1.

(1) $\frac{dy}{dx}=\frac{x+1}{y^4+1}$

$(y^4+1)dy=(x+1)dx$

$\int(y^4+1)dy=\int(x+1)dx$

$\frac{1}{2}x^2+x=\frac{1}{5}y^5+y+C$

(2). $\frac{1}{(y-1)^2+1}dy=dx$

$\int\frac{1}{(y-1)^2+1}dy=\int 1\,dx$

$\arctan(y-1)=x+C$

$y-1=\tan(x+c)$

(3). $\frac{1}{1+y}dy=\frac{1}{\tan x}dx$

$\int\frac{1}{1+y}dy=\int\frac{1}{\tan x}dx$

$\ln|1+y|=\ln|\sin x|+C$

$1+y=\pm e^c\cdot\sin x\quad(C\in R)$

$y=C_0\cdot\sin x-1\quad(\pm C_0=e^c)$

(4). $x^2y\,dx-(1+x^2)(1-y^2)dy=0$

$\int(\frac{1}{y}-y)dy=\int(1-\frac{1}{1+x^2})dx\quad(y\neq0)$

$2\ln|y|-y^2=2x-2\arctan x+C$

当$y=0$时，$x^2y\,dx-(1+x^2)(1-y^2)dy=0$成立.

$y=0$也是方程的解.

(5). $\frac{dy}{dx}=\sin\frac{x-y}{2}-\sin\frac{x+y}{2}$

$=\sin\frac{x}{2}\cos\frac{y}{2}-\sin\frac{y}{2}\cos\frac{x}{2}-\sin\frac{x}{2}\cos\frac{y}{2}-\cos\frac{x}{2}\sin\frac{y}{2}$

$=-2\sin\frac{y}{2}\cos\frac{x}{2}$

$\frac{1}{\sin\frac{y}{2}}dy=(-2)\times\cos\frac{x}{2}\,dx\quad(\sin\frac{y}{2}\neq0)$

$2\ln|\tan\frac{y}{4}|=-4\sin\frac{x}{2}+C$ (通解).

当$\sin\frac{y}{2}=0$，$y=2k\pi\ (k\in Z)$时

$\frac{dy}{dx}=\sin\frac{x-y}{2}-\sin\frac{x+y}{2}$成立

$\therefore y=2k\pi\ (k\in Z)$ (通解)

(6) 原方程可化成：

$\tan y\,dy=-\tan x\,dx$

$-\ln|\cos y|=\ln|\cos x|+C$

$\ln|\cos x\cdot\cos y|=-C\quad(C\in R)$

$\cos x\cos y=C'\quad(C'\in R)$

(7). $(1+e^x)y\cdot\frac{dy}{dx}=e^x$

$\int y\cdot dy=\int\frac{e^x}{1+e^x}dx$

$\frac{1}{2}y^2=\ln(1+e^x)+C$ (通解)

代入$y(0)=1\Rightarrow C=-\ln2+\frac{1}{2}$

特解：$\frac{1}{2}y^2=\ln(1+e^x)-\ln2+\frac{1}{2}$

2.

(8) $\cot x\,dy = -\cot y\,dx$

$-\int \tan y\,dy = \int \tan x\,dx$

$-\ln|\cos y| = \ln|\cos x| + C$

$\cos x \cos y = C'$ (通解)

代入 $y(0)=0 \Rightarrow C=1$

$\cos y = \frac{1}{\cos x} = \sec x$ (特解)

(9). $\frac{1}{2}e^{x^2}dx^2 = y(1-y^5)dy$

$\int \frac{1}{2}e^{x^2}dx^2 = \int (1-y^5)dy$

$y - \frac{1}{6}y^6 = \frac{1}{2}e^{x^2} + C$ (通解)

代入 $y(0)=0 \Rightarrow C = -\frac{1}{2}$

$\frac{1}{2}e^{x^2} + \frac{1}{6}y^6 - y = \frac{1}{2}$ (特解)

(10). $\frac{dy}{dx} = \frac{x^2y - y}{y+1}$

$\frac{y(x^2-1)}{y+1} = \frac{dy}{dx}$

$\int (x^2-1)dx = \int (1+\frac{1}{y})dy$

$y + \ln|y| = \frac{1}{3}x^3 - x + C$ (通解)

代入 $y(3)=-1 \Rightarrow C=-7$

$\frac{1}{3}x^3 - x - y - \ln|y| = 7$ (特解)

