6.4

1.解：(1) $\int_0^1 xe^x dx$

$= xe^x\big|_0^1 - \int_0^1 e^x dx$

$= xe^x\big|_0^1 - e^x\big|_0^1$

$= 1.$

(2) $\int_0^{\frac12} \arcsin x\, dx$

$= \arcsin x\cdot x\big|_0^{\frac12} - \int_0^{\frac12} \frac{x}{\sqrt{1-x^2}} dx$

$= \arcsin x\cdot x\big|_0^{\frac12} + \frac12\int_0^{\frac12} \frac{d(1-x^2)}{\sqrt{1-x^2}}$

$= \arcsin x\cdot x\big|_0^{\frac12} + \sqrt{1-x^2}\big|_0^{\frac12}$

$= \frac{\pi}{12} + \frac{\sqrt3}{2} - 1$

小结论推导：

$I_n = \int_0^{\frac{\pi}{2}} \sin^n x\, dx \ (n\geq 2)$

$= -\int_0^{\frac{\pi}{2}} (\sin x)^{n-1} d(\cos x)$

$= -\left[(\sin x)^{n-1}\cos x\big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \cos x\, d[(\sin x)^{n-1}]\right]$

$= \int_0^{\frac{\pi}{2}} \cos x\cdot(n-1)(\sin x)^{n-2}\cos x\, dx$

$= (n-1)\int_0^{\frac{\pi}{2}} [(\sin x)^{n-2} - \sin^n x]\, dx$

$= (n-1)(I_{n-2} - I_n)$

$\therefore nI_n = (n-1)I_{n-2}$

$\Rightarrow I_n = \frac{n-1}{n} I_{n-2}$

$\because I_0 = \int_0^{\frac{\pi}{2}} (\sin x)^0 dx = \frac{\pi}{2}$

$$\boxed{\therefore I_n = \begin{cases} \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\cdots\cdot\frac12\cdot I_0, & n\text{为偶数} \\ \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac23, & n\text{为奇数}\end{cases}}$$

(3) 由右边推导结论：

$\int_0^{\frac{\pi}{2}} \cos^7 x\, dx$

$= \frac67\times\frac45\times\frac23$

$= \frac{16}{35}$

(4) 同理：

$\int_0^{\frac{\pi}{2}} \sin^6 x\, dx$

$= \frac56\times\frac34\times\frac12\times\frac{\pi}{2}$

$= \frac{5\pi}{32}$

$$\boxed{\int_0^{\frac{\pi}{2}} \sin^n x\, dx = \int_0^{\frac{\pi}{2}} \cos^n x\, dx}$$

证：令 $x = \frac{\pi}{2} - t$

$dx = -dt$

$\therefore \int_0^{\frac{\pi}{2}} \sin^n x\, dx = -\int_{\frac{\pi}{2}}^0 \cos^n t\, dt$

$= \int_0^{\frac{\pi}{2}} \cos^n t\, dt$

$= \int_0^{\frac{\pi}{2}} \cos^n x\, dx$

2.解 (1) $\int_0^{\pi} x\sin\frac{x}{2}dx$

$= -2\int_0^{\pi} x\,d\cos\frac{x}{2}$

$= -2\left(x\cos\frac{x}{2}\Big|_0^{\pi} - \int_0^{\pi}\cos\frac{x}{2}dx\right)$

$= 2\left(2\sin\frac{x}{2}\Big|_0^{\pi} - x\cos\frac{x}{2}\Big|_0^{\pi}\right)$

$= 4.$

(2) $\int_0^{e} x\ln^2 x\,dx$

$= \frac{1}{2}\int_0^{e}\ln^2 x\,dx^2$

$= \frac{1}{2}\left(x^2\ln^2 x\Big|_0^{e} - \int_0^{e} 2x\ln x\,dx\right)$

$= \frac{1}{2}\left[x^2\ln^2 x - \left(x^2\ln x - \int_0^{e} x\,dx\right)\right]$

$= \frac{1}{2}\left(x^2\ln^2 x + \frac{1}{2}x^2 - x^2\ln x\right)\Big|_0^{e}$

$\because \lim\limits_{x\to 0}\frac{1}{2}\left(x^2\ln^2 x + \frac{1}{2}x^2 - x^2\ln x\right) = 0$ (通过广义积分求)

$\therefore$ 原式 $= \frac{e^2}{4}$ (书后答案错误)

$\int_1^{e} x\ln^2 x\,dx = \frac{1}{4}(e^2-1)$

(3) $\int_0^{1} x\arctan x\,dx$

$= \frac{1}{2}\int_0^{1}\arctan x\,dx^2$

$= \frac{1}{2}\left(x^2\cdot\arctan x\Big|_0^{1} - \int_0^{1}\frac{x^2}{x^2+1}dx\right)$

