frac{\beta x}{\sin \beta x} \cdot \lim_{x \rightarrow 0} \frac{\alpha}{\beta} = \frac{\alpha}{\beta}$$

$$(2) \lim_{x \rightarrow 0} \frac{\tan \alpha x}{\tan \beta x} (\beta \neq 0) = \lim_{x \rightarrow 0} \frac{\alpha x}{\beta x} = \frac{\alpha}{\beta}$$

$$(3) \lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} = \lim_{x \rightarrow 0} \frac{\frac{1}{2}x^2}{x^2} = \frac{1}{2}$$

$$\begin{aligned}
 (4) \lim_{x \rightarrow \frac{\pi}{4}} \frac{\sqrt{2} - 2 \cos x}{\sin \left( x - \frac{\pi}{4} \right)} \\
 = \lim_{x \rightarrow \frac{\pi}{4}} \frac{2 \sin x}{\cos \left( x - \frac{\pi}{4} \right)} = \frac{2 \sin x}{\frac{\sqrt{2}}{2} (\cos x + \sin x)} = \frac{\sqrt{2}}{1} = \sqrt{2}
 \end{aligned}$$

5.

$$\begin{aligned}
 (1) \lim_{x \rightarrow 0} (1 - 3x)^{\frac{1}{x}} \\
 = \lim_{x \rightarrow 0} [1 + (-3x)]^{\frac{1}{-3x} \cdot (-3)} = e^{-3}
 \end{aligned}$$

$$\begin{aligned}
 (2) \lim_{x \rightarrow \infty} \left( \frac{1+x}{2+x} \right)^{\frac{1-x^2}{1-x}} \\
 = \lim_{x \rightarrow \infty} \left[ 1 + \left( -\frac{1}{x+2} \right) \right]^{-(x+2) \cdot \frac{1+x}{-(x+2)}} \\
 = \lim_{x \rightarrow \infty} e^{\frac{1+x}{-(x+2)}} = e^{-1}
 \end{aligned}$$

$$(3) \lim_{x \rightarrow 0} (1 + \sin x)^{3 \csc x} = \lim_{x \rightarrow 0} (1 + \sin x)^{\frac{1}{\sin x} \cdot 3} = e^3$$

$$\begin{aligned}
 (4) \lim_{x \rightarrow \infty} \left( \frac{x+1}{x-1} \right)^x \\
 = \lim_{x \rightarrow \infty} \left( 1 + \frac{2}{x-1} \right)^{\frac{x-1}{2} \cdot \frac{2x}{x-1}} \\
 = \lim_{x \rightarrow \infty} e^{\frac{2x}{x-1}} \\
 = e^2
 \end{aligned}$$

## 习题 2.4

1. (1)  $\lim_{x \rightarrow 0} \frac{3x^2 - 4x}{x} = \lim_{x \rightarrow 0} (3x - 4) = -4 \neq 0 \quad \therefore 3x^2 - 4x = O(x)$

(2)  $\lim_{x \rightarrow 0} \frac{x^2 \sin \frac{1}{x}}{x} = \lim_{x \rightarrow 0} x \sin \frac{1}{x} = 0 \quad \therefore x^2 \sin \frac{1}{x} = o(x)$

(3)  $\lim_{x \rightarrow 0} \frac{x \sin x^2}{x^3} = \lim_{x \rightarrow 0} \frac{\sin x^2}{x^2} = \lim_{x \rightarrow 0} \frac{\sin x^2}{x^2} = 1 \quad \therefore x \sin x^2 \sim x^3$

(4)  $\lim_{x \rightarrow 0} \frac{(1+x)^2 - 1 - 2x}{x^2} = \lim_{x \rightarrow 0} \frac{x^2}{x^2} = 1 \quad \therefore (1+x)^2 - 1 - 2x \sim x^2$

2. (1)  $\lim_{x \rightarrow +\infty} \frac{x+1}{x^2+1} * x = \lim_{x \rightarrow +\infty} \frac{x^2+x}{x^2+1} = \lim_{x \rightarrow \infty} (1 + \frac{x-1}{x^2+1}) = 1 \quad \therefore \frac{x+1}{x^2+1} \sim \frac{1}{x}$

(2) 令  $t = \frac{1}{x}$ ,  $\lim_{t \rightarrow 0} \frac{t^2 \sin \frac{1}{t}}{t} = 0$  (同 1. (2))  $\therefore t^2 \sin \frac{1}{t} = o(t) \quad \therefore \frac{1}{x^2} \sin x = o(\frac{1}{x})$

(3) 令  $t = \frac{1}{x}$ ,  $\lim_{t \rightarrow 0} \frac{2t \sin t}{t^2} = \lim_{t \rightarrow 0} \frac{2 \sin t}{t} = 2 \neq 0, \quad \therefore 2t \sin t = O(t^2)$ , 即  $\frac{2}{x} \sin \frac{1}{x} = O(\frac{1}{x^2})$

(4) 令  $t = \frac{1}{x}$ ,  $\lim_{t \rightarrow 0} \frac{(1+t)^2 - 1 - 2t}{t^2} = 1$  (同 1. (4))  $\therefore (1+t)^2 - 1 - 2t \sim t^2$

$$\therefore (1 + \frac{1}{x})^2 - 1 - \frac{2}{x} \sim \frac{1}{x^2}$$

3. (1) 原式  $= \lim_{x \rightarrow 0} \frac{\alpha x}{\beta x} = \lim_{x \rightarrow 0} \frac{\alpha}{\beta} = \frac{\alpha}{\beta}$

(2) 原式  $= \lim_{x \rightarrow 0} \frac{x^m}{x^m} = 1$

(3) 原式  $= \lim_{x \rightarrow 0} \frac{\frac{1}{2}x}{x} = \frac{1}{2}$

(4) 原式  $= \lim_{x \rightarrow 0} \frac{\tan x}{x} = \lim_{x \rightarrow 0} \frac{x}{x} = 1$

(5) 原式  $= \lim_{x \rightarrow 0} \frac{\frac{1}{2}x^2}{x^2} = \frac{1}{2}$

(6) 原式  $= \lim_{x \rightarrow 0} \frac{\frac{1}{n} \sin x}{\tan x} = \lim_{x \rightarrow 0} \frac{\frac{1}{n}x}{x} = \frac{1}{n}$

(7) 原式  $= \lim_{x \rightarrow 0} \frac{x^2}{\frac{1}{2}x^2} = 2$

(8) 原式  $= \lim_{x \rightarrow 0} \frac{\sin x}{\sin \beta x} = \lim_{x \rightarrow 0} \frac{x}{\beta x} = \frac{1}{\beta}$

