第 3 章复习题

1. $\lim\limits_{h\to 0}\frac{f(1-h)-f(1)}{2h}=\lim\limits_{h\to 0}\frac{f(1+h)-f(1)}{-2h}=-\frac{1}{2}\lim\limits_{h\to 0}\frac{f(1+h)-f(1)}{h}=-\frac{1}{2}f'(1)=-1$

2. f(x)=x-[x], f(0)=0.($\lim\limits_{x\to 0+}[x]=0\ \ \therefore\ \lim\limits_{x\to 0+}(x-[x])=\lim\limits_{x\to 0+}x=0$)

   $f'_+(0)=\lim\limits_{x\to 0^+}\frac{f(x)-f(0)}{x}=1$

   $f'_-(0)=\lim\limits_{x\to 0^-}\frac{f(x)-f(0)}{x}=\lim\limits_{x\to 0^-}\frac{x-[x]}{x}=\infty$

   ($\lim\limits_{x\to 0^-}[x]=1,\quad \therefore \lim\limits_{x\to 0^-}(x-[x])=\lim\limits_{x\to 0^-}(x+1)=1$)

   $f'_-(0)\neq f'_+(0)\qquad \therefore f'(0)$不存在.

   x∈ $(0,1)$时，$f'(x)=(x-[x])'=1$　x∈ $(-1,0)$时，$f'(x)=(x-[x])'=1$

   $\therefore \lim\limits_{x\to 0}f'(x)=1$.

3. $e^y+6xy+x^2-1=0$,x=0 时，y=0.

   $y'e^y+6y+6xy'+2x=0$，y'(0)=0

   $y''e^y+(y')^2e^y+6y'+6y'+6xy''y'+2=0$
   $\therefore y''+2=0$

   $\therefore y''(0)=-2$.

4. 2ysin $x+xlny=0$.

   两边对 x 求导：$2y'sinx+2ycosx+lny+x\frac{y'}{y}=0$.

   $y'=-\frac{2y^2cosx+ylny}{x+2ysinx}$.

   再对 x 求导：

   $2y''sinx+2y'cosx-2ysinx+2y'cosx+2\frac{y'}{y}+\frac{xyy''-x(y')^2}{y^2}=0$.

   $y''=\frac{2y^3sinx-4y'y^2cosx-2yy'+x(y')^2}{xy+2y^2sinx}$

5. (1)　$y'=[(1+x^2+x^4)^{\frac{1}{2}}]'=\frac{1}{2}(2x+4x^3)(1+x^2+x^4)^{-\frac{1}{2}}=x(1+2x^2)(1+x^2+x^4)^{-\frac{1}{2}}$

(2) $y = x^{sinx+2cosx}$

两边取对数: $ln|x|(sinx + 2cosx) = ln|y|$

两边对 X 求导: $\frac{y'}{y} = (cosx - 2sinx)ln|x| + \frac{1}{x}(sinx + 2cosx)$

$$y' = x^{sinx+2cosx}[(cosx - 2sinx)ln|x| + \frac{1}{x}(sinx + 2cosx)]$$

(3) $y = (1 + \frac{1}{x})^x$

两边取对数: $ln|y| = xln\left|1 + \frac{1}{x}\right|$

两边对 X 求导: $\frac{y'}{y} = ln\left|1 + \frac{1}{x}\right| + \frac{x^2}{1+x}(-\frac{1}{x^2})$

$$y' = (1 + \frac{1}{x})^x[ln\left|1 + \frac{1}{x}\right| - \frac{1}{1+x}]$$

(4) $y = \sqrt[2]{\frac{sin^2x(1+cos^2x)}{1+sin^2x}}$

两边取对数: $lny = \frac{1}{2}ln\frac{sin^2x(1+cos^2x)}{1+sin^2x}$

两边对 X 求导: $\frac{y'}{y} = \frac{1}{2}[\frac{2sinxcosx+2sinxcos^3x-2sin^3xcosx}{sin^2x(1+cos^2x)} - \frac{2sinxcosx}{1+sin^2x}]$

将$y = \sqrt[2]{\frac{sin^2x(1+cos^2x)}{1+sin^2x}}$代入上式:

$$y' = \sqrt[2]{\frac{sin^2x(1+cos^2x)}{1+sin^2x}}\frac{cosx(2+cos^2xsin^2x+sin^4x-cos^4x)}{sinx(1+cos^2x)(1+sin^2x)}$$

$$= \frac{cosx(2+cos^2xsin^2x+sin^4x-cos^4x)}{(1+cos^2x)^{\frac{1}{2}}(1+sin^2x)^{\frac{1}{2}}}$$

