

习题 6.1

1. (1) ✕ ($f(x)$ 在 $[a, b]$ 上可积一定有界, 但 $f(x)$ 有界为 $f(x)$ 在 $[a, b]$ 上可积最基本条件)

(2) ✕ ($\lambda \rightarrow 0 \neq n \rightarrow \infty$)

(3) ✓

(4) ✓

2. (1) C

(2) D (由施瓦茨不等式得 $\left(\int_a^b f(x)g(x) dx\right)^2 \leq \int_a^b f^2(x) dx \int_a^b g^2(x) dx$

其中令 $f(x) = x, g(x) = 1$ 则 $\left(\int_a^b x dx\right)^2 \leq \int_a^b x^2 dx (b-a)$

即 $\left(\int_0^1 x dx\right)^2 \leq \int_0^1 x^2 dx$)

(3) D

3. $S = \int_0^{10} (10t + 1) dt = 510m$

4. (1) 解: 令 $f(x) = x, x \in [0, 1]$

将 $x \in [0, 1], n$ 等分, $\Delta x_i = \frac{1}{n}$

取 $\xi_i = \frac{i}{n} (i = 1, 2, \dots, n)$

则 $\sum_{i=1}^n f(\xi_i) \Delta x_i = \frac{1}{n^2} (1 + 2 + \dots + n)$

$$= \frac{1}{n^2} \cdot \frac{n(n+1)}{2} = \frac{n+1}{2n}$$

当 $\lambda \rightarrow 0$ 时, $\pi \rightarrow \infty$

故 $\int_0^1 x dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(\xi_i) \Delta x_i = \frac{1}{2}$

(2) 解: 令 $g(x) = e^x, x \in [0, 1]$

将 $x \in [0, 1] n$ 等分, $\Delta x_i = \frac{1}{n}$

取 $\xi_i = \frac{i}{n} (i = 1, 2, \dots, n)$

则 $\sum_{i=1}^n f(\xi_i) \Delta x_i = \frac{1}{n} e^{\frac{1}{n}} + \frac{1}{n} e^{\frac{2}{n}} + \dots + \frac{1}{n} e^{\frac{n}{n}}$

$$= \frac{1}{n} \left(e^{\frac{1}{n}} + e^{\frac{2}{n}} + e^{\frac{3}{n}} + \dots + e^1 \right)$$

$$= \frac{1}{n} \cdot \frac{e^{\frac{1}{n}}(1-e)}{1-e^{\frac{1}{n}}} \quad (\text{等比数列})$$

由 $\lambda \rightarrow 0$ 时, $n \rightarrow \infty$ 时

$$\begin{aligned}\lim_{n \rightarrow \infty} \sum_{i=1}^n f(\xi_i) \Delta x_i &= \lim_{n \rightarrow \infty} \frac{1}{n} \cdot \frac{e^{\frac{1}{n}}(1-e)}{-\frac{1}{n}} \quad (\text{等价无穷小}) \\ &= \lim_{n \rightarrow \infty} (e-1)e^{\frac{1}{n}} = e-1\end{aligned}$$

5. 由定积分几何意义得 $\frac{1}{2} \cdot 2(k+8+k) = 10$

$$\Rightarrow 2k = 2 \Rightarrow k = 1$$

6. (1) 解: 令 $f(x) = \frac{1}{(1+x)^2}$, $x \in [0,1]$

对 $x \in [0,1]$ n 等分, $\Delta x_i = \frac{1}{n}$, 取 $\xi_i = \frac{2}{n} (i = 1, 2, \dots, n)$

$$\text{则 } \sum_{i=1}^n f(\xi_i) \Delta x_i = \frac{1}{\left(1+\frac{1}{n}\right)^2} \cdot \frac{1}{n} + \frac{1}{\left(1+\frac{2}{n}\right)^2} \cdot \frac{1}{n} + \dots + \frac{1}{\left(1+\frac{n}{n}\right)^2} \cdot \frac{1}{n}$$

$$= \sum_{i=1}^n \frac{n}{(n+i)^2}$$

当 $n \rightarrow \infty$, $\lambda \rightarrow 0$

$$\text{则原式} = \int_0^1 \frac{1}{(1+x)^2} dx$$

(2) 解: 令 $f(x) = \sin x$

在 $x \in \left[0, \frac{\pi}{2}\right]$ 分成 n 等份, 则 $\Delta x_i = \frac{\pi}{2n}$

取 $\xi_i = \frac{i\pi}{2n} (i = 0, 1, 2, \dots, n-1)$

$$\sum_{i=1}^n f(\xi_i) \Delta x_i = \frac{\pi}{2n} \left[\sin 0 + \sin \frac{\pi}{2n} + \dots + \frac{\sin(n-1)\pi}{2n} \right]$$

当 $n \rightarrow \infty$, $\lambda \rightarrow 0$

$$\text{则原式} = \int_0^{\frac{\pi}{2}} \sin x dx$$

7. (1) 解: 原式 $= \int_c^b f(x) dx + \int_b^a f(x) dx = \int_c^a f(x) dx$

(2) 由积分中值定理得

$$\text{平均值为 } f(\xi) = \frac{1}{1-0} \int_0^1 f(x) dx = \int_0^1 e^x dx$$

8. (1) 由 $f(x) = x$, $g(x) = \sqrt{x}$ 在区间 $[0,1]$ 可积, 且在 $[0,1]$ 上 $x \leq \sqrt{x}$

$$\text{由保序性 } \int_0^1 x dx \leq \int_0^1 \sqrt{x} dx$$

(2) 同理 $x \geq \sin x$

$$\int_0^{\frac{\pi}{2}} x dx \geq \int_0^{\frac{\pi}{2}} \sin x dx$$

(3) 因为在 $[-1,0]$ 上 $e^{2x} \leq e^x$ 所以 $\int_{-1}^0 e^{2x} dx \leq \int_{-1}^0 e^x dx$

两边加负号即 $\int_0^{-1} e^{2x} dx \geq \int_0^{-1} e^x dx$

(4) 由 $\ln x \geq (\ln x)^2$ 在 $[1, 2]$ 上

$$\text{同理 } \int_1^2 \ln x dx \geq \int_1^2 (\ln x)^2 dx$$

(5) 由在 $\left[0, \frac{\pi}{2}\right]$ 上 $x \leq \tan x$

$$\text{同理 } \int_0^{\frac{\pi}{2}} x dx \leq \int_0^{\frac{\pi}{2}} \tan x dx$$

9. (1) 由在 $x \in [0, 1]$ 内 $\frac{1}{1+x^2} \in \left[\frac{1}{2}, 1\right]$

$$\text{则 } \frac{1}{2} \cdot 1 \leq \int \frac{dx}{1+x^2} \leq 1 \cdot 1 \quad (\text{推论 6.1.1})$$

$$\text{即 } \frac{1}{2} \leq \int \frac{dx}{1+x^2} \leq 1$$

(2) 由于在 $x \in [0, 2]$, $x^2 - 2x \in [-1, 0]$

$$\text{则 } e^{x^2-2x} \in \left[\frac{1}{e}, 1\right]$$

$$\frac{2}{e} \leq \int_0^2 e^{x^2-2x} dx \leq 2$$

(3) 在 $x \in [0, 2\pi]$ 内 $\frac{1}{1+0.5 \cos x} \in \left[\frac{2}{3}, 2\right]$

$$\text{则 } \frac{4\pi}{3} \leq \int_0^{2\pi} \frac{dx}{1+0.5 \cos x} \leq 4\pi$$

(4) 在 $x \in [0, 100]$ 内 则 $\frac{e^{-x}}{x+100} \in \left[\frac{\frac{1}{e^{100}}}{200}, \frac{1}{100}\right]$

$$\frac{1}{2e^{100}} \leq \int_0^{100} \frac{e^{-x}}{x+100} dx \leq 1$$

10. 证明: (1) $\int_0^2 \frac{1}{\sqrt{1+x^3}} dx = \int_0^1 \frac{1}{\sqrt{1+x^3}} dx + \int_1^2 \frac{1}{\sqrt{1+x^3}} dx$

由积分中值定理得

$$\text{原式} = f(\xi_1)(1-0) + f(\xi_2)(2-1) \quad (\text{其中 } \xi_1 \in [0, 1], \xi_2 \in [1, 2])$$

$$= \frac{1}{\sqrt{1+\xi_1^3}} + \frac{1}{\sqrt{1+\xi_2^3}}$$

$$\text{则 } \frac{1}{\sqrt{1+0}} + \frac{1}{\sqrt{1+1}} \leq \int_0^2 \frac{1}{\sqrt{1+x^3}} dx \leq \frac{1}{\sqrt{1+1}} + \frac{1}{\sqrt{1+2^3}}$$

$$\text{即 } \frac{1}{3} + \frac{\sqrt{2}}{2} \leq \int_0^2 \frac{1}{\sqrt{1+x^3}} dx \leq 1 + \frac{\sqrt{2}}{2}$$

(2) $\int_0^{\frac{\pi}{2}} \sin^2 x dx = \int_0^{\frac{\pi}{6}} \sin^2 x dx + \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin^2 x dx + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin^2 x dx$

积分中值定理

$$= \sin^2 \xi_1 \left(\frac{\pi}{6} - 0\right) + \sin^2 \xi_2 \left(\frac{\pi}{3} - \frac{\pi}{6}\right) + \sin^2 \xi_3 \left(\frac{\pi}{2} - \frac{\pi}{3}\right) \quad (\xi_1 \in \left[0, \frac{\pi}{6}\right], \xi_2 \in \left[\frac{\pi}{6}, \frac{\pi}{3}\right], \xi_3 \in \left[\frac{\pi}{3}, \frac{\pi}{2}\right])$$

$$\text{则} \frac{\pi}{6} \sin^2 0 + \frac{\pi}{6} \sin^2 \frac{\pi}{6} + \frac{\pi}{6} \sin^2 \frac{\pi}{3} \leq \int_0^{\frac{\pi}{2}} \sin^2 x \, dx \leq \frac{\pi}{6} \left(\sin^2 \frac{\pi}{6} + \sin^2 \frac{\pi}{3} + \sin^2 \frac{\pi}{2} \right)$$

$$\text{即} \frac{\pi}{6} \leq \int_0^{\frac{\pi}{2}} \sin^2 x \, dx \leq \frac{\pi}{6} \left(\frac{1}{4} + \frac{3}{4} + 1 \right) = \frac{\pi}{3} \text{ 得证}$$

