$$\tan\varphi=\frac{\frac{r'(\theta)\tan\theta+r(\theta)}{r'(\theta)-r(\theta)\tan\theta}-\tan\theta}{1+\frac{r'(\theta)\tan\theta+r(\theta)}{r'(\theta)-r(\theta)\tan\theta}\cdot\tan\theta}$$

$$=\frac{r'(\theta)\tan\theta+r(\theta)-\tan\theta\,(r'(\theta)-r(\theta)\tan\theta)}{r'(\theta)-r(\theta)\tan\theta+r'(\theta)\tan^2\theta+r(\theta)\tan\theta}$$

$$=\frac{r'(\theta)\tan\theta+r(\theta)-r'(\theta)\tan\theta+r(\theta)\tan^2\theta}{r'(\theta)+r'(\theta)\tan^2\theta}$$

$$=\frac{r(\theta)+r(\theta)\tan^2\theta}{r'(\theta)+r'(\theta)\tan^2\theta}$$

$$=\frac{r(\theta)\sec^2\theta}{r'(\theta)\sec^2\theta}$$

$$=\frac{r(\theta)}{r'(\theta)}$$ ∴得证

9. 解: $\begin{cases}x=r\cos\theta=e^\theta\cos\theta\\ y=r\sin\theta=e^\theta\sin\theta\end{cases}$

$$k=\frac{dy}{dx}=\frac{e^\theta\sin\theta+e^\theta\cos\theta}{e^\theta\cos\theta-e^\theta\sin\theta}$$

$$=\frac{\sin\theta+\cos\theta}{\cos\theta-\sin\theta}$$

$$k\big|_{\theta=\frac{\pi}{2}}=\frac{1+0}{0-1}=-1$$

$$x_0=r\cos\theta=e^{\frac{\pi}{2}}\times0=0$$

$$y_0=r\sin\theta=e^{\frac{\pi}{2}}\times1=e^{\frac{\pi}{2}}$$

切线方程: $y-e^{\frac{\pi}{2}}=-(x-0)$

$$y=e^{\frac{\pi}{2}}-x$$

习题 6.2

1. (1) $F'(x)=\sqrt{1+x^2}\qquad F'(0)=1$

(2) $F'(x)=2-\frac{1}{\sqrt{x}}<0\Rightarrow x<\frac{1}{4}$ 区间为 $(0,\frac{1}{4})$

(3) $F'(x)=f(e^{-x})\cdot e^{-x}\cdot(-1)-f(x)=-e^{-x}f(e^{-x})-f(x)$

(4) 令 $\int_0^y e^{-t^2}dt+\int_0^x\sin^2t\,dt=F(x)$

$F'(x)=e^{-y^2}y'+\sin^2x=0\Rightarrow y'=-e^{y^2}\sin^2x$

(5) ∵ $[-\pi,\pi]$ 关于原点对称

又 $|\sin x|$ 为偶函数

∴原式 $=2\int_0^\pi\sin x\,dx=-2\cos x\big|_0^\pi=4$

2. (1) 原式 $=\lim\limits_{x\to0}\frac{\cos x^2}{1}=1$

(2) 原式 $=\lim\limits_{x\to0}\frac{2\int_0^x e^t dt\cdot e^x}{xe^{2x^2}}=\lim\limits_{x\to0}\frac{2\int_0^x e^t dt}{xe^{2x^2-x}}$

$=\lim\limits_{x\to0}\frac{2e^x}{e^{2x^2-x}+xe^{2x^2-x}\cdot(4x-1)}=\frac{2\times1}{1+0\times1\times(-1)}=2$

3. (1) $\int_0^1\sqrt{x}(1-\sqrt{x})^2dx=\int_0^1\sqrt{x}(1+x-2\sqrt{x})dx$

$=\int_0^1(\sqrt{x}+x^{\frac{3}{2}}-2x)dx=(\frac{2}{3}x^{\frac{3}{2}}+\frac{2}{5}x^{\frac{5}{2}}-x^2)\big|_0^1$

$=\frac{2}{3}+\frac{2}{5}-1=\frac{1}{15}$

(2) 原式 $=\int_0^1\frac{-(x^2+1)+2}{1+x^2}dx=\int_0^1(-1+\frac{2}{1+x^2})dx$

$=(-x+2\arctan x)\big|_0^1=-1+2\times\frac{\pi}{4}=\frac{\pi}{2}-1$

(3) 令 $1-x=t\Rightarrow x=1-t\qquad dx=-dt\qquad x\big|_0^1\to t\big|_1^0$

原式 $=\int_1^0e^t(-dt)=\int_0^1e^tdt=e^x\big|_0^1=e-1$

(4) 原式 $=\int_0^1\frac{\frac{1}{2}}{\sqrt{1-(\frac{x}{2})^2}}dx=\int_0^1\frac{1}{\sqrt{1-(\frac{x}{2})^2}}d(\frac{1}{2}x)$