(1) 设x是关于y的函数

$\frac{dy}{dx} = \frac{2xy}{x^2-y^2}$, $\frac{dx}{dy} = \frac{x}{2y} - \frac{y}{2x}$

设 $\frac{x}{y} = u$

则: $u + y\frac{du}{dy} = \frac{1}{2}(u - \frac{1}{u})$

$\frac{1}{y}dy = \frac{2}{-u-\frac{1}{u}}du$

$\int \frac{1}{y}dy = \int \frac{2u}{-u^2-1}du$

$\ln|y| + C = -\ln(1+u^2)$

$\frac{1}{1+u^2} = Cy$

代入 $u = \frac{x}{y} \Rightarrow x^2 + y^2 = Cy$

(2) 令 $u = \frac{y}{x}$. $\frac{dy}{dx} = u + \frac{du}{dx}\cdot x$

$\frac{dy}{dx} = \frac{2(\frac{y}{x})^4+1}{(\frac{y}{x})^3}$

$u + x\cdot\frac{du}{dx} = \frac{2u^4+1}{u^3}$

$\frac{4}{x}dx = \frac{1}{u+u^3}du$

对式子两边积分.

$4\ln|x| + C = \ln|u^4+1|$

代入 $u = \frac{y}{x}$

得: $y^4 = Cx^8 - x^4$

(3) 令 $u=\frac{y}{x}$

$$y'=\frac{\frac{y}{x}}{1+\sqrt{\frac{y}{x}}}$$

$$u+x\frac{du}{dx}=\frac{u}{1+\sqrt{u}}$$

$$\int\frac{1}{x}dx=\int\left(-\frac{1}{u\sqrt{u}}-\frac{1}{u}\right)du$$

$$\ln|x|+C=\frac{2}{\sqrt{u}}-\ln u$$

代入 $u=\frac{y}{x}$，得：

$$\ln|y|+C=2\sqrt{\frac{x}{y}}$$

(4) 令 $u=\frac{y}{x}$

$$u+x\frac{du}{dx}=\frac{1+u^4+3u^2}{u}$$

$$\frac{1}{x}dx=\frac{1}{2}\cdot\frac{1}{(u^2+1)^2}\,d(u^2+1)$$

对两边积分.

$$\ln|x|+C=-\frac{1}{1+u^2}\times\frac{1}{2}$$

$$\ln|x|+C=-\frac{x^2}{2(x^2+y^2)}.$$

(5) 令 $u=\frac{y}{x}$

$$(1+u\cos u)\,dx=\cos u\,dy$$

$$\left(\frac{1}{\cos u}+u\right)=u+x\frac{du}{dx}$$

$$\int\frac{1}{x}dx=\int\cos u\,du$$

$$\ln|x|+C=\sin\frac{y}{x}.$$

(6) 原方程可化成：

$$\frac{dy}{dx}=\frac{(x-1)-2(y+2)}{(y+2)-2(x-1)}$$

令 $m=y+2$，$n=x-1$，$u=\frac{m}{n}$

$$\frac{dm}{dn}=\frac{dy}{dx}=\frac{n-2m}{m-2n}$$

$$\frac{dy}{dx}=u+(x-1)\frac{du}{dx}=\frac{n-2m}{m-2n}=\frac{1-2u}{u-2}$$

整理得：$\left(\frac{1}{x-1}\right)dx=\left[\frac{u-1}{1-u^2}+\frac{1}{2}\left(\frac{1}{u-1}-\frac{1}{u+1}\right)\right]du$

对两边积分：

$$\ln|1-x|+C=\frac{1}{2}\ln|u-1|-\frac{3}{2}\ln|u+1|$$

代入 $u=\frac{m}{n}$，得：

$$(y-x+3)=C(y+x+1)^3$$

(7). 令 $u=\frac{y}{x}$，原方程可化为：

$$\frac{x^2+2xy-y^2}{x^2-2xy-y^2}=\frac{dy}{dx}$$

$$\frac{1+2u-u^2}{1-2u-u^2}=u+x\cdot\frac{du}{dx}$$

$$\frac{1+u^2+u(u^2+1)}{1-2u-u^2}=x\frac{du}{dx}$$

$$\frac{1-2u-u^2}{1+u^2+u(u^2+1)}\,du=\frac{1}{x}dx$$

(6). 将x看成关于y的函数.

