习题 5.3

1. 求下列不定积分

(1) $\int x\cos x\,dx=\int x\,d\sin x=x\sin x-\int \sin x\,d\,x$

$=x\sin x+\cos x+C$

(2) $\int \ln x\,d\,x=\int x\ln x\,dx$

$=x\ln x-\int x\,d\ln x$

$=x\ln x-\int x\cdot\frac{1}{x}dx$

$=x\ln x-x+C$

(3) $\int x^2e^x d\,x=\int x^2(e^x)'d\,x=x^2e^x-\int e^x\cdot 2xdx$

$=x^2e^x-\left(e^x\cdot 2x-\int e^x\cdot 2dx\right)$

$=e^x(x^2-2x+2)+C$

(4) $\int \arcsin x\,d\,x=\int x'\arcsin x\,dx$

$$=x\arcsin x-\int x\cdot\frac{1}{\sqrt{1-x^2}}dx$$

$=x\arcsin x+\sqrt{1-x^2}+C$

(5) $\int \frac{\ln(\ln x)}{x}dx=\int(\ln x)'\ln(\ln x)\,dx$

$=\ln x\cdot\ln(\ln x)-\int \ln x\cdot\frac{\frac{1}{x}}{\ln x}dx$

$= ln\,x \cdot ln(ln\,x) - ln\,x + C$

(6) $\int e^{2x} \cos x\, dx = \int \frac{1}{2}(e^{2x})' \cos x\, dx$

$= \frac{1}{2}\left(e^{2x} \cos x + \int e^{2x} \sin x\, dx\right)$

$= \frac{1}{2}\left[e^{x} \cos x + \frac{1}{2}\left(e^{2x} \sin x - \int e^{2x} \cos x\, dx\right)\right]$

移项可得$\int e^{2x} \cos x\, dx = \frac{1}{5} e^{2x}(\sin x + 2\cos 2\,x) + C$

(7) $\int x \sin x \cos x\, dx = \frac{1}{2}\int x \sin 2\,x dx = -\frac{1}{4}\int x(\cos 2\,x)' d\,x$

$= -\frac{1}{4}\left(x \cos 2\,x - \int \cos 2\,x\, dx\right)$

$= -\frac{1}{4}\left(x \cos 2\,x - \frac{1}{2}\sin 2\,x\right) + C$

(8) $\int x\, f''(x) dx = \int x\left(f'(x)\right)' d\,x$

$= xf'(x) - \int f'(x) d\,x$

$= xf'(x) - f(x) + C$

(9) $\int x \sin^2 x\, dx = \int x \frac{1 - \cos 2\,x}{2} dx$

$= \frac{1}{2}\int x\left(x - \frac{1}{2}\sin 2\,x\right)' d\,x$

$= \frac{1}{2}\left[x^2 - \frac{1}{2} x \sin 2\,x - \int \left(x - \frac{1}{2}\sin 2\,x\right) d\,x\right]$

$= \frac{1}{2}\left[x^2 - \frac{1}{2} x \sin 2\,x - \frac{1}{2} x^2 + \left(-\frac{1}{4}\cos 2\,x\right)\right]$

$$=\frac{1}{4}x^2-\frac{x}{4}sin\,2\,x-\frac{1}{8}cos\,2\,x+C$$

$$(10)\int x(arctan\,x)^2d\,x$$

$$=\int\frac{1}{2}(x^2)'(arctan\,x)^2dx$$

$$=\frac{1}{2}[x^2\,arctan\,x)^2-\int 2\,arctan\,x\left(1-\frac{1}{1+x^2}\right)dx$$

$$=\frac{1}{2}x^2(arctan\,x)^2-\int x'\,arctan\,x\,dx+\frac{1}{2}[(arctan\,x)^2]'dx$$

$$=\frac{1}{2}x^2(arctan\,x)^2-\left(x\,arctan\,x-\int\frac{x}{1+x^2}dx\right)+\frac{1}{2}(arctan\,x)^2$$