$=\arcsin\frac{x}{2}\big|_0^1=\frac{\pi}{6}-0=\frac{\pi}{6}$

(5) 原式 $=\int_{-1}^2\sqrt{2+x}\,d(x+2)=\frac{2}{3}(x+2)^{\frac{3}{2}}\big|_{-1}^2$

$=\frac{2}{3}\times4^{\frac{3}{2}}-\frac{2}{3}=\frac{2}{3}\times7=\frac{14}{3}$

(6) 原式 $=\int_0^\pi\frac{1-\cos2x}{2}dx=\int_0^\pi\frac{1}{2}dx-\frac{1}{4}\int_0^\pi\cos2x\,d(2x)$

$=\frac{x}{2}\big|_0^\pi-(\frac{1}{4}\sin2x)\big|_0^\pi=\frac{\pi}{2}-0=\frac{\pi}{2}$

(7) 原式 $=\int_0^{\frac{\pi}{4}}(\frac{\sin x+\cos x}{\cos x})^2dx=\int_0^{\frac{\pi}{4}}(\tan x+1)^2dx$

$=\int_0^{\frac{\pi}{4}}(\tan^2x+1+2\tan x)dx$

$=\int_0^{\frac{\pi}{4}}(\sec^2x+2\tan x)dx$

$=(\tan x-2\ln|\cos x|)\big|_0^{\frac{\pi}{4}}=1-2\ln\frac{\sqrt{2}}{2}$

$=1+\ln(\frac{\sqrt{2}}{2})^{-2}=1+\ln2$

(8) 原式 $=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sqrt{2\sin^2x}\,dx$

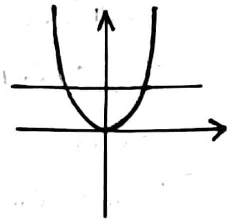
∵ $[-\frac{\pi}{2},\frac{\pi}{2}]$ 关于原点对称，又 $\sqrt{2\sin^2x}$ 为偶函数

∴原式 $=2\int_0^{\frac{\pi}{2}}\sqrt{2}\sin x\,dx=-2\sqrt{2}\cos x\big|_0^{\frac{\pi}{2}}=2\sqrt{2}$

(9) 原式 $=\int_0^1x^2dx+\int_1^2 1dx$

$=\frac{x^3}{3}\big|_0^1+x\big|_1^2$

$=\frac{1}{3}+2-1=\frac{4}{3}$



(10) 原式 $=\int_0^1 0\,dx+\int_1^2\sin x\,dx+\int_2^3 2\sin x\,dx+\int_3^\pi 3\sin x\,dx$

$=-\cos x\big|_1^2-2\cos x\big|_2^3-3\cos x\big|_3^\pi$

$=-\cos2+\cos1-2\cos3+2\cos2+3+3\cos3$

$=\cos1+\cos2+\cos3+3$

4. 解: $\frac{dy}{dx}=\frac{f^2(t)f'(t)}{f(t)f'(t)}=f(t)$

$$\frac{d^2y}{dx^2}=\frac{f'(t)}{f(t)\,f'(t)}=\frac{1}{f(t)}$$

5. 证：$y' = xf(x)$

∵ $f(x)>0$ 当 $x>0$ 时，$y'>0$，$y\uparrow$

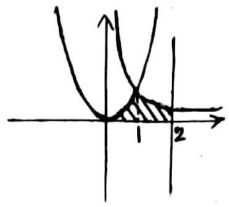
当 $x<0$ 时，$y'<0$，$y\downarrow$

∴ 当 $x=0$ 时，y 取最小值，得证

6. 解：$S=\int_0^1 x^2\,dx+\int_1^2 \frac{1}{x}\,dx$

$=\frac{x^3}{3}\Big|_0^1 + \ln x\Big|_1^2$

$=\frac{1}{3}+\ln 2$



7. (1) 证：$\int_{-\pi}^{\pi}\cos nx\,dx = \int_{-\pi}^{\pi}\frac{1}{n}\cos nx\,d(nx)$

$=\frac{1}{n}\sin(nx)\Big|_{-\pi}^{\pi} = \frac{1}{n}[\sin n\pi - \sin(-n\pi)]=0$ ∴得证

