习题 5.3

1.

(1) $\int \mathrm{x}\cos\mathrm{x}\,dx = \int \mathrm{x}\,d\sin\mathrm{x} = x\sin\mathrm{x} - \int \sin\mathrm{x}\,\mathrm{d}\,x$

$$= x\sin\mathrm{x} + \cos\mathrm{x} + C$$

(2) $\int \ln\mathrm{x}\,\mathrm{d}\,x = x\ln\mathrm{x} - \int \mathrm{x}\,d\ln\mathrm{x}$

$= x\ln\mathrm{x} - \int \mathrm{x}\cdot\frac{1}{\mathrm{x}}dx$

$= x\ln\mathrm{x} - x + C$

(3) $\int \mathrm{x}^2\mathrm{e}^{\mathrm{x}}\mathrm{d}\,x = \int \mathrm{x}^2(\mathrm{e}^{\mathrm{x}})'\mathrm{d}\,x = \mathrm{x}^2\mathrm{e}^{\mathrm{x}} - \int \mathrm{e}^{\mathrm{x}}\cdot 2x dx$

$= \mathrm{x}^2\mathrm{e}^{\mathrm{x}} - \left(\mathrm{e}^{\mathrm{x}}\cdot 2\mathrm{x} - \int \mathrm{e}^{\mathrm{x}}\cdot 2\mathrm{dx}\right)$

$= \mathrm{e}^{\mathrm{x}}(\mathrm{x}^2 - 2\mathrm{x} + 2) + C$

(4) $\int \arcsin\mathrm{x}\,\mathrm{d}\,x = \int \mathrm{x}'\arcsin\mathrm{x}\,dx = x\arcsin\mathrm{x} - \int \mathrm{x}\cdot\frac{1}{\sqrt{1-\mathrm{x}^2}}dx$

$= x\arcsin\mathrm{x} + \sqrt{1-\mathrm{x}^2} + C$

(5) $\int \frac{\ln(\ln\mathrm{x})}{\mathrm{x}}dx = \int (\ln\mathrm{x})'\ln(\ln\mathrm{x})\,dx$

$= \ln\mathrm{x}\cdot\ln(\ln\mathrm{x}) - \int \ln\mathrm{x}\cdot\frac{\frac{1}{\mathrm{x}}}{\ln\mathrm{x}}dx$

$= \ln\mathrm{x}\cdot\ln(\ln\mathrm{x}) - \ln\mathrm{x} + C$

(6) $\int \mathrm{e}^{2\mathrm{x}}\cos\mathrm{x}\,dx = \int \frac{1}{2}(\mathrm{e}^{2\mathrm{x}})'\cos\mathrm{x}\,dx$

$$= \frac{1}{2}\left(e^{2x}\cos x + \int e^{2x}\sin x\,dx\right)$$

$$= \frac{1}{2}\left[e^{x}\cos x + \frac{1}{2}\left(e^{2x}\sin x - \int e^{2x}\cos x\,dx\right)\right]$$

移项可得 $\int e^{2x}\cos x\,dx = \frac{1}{5}e^{2x}(\sin x + 2\cos 2x) + C$

$$(7)\int x\sin x\cos x\,dx = \frac{1}{2}\int x\sin 2x\,dx = -\frac{1}{4}\int x(\cos 2x)'\,dx$$

$$= -\frac{1}{4}\left(x\cos 2x - \int \cos 2x\,dx\right)$$

$$= -\frac{1}{4}\left(x\cos 2x - \frac{1}{2}\sin 2x\right) + C$$

$$(8)\int x f''(x)dx = \int x\left(f'(x)\right)'dx$$

$$= xf'(x) - \int f'(x)dx$$

$$= xf'(x) - f(x) + C$$

$$(9)\int x\sin^2 x\,dx = \int x\frac{1-\cos 2x}{2}dx$$

$$= \frac{1}{2}\int x\left(x - \frac{1}{2}\sin 2x\right)'dx$$

$$= \frac{1}{2}\left[x^2 - \frac{1}{2}x\sin 2x - \int\left(x - \frac{1}{2}\sin 2x\right)dx\right]$$

$$= \frac{1}{2}\left[x^2 - \frac{1}{2}x\sin 2x - \frac{1}{2}x^2 + \left(-\frac{1}{4}\cos 2x\right)\right]$$

$$= \frac{1}{4}x^2 - \frac{x}{4}\sin 2x - \frac{1}{8}\cos 2x + C$$

$$(10)\int x(\arctan x)^2 dx$$

$$= \int \frac{1}{2}(x^2)'(\arctan x)^2 dx$$

$$= \frac{1}{2}\left[x^2 \arctan x)^2 - \int 2\arctan x\left(1 - \frac{1}{1+x^2}\right)dx\right]$$

$$= \frac{1}{2}x^2(\arctan x)^2 - \int x' \arctan x\, dx + \frac{1}{2}[[(\arctan x)^2]'dx$$

$$= \frac{1}{2}x^2(\arctan x)^2 - \left(x\arctan x - \int \frac{x}{1+x^2}dx\right) + \frac{1}{2}(\arctan x)^2$$