$$=\frac{1+x^2}{2}(arctan\,x)^2-x\,arctan\,x+\frac{ln(1+x^2)}{2}+C$$

$$(11)\int ln\left(x+\sqrt{1+x^2}\right)d\,x$$

$$=\int x'\,ln\left(x+\sqrt{1+x^2}\right)dx$$

$$=x\,ln\left(x+\sqrt{1+x^2}\right)-\int\frac{1+\frac{2x}{2\sqrt{1+x^2}}}{x+\sqrt{1+x^2}}\cdot xdx$$

$$=x\,ln\left(x+\sqrt{1+x^2}\right)-\int\frac{x}{\sqrt{1+1^2}}dx$$

$$=x\,ln\left(x+\sqrt{1+x^2}\right)-\sqrt{1+x^2}+C$$

$$(12)\int\frac{x\,cos\,x}{sin^3\,x}dx=\int -x\cdot\frac{1}{2}\left(\frac{1}{sin^2\,x}\right)'dx$$

$$=-\left(\frac{x}{sin^2\,x}-\int\frac{1}{sin^2\,x}dx\right)$$

$$=-\frac{x}{2\,sin^2\,x}-\frac{1}{2}cot\,x+C$$

$$(13)\int sec^5 x\, d\, x = \int (tan\, x)'\, sec^3 x\, dx$$

$$= tan\, x\, sec^3 x - 3\int tan\, x\, sec^4 \sin x\, dx$$

得①$4\int sec^5 x\, d\, x = tan\, x\, sec^3 x + 3\int sec^3 x\, d\, x$

由$\int sec^3 x\, d\, x = tan\, x\, sec\, x - \int tan\, x \frac{\sin x}{\cos^2 x} dx$

得②$2\int sec^3 x\, d\, x = tan\, x\, sec\, x + \int \frac{1}{\cos x} dx$

$$= tan\, x\, sec\, x + ln|sec\, x + tan\, x| + C$$

②代入①得

$$\int sec^5 x\, d\, x = \frac{1}{4} tan\, x\, sec^3 x + \frac{3}{8} tan\, x\, sec\, x + \frac{3}{8} ln|sec\, x + tan\, x| + C$$

$$(14)\int \frac{x^2 \arctan x}{1+x^2} dx$$

$$= \int \left(1 - \frac{1}{1+x^2}\right) \arctan x\, dx$$

$$= (x - \arctan x) \arctan x - \int (x - \arctan x) \frac{1}{1+x^2} dx$$

$$= (x - \arctan x) \arctan x - \frac{1}{2} ln(1+x^2) + \int \frac{\arctan x}{1+x^2} dx$$

由$\int \frac{\arctan x}{1+x^2} dx = (\arctan x)^2 - \int \frac{\arctan x}{1+x^2} dx$

得$\int \frac{\arctan x}{1+x^2} = \frac{1}{2}(\arctan x)^2 + C$

$$\int \frac{x^2 \arctan x}{1+x^2} dx = x \arctan x - \frac{1}{2} ln(1+x^2) - \frac{1}{2}(\arctan x)^2 + C$$

2.对于正整数 n≥2,建立$\boldsymbol{I_n = \int \sin^n x\, d\, x}$的递推公式

$$I_n = \int sin^{n-1} x \cdot sin\, x\, dx = -\int sin^{n-1} x\, (cos\, x)' d\, x$$

$$= -sin^{n-1} x\, cos\, x + \int cos\, x\, (n-1)\, sin^{n-2} x\, cos\, x\, dx$$

$$= -sin^{n-1} x\, cos\, x + \int (1 - sin^2 x)(n-1)\, sin^{n-2} x\, dx$$

$$= -sin^{n-1} x\, cos\, x + (n-1)\left(\int sin^{n-2} x\, d\, x - \int sin^n x\, d\, x\right)$$

整理可得$I_n = -\frac{1}{n} sin^{n-1} x\, cos\, x + \frac{n-1}{n} I_{n-2}$