(9) 原式  $= \lim_{x \rightarrow 0} \frac{\tan x (1 - \cos x)}{x (\sin x)^2} = \lim_{x \rightarrow 0} \frac{x^{\frac{1}{2}}}{x^{\frac{3}{2}}} x^2 = \frac{1}{2}$

(10) 原式  $= \lim_{x \rightarrow 0} \frac{x^2}{x^2} = 1$

4. 均设为关于  $x$  的  $k$  阶无穷小量

$$(1) \lim_{x \rightarrow 0} \frac{x^3 + \sin x^2}{x^k} = \lim_{x \rightarrow 0} (x^{3-k} + 100x^{2-k})$$

当  $k=2$  时, 原式  $= 100 \neq 0 \therefore$  是  $x$  的二阶无穷小量

$$(2) \lim_{x \rightarrow 0} \frac{x^2 + \sin x^2}{x^k} = \lim_{x \rightarrow 0} (x^{2-k} + \frac{\sin x^2}{x^k})$$

当  $k=2$  时, 原式  $= 2 \neq 0 \therefore$  是  $x$  的二阶无穷小量

$$(3) \lim_{x \rightarrow 0} \frac{x^2(1+x)}{x^k(1+\sqrt[3]{x})} = \lim_{x \rightarrow 0} \frac{1+x}{1+\sqrt[3]{x}} = 1$$

当  $k=2$  时, 原式  $= 1 \neq 0 \therefore$  是  $x$  的二阶无穷小量

$$(4) \lim_{x \rightarrow 0} \frac{\ln(1+x^3)}{x^k} = \lim_{x \rightarrow 0} x^{3-k}$$

当  $k=3$  时, 原式  $= 1 \neq 0 \therefore$  是  $x$  的三阶无穷小量

附: 额外三角等价无穷小替换

$$\tan x - x \sim \frac{1}{3}x^3$$

$$x - \sin x \sim \frac{1}{6}x^3$$

$$\tan x - \sin x \sim \frac{1}{2}x^3$$

## 习题 2.5

$f(x_0) = f(x_0^-) = f(x_0^+) \Rightarrow f(x)$  在  $x_0$  处连续

1.(1) 证明:  $f(x_0^-) = \lim_{x \rightarrow x_0^-} f(x) = \cos x_0$

$$f(x_0^+) = \lim_{x \rightarrow x_0^+} f(x) = \cos x_0$$

$$f(x_0) = \cos x_0 = f(x_0^-) = f(x_0^+)$$

$\therefore f(x)$  在  $x_0$  处连续.

(2) 证明:  $f(x_0^-) = \lim_{x \rightarrow x_0^-} f(x) = a^{x_0}$

$$f(x_0^+) = \lim_{x \rightarrow x_0^+} f(x) = a^{x_0}$$

$$f(x_0) = a^{x_0} = f(x_0^-) = f(x_0^+)$$

$\therefore f(x)$  在  $x_0$  处连续.

2.(1)  $f(x) = \begin{cases} 1+x, & x \geq 0 \\ x, & x < 0 \end{cases}$

分段点:  $x=0$        $f(0)=1$

$$f(0^+) = \lim_{x \rightarrow 0^+} f(x) = 1 \quad f(0^-) = \lim_{x \rightarrow 0^-} f(x) = 0$$

$\therefore f(x)$  在分段点处不连续.

(2)  $f(x) = \begin{cases} x \sin \frac{1}{x}, & x > 0 \\ 1, & x = 0 \\ 2+x, & x < 0 \end{cases}$

分段点:  $x=0$        $f(0)=1$

$$f(0^+) = \lim_{x \rightarrow 0^+} f(x) = 0 \quad f(0^-) = \lim_{x \rightarrow 0^-} f(x) = 2$$

$\therefore f(x)$  在分段点处不连续.

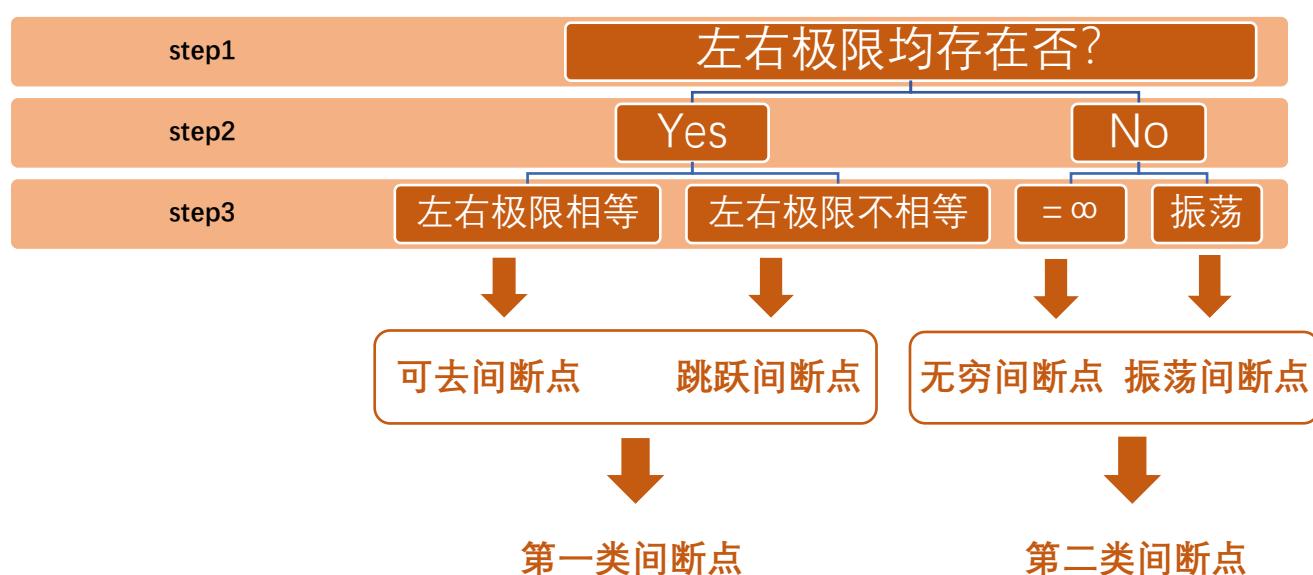
3.[补充 1]  $x_0$  为  $f(x)$  的间断点的三种情况:

①  $f(x)$  在  $x_0$  处无定义

②  $\lim_{x \rightarrow x_0} f(x)$  不存在

③  $\lim_{x \rightarrow x_0} f(x) \neq f(x_0)$

[补充 2] 判断间断点类型



(1)  $f(x) = \frac{x}{\sin x}$

$$f(0^+) = \lim_{x \rightarrow 0^+} \frac{x}{\sin x} = 1 \quad f(0^-) = \lim_{x \rightarrow 0^-} \frac{x}{\sin x} = 1$$

$\therefore x=0$  为  $f(x)$  的可去间断点.