6.证明: $\lim_{x\to 0}\frac{f(x)}{x} = A$

$\lim_{X\to 0} x = 0$　　$\therefore \lim_{x\to 0} f(x) = 0$, 　且$f(x)$在 x=0 处连续。

$\therefore f(0) = \lim_{x\to 0} f(x) = 0$

$f'(0) = \lim_{x\to 0}\frac{f(x)-f(0)}{x} = A$

7.证明:

f(x)=x(x+1)(x+2)…(x+n+1)

f '(x)=(x+1)(x+2)…(x+n+1)+x(x+2)…(x+n+1)+x(x+1)(x+3)…(x+n+1)+x(x+1)(x+2)(x+4)…(x+n+1)+…+x(x+1)…(x+n)

$f'(-1)$=x(x+2)(x+3)...(x+n+1)

=(-1)x1x2x3...xn

=-n!

8.

y=$\sin^4 x+\cos^4 x$

=$(\sin^2 x+\cos^2 x)^2$-2$\sin^2 x\cos^2 x$

=1$-\frac{1}{2}\sin^2 2x$=$\frac{3}{4}+\frac{1}{4}\cos 4x$

y' =$-\sin 4x$

=$\cos(4x+\frac{\pi}{2})$

$\because(\cos\omega x)^{(n)}=\omega^n\cos(\omega x+\frac{n\pi}{2})$

$\therefore y^{(n)}=4^{n-1}\cos(4x+\frac{n\pi}{2})$

9.证明

f(x)=$(x-a)^n\varphi$(x)

$\because\varphi$(x)在点 a 的某领域内有（n-1）阶连续导函数

$\therefore$

$\therefore$ $f^{(n-1)}$(x) = $C_{n-1}^0[(x-a)^n]^{(0)}\varphi^{(n-1)}(x)$ + $C_{n-1}^1[(x-a)^n]^{(1)}\varphi^{(n-2)}(x)$ +...+ $C_{n-1}^{n-1}[(x-a)^n]^{(n-1)}\varphi^{(0)}(x)$

$\therefore f^{(n-1)}$(a)=0

$f^{(n)}$(a)$\lim\limits_{x\to a}\frac{f^{(n-1)}(x)-f^{(n-1)}(a)}{x-a}$,　将上式带入

$\therefore$ $f^{(n)}$(a)=φ(a)n!

10.

(1) f(x)在 x=0 连续：

$f(0)=\lim\limits_{x\to 0} f(x)=0$

$\therefore \lim\limits_{x\to 0} x^m \sin\frac{1}{x}=0$

$\therefore \lim\limits_{x\to 0} x^m=0 \to m>0$

(2) f(x)在 x=0 可导:

在 m>0 前提下，有$f'(0)$存在

$f'(0)=\lim\limits_{x\to 0}\frac{x^m \sin\frac{1}{x}}{x}=\lim\limits_{x\to 0} x^{m-1}\sin\frac{1}{x}$

$\therefore m>1$

(3) f'(x)在 x=0 连续:

$f'(0)=\lim\limits_{x\to 0} f'(x)$

由(2)知$f'(0)$若存在则为 0

$\therefore \lim\limits_{x\to 0} f'(x)=0=\lim\limits_{x\to 0}(mx^{m-1}\sin\frac{1}{x}-x^{m-2}\cos\frac{1}{x})$

$$\begin{cases} m-2>0 \\ m-1>0 \end{cases} \Longrightarrow m>2$$

11.证明

当 x≠0 时，$f'(x)=e^{\frac{-1}{x^2}}(\frac{2}{x^3})$

又$\because f'(0)=\lim\limits_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim\limits_{x\to 0}\frac{1}{xe^{\frac{1}{x^2}}}=0$，$\lim\limits_{x\to 0} f'(x)=\lim\limits_{x\to 0}\frac{2}{x^3 e^{\frac{1}{x^2}}}=0$

$\therefore f'(0)=0$且f'(x)在 x=0 连续

$\therefore f'(x)=\begin{cases} \frac{2}{x^3}e^{\frac{-1}{x^2}}, x\neq 0 \\ 0, x=0 \end{cases}$，$f''(0)=\lim\limits_{x\to 0}\frac{f'(x)-f'(0)}{x-0}=0$

