习题 2.1

1. 利用数列极限定义证明下列各式

(1) $\lim\limits_{n\to\infty}\frac{n}{n+1}=1$

对于 $\forall\varepsilon>0$, 要使 $\left|\frac{n}{n+1}-1\right|=\frac{1}{n+1}<\varepsilon$

只需要 $\frac{1}{n}<\varepsilon$ 即 $n>\frac{1}{\varepsilon}$

取 $N=\left[\frac{1}{\varepsilon}\right]$，则当 $n>N$ 时

有 $\left|\frac{n}{n+1}-1\right|=\frac{1}{n+1}<\frac{1}{n}<\varepsilon$

由定义知 $\lim\limits_{n\to\infty}\frac{n}{n+1}=1$

(2) $\lim\limits_{n\to\infty}\left[1+\frac{(-1)}{n}\right]^n=1$

对于 $\forall\varepsilon>0$ 要使 $\left|1+\frac{(-1)^n}{n}-1\right|=\left|\frac{(-1)^n}{n}\right|<\varepsilon$

即使 $\frac{1}{n}<\varepsilon$ 即 $n>\frac{1}{\varepsilon}$

取 $N=\left[\frac{1}{\varepsilon}\right]+1$，则当 $n>N$ 时

有 $\left|1+\frac{(-1)^n}{n}-1\right|<\varepsilon$

由定义知 $\lim\limits_{n\to\infty}\left[1+\frac{(-1)}{n}\right]^n=1$

(3) $\lim\limits_{n\to\infty}\frac{1}{\sqrt{n+1}}=0$

对于 $\forall\varepsilon>0$ 要使 $\left|\frac{1}{\sqrt{n+1}}-0\right|=\frac{1}{\sqrt{n+1}}<\varepsilon$

只需要 $\frac{1}{\sqrt{n}}<\varepsilon$ 即 $n>\frac{1}{\varepsilon^2}$

取 $N=\left[\frac{1}{\varepsilon^2}\right]$,则当 $n>N$ 时

有 $\left|\frac{1}{\sqrt{n+1}}-0\right|=\frac{1}{\sqrt{n+1}}<\frac{1}{\sqrt{n}}<\varepsilon$

由定义知：$\lim\limits_{n\to\infty}\frac{1}{\sqrt{n+1}}=0$

(4) $\lim\limits_{n\to\infty}\frac{1}{n^\alpha}=0$($\alpha$ 为正常数)

对于 $\forall\varepsilon>0$,要使 $\left|\frac{1}{n^\alpha}-0\right|=\frac{1}{n^\alpha}<\varepsilon$

只需要 $n>\sqrt[\alpha]{\frac{1}{\varepsilon}}$

取 $N=\left[\sqrt[\alpha]{\frac{1}{\varepsilon}}\right]+1$,则当 $n>N$

有 $\left|\frac{1}{n^\alpha}-0\right|<\varepsilon$

由定义知 $\lim\limits_{n\to\infty}\frac{1}{n^\alpha}=0$

2、

(1)若 $\lim\limits_{n\to\infty}a_n=a$，其中 $a\neq 0$，则 $\lim\limits_{n\to\infty}|a_n|=|a|$；问反之是否成立

因为 $\lim\limits_{n\to\infty}a_n=a$，$\forall\varepsilon>0,\exists N_0$

当 $n>N_0$时，$|a_n-a|<\varepsilon$

则$\forall\varepsilon>0$，取 $N=N_0$，当 $n>N$ 时，$\big||a_n|-|a|\big|<|a_n-a|<\varepsilon$

因此 $\lim\limits_{n\to\infty}|a_n| = |a|$

反之不成立，如：$a_n = (-1)^n$

$\lim\limits_{n\to\infty}|a_n| = \lim\limits_{n\to\infty}|(-1)^n| = 1$

但 $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty}(-1)^n \neq 1$

(2)试证明 $\lim\limits_{n\to\infty} a_n = 0$ 当且仅当 $\lim\limits_{n\to\infty}|a_n| = 0$

$\because \lim\limits_{n\to\infty} a_n = 0, \forall\varepsilon > 0, \exists N$

当 $n > N$ 时, $|a_n - 0| < \varepsilon$

则 $\forall\varepsilon > 0, \exists N$, 当 $n > N$ 时, $\big||a_n| - 0\big| < \varepsilon$

