习题 3.3

1. 求函数 y=x^2在点 x=1 的微分，其中自变量 x 的增量Δx分别如下：

$\Delta x = 0.1$；$\Delta x = 0.01$；$\Delta x = 0.001$；

解：由公式可得$d\,y|_{\{x=1\}} = f'(1)\Delta x$，分别代入△x 可得：

dy=0.2；dy=0.02；dy=0.002

2.求下列函数的微分

（1）$y = \frac{x^2-1}{x^2+1}$

解：$y = \frac{x^2+1-2}{x^2+1} = 1 - \frac{2}{x^2+1}$　$\therefore y' = \left(0 - \frac{-2\cdot 2x}{(x^2+1)^2}\right) = \frac{4x}{(x^2+1)^2}$

$$\therefore dy = \frac{4x}{(x^2+1)}dx$$

(2) $y = \tan x + \sec x$

解：$y' = \frac{\sin x}{\cos x} + \frac{1}{\cos x}$　$\therefore y' = \frac{\sin^2 x + \cos^2 x}{\cos^2 x} + \frac{\sin x}{\cos^2 x} = \sec^2 x + \sec x \tan x$

$$\therefore dy = (\sec^2 x + \sec x \tan x)dx$$

(3) $y = \arccos\ \frac{1}{x}$

解：$y' = \frac{-1}{\sqrt{1-\frac{1}{x^2}}} \cdot \left(\frac{1}{-x^2}\right) = \frac{1}{|x|\cdot\sqrt{x^2-1}}$

$\therefore dy = \frac{1}{|x|\cdot\sqrt{x^2-1}}dx$ 且（|x|>1）

（4） $y = \arcsin\sqrt{1-x^2}$

解：$y' = \frac{1}{\sqrt{1-\left(\sqrt{1-x^2}\right)^2}} \cdot \left(\frac{1}{2}\cdot\frac{1}{\sqrt{1-x^2}}\right)\cdot(-2x)$

$$= \frac{1}{\sqrt{x^2}} \cdot \frac{-x}{\sqrt{1-x^2}}$$

$$= \frac{-x}{|x|\sqrt{1-x^2}}$$

$\therefore dy = \frac{-x}{|x|\sqrt{1-x^2}}dx$且（|x|<1）

(5)$y = \arctan\ \frac{x^2-1}{x^2+1}$

解：$y' = \dfrac{1}{1+\left(\dfrac{x^2-1}{x^2+1}\right)^2}\cdot\left(\dfrac{x^2-1}{x^2+1}\right)'$　　由(1)知$\left(\dfrac{x^2-1}{x^2+1}\right)' = \dfrac{4x}{(x^2+1)^2}$

$$= \frac{1}{2x^4+2}\cdot 4x = \frac{2x}{x^4+1}$$

$$\therefore dy = \frac{2x}{x^4+1}dx$$

（6）$y = (x^2+4x+1)\left(x^2-\sqrt{x}\right)$

解：$y' = (2x+4)\left(x^2-\sqrt{x}\right) + (x^2+4x+1)\left(2x-\frac{1}{2\sqrt{x}}\right)$

$$= 2x^3 - 2x^{\frac{3}{2}} + 4x^2 - 4x^{\frac{1}{2}} + 2x^3 - \frac{1}{2}x^{\frac{3}{2}} + 8x^2 - 2x^{\frac{1}{2}} + 2x - 2x^{-\frac{1}{2}}$$

$$= 4x^3 + 12x^2 - \frac{5}{2}x^{\frac{3}{2}} + 2x - 6x^{\frac{1}{2}} - \frac{1}{2}x^{-\frac{1}{2}}$$

$$\therefore dy = \left(4x^3 + 12x^2 - \frac{5}{2}x^{\frac{3}{2}} + 2x - 6x^{\frac{1}{2}} - \frac{1}{2}x^{-\frac{1}{2}}\right)dx$$

3.求下列复合函数的微分

（1）$y = \ln\sqrt{1+x^2}$

解：$y' = \dfrac{1}{\sqrt{1+x^2}}\cdot\dfrac{1}{2\sqrt{1+x^2}}\cdot 2x = \dfrac{x}{1+x^2}$

$$\therefore dy = \frac{x}{1+x^2}dx$$

（2）$y = \arcsin\frac{1}{x}$

解：$y' = \dfrac{1}{\sqrt{1-\left(\frac{1}{x}\right)^2}}\cdot\left(-\dfrac{1}{x^2}\right)$

$$= \sqrt{\frac{x^2}{x^2-1}}\cdot\left(-\frac{1}{x^2}\right)$$

$$= \frac{|x|}{\sqrt{x^2-1}}\cdot\frac{1}{-|x|^2} = \frac{1}{|x|\cdot\sqrt{x^2-1}}$$

$$\therefore dy = \frac{1}{|x|\sqrt{x^2-1}}dx\left(\frac{1}{\sqrt{x^4-x^2}}dx\right)$$

（3）$y = \arctan\sqrt{x}$

解：$y' = \dfrac{1}{1+\left(\sqrt{x}\right)^2}\cdot\dfrac{1}{2\sqrt{x}} = \dfrac{1}{2(1+x)\sqrt{x}}$

$$\therefore dy = \frac{1}{2(1+x)\sqrt{x}}dx$$

（4）$y = e^{\sin x}$

解：$y' = e^{\sin x} \cdot \cos x$

$$\therefore dy = \left(e^{\sin x} \cdot \cos x\right)dx$$

4.求下列各式的近似值

解：由微分定义：$\Delta y = f'(x)\Delta x + o(\Delta x) \quad (\Delta x \to 0)$，即 $f(x + \Delta x) - f(x) \approx f'(x)\Delta x$

$\therefore f(x + \Delta x) \approx f(x) + f'(x)\Delta x$

（1）令 $\sqrt[3]{1.02} = \sqrt[3]{1 + 0.02}$ $\quad \therefore f(x) = x^{\frac{1}{3}} \quad f'(x) = \frac{1}{3}x^{-\frac{2}{3}} \quad \Delta x = 0.02$

$\therefore f(1.02) \approx f(1) + f'(1) \cdot 0.02 = 1 + \frac{1}{3} \times 0.02 = 1.00666 \ldots \approx 1.007$

（2）令 $\text{In } 1.005 = \ln(1 + 0.005)$ $\quad \therefore f(x) = \ln x$，$f'(x) = \frac{1}{x} \quad \Delta x = 0.005$

$\therefore f(1.005) \approx f(1) + f'(1) \cdot 0.005 = 0 + 1 \times 0.005 = 0.005$