习题<3.2>答案

1. (1) $(x^n)' = nx^{n-1}$

$\Rightarrow y = x^3 - 2x^2 + 3x - 4$

$y' = 3x^2 - 4x + 3$

(2) $(uv)' = u'v + uv'$

$y = (x^2+3x+2)(x^2-3x+2)$

方a. $y' = (2x+3)(x^2-3x+2) + (x^2+3x-2)(2x-3)$

$\Rightarrow y' = 4x^3 - 10x$

方b. $y = (x^2+2)^2 - (3x)^2 = x^4 - 5x^2 + 4$

$\Rightarrow y' = 4x^3 - 10x$

(3) $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$

$y = \frac{\cos x}{x} + \frac{x}{\cos x}$

$\Rightarrow y' = \frac{-\sin x \cdot x - \cos x}{x^2} + \frac{\cos x + x\sin x}{\cos^2 x}$

$\Rightarrow y' = -x^{-1}\sin x - x^{-2}\cos x + \sec x + x\tan x \cdot \sec x$

Tip: $\frac{1}{\sin x} = \csc x$, $\frac{1}{\cos x} = \sec x$

(4) $(\ln x)' = \frac{1}{x}$ / $(\ln|x|)' = \frac{1}{x}$

$y = x\ln x$

$\Rightarrow y' = \ln x + x \cdot \frac{1}{x}$

$\Rightarrow y' = \ln x + 1$

(5) $y = x^2 + x^{-2}$

$\Rightarrow y' = 2x - 2x^{-3}$

(6) $(e^x)' = e^x$

$y = e^x \cos x$

$\Rightarrow y' = e^x \cos x + e^x(-\sin x)$

$\Rightarrow y' = e^x(\cos x - \sin x)$

(7) $y = e^x \sin x$

$\Rightarrow y' = e^x \sin x + e^x \cos x$

$\Rightarrow y' = e^x(\sin x + \cos x)$

(8) $y = e^x \ln x$

$\Rightarrow y' = e^x \ln x + e^x \cdot \frac{1}{x}$

$\Rightarrow y' = e^x(\ln x + \frac{1}{x})$

2. (1) △ $(f(g(x)))' = f'(g(x)) \cdot g'(x)$

$y = e^{x^2 + \sin x}$

$\Rightarrow y' = e^{x^2 + \sin x} \cdot (2x + \cos x)$

(2) $y = x\ln(x^2 + e^x)$

$\Rightarrow y' = \ln(x^2+e^x) + x \cdot \frac{1}{x^2+e^x} \cdot (2x + e^x)$

$\Rightarrow y' = \ln(x^2+e^x) + \frac{2x^2 + xe^x}{x^2+e^x}$

(3) $y = \sin 2x$

$\Rightarrow y' = \cos 2x \times 2$

$\Rightarrow y' = 2\cos 2x$

(4) $y = \cos 2x$

$\Rightarrow y' = -\sin 2x \times 2$

$\Rightarrow y' = -2\sin 2x$

(5) $y=\sqrt{x}\arcsin\sqrt{x}$

Tip: $(\arcsin x)'=\frac{1}{\sqrt{1-x^2}}$;

$(\arccos x)'=-\frac{1}{\sqrt{1-x^2}}$;

$(\arctan x)'=\frac{1}{1+x^2}$;

$(\text{arccot}\, x)'=-\frac{1}{1+x^2}$

$\Rightarrow y'=\frac{1}{2\sqrt{x}}\cdot\arcsin\sqrt{x}+\sqrt{x}\cdot\frac{1}{\sqrt{1-x}}\cdot\frac{1}{2\sqrt{x}}$

$\Rightarrow y'=\frac{\arcsin\sqrt{x}}{2\sqrt{x}}+\frac{1}{2\sqrt{1-x}}$ $(x\neq 0)$

(6) $y=\sqrt{x}\arccos\sqrt{x}$

$\Rightarrow y'=\frac{1}{2\sqrt{x}}\cdot\arccos\sqrt{x}+\sqrt{x}\cdot\left(-\frac{1}{\sqrt{1-x}}\right)\cdot\frac{1}{2\sqrt{x}}$