(2)  $f(x) = [x]$

$$f(0^+) = \lim_{x \rightarrow 0^+} [x] = 0 \quad f(0^-) = \lim_{x \rightarrow 0^-} [x] = -1$$

$\therefore x=0$  为  $f(x)$  的跳跃间断点.

(3)  $f(x) = \frac{1}{\sin x}$

$$f(0^+) = \lim_{x \rightarrow 0^+} \frac{1}{\sin x} = \infty \quad f(0^-) = \lim_{x \rightarrow 0^-} \frac{1}{\sin x} = \infty$$

$\therefore x=0$  为  $f(x)$  的无穷间断点.

$$(4) \quad f(x) = \sin \frac{1}{x}$$

$$f(0) = \lim_{x \rightarrow 0} \sin \frac{1}{x}, \text{ 不存在}$$

$\therefore x=0$  为  $f(x)$  的振荡间断点.

$$(5) \quad f(x) = \frac{1}{1 - e^{\frac{x}{1-x}}}$$

$$f(0) = \lim_{x \rightarrow 0} \frac{1}{1 - e^{\frac{x}{1-x}}} \text{ 不存在}$$

$\therefore x=0$  为  $f(x)$  的振荡间断点.

$$(6) \quad f(x) = \frac{\tan x}{x}$$

$$f(0^+) = \lim_{x \rightarrow 0^+} \frac{\tan x}{x} = 1 \quad f(0^-) = \lim_{x \rightarrow 0^-} \frac{\tan x}{x} = 1$$

$\therefore x=0$  为  $f(x)$  的可去间断点.

$$4.(1) \quad f(x) = x - [x]$$

$$\text{对 } \forall x_0 \in \mathbb{Z} : f(x_0^+) = \lim_{x \rightarrow x_0^+} (x - [x]) = 0$$

$$f(x_0^-) = \lim_{x \rightarrow x_0^-} (x - [x]) = 1$$

$\therefore f(x)$  在所有整数点处不连续, 而在其他点处是连续的.

$$(2) \quad f(x) = \frac{x}{\sin x}$$

间断点:  $x=n\pi, n \in \mathbb{Z}$  (无定义)

$\therefore f(x)$  在  $x=n\pi$  ( $n \in \mathbb{Z}$ ) 处不连续, 而在其他点是连续的.

$$(3) \quad f(x) = \cot x = \frac{\cos x}{\sin x}$$

$x=n\pi, n \in \mathbb{Z}$  时无定义, 同上

$\therefore f(x)$  在  $x=n\pi$  ( $n \in \mathbb{Z}$ ) 处不连续, 而在其他点是连续的.

$$(4) \quad f(x) = \sqrt{\frac{(x-1)(x-3)}{x+1}}$$

$$\frac{(x-1)(x-3)}{x+1} \geq 0 \Rightarrow \text{定义域: } [-1, 1] \cup [3, +\infty)$$

$\therefore f(x)$  在其定义域上连续.



5.

$$\begin{aligned}
 (1) \quad & \lim_{x \rightarrow 0^+} \arcsin \frac{1-x}{1-x^2} \\
 &= \lim_{x \rightarrow 0^+} \arcsin \frac{1-x}{(1-x)(1+x)} \\
 &= \lim_{x \rightarrow 0^+} \arcsin \frac{1}{1+x} \\
 &= \frac{\pi}{2}
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad & \lim_{x \rightarrow 0} \ln(1 + e^x) \\
 &= \ln \lim_{x \rightarrow 0} (1 + e^x) \\
 &= \ln 2
 \end{aligned}$$

$$(3) \text{ 令 } F(x) = \frac{\sqrt[3]{x+1} \ln(2+x^2)}{(1-x^3) + \cos x}$$

$$(4) \text{ 令 } F(x) = \frac{x^2 + e^{1-x}}{\ln(2+x)}$$

由于初等函数在其定义域内连续

$$\text{同 (3) } \lim_{x \rightarrow 1} F(x) = F(1) = \frac{2}{\ln 3}$$

$$\text{故 } \lim_{x \rightarrow 0} F(x) = F(0) = \frac{\ln 2}{2}$$

$$\begin{aligned}
 (5) \quad & \lim_{x \rightarrow 0} \sqrt{\frac{1+x}{1-x}} = \lim_{x \rightarrow 0} \sqrt{\frac{2}{1-x}} - 1 \\
 &= 1
 \end{aligned}$$

$$\begin{aligned}
 (6) \quad & \lim_{x \rightarrow 2} \frac{1}{\sin(\pi x + \frac{\pi}{2})} \\
 &= \frac{1}{\sin \frac{\pi}{2}} = 1
 \end{aligned}$$

6.证明: (1)  $\because f(x)$  在  $x_0$  处连续  $\therefore \lim_{x \rightarrow x_0} f(x) = f(x_0)$

$$\therefore \forall \varepsilon > 0, \exists \delta > 0,$$

$$\text{当 } 0 < |x - x_0| < \delta, \text{ 有 } |f(x) - f(x_0)| < \varepsilon,$$

$$\therefore |f(x) - f(x_0)| \geq ||f(x)| - |f(x_0)||$$

$$\therefore ||f(x)| - |f(x_0)|| < \varepsilon, \text{ 即 } |f(x)| \text{ 在 } x_0 \text{ 处连续}$$

$$\text{故 } |f(x)^2 - f(x_0)^2| = [f(x) + f(x_0)][f(x) - f(x_0)] < \varepsilon$$

$$\therefore f(x)^2 \text{ 在 } x_0 \text{ 处连续}$$

(2) 反之不成立

例如  $f(x) = \begin{cases} 1, & x \geq 0 \\ -1, & x < 0 \end{cases}$  在  $x=0$  处不连续

$$7. |x| > 1 \text{ 时, } \lim_{n \rightarrow \infty} \frac{x^{2n+1} + (a-1)x^{n-1}}{x^{2n} - ax^{n-1}}$$

$$= \lim_{n \rightarrow \infty} \frac{x + \frac{a-1}{x^n} - \frac{1}{x^{2n}}}{1 - \frac{a}{x^n} - \frac{1}{x^{2n}}} = x$$

$$|x| < 1 \text{ 时, 原式} = \lim_{n \rightarrow \infty} \frac{0+0-1}{0-0-1} = 1 (|x| < 1, n \rightarrow \infty \text{ 时, } x^n \rightarrow 0)$$

$$x = -1 \text{ 时, } f(-1) = \lim_{n \rightarrow \infty} \frac{-1 + (-1)^n(a-1) - 1}{1 - (-1)^n a - 1}$$

$$= \lim_{n \rightarrow \infty} \frac{-2 + (-1)^n(a-1)}{(-1)^{n+1} a}$$

$$= \begin{cases} -\frac{a+1}{a}, & n \text{ 为奇} \\ \frac{3-a}{a}, & n \text{ 为偶} \end{cases} \text{ 故 } f(-1) \text{ 不存在}$$

$$x = 1 \text{ 时, } f(-1) = \lim_{n \rightarrow \infty} \frac{1+a-1-1}{1-a-1} = \frac{1-a}{a}$$

$$\therefore f(x) \begin{cases} 1, & |x| < 1 \\ \frac{1-a}{a}, & x = 1 \\ x, & |x| > 1 \end{cases} \quad \text{在 } x = -1 \text{ 无定义}$$