$\therefore \lim\limits_{n\to\infty}|a_n| = 0$

而当 $\lim\limits_{n\to\infty}|a_n| = 0, \forall\varepsilon > 0, \exists N$

当 $n > N$ 时, $\big||a_n| - 0\big| < \varepsilon$

则 $\forall\varepsilon > 0, \exists N$, 当 $n > N$ 时, $|a_n - 0| < \varepsilon$

$\therefore \lim\limits_{n\to\infty} a_n = 0$

综上所述，$\lim\limits_{n\to\infty} a_n = 0$ 当且仅当 $\lim\limits_{n\to\infty}|a_n| = 0$

3. 求下列极限

(1) $\lim\limits_{n\to\infty}\dfrac{3n^5 - 4n^3 + 5n}{n^6 + 4n + 1}$

解：原式 $= \lim\limits_{n\to\infty}\dfrac{\frac{3}{n} - \frac{4}{n^3} + \frac{5}{n^5}}{1 + \frac{4}{n^5} + \frac{1}{n^6}} = \dfrac{3\times 0 - 4\times 0 + 5\times 0}{1 + 4\times 0 + 0} = 0$

(2) $\lim\limits_{n\to\infty}\dfrac{n^3 + 3n^2 + 1}{n^3 + 1}$

解：原式 $= \lim\limits_{n\to\infty}\dfrac{1 + \frac{3}{n} + \frac{1}{n^3}}{1 + \frac{1}{n^3}} = \dfrac{1 + 3\times 0 + 0}{1 + 0} = 1$

(3) $\lim\limits_{n\to\infty}\dfrac{(-2)^n+3^n}{(-2)^{n+1}+3^{n+1}}$

解：原式 $=\lim\limits_{n\to\infty}\dfrac{\frac{1}{3}\left(\frac{-2}{3}\right)^n+\frac{1}{3}}{\left(\frac{-2}{3}\right)^{n+1}+1}=\dfrac{\frac{1}{3}\times 0+\frac{1}{3}}{0+1}=\dfrac{1}{3}$

(4) $\lim\limits_{n\to\infty}\dfrac{1}{n^2}(1+2+\cdots+n)$

解：原式 $=\lim\limits_{n\to\infty}\dfrac{\frac{n(n+1)}{2}}{n^2}=\lim\limits_{n\to\infty}\dfrac{n^2+n}{2n^2}=\lim\limits_{n\to\infty}\dfrac{1+\frac{1}{n}}{2}=\dfrac{1}{2}$

(5) $\lim\limits_{n\to\infty}\left[\dfrac{1}{1\cdot 2}+\dfrac{1}{2\cdot 3}+\cdots+\dfrac{1}{(n-1)n}\right]$

解：原式 $=\lim\limits_{n\to\infty}\left(1-\dfrac{1}{2}+\dfrac{1}{2}-\dfrac{1}{3}+\cdots+\dfrac{1}{n-1}-\dfrac{1}{n}\right)$

$=\lim\limits_{n\to\infty}\left(1-\dfrac{1}{n}\right)=1$

(6) $\lim\limits_{n\to\infty}\left(\dfrac{1+2+\cdots+n}{n+2}-\dfrac{n}{2}\right)$

解：原式 $=\lim\limits_{n\to\infty}\left(\dfrac{\frac{n(n+1)}{n}}{n+2}-\dfrac{n}{2}\right)=\lim\limits_{n\to\infty}\dfrac{-n}{2n+4}=-\dfrac{1}{2}$

(7) $\lim\limits_{n\to\infty}(1+2+3+\cdots+K)^{\frac{1}{n}}$，$\left(K\text{ 为正整数}\right)$

解：$\because \lim\limits_{n\to\infty}a^{\frac{1}{n}}=1$

$\therefore$ 原式 $=\lim\limits_{n\to\infty}\left[\dfrac{k(k+1)}{2}\right]^{\frac{1}{h}}=1$

(8) $\lim\limits_{n\to\infty}\left(\sqrt{(n+1)(n+2)}-n\right)$

解：原式 $=\lim\limits_{n\to\infty}\dfrac{(n+1)(n+2)-n^2}{\sqrt{(n+1)(n+2)}+n}=\lim\limits_{n\to\infty}\dfrac{(1+2)n+2}{\sqrt{(n+1)(n+2)}+n}$

$=\lim\limits_{n\to\infty}\dfrac{3+\frac{2}{n}}{\sqrt{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}+1}=\dfrac{3}{2}$

4. 利用单调有界原理求下列数列的极限

(1)$a_1=\frac{1}{5}$，$a_{n+1}=\frac{n}{3n+2}a_n$，$n=1,2,3,\cdots$

解：$\because 0<\frac{a_{n+1}}{a_n}=\frac{n}{3n+2}<1$

$a_1=\frac{1}{5}$，由数学归纳法知：$a_n>0$

又$\frac{a_{n+1}}{a_n}<1$，则$\{a_n\}$是单调递减的且有下界 0

$\therefore \{a_n\}$ 有极限

对 $a_{n+1}=\frac{n}{3n+2}a_n$两边同时取极限

则 $\lim\limits_{n\to\infty}a_{n+1}=\lim\limits_{n\to\infty}a_n=\frac{1}{3}\lim\limits_{n\to\infty}a_n\Rightarrow\lim\limits_{n\to\infty}a_n=0$