$f''(x)=\begin{cases} (\frac{-6}{x^4}+\frac{4}{x^6})e^{\frac{-1}{x^2}}, x\neq 0 \\ 0, x=0 \end{cases}$ (关于x^{-1}的六次多项式)

设$f^{(n)}$(x)=$\begin{cases} P_n(\frac{1}{x})e^{-\frac{1}{x^2}}, x\neq 0 \\ 0, x=0 \end{cases}$ （P_n（x^{-1}）是关于x^{-1}的3n 次多项式)

则$f^{(n+1)}$(0)=$\lim\limits_{x\to 0}\frac{f^{(n)}(x)-f^{(n)}(0)}{x-0}$=$\frac{x^{-1}P_n(x^{-1})}{e^{\frac{1}{x^2}}}$=0

$f^{(n+1)}$(x)=($\frac{2}{x^3}P_n$（x^{-1}）－$\frac{1}{x^2}P'_n$（x^{-1}））$e^{\frac{-1}{x^2}}$

=$P_{n+1}(x^{-1})e^{\frac{-1}{x^2}}$(x≠0)

显然$P_{n+1}(x^{-1})$是关于 x 的 3(n+1)次多项式

∴$f^{(n+1)}$(x)=$\begin{cases} P_{n+1}(\frac{1}{x})e^{\frac{-1}{x^2}}, x\neq 0 \\ 0, x=0 \end{cases}$

由数学归纳法可知f(x)在 x=0 处 n 阶可导且$f^{(n)}$(0)=0

12.

(1) 证明

∵φ(x)在 x=a 连续

∴f(x)在 x=a 连续

$\lim\limits_{x\to a+}\frac{f(x)-f(a)}{x-a}=\lim\limits_{x\to a+}\varphi(a)=\varphi(a)$

$\lim\limits_{x\to a-}\frac{f(x)-f(a)}{x-a}=\lim\limits_{x\to a-}\varphi(a)=\varphi(a)$

∴f(x)在 x=a 可导，且$f'(a)=\varphi(a)$

(2)

g'_+(a)=$\lim\limits_{x\to a+}\frac{|x-a|\varphi(x)}{x-a}$=φ(a)

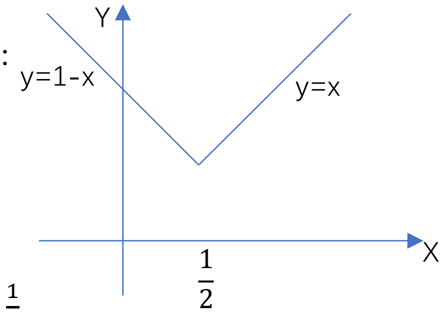
g'_-(a)=$\lim\limits_{x\to a-}\frac{|x-a|\varphi(x)}{x-a}$=−φ(a)

要使 g(x)在 x=a 可导

则g'_+(a)=g'_-(a)

即φ(a)=0

13. 解：



因为$f(x)$为多项式函数

所以$f(x)$可导

由于$f(x) \geq x$ 所以 $f\left(\frac{1}{2}\right) \geq \frac{1}{2}$

假设$f\left(\frac{1}{2}\right) = \frac{1}{2}$ 所以 $f\left(\frac{1}{2}\right)$为较小值

由费马定理，$f'\left(\frac{1}{2}\right) = 0$

当$x > \frac{1}{2}$时，$f(x) \geq x$ 则$f'_+\left(\frac{1}{2}\right) \geq 1$，与$f'\left(\frac{1}{2}\right) = 0$相矛盾

所以$f\left(\frac{1}{2}\right) \neq \frac{1}{2}$ 故$f\left(\frac{1}{2}\right) > \frac{1}{2}$

14. 解 ： $\lim\limits_{x\to\infty}\left(\frac{f\left(\frac{1}{x}\right)}{f(0)}\right)^x = \lim\limits_{x\to\infty}\left(1+\frac{f\left(\frac{1}{x}\right)-f(0)}{f(0)}\right)^x = \lim\limits_{x\to\infty}\left(1+\frac{f\left(\frac{1}{x}\right)-f(0)}{f(0)}\right)^{\frac{f\left(\frac{1}{x}\right)-f(0)}{f(0)}\cdot x\cdot\frac{f(0)}{f\left(\frac{1}{x}\right)-f(0)}} =$