$= \frac{1}{2}\left(x^2\arctan x\Big|_0^{1} - \int_0^{1}dx + \int_0^{1}\frac{dx}{x^2+1}\right)$

$= \frac{1}{2}\left(x^2\arctan x\Big|_0^{1} - x\Big|_0^{1} + \arctan x\Big|_0^{1}\right)$

$= \frac{\pi}{4} - \frac{1}{2}.$

(4). $\int_0^{\frac{\pi}{2}} e^{2x}\cos x\,dx$

$= \int_0^{\frac{\pi}{2}} e^{2x}\,d\sin x$

$= e^{2x}\cdot\sin x\Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} 2e^{2x}\sin x\,dx$

$= e^{2x}\cdot\sin x\Big|_0^{\frac{\pi}{2}} + 2\int_0^{\frac{\pi}{2}} e^{2x}\,d\cos x$

$= e^{2x}\cdot\sin x\Big|_0^{\frac{\pi}{2}} + 2\left(e^{2x}\cdot\cos x\Big|_0^{\frac{\pi}{2}} - 2\int_0^{\frac{\pi}{2}} e^{2x}\cos x\,dx\right)$

移项得: $\int_0^{\frac{\pi}{2}} e^{2x}\cos x\,dx = \frac{1}{5}\left(e^{2x}\cdot\sin x\Big|_0^{\frac{\pi}{2}} + 2e^{2x}\cos x\Big|_0^{\frac{\pi}{2}}\right)$

$= \frac{1}{5}(e^{\pi}-2).$

(5). $\int_0^{1} e^{-\sqrt{x}}dx$

令 $-\sqrt{x} = t \Rightarrow x = t^2$

$dx = 2t\,dt$

$\therefore$ 原式 $= \int_0^{-1} e^{t}\cdot 2t\,dt$

$= -2\int_{-1}^{0} t\cdot e^{t}dt$

$= -2e^{t}(t-1)\Big|_{-1}^{0}$

$= 2 - 4e^{-1}.$

(6) $\int_{\frac{1}{e}}^{e}|\ln x|\,dx$

$\int\ln x\,dx = x\ln x - x + C$

$= \int_1^{e}\ln x\,dx - \int_{\frac{1}{e}}^{1}\ln x\,dx$

$= (x\ln x - x)\Big|_1^{e} - (x\ln x - x)\Big|_{\frac{1}{e}}^{1}$

$= 2 - \frac{2}{e}$

(7) $\int_1^e \sin(\ln x)dx$

$=[x\cdot\sin(\ln x)]|_1^e - \int_1^e \cos(\ln x)dx$

$=[x\sin x(\ln x)]|_1^e - [x\cos(\ln x)]|_1^e - \int_1^e \sin(\ln x)dx$

移项得 $\int_1^e \sin(\ln x)dx = \frac{1}{2}[x\sin x(\ln x)|_1^e - x\cos(\ln x)|_1^e]$

$=\frac{1}{2}(e\sin 1 - e\cos 1 + 1)$

~~(8) $\int_0^{\frac{1}{2}} \frac{x\arcsin x}{\sqrt{1-x^2}}dx$~~

~~$=\int_0^{\frac{1}{2}} x\arcsin x\, d\arcsin x$~~

~~令 $\arcsin x = t \Rightarrow x =$~~

~~原式 $=\int_0^{\frac{\pi}{6}} x\sin x$~~

~~$=(\sin x - x\cos$~~

(8) $\int_0^{\frac{1}{2}} \frac{x\arcsin x}{\sqrt{1-x^2}}dx$

$=\int_0^{\frac{1}{2}} x\arcsin x\, d\arcsin x$

令 $\arcsin x = t \Rightarrow x = \sin t$

$\therefore$ 原式 $=\int_0^{\frac{\pi}{6}} t\sin t\, dt$

$=(\sin t - t\cos t)|_0^{\frac{\pi}{6}}$

$=\frac{1}{2} - \frac{\sqrt{3}\pi}{12}.$

$$\boxed{\int x\sin ax\, dx = \frac{1}{a^2}\sin ax - \frac{x}{a}\cos ax + C}$$

2.(1) 也可用

可省

3. 解: $\int_0^2 x^2 f''(x)dx$

$=\int_0^2 x^2 d(f'(x))$

$=x^2 f'(x)|_0^2 - 2(\int_0^2 x f'(x)dx)$

$=x^2 f'(x)|_0^2 + 2\int_0^2 f(x) - 2xf(x)|_0^2$

将题中数据代入,得原式 $=0$.