则: $y^3dx=(1-2xy^2)dy$

$$\frac{dx}{dy}=\frac{1}{y^3}-\frac{2}{y}x$$

$$\frac{dx}{dy}+\frac{2}{y}\cdot x=\frac{1}{y^3}$$

当 $x'+\frac{2}{y}x=0$ 时，解得: $x=\frac{C}{y^2}$

~~由~~令得 $x=\frac{C(y)}{y^2}$

$$x'=\frac{C'(y)}{y^2}-\frac{2C(y)}{y^3}$$

代入 $x'+\frac{2}{y}\cdot x=\frac{1}{y^3}$

得 $C'(y)=\frac{1}{y}$

$$\therefore C(y)=\ln|y|+C$$

$\therefore x=(\ln|y|+C)\cdot\frac{1}{y^2}$ (通解).

接2.(7).

①

$\frac{1}{x}dx = \frac{(1-u) - u(1+u)}{(1+u^2)(1+u)} du$

$= \frac{1+u^2-u-u^2}{(1+u^2)(1+u)} du - \frac{u}{1+u^2} du$

$= \frac{1}{1+u} du - \frac{2u}{1+u^2} du$ ②

对①②积分:

$\ln|1+u| - \ln|1+u^2| = \ln|x| + C$

代入 $u = \frac{y}{x}$, 得:

$\frac{y+x}{y^2+x^2} = C$ (通解)

代入 $y(1) = 1 \Rightarrow C = 1$

$\frac{y+x}{y^2+x^2} = 1$ (特解)

(8) 令 $u = \frac{y}{x}$

$y' = \frac{x}{y} + \frac{y}{x}$

$y' = \frac{1}{u} + u$

$u + x\frac{du}{dx} = \frac{1}{u} + u$

$\int \frac{1}{x} dx = \int u du$

$\ln|x| + C = \frac{1}{2}u^2$

2(8)

代入 $y(1) = 1$, $\Rightarrow C = 2$

代入 $u = \frac{y}{x}$

特解: $x^2\ln x^2 + 4x^2 = y^2$

3. 设所求曲线为 $y = y(x)$

由题: $\frac{dy}{dx} \cdot \frac{y}{x} = -1$

$y(0) = +1$

$y dy = -x dx$

积分得: $+x^2 + y^2 = C$

代入 $y(0) = 1 \Rightarrow C = 1$

$\therefore y^2 + x^2 = 1$

4. 由题意: $\frac{dy}{dt} = ky(1000-y)$ k为比例系数

$y^{-1} \times (1000-y)^{-1} \cdot dy = k dt$

$\frac{1}{1000}\left(\frac{1}{y} + \frac{1}{y-1000}\right) dy = k dt$

积分得: $\ln|y| - \ln|1000-y| = 1000kt + C$

$\frac{y}{1000-y} = Ce^{1000kt}$

代入 $y(0) = 100$, $y(3) = 250$

$\Rightarrow \begin{cases} C = \frac{1}{9} \\ 1000k = \frac{\ln 3}{3} \end{cases}$

$\therefore y = \frac{1000 \cdot 3^{\frac{t}{3}}}{9 + 3^{\frac{t}{3}}}$ (始终小于1000尾).

5.