$e^{\lim\limits_{x\to\infty} x\cdot\frac{f\left(\frac{1}{x}\right)-f(0)}{f(0)}}$ 令$x = \frac{1}{t}$ $x \to \infty$，$t \to 0$

$$\therefore \lim_{x\to\infty} x\cdot\frac{f\left(\frac{1}{x}\right)-f(0)}{f(0)} = \lim_{t\to 0}\frac{f(t)-f(0)}{tf(0)} = \frac{f'(0)}{f(0)}$$

$\therefore$ 原式 $= e^{\frac{f'(0)}{f(0)}}$

15. 解：$f(x) = -x^3 + x$ $f(x+1) = -x^3 - 3x^2 - 2x = af(x)$ $x \in [-1,0)$

$f(x)$中$x \in [0,1)$ $f(x+1)$中$x \in [-1,0)$

$$\therefore f(x) = \begin{cases} -x^3 + x, & x \in [0,1) \\ \frac{1}{a}(-x^2 - 3x - 2), & x \in [-1,0) \end{cases}$$

因为$f(0) = 0$ $\lim\limits_{x\to 0} f(x) = f(0)$ $\therefore f(x)$在$x = 0$处连续

$$f'_-(0) = \lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-}\frac{1}{a}(-x^2 - 3x - 2) = \frac{-2}{a}$$

$$f_+'(0) = \lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+}(-x^2+1) = 1$$

又因为$f(x)$在$x=0$处可导　$\therefore f_-'(0) = f_+'(0)$　$\therefore a=-2, f'(0)=1$

16. 解：因为$f'(x)=f^2(x), f(0)=2$　$\therefore f'(0)=f^2(0)=4$

$$f''(0) = \left(f^2(0)\right)' = 2f(0)f'(0) = 2f^3(0) = 2\times 2^3$$

$$f'''(0) = 6f^4(0) = 6\times 2^4$$

设$f^{(n)}(0) = n!\ 2^{n+1} = n!\, f^{n+1}(0)$

则　$f^{(n+1)}(0) = [n!\, f^{n+1}(0)]' = (n+1)!\, f^n(0)\cdot f^2(0) = (n+1)!\, f^{n+2}(0)$

$\therefore$ 由数学归纳法可知$f^{(n)}(0) = n!\ 2^{n+1}$

17. 解：因为$f(xy)=f(x)+f(y)$　$\therefore f(x)=f(x)+f(1)$　$\therefore f(1)=0$

又因为$f(1) = f(x) + f\left(\frac{1}{x}\right) = 0$　$\therefore f(x) = -f\left(\frac{1}{x}\right)$

$$f'(x_0) = \lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h} = \lim_{h\to 0}\frac{f(x_0+h)+f\left(\frac{1}{x_0}\right)}{h}$$

$$= \lim_{h\to 0}\frac{f\left(1+\frac{h}{x_0}\right)}{h}$$

由洛必达法则可知　$f'(x_0) = \lim_{h\to 0} f'\left(1+\frac{h}{x_0}\right)\cdot\frac{1}{x_0} = \frac{f'(1)}{x_0} = \frac{a}{x_0}$

$\therefore f'(x) = \frac{a}{x}$，$x\in\left(0,\ +\infty\right)$

18. 解：充分性：若$f(x)$在$x=a$处可导且$f'(a)=0$，$f(a)=0$，则$|f(x)|$在$x=a$处可导

因为 $f(a)=0$　$f'(a)=0$　　$\therefore \lim_{x\to a}\frac{f(x)}{x-a}=0$，$\therefore |f(a)|=0$

$$|f_+'(a)| = \lim_{x\to a^+}\frac{|f(x)|-|f(a)|}{x-a} = \lim_{x\to a^+}\frac{|f(x)|}{x-a} = 0$$

$$|f_-'(a)| = \lim_{x\to a^-}\frac{|f(x)|-|f(a)|}{x-a} = \lim_{x\to a^-}\frac{|f(x)|}{x-a} = 0$$

$\therefore |f(x)|$在$x=a$处可导且$|f(a)|'=0$

必要性：若$|f(x)|$在$x=a$处可导且$f(a)=0$，则$f'(a)=0$

因为$|f(x)|$在$x=a$处可导　　$\lim\limits_{x\to a^+}\frac{|f(x)|}{x-a}=-\lim\limits_{x\to a^-}\frac{|f(x)|}{x-a}$

$\therefore \lim\limits_{x\to a}\frac{|f(x)|}{x-a}=0$，$\therefore f'(a)=0$