11. 解：由题意得 $\int_0^1 f(x) \, dx$ 为具体值

$$\text{设} f(x) = x + t$$

$$\text{则} 2 \int_0^1 f(x) \, dx = 2 \int_0^1 (x + t) \, dx = t$$

$$\text{则} 2 \int_0^1 x \, dx + 2 \int_0^1 t \, dx = t$$

$$\text{则} 2 \int_0^1 x \, dx = -t$$

$$t = -2 \cdot \frac{1}{2} \cdot 1 = -1$$

$$\text{故} f(x) = x - 1$$

习题 6.2

1. (1) $F'(x) = \sqrt{1+x^2} \quad F'(0) = 1$

(2) $F'(x) = 2 - \frac{1}{\sqrt{x}} < 0 \Rightarrow x < \frac{1}{4}$ 区间为 $(0, \frac{1}{4})$

(3) $F'(x) = f(e^{-x}) \cdot e^{-x}(-1) - f(x) = -f(e^{-x}) \cdot e^{-x} - f(x)$

(4) 令 $\int_0^y e^{-t^2} dt + \int_0^x \sin^2 t dt = F(x)$

$$F'(x) = e^{-y^2} y' + \sin^2 x = 0 \Rightarrow y' = -e^{y^2} \sin^2 x$$

(5) 因为 $[-\pi, \pi]$ 关于原点对称 又 $|\sin x|$ 为偶函数

所以 原式 $= 2 \int_0^\pi \sin x dx = -2 \cos x \Big|_0^\pi = 4$

2. (1) 原式 $= \lim_{x \rightarrow 0} \frac{\cos x^2}{1} = 1$

$$\begin{aligned} (2) \text{ 原式} &= \lim_{x \rightarrow 0} \frac{2 \int_0^x e^t dt \cdot e^x}{x e^{2x^2}} = \lim_{x \rightarrow 0} \frac{2 \int_0^x e^t dt}{x e^{2x^2 - x}} \\ &= \lim_{x \rightarrow 0} \frac{2e^x}{e^{2x^2 - x} + x e^{2x^2 - x} \cdot (4x - 1)} = \frac{2 \times 1}{1 + 0 \times 1 \times (-1)} = 2 \end{aligned}$$

$$\begin{aligned} 3. (1) \int_0^1 \sqrt{x} (1 - \sqrt{x})^2 dx &= \int_0^1 \sqrt{x} (1 + x - 2\sqrt{x}) dx \\ &= \int_0^1 \left(\sqrt{x} + x^{\frac{3}{2}} - 2x \right) dx = \left(\frac{2}{3} x^{\frac{3}{2}} + \frac{2}{5} x^{\frac{5}{2}} - x^2 \right) \Big|_0^1 \\ &= \frac{2}{3} + \frac{2}{5} - 1 = \frac{1}{15} \end{aligned}$$

$$\begin{aligned} (2) \text{ 原式} &= \int_0^1 \frac{-(x^2+1)+2}{1+x^2} dx = \int_0^1 \left(-1 + \frac{2}{1+x^2} \right) dx \\ &= (-x + 2 \arctan x) \Big|_0^1 = -1 + 2 \times \frac{\pi}{4} = \frac{\pi}{2} - 1 \end{aligned}$$

$$\begin{aligned} (3) \text{ 令 } 1-x &= t \Rightarrow x = 1-t \\ dx &= -dt \quad x \Big|_0^1 \rightarrow t \Big|_1^0 \\ \text{原式} \int_1^0 e^t (-dt) &= \int_0^1 e^t dt = e^x \Big|_0^1 = e - 1 \end{aligned}$$

$$\begin{aligned} (4) \text{ 原式} &= \int_0^1 \frac{\frac{1}{2}}{\sqrt{1 - \left(\frac{x}{2}\right)^2}} dx = \int_0^1 \frac{1}{\sqrt{1 - \left(\frac{x}{2}\right)^2}} d\left(\frac{1}{2}x\right) \\ &= \arcsin \frac{x}{2} \Big|_0^1 = \frac{\pi}{6} - 0 = \frac{\pi}{6} \end{aligned}$$

$$\begin{aligned} (5) \text{ 原式} &= \int_{-1}^2 \sqrt{2+x} d(x+2) = \frac{2}{3} (x+2)^{\frac{3}{2}} \Big|_{-1}^2 \\ &= \frac{2}{3} \times 4^{\frac{3}{2}} - \frac{2}{3} = \frac{2}{3} \times 7 = \frac{14}{3} \end{aligned}$$

$$(6) \text{ 原式} = \int_0^{\pi} \frac{1-\cos 2x}{2} dx = \int_0^{\pi} \frac{1}{2} dx - \frac{1}{4} \int_0^{\pi} \cos 2x d(2x)$$

$$= \frac{x}{2} \Big|_0^{\pi} - \left(\frac{1}{4} \sin 2x \right) \Big|_0^{\pi} = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

$$(7) \text{ 原式} = \int_0^{\frac{\pi}{4}} \left(\frac{\sin x + \cos x}{\cos x} \right)^2 dx = \int_0^{\frac{\pi}{4}} (\tan x + 1)^2 dx$$

$$= \int_0^{\frac{\pi}{4}} (\tan^2 x + 1 + 2 \tan x) dx$$

$$= \int_0^{\frac{\pi}{4}} (\sec^2 x + 2 \tan x) dx$$

$$= (\tan x - 2 \ln |\cos x|) \Big|_0^{\frac{\pi}{4}} = 1 - 2 \ln \frac{\sqrt{2}}{2}$$

$$= 1 + \ln \left(\frac{\sqrt{2}}{2} \right)^{-2} = 1 + \ln 2$$

$$(8) \text{ 原式} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{2 \sin^2 x} dx$$

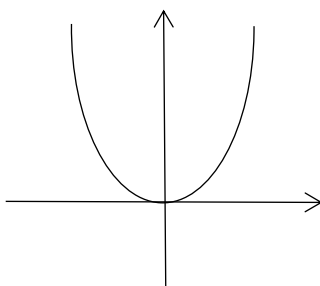
因为 $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ 关于原点对称, 又 $\sqrt{2 \sin^2 x}$ 为偶函数

$$\text{所以原式} = 2 \int_0^{\frac{\pi}{2}} \sqrt{2} \sin x dx = -2\sqrt{2} \cos x \Big|_0^{\frac{\pi}{2}} = 2\sqrt{2}$$

$$(9) \text{ 原式} = \int_0^1 x^2 dx + \int_1^2 1 dx$$

$$= \frac{x^3}{3} \Big|_0^1 + x \Big|_1^2$$

$$= \frac{1}{3} + 2 - 1 = \frac{4}{3}$$



$$(10) \text{ 原式} = \int_0^1 0 dx + \int_1^2 \sin x dx + \int_2^3 2 \sin x dx + \int_3^{\pi} 3 \sin x dx$$

$$= -\cos x \Big|_1^2 - 2 \cos x \Big|_2^3 - 3 \cos x \Big|_3^{\pi}$$

$$= -\cos 2 + \cos 1 - 2 \cos 3 + 2 \cos 2 + 3 + 3 \cos 3$$

$$= \cos 1 + \cos 2 + \cos 3 + 3$$

$$4. \text{ 解: } \frac{dy}{dx} = \frac{f^2(t)f'(t)}{f(x)f'(t)} = f(t)$$

$$\frac{d^2 y}{dx^2} = \frac{f'(t)}{f(t)f'(t)} = \frac{1}{f(t)}$$

$$5. \text{ 证: } y' = xf(x)$$

因为 $f(x) > 0$ 当 $x > 0$ 时, $y' > 0$, $y \uparrow$

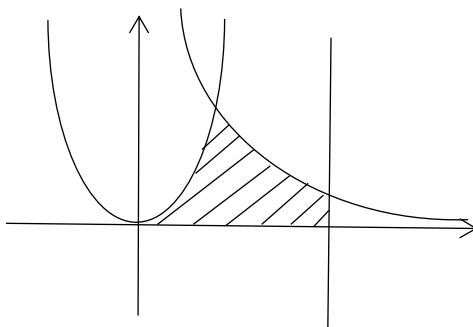
当 $x < 0$ 时, $y' < 0$, $y \downarrow$

所以当 $x = 0$ 时, y 取最小值, 得证

$$6. \text{ 解: } S = \int_0^1 x^2 dx + \int_1^2 \frac{1}{x} dx$$

$$= \frac{x^3}{3} \Big|_0^1 + \ln x \Big|_1^2$$

$$= \frac{1}{3} + \ln 2$$



$$7. (1) \text{ 证: } \int_{-\pi}^{\pi} \cos nx dx = \int_{-\pi}^{\pi} \frac{1}{n} \cos nx d(nx)$$

$$= \frac{1}{n} \sin(nx) \Big|_{-\pi}^{\pi} = \frac{1}{n} [\sin n\pi - \sin(-n\pi)] = 0$$

所以得证

$$(2) \text{ 证: } = \int_{-\pi}^{\pi} \sin nx dx = \frac{1}{n} \int_{-\pi}^{\pi} \frac{1}{n} \sin nx d(nx)$$

$$= -\frac{1}{n} \cos nx \Big|_{-\pi}^{\pi} = -\frac{1}{n} [\cos n\pi - \cos(-n\pi)]$$

$$= -\frac{1}{n} (\cos n\pi - \cos n\pi) = 0$$

所以得证

$$(3) \text{ 证: } \int_{-\pi}^{\pi} \cos^2 nx dx = \int_{-\pi}^{\pi} \frac{1 + \cos(2nx)}{2} dx$$

$$= \frac{1}{2} \int_{-\pi}^{\pi} 1 dx + \frac{1}{2} \int_{-\pi}^{\pi} \frac{1}{2n} \cos(2nx) d(2nx)$$

$$= \frac{1}{2} x \Big|_{-\pi}^{\pi} + \frac{1}{4n} \sin(2nx) \Big|_{-\pi}^{\pi}$$

$$= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) + 0 = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

所以得证

$$(4) \text{ 证: } \int_{-\pi}^{\pi} \sin^2 nx dx = \int_{-\pi}^{\pi} \frac{1 - \cos(2nx)}{2} dx$$

$$= \int_{-\pi}^{\pi} \frac{1}{2} dx - \int_{-\pi}^{\pi} \frac{1}{4n} \cos(2nx) d(2nx)$$

$$= \frac{x}{2} \Big|_{-\pi}^{\pi} - \frac{1}{4n} \sin(2nx) \Big|_{-\pi}^{\pi}$$

$$= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) - 0 = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