$$= \frac{1+x^2}{2}(\arctan x)^2 - x\arctan x + \sqrt{1+x^2} + C$$

$$(11)\int \ln\left(x + \sqrt{1+x^2}\right)dx$$

$$= \int x' \ln\left(x + \sqrt{1+x^2}\right)dx$$

$$= x\ln\left(x + \sqrt{1+x^2}\right) - \int \frac{1 + \frac{2x}{2\sqrt{1+x^2}}}{x + \sqrt{1+x^2}} \cdot x dx$$

$$= x\ln\left(x + \sqrt{1+x^2}\right) - \int \frac{x}{\sqrt{1+1^2}}dx$$

$$= x\ln\left(x + \sqrt{1+x^2}\right) - \sqrt{1+x^2} + C$$

$$(12)\int \frac{x\cos x}{\sin^3 x}dx = \int -x \cdot \frac{1}{2}\left(\frac{1}{\sin^2 x}\right)' dx$$

$$= -\left(\frac{x}{\sin^2 x} - \int \frac{1}{\sin^2 x}dx\right)$$

$$= -\frac{x}{2\sin^2 x} - \frac{1}{2}\cot x + C$$

$$(13)\int \sec^5 x\, dx = \int (\tan x)' \sec^3 x\, dx$$

$$= \tan x \sec^3 x - 3\int \tan x \sec^4 \sin x\, dx$$

得①$4\int \sec^5 \mathrm{x}\,\mathrm{d}\,x = \tan \mathrm{x}\sec^3 \mathrm{x} + 3\int \sec^3 \mathrm{x}\,\mathrm{d}\,x$

由$\int \sec^3 \mathrm{x}\,\mathrm{d}\,x = \tan \mathrm{x}\sec \mathrm{x} - \int \tan \mathrm{x}\dfrac{\sin \mathrm{x}}{\cos^2 \mathrm{x}}\,dx$

得②$2\int \sec^3 \mathrm{x}\,\mathrm{d}\,x = \tan \mathrm{x}\sec \mathrm{x} + \int \frac{1}{\cos \mathrm{x}}\,dx$

$= \tan \mathrm{x}\sec \mathrm{x} + \ln|\sec \mathrm{x} + \tan \mathrm{x}| + C$

②代入①得

$$\int \sec^5 \mathrm{x}\,\mathrm{d}\,x = \frac{1}{4}\tan \mathrm{x}\sec^3 \mathrm{x} + \frac{3}{8}\tan \mathrm{x}\sec \mathrm{x} + \frac{3}{8}\ln|\sec \mathrm{x} + \tan \mathrm{x}| + C$$

$$(14)\int \frac{\mathrm{x}^2 \arctan \mathrm{x}}{1 + \mathrm{x}^2}\,dx$$

$$= \int \left(1 - \frac{1}{1 + \mathrm{x}^2}\right)\arctan \mathrm{x}\,dx$$

$$= (\mathrm{x} - \arctan \mathrm{x})\arctan \mathrm{x} - \int (\mathrm{x} - \arctan \mathrm{x})\frac{1}{1 + \mathrm{x}^2}\,dx$$

$$= (\mathrm{x} - \arctan \mathrm{x})\arctan \mathrm{x} - \frac{1}{2}\ln(1 + \mathrm{x}^2) + \int \frac{\arctan \mathrm{x}}{1 + \mathrm{x}^2}\,dx$$

由$\int \dfrac{\arctan \mathrm{x}}{1 + \mathrm{x}^2}\,dx = (\arctan \mathrm{x})^2 - \int \dfrac{\arctan \mathrm{x}}{1 + \mathrm{x}^2}\,dx$

得$\int \dfrac{\arctan \mathrm{x}}{1 + \mathrm{x}^2} = \dfrac{1}{2}(\arctan \mathrm{x})^2 + C$

$$\int \frac{\mathrm{x}^2 \arctan \mathrm{x}}{1 + \mathrm{x}^2}\,dx = x\arctan \mathrm{x} - \frac{1}{2}\ln(1 + \mathrm{x}^2) - \frac{1}{2}(\arctan \mathrm{x})^2 + C$$

2.对于正整数 n≥2,建立$\mathrm{I_n} = \int \sin^{\mathrm{n}} \mathrm{x}\,\mathrm{d}\,x$的递推公式

$$\mathrm{I_n} = \int \sin^{\mathrm{n}-1} \mathrm{x} \cdot \sin \mathrm{x}\,dx = -\int \sin^{\mathrm{n}-1} \mathrm{x}\,(\cos \mathrm{x})'\mathrm{d}\,x$$

$$= -\sin^{\mathrm{n}-1} \mathrm{x}\cos \mathrm{x} + \int \cos \mathrm{x}\,(\mathrm{n} - 1)\sin^{\mathrm{n}-2} \mathrm{x}\cos \mathrm{x}\,dx$$

$$= -\sin^{n-1} x \cos x + \int (1 - \sin^2 x)(n-1) \sin^{n-2} x \, dx$$

$$= -\sin^{n-1} x \cos x + (n-1)\left(\int \sin^{n-2} x \, dx - \int \sin^n x \, dx\right)$$

整理可得$I_n = -\frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n} I_{n-2}$