4.解:(1) $\because (f^2(x))' = 2f(x)f'(x)$

$\therefore \int_a^b x f(x) f'(x)\,dx$

$= \frac{1}{2}\int_a^b x\,d(f^2(x))$

$= \frac{x}{2} f^2(x)\Big|_a^b - \frac{1}{2}\int_a^b f^2(x)\,dx$

$= \frac{b}{2} f^2(b) - \frac{a}{2} f^2(a) - \frac{1}{2}\times 1$

$= -\frac{1}{2}$

(2) 由柯西瓦茨不等式得: P169

$\left(\int_a^b (f(x)\cdot x f'(x))\,dx\right)^2 \le \int_a^b f^2(x)\,dx \cdot \int_a^b (x\cdot f'(x))^2\,dx$

$\Rightarrow \frac{1}{4} \le \int_a^b x^2 (f'(x))^2\,dx$ (得证)

5.解:证明:(1). $\int_0^{\frac{\pi}{2}} [f(x)+f''(x)]\sin x\,dx$

$= \int_0^{\frac{\pi}{2}} f(x)\sin x\,dx + \int_0^{\frac{\pi}{2}} f''(x)\sin x\,dx$

$= \int_0^{\frac{\pi}{2}} f(x)\sin x\,dx + \sin x\cdot f'(x)\Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} f'(x)\cos x\,dx$

$= f'(\frac{\pi}{2}) + \int_0^{\frac{\pi}{2}} f(x)\sin x\,dx - \int_0^{\frac{\pi}{2}} \cos x\,d f(x)$

$= f'(\frac{\pi}{2}) + \int_0^{\frac{\pi}{2}} f(x)\sin x\,dx - \left(\cos x\cdot f(x)\Big|_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} f(x)\sin x\,dx\right)$

$= f(0) + f'(\frac{\pi}{2})$.

(2). $\int_0^{\frac{\pi}{2}} [f(x)+f''(x)]\cos x\,dx$

$= \int_0^{\frac{\pi}{2}} f(x)\,\cancel{\sin x}\,\cos x\,dx + \int_0^{\frac{\pi}{2}} f''(x)\cos x\,dx$

$= \int_0^{\frac{\pi}{2}} f(x)\cos x\,dx + \int_0^{\frac{\pi}{2}} \cos x\,d f'(x)$

$= \int_0^{\frac{\pi}{2}} f(x)\cos x\,dx + \cos x f'(x)\Big|_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} f'(x)\sin x\,dx$

$= -f'(0) + \int_0^{\frac{\pi}{2}} f(x)\cos x\,dx + \int_0^{\frac{\pi}{2}} \sin x\,d f(x)$

$= -f'(0) + \int_0^{\frac{\pi}{2}} f(x)\cos x\,dx + \sin x f(x)\Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} f(x)\cos x\,dx$

$= f(\frac{\pi}{2}) - f'(0)$.

6.解：(1) $\cancel{\int_0^{\frac{\pi}{2}} x^2\cos}$

$\int_0^{\frac{\pi}{2}} x^2\sin x\,dx$ $\qquad f(x)=x^2,\ f'(x)=2x$

$=\int_0^{\frac{\pi}{2}}(x^2+2)\sin x\,dx-2\int_0^{\frac{\pi}{2}}\sin x\,dx$ $\qquad f(0)=0,\ f'(\frac{\pi}{2})=\pi$

$=\pi+2\cos x\Big|_0^{\frac{\pi}{2}}$

$=\pi-2$

(2). $\int_0^{\frac{\pi}{2}} x^4\cos x\,dx$ $\qquad f(x)=x^4,\ f'(x)=4x^3$

$=\int_0^{\frac{\pi}{2}}(x^4+12x^2)\cos x\,dx-12\int_0^{\frac{\pi}{2}}x^2\cos x\,dx$ $\qquad f(\frac{\pi}{2})=\frac{\pi^4}{16},\ f'(0)=0$

$=\int_0^{\frac{\pi}{2}}(x^4+12x^2)\cos x\,dx-12\left(\int_0^{\frac{\pi}{2}}(x^2+2)\cos x\,dx-\int_0^{\frac{\pi}{2}}2\cos x\,dx\right)$ $\qquad g(x)=x^2,\ g'(x)=2x$

$=\frac{\pi^4}{16}-3\pi^2+24$.（课本后面答案错误） $\qquad g(\frac{\pi}{2})=\frac{\pi^2}{4},\ g'(0)=0$