所以得证

8. (1) 证: $\int_{-\pi}^{\pi} \cos mx \cos nx dx = \int_{-\pi}^{\pi} \frac{1}{2} [\cos(m+n)x + \cos(m-n)x] dx$ (积化和差公式)

$$\begin{aligned} &= \frac{1}{2} \int_{-\pi}^{\pi} \cos(m+n)x dx + \frac{1}{2} \int_{-\pi}^{\pi} \cos(m-n)x dx \\ &= \frac{1}{2(m+n)} \int_{-\pi}^{\pi} \cos(m+n)x d(m+n)x + \frac{1}{2(m-n)} \int_{-\pi}^{\pi} \cos(m-n)x d(m-n)x \\ &= \frac{1}{2(m+n)} \sin(m+n)x \Big|_{-\pi}^{\pi} + \frac{1}{2(m-n)} \sin(m-n)x \Big|_{-\pi}^{\pi} \\ &= 0 + 0 = 0 \end{aligned}$$

所以得证

(2) 证: $\int_{-\pi}^{\pi} \sin mx \sin nx dx = \int_{-\pi}^{\pi} \left[-\frac{1}{2} \cos(m+n)x + \frac{1}{2} \cos(m-n)x \right] dx$

$$\begin{aligned} &= -\frac{1}{2} \int_{-\pi}^{\pi} \cos(m+n)x dx + \frac{1}{2} \int_{-\pi}^{\pi} \cos(m-n)x dx \\ &= -\frac{1}{2(m+n)} \int_{-\pi}^{\pi} \cos(m+n)x d(m+n)x + \frac{1}{2(m-n)} \int_{-\pi}^{\pi} \cos(m-n)x d(m-n)x \\ &= -\frac{1}{2(m+n)} \sin(m+n)x \Big|_{-\pi}^{\pi} + \frac{1}{2(m-n)} \sin(m-n)x \Big|_{-\pi}^{\pi} \\ &= 0 + 0 = 0 \end{aligned}$$

所以得证

(3) 证: ① $m \neq n$ 时

$$\begin{aligned} \int_{-\pi}^{\pi} \sin mx \cos nx dx &= \int_{-\pi}^{\pi} \frac{1}{2} [\sin(m+n)x + \sin(m-n)x] dx \\ &= \frac{1}{2} \int_{-\pi}^{\pi} \sin(m+n)x dx + \frac{1}{2} \int_{-\pi}^{\pi} \sin(m-n)x dx \\ &= -\frac{1}{2(m+n)} \cos(m+n)x \Big|_{-\pi}^{\pi} - \frac{1}{2(m-n)} \cos(m-n)x \Big|_{-\pi}^{\pi} \\ &= -\frac{1}{2(m+n)} [\cos(m+n)\pi - \cos(-(m+n)\pi)] \\ &\quad - \frac{1}{2(m-n)} [\cos(m-n)\pi - \cos(-(m-n)\pi)] \\ &= 0 - 0 = 0 \end{aligned}$$

② $m = n$ 时

$$\int_{-\pi}^{\pi} \sin mx \cos nx dx = \frac{1}{2} \int_{-\pi}^{\pi} \sin 2mx dx = 0 \quad (\text{第 7. (2) 的结论})$$

习题 6.3

1. (1) $\int_0^1 (2x-3)^2 dx$

$$\text{令 } 2x-3 = t \Rightarrow x = \frac{3+t}{2}$$

(积分变量变化时, 积分区间也要相应变化)

$$\Rightarrow \int_0^1 (2x-3)^2 dx = \int_{-3}^{-1} t^2 d\left(\frac{3+t}{2}\right) = \frac{1}{2} \cdot \frac{t^3}{3} \Big|_{-3}^{-1} = \frac{1}{2} \left(\frac{-1}{3} - \frac{-27}{3} \right) = \frac{13}{3}$$

(2) $f(x)$ 在 $\left[0, \frac{1}{2}\right]$ 上连续可导 $\Rightarrow f(x)$ 在 $\left[0, \frac{1}{2}\right]$ 上可积

$$\int_0^1 f'\left(\frac{1-x}{2}\right) dx \quad \text{令 } \frac{1-x}{2} = t \Rightarrow x = 1-2t, \quad \frac{1-x}{2} \Big|_0^1 \rightarrow t \Big|_{\frac{1}{2}}^0$$

$$\Rightarrow \int_0^1 f'\left(\frac{1-x}{2}\right) dx = \int_{\frac{1}{2}}^0 f'(t) d(1-2t) = 2 \int_{\frac{1}{2}}^0 f'(t) dt = 2f(t) \Big|_{\frac{1}{2}}^0 = 2\left(f\left(\frac{1}{2}\right) - f(0)\right)$$

2. (1) $\int_0^1 x\sqrt{1-x} dx$ (令 $\sqrt{1-x} = t$)

$$= \int_1^0 t \cdot (1-t^2) d(1-t^2)$$

$$= 2 \int_0^1 (t^2 - t^4) dt$$

$$= 2 \cdot \frac{t^3}{3} \Big|_0^1 - 2 \cdot \frac{t^5}{5} \Big|_0^1$$

$$= 2\left(\frac{1}{3} - 0\right) - 2\left(\frac{1}{5} - 0\right)$$

$$= \frac{2}{3} - \frac{2}{5} = \frac{4}{15}$$

(2) $\int_0^1 x(2-x^2)^5 dx$

$$= -\frac{1}{2} \int_0^1 (2-x^2)^5 d(2-x^2)$$

$$= -\left(\frac{1}{12} - \frac{2^6}{12}\right) = \frac{21}{4}$$

(3) $\int_1^{\sqrt{3}} \frac{dx}{x^2\sqrt{1+x^2}}$ (令 $x = \tan t$, 积分上下限改变为 $\frac{\pi}{3}, \frac{\pi}{4}$)

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{\tan^2 t \cdot \sec t} \cdot \sec^2 t \cdot dt$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\cos t}{\sin^2 t} dt$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{\sin^2 t} d \sin t$$

$$= -\frac{1}{\sin t} \Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$

$$= -\left(\frac{2\sqrt{3}}{3} - \sqrt{2}\right) = \sqrt{2} - \frac{2\sqrt{3}}{3}$$

(4) $\int_0^1 \frac{dx}{e^x + e^{-x}}$ (令 $e^x = t$, 则积分上下限改变为 $e, 1$)

$$= \int_1^e \frac{1}{t+\frac{1}{t}} \cdot \frac{1}{t} dt$$

$$= \int_1^e \frac{1}{1+t^2} dt$$

$$= \arctan t \Big|_1^e$$

$$= \arctan e - \frac{\pi}{4}$$

(5) $\int_0^1 \frac{1}{e^x+1} dx$ (令 $e^x = t$, 则积分上下限改变为 $e, 1$)

$$= \int_1^e \frac{1}{1+t} \cdot \frac{1}{t} dt$$

$$= \int_1^e \left(\frac{1}{t} - \frac{1}{1+t}\right) dt$$

$$= \ln t \Big|_1^e - \ln(1+t) \Big|_1^e$$

$$= 1 - 0 - \ln(1+e) + \ln 2$$

$$= 1 + \ln 2 - \ln(1+e)$$

$$= \ln \frac{2e}{1+e}$$

(6) $\int_{\frac{1}{2}}^1 \frac{\sqrt{1-x^2}}{x^2} dx$ (令 $x = \sin t$, 则积分上下限改变为 $\frac{\pi}{2}, \frac{\pi}{4}$)

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos t}{\sin^2 t} \cdot \cos t dt$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos^2 t}{\sin^2 t} dt$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1-\sin^2 t}{\sin^2 t} dt$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\csc^2 t - 1) dt$$

$$= -\cot t \Big|_{\frac{\pi}{2}}^{\frac{\pi}{4}} - t \Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= -(0-1) - \left(\frac{\pi}{2} - \frac{\pi}{4}\right)$$

$$= 1 - \frac{\pi}{4}$$

(7) $\int_0^{\frac{\pi}{2}} \frac{\cos t}{\sin t + \cos t} dt$ (令 $x = a \sin t$, 则积分上下限改变为 $\frac{\pi}{2}, 0$)

$$= \int_0^{\frac{\pi}{2}} \frac{\frac{1}{2}(\sin t + \cos t) + \frac{1}{2}(\cos t - \sin t)}{\sin t + \cos t} dt$$

$$\begin{aligned}
&= \int_0^{\frac{\pi}{2}} \left(\frac{1}{2} + \frac{1}{2} \cdot \frac{\cos t - \sin t}{\sin t + \cos t} \right) dt \\
&= \int_0^{\frac{\pi}{2}} \left(\frac{1}{2} + \frac{1}{2} \cdot \frac{\cos t - \sin t}{\sin t + \cos t} \right) dt \\
&= \frac{\pi}{4} + \frac{1}{2} \ln(\sin t + \cos t) \Big|_0^{\frac{\pi}{2}} \\
&= \frac{\pi}{4} + (|n| - |n|) \\
&= \frac{\pi}{4}
\end{aligned}$$

$$(8) \quad I = \int_0^{\frac{\pi}{2}} \frac{\sin \theta}{\sin \theta + \cos \theta} d\theta, \quad J = \int_0^{\frac{\pi}{2}} \frac{\cos \theta}{\sin \theta + \cos \theta} d\theta \text{ (半余法)}$$

$$a. (I + J) = \int_0^{\frac{\pi}{2}} \frac{\sin \theta + \cos \theta}{\sin \theta + \cos \theta} d\theta = \int_0^{\frac{\pi}{2}} 1 d\theta = \frac{\pi}{2}$$

$$b. [I - J] = \int_0^{\frac{\pi}{2}} \frac{\sin \theta - \cos \theta}{\sin \theta + \cos \theta} d\theta = - \int_0^{\frac{\pi}{2}} \frac{1}{\sin \theta + \cos \theta} d(\sin \theta + \cos \theta)$$

$$\Rightarrow I - J = -\ln(\sin \theta + \cos \theta) \Big|_0^{\frac{\pi}{2}} = -(\ln 1 - \ln 1) = 0$$

$$\Rightarrow \frac{a+b}{2} = I = \frac{\pi}{4}, \quad \frac{a-b}{2} = J = \frac{\pi}{4}$$

