6.6.

(1). $S=\int_0^4 \sqrt{1+(y')^2}\,dx = \int_0^4 \sqrt{1+\frac{9}{4}x}\,dx = \left[\frac{4}{9}\times\frac{2}{3}\times(1+\frac{9}{4}x)^{\frac{3}{2}}\right]_0^4$

$=\frac{4}{9}\times\frac{2}{3}\times\left[(1+\frac{9}{4}\times 4)^{\frac{3}{2}}-1\right]=\frac{8}{27}(10^{\frac{3}{2}}-1)$

(2). $x'=\frac{y}{2}-\frac{1}{2y}$

$S=\int_1^e \sqrt{1+(x')^2}\,dy = \int_1^e \sqrt{1+(\frac{y}{2}-\frac{1}{2y})^2}\,dy$

$=\int_1^e \frac{1}{2}(y+\frac{1}{y})dy = \frac{1}{2}\left[\frac{1}{2}y^2+\ln y\right]_1^e$

$=\frac{1}{2}\times(\frac{1}{2}e^2+1-\frac{1}{2}) = \frac{1}{4}(e^2+1)$

(3). 由题目可知，$x\geq 0$，$y\geq 0$.

$y=(1-\sqrt{x})^2$ $\quad y'=\frac{dy}{dx}=2(1-\sqrt{x})(-\frac{1}{2\sqrt{x}})=1-\frac{1}{\sqrt{x}}$.

$S=\int_0^1 \sqrt{1+(y')^2}\,dx = \int_0^1 \sqrt{1+(1-\frac{1}{\sqrt{x}})^2}\,dx$

$=2\int_0^1 \sqrt{2x-2\sqrt{x}+1}\,d\sqrt{x} = 1+\frac{\sqrt{2}}{2}\ln(1+\sqrt{2})$.

(4). 设 $x=a\cos^3 t \quad y=a\sin^3 t \ (0\le t\le 2\pi)$

$$S=4\int_0^{\frac{\pi}{2}}\sqrt{[(a\cos^3 t)']^2+[(a\sin^3 t)']^2}\,dt$$

$$=4a\int_0^{\frac{\pi}{2}}\sqrt{(-3\cos^2 t\sin t)^2+(3\sin^2 t\cos t)^2}\,dt$$

$$=12a\int_0^{\frac{\pi}{2}}\sin t\cos t\,dt=6a.$$

(5). $$S=\int_0^{2\pi}\sqrt{[(a(\cos t+t\sin t))']^2+[(a(\sin t-t\cos t))']^2}\,dt$$

$$=|a|\int_0^{2\pi}\sqrt{(t\cos t)^2+(t\sin t)^2}\,dt$$

$$=|a|\int_0^{2\pi}t\,dt=2\pi^2|a|$$

(6). $$S=\int_0^{2\pi}\sqrt{r^2+(r')^2}\,d\theta=\int_0^{2\pi}\sqrt{a^2(1+\cos\theta)^2+a^2\sin^2\theta}\,d\theta$$

$$=4a\int_0^{\pi}\sqrt{\frac{1+\cos\theta}{2}}\,d\theta=4a\int_0^{\pi}\cos\frac{\theta}{2}\,d\theta=8a$$

2.(1). $\int_a^b |f(x)-g(x)|dx$.

面积 $A\geq 0$，且 $f(x)$、$g(x)$ 的大小无法确定

故面积为 $\int_a^b |f(x)-g(x)|dx$.

(2). $\pi\int_a^b |f^2(x)-g^2(x)|dx$.

在区间 $[a,b]$ 上，由曲线 $y=f(x)$，$y=g(x)$ 所围成的平面绕 x 轴旋转一周所成的旋转体的体积微元为

$dV=\pi|f^2(x)-g^2(x)|dx$

$\therefore V=\pi\int_a^b |f^2(x)-g^2(x)|dx$.

3. (1) 由 $\begin{cases} y=x^2 \\ x+y=2 \end{cases}$ 得 $x=1$ 或 $x=-2$.

$S=\int_{-2}^{1}(2-x-x^2)dx=(2x-\frac{1}{2}x^2-\frac{1}{3}x^3)\Big|_{-2}^{1}=\frac{9}{2}$

(2). $S=\int_{\frac{1}{e}}^{1}|\ln x|dx+\int_1^e \ln x\,dx$.

~~$=\int_1^{\frac{1}{e}}\ln x\,dx+\int_1^e \ln x\,dx$~~

$=(x\ln x-x)\Big|_{1}^{\frac{1}{e}}+(x\ln x-x)\Big|_1^e$

$=2-\frac{2}{e}$

(3). 令 $x=a\sin t$　$y=b\cos t$.