$\Rightarrow y'=\frac{\arccos\sqrt{x}}{2\sqrt{x}}-\frac{1}{2\sqrt{1-x}}$ $(x\neq 0)$

(7) $y=x^2\arctan\frac{1}{x}$

$\Rightarrow y'=2x\cdot\arctan\frac{1}{x}+x^2\frac{1}{1+(\frac{1}{x})^2}\cdot\left(-\frac{1}{x^2}\right)$

$\Rightarrow y'=2x\arctan\frac{1}{x}-\frac{x^2}{x^2+1}$

(8) $y=x^2\,\text{arccot}\frac{1}{x}$

$\Rightarrow y'=2x\cdot\text{arccot}\frac{1}{x}+x^2\left(-\frac{1}{1+(\frac{1}{x})^2}\right)\cdot\left(\frac{1}{x^2}\right)$

$\Rightarrow y'=2x\,\text{arccot}\frac{1}{x}-\frac{x^2}{x^2+1}$

(9) $(\sec x)'=\tan x\cdot\sec x$

$(\csc x)'=-\cot x\csc x$

$(\tan x)'=\sec^2 x$

$(\cot x)'=-\csc^2 x$

$y=\sec x^2$

$\Rightarrow y'=\tan x^2\cdot\sec x^2\cdot(2x)$

$\Rightarrow y'=2x\tan x^2\sec x^2$

(10) $y=\csc\sqrt{x}$

$\Rightarrow y'=-\cot\sqrt{x}\csc\sqrt{x}\cdot\frac{1}{2\sqrt{x}}$

$\Rightarrow y'=-\frac{\cot\sqrt{x}\csc\sqrt{x}}{2\sqrt{x}}$

(11) $y=x^{\frac{1}{2}}+x^{-\frac{1}{2}}$

$\Rightarrow y'=\frac{1}{2}x^{-\frac{1}{2}}-\frac{1}{2}x^{-\frac{3}{2}}$

$\Rightarrow y'=\frac{1}{2}(x^{-\frac{1}{2}}-x^{-\frac{3}{2}})$

(12) $y=\dfrac{e^{\sqrt{x}}-e^{-\sqrt{x}}}{e^{\sqrt{x}}+e^{-\sqrt{x}}}$

$\Rightarrow y=\dfrac{e^{\sqrt{x}}+e^{-\sqrt{x}}-2e^{-\sqrt{x}}}{e^{\sqrt{x}}+e^{-\sqrt{x}}}$

$y=1-2\dfrac{1}{e^{2\sqrt{x}}+1}$

$\Rightarrow y'=-2\dfrac{-e^{2\sqrt{x}}\cdot\frac{1}{\sqrt{x}}}{(e^{2\sqrt{x}}+1)^2}$

$\Rightarrow y'=\dfrac{2e^{2\sqrt{x}}}{\sqrt{x}(e^{2\sqrt{x}}+1)^2}$ $(x\neq 0)$

3. 解: (1) $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$

对两边关于x求导

$\Rightarrow \frac{1}{a^2}\cdot 2x+\frac{1}{b^2}\cdot 2y\cdot y'=0$

$\Rightarrow y'=-\dfrac{b^2x}{a^2y}$ $(y\neq 0)$

(2) $x^2+2xy-y^2=2x$

对两边关于x求导

$\Rightarrow 2x+2y+2xy'-2y\cdot y'=2$

$\Rightarrow (x-y)y'=1-x-y$

$\Rightarrow y'=\dfrac{1-x-y}{x-y}$ $(x\neq y)$

(3) $\sqrt{x}+\sqrt{y}=\sqrt{a}$

对两边关于x求导

$\Rightarrow \dfrac{1}{2\sqrt{x}}+\dfrac{1}{2\sqrt{y}}\cdot y'=0$

$\Rightarrow y'=-\dfrac{\sqrt{xy}}{x}$ $(x>0, y>0)$

(4) $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$

对两边关于x求导

$\Rightarrow \frac{2}{3}x^{-\frac{1}{3}}+\frac{2}{3}y^{-\frac{1}{3}}\cdot y'=0$

$\Rightarrow y'=-(\frac{y}{x})^{\frac{1}{3}}$ $(x\neq 0)$

(5) $\arctan\frac{y}{x}=\ln\sqrt{x^2+y^2}$

对两边关于x求导

$\Rightarrow \dfrac{1}{1+(\frac{y}{x})^2}\cdot\dfrac{y'x-y}{x^2}=\dfrac{1}{\sqrt{x^2+y^2}}\cdot\dfrac{2x+2y\cdot y'}{2\sqrt{x^2+y^2}}$