$$(9) \quad \int_0^{\frac{\pi}{2}} \frac{dx}{a^2 \sin^2 x + b^2 \cos^2 x} \quad (a, b > 0) \quad (\text{令 } u = \tan x, \text{ 则积分上下限改变为 } +\infty, 0)$$

$$= \int_0^{+\infty} \frac{1}{a^2 \cdot \frac{u^2}{1+u^2} + b^2 \cdot \frac{1}{1+u^2}} d\arctan u$$

$$= \int_0^{+\infty} \frac{1+u^2}{a^2 u^2 + b^2} \cdot \frac{1}{1+u^2} du$$

$$= \int_0^{+\infty} \frac{1}{a^2 u^2 + b^2} du$$

$$= \frac{1}{b^2} \int_0^{+\infty} \frac{1}{1 + \left(\frac{a}{b}u\right)^2} \left(d\left(\frac{a}{b}u\right)\right) \cdot \frac{b}{a}$$

$$= \frac{1}{ab} \arctan \frac{a}{b}u \Big|_0^{+\infty} \quad \left(\lim_{x \rightarrow +\infty} \arctan \frac{a}{b}x = \frac{\pi}{2}, \text{ 可认为 } \arctan \frac{a}{b}u = \frac{\pi}{2} \quad (u \rightarrow +\infty) \right)$$

$$= \frac{1}{ab} \left(\frac{\pi}{2} - 0 \right)$$

$$(10) \quad f(x) = \begin{cases} 1+x^2, & x \leq 0 \\ e^{-x}, & x > 0 \end{cases}$$

$$\int_1^3 f(x-2) dx \quad (\text{令 } x-2=t, \text{ 则积分上下限改变为 } 1, -1)$$

$$= \int_{-1}^1 f(t) dt$$

$$= \int_{-1}^0 f(t) dt + \int_0^1 f(t) dt \quad (\text{分段函数将积分区间相应分段})$$

$$= \int_{-1}^0 (1+t^2) dt + \int_0^1 e^{-t} dt$$

$$= t|_{-1}^0 + \frac{t^3}{3} \Big|_{-1}^0 - e^{-t} \Big|_0^1$$

$$= 1 + \frac{1}{3} - e^{-1} + 1$$

$$= \frac{7}{3} - \frac{1}{e}$$

3. 证明: 因为 $f(x)$ 在 $[-a, a]$ 上连续

$\Rightarrow f(x)$ 在 $[-a, a]$ 上可积

$$\int_{-a}^a x(f(x) + f(-x))dx = \int_{-a}^a x f(x)dx + \int_{-a}^a x f(-x)dx$$

$$\Rightarrow \int_{-a}^a x f(-x)dx \quad (\text{令 } -x = t, \text{ 则积分上下限改变为 } -a, a) \quad \int_a^{-a} (-t)f(t)d(-t) =$$

$$\int_a^{-a} t f(t)dt = \int_a^{-a} x f(x)dx$$

$$\Rightarrow \int_{-a}^a x(f(x) + f(-x))dx = \int_{-a}^a x f(x)dx + \int_a^{-a} x f(x)dx = 0$$

4 证明:

$$\int_0^1 x^m (1-x)^n dx \stackrel{t=1-x}{=} \int_1^0 (1-t)^m t^n d(1-t) = \int_0^1 t^n (1-t)^m dt = \int_0^1 x^n (1-x)^m dx \quad (\text{等于右式})$$

$$\text{综上: } \int_0^1 x^m (1-x)^n dx = \int_0^1 x^n (1-x)^m dx$$

5. 证明: 令 $t = \frac{1}{u}$ ($x > 0$) ($x > 0 \Rightarrow t > 0$ 即可 $\Rightarrow t = \frac{1}{u}$ ($u > 0$))

$$t|_x^1 \Rightarrow u|_{\frac{1}{x}}^1$$

$$\Rightarrow \int_x^1 \frac{1}{1+t^2} dt = \int_{\frac{1}{x}}^1 \frac{1}{1+(\frac{1}{u})^2} \cdot \left(-\frac{1}{u^2}\right) du = \int_1^{\frac{1}{x}} \frac{1}{1+u^2} du = \int_1^{\frac{1}{x}} \frac{1}{1+t^2} dt$$

$$\text{综上: } \int_x^1 \frac{1}{1+t^2} dt = \int_1^{\frac{1}{x}} \frac{1}{1+t^2} dt \quad (x > 0)$$

6. 证明: $f(x)$ 为连续函数 \Rightarrow 在 $x \in D$ 时可积

(1) 因为 $f(x)$ 为奇函数

$$\text{所以 } f(x) = -f(-x)$$

$$\text{令 } F(x) = \int_0^x f(t)dt, \text{ 则 } F(-x) = \int_0^{-x} f(t)dt$$

$$\text{令 } t = -u, \quad t|_0^{-x} \rightarrow u|_0^x$$

$$\Rightarrow F(-x) = \int_0^x f(t)dt = \int_0^x f_0 f(-u)d(-u) = \int_0^x -f(-u)du = \int_0^x f(u)dx =$$

$$\int_0^x f(t)dt = F(x)$$

故 $\int_0^{-x} f(t)dt$ 在 $f(x)$ 为奇函数时, 为偶函数。

(2) 因为 $f(x)$ 为偶函数

$$\text{所以 } f(x) = f(-x)$$

$$\text{令 } G(x) = \int_0^x f(t) dt, \quad G(-x) = \int_0^{-x} f(t) dt$$

$$\text{令 } t = -k, \quad t|_0^{-x} \rightarrow k|_0^x$$

$$\Rightarrow G(-x) = \int_0^x f(t) dt = \int_0^x f(-k) d(-k) = -\int_0^x f(k) dk = -G(x)$$

故当 $f(x)$ 为偶函数时, $\int_0^x f(t) dt$ 为奇函数。

习题 6.4

1. (1) 解:

$$\begin{aligned}& \int_0^1 x e^x dx \\&= x e^x \Big|_0^1 - \int_0^1 e^x dx \\&= x e^x \Big|_0^1 - e^x \Big|_0^1 \\&= 1\end{aligned}$$

(2) 解:

$$\begin{aligned}& \int_0^{\frac{1}{2}} \arcsin x dx \\&= x \cdot \arcsin x \Big|_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} \frac{x}{\sqrt{1-x^2}} dx \\&= x \cdot \arcsin x \Big|_0^{\frac{1}{2}} + \frac{1}{2} \int_0^{\frac{1}{2}} \frac{d(1-x^2)}{\sqrt{1-x^2}} \\&= x \cdot \arcsin x \Big|_0^{\frac{1}{2}} + \sqrt{1-x^2} \Big|_0^{\frac{1}{2}} \\&= \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1\end{aligned}$$

(3) 解:

由推导结论:

$$\begin{aligned}& \int_0^{\frac{\pi}{2}} x dx \cos^2 x dx \\&= \frac{6}{7} \times \frac{4}{5} \times \frac{2}{3} \\&= \frac{16}{35}\end{aligned}$$

(4) 解:

同理:

$$\begin{aligned}& \int_0^{\frac{\pi}{2}} \sin x^6 dx \\&= \frac{5}{6} \times \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2} \\&= \frac{5\pi}{32}\end{aligned}$$

结论推导:

$$\begin{aligned}
 A_n &= \int_0^{\frac{\pi}{2}} \sin^n x dx \quad (n \geq 2) \\
 &= - \int_0^{\frac{\pi}{2}} (\sin x)^{n-1} d(\cos x) \\
 &= -(\sin x)^{n-1} \cos x \Big|_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos x d(\sin x)^{n-1} \\
 &= (n-1) \int_0^{\frac{\pi}{2}} \cos x (\sin x)^{n-2} \cos x dx \\
 &= (n-1) (A_{n-2} - A_n) \\
 \therefore n A_n &= (n-1) A_{n-2} \\
 \Rightarrow A_n &= \frac{n-1}{n} A_{n-2} \\
 \because A_0 &= \int_0^{\frac{\pi}{2}} (\sin x)^0 dx = \frac{\pi}{2} \\
 A_1 &= \int_0^{\frac{\pi}{2}} (\sin x)^1 dx = 1
 \end{aligned}$$

$$\therefore A_n = \begin{cases} \frac{(n-1)!!}{n!!} \cdot \frac{\pi}{2} & (n = 2k) \\ \frac{(n-1)!!}{n!!} & (n = 2k+1) \end{cases}$$

易证:

$$\int_0^{\frac{\pi}{2}} \sin^n x dx = \int_0^{\frac{\pi}{2}} \cos^n x dx$$

2. (1) 解:

$$\begin{aligned}
 &\int_0^{\pi} x \sin \frac{x}{2} dx \\
 &= -2 \int_0^{\pi} x d \cos \frac{x}{2} \\
 &= -2 \left(x \cos \frac{x}{2} \Big|_0^{\pi} - \int_0^{\pi} \cos \frac{x}{2} dx \right) \\
 &= 2 \left(2 \sin \frac{x}{2} \Big|_0^{\pi} - x \cos \frac{x}{2} \Big|_0^{\pi} \right) \\
 &= 4
 \end{aligned}$$

(2) 解:

$$\begin{aligned}& \int_0^e x \ln^2 x \, dx \\&= \frac{1}{2} \int_0^e \ln^2 x \, dx^2 \\&= \frac{1}{2} (x^2 \ln^2 x \big|_0^e - \int_0^e 2x \ln x \, dx) \\&= \frac{1}{2} [x^2 \ln^2 x \big|_0^e - (x^2 \ln x - \int_0^e x \, dx)] \\&= \frac{1}{2} [x^2 \ln^2 x + \frac{1}{2} x^2 - x^2 \ln x] \big|_0^e \\&= \frac{e^2}{4} \quad (\text{正确答案}) \\&\int_1^e x \ln^2 x \, dx = \frac{1}{4} (e^2 - 1) \quad (\text{改正后的题目答案})\end{aligned}$$

(3)

$$\begin{aligned}& \int_0^1 x \arctan x \, dx \\&= \frac{1}{2} \int_0^1 \arctan x \, dx^2 \\&= \frac{1}{2} (x^2 \arctan x \big|_0^1 - \int_0^1 \frac{x^2}{x^2 + 1} \, dx) \\&= \frac{1}{2} (x^2 \arctan x \big|_0^1 - \int_0^1 dx + \int_0^1 \frac{dx}{x^2 + 1}) \\&= \frac{1}{2} (x^2 \arctan x \big|_0^1 - x \big|_0^1 + \arctan x \big|_0^1) \\&= \frac{\pi}{4} - \frac{1}{2}\end{aligned}$$