要使  $f(x)$  在  $[0, +\infty)$  上连续,  $\frac{1-a}{a} = 1, \therefore a = \frac{1}{2}$

## 习题 2.6

1. (1) 证明:

$$\forall x_0 \in [0,2] (x \neq 1) \\ \lim_{x \rightarrow x_0} f(x_0) = \lim_{x \rightarrow x_0} \frac{1}{x-1} = \frac{1}{x_0-1} = f(x_0)$$

$$\text{当 } x_0 = 1 \text{ 时有 } \lim_{x \rightarrow 1^+} -f(x) = \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} \frac{1}{x-1} = \infty$$

故  $f(x)$  在闭区间  $[0,2]$  上除点  $x = 1$  外时连续的。

(2) 证明: 1. 假设  $\exists x_0 \in [0,2]$  使  $f(x_0)$  为最大值

$$\text{则 } f(x_0) = \frac{1}{x_0-1} \text{ 且 } f(x_0) > 0$$

$$\text{不妨取 } x_1 = \frac{1}{f(x_0)+1} + 1 \in [0,2] \text{ 且 } x_1 \neq 1$$

$$f(x_1) = \frac{1}{x_1-1} = \frac{1}{\frac{1}{f(x_0)+1}+1-1} = f(x_0) + 1 > f(x_0) \text{ 这与条件矛盾。}$$

2. 假设  $\exists x'_0 \in [0,2]$  使  $f(x'_0)$  为最小值

$$\text{则 } f(x'_0) = \frac{1}{x'_0-1} \text{ 且 } f(x'_0) < 0$$

$$\text{不妨取 } x_2 = \frac{1}{f(x'_0)-1} + 1 \in [0,2] \text{ 且 } x_2 \neq 1$$

$$f(x_2) = \frac{1}{x_2-1} = \frac{1}{\frac{1}{f(x'_0)-1}+1-1} = f(x'_0) - 1 < f(x'_0) \text{ 这与条件矛盾}$$

综上  $f(x)$  在闭区间  $[0,2]$  上既无最大值也无最小值。

2. 证明:

1. 假设  $\exists x_0 \in (0,1)$  使  $f(x_0)$  为最大值

$$\text{则 } f(x_0) = \frac{1}{x_0} \text{ (} f(x_0) > 0 \text{), 不妨取 } x_1 = \frac{1}{f(x_0)+1} \in (0,1)$$

$$\text{此时 } f(x_1) = f(x_0) + 1 > f(x_0) \text{ (矛盾)}$$

2. 同理假设  $\exists x'_0 \in (0,1)$  使  $f(x'_0)$  为最小值 ( $f(x'_0) > 1$ )

$$\text{取 } x_2 = \frac{1}{\frac{1}{2}(f(x'_0)+1)} \in (0,1)$$

$$\text{此时 } f(x_2) = \frac{1}{2}(f(x'_0) + 1) < \frac{1}{2} \cdot 2f(x'_0) = f(x'_0) \text{ (矛盾)}$$

综上  $f(x)$  在闭区间  $[0,2]$  上既无最大值也无最小值。

3. 证明: (1) 记  $f(x) = x^3 - 5x + 1$

由所有基本初等函数在其定义域内均连续得 $f(x)$ 在闭区间 $[0,1]$ 上连续。

$$\text{由 } f(0) = 1, f(1) = -3 \quad f(0) \cdot f(1) < 0$$

故由零点定理得至少存在一点  $\xi \in (0,1)$  使得  $f(\xi) = 0$  即方程  $x^3 - 5x + 1 = 0$  在  $(0, 1)$  内至少有一根。

(2) 记  $g(x) = x - 2\sin x$  同 (1) 中论述  $g(x)$  在闭区间  $[\frac{\pi}{2}, \pi]$  上连续

$$\text{由 } g(\frac{\pi}{2}) = \frac{\pi}{2} - 2 < 0, g(\pi) = \pi > 0 \quad g(\frac{\pi}{2}) \cdot g(\pi) < 0$$

故由零点定理得至少存在一点  $\xi \in (\frac{\pi}{2}, \pi)$  使得  $g(\xi) = 0$  即方程  $x - 2\sin x = 0$  有根。

4. 证明: 记  $g(x) = f(x) - x$

由  $f(x)$ ,  $x$  均在  $[a, b]$  上连续得  $g(x)$  在  $[a, b]$  上连续

$$g(a) = f(a) - a \geq 0, g(b) = f(b) - b \leq 0$$

1. 当  $g(a) = 0$  或  $g(b) = 0$  时  $f(a) = a$  或  $f(b) = b$  (原式显然成立)

2. 当  $g(a) > 0, g(b) < 0$  时  $g(a) \cdot g(b) < 0$

由零点定理得在  $(a, b)$  上存在一点  $\xi$  使得  $g(\xi) = 0$  综上原式得

证。

5. 证明: 由  $AB < 0$  不妨设  $A > 0, B < 0$

则  $f(a) = A > 0$  由  $\lim_{x \rightarrow +\infty} f(x) = B < 0$  (极限的局部保号性得)

一定  $\exists X > a$ , 当  $x > X$  时  $f(x) < 0$  成立

不妨取  $b = x + 1$ , 则  $f(b) < 0$ , 则  $f(a) \cdot f(b) < 0$

由  $f(x)$  在  $[a, +\infty)$  上连续得  $f(x)$  在  $[a, b]$  上连续

故由零点定理得至少存在一点  $\xi \in (a, b)$ , 使得  $f(\xi) = 0$

即  $f(x)$  在  $[a, +\infty)$  上至少有一个零点。

6.证明: 由 $|f(x)| \leq e^{\sin x} - 1$ 得

$$|f(0)| \leq e^{\sin 0} - 1 = 0 \Rightarrow f(0) = 0$$

由 $0 \leq |f(x)| \leq e^{\sin x} - 1$ 得

$$\lim_{x \rightarrow 0} |f(x)| = 0 \quad (\text{夹逼定理})$$

即 $(\lim_{x \rightarrow 0} f(x) = 0) = f(0)$  则函数 $f(x)$ 在 $x = 0$ 处连续。

(由 $\lim_{x \rightarrow 0} |f(x)| = 0$ 推得 $\lim_{x \rightarrow 0} f(x) = 0$ 为教材 2.1 习题 2. (2) 结论, 使