(2) 证：$\int_{-\pi}^{\pi}\sin nx\,dx=\frac{1}{n}\int_{-\pi}^{\pi}\sin nx\,d(nx)$

$=-\frac{1}{n}\cos nx\Big|_{-\pi}^{\pi} = -\frac{1}{n}[\cos n\pi-\cos(-n\pi)]$

$=-\frac{1}{n}(\cos n\pi-\cos n\pi)=0$ ∴得证

(3) 证：$\int_{-\pi}^{\pi}\cos^2 nx\,dx=\int_{-\pi}^{\pi}\frac{1+\cos(2nx)}{2}\,dx$

$=\frac{1}{2}\int_{-\pi}^{\pi}1\,dx+\frac{1}{2}\int_{-\pi}^{\pi}\frac{1}{2n}\cos(2nx)\,d(2nx)$

$=\frac{1}{2}x\Big|_{-\pi}^{\pi}+\frac{1}{4n}\sin(2nx)\Big|_{-\pi}^{\pi}$

$=\frac{\pi}{2}-(\frac{-\pi}{2})+0=\frac{\pi}{2}+\frac{\pi}{2}=\pi$ ∴得证

(4) 证：$\int_{-\pi}^{\pi}\sin^2 nx\,dx=\int_{-\pi}^{\pi}\frac{1-\cos(2nx)}{2}\,dx$

$=\int_{-\pi}^{\pi}\frac{1}{2}\,dx-\int_{-\pi}^{\pi}\frac{1}{4n}\cos(2nx)\,d(2nx)$

$=\frac{x}{2}\Big|_{-\pi}^{\pi}-\frac{1}{4n}\sin(2nx)\Big|_{-\pi}^{\pi}$

$=\frac{\pi}{2}-\frac{-\pi}{2}-0=\frac{\pi}{2}+\frac{\pi}{2}=\pi$ ∴得证

8.(1) 证：$\int_{-\pi}^{\pi}\cos mx\cos nx\,dx=\int_{-\pi}^{\pi}\frac{1}{2}[\cos(m+n)x+\cos(m-n)x]\,dx$

$=\frac{1}{2}\int_{-\pi}^{\pi}\cos(m+n)x\,dx+\frac{1}{2}\int_{-\pi}^{\pi}\cos(m-n)x\,dx$ 积化和差公式

$=\frac{1}{2(m+n)}\int_{-\pi}^{\pi}\cos(m+n)x\,d(m+n)x+\frac{1}{2(m-n)}\int_{-\pi}^{\pi}\cos(m-n)x\,d(m-n)x$

$=\frac{1}{2(m+n)}\sin(m+n)x\Big|_{-\pi}^{\pi}+\frac{1}{2(m-n)}\sin(m-n)x\Big|_{-\pi}^{\pi}$

$=0+0=0$ ∴得证

(2) 证：$\int_{-\pi}^{\pi}\sin mx\sin nx\,dx=\int_{-\pi}^{\pi}[-\frac{1}{2}\cos(m+n)x+\frac{1}{2}\cos(m-n)x]\,dx$

$=-\frac{1}{2}\int_{-\pi}^{\pi}\cos(m+n)x\,dx+\frac{1}{2}\int_{-\pi}^{\pi}\cos(m-n)x\,dx$

$=-\frac{1}{2(m+n)}\int_{-\pi}^{\pi}\cos(m+n)x\,d(m+n)x+\frac{1}{2(m-n)}\int_{-\pi}^{\pi}\cos(m-n)x\,d(m-n)x$

$=-\frac{1}{2(m+n)}\sin(m+n)x\Big|_{-\pi}^{\pi}+\frac{1}{2(m-n)}\sin(m-n)x\Big|_{-\pi}^{\pi}$

$=0+0=0$ ∴得证

(3) 证：$\int_{-\pi}^{\pi}\sin mx\cos nx\,dx=\int_{-\pi}^{\pi}\frac{1}{2}[\sin(m+n)x+\sin(m-n)x]\,dx$

① $m\neq n$ 时 $=\frac{1}{2}\int_{-\pi}^{\pi}\sin(m+n)x\,dx+\frac{1}{2}\int_{-\pi}^{\pi}\sin(m-n)x\,dx$

$=-\frac{1}{2(m+n)}\cos(m+n)x\Big|_{-\pi}^{\pi}-\frac{1}{2(m-n)}\cos(m-n)x\Big|_{-\pi}^{\pi}$

$=-\frac{1}{2(m+n)}[\cos(m+n)\pi-\cos(-(m+n)\pi)]-\frac{1}{2(m+n)}[\cos(m-n)\pi-\cos(-(m-n)\pi)]=0-0=0$

② $m=n$ 时，$\int_{-\pi}^{\pi}\sin mx\cos nx\,dx=\frac{1}{2}\int_{-\pi}^{\pi}\sin 2mx\,dx=0$ 第7.(2)的结论