(4)

$$\begin{aligned}
& \int_0^{\frac{\pi}{2}} e^{2x} \cos x \, dx \\
&= \int_0^{\frac{\pi}{2}} e^{2x} d \sin x \\
&= e^{2x} \sin x \Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} 2e^{2x} \sin x \, dx \\
&= e^{2x} \sin x \Big|_0^{\frac{\pi}{2}} + 2 \int_0^{\frac{\pi}{2}} e^{2x} d \cos x \\
&= e^{2x} \sin x \Big|_0^{\frac{\pi}{2}} + 2(e^{2x} \cos x \Big|_0^{\frac{\pi}{2}} - 2 \int_0^{\frac{\pi}{2}} e^{2x} \cos x \, dx)
\end{aligned}$$

移项得:

$$\text{原式} = \frac{1}{5}(e^{\pi} - 2)$$

(5)

$$\int_0^1 e^{\sqrt{x}} \, dx$$

$$\text{令 } \sqrt{x} = t$$

$$\Rightarrow x = t^2$$

$$dx = 2t \, dt$$

$$\text{原式} = \int_0^1 2te^t \, dt$$

$$= -2 \int_{-1}^0 te^t \, dt$$

$$= -2e^t(t-1) \Big|_{-1}^0$$

$$= 2 - 4e^{-1}$$

(6)

$$\begin{aligned}
& \int_{\frac{1}{e}}^e |\ln x| dx \\
&= \int_1^e \ln x dx - \int_{\frac{1}{e}}^1 \ln x dx \\
&= (x \ln x - x) \Big|_1^e - (x \ln x - x) \Big|_{\frac{1}{e}}^1 \\
&= 2 - \frac{2}{e}
\end{aligned}$$

$$\int \ln x dx = x \ln x - x + c$$

(7)

$$\begin{aligned}
& \int_1^e \sin(\ln x) dx \\
&= [x \sin(\ln x)] \Big|_1^e - \int_1^e \cos(\ln x) dx \\
&= [x \sin(\ln x)] \Big|_1^e - [x \cos(\ln x)] \Big|_1^e - \int_1^e \sin(\ln x) dx
\end{aligned}$$

移项得:

$$\text{原式} = \frac{1}{2} (e \sin 1 - e \cos 1 + 1)$$

(8)

$$\begin{aligned}
& \int_0^{\frac{1}{2}} \frac{x \arcsin x}{\sqrt{1-x^2}} dx \\
&= \int_0^{\frac{1}{2}} x \arcsin x d \arcsin x
\end{aligned}$$

$$\text{令 } \arcsin x = t \Rightarrow x = \sin t$$

原式

$$= \int_0^{\frac{\pi}{6}} t \sin t dt$$

$$\int x \sin ax dx$$

$$= \frac{1}{a^2} \sin ax - \frac{x}{a} \cos ax + C$$

$$= \frac{1}{2} - \frac{\sqrt{3} \pi}{12}$$

3. 解:

$$\begin{aligned} & \int_0^2 x^2 f''(x) dx \\ &= \int_0^2 x^2 d(f'(x)) \\ &= x^2 f'(x) \Big|_0^2 - 2 \left(\int_0^2 x f'(x) dx \right) \\ &= x^2 f'(x) \Big|_0^2 + 2 \int_0^2 f(x) - 2xf(x) \Big|_0^2 \end{aligned}$$

代入数据得原式 = 0

4. 证明:

(1)

$$\begin{aligned} & \because (f^2(x))' = 2f(x)f'(x) \\ & \therefore \int_a^b xf(x)f'(x)dx \\ &= \frac{1}{2} \int_a^b x d(f^2(x)) \\ &= \frac{x}{2} f^2(x) \Big|_a^b - \frac{1}{2} \int_a^b f^2(x) dx \\ &= \frac{b}{2} f^2(b) \Big|_a^b - \frac{a}{2} f^2(a) - \frac{1}{2} \\ &= -\frac{1}{2} \end{aligned}$$

(2)

由施瓦茨不等式得(P169)

$$\begin{aligned} \left(\int_a^b (f(x) \cdot x f'(x)) dx \right)^2 &\leq \int_a^b f^2(x) dx \cdot \int_a^b (x f'(x))^2 dx \\ \Rightarrow \frac{1}{4} &\leq \int_a^b x^2 (f'(x))^2 dx \text{ 得证} \end{aligned}$$

5. 证明:

(1)

$$\begin{aligned} &\int_0^{\frac{\pi}{2}} [f(x) + f''(x)] \sin x \, dx \\ &= \int_0^{\frac{\pi}{2}} f(x) \sin x \, dx + \int_0^{\frac{\pi}{2}} f''(x) \sin x \, dx \\ &= \int_0^{\frac{\pi}{2}} f(x) \sin x \, dx + \sin x \cdot f'(x) \Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} f'(x) \cos x \, dx \\ &= f'(\frac{\pi}{2}) + \int_0^{\frac{\pi}{2}} f(x) \sin x \, dx - \int_0^{\frac{\pi}{2}} \cos x \, df(x) \\ &= f'(\frac{\pi}{2}) + \int_0^{\frac{\pi}{2}} f(x) \sin x \, dx - (\cos x \cdot f(x) \Big|_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} f(x) \sin x \, dx) \\ &= f(0) + f'(\frac{\pi}{2}) \end{aligned}$$

(2)

$$\begin{aligned} &\int_0^{\frac{\pi}{2}} [f(x) + f''(x)] \cos x \, dx \\ &= \int_0^{\frac{\pi}{2}} f(x) \cos x \, dx + \int_0^{\frac{\pi}{2}} \cos x \, df'(x) \\ &= \int_0^{\frac{\pi}{2}} f(x) \cos x \, dx + \cos x \, f'(x) \Big|_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} f'(x) \sin x \, dx \\ &= -f'(0) + \int_0^{\frac{\pi}{2}} f(x) \cos x \, dx + \int_0^{\frac{\pi}{2}} \sin x \, df(x) \end{aligned}$$

$$\begin{aligned}
&= -f'(0) + \int_0^{\frac{\pi}{2}} f(x) \cos x \, dx + \sin x f(x) \Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} f(x) \cos x \, dx \\
&= f\left(\frac{\pi}{2}\right) - f'(0)
\end{aligned}$$

6. 解 (1)

$$f(x) = x^2, f'(x) = 2x, f(0) = 0, f'\left(\frac{\pi}{2}\right) = \pi$$

$$\begin{aligned}
&\int_0^{\frac{\pi}{2}} x^2 \sin x \, dx \\
&= \int_0^{\frac{\pi}{2}} (x^2 + 2) \sin x \, dx - 2 \int_0^{\frac{\pi}{2}} \sin x \, dx \\
&= \pi + 2 \cos x \Big|_0^{\frac{\pi}{2}} \\
&= \pi - 2
\end{aligned}$$

(2)

$$f(x) = x^4, f'(x) = 4x^3, f\left(\frac{\pi}{2}\right) = \frac{\pi^4}{16}, f'(0) = 0$$

$$g(x) = x^2, g'(x) = 2x, g\left(\frac{\pi}{2}\right) = \frac{\pi^2}{4}, g'(0) = 0$$

$$\begin{aligned}
&\int_0^{\frac{\pi}{2}} x^4 \cos x \, dx \\
&= \int_0^{\frac{\pi}{2}} (x^4 + 12x^2) \cos x \, dx - 12 \int_0^{\frac{\pi}{2}} x^2 \cos x \, dx \\
&= \int_0^{\frac{\pi}{2}} (x^4 + 12x^2) \cos x \, dx - 12 \left(\int_0^{\frac{\pi}{2}} (x^2 + 2) \cos x \, dx - \int_0^{\frac{\pi}{2}} 2 \cos x \, dx \right) \\
&= \frac{\pi^4}{16} - 3\pi^2 + 24 \quad (\text{课本后答案错误})
\end{aligned}$$

习题 6.5

1. (1) × $x = 0$ 为奇点

$$\int_{-1}^1 \frac{1}{x} dx = \int_{-1}^0 \frac{1}{x} dx + \int_0^1 \frac{1}{x} dx = \ln|x| \Big|_{-1}^{0+\varepsilon} \textcircled{1} + \ln|x| \Big|_{0+\varepsilon}^1 \textcircled{2} \quad (\varepsilon \rightarrow 0)$$

$$\textcircled{1} \rightarrow +\infty \quad \textcircled{2} \rightarrow -\infty$$

①②左右两边极限均不存在，所以原式极限不存在 (P189)

- (2) × $x = 0$ 为 $\int_{-\infty}^{+\infty} \frac{1}{x^2} dx$ 奇点

$$\int_{-\infty}^{+\infty} \frac{1}{x^2} dx = \int_{-\infty}^0 \frac{1}{x^2} dx + \int_0^{+\infty} \frac{1}{x^2} dx = -\left(\frac{1}{x} \Big|_{-\infty}^0 + \frac{1}{x} \Big|_0^{+\infty}\right)$$

$$\textcircled{1} \quad \textcircled{2}$$

$$\textcircled{1} \rightarrow +\infty \quad \textcircled{2} \rightarrow -\infty$$

左右两边极限不存在，原式极限不存在 (P189 课本)

- (3) × $\int \sin x dx = -\cos x$ 当 $x \rightarrow \infty$ 时, $-\cos x$ 无极限, 发散

- (4) × 反例: 令 $p = 0$ $\int_0^2 \frac{1}{1-x} dx = \int_0^1 \frac{1}{1-x} dx + \int_1^2 \frac{1}{1-x} dx$

