习题 6.6

1.

(1)$S=\int_0^4\sqrt{1+(y')^2}\,dx=\int_0^4\sqrt{1+\frac{9}{4}x}\,dx=\left[\frac{4}{9}\times\frac{2}{3}\times\left(1+\frac{9}{4}x\right)^{\frac{3}{2}}\right]_0^4$

$=\frac{4}{9}\times\frac{2}{3}\times\left[\left(1+\frac{9}{4}\times 4\right)^{\frac{3}{2}}-1\right]=\frac{8}{27}\left(10^{\frac{3}{2}}-1\right)$

(2)$x'=\frac{y}{2}-\frac{1}{2y}$

$S=\int_1^e\sqrt{1+(x')^2}\,dy=\int_1^e\sqrt{1+\left(\frac{y}{2}-\frac{1}{2y}\right)^2}\,dy$

$=\int_1^e\frac{1}{2}\left(y+\frac{1}{y}\right)d\,y=\frac{1}{2}\left[\frac{1}{2}y^2+ln\,y\right]_1^e$

$=\frac{1}{2}\times\left(\frac{1}{2}e^2+1-\frac{1}{2}\right)=\frac{1}{4}(e^2+1)$

(3)由题目可知，$x\geq 0$，$y\geq 0$

$y=\left(1-\sqrt{x}\right)^2\quad y'=\frac{dy}{dx}=2\left(1-\sqrt{x}\right)\left(-\frac{1}{2\sqrt{x}}\right)=1-\frac{1}{\sqrt{x}}.$

$S=\int_0^1\sqrt{1+(y')^2}\,dx=\int_0^1\sqrt{1+\left(1-\frac{1}{\sqrt{x}}\right)^2}\,dx$

$=2\int_0^1\sqrt{2x-2\sqrt{x}+1}\,d\sqrt{x}=1+\frac{\sqrt{2}}{2}ln\left(1+\sqrt{2}\right).$

(4)设$x=a\cos^3 t\quad y=a\sin^3 t\,(0\leq t\leq 2\pi)$

$S=4\int_0^{\frac{\pi}{2}}\sqrt{[(a\cos^3 t)']^2+[(a\sin^3 t)']^2}\,dt$

$$= 4a\int_0^{\frac{\pi}{2}}\sqrt{(-3\cos^2 t\sin t)^2+(3\sin^2 t\cos t)^2}\,dt$$

$$= 12a\int_0^{\frac{\pi}{2}}\sin t\cos t\,dt = 6a.$$

$$(5)S=\int_0^{2\pi}\sqrt{\left[\left(a(\cos t+t\sin t)\right)'\right]^2+\left[\left(a(\sin t-t\cos t)\right)'\right]^2}\,dt$$

$$= |a|\int_0^{2\pi}\sqrt{(t\cos t)^2+(t\sin t)^2}\,dt$$

$$= |a|\int_0^{2\pi}t\,dt = 2\pi^2|a|$$

$$(6)S=\int_0^{2\pi}\sqrt{r^2+(r')^2}\,d\theta=\int_0^{2\pi}\sqrt{a^2(1+\cos\theta)^2+a^2\sin^2\theta}\,d\theta$$

$$= 4a\int_0^{\pi}\sqrt{\frac{1+\cos\theta}{2}}\,d\theta = 4a\int_0^{\pi}\cos\frac{\theta}{2}\,d\theta = 8a$$

2.

$$(1)\int_a^b |f(x)-g(x)|dx.$$

面积$A\geq 0$，且$f(x)$、$g(x)$的大小无法确定

故面积为$\int_a^b |f(x)-g(x)|dx$

$$(2)\pi\int_a^b |f^2(x)-g^2(x)|dx$$

在区间$[a,b]$上，由曲线$y=f(x)$，y

$= g(x)$所围成的平面绕 x 轴旋转

一周所成的旋转体的体积微元为

$$dV=\pi|f^2(x)-g^2(x)|dx$$

$$\therefore V=\pi\int_a^b|f^2(x)-g^2(x)|dx$$

3.