$\Rightarrow \dfrac{y'x-y}{x^2+y^2}=\dfrac{x+y\cdot y'}{x^2+y^2}$

$\Rightarrow y'x-y=x+y\cdot y'$

$\Rightarrow y'=\dfrac{x+y}{x-y}$ $(x\neq y, x\neq 0)$

(6) $x^y=y^x$ $(x>0, y>0)$

$\Rightarrow y\ln x=x\ln y$ 对数求导法

对两边关于x求导

$\Rightarrow y'\ln x+y\frac{1}{x}=\ln y+x\frac{1}{y}\cdot y'$

1°. 由 $y\ln x=x\ln y$ 变形得

$\Rightarrow y'(\ln x-\frac{x}{y})=y'(\ln y-1)\cdot\frac{x}{y}$

$\ln y-\frac{y}{x}=\frac{y}{x}(\ln x-1)$

$\Rightarrow y'=\dfrac{y^2(\ln x-1)}{x^2(\ln y-1)}$ $(x>0, y>0)$

2°. $\Rightarrow y'=\dfrac{xy\ln y-y^2}{xy\ln x-x^2}$ $(x>0, y>0)$

(7) $x-y+\varepsilon\sin y=0$ (ε为参数)

对两边关于x求导

$\Rightarrow 1-y'+\varepsilon\cos y\cdot y'=0$

$\Rightarrow y'=\dfrac{1}{1-\varepsilon\cos y}$

4. 解：(1) $y = x^{\sin x}$

对数求导法

对两边取对数

$\Rightarrow \ln y = \sin x \ln x$

$\Rightarrow \frac{1}{y} \cdot y' = \cos x \ln x + \sin x \cdot \frac{1}{x}$

$\Rightarrow y' = y \cos x \ln x + \frac{y \sin x}{x}$

$\Rightarrow y' = x^{\sin x}(\cos x \ln x + \frac{\sin x}{x})$

(2) $y = x^{\ln x}$

对两边取对数

$\Rightarrow \ln y = \ln x \cdot \ln x = \ln^2 x$

$\Rightarrow \frac{1}{y} \cdot y' = 2\ln x \cdot \frac{1}{x}$

$\Rightarrow y' = \frac{2y\ln x}{x}$

$\Rightarrow y' = 2x^{\ln x - 1} \ln x$

(3) $y = \sqrt[3]{\frac{(x-1)(x-2)}{(x-3)(x-4)}}$

若题中y关于x的表达式已给出，则需将y=f(x)代入结果

对两边取对数

$\Rightarrow \ln y = \frac{1}{3} \ln \frac{(x-1)(x-2)}{(x-3)(x-4)} = \frac{1}{3}\ln(x-1) + \frac{1}{3}\ln(x-2) - \frac{1}{3}\ln(x-3) - \frac{1}{3}\ln(x-4)$

$\Rightarrow \frac{1}{y} \cdot y' = \frac{1}{3} \frac{1}{x-1} + \frac{1}{3} \cdot \frac{1}{x-2} - \frac{1}{3} \cdot \frac{1}{x-3} - \frac{1}{3} \cdot \frac{1}{x-4}$

$\Rightarrow y' = \frac{-4x^2+20x-22}{3(x-1)(x-2)(x-3)(x-4)} \cdot y$

$\Rightarrow y' = \frac{-4x^2+20x-22}{3(x-1)(x-2)(x-3)(x-4)} \cdot \sqrt[3]{\frac{(x-1)(x-2)}{(x-3)(x-4)}}$

5. 解：$y' = \frac{dy}{dx}$，$y = y(t)$，$x = x(t)$ $\Rightarrow y' = \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$ $\Rightarrow$ $x = x(t)$ 在 $t \in D$ 时单调.