等式右侧两极限均不存在

- (5) ✓ $\int \frac{1}{x^p} dx = \frac{1}{1-p} x^{1-p}$

$$\int_0^{+\infty} \frac{1}{x^p} dx = \frac{1}{1-p} x^{1-p} \Big|_{0+\varepsilon}^{+\infty} \quad (\varepsilon \rightarrow 0)$$

$$\textcircled{1} 1-p < 0 \text{ 时, } x \rightarrow +\infty \text{ 时 } x^{1-p} \text{ 为 } 0,$$

$$x \rightarrow 0, x^{1-p} \rightarrow +\infty, \text{ 发散}$$

$$\textcircled{2} 1-p > 0 \text{ 且 } 1-p < 1 \text{ 时 } x \rightarrow +\infty, x^{1-p} \rightarrow +\infty,$$

$$x \rightarrow 0, x^{1-p} \rightarrow 0, \text{ 发散}$$

2. (1) $\int_1^{+\infty} \frac{\ln x}{x^2} dx = \int_1^{+\infty} \frac{\ln x - 1}{x^2} dx + \int_1^{+\infty} \frac{1}{x^2} dx$

$$= -\frac{\ln x}{x} \Big|_1^{+\infty} - \frac{1}{x} \Big|_1^{+\infty}$$

$$= -(0-0) - (0-1) = 1$$

$$(2) \int_0^1 \frac{1}{\sqrt{1-x^2}} dx = \arcsin x \Big|_0^1 = \frac{\pi}{2}$$

$$(3) \int_0^1 \frac{e^{\sqrt{x}}}{\sqrt{x}} dx = 2 \int_0^1 e^{\sqrt{x}} d\sqrt{x} = 2e^{\sqrt{x}} \Big|_0^1 = 2(e-1)$$

$$(4) \int_{-\infty}^{+\infty} e^{-a|x|} dx = \int_{-\infty}^0 e^{ax} dx + \int_0^{+\infty} e^{-ax} dx$$

$$= \frac{1}{a} e^{ax} \Big|_{-\infty}^0 - \frac{1}{a} e^{-ax} \Big|_0^{+\infty}$$

$$= \frac{1}{a} - \lim_{x \rightarrow -\infty} \frac{1}{a} e^{ax} - \lim_{x \rightarrow +\infty} \frac{1}{a} e^{-ax} + \frac{1}{a}$$

↓

↓

$$= 0 \quad = 0$$

$$= \frac{2}{a}$$

$$3. \quad (1) \int_{-\infty}^{+\infty} \frac{dx}{x^2+x+1} = \int_{-\infty}^{+\infty} \frac{1}{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}} dx \quad (\text{配方})$$

$$= \frac{4}{3} \int_{-\infty}^{+\infty} \frac{1}{\left[\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right]^2 + 1} dx$$

$$= \frac{2}{\sqrt{3}} \int_{-\infty}^{+\infty} \frac{1}{\left[\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right]^2 + 1} d\left[\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right]$$

$$= \frac{2}{\sqrt{3}} \arctan \left[\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right] \Big|_{-\infty}^{+\infty}$$

$$= \frac{2}{\sqrt{3}} \left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] = \frac{2\sqrt{3}}{3} \pi$$

$$(2) \int_2^{+\infty} \frac{dx}{x^2+x-2} = \int_2^{+\infty} \frac{dx}{(x-1)(x+2)}$$

$$= \frac{1}{3} \int_2^{+\infty} \left(\frac{1}{x-1} - \frac{1}{x+2}\right) dx \quad (\text{裂项})$$

$$= \frac{1}{3} (\ln|x-1| \Big|_2^{+\infty} - \ln|x+2| \Big|_2^{+\infty})$$

$$= \frac{2}{3} \ln 2 \quad (\text{注: } \lim_{x \rightarrow +\infty} \ln|x-1| = \lim_{x \rightarrow +\infty} \ln|x+2|)$$

$$(3) \int_0^1 x^2 \ln x dx = \frac{1}{3} \int_0^1 \ln x dx^3$$

$$= \frac{1}{3} (x^2 \ln x \Big|_0^1 - \int_0^1 x^3 d \ln x)$$

$$= \frac{1}{3} x^3 \left(\ln x - \frac{1}{3}\right) \Big|_0^1 = -\frac{1}{9}$$

$$(4) \text{ 令 } \sqrt{x} = t$$

$$\int_0^{+\infty} e^{-t} dt^2 = \int_0^{+\infty} 2t \cdot e^{-t} dt$$

$$= -2(1+t)e^{-t} \Big|_0^{+\infty}$$

$$= -2 \cdot (-1) = 2$$

$$(5) \text{ 题目错误, 原题应为: } \int_1^2 \frac{x}{\sqrt{x-1}} dx$$

$$\text{令 } \sqrt{x-1} = t, x = t^2 + 1$$

$$\text{原式} = \int_0^1 (1+t^2) dt = \left(t + \frac{t^3}{3}\right) \Big|_0^1 = \frac{8}{3}$$

$$(6) \int_0^1 \sqrt{\frac{x}{1-x}} dx$$

$$\text{令 } \sqrt{1-x} = t, \sqrt{x} = \sqrt{1-t^2}$$

$$\text{原式} = \int_0^1 \frac{\sqrt{1-t^2}}{t} d(1-t^2)$$

$$= 2 \int_0^1 \sqrt{1-t^2} dt$$

$$= 2 \times \frac{\pi}{4} = \frac{\pi}{2}$$

(令 $f(t) = \sqrt{1-t^2} \Rightarrow f^2(t) + t^2 = 1$, $(x^2 + y^2) = 1$ 为圆方程而 $\sqrt{1-t^2} \geq 0$, 所以

为半圆, 那么从 -1 积到 1 的面积为 $\frac{1}{2} \times \pi \times 1^2 = \frac{\pi}{2}$)

$$(7) \int_a^b \frac{1}{\sqrt{(x-a)(b-x)}} dx$$

$$(x-a)(b-x) = bx - x^2 - ab + ax = -x^2 + (a+b)x - ab$$

$$= -\left(x - \frac{a+b}{2}\right)^2 + \left(\frac{a-b}{2}\right)^2 \quad (\text{这一步就是配方})$$

$$= \frac{|b-a|}{2} \left[1 - \left(\frac{x - \frac{a+b}{2}}{\frac{|a-b|}{2}} \right)^2 \right]$$

$$\text{所以原式} = \int_a^b \frac{1}{\sqrt{1 - \left(\frac{x - \frac{a+b}{2}}{\frac{|a-b|}{2}} \right)^2}} dx \cdot \frac{2}{|b-a|}$$

$$\text{令} \left(\frac{x - \frac{a+b}{2}}{\frac{|a-b|}{2}} \right)^2 \text{ 为} \textcircled{1}, \text{ 把} \textcircled{1} \text{ 看成整体}$$

$$\text{发现} \frac{2}{|a-b|} dx = d\left(\frac{2}{b-a}x - \frac{a+b}{b-a}\right) \quad (\text{任意常数, 根据整体配})$$

$$\text{所以原式} \int_a^b \frac{1}{\sqrt{1-\textcircled{1}^2}} d\textcircled{1}$$

$$= \arcsin \textcircled{1} \Big|_a^b \quad x = b \rightarrow \textcircled{1} = +1$$

$$= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi \quad x = a \rightarrow \textcircled{1} = -1$$

注: 3. (7) 看似复杂, 实则是对关于 x 的一元二次多项式配方, 化成 $\frac{1}{\sqrt{1-a^2}}$ 或 $\frac{1}{\sqrt{1+a^2}}$, 达到求积分的目的

$$(8) \int_0^1 (\ln x)^2 dx = x(\ln x)^2 \Big|_0^1 - \int_0^1 x d[(\ln x)^2]$$

$$= x(\ln x)^2 \Big|_0^1 - 2 \int_0^1 \ln x dx$$

$$= [(\ln x)^2 - 2\ln x + 2] \Big|_0^1 = 2$$

$$4. \int_2^{+\infty} \frac{1}{x(\ln x)^k} dx = \int_2^{+\infty} \frac{1}{(\ln x)^k} d\ln x$$

$$\text{令} \ln x = m$$

$$\int_{\ln 2}^{+\infty} \frac{1}{m^k} dm = \frac{1}{1-k} \cdot m^{1-k} \Big|_{\ln 2}^{+\infty}$$

$$= \lim_{m \rightarrow +\infty} \frac{m^{1-k}}{1-k} \cdot \frac{(\ln 2)^{1-k}}{1-k} \quad (k \neq 1 \text{ 时, 该项为常值})$$

若收敛, 则 $1-k < 0$, 才能使 $m^{1-k} \rightarrow 0, k > 1$

$$5. (1) \text{ 因为 } \lim_{x \rightarrow +\infty} x^2 \cdot \frac{x^2}{x^4 - x^2 + 1} = \lim_{x \rightarrow +\infty} \frac{1}{1 - \frac{1}{x^2} + \frac{1}{x^4}} = 1$$

$\begin{array}{ccc} \downarrow & & \downarrow \\ 0 & & 0 \end{array}$

所以 $\int_0^{+\infty} \frac{x^2}{x^4 - x^2 + 1} dx$ 收敛

注：用来判断敛散性的一种方法：极限审敛法（与书上不太相同）

1. 对于无穷限广义积分 $\int_a^{+\infty} f(x) dx$: $f(x)$ 在 $[a, +\infty)$ 连续且非负

① 若 $\exists P > 1, \lim_{x \rightarrow +\infty} x^P f(x) = C < +\infty$, 则 $\int_a^{+\infty} f(x) dx$ 收敛

② 若 $\lim_{x \rightarrow +\infty} x f(x) = d > 0$ (或 $= +\infty$) 则 $\int_a^{+\infty} f(x) dx$ 发散

2. 对于非负积分 $\int_a^b f(x) dx$, a 为瑕点, $f(x)$ 在 $[a, b]$ 连续且非负

① 若 $\exists 0 < q < 1, \lim_{x \rightarrow a^+} (x - a)^q f(x)$ 存在, 则 $\int_a^b f(x) dx$ 收敛

② 若 $\lim_{x \rightarrow a^+} (x - a) f(x) = d > 0$ (或 $= +\infty$), 则 $\int_a^b f(x) dx$ 发散

$$(2) \text{ 因为 } x \cdot \sqrt[3]{x^2 + 1} > x \cdot x^{\frac{2}{3}} = x^{\frac{5}{3}}$$

$$\text{所以 } \int_1^{+\infty} \frac{1}{x \sqrt[3]{x^2 + 1}} dx < \int_1^{+\infty} \frac{1}{x^{\frac{5}{3}}} dx$$