(1)由$\begin{cases}y=x^2\\x+y=2\end{cases}$得$x=1$或$x=-2$

$$S=\int_{-2}^{1}(2-x-x^2)d\,x=\left(2x-\frac{1}{2}x^2-\frac{1}{3}x^3\right)\Big|_{-2}^{1}=\frac{9}{2}$$

$$(2)S=\int_{\frac{1}{e}}^{1}|ln\,x|dx+\int_1^e ln\,x\cdot dx$$

$$=\int_1^{\frac{1}{e}}ln\,x\,dx+\int_1^e ln\,x\,dx$$

$$=(x\,ln\,x-x)|_1^{\frac{1}{e}}+(x\,ln\,x-x)|_1^e$$

$$=2-\frac{2}{e}$$

(3)令$x=a\sin t$，$y=b\cos t$

则$S=4\int_0^a y\,dx=\frac{4b}{a}\int_0^a\sqrt{a^2-x^2}\,dx=\int_a^{4b}\int_0^{\frac{\pi}{2}}a^2\cos^2 t\,dt$

$$=4ab\int_0^{\frac{\pi}{2}}\frac{1+cos\,2\,t}{2}dt=2ab\left[t+\frac{1}{2}sin\,2\,t\right]_0^{\frac{\pi}{2}}=ab\pi$$

$$(4)S=\int_0^1(e^x-e^{-x})dx=[e^x-(-e^{-x})]_0^1$$

$$=e+e^{-1}-2$$

$$(5)S=\int_0^{\frac{\pi}{4}}(cos\,x-sin\,x)dx+\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}(cos\,x-sin\,x)dx$$

$$=(sin\,x+cos\,x)|_0^{\frac{\pi}{4}}+(-cos\,x-sin\,x)|_{\frac{\pi}{4}}^{\frac{\pi}{2}}=2\sqrt{2}-2$$

(6)由$\begin{cases} y = \frac{1}{2}x^2 \\ x^2 + y^2 = 8 \end{cases}$ 得两曲线的交点为$(-2,2), (2,2)$

则$S_1 = \int_{-2}^{2}\left(\sqrt{8-x^2} - \frac{1}{2}x^2\right)dx = 2\int_{0}^{2}\left(\sqrt{8-x^2} - \frac{1}{2}x^2\right)dx$

$= 2\left[4\,arcsin\frac{x}{\sqrt{8}} + \frac{1}{2}x\sqrt{8-x^2} - \frac{1}{6}x^3\right]\Big|_0^2$

$= 2\pi + \frac{4}{3}$

$S_2 = S - S_1 = \pi\left(2\sqrt{2}\right)^2 - 2\pi - \frac{4}{3} = 6\pi - \frac{4}{3}$

4.

(1)$S = \frac{1}{2}\int_0^{\frac{\pi}{2}} r^2 d\theta$

$x_2 = 4\int_0^{\frac{\pi}{2}} \frac{cos\,2\,\theta + 1}{2} d\theta$

$= (sin\,2\,\theta + 2x)|_0^{\frac{\pi}{2}} = \pi$

(2)$S = \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} r^2 d\theta = \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2a^2\,cos\,2\,\theta d\theta$

$= \left(\frac{1}{2}a^2\,sin\,2\theta\right)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 4 \times \frac{a^2}{2}sin\,2\theta\Big|_0^{\frac{\pi}{4}} = 2a^2$

(3)$0 \le \mathrm{t} \le 2\pi$，该图形关于$x$轴与$y$轴都对称

$x' = -3a\,cos^2\,t\,sin\,t$

$S = 4\int_0^{\frac{\pi}{2}} |a\,sin^3\,t\,(-3a\,cos^2\,t\,sin\,t)|dt$

$= 12a^2\int_0^{\frac{\pi}{2}} sin^4\,t\,cos^2\,t\,dt = \frac{3}{8}\pi a^2$

5.

(1)两曲线交点为(0,0)与(1,1)的旋转体体积

$$V=\pi\int_0^1\left[\left(\sqrt{x}\right)^2-(x^2)^2\right]d\,x=\pi\int_0^1(x-x^4)d\,x=\frac{3}{10}$$

$$(2)V=\pi\int_{-a}^{a}\left[\left(b+\sqrt{a^2-x^2}\right)^2-\left(b-\sqrt{a^2-x^2}\right)^2\right]dx$$

$$=4\pi b\int_{-a}^{a}\sqrt{a^2-x^2}\,dx=2\pi^2a^2b$$

$$(3)V=\pi\int_{-a}^{a}y^2dx=3\pi a^3\int_0^{\pi}\sin^7 t\cos^2 t\,dt$$

$$=6\pi a^3\int_0^{\frac{\pi}{2}}(\sin^7 t-\sin^9 t)dt=\frac{32}{105}\pi a^3$$

6. 证明：旋转曲面的方程为$\pm\sqrt{y^2+z^2}=f(x)$，由旋转曲面的对称性，取次曲面的上半部分$\Sigma: z=\sqrt{f^2(x)-y^2}$