用 定义很好证明)

## 第 2 章复习题

1. 证明：反证法：

假设  $\{a_n + b_n\}$  收敛

因为：  $b_n = a_n + b_n - a_n$  又  $\{a_n\}$  收敛

则  $\{b_n\}$  收敛，与  $\{b_n\}$  发散矛盾

则假设不成立，即  $\{a_n + b_n\}$  发散

$\{a_n b_n\}$  不一定发散，如：  $a_n = 0$ ,  $b_n = n$ ,  $a_n b_n = 0$ ,  $\lim_{n \rightarrow \infty} a_n b_n = 0$

2. 不能，如：  $a_n = n$ ,  $b_n = -n$ ,  $a_n + b_n = 0$ ,  $\{a_n + b_n\}$  收敛

$a_n = (-1)^n$ ,  $b_n = (-1)^n$ ,  $a_n b_n = 1$ ,  $\{a_n b_n\}$  收敛

3. 不能，如：  $a_n = \frac{1}{\sqrt{n}}$ ,  $b_n = n$ ,  $\lim_{n \rightarrow \infty} a_n = 0$

但  $a_n b_n = \sqrt{n}$  不收敛

所以  $\lim_{n \rightarrow \infty} a_n b_n$  不存在

4. 不能，如：  $a_n = 2$  (n 为奇),  $0$  (n 为偶)

$b_n = 0$  (n 为奇),  $2$  (n 为偶)

$\lim_{n \rightarrow \infty} a_n b_n = 0$ , 但  $\lim_{n \rightarrow \infty} a_n$  和  $\lim_{n \rightarrow \infty} b_n$  都不存在

5.

(1)

$$a_n \geq \frac{1}{\sqrt{n^2+n}} + \frac{1}{\sqrt{n^2+n}} + \dots + \frac{1}{\sqrt{n^2+n}} = \frac{n}{\sqrt{n^2+n}}$$

$$a_n \leq \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+1}} + \dots + \frac{1}{\sqrt{n^2+1}} = \frac{n}{\sqrt{n^2+1}}$$

$$\text{又} \lim_{n \rightarrow \infty} \frac{n}{\sqrt{n^2+n}} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{\frac{1}{n}+1}} = 1$$

$$\lim_{n \rightarrow \infty} \frac{n}{\sqrt{n^2+1}} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{\frac{1}{n^2}+1}} = 1$$

则由夹逼定理知:  $\lim_{n \rightarrow \infty} a_n = 1$

(2) 令  $\max\{A, B, C, D\} = a$  则:

$$\sqrt[n]{a^n} \leq \sqrt[n]{A^n + B^n + C^n + D^n} \leq \sqrt[n]{4a^n}$$

$$a \leq \sqrt[n]{A^n + B^n + C^n + D^n} \leq a\sqrt[n]{4}$$

又  $\lim_{n \rightarrow \infty} (a\sqrt[n]{4}) = a$ , 所以  $\lim_{n \rightarrow \infty} a_n = \max\{A, B, C, D\}$

## 6

(1) 证明: 单调性:  $\because 0 < a_1 < 1$

由数学归纳法知  $a_n > 0$

则  $a_{n+1} - a_n = -a_n^2 < 0$

$\therefore 0 < a_{n+1} < a_n$

则  $\{a_n\}$  单调递减

有界性:  $\because a_n > 0$

$\therefore \{a_n\}$  收敛

$$\text{令} \lim_{n \rightarrow \infty} a_n = a, \text{则} \lim_{n \rightarrow \infty} a_{n+1} = \lim_{n \rightarrow \infty} a_n(1 - a_n)$$

$$\therefore a = a(1 - a)$$

$$\therefore a = 0$$

$$\text{则} \lim_{n \rightarrow \infty} a_n = 0$$

(2) 证明: 单调性:  $a_1 = \sqrt{2}, a_2 = \sqrt{3 + 2\sqrt{2}}$ , 则  $a_2 > a_1$

$$\text{设 } a_{k+1} > a_k, \text{则 } a_{k+2} = \sqrt{3 + 2a_{k+1}} > \sqrt{3 + 2a_k} = a_{k+1}$$

由数学归纳法知,  $\{a_n\}$  单调递增

有界性:  $n=1, a_1=\sqrt{2}<3$   
 假设  $n=k, a_k<3$

则  $n=k+1$  时,  $a_{k+1}=\sqrt{3+2a_k}<3$  成立

$\therefore a_n<3$

$\therefore \{a_n\}$  收敛

令  $\lim_{n \rightarrow \infty} a_n = a$ , 则  $\lim_{n \rightarrow \infty} a_{n+1} = \lim_{n \rightarrow \infty} \sqrt{3+2a_n}$

$a = \sqrt{3+2a}$

有极限的保号性知  $a=3$

则  $\lim_{n \rightarrow \infty} a_n = 3$

## 7、求下列数列极限

$$(1) \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n-2}\right)^{n+1}$$

解: 原式  $= \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n-2}\right)^{-(n-2) \cdot \frac{n+1}{-(n-2)}} = e^{-1} \cdot 1^\infty$

$$(2) \lim_{n \rightarrow \infty} \left(\frac{1+n}{2+n}\right)^n$$

解: 原式  $= \lim_{n \rightarrow \infty} \left(1 + \frac{-1}{n+2}\right)^{-(n+2) \cdot \frac{n}{-(n+2)}} = e^{-1} \cdot 1^\infty$

$$(3) \lim_{n \rightarrow \infty} n \sin \frac{1}{n}$$

解: 原式  $= \lim_{n \rightarrow \infty} \frac{\sin \frac{1}{n}}{\frac{1}{n}} = 1$

$$(4) \lim_{n \rightarrow \infty} (\sqrt{n+2} - 2\sqrt{n+1} + \sqrt{n}) * \sqrt{n}$$

解: 原式  $= \lim_{n \rightarrow \infty} [(\sqrt{n+2} - \sqrt{n+1}) - (\sqrt{n+1} - \sqrt{n})] * \sqrt{n}$

$$= \lim_{n \rightarrow \infty} \left( \frac{\sqrt{n}}{\sqrt{n+2} + \sqrt{n+1}} - \frac{\sqrt{n}}{\sqrt{n+1} + \sqrt{n}} \right)$$

$$= \frac{1}{2} - \frac{1}{2}$$

$$= 0$$

$$(5) \lim_{n \rightarrow \infty} \tan^n \left( \frac{\pi}{4} + \frac{2}{n} \right)$$