(1) $\begin{cases} x = 1 - t^2 \\ y = 1 - t^3 \end{cases}$ $\Rightarrow \frac{dx}{dt} = -2t$，$\frac{dy}{dt} = -3t^2$ $\Rightarrow y' = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3}{2}t$

(2) $\begin{cases} x = \ln(1+t^2) \\ y = t - \arctan t \end{cases}$ $\Rightarrow \frac{dx}{dt} = \frac{2t}{1+t^2}$，$\frac{dy}{dt} = 1 - \frac{1}{1+t^2} = \frac{t^2}{1+t^2}$ $\Rightarrow y' = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{t}{2}$

(3) $\begin{cases} x = a\cos^3 t \\ y = a\sin^3 t \end{cases}$ $\Rightarrow \frac{dx}{dt} = 3a\cos^2 t(-\sin t)$，$\frac{dy}{dt} = 3a\sin^2 t \cos t$

$\Rightarrow y' = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = -\tan t$

6. 证：$\because \sqrt{x}+\sqrt{y}=\sqrt{a}\ (a>0)$ $(x\geqslant 0, y\geqslant 0)$

$\therefore$ 抛物线与x轴交点P_1为$(a,0)$

与y轴交点P_2为$(0,a)$

对$\sqrt{x}+\sqrt{y}=\sqrt{a}\ (a>0)$两边关于x求导

$\Rightarrow \frac{1}{2\sqrt{x}}+\frac{y'}{2\sqrt{y}}=0$

$\Rightarrow y'=-\frac{\sqrt{xy}}{x}=k_{切}$

$\Rightarrow l_{切}: y-y_0=-\frac{\sqrt{x_0y_0}}{x_0}(x-x_0)$ $\Rightarrow$ 抛物线在x_0点处的切线

在点x_0处，$\sqrt{x_0}+\sqrt{y_0}=\sqrt{a}$ $\Rightarrow x_0+2\sqrt{x_0y_0}+y_0=a$

$\Rightarrow l_{切}$与x轴交点P_1为$(x_0+\sqrt{x_0y_0}, 0)$ $x_0y_0=\sqrt{x_0y_0}(x-x_0)$

与y轴交点P_2为$(0, \sqrt{x_0y_0}+y_0)$ $y-y_0=\sqrt{x_0y_0}$

$\Rightarrow x_0+\sqrt{x_0y_0}+y_0+\sqrt{x_0y_0}=x_0+2\sqrt{x_0y_0}+y_0=a$

故抛物线$\sqrt{x}+\sqrt{y}=\sqrt{a}\ (a>0)$上任一点的切线截两个坐标轴的截距之和为a.

7. 证: $\begin{cases} x = a(\cos t + t\sin t) \\ y = a(\sin t - t\cos t) \end{cases} \Rightarrow \begin{array}{l} \dfrac{dx}{dt} = a(-\sin t + \sin t + t\cos t) = at\cos t \\ \dfrac{dy}{dt} = a(\cos t - \cos t + t\sin t) = at\sin t \end{array}$

$\Rightarrow y' = \dfrac{dy}{dx} = \dfrac{dy}{dt} \cdot \dfrac{dt}{dx} = \dfrac{\sin t}{\cos t} = \tan t$

$\Rightarrow k_{法} = -\dfrac{1}{y'} = -\cot t$

点(x_0, y_0)到 $l: Ax+By+C=0$ 的距离: $d = \dfrac{|Ax_0+By_0+C|}{\sqrt{A^2+B^2}}$

$\Rightarrow$ 曲线上点x_0 的法线为 : $y - y_0 = -\cot t_0 (x - x_0)$

切线

$l_{法}$: $\cos t_0\, x + \sin t_0\, y - x_0\cos t_0 - y_0 \sin t_0 = 0$

$\Rightarrow d = \dfrac{|\cos t_0 \cdot x_1 + \sin t_0 \cdot y_1 - x_0\cos t_0 - y_0 \sin t_0|}{\sqrt{\cos^2 t_0 + \sin^2 t_0}}$ $((x_1, y_1)$为原点$)$ $= a\cos^2 t_0 + at_0\sin t_0\cos t_0 + a\sin^2 t_0 - at_0\sin t_0\cos t_0$

$= a(\cos^2 t_0 + \sin^2 t_0)$

$= a$

故原点到曲线上任意一点的法线的距离等于a.