(2)$a_1=\sqrt{2},a_{n+1}=\sqrt{2+a_n},n=1,2,\ldots$

解：$\because a_1=\sqrt{2}\quad a_{n+1}=\sqrt{2+a_n}$

由数学归纳法知：$a_{n+1}>a_n$

$\therefore \{a_n\}$单调递增

则$a_{n+1}=\sqrt{2+a_n}>a_n\Rightarrow a_n<2$

且$\{a_n\}$有界 $\therefore \{a_n\}$有极限

两边取极限, $\lim\limits_{n\to\infty}a_{n+1}=\lim\limits_{n\to\infty}\sqrt{2+a_n}$

则 $\lim\limits_{n\to\infty}a_n=2$

5. 利用夹逼定理求下列极限

(1) $\lim\limits_{n\to\infty}(1+2^n+3^n+4^n)^{\frac{1}{n}}$

解：$4^n\le 1+2^n+3^n+4^n\le 4\cdot 4^n$

$(4^n)^{\frac{1}{n}} \le (1+2^n+3^n+4^n)^{\frac{1}{n}} \le (4\cdot 4^n)^{\frac{1}{n}}$

又 $\lim\limits_{n\to\infty}(4^n)^{\frac{1}{n}} = \lim\limits_{n\to\infty}(4\cdot 4^n)^{\frac{1}{n}} = 4$

则由夹逼定理知：$\lim\limits_{n\to\infty}(1+2^n+3^n+4^n)^{\frac{1}{n}} = 4$

(2) $\lim\limits_{n\to\infty}[(n+1)^\alpha - n^\alpha]$, 其中常数 $\alpha \in (0,1)$

解：$0 \le (n+1)^\alpha - n^\alpha = n^\alpha\left[\left(1+\frac{1}{n}\right)^\alpha - 1\right]$

$\le n^\alpha\left[\left(1+\frac{1}{n}\right)^1 - 1\right] = n^\alpha \cdot \frac{1}{n} = \frac{1}{n^{1-\alpha}}$

又 $\lim\limits_{n\to\infty}\left(\frac{1}{n^{1-\alpha}}\right) = 0$

$\therefore \lim\limits_{n\to\infty}[(n+1)^\alpha - n^\alpha] = 0$

6. 试用子列证明下列数列发散

(1)$a_n = (-1)^n \frac{n}{n+1}$

证明：$a_{2k-1} = (-1)\cdot\frac{2k-1}{2k}$　$a_{2k} = \frac{2k}{2k+1}$

$\because \lim\limits_{k\to\infty} a_{2k-1} = -1$，$\lim\limits_{k\to\infty} a_{2k-1} \ne \lim\limits_{k\to\infty} a_{2k}$，$\lim\limits_{k\to\infty} a_{2k} = 1$

$\therefore \{a_n\}$ 发散

(2)$a_n = 2+(-1)^n$

证明：$a_{2k-1} = 2-1 = 1$　$a_{2k} = 2+1 = 3$

$\because \lim\limits_{k\to\infty} a_{2k-1} \ne \lim\limits_{k\to\infty} a_{2k}$

$\therefore \{a_n\}$ 发散

(3) $\lim\limits_{n\to\infty}\left(\frac{1}{n}-\frac{2}{n}+\frac{3}{n}-\frac{4}{n}+\cdots+\frac{(-1)^{n-1}n}{n}\right)$

证明：令$a_n = \frac{1}{n}-\frac{2}{n}+\frac{3}{n}-\frac{4}{n}+\cdots+\frac{(-1)^{n-1}n}{n}$

则$a_{2k}=\frac{1-2+3-4+\cdots-2k}{2k}=\frac{-k}{2k}=-\frac{1}{2}$

$a_{2k+1}=\frac{1-2+3-4+\cdots+2k+1}{2k+1}=\frac{k+1}{2k+1}$

$\lim\limits_{k\to\infty}a_{2k}=-\frac{1}{2}\quad \lim\limits_{k\to\infty}a_{2k+1}=\lim\limits_{k\to\infty}\frac{k+1}{2k+1}=\frac{1}{2}$

$\therefore\{a_n\}$ 发散

7. 试证明：对于数列$\{a_n\}$，$\lim\limits_{n\to\infty}a_n=a$的充要条件是$\{a_n\}$的奇子列和偶子列均收敛于 a，即$\lim\limits_{k\to\infty}a_{2k-1}=\lim\limits_{k\to\infty}a_{2k}=a$

