习题 2.3

1.

$\forall\delta>0$,当 $0<|x-x_0|<\delta$ 时

$\lim\limits_{x\to x_0} g(x)=A\Rightarrow\quad |g(x)-A|<\varepsilon\Rightarrow A-\varepsilon<g(x)<A+\varepsilon$

同理：$\lim\limits_{x\to x_0} h(x)=A\Rightarrow A-\varepsilon<h(x)<A+\varepsilon$

$\because g(x)\le f(x)\le h(x)$

$\therefore A-\varepsilon<g(x)\le f(x)\le h(x)<A+\varepsilon$

$\Rightarrow A-\varepsilon<f(x)<A+\varepsilon\Rightarrow\quad |f(x)-A|<\varepsilon\Rightarrow \lim\limits_{x\to x_0} f(x)=A$

2. (1) $\lim\limits_{x\to\infty}\frac{[x]}{x}$

$$x-1<[x]\le x\Rightarrow\quad \frac{x-1}{x}<\frac{[x]}{x}\le 1$$

$$\because \lim_{x\to\infty}\frac{x-1}{x}=1,\lim_{x\to\infty}1=1\quad \therefore \lim_{x\to\infty}\frac{[x]}{x}=1$$

(2) $\lim\limits_{x\to+\infty}\sqrt{1+\dfrac{1}{x^\alpha}}(\alpha>0)$

$$\sqrt{1}<\sqrt{1+\frac{1}{x^\alpha}}<1+\frac{1}{x^\alpha}$$

$$\because \lim_{x\to+\infty}\sqrt{1}=1\quad \lim_{x\to+\infty}1+\frac{1}{x^\alpha}=1$$

$$\therefore \lim_{x\to+\infty}\sqrt{1+\frac{1}{x^\alpha}}=1$$

$$\frac{1}{x}-1<\left[\frac{1}{x}\right]\le\frac{1}{x}\Rightarrow 1-x<x\left[\frac{1}{x}\right]\le 1$$

$\because \lim\limits_{x\to 0} 1-x=1 \quad \lim\limits_{x\to 0} 1=1 \quad \therefore \lim\limits_{x\to 0} x\left[\frac{1}{x}\right]=1$

3.

(1) $\lim\limits_{x\to 0} \sin\frac{1}{x}$

取$x_1=\frac{1}{2n\pi}$， $x_2=\frac{1}{2n\pi+\frac{\pi}{2}} \quad n\in N^*$

易知 $\lim\limits_{n\to\infty} x_n'=\lim\limits_{n\to\infty} x_2=0, \quad x_n''\neq x_2\neq 0$

$\because \lim\limits_{n\to\infty} \sin\frac{1}{x_n'}=\lim\limits_{n\to\infty} \sin 2\,n\pi=0$

$\lim\limits_{n\to\infty} \sin\frac{1}{x_n''}=\lim\limits_{n\to\infty} \sin\left(2n\pi+\frac{\pi}{2}\right)=1$

$\therefore \lim\limits_{x\to 0} \sin\frac{1}{x}$不存在

(2) $\lim\limits_{x\to 0} \cos\frac{1}{x}$

证明过程同(1)

4.

(1) $\lim\limits_{x\to 0} \frac{\sin\alpha\,x}{\sin\beta\,x}(\beta\neq 0)$

$=\lim\limits_{x\to 0} \frac{\sin\alpha\,x}{\alpha x}\cdot\frac{\beta x}{\sin\beta\,x}\cdot\frac{\alpha}{\beta}$

$=\lim\limits_{x\to 0} \frac{\sin\alpha\,x}{\alpha x}\cdot\lim\limits_{x\to 0} \frac{\beta x}{\sin\beta\,x}\cdot\lim\limits_{x\to 0} \frac{\alpha}{\beta}=\frac{\alpha}{\beta}$

(2) $\lim\limits_{x\to 0} \frac{\tan\alpha\,x}{\tan\beta\,x}(\beta\neq 0)=\lim\limits_{x\to 0} \frac{\alpha x}{\beta x}=\frac{\alpha}{\beta}$

(3) $\lim\limits_{x\to 0} \frac{1-\cos x}{x^2}=\lim\limits_{x\to 0} \frac{\frac{1}{2}x^2}{x^2}=\frac{1}{2}$

(4) $\lim\limits_{x \to \frac{\pi}{4}} \dfrac{\sqrt{2} - 2\cos x}{\sin\left(x - \frac{\pi}{4}\right)}$

$= \lim\limits_{x \to \frac{\pi}{4}} \dfrac{2\sin x}{\cos\left(x - \frac{\pi}{4}\right)} = \dfrac{2\sin x}{\frac{\sqrt{2}}{2}(\cos x + \sin x)} = \dfrac{\sqrt{2}}{1} = \sqrt{2}$

5.

(1) $\lim\limits_{x \to 0}(1 - 3x)^{\frac{1}{x}}$

$= \lim\limits_{x \to 0}[1 + (-3x)]^{\frac{1}{-3x}\cdot(-3)} = e^{-3}$

(2) $\lim\limits_{x \to \infty}\left(\dfrac{1 + x}{2 + x}\right)^{\frac{1-x^2}{1-x}}$

$= \lim\limits_{x \to \infty}\left[1 + \left(-\dfrac{1}{x + 2}\right)\right]^{-(x+2)\cdot\frac{1+x}{-(x+2)}}$

$= \lim\limits_{x \to \infty} e^{\frac{1+x}{-(x+2)}} = e^{-1}$

(3) $\lim\limits_{x \to 0}(1 + \sin x)^{3\csc x} = \lim\limits_{x \to 0}(1 + \sin x)^{\frac{1}{\sin x}\cdot 3} = e^{3}$

(4) $\lim\limits_{x \to \infty}\left(\dfrac{x + 1}{x - 1}\right)^{x}$

$= \lim\limits_{x \to \infty}\left(1 + \dfrac{2}{x - 1}\right)^{\frac{x-1}{2}\cdot\frac{2x}{x-1}}$

$= \lim\limits_{x \to \infty} e^{\frac{2x}{x-1}}$

$= e^{2}$