解: 原式  $= \lim_{n \rightarrow \infty} \left( \frac{1 + \tan^2 \frac{2}{n}}{1 - \tan^2 \frac{2}{n}} \right)^n = \lim_{n \rightarrow \infty} \left( \frac{1 - \tan^2 \frac{2}{n} + 2 \tan^2 \frac{2}{n}}{1 - \tan^2 \frac{2}{n}} \right)^n$



$$\begin{aligned}
&= \lim_{n \rightarrow \infty} \left( 1 + \frac{2 \tan^2 \frac{2}{n}}{1 - \tan^2 \frac{2}{n}} \right)^{\frac{1 - \tan^2 \frac{2}{n}}{2 \tan^2 \frac{2}{n}} \cdot \frac{2 \tan^2 \frac{2}{n}}{1 - \tan^2 \frac{2}{n}}} \\
&= e^{\lim_{n \rightarrow \infty} \left( \frac{2n \tan^2 \frac{2}{n}}{1 - \tan^2 \frac{2}{n}} \right)} = e^{\lim_{n \rightarrow \infty} \frac{2n \cdot \frac{2}{n}}{1 - \tan^2 \frac{2}{n}}} \\
&= e^{\frac{4}{1-0}} = e^4
\end{aligned}$$

$$(6) \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{n+k}{n^2+k}$$

解：令  $a_n = \sum_{k=1}^n \frac{n+k}{n^2+k}$  由于  $\forall 1 \leq k \leq n$  有  $\frac{n+k}{n^2+n} \leq \frac{n+k}{n^2+k} \leq \frac{n+k}{n^2+1}$

$$\text{则 } \sum_{k=1}^n \frac{n+k}{n^2+n} \leq a_n \leq \sum_{k=1}^n \frac{n+k}{n^2+1}$$

$$\text{即 } \frac{n^2 + \frac{n(n+1)}{2}}{n^2+n} \leq a_n \leq \frac{n^2 + \frac{n(n+1)}{2}}{n^2+1}$$

$$\text{因为 } \lim_{n \rightarrow \infty} \frac{n^2 + \frac{n(n+1)}{2}}{n^2+n} = \lim_{n \rightarrow \infty} \frac{n^2 + \frac{n(n+1)}{2}}{n^2+1}$$

$$\text{由夹逼定理知 } \lim_{n \rightarrow \infty} a_n = \frac{3}{2} \text{ 即 } \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{n+k}{n^2+k} = \frac{3}{2}$$

## 8、求下列函数极限

$$(1) \text{ 解：原式} = \lim_{n \rightarrow \infty} \frac{1 + \frac{1}{x}}{x + \sqrt{1 + \frac{1}{x^2}}} = \frac{1}{2}$$

$$(2) \text{ 解：原式} = \lim_{n \rightarrow \infty} \frac{\frac{(2x+3)^{20} \cdot (3x+2)^{30}}{(3x)^{20} \cdot (3x)^{30}}}{\frac{(2x+1)^{50}}{(3x)^{50}}} = \lim_{n \rightarrow \infty} \frac{\left(\frac{2}{3} + \frac{1}{x}\right)^{20} \cdot \left(1 + \frac{2}{3x}\right)^{30}}{\left(\frac{2}{3} + \frac{1}{3x}\right)^{50}} = \lim_{n \rightarrow \infty} \frac{\left(\frac{2}{3}\right)^{20} \cdot 1}{\left(\frac{2}{3}\right)^{50}} = \left(\frac{2}{3}\right)^{30}$$

(3)解：当  $m=n$  时，原式=1

$$\text{当 } m > n \text{ 时，原式} = \lim_{x \rightarrow \infty} \frac{x^{\frac{1}{m} - \frac{1}{n}} - \frac{1}{\sqrt[n]{x}}}{1 - \frac{1}{\sqrt[n]{x}}} = \frac{0-0}{1-0} = 0$$

当  $m < n$  时，原式=+∞

$$(4) \text{解：原式} = \lim_{x \rightarrow 0} \frac{\cos x - 1 + 1 \cos 3x}{x^2} = \lim_{x \rightarrow 0} \frac{\cos x - 1}{x^2} + \lim_{x \rightarrow 0} \frac{\cos 3x}{x^2} = \lim_{x \rightarrow 0} \frac{-\frac{1}{2}x^2}{x^2} + \lim_{x \rightarrow 0} \frac{9x^2}{x^2} = 4$$

(5)解：令  $t=1-x$ , 则当  $x \rightarrow 1$  时,  $t \rightarrow 0$

$$\text{原式} = \lim_{t \rightarrow 0} t \tan \frac{\pi}{2} (1-t) = \lim_{t \rightarrow 0} t \cos \frac{t}{2} = \lim_{t \rightarrow 0} t \frac{\cos \frac{\pi}{2} t}{\sin \frac{\pi}{2} t} = \lim_{t \rightarrow 0} t \frac{\cos \frac{\pi}{2} t}{\frac{\pi}{2} t} = \frac{2}{\pi}$$

$$(6) \text{解：原式} = \lim_{x \rightarrow 0} \frac{2+e^{\frac{1}{x}}}{1+e^{\frac{1}{x}}} - \lim_{x \rightarrow 0} \frac{\sin x}{x} = \frac{2+0}{1+0} - 1 = 1$$

## 9

解:  $\lim_{x \rightarrow 1} f(x) = \lim_{x \rightarrow 1} [3x^2 + 2x \lim_{x \rightarrow 1} f(x)] = 3 + 2 \lim_{x \rightarrow 1} f(x)$

可以解得  $\lim_{x \rightarrow 1} f(x) = -3$

代入原式得  $f(x) = 3x^2 - 6x$

## 10

(1) 解: 显然  $3x$  和  $\sqrt{ax^2 + bx + 1}$  是同阶无穷大量

$$\therefore a = 9$$

$$\lim_{x \rightarrow \infty} \frac{-b - \frac{1}{x}}{3 + \sqrt{a + \frac{b}{x} + \frac{1}{x^2}}} = 2$$

$$\therefore b = -12$$

(2) 解:  $x^2 + ax + b$  和  $x - 1$  是同阶无穷小

$$\text{当 } x \rightarrow 1, x^2 + ax + b \rightarrow 1 + a + b \rightarrow 0$$

$$\text{再洛必达 } 2x + a = 5, \therefore a = 3, b = -4$$

11. (1) 解: 因为  $\lim_{x \rightarrow 0} \frac{f(x)}{1 - \cos x} = \lim_{x \rightarrow 0} \frac{f(x)}{\frac{1}{2}x^2}$

$$\text{则 } \lim_{x \rightarrow 0} \frac{f(x)}{x^2} = 2$$

$$\text{所以 } \lim_{x \rightarrow 0} \left(1 + \frac{f(x)}{x}\right)^{\frac{1}{x}} = \lim_{x \rightarrow 0} \left(1 + \frac{f(x)}{x}\right)^{\frac{x}{f(x)} \times \frac{f(x)}{x^2}} = \lim_{x \rightarrow 0} e^{\frac{f(x)}{x^2}} = e^2$$