(1). $\frac{dy}{dx}=x(1+2y)$

$\int\frac{1}{1+2y}dy=\int x\,dx$

$\frac{1}{2}\ln(1+2y)=\frac{1}{2}x^2+C\ (C\in R)$

$1+2y=\pm e^{2C}\cdot e^{x^2}\ (\pm e^{2C}\in R)$

$y=\pm\frac{1}{2}e^{2C}\cdot e^{x^2}-\frac{1}{2}\ (\pm\frac{1}{2}e^{2C}\in R)$

$y=C_0e^{x^2}-\frac{1}{2}\ (C_0\in R)$

(2) 当 $y'+y=0$ 解得：$y=Ce^{-x}$

由常数变易法，

$y=C(x)e^{-x}$

$y'=C'(x)e^{-x}-e^{-x}C(x)$

代入 $y'+y=\sin x \Rightarrow C'(x)=e^{x}\sin x$

$C(x)=\int e^{x}\sin x\,dx=\frac{1}{2}e^{x}(\sin x-\cos x)+C$

↳ 两次分部积分，再解方程

方程通解：

$y=\frac{1}{2}(\sin x-\cos x)+Ce^{-x}$

(3). 当 $y'-\frac{2}{x}y=0$ 解得：$y=Cx^2$

由常数变易法：$y=C(x)\cdot x^2$

$y'=C'(x)x^2+2x\cdot C(x)$

代入 $y'-\frac{2}{x}y=\frac{2}{3}x^4$

$C'(x)=\frac{2}{3}x^2$

5·(3)·

$C(x)=\frac{2}{9}x^3+C$

$y=\frac{2}{9}x^5+Cx^2$ (通解)·

(4)· 当 $y'-\frac{3}{x^2}y=0$ 时·

$\frac{dy}{dx}=\frac{3}{x^2}y$

$y=Ce^{-\frac{3}{x}}$

由常数变易法：

$y=C(x)e^{-\frac{3}{x}}$

$y'=C'(x)e^{-\frac{3}{x}}+C(x)e^{-\frac{3}{x}}$

代入 $y'-\frac{3}{x^2}y=\frac{1}{x^2}$

$\Rightarrow C'(x)=\frac{1}{x^2}\times e^{\frac{3}{x}}$

$\int C'(x)=\int e^{\frac{3}{x}}\,d\frac{1}{x}\times(-1)$

$=-\frac{1}{3}e^{\frac{3}{x}}+C$

$\therefore y=Ce^{-\frac{3}{x}}-\frac{1}{3}$ (通解)·

(5). 当 $y'+\frac{1}{x}\cdot y=0$ 时，解得：$y=\frac{C}{x}$

由常数变易法：$y=\frac{C(x)}{x}$

$y'=\frac{C'(x)\cdot x-C(x)}{x^2}$

代入 $y'+\frac{1}{x}y=\frac{\sin x}{x}$

$C'(x)=\sin x$

$\therefore y=(-\cos x+C)\cdot\frac{1}{x}$ (通解)·

(7). 将方程改写为 $\frac{dx}{dy}=x\cos y+\sin 2y$ 即 $\frac{dx}{dy}-\cos y\cdot x=\sin 2y$

故原方程的通解为 $x=e^{\int \cos y dy}\left[\int \sin 2y\cdot e^{-\int \cos y dy}dy+C\right]$

$$=e^{\sin y}\left[\int \sin 2y\cdot e^{-\sin y}dy+C\right]$$

因为 $\int \sin 2y\cdot e^{-\sin y}dy=2\int \sin y\, e^{-\sin y}d\sin y=-2\int \sin y\, de^{-\sin y}$

$$=-2\sin y\, e^{-\sin y}+2\int e^{-\sin y}d\sin y$$

$$=-2\sin y\, e^{-\sin y}-2e^{-\sin y}+C$$

所以 $x=Ce^{\sin y}-2(\sin y+1)$. 其中C是任意常数.

(8). 将原方程变形可得 $\frac{dx}{dy}+\frac{1+y}{y}x=\frac{e^y}{y}$

所求通解为 $x=e^{-\int \frac{1+y}{y}dy}\left(C+\int \frac{e^y}{y}e^{\int \frac{1+y}{y}dy}dy\right)$

$$=e^{-(\ln y+y)}\left(C+\int \frac{e^y}{y}e^{\ln y+y}dy\right)$$

$$=\frac{e^{-y}}{y}\left(C+\int e^{2y}dy\right)=\frac{e^{-y}}{y}\left(C+\frac{1}{2}e^{2y}\right)$$

$$=\frac{Ce^{-y}}{y}+\frac{e^y}{2y}$$ （C为任意常数）.

(9) 原式可写成 $\frac{dy}{dx}-2yx=e^{x^2}\cos x$.

其对应的齐次方程为 $\frac{dy}{dx}-2xy=0$

变形为 $\frac{dy}{y}=2x\,dx$.

求得通解为 $y=Ce^{x^2}$

令 $y=C(x)e^{x^2}$，代入原式，得

$2xe^{x^2}C(x)+e^{x^2}C'(x)-2xe^{x^2}C(x)=e^{x^2}\cos x$ （C为常数）

化简得 $y=(\sin x+C)e^{x^2}$

即原式通解为 $y=(\sin x+c)e^{x^2}$ （C为任意常数）.

(10). 原方程可写为 $y^{-4}y'+\frac{1}{3}y^{-3}=\frac{1}{3}(1-2x)$

令 $z=y^{-3}$，则 $z'=-3y^{-4}y'$

原方程可化为 $z'-z=1-2x$.

$z=e^{\int dx}[\int(1-2x)e^{-\int dx}dx+C]$

$=e^{x}[\int(1-2x)e^{-x}dx+C]$

$=e^{x}[(-2x-1)e^{-x}+C]=-2x-1+Ce^{x}$ （C为任意常数）

即 $y^{-3}=-2x-1+Ce^{x}$ 为原方程通解.