证明： $\because\lim\limits_{n\to a}a_n=a$

则 $\forall\varepsilon>0,\exists N>0,\forall n\geq N$ 得 $|a_n-a|<\varepsilon$, 当 $k>N$ 时,

$n_k\geq K>N$

则 $|a_{nk}-a|<\varepsilon$ 即 $\lim\limits_{n\to\infty}a_{nk}=a$

$\therefore\lim\limits_{k\to\infty}a_{2k-1}=\lim\limits_{k\to\infty}a_{2k}=a$

又$\{a_{2k-1}\}$、$\{a_{2k}\}$包含了$\{a_n\}$的所有项

$\therefore\lim\limits_{n\to\infty}a_n=a$

则 $\lim\limits_{k\to\infty}a_{2k-1}=\lim\limits_{k\to\infty}a_{2k}=a\Leftrightarrow\lim\limits_{n\to\infty}a_n=a$

8. 利用柯西收敛准则证明下列数列是收敛的

$(1)a_n=\frac{\sin 1}{1^2}+\frac{\sin 2}{2^2}+\cdots+\frac{\sin n}{n^2}$

证明：令 $n>m$

则$|a_n-a_m|=\left|\frac{\sin\ (m+1)}{(m+1)^2}+\frac{\sin\ (m+2)}{(m+2)^2}+\cdots+\frac{\sin n}{n^2}\right|$

$<\left|\frac{1}{(m+1)^2}+\frac{1}{(m+2)^2}+\cdots+\frac{1}{n^2}\right|$

$$< \left|\frac{1}{m(m+1)}+\frac{1}{(m+1)(m+2)}+\cdots+\frac{1}{(n-1)n}\right|$$

$$= \left(\frac{1}{m}-\frac{1}{m+1}\right)+\left(\frac{1}{m+1}-\frac{1}{m+2}\right)+\cdots+(\frac{1}{n-1}+\frac{1}{n})$$

$$= \frac{1}{m}-\frac{1}{n}$$

$$< \frac{1}{m}$$

又$\forall \varepsilon < \frac{1}{2}$，存在 $N = \left[\frac{1}{\varepsilon}\right]$，当 $m > N$ 时，$\frac{1}{m} < \varepsilon$

即 $N = \left[\frac{1}{\varepsilon}\right]$，当 $n > m > N$ 时，$|a_n - a_m| < \varepsilon$

由柯西收敛准则，$\{a_n\}$收敛。

$(2)a_n = \frac{\cos 1\,!}{1\cdot 2}+\frac{\cos 2\,!}{2\cdot 3}+\cdots+\frac{\cos n\,!}{n(n+1)}$

证明：令 $m > n$

则

$$|a_m - a_n| < \left|\frac{1}{(m+1)m}+\cdots+\frac{1}{(n+2)(n+3)}+\frac{1}{(n+1)(n+2)}\right|$$

$$= \frac{1}{n+1}-\frac{1}{m+1} < \frac{1}{n+1} < \frac{1}{n}$$

$\forall \varepsilon > 0,$ 取 $N = \left[\frac{1}{\varepsilon}\right]$ 当 $n > N$ 时, $|a_m - a_n| < \varepsilon$

由柯西收敛准则，$\{a_n\}$收敛。

9. 利用柯西收敛准则证明下列数列是发散的

$(1)a_n = 1+\frac{1}{2}+\cdots+\frac{1}{n}$

证明：取 $\varepsilon = \frac{1}{4}, \forall N \in N^{+}$取 $n = N+1, m = 2N+2$

则 有 $n, m > N$

则$|a_m - a_n| = \frac{1}{N+2} + \frac{1}{N+3} + \cdots + \frac{1}{2N+2} > \frac{N+1}{2N+2} = \frac{1}{2} > \varepsilon$

由柯西收敛准则，$\{a_n\}$发散

$(2) a_n = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}}$

证明：取 $\varepsilon = \frac{1}{4}, \forall N \in N^+$取 $n = N+1, m = 2N+2$

则有 $n, m > N$

$$
\begin{aligned}
\text{则 } |a_m - a_n| &= \frac{1}{\sqrt{N+2}} + \frac{1}{\sqrt{N+3}} + \cdots + \frac{1}{\sqrt{2N+2}} \\
&> \frac{1}{N+2} + \frac{1}{N+3} + \cdots + \frac{1}{2N+2} > \frac{N+1}{2N+2} = \frac{1}{2} > \varepsilon
\end{aligned}
$$

由柯西收敛准则，$\{a_n\}$发散