Σ在 x-O-y 面上的投影区域为

$$D=\{(x,y)\mid -f(x)\le y\le f(x)\quad a\le x\le b\}$$

$$S=2\iint_D\sqrt{1+z_x^2+z_y^2}\,dxdy$$

$$=2\iint_D\sqrt{1+\left[\frac{f(x)f'(x)}{\sqrt{f^2(x)-y^2}}\right]^2+\left[\frac{-y}{\sqrt{f^2(x)-y^2}}\right]^2}\,dxdy$$

$$=2\int_a^b f(x)\sqrt{1+[f'(x)]^2}\,dx\int_{-f(x)}^{f(x)}\frac{1}{\sqrt{f^2(x)-y^2}}dy$$

$$=2\int_a^b f(x)\sqrt{1+[f'(x)]^2}\cdot\left[arcsin\frac{y}{f(x)}\right]_{-f(x)}^{f(x)}dx$$

$$=2\pi\int_a^b f(x)\sqrt{1+[f'(x)]^2}\,dx$$

$(1) y' = 2x^{-\frac{1}{2}}$

$$S = 2\pi \int_0^1 2\,x^{\frac{1}{2}} \sqrt{1 + \left(2x^{-\frac{1}{2}}\right)^2}\, dx$$

$$= 4\pi \int_0^1 \sqrt{x+4}\, dx = 4\pi \times \frac{2}{3}(x+4)^{\frac{3}{2}}\Big|_0^1$$

$$= \frac{8\pi}{3}\left(5\sqrt{5} - 8\right)$$

$$(2) y' = \frac{dy}{dx} = \frac{3a\sin^2 t\cos t}{-3a\cos^2 t\sin t} = -\tan t$$

由对称性知

$$S = 2\int_0^a 2\,\pi y \cdot \sqrt{1+(y')^2}dx$$

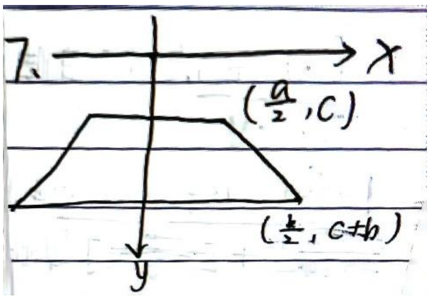
$$= -4\pi \int_0^{\frac{\pi}{2}} a\sin^3 t\sqrt{1+\tan^2 t}\,(-3a\cos^2 t\sin t)dt$$

$$= 12a^2\pi \int_0^{\frac{\pi}{2}} \sin^4 t\sec t\cos^2 t\, dt$$

$$= 12a^2\pi \int_0^{\frac{\pi}{2}} \sin^4 t\, d\sin t$$

$$= \frac{12}{5}a^2\pi$$

7.



如图，AB 的方程为$y = \frac{2h}{b-a}\left(x - \frac{a}{2}\right) + c$对于薄板上每一点$(x, y)$的压力$dF = \rho g y \cdot x dy$

由对称性可知

$$P=2\int_{c}^{c+h} d\,F=\int_{c}^{c+h}\left[a+\frac{b-a}{h}(y-c)\right]\rho g y\,dy$$

$$=\frac{1}{6}\rho g h(3ac+3bc+ab+2bh)$$

8. 球的密度与水相同⟹球在水中移动时不做功，x 为积分变量，x∈[0,2r]。把球体分为很多薄层，将相应于$[x,x+dx]$的那一层球体抬到水面时不做功，从离开水面时开始做功且 x−O−y 面上方圆的方程为$(x-r)^2+y^2=r^2$，可知，将相应于$[x,x+dx]$的那一薄层球体提升到$[x-2r,x+dx-2r]$位置时所做的功微元为(ρ为水密度)

$$dW=\rho g(2r-x)\pi y^2dx=\rho g\pi(2r-x)[r^2-(x-r)^2]dx$$

$$=\rho g\pi(2r-x)(2rx-x^2)dx=\rho g\pi(x^3-4rx^2+4r^2x)dx$$

故$W=\int_{0}^{2r} dw=\rho g\pi\int_{0}^{2r}(x^3-4rx^2+4r^2x)dx$

$$=\rho g\pi\left(\frac{x^4}{4}-\frac{4rx^3}{3}+2r^2x^2\right)\Bigg|_0^{2r}=\frac{4}{3}\rho g\pi r^4$$