(11). $P(x)=-\tan x$，$Q(x)=\sec x$

于是所求通解为

$y=e^{\int\tan x\,dx}(\int\sec x\cdot e^{-\int\tan x\,dx}dx+c)$

$=e^{-\ln\cos x}(\int\sec x\cdot e^{\ln\cos x}dx+c)$

$=\frac{1}{\cos x}(\int\sec x\cdot\cos x\,dx+c)$

$=\frac{1}{\cos x}(x+c)$. （C为任意常数）

将 $y(0)=0$ 代入，得 $C=0$

故原方程的特解为 $y=\frac{x}{\cos x}$

(12). 原方程对应的齐次方程 为 $y'+2xy=0$.

得其通解为 $y=Ce^{-x^2}$. （C为任意常数）.

令 $y=C(x)e^{-x^2}$. 则 $y'=C'(x)e^{-x^2}-2xC(x)e^{-x^2}$

代入原方程得 $C'(x)=2e^{x^2}x^3$.

两边同时积分得

$$C(x)=\int 2e^{x^2}x^3dx=\int x^2de^{x^2}=x^2e^{x^2}-\int e^{x^2}dx^2$$
$$=x^2e^{x^2}-e^{x^2}+C_0.$$ （C_0为任意常数）.

则原方程通解 $y=x^2-1+C_0e^{-x^2}$.

将 $y(0)=1$ 代入，得 $C_0=2$.

故原方程对应的特解为 $y=2e^{-x^2}+x^2-1$.

(13). 其对应的齐次方程 $y'-\frac{y}{x}=0$.

将其化为 $\frac{dy}{y}=\frac{dx}{x}$，得到的通解 $y=Cx$. （C为任意常数）

设 $y=C(x)x$. 则 $y'=C'(x)x+C(x)$.

代入原方程，得 $C'(x)=\frac{-\ln x}{x^2}$.

通过分部积分得 $C(x)=\frac{\ln x}{x}+\frac{1}{x}+C_0$

故原方程通解为 $y=C(x)x=\ln x+1+C_0x$ （C_0为任意常数）.

代入 $y(1)=1$，得 $C_0=0$. 故原方程对应的特解为 $y=\ln x+1$

(14). 原方程可变形为 $y'-\frac{1}{2x}y=\frac{-x^2}{2}$

$P(x)=-\frac{1}{2x}$，$Q(x)=\frac{-x^2}{2}$，于是所求通解为

$$y=e^{\int\frac{1}{2x}dx}\left[\int\left(-\frac{x^2}{2}\right)\cdot e^{-\int\frac{1}{2x}dx}dx+C\right]$$
$$=e^{\frac{1}{2}\ln x}\left[\int\left(-\frac{x^2}{2}\right)\cdot e^{-\frac{1}{2}\ln x}dx+C\right]$$
$$=\sqrt{x}\left[\int\left(-\frac{x^2}{2}\right)\frac{1}{\sqrt{x}}dx+C\right]=\sqrt{x}\left(-\frac{x^{\frac{5}{2}}}{5}+C\right)$$ （C为任意常数）

代入 $y(1)=0$，得 $C=\frac{1}{5}$

故原方程对应的特解为 $y=\sqrt{x}\left(\frac{1-x^{\frac{5}{2}}}{5}\right)=\frac{\sqrt{x}-x^3}{5}$

6. 解：由 $\frac{dx}{dt}-2te^{-x}=0$ 得 $e^{x}dx=2tdt$

两边同时积分得 $e^{x}=t^{2}+C$.

~~代入~~将 $x|_{t=0}=0$ 代入，得 $C=1$ 故 $e^{x}=t^{2}+1$

即 $x=\ln(1+t^{2})$

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{\ln(1+t^{2})\cdot 2t}{\frac{2t}{1+t^{2}}}=(1+t^{2})\ln(1+t^{2})$$

7.(1) 设细菌数量为 y_t，时间为 t，增长速度为 $y_{t-1}\cdot k$.

则 $y_1=y_0(k+1)$. $y_4=y_0(k+1)^4$.

则 $\frac{y_4}{y_1}=(k+1)^3=\frac{3000}{1000}=3$.

所以 $(k+1)^3=3$.

故 $y_t=y_0(k+1)^t=y_1(k+1)^{t-1}=1000(k+1)^{t-1}=1000\times3^{\frac{t-1}{3}}$

(2) 当 $t=0$ 时，$y_0=1000\times3^{-\frac{1}{3}}=\frac{1000}{\sqrt[3]{3}}\approx693$.

故最初有693个细菌.

