习题3.1.

1. (1) $f(x)=x^2$, $x_0=1$.

$$f'(x_0)=\lim_{\Delta x\to 0}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$$
$$=\lim_{\Delta x\to 0}\frac{(x_0+\Delta x)^2-x_0^2}{\Delta x}$$
$$=\lim_{\Delta x\to 0}(2+\Delta x)$$
$$=2.$$

(2) $f(x)=\frac{1}{x^2}$, $x_0=2$.

$$f'(x_0)=\lim_{\Delta x\to 0}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$$
$$=\lim_{\Delta x\to 0}\frac{\frac{1}{(x_0+\Delta x)^2}-\frac{1}{x_0^2}}{\Delta x}$$
$$=-\lim_{\Delta x\to 0}\frac{2+\Delta x}{x_0^2(x_0+\Delta x)^2}$$
$$=-\frac{2}{x_0^3}$$
$$=-\frac{1}{4}.$$

(3) $f(x)=x(x+1)\cdots(x+2020)$, $x_0=0$.

$$\lim_{\Delta x\to 0}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$$
$$=\lim_{\Delta x\to 0}\frac{(x_0+\Delta x)(x_0+1+\Delta x)\cdots(x_0+2020+\Delta x)-x_0(x_0+1)\cdots(x_0+2020)}{\Delta x}$$
$$=\lim_{\Delta x\to 0}\frac{x_0[(x_0+1+\Delta x)\cdots(x_0+2020+\Delta x)-(x_0+1)\cdots(x_0+2020)]}{\Delta x}$$
$$+\lim_{\Delta x\to 0}(x_0+1+\Delta x)\cdots(x_0+2020+\Delta x)$$
$$=\lim_{\Delta x\to 0}(x_0+1+\Delta x)\cdots(x_0+2020+\Delta x)$$
$$=2020!$$

2. (1) $f'_-(0)=\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0}$

$$=\lim_{x\to 0^-}\frac{f(x)}{x}=+\infty.$$

$$f'_+(0)=\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0}$$
$$=\lim_{x\to 0^+}\frac{f(x)}{x}=+\infty.$$

$f(x)$在$x=0$不可导.

(2) $f'_-(0)=\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0}$

$$=\lim_{x\to 0^-}\frac{f(x)-1}{x}$$
$$\cancel{=\lim_{x\to 0}\left(x+\frac{1}{x}\right)}$$
$$=\lim_{x\to 0^-}x^2=0.$$

$$f'_+(0)=\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0}$$
$$=\lim_{x\to 0^+}\frac{f(x)-1}{x}$$
$$=\lim_{x\to 0^+}x=0.$$

$f'(0)=f'_-(0)=f'_+(0)=0$

$\therefore f(x)$在$x=0$可导.

3. (1) $y'|_{x=0}=e^x|_{x=0}=1$.

$\therefore k_切=1$, $k_法=-1$.

$l_切$: $y=x+1$.

$l_法$: $y=-x+1$.

(2) 设 $P(x_0, \ln x_0)$.

则 $y'|_{x=x_0} = \frac{1}{x_0}$.

令 $\frac{1}{x_0} = \frac{1}{2}$

解得: $x_0 = 2$.

即 $P(2, \ln 2)$.

4. 可导 => 连续.

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1)$.

即 $\lim_{x\to 1^-} x^2 = \lim_{x\to 1^+}(ax+b) = a+b$. ①

可导 <=> 左右导存在且相等.

$\lim_{x\to 1^-} f'(x) = \lim_{x\to 1^+} f'(x)$

即: $\lim_{x\to 1^-} 2x = \lim_{x\to 1^+} a$. ②

解①②得: $\begin{cases} a = 2. \\ b = -1. \end{cases}$

5. 证明:

$$\text{左边} = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)+f(x_0)-f(x_0-h)}{h}$$

$$= \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{(x_0+h)-x_0} + \lim_{h\to 0} \frac{f(x_0)-f(x_0-h)}{x_0-(x_0-h)}$$

$$= f'(x_0) + f'(x_0)$$

$$= 2f'(x_0) = \text{右边}.$$

6. 证明 ①偶函数满足:

$f(x) = f(-x)$.

两边同时求导:

$f'(x) = -f'(-x)$

即偶函数导数为奇函数.

②奇函数满足:

$-f(x) = f(-x)$.

两边同时求导:

$-f'(x) = \cancel{f'(-x)}\ -f'(-x)$.

$\Rightarrow f'(x) = f'(-x)$.

即奇函数导数为偶函数.

③周期函数满足:

$f(x) = f(x+T)$.

两边同时求导:

$f'(x) = f'(x+T)$

即周期函数导数为周期函数.

7. ① $f'_-(0) = \lim_{\Delta x\to 0^-} \frac{f(0+\Delta x)-f(0)}{\Delta x}$

$= \lim_{\Delta x\to 0^-} -\frac{1}{\Delta x} = +\infty$.

② $f'_+(0) = \lim_{\Delta x\to 0^+} \frac{f(0+\Delta x)-f(0)}{\Delta x} = 0$.

$\because f'_-(0) \neq f'_+(0)$, 故 $f'(0)$ 不存在.

③ $f'(0)$ 不存在.

④ $\lim_{x\to 0} f'(x) = \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = 0$.

8. $|f(0)| \leq 1-\cos 0 = 0$. 即 $f(0) = 0$.

①证连续:

$\cos x - 1 \leq f(x) \leq 1-\cos x$.

$\because \lim_{x\to 0^-}(\cos x - 1) = \lim_{x\to 0^+}(1-\cos x) = 0 = f(0)$

$\therefore$ ~~由夹逼定理~~: $f(0)$ 在 $x=0$ 连续

②证左右导相等:

$\lim_{x\to 0^-} \frac{-(\cos x-1)}{x} \leq \lim_{x\to 0^-} \frac{f(x)}{x} \leq \lim_{x\to 0^-} \frac{\cos x - 1}{x}$.

$\because \lim_{x\to 0^-} \frac{1-\cos x}{x} = \lim_{x\to 0^-} \frac{\cos x - 1}{x} = 0$.

$\therefore$ 由夹逼定理: $f'_-(x_0) = f'_+(x_0) = f(x_0) = 0$ (x

$\therefore$ $f(x)$ 在 $x_0 = 0$ 处可导.