8. 证: (1) 由题得: $L: r = r(\theta)$. $L: \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$, $r = r(\theta)$ 关于θ可导

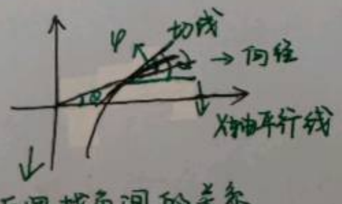
$\Rightarrow \dfrac{dx}{d\theta} = r'\cos\theta - r\sin\theta = r'(\theta)\cos\theta - r(\theta)\sin\theta$

$\dfrac{dy}{d\theta} = r'\sin\theta + r\cos\theta = r'(\theta)\sin\theta + r(\theta)\cos\theta$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \dfrac{r'(\theta)\sin\theta + r(\theta)\cos\theta}{r'(\theta)\cos\theta - r(\theta)\sin\theta} = \dfrac{r'(\theta)\tan\theta + r(\theta)}{r'(\theta) - r(\theta)\tan\theta}$ 上下同除以$\cos\theta$

(2) 设点$P(r, \theta)$处切线与x轴夹角为α.

则



$\Rightarrow$ 同位角: $\theta = \alpha - \varphi$

即 $\varphi = \alpha - \theta$

利用画图找角间的关系

$\Rightarrow \tan\varphi = \tan(\alpha - \theta) = \dfrac{\tan\alpha - \tan\theta}{1 + \tan\alpha\tan\theta}$

$\tan\alpha = k_{切} = \dfrac{dy}{dx} = \dfrac{r'(\theta)\tan\theta + r(\theta)}{r'(\theta) - r(\theta)\tan\theta}$

上下同时乘$[r'(\theta) - r(\theta)\tan\theta]$ $\Rightarrow \tan\varphi = \dfrac{\dfrac{r'(\theta)\tan\theta + r(\theta)}{r'(\theta) - r(\theta)\tan\theta} - \tan\theta}{1 + \dfrac{r'(\theta)\tan\theta + r(\theta)}{r'(\theta) - r(\theta)\tan\theta}\tan\theta}$

$\Rightarrow \tan\varphi = \dfrac{r'(\theta)\tan\theta + r(\theta) - r'(\theta)\tan\theta + r(\theta)\tan^2\theta}{r'(\theta) - r(\theta)\tan\theta + r'(\theta)\tan^2\theta + r(\theta)\tan\theta}$

$\Rightarrow \tan\varphi = \dfrac{r(\theta)(1 + \tan^2\theta)}{r'(\theta)(1 + \tan^2\theta)} = \dfrac{r(\theta)}{r'(\theta)}$

9.解：∵极坐标曲线 $r=e^{\theta}$

∴其可用极角θ作为参数

表示如下

$$\begin{cases} x = r\cos\theta = e^{\theta}\cos\theta \\ y = r\sin\theta = e^{\theta}\sin\theta \end{cases}$$

$$\Rightarrow y' = \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

$$\frac{dy}{d\theta} = e^{\theta}\sin\theta + e^{\theta}\cos\theta$$

$$\frac{dx}{d\theta} = e^{\theta}\cos\theta - e^{\theta}\sin\theta$$

$$\Rightarrow y' = \frac{\sin\theta + \cos\theta}{\cos\theta - \sin\theta}$$

极坐标中的点、

点$(e^{\frac{\pi}{2}}, \frac{\pi}{2})$处的极角为$\frac{\pi}{2}$

$$\Rightarrow y_1' = \frac{1+0}{0-1} = -1, \quad y_1 = e^{\frac{\pi}{2}}, \quad x_1 = 0$$

$\Rightarrow l_{切}$: $y - y_1 = y_1'(x - x_1)$

$\Rightarrow l_{切}$: $y = -x + e^{\frac{\pi}{2}}$

$\Rightarrow l_{切}$: $x + y - e^{\frac{\pi}{2}} = 0$