8. 解：由题设，飞机的质量 $m=9000\ kg$，着陆时的水平速度 $V_0=700km/h$

从飞机着陆记起，设 t 时刻飞机的滑行距离为 $x(t)$，速度 $V(t)$.

由牛顿第二定律可得 $m\frac{dv}{dt}=-kv$.

又因为 $\frac{dv}{dt}=\frac{dv}{dx}\cdot\frac{dx}{dt}=v\frac{dv}{dx}$.

由上述两式可得，$dx=-\frac{m}{k}dv$.

积分得 $x(t)=-\frac{m}{k}v+C$，由于 $V(0)=V_0$，$x(0)=0$. 得 $C=\frac{m}{k}V_0$

故 $x(t)=\frac{m}{k}(V_0-V(t))$.

当 $V(t)\to0$ 时，$x(t)\to\frac{mV_0}{k}=\frac{9000\times700}{6.0\times10^{6}}=1.05(km)$.

故飞机滑行的最长距离是1.05km.

9. (1) $y' = \frac{1}{3}e^{3x} - \cos x + C$

$y = \frac{1}{9}e^{3x} - \sin x + C_1 x + C_2$

(2) 令 $y' = p$, $y'' = p'$

$p' - p - x = 0 \Rightarrow p' - p = x \Rightarrow p = (\int x e^{\int -1dx} dx + C) e^{\int 1dx}$

$= [-(x+1)e^{-x} + C]e^{x}$

$= -(x+1) + Ce^{x} = y'$

$\Rightarrow y = -\frac{x^2}{2} - x + C_1 e^{x} + C_2$

(3) 令 $y' = p$, $y'' = p'$

$(1+x^2)p' = 2xp = (1+x^2)\frac{dp}{dx}$

$\frac{dp}{p} = \frac{2x}{1+x^2}dx$

$\ln p = \ln(1+x^2) + C$

$y' = p = C(1+x^2)$

$y = C_1(x + \frac{x^3}{3}) + C_2$

(4) 令 $y' = p(y) \Rightarrow y'' = p'$, $\frac{dy'}{dx} = p\frac{dp}{dy}$

原式 $= yp\frac{dp}{dy} - p^2 = 0$

当 $p=0$ 时，$y = C$ 显然为方程解

$p \neq 0$ 时，$y\frac{dp}{dy} - p = 0 \Rightarrow p = C_1 y = \frac{dy}{dx}$

$\Rightarrow \frac{dy}{y} = C_1 dx \Rightarrow y = C_2 e^{C_1 x}$

(5) 令 $y' = p$ $y'' = \frac{dp}{dx}$

$\frac{dp}{dx} = p^2 + 1 \Rightarrow \frac{dp}{p^2+1} = dx$

$\Rightarrow p = \tan(x + C_1) = y'$

$\Rightarrow y = -\ln|\cos(x + C_1)| + C_2$

(6) 令 $p = y'$，$y'' = p\frac{dp}{dy}$

原式 $= p\frac{dp}{dy} + \frac{p^2}{1-y} = 0 \Rightarrow \frac{dp}{dy} = -\frac{p}{1-y}$

$\Rightarrow y' = p = C_1(y-1)$, $y \neq 1$

$\Rightarrow y = 1 + C_2 e^{C_1 x}$ $(C_2 \neq 0)$

10. (1) $\lambda^2 + 5\lambda + 6 = 0 \Rightarrow \lambda_1 = -2$ $\lambda_2 = -3$

$\therefore$ 通解：$y = C_1 e^{-2x} + C_2 e^{-3x}$

(2) $\lambda^2 - 4\lambda + 4 = 0 \Rightarrow \lambda_1 = \lambda_2 = 2$

$\therefore$ 通解：$y = (C_1 + C_2 x)e^{2x}$

(3) $\lambda^2 + 8\lambda + 25 = 0 \Rightarrow \lambda_{1,2} = \frac{-8 \pm \sqrt{-36}}{2} = -4 \pm 3i$

$\alpha = -4$, $\beta = 3$

$\therefore$ 通解：$y = e^{-4x}(C_1\cos 3x + C_2 \sin 3x)$

(4) $\lambda^2 - 3\lambda + 4 = 0 \Rightarrow \lambda_{1,2} = \frac{3 \pm \sqrt{7}i}{2}$

$\alpha = \frac{3}{2}$ $\beta = \frac{\sqrt{7}}{2}$

$\therefore$ 通解：$y = e^{\frac{3}{2}x}(C_1 \cos\frac{\sqrt{7}}{2}x + C_2 \sin\frac{\sqrt{7}}{2}x)$