因为 $\int_1^{+\infty} \frac{1}{x^{\frac{5}{3}}} dx$ 收敛（比较判别法）

$$\text{所以 } \int_1^{+\infty} \frac{1}{x \sqrt[3]{x^2 + 1}} dx \text{ 收敛}$$

$$(3) \int_0^2 \frac{dx}{\ln x}$$

$x = 0, x = 1$ 为可能瑕点

$$\text{原式} = \int_0^1 \frac{dx}{\ln x} + \int_1^2 \frac{dx}{\ln x}$$

主要上述两式任意一式不收敛, 原式不收敛, 否则收敛

$$\int_1^2 \frac{dx}{\ln x} \text{ 利用极限审敛法}$$

$$\lim_{x \rightarrow 1^+} \frac{(x-1)}{\ln x} = \lim_{x \rightarrow 1^+} \frac{x-1}{\underset{\downarrow}{x-1}} = 1 \text{ 存在极限}$$

等价无穷小

$$\int_1^2 \frac{dx}{\ln x} \text{ 发散, 所以原式发散}$$

$$(4) \int_0^{+\infty} \frac{x^m}{x^{n+1}} dx \leq \int_0^{+\infty} \frac{x^m}{x^n} dx$$

$$\int_0^{+\infty} x^{m+n} dx = \frac{1}{m-n+1} x^{m-n+1} \Big|_0^{+\infty}$$

所以当 $m - n + 1 < 0$ 时, 原式收敛, 否则不收敛

(5) 利用极限审敛法:

$$\text{若 } n = p > 1, x^p \cdot \frac{\arctan x}{x^p} = \arctan x$$

$x \rightarrow +\infty, \arctan x \rightarrow \frac{\pi}{2}$ 存在极限为满足原式收敛的一个必要条件

$$\text{另: } \lim_{x \rightarrow 0} (x-0)^q \cdot \frac{\arctan x}{x^n} = \frac{\arctan x}{x^{n-q}} \quad (0 \text{ 为可能奇点})$$

因为 $n > 1, n > q (q \in (0,1))$

所以该极限为 $\frac{0}{0}$ 型, 用洛必达:

$$\lim_{x \rightarrow 0} \frac{\frac{1}{1+x^2}}{\frac{x^{n-q-1}}{n-q}} = \lim_{x \rightarrow 0} \frac{n-q}{x^{n-q-1}}$$

要使该极限存在, 则 $n - q - 1 \Rightarrow n < q + 1$

所以 $n < 2$

综上, $1 < n < 2$ 时, 原式收敛, 否则不收敛

(6) 用比较判别法的极限形式:

$$\int_0^{+\infty} \frac{\sin^2 x}{x} dx \leq \int_0^{+\infty} \frac{1}{x} dx = \ln x \Big|_0^{+\infty}$$

① ②

因为 $\tau = 1$ 所以 ① 与 ② 有相同的敛散性 (P189~P191)

因为 $\ln x \Big|_0^{+\infty}$ 极限不存在

所以原式发散

$$(8) \text{ 因为 } \lim_{x \rightarrow 1^-} \frac{\ln x}{1-x^2} = \lim_{x \rightarrow 1^-} \frac{x-1}{1-x^2} = \lim_{x \rightarrow 1^-} \frac{-1}{1+x} = -\frac{1}{2}$$

所以只有 $x = 0$ 为 $\int_0^1 \frac{\ln x}{1-x^2} dx$ 的可能奇点、瑕点

利用极限审敛法:

$$\lim_{x \rightarrow 0} \frac{(x-0)^q \cdot \ln x}{1-x^2} = \lim_{x \rightarrow 0} \frac{\ln x}{x^{-q} - x^{2-q}} \text{ 属于 } \frac{\infty}{\infty} \text{ 型}$$

$$\text{洛必达法则: } \exists q \in (0,1) \text{ 则 } \lim_{x \rightarrow 0} \frac{\frac{1}{x}}{\underset{\substack{\downarrow \\ q \in (0,1) \text{ 则该项为 } 0}}{-q \cdot x^{-q-1} - (2-q)x^{1-q}}} = \lim_{x \rightarrow 0} \frac{\frac{1}{x}}{-q x^{-q-1}} = -\frac{1}{q} x^q = 0$$

所以原式收敛

注: 对于本题判断敛散性, 用比较判别法和极限审敛法, 若用极限审敛法, 则找到所用瑕点, 再去判断

$$6. (1) \int_x^1 \frac{\cos t}{t^2} dt \leq \int_x^1 \frac{1}{t^2} dt$$

由等价无穷小 $x \rightarrow 0$ 时 $1 - \cos x \sim \frac{1}{2} x^2$

所以 $\cos t \sim 1 - \frac{1}{2} t^2$

$$\text{所以 } \int_x^1 \frac{1 - \frac{t^2}{2}}{t^2} dt = \int_x^1 \frac{\cos t}{t^2} dt \leq \int_x^1 \frac{1}{t^2} dt$$

$$\text{由夹逼定理: } -\frac{3}{2} + \frac{x}{2} + \frac{1}{x} \leq \int_x^1 \frac{\cos t}{t^2} dt \leq -1 + \frac{1}{x} \quad (x \rightarrow 0)$$

$$\text{所以} \lim_{x \rightarrow 0} \int_x^1 \frac{\cos t}{t^2} dt = 1$$

$$(2) \int_0^x \sqrt{1+t^4} dt > \int_0^x \sqrt{t^4} dt = \int_0^x t^2 dt = \frac{1}{3} t^3 \Big|_0^x$$

因为 $x \rightarrow \infty$

$$\text{所以} \lim_{x \rightarrow +\infty} \frac{1}{3} t^3 \Big|_0^x \rightarrow +\infty$$

所以原式为 $\frac{\infty}{\infty}$ 型，用洛必达法则

$$\lim_{x \rightarrow +\infty} \frac{\sqrt{1+t^4}}{3x^2} = \frac{1}{3} \lim_{x \rightarrow +\infty} \frac{\sqrt{1+x^4}}{\sqrt{x^4}} = \frac{1}{3}$$

$$7. \int_1^{+\infty} \frac{1}{x^2} dx = -\frac{1}{x} \Big|_1^{+\infty} = -0 - (-1) = 1 \text{ 收敛}$$

$$\text{所以} \int_1^{+\infty} \left[f^2(x) + \frac{1}{x^2} \right] dx \text{ 收敛}$$

$$f^2(x) \rightarrow a^2, \frac{1}{x^2} \rightarrow b^2$$

因为 $ab < \frac{1}{2}(a^2 + b^2)$ 基本不等式

$$\text{所以} \left| f(x) \cdot \frac{1}{x} \right| < \frac{1}{2} \left[f^2(x) + \frac{1}{x^2} \right]$$

$$\text{所以} \frac{\left| \frac{f(x)}{x} \right|}{f^2(x) + \frac{1}{x^2}} < \frac{1}{2} = \tau$$

$$\text{由比较判别法} \tau \in \left(0, \frac{1}{2} \right)$$

$$\text{又因为} \int_1^{+\infty} \left[f^2(x) + \frac{1}{x^2} \right] dx \text{ 收敛}$$

$$\text{所以} \int_1^{+\infty} \left| \frac{f(x)}{x} \right| dx \text{ 收敛}$$

$$\text{即} \int_1^{+\infty} \frac{f(x)}{x} dx \text{ 绝对收敛}$$

习题 6.6

1.

$$\begin{aligned}(1) S &= \int_0^4 \sqrt{1 + (y')^2} dx = \int_0^4 \sqrt{1 + \frac{9}{4}x} dx = \left[\frac{4}{9} \times \frac{2}{3} \times \left(1 + \frac{9}{4}x\right)^{\frac{3}{2}} \right]_0^4 \\ &= \frac{4}{9} \times \frac{2}{3} \times \left[\left(1 + \frac{9}{4} \times 4\right)^{\frac{3}{2}} - 1 \right] = \frac{8}{27} \left(10^{\frac{3}{2}} - 1\right)\end{aligned}$$

$$(2) x' = \frac{y}{2} - \frac{1}{2y}$$

$$\begin{aligned}S &= \int_1^e \sqrt{1 + (x')^2} dy = \int_1^e \sqrt{1 + \left(\frac{y}{2} - \frac{1}{2y}\right)^2} dy \\ &= \int_1^e \frac{1}{2} \left(y + \frac{1}{y}\right) dy = \frac{1}{2} \left[\frac{1}{2} y^2 + \ln y \right]_1^e \\ &= \frac{1}{2} \times \left(\frac{1}{2} e^2 + 1 - \frac{1}{2} \right) = \frac{1}{4} (e^2 + 1)\end{aligned}$$

(3) 由题目可知, $x \geq 0$, $y \geq 0$

$$y = (1 - \sqrt{x})^2 \quad y' = \frac{dy}{dx} = 2(1 - \sqrt{x}) \left(-\frac{1}{2\sqrt{x}}\right) = 1 - \frac{1}{\sqrt{x}}.$$

$$\begin{aligned}S &= \int_0^1 \sqrt{1 + (y')^2} dx = \int_0^1 \sqrt{1 + \left(1 - \frac{1}{\sqrt{x}}\right)^2} dx \\ &= 2 \int_0^1 \sqrt{2x - 2\sqrt{x} + 1} d\sqrt{x} = 1 + \frac{\sqrt{2}}{2} \ln(1 + \sqrt{2}).\end{aligned}$$

(4) 设 $x = a \cos^3 t$ $y = a \sin^3 t$ ($0 \leq t \leq 2\pi$)

$$S = 4 \int_0^{\frac{\pi}{2}} \sqrt{[(a \cos^3 t)']^2 + [(a \sin^3 t)']^2} dt$$

$$= 4a \int_0^{\frac{\pi}{2}} \sqrt{(-3 \cos^2 t \sin t)^2 + (3 \sin^2 t \cos t)^2} dt$$

$$= 12a \int_0^{\frac{\pi}{2}} \sin t \cos t dt = 6a.$$

$$(5) S = \int_0^{2\pi} \sqrt{\left[(a(\cos t + t \sin t))'\right]^2 + \left[(a(\sin t - t \cos t))'\right]^2} dt$$

$$= |a| \int_0^{2\pi} \sqrt{(t \cos t)^2 + (t \sin t)^2} dt$$

$$= |a| \int_0^{2\pi} t dt = 2\pi^2 |a|$$

$$(6) S = \int_0^{2\pi} \sqrt{r^2 + (r')^2} d\theta = \int_0^{2\pi} \sqrt{a^2(1 + \cos \theta)^2 + a^2 \sin^2 \theta} d\theta$$

$$= 4a \int_0^{\pi} \sqrt{\frac{1 + \cos \theta}{2}} d\theta = 4a \int_0^{\pi} \cos \frac{\theta}{2} d\theta = 8a$$

