习题 7.3

1. 解:（1）$y''' = e^{2x} - cos\,x$

对所给方程积分 3 次，得:

$$y'' = \frac{1}{2}e^{2x} - sin\,x + c_1$$

$$y' = \frac{1}{4}e^{2x} + cos\,x + c_1 x + c_2$$

$$y = \frac{1}{8}e^{2x} + sin\,x + \frac{1}{2}c_1 x^2 + c_2 x + c_3$$ (C_1, C_2为任意常数)

（2）$y'' = x + sin\,x$

同理，得:

$$y' = \frac{1}{2}x^2 - cos\,x + c_1$$

$$y = \frac{1}{6}x^3 - sin\,x + c_1 x + c_2$$ (C_1, C_2为任意常数)

（3） $xy'' + y' = 0$

令$y' = P(x)$则$y'' = P'(x)$

$\therefore$原方程可化为$xP' + P = 0$

$$-\frac{1}{P}dP = \frac{dx}{x}$$

$$-ln|P| = ln\,x - ln\,c_1$$

$$y' = P = \pm\frac{c_1}{x}$$

$\therefore y = C\,ln|x| + C_2$(C, C_2为任意常数)

（4）$y'' - 4y = x + 1$

齐次方程为:$y'' - 4y = 0$ $\therefore$特征方程为:$r^2 - 4r = 0$

解得$r_1 = 2, r_2 = -2$ $\therefore$齐次方程通解为$y = c_1 e^{2x} + c_2 e^{-2x}$

设特解$y^* = Ax + B$ $\therefore (y^*)'' = 0$

$\therefore 0 - 4(Ax + B) = x + 1$

$\begin{cases} A = -\frac{1}{4} \\ B = -\frac{1}{4} \end{cases}$ $\therefore$微分方程通解为:$y = c_1 e^{2x} + c_2 e^{-2x} - \frac{1}{4}x - \frac{1}{4}$

（5）$y^3y''-1=0$

令$y'=p(x)$，$\therefore y''=p\frac{dp}{dy}$

代入原式得:$y^3p\frac{dp}{dy}=1$

$p\,dp=\frac{dy}{y^3}$

$p^2=-\frac{1}{y^2}+c_1$

$\frac{dy}{dx}=\sqrt{c_1-\frac{1}{y^2}}$

$\frac{y\,dy}{\sqrt{c_1y^2-1}}=dx$

$\therefore\sqrt{c_1y^2-1}=c_1x+c_2$

$\therefore$原式通解为:

$c_1y^2-1=(c_1x+c_2)^2$

（6）$y'^1=(y')^3+y'$

令$y'=p(x)$，则$y''=p\frac{dp}{dy}$

$\therefore$原式可化为:$p\frac{dp}{dy}=p^3+p$

$\frac{dp}{p^2+1}=dy$

$p=tan(y+c_1)$

$\frac{dy}{tan(y+c_1)}=dx$

$ln|sin(y+c_1)|=x+ln\,c_2$

$\therefore$原式通解为:$y=arcsin(c_2e^x)-c_1$

2.解:（1）令$y'=p(x)$　　则$y''=p'(x)$

$\therefore$原式可化为$(1+x^2)p'=2xp$

$\frac{dp}{p}=\frac{2x\,dx}{x^2+1}$

$\frac{dp}{p}=\frac{d(x^2+1)}{x^2+1}$

$ln\,p=ln(x^2+1)+ln\,c_1$

$p=c_1(x^2+1)$

$\therefore y' = c_1(x^2+1)$
$\because y'(0) = 3$
$\therefore c_1 = 3$
$\therefore y' = 3(x^2+1)$
$\therefore y = x^3 + 3x + c_2$
$\because y(0) = 1$
$\therefore$ 微积分方程的特解为$y = x^3 + 3x + 1$

（2）令$y' = p(y)$，$y'' = p\frac{dp}{dy}$

$\therefore$ 原式可化为$p\frac{dp}{dy} = \frac{3}{2}y^2$

$$p\,dp = \frac{3}{2}y^2\,dy$$

$$p^2 = y^3 + c_1$$

$\because y(0) = y'(0) = 1 \therefore c_1 = 0$

$\therefore \frac{dy}{dx} = y^{\frac{3}{2}}$

$y^{-\frac{3}{2}}dy = dx$

$-2y^{\frac{1}{2}} = x + c_2$

$\because y(0) = 1 \therefore c_2 = -2$

$\therefore y^{-\frac{1}{2}} = -\frac{1}{2}x + 1$

$\therefore$ 微分方程的特解为:$y = \frac{1}{\left(1-\frac{1}{2}x\right)^2}$

3、解:$\because \frac{d\alpha}{dx} = \frac{dy}{dx} = tan\,\alpha$

$$\therefore \begin{cases} \alpha = y + c_1 ① \\ \frac{d\alpha}{tan\,\alpha} = dx \Rightarrow \ln|\sin\alpha| = x + ln\,c_2 \end{cases}$$

$$sin\,\alpha = c_2e^x$$
$$\alpha = arc\,sin(c_2e^x) \quad ②$$

联立①②式:$y + c_1 = arc\,sin(c_2e^x)$

$\because y(0) = 0, \alpha(0) = \frac{\pi}{4}$, 代入①式得$c_1 = \frac{\pi}{4}$

联立上式可得$c_2 = \frac{\sqrt{2}}{2}$

$\therefore y(x) = arc\,sin\frac{\sqrt{2}e^x}{2} - \frac{\pi}{4}$