(2) 解: 因为  $\sqrt{1 + f(x) \sin x^2} - 1 \rightarrow 0 \quad (x \rightarrow 0)$

$$f(x) \sin x^2 \rightarrow 0 \quad (x \rightarrow 0)$$

$$\text{所以 } \lim_{x \rightarrow 0} \frac{\sqrt{1 + f(x) \sin x^2} - 1}{1 - \cos x} = \lim_{x \rightarrow 0} \frac{\frac{1}{2} f(x) \sin x^2}{\frac{1}{2} x^2} = \lim_{x \rightarrow 0} \frac{f(x) x^2}{x^2} = 3$$

$$\text{即 } \lim_{x \rightarrow 0} f(x) = 3$$

12. (1) 解:  $\lim_{x \rightarrow 0^+} f(x) = \frac{1}{x+1} = 1$

$$\lim_{x \rightarrow 0^-} f(x) = -\frac{1}{x+1} = -1$$

$x=0$  为第一类间断点中的跳跃间断点

$$\lim_{x \rightarrow 1^+} f(x) = \frac{1}{x+1} = \frac{1}{2}$$

$$\lim_{x \rightarrow 1^-} f(x) = \frac{1}{x+1} = \frac{1}{2}$$

而  $f(1) = 0$ ,  $x=1$  为第一类间断点中的可去间断点

同理,  $x = -1$  为第一类间断点中的可去间断点

(2)解: 当  $x=k\pi$  时,  $\sin x = 0$

$$\lim_{x \rightarrow 2k\pi^+} f(x) = 1$$

$$\lim_{x \rightarrow 2k\pi^-} f(x) = -1$$

$x=2k\pi$  为第一类间断点中的跳跃间断点

同理,  $x = (2k+1)\pi$  为第一类间断点中的跳跃间断点

**13.**证: 反证法: 假设  $f(x)$  在  $R$  上无界

①  $f(x)$  在  $x=x_0$  时, 有  $\lim_{x \rightarrow x_0} f(x) = \infty$

则  $x=x_0$  是  $f(x)$  的无穷间断点

不满足连续条件

②  $f(x)$  在  $x \rightarrow \infty$  时, 有  $\lim_{x \rightarrow \infty} f(x) = \infty$

则  $f(x)$  不满足周期条件

故  $f(x)$  有界

**14.**把分段点找到, 令其左右相等即可

① 当  $|x| < 1$  时,  $\lim_{x \rightarrow \infty} |x^n| = 0$

$$f(x) = ax^2 + bx$$

② 当  $|x|=1$  时,  $f(x) = a+b|x|$

③ 当  $|x| > 1$  时,  $f(x) = \frac{1}{x}$

$$\text{即 } f(x) = \begin{cases} \frac{1}{x}, & x < -1 \\ a-b, & x = -1 \\ ax^2 + bx, & -1 < x < 1 \\ a+b, & x = 1 \\ \frac{1}{x}, & x > 1 \end{cases}$$

任意取分段点左右极限相等, 联立方程

这里取  $-1$  和  $1$

$$\begin{cases} \lim_{x \rightarrow -1^-} f(x) = f(-1) \\ \lim_{x \rightarrow 1^+} f(x) = f(1) \end{cases}$$

$$\text{则} \begin{cases} a - b = -1 \\ a + b = 1 \end{cases} \text{解得: } \begin{cases} a = 0 \\ b = 1 \end{cases}$$

## 15.

证：因为  $f(x+y) = f(x) + f(y)$ ，令  $y = \Delta x$  且  $\Delta x \rightarrow 0$

$$\text{原式} = f(x + \Delta x) = f(x) + f(\Delta x)$$

$$\text{两边同时取极限: } \lim_{\Delta x \rightarrow 0} f(x + \Delta x) = \lim_{\Delta x \rightarrow 0} f(x) + \lim_{\Delta x \rightarrow 0} f(\Delta x)$$

$$\text{又因为 } \lim_{\Delta x \rightarrow 0} f(\Delta x) = 0, \lim_{\Delta x \rightarrow 0} f(x) = f(x)$$

$$\text{所以 } \lim_{\Delta x \rightarrow 0} f(x + \Delta x) = f(x)$$

则  $f(x)$  连续

## 16.

设  $f(x)$  在  $(0, +\infty)$  上连续，且满足  $f(x^2) = f(x), x \in (0, +\infty)$ . 证明  $f(x)$  在  $(0, +\infty)$  上为常值函数.

$$\text{证明: } \because f(x^2) = f(x)$$

$$\therefore f(x) = f(x^{\frac{1}{2}}) = f(x^{\frac{1}{4}}) = f(x^{\frac{1}{8}}) = \dots = f(x^{\frac{1}{2^n}})$$

$$\because n \rightarrow \infty \text{ 时, } \lim_{n \rightarrow \infty} x^{\frac{1}{2^n}} = 1$$

$$\therefore \text{当 } n \rightarrow \infty \text{ 时, } f(x) = f(x^{\frac{1}{2}}) = f(x^{\frac{1}{4}}) = f(x^{\frac{1}{8}}) = \dots = f(x^{\frac{1}{2^n}}) = f(\lim_{n \rightarrow \infty} x^{\frac{1}{2^n}}) = f(1)$$

$$\therefore f(x) = f(1) \text{ 为常值函数}$$

## 17.

设  $f(x)$  在  $[a, b]$  上有定义，满足  $a \leq f(x) \leq b, x \in [a, b]$ ，假设存在常数  $L \in [0, 1)$ ，使得任意  $x', x'' \in [a, b], |f(x') - f(x'')| \leq L|x' - x''|$ .