2.

$$(1) \int_a^b |f(x) - g(x)| dx.$$

面积 $A \geq 0$, 且 $f(x)$ 、 $g(x)$ 的大小无法确定

$$\text{故面积为 } \int_a^b |f(x) - g(x)| dx$$

$$(2) \pi \int_a^b |f^2(x) - g^2(x)| dx$$

在区间 $[a, b]$ 上, 由曲线 $y = f(x)$, y

$= g(x)$ 所围成的平面绕 x 轴旋转

一周所成的旋转体的体积微元为

$$dV = \pi |f^2(x) - g^2(x)| dx$$

$$\therefore V = \pi \int_a^b |f^2(x) - g^2(x)| dx$$

3.

$$(1) \text{ 由 } \begin{cases} y = x^2 \\ x + y = 2 \end{cases} \text{ 得 } x = 1 \text{ 或 } x = -2$$

$$S = \int_{-2}^1 (2 - x - x^2) dx = \left(2x - \frac{1}{2}x^2 - \frac{1}{3}x^3 \right) \Big|_{-2}^1 = \frac{9}{2}$$

$$(2) S = \int_{\frac{1}{e}}^1 |\ln x| dx + \int_1^e \ln x \cdot dx$$

$$= \int_{\frac{1}{e}}^1 \ln x dx + \int_1^e \ln x dx$$

$$= (x \ln x - x) \Big|_{\frac{1}{e}}^1 + (x \ln x - x) \Big|_1^e$$

$$= 2 - \frac{2}{e}$$

$$(3) \text{ 令 } x = a \sin t, \quad y = b \cos t$$

$$\text{则 } S = 4 \int_0^a y dx = \frac{4b}{a} \int_0^a \sqrt{a^2 - x^2} dx = \int_a^{4b} \int_0^{\frac{\pi}{2}} a^2 \cos^2 t dt$$

$$= 4ab \int_0^{\frac{\pi}{2}} \frac{1 + \cos 2t}{2} dt = 2ab \left[t + \frac{1}{2} \sin 2t \right]_0^{\frac{\pi}{2}} = ab\pi$$

$$(4) S = \int_0^1 (e^x - e^{-x}) dx = [e^x - (-e^{-x})]_0^1$$

$$= e + e^{-1} - 2$$

$$(5) S = \int_0^{\frac{\pi}{4}} (\cos x - \sin x) dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\cos x - \sin x) dx$$

$$= (\sin x + \cos x) \Big|_0^{\frac{\pi}{4}} + (-\cos x - \sin x) \Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} = 2\sqrt{2} - 2$$

$$(6) \text{ 由 } \begin{cases} y = \frac{1}{2}x^2 \\ x^2 + y^2 = 8 \end{cases} \text{ 得两曲线的交点为 } (-2, 2), (2, 2)$$

$$\begin{aligned} \text{则 } S_1 &= \int_{-2}^2 \left(\sqrt{8-x^2} - \frac{1}{2}x^2 \right) dx = 2 \int_0^2 \left(\sqrt{8-x^2} - \frac{1}{2}x^2 \right) dx \\ &= 2 \left[4 \arcsin \frac{x}{\sqrt{8}} + \frac{1}{2}x\sqrt{8-x^2} - \frac{1}{6}x^3 \right] \Big|_0^2 \\ &= 2\pi + \frac{4}{3} \end{aligned}$$

$$S_2 = S - S_1 = \pi(2\sqrt{2})^2 - 2\pi - \frac{4}{3} = 6\pi - \frac{4}{3}$$

4.

$$(1) S = \frac{1}{2} \int_0^{\frac{\pi}{2}} r^2 d\theta$$

$$\begin{aligned} x_2 &= 4 \int_0^{\frac{\pi}{2}} \frac{\cos 2\theta + 1}{2} d\theta \\ &= (\sin 2\theta + 2x) \Big|_0^{\frac{\pi}{2}} = \pi \end{aligned}$$

$$(2) S = \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} r^2 d\theta = \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2a^2 \cos 2\theta d\theta$$

$$= \left(\frac{1}{2} a^2 \sin 2\theta \right) \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 4 \times \frac{a^2}{2} \sin 2\theta \Big|_0^{\frac{\pi}{4}} = 2a^2$$

(3) $0 \leq t \leq 2\pi$, 该图形关于x轴与y轴都对称

$$x' = -3a \cos^2 t \sin t$$

$$\begin{aligned} S &= 4 \int_0^{\frac{\pi}{2}} |a \sin^3 t (-3a \cos^2 t \sin t)| dt \\ &= 12a^2 \int_0^{\frac{\pi}{2}} \sin^4 t \cos^2 t dt = \frac{3}{8} \pi a^2 \end{aligned}$$

5.

(1) 两曲线交点为(0,0)与(1,1)的旋转体体积

$$V = \pi \int_0^1 [(\sqrt{x})^2 - (x^2)^2] dx = \pi \int_0^1 (x - x^4) dx = \frac{3}{10}$$

$$(2) V = \pi \int_{-a}^a \left[\left(b + \sqrt{a^2 - x^2} \right)^2 - \left(b - \sqrt{a^2 - x^2} \right)^2 \right] dx$$

$$= 4\pi b \int_{-a}^a \sqrt{a^2 - x^2} dx = 2\pi^2 a^2 b$$

$$(3) V = \pi \int_{-a}^a y^2 dx = 3\pi a^3 \int_0^\pi \sin^7 t \cos^2 t dt$$

$$= 6\pi a^3 \int_0^{\frac{\pi}{2}} (\sin^7 t - \sin^9 t) dt = \frac{32}{105} \pi a^3$$

6. 证明: 旋转曲面的方程为 $\pm\sqrt{y^2 + z^2} = f(x)$, 由旋转曲面的对称

性, 取次曲面的上半部分 $\Sigma: z = \sqrt{f^2(x) - y^2}$

Σ 在 x - O - y 面上的投影区域为

$$D = \{(x, y) \mid -f(x) \leq y \leq f(x) \quad a \leq x \leq b\}$$

$$S = 2 \iint_D \sqrt{1 + z_x^2 + z_y^2} dx dy$$

$$= 2 \iint_D \sqrt{1 + \left[\frac{f(x)f'(x)}{\sqrt{f^2(x) - y^2}} \right]^2 + \left[\frac{-y}{\sqrt{f^2(x) - y^2}} \right]^2} dx dy$$

$$= 2 \int_a^b f(x) \sqrt{1 + [f'(x)]^2} dx \int_{-f(x)}^{f(x)} \frac{1}{\sqrt{f^2(x) - y^2}} dy$$

$$= 2 \int_a^b f(x) \sqrt{1 + [f'(x)]^2} \cdot \left[\arcsin \frac{y}{f(x)} \right]_{-f(x)}^{f(x)} dx$$

$$= 2\pi \int_a^b f(x) \sqrt{1 + [f'(x)]^2} dx$$

$$(1)y' = 2x^{-\frac{1}{2}}$$

$$S = 2\pi \int_0^1 2x^{\frac{1}{2}} \sqrt{1 + \left(2x^{-\frac{1}{2}}\right)^2} dx$$

$$= 4\pi \int_0^1 \sqrt{x+4} dx = 4\pi \times \frac{2}{3} (x+4)^{\frac{3}{2}} \Big|_0^1$$

$$= \frac{8\pi}{3} (5\sqrt{5} - 8)$$

$$(2)y' = \frac{dy}{dx} = \frac{3a \sin^2 t \cos t}{-3a \cos^2 t \sin t} = -\tan t$$

由对称性知

$$S = 2 \int_0^a 2\pi y \cdot \sqrt{1 + (y')^2} dx$$

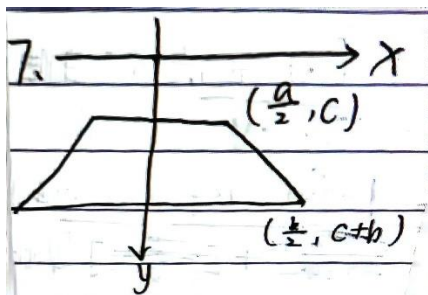
$$= -4\pi \int_0^{\frac{\pi}{2}} a \sin^3 t \sqrt{1 + \tan^2 t} (-3a \cos^2 t \sin t) dt$$

$$= 12a^2\pi \int_0^{\frac{\pi}{2}} \sin^4 t \sec t \cos^2 t dt$$

$$= 12a^2\pi \int_0^{\frac{\pi}{2}} \sin^4 t d \sin t$$

$$= \frac{12}{5} a^2\pi$$

7.



如图, AB 的方程为 $y = \frac{2h}{b-a} \left(x - \frac{a}{2}\right) + c$ 对于薄板上每一点 (x, y) 的

压力 $dF = \rho g y \cdot x dy$

由对称性可知

$$\begin{aligned}P &= 2 \int_c^{c+h} dF = \int_c^{c+h} \left[a + \frac{b-a}{h}(y-c) \right] \rho g y dy \\&= \frac{1}{6} \rho g h (3ac + 3bc + ab + 2bh)\end{aligned}$$

8. 球的密度与水相同 \Rightarrow 球在水中移动时不做功, x 为积分变量,

$x \in [0, 2r]$ 。把球体分为很多薄层, 将相应于 $[x, x+dx]$ 的那一层球体

抬到水面时不做功, 从离开水面时开始做功且 x - O - y 面上方圆的方

程为 $(x-r)^2 + y^2 = r^2$, 可知, 将相应于 $[x, x+dx]$ 的那一薄层球体

提升到 $[x-2r, x+dx-2r]$ 位置时所做的功微元为 (ρ 为水密度)

$$\begin{aligned}dW &= \rho g (2r-x) \pi y^2 dx = \rho g \pi (2r-x) [r^2 - (x-r)^2] dx \\&= \rho g \pi (2r-x) (2rx - x^2) dx = \rho g \pi (x^3 - 4rx^2 + 4r^2x) dx\end{aligned}$$

$$\text{故 } W = \int_0^{2r} dw = \rho g \pi \int_0^{2r} (x^3 - 4rx^2 + 4r^2x) dx$$

$$= \rho g \pi \left(\frac{x^4}{4} - \frac{4rx^3}{3} + 2r^2x^2 \right) \Big|_0^{2r} = \frac{4}{3} \rho g \pi r^4$$