则 $S=4\int_0^{a}y\,dx=\frac{4b}{a}\int_0^a\sqrt{a^2-x^2}\,dx=\frac{4b}{a}\int_0^{\frac{\pi}{2}}a^2\cos^2t\,dt$

$=4ab\int_0^{\frac{\pi}{2}}\frac{1+\cos 2t}{2}dt=2ab[t+\frac{1}{2}\sin 2t]_0^{\frac{\pi}{2}}=ab\pi$

(4). $S=\int_0^1 (e^x-e^{-x})dx=[e^x-(-e^{-x})]_0^1$

$=e+e^{-1}-2.$

(5). $S=\int_0^{\frac{\pi}{4}}(\cos x-\sin x)dx+\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}(\sin x-\cos x)dx$

$=(\sin x+\cos x)\Big|_0^{\frac{\pi}{4}}+(-\cos x-\sin x)\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}}=2\sqrt{2}-2$

(6). 由$\begin{cases} y=\frac{1}{2}x^2 \\ x^2+y^2=8 \end{cases}$ 得两曲线的交点为 $(-2,2)$，$(2,2)$

则 $S_1=\int_{-2}^2(\sqrt{8-x^2}-\frac{1}{2}x^2)dx=2\int_0^2(\sqrt{8-x^2}-\frac{1}{2}x^2)dx$

$=2[4\arcsin\frac{x}{\sqrt{8}}+\frac{1}{2}x\sqrt{8-x^2}-\frac{1}{6}x^3]\Big|_0^2$

$=2\pi+\frac{4}{3}.$

$S_2=S-S_1=\pi(2\sqrt{2})^2-2\pi-\frac{4}{3}=6\pi-\frac{4}{3}.$

4. (1). $S=\frac{1}{2}\int_0^{\frac{\pi}{2}} r^2 d\theta \times 2 = 4\int_0^{\frac{\pi}{2}} \frac{\cos 2\theta+1}{2} d\theta$

$= (\sin 2\theta + 2x)\Big|_0^{\frac{\pi}{2}} = \pi$

(2). $S = \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} r^2 d\theta = \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2a^2\cos 2\theta d\theta$

$=(\frac{1}{2}a^2\sin 2\theta)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 4\times\frac{a^2}{2}\sin 2\theta\Big|_0^{\frac{\pi}{4}} = 2a^2$

(3) $0\le t\le 2\pi$，该图形关于x轴与y轴都对称

$x'=-3a\cos^2 t\sin t$

$S=4\int_0^{\frac{\pi}{2}} |a\sin^3 t(-3a\cos^2 t\sin t)|dt$

$=12a^2\int_0^{\frac{\pi}{2}}\sin^4 t\cos^2 t dt = \frac{3}{8}\pi a^2$

5. (1) 两曲线交点为(0,0)与(1,1)，旋转体的体积

$V=\pi\int_0^1[(\sqrt{x})^2-(x^2)^2]dx = \pi\int_0^1(x-x^4)dx = \frac{3}{10}\pi$

(2). $V=\pi\int_{-a}^{a}[(b+\sqrt{a^2-x^2})^2-(b-\sqrt{a^2-x^2})^2]dx$

$=4\pi b\int_{-a}^{a}\sqrt{a^2-x^2}dx = 2\pi^2a^2b$

(3). $V=\pi\int_{-a}^{a}y^2dx = 3\pi a^3\int_0^{\pi}\sin^7 t\cos^2 t dt$

$=6\pi a^3\int_0^{\frac{\pi}{2}}(\sin^7 t-\sin^9 t)dt = \frac{32}{105}\pi a^3$

6、~~七#~~证明：旋转曲面的方程为 $\pm\sqrt{y^2+z^2}=f(x)$，由旋转曲面的对称性，取此曲面的上半部分 $\Sigma: z=\sqrt{f^2(x)-y^2}$

Σ 在 xOy 面上的投影区域为 $D=\{(x,y)\mid -f(x)\le y\le f(x),\ a\le x\le b\}$

$$\therefore S=2\iint_D\sqrt{1+z_x^2+z_y^2}\,dxdy=2\iint_D\sqrt{1+\left[\frac{f(x)f'(x)}{\sqrt{f^2(x)-y^2}}\right]^2+\left[\frac{-y}{\sqrt{f^2(x)-y^2}}\right]^2}\,dxdy$$

$$=2\int_a^b f(x)\sqrt{1+[f'(x)]^2}\,dx\int_{-f(x)}^{f(x)}\frac{1}{\sqrt{f^2(x)-y^2}}\,dy$$