试证明：(1)  $f(x)$  在  $[a, b]$  上连续。

(2) 存在唯一  $\xi \in [a, b]$ ，使得  $f(\xi) = \xi$

(3) 对于任意的  $x_1 \in [a, b]$ ，定义迭代序列  $x_{n+1} = f(x_n), n = 1, 2, 3, \dots, \lim_{n \rightarrow \infty} x_n = \xi$

证明:

$$(1) \because |f(x') - f(x'')| \leq L|x' - x''|, \text{不妨令 } x'' = x_0, x' \rightarrow x_0 \in [a, b]$$

$$\therefore L|x' - x_0| \rightarrow 0$$

$$\text{又} \because |f(x') - f(x_0)| \geq 0, \lim_{x' \rightarrow x_0} |f(x') - f(x_0)| \leq \lim_{x' \rightarrow x_0} L|x' - x_0| = 0$$

$$\therefore \lim_{x' \rightarrow x_0} |f(x') - f(x_0)| = 0, \therefore \lim_{x' \rightarrow x_0} f(x') = f(x_0)$$

故连续.

$$(2) \text{根据题意设 } F(x) = f(x) - x,$$

$$\because a \leq f(x) \leq b, \text{所以 } F(a) \geq 0, F(b) \leq 0$$

并且  $F(a) \cdot F(b) \leq 0$ , 由零点存在定理: 必存在唯一  $\xi \in [a, b]$ , 使得  $F(\xi) = 0$ ,

$$\therefore f(\xi) = \xi$$

$$(3) \text{由题意: } |f(x_n) - f(\xi)| \leq L|x_n - \xi|$$

$$\text{由 (2): } f(\xi) = \xi, \therefore |f(x_n) - \xi| \leq L|x_n - \xi|$$

$$\therefore |x_{n+1} - \xi| \leq L|x_n - \xi|, \text{设数列 } x_n \text{ 存在且为 } A, \text{则有 } \lim_{n \rightarrow \infty} x_n = A$$

$$\therefore \text{当 } n \rightarrow \infty \text{ 时, } |A - \xi| \leq L|A - \xi|, \text{又因为 } L \neq 1, \text{则 } A = \xi$$

$$\text{故假设成立, } \lim_{n \rightarrow \infty} x_n = A = \xi$$

证毕.

## 18.

函数  $f(x)$  在  $[0, 1]$  上连续,  $f(0) = f(1)$ , 证明: 对于任意的自然数  $n \geq 2$ , 存在  $\xi_n$ , 使得  $f(\xi_n) = f(\frac{1}{n} + \xi_n)$ .

$$\text{证: 令 } F(x) = f(x) - f\left(\frac{1}{n} + x\right) \because f(x) \text{ 在 } x \in [0, 1] \text{ 上连续, } \therefore \frac{1}{n} + x \in [0, 1]$$

$$\therefore F(x) = f(x) - f\left(\frac{1}{n} + x\right) \text{ 在 } [0, 1 - \frac{1}{n}] \text{ 上连续, 即 } F(x) \text{ 在 } [0, \frac{n-1}{n}] \text{ 上连续}$$

$$\because \text{函数 } F(x) \text{ 连续, } \therefore \text{由连续函数性质: 存在 } m, M \text{ 满足 } m \leq F(x) \leq M,$$

$$\text{令 } x = \frac{k}{n}, k = 0, 1, 2, 3, \dots, n-1 \text{ (注意自变量范围),}$$

$$\therefore m \leq F\left(\frac{0}{n}\right) \leq M$$

$$m \leq F\left(\frac{1}{n}\right) \leq M$$

.....

.....

.....

$$m \leq F\left(\frac{n-1}{n}\right) \leq M$$

将n个式子相加, 得  $nm \leq F\left(\frac{0}{n}\right) + F\left(\frac{1}{n}\right) + \cdots + F\left(\frac{n-1}{n}\right) \leq nM$

$$\therefore m \leq \frac{\sum_{k=0}^{n-1} F\left(\frac{k}{n}\right)}{n} \leq M, \text{ 由介值定理, 存在 } \xi_n \in [0, 1], \text{ 使得 } \frac{\sum_{k=0}^{n-1} F\left(\frac{k}{n}\right)}{n} = F(\xi_n)$$

展开:

$$nF(\xi_n) = F\left(\frac{0}{n}\right) + F\left(\frac{1}{n}\right) + \cdots + F\left(\frac{n-1}{n}\right)$$

$$= f(0) - f\left(\frac{1}{n} + 0\right) + f\left(\frac{1}{n}\right) - f\left(\frac{1}{n} + \frac{1}{n}\right) + f\left(\frac{2}{n}\right) - f\left(\frac{1}{n} + \frac{2}{n}\right) \cdots + f\left(\frac{n-1}{n}\right) - f\left(\frac{1}{n} + \frac{n-1}{n}\right)$$

$$= f(0) - f\left(\frac{1}{n}\right) + f\left(\frac{1}{n}\right) - f\left(\frac{2}{n}\right) + f\left(\frac{2}{n}\right) - f\left(\frac{3}{n}\right) \cdots + f\left(\frac{n-1}{n}\right) - f\left(\frac{n}{n}\right)$$

$$= f(0) - f(1) = 0$$

$$\therefore nF(\xi_n) = 0, \quad F(\xi_n) = 0, \quad \text{即 } F(\xi_n) = f(\xi_n) - f\left(\frac{1}{n} + \xi_n\right) = 0,$$

$$\therefore f(\xi_n) = f\left(\frac{1}{n} + \xi_n\right).$$

证毕.

## 19.

对于任意的x, 函数满足  $f(x) = f(2x)$ , 且  $f(x)$  在  $x = 0$  处连续, 证明:  $f(x)$  为常值函数

证明:  $\because f(x) = f(2x)$

$$\therefore f(x) = f\left(\frac{x}{2}\right) = f\left(\frac{x}{4}\right) = f\left(\frac{x}{8}\right) = \cdots = f\left(\frac{x}{2^n}\right)$$

$$\text{当 } n \rightarrow \infty \text{ 时, } \lim_{n \rightarrow \infty} \frac{x}{2^n} = 0$$

$$\therefore \text{当 } n \rightarrow \infty \text{ 时, } f(x) = f\left(\frac{x}{2}\right) = f\left(\frac{x}{4}\right) = f\left(\frac{x}{8}\right) = \cdots = f\left(\frac{x}{2^n}\right) = f\left(\lim_{n \rightarrow \infty} \frac{x}{2^n}\right) = f(0)$$

$\because f(x)$  在  $x = 0$  处连续,  $\therefore f(0)$  存在

故  $f(x) = f(0)$  恒为常数.

## 20.

对于任意的 $x$ , 总有 $\varphi(x) \leq f(x) \leq \psi(x)$ , 且 $\lim_{x \rightarrow \infty} (\varphi(x) - \psi(x)) = 0$ . 问: 极限 $\lim_{x \rightarrow \infty} f(x)$ 是否存在, 给出理由.

解:  $\lim_{x \rightarrow \infty} f(x)$ 不一定存在, 理由如下

$\because \lim_{x \rightarrow \infty} (\varphi(x) - \psi(x)) = 0$  不等价于 $\varphi(x)$ ,  $\psi(x)$ 极限存在,

$\therefore$  不满足夹逼准则,  $\lim_{x \rightarrow \infty} f(x)$ 不一定存在

反例如:  $\varphi(x) = \sin x - \frac{1}{x}$ ,  $f(x) = \sin x$ ,  $\psi(x) = \sin x + \frac{1}{x}$