(5) $\lambda^2 + 4\lambda + 29 = 0 \Rightarrow \lambda_{1,2} = -2 \pm 5i$

$\alpha = -2$, $\beta = 5$

$\therefore y = e^{-2x}(C_1 \cos 5x + C_2 \sin 5x)$

$x = 0$, $y = 0 \Rightarrow C_1 = 0$

$y' = C_2(-2e^{-2x}\sin 5x + 5e^{-2x}\cos 5x)$

$x = 0$, $y' = 15 \Rightarrow C_2 = 3$

$\therefore y = 3e^{-2x}\sin 5x$

(6) $4\lambda^2 + 4\lambda + 1 = 0 \Rightarrow \lambda_1 = \lambda_2 = -\frac{1}{2}$

$\therefore y = (C_1 + C_2 x)e^{-\frac{1}{2}x}$

$y' = -\frac{1}{2}C_1 e^{-\frac{1}{2}x} + C_2 e^{-\frac{1}{2}x} - \frac{1}{2}C_2 x e^{-\frac{1}{2}x}$

$\because y(0) = 2$, $y'(0) = 0 \Rightarrow \begin{cases} C_1 = 2 \\ -\frac{1}{2}C_1 + C_2 = 0 \end{cases}$

$\Rightarrow C_1 = 2$ $C_2 = 1$ $\therefore y = 2e^{-\frac{1}{2}x} + xe^{-\frac{1}{2}x}$