$$=2\int_a^b f(x)\sqrt{1+[f'(x)]^2}\cdot\left[\arcsin\frac{y}{f(x)}\right]_{-f(x)}^{f(x)}dx$$

$$=2\pi\int_a^b f(x)\sqrt{1+[f'(x)]^2}\,dx$$

(1). $y'=2x^{-\frac{1}{2}}$

$$S=2\pi\int_0^1 2x^{\frac{1}{2}}\sqrt{1+(2x^{-\frac{1}{2}})^2}\,dx$$

$$=4\pi\int_0^1\sqrt{x+4}\,dx=4\pi\times\frac{2}{3}(x+4)^{\frac{3}{2}}\Big|_0^1$$

$$=\frac{8\pi}{3}(5\sqrt{5}-8)$$

(2). $y'=\dfrac{dy}{dx}=\dfrac{3a\sin^2t\cos t}{-3a\cos^2t\sin t}=-\tan t$

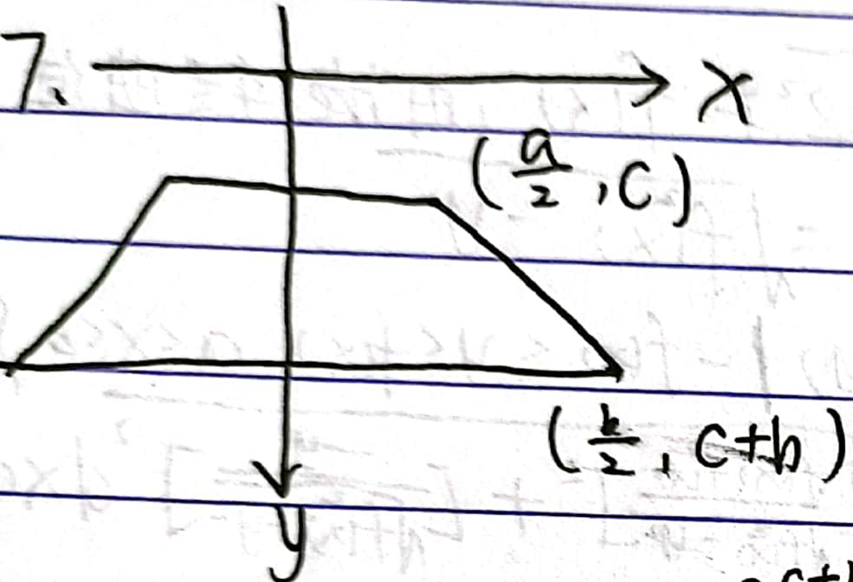
由对称性可知，$S=2\int_0^a 2\pi y\sqrt{1+(y')^2}\,dx$

$$=-4\pi\int_0^{\frac{\pi}{2}}a\sin^3t\sqrt{1+\tan^2t}\,(-3a\cos^2t\sin t)dt$$

$$=12a^2\pi\int_0^{\frac{\pi}{2}}\sin^4t\sec t\cos^2t\,dt$$

$$=12a^2\pi\int_0^{\frac{\pi}{2}}\sin^4t\,d\sin t$$

$$=\frac{12}{5}a^2\pi$$

7. 

如图，AB的方程为 $y=\frac{2h}{b-a}(x-\frac{a}{2})+c$

对于薄板上每一点 (x,y) 的压力

$dF=\rho g y\cdot x dy$

由对称性可知.

$$P=2\int_c^{c+h}dF=\int_c^{c+h}\left[a+\frac{b-a}{h}(y-c)\right]\rho g y\,dy$$

$$=\frac{1}{6}\rho g h(3ac+3bc+ab+2bh)$$

8. 球的密度与水相同，球在水中移动时不做功，x为积分变量，$x\in[0,2r]$

把球体分为很多薄层，将相应于$[x, x+dx]$的那一层球体抬到水面时不做功，从离开水面时开始做功，且xOy面上方圆的方程为$(x-r)^2+y^2=r^2$，可知，将相应于$[x, x+dx]$的那一薄层球体提升到$[x-2r, x+dx-2r]$的位置时所做的功微元为（ρ为水密度）$dW=\rho g(2r-x)\pi y^2dx=\rho g\pi(2r-x)[r^2-(x-r)^2]dx$

$$=\rho g\pi(2r-x)(2rx-x^2)dx=\rho g\pi(x^3-4rx^2+4r^2x)dx$$

故 $$W=\int_0^{2r}dW=\rho g\pi\int_0^{2r}(x^3-4rx^2+4r^2x)dx$$

$$=\rho g\pi\left(\frac{x^4}{4}-\frac{4rx^3}{3}+2r^2x^2\right)\Big|_0^{2r}=\frac{4}{3}\rho g\pi r^4$$