11. (1) $\lambda^2 - \lambda - 2 = 0 \Rightarrow \lambda_{1,2} = 2, -1$

通解：$y = C_1 e^{2x} + C_2 e^{-x}$

$\lambda_0 = 0$ 不是 $\lambda^2 - \lambda - 2 = 0$ 的根

$\therefore$ 特解 $y^* = ax^2 + bx + c$

$2a - 2ax - b - 2ax^2 - 2bx - 2c = 4x^2$

$\Rightarrow \begin{cases} a = -2 \\ b = 2 \\ c = -3 \end{cases}$

$\therefore$ 解：$y = C_1 e^{2x} + C_2 e^{-x} + 2x - 2x^2 - 3$

(2) $\lambda^2 - \lambda - 2 = 0 \Rightarrow \lambda_1 = -1$ $\lambda_2 = 2$

通解：$y = C_1 e^{-x} + C_2 e^{2x}$

设特解：$y^* = axe^{2x}$

$4ae^{2x} + 4axe^{2x} - (Ae^{2x} + 2Axe^{2x}) - 2axe^{2x} = e^{2x}$

$\Rightarrow a = \frac{1}{3}$

$\therefore$ 解：$y = C_1 e^{-x} + C_2 e^{2x} + \frac{x}{3}e^{2x}$

(3) $\lambda^2-\lambda-2=0 \quad \lambda_1=2,\ \lambda_2=-1$

通解：$y=C_1e^{2x}+C_2e^{-x}$

$f(x)=\sin 2x=e^{\alpha x}(A_1\cos\beta x+A_2\sin\beta x)$

$\Rightarrow \quad \alpha=0 \quad \beta=2$

$\pm 2i$不为特征方程根 $\Rightarrow k=0$

$\therefore y^*=Q_1\cos 2x+Q_2\sin 2x$

将y^*带入原式 $\Rightarrow \begin{cases}2Q_1-6Q_2=1\\6Q_1+2Q_2=0\end{cases}$

$\Rightarrow \begin{cases}Q_1=\frac{1}{20}\\Q_2=-\frac{3}{20}\end{cases}$

$\therefore$ 解：$y=C_1e^{2x}+C_2e^{-x}+\frac{1}{20}\cos 2x-\frac{3}{20}\sin 2x$

(4) $\lambda^2-2\lambda+1=0 \Rightarrow \lambda_1=\lambda_2=1$

$\therefore$ 通解：$y=(C_1+C_2x)e^x$

$\because \lambda_0=0$不为特征方程根

$\therefore \quad y^*=ax^2+bx+c$

将y^*带入原式 $\Rightarrow \begin{cases}a=1\\b=4\\c=5\end{cases}$

$\therefore$ 解：$y=(C_1+C_2x)e^x+x^2+4x+5$

(5) $\lambda^2-6\lambda+9=0 \Rightarrow \lambda_1=\lambda_2=3$

通解：$y=(C_1+C_2x)e^{3x}$

$\alpha=1,\ \beta=1 \qquad 1\pm i$不为特征方程根

$\therefore y^*=e^x(A\cos x+B\sin x)$

将y^*带入原式 $\Rightarrow \begin{cases}A=\frac{3}{25}\\B=-\frac{4}{25}\end{cases}$

$\therefore$ 解：$y=(C_1+C_2x)e^{3x}+(\frac{3}{25}\cos x-\frac{4}{25}\sin x)e^x$

(6) 令$x=e^t \Rightarrow t=\ln x$

$D(D-1)y-2Dy+2y=t^2-2t$

$\Rightarrow (D^2-3D+2)y=t^2-2t$

$\Rightarrow \frac{d^2y}{dt^2}-3\frac{dy}{dt}+2y=t^2-2t$

$\lambda^2-3\lambda+2=0 \Rightarrow \lambda_1=1,\ \lambda_2=2$

$\therefore$ 通解：$y=C_1e^t+C_2e^{2t}$

设$y^*=at^2+bt+c$

将y^*带入原式 $\Rightarrow \begin{cases}a=\frac{1}{2}\\b=\frac{1}{2}\\c=\frac{1}{4}\end{cases}$

$\therefore$ 解：$y=C_1x+C_2x^2+\frac{1}{2}\ln^2x+\frac{1}{2}\ln x+\frac{1}{4}$

(7) $\lambda^2-4=0 \quad \lambda_1=2 \quad \lambda_2=-2$

通解：$y=C_1e^{2x}+C_2e^{-2x}$

$\lambda_0=0$不是特征方程根

$\therefore y^*=a$

带入原式 $\Rightarrow -4a=4 \Rightarrow a=-1$

$y(0)=0 \Rightarrow C_1=C_2$

$y(0)=1 \Rightarrow 2C_1-1=1 \Rightarrow C_1=C_2=1$

$\therefore y=e^{2x}+e^{-2x}-1$

(8) $\lambda^2-1=0 \quad \lambda_1=1,\ \lambda_2=-1$

通解：$y=C_1e^x+C_2e^{-x}$

$\because \lambda_0=1$为特征方程单根

$\Rightarrow \quad y^*=(ax^2+bx)e^x$

带入原式 $\Rightarrow 4ax+2(a+b)=4x \Rightarrow \begin{cases}a=1\\b=1\end{cases}$

$\therefore$ 解：$y=C_1e^x+C_2e^{-x}+(x^2-x)e^x$

$\because y(0)=0,\ y'(0)=1 \Rightarrow C_1=1,\ C_2=-1$

$\therefore \quad y=(x^2-x+1)e^x+e^{-x}$

12. $f'(x)=\cos x-\int_0^x f(t)dt$

$f''(x)=-\sin x-f(x)$

$\Rightarrow \begin{cases}f''(x)+f(x)=-\sin x\\f(0)=0,\ f'(0)=1\end{cases}$ ①

$\lambda^2+1=0 \quad \lambda_{1,2}=\pm i$

通解：$y=C_1\cos x+C_2\sin x$

$y^*=x(a\cos x+b\sin x)$，带入原式 $\Rightarrow a=\frac{1}{2},\ b=0 \Rightarrow$ 解：$f(x)=C_1\cos x+C_2\sin x+\frac{x}{2}\cos x$

由① $\Rightarrow C_1=0,\ C_2=\frac{1}{2} \quad \therefore f(x)=\frac{1}{2}(x\cos x+\sin x)$

13. ① $x\in(-\pi,0)$，$y'=-\frac{x}{y} \Rightarrow ydy=-xdx \Rightarrow y^2=-x^2+C \Rightarrow x^2+y^2=\pi^2$

$\because$ 曲线过点$(-\frac{\pi}{\sqrt{2}},\frac{\pi}{\sqrt{2}})$ $\therefore y=\sqrt{\pi^2-x^2}$

② $x\in[0,\pi)$，$y''+y=0$ 易知$y^*=C_1\cos x+C_2\sin x$，设特解$Y=ax+b$

$y''+y+x=0$

$ax+b+x=0 \Rightarrow a=-1,\ b=0$

$\therefore y=C_1\cos x+C_2\sin x-x \quad \because y_-(0)=y_+(0) \Rightarrow C_1=\pi \quad y_-'(0)=y_+'(0)$

$\Rightarrow (\sqrt{\pi^2-x^2})'|_{x=0}=0=C_2-1 \Rightarrow C_2=1$

$\therefore y=\begin{cases}\sqrt{\pi^2-x^2}, & -\pi<x<0\\ \pi\cos x+\sin x-x, & 0\le x<\pi\end{cases}$