习题 6.3

1.（1）$\int_0^1(2x-3)^2dx$

令$2x-3=t \Rightarrow x=\frac{3+t}{2}$

（积分变量变化时，积分区间也要相应变化）

$\Rightarrow \int_0^1(2x-3)^2dx=\int_{-3}^{-1}t^2d\left(\frac{3+t}{2}\right)=\frac{1}{2}\cdot\frac{t^3}{3}\Big|_{-3}^{-1}=\frac{1}{2}\left(\frac{-1}{3}-\frac{-27}{3}\right)=\frac{13}{3}$

（2）$f(x)$在$\left[0,\frac{1}{2}\right]$上连续可导$\Rightarrow f(x)$在$\left[0,\frac{1}{2}\right]$上可积

$\int_0^1 f'\left(\frac{1-x}{2}\right)dx$　令$\frac{1-x}{2}=t \Rightarrow x=1-2t$，　$\frac{1-x}{2}\Big|_0^1 \to t\Big|_{\frac{1}{2}}^0$

$\Rightarrow \int_0^1 f'\left(\frac{1-x}{2}\right)dx=\int_{\frac{1}{2}}^0 f'(t)d(1-2t)=2\int_0^{\frac{1}{2}}f'(t)dt=2f(t)\Big|_0^{\frac{1}{2}}=2\left(f\left(\frac{1}{2}\right)-f(0)\right)$

2.（1）$\int_0^1 x\sqrt{1-x}\,dx$（令$\sqrt{1-x}=t$）

$=\int_1^0 t\cdot(1-t^2)d(1-t^2)$

$=2\int_0^1(t^2-t^4)dt$

$=2\cdot\frac{t^3}{3}\Big|_0^1-2\cdot\frac{t^4}{5}\Big|_0^1$

$=2\left(\frac{1}{3}-0\right)-2\left(\frac{1}{5}-0\right)$

$=\frac{2}{3}-\frac{2}{5}=\frac{4}{15}$

（2）$\int_0^1 x(2-x^2)^5dx$

$=-\frac{1}{2}\int_0^1(2-x^2)^5d(2-x^2)$

$=-\left(\frac{1}{12}-\frac{2^6}{12}\right)=\frac{21}{4}$

(3)$\int_1^{\sqrt{3}}\frac{dx}{x^2\sqrt{1+x^2}}$(令$x=\tan t$，积分上下限改变为$\frac{\pi}{3}$，$\frac{\pi}{4}$)

$=\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{1}{\tan^2 t\cdot\sec t}\cdot\sec^2 t\cdot dt$

$=\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{\cos t}{\sin^2 t}dt$

$=\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{1}{\sin^2 t}d\sin t$

$$= -\frac{1}{\sin t}\bigg|_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$

$$= -\left(\frac{2\sqrt{3}}{3} - \sqrt{2}\right) = \sqrt{2} - \frac{2\sqrt{3}}{3}$$

（4）$\int_0^1 \frac{dx}{e^x+e^{-x}}$（令$e^x =$ t，则积分上下限改变为e，1）

$$= \int_1^e \frac{1}{t+\frac{1}{t}} \cdot \frac{1}{t} \mathrm{dt}$$

$$= \int_1^e \frac{1}{1+t^2} \mathrm{dt}$$

$$= \arctan t|_1^e$$

$$= \arctan e - \frac{\pi}{4}$$

（5）$\int_0^1 \frac{1}{e^x+1} \mathrm{d\,x}$（令$e^x = \mathrm{t}$，则积分上下限改变为e，1）

$$= \int_1^e \frac{1}{1+t} \cdot \frac{1}{t} \mathrm{dt}$$

$$= \int_1^e \left(\frac{1}{t} - \frac{1}{1+t}\right) d\mathrm{t}$$

$$= \ln t\,|_1^e - \ln(1+t)_1^e$$

$$= 1 - 0 - \ln(1+e) + \ln 2$$

$$= 1 + \ln 2 - \ln(1+e)$$

$$= \ln \frac{2e}{1+e}$$

（6）$\int_{\frac{1}{2}}^1 \frac{\sqrt{1-x^2}}{x^2} \mathrm{dx}$(令$\mathrm{x} = \sin t$，则积分上下限改变为$\frac{\pi}{2}$，$\frac{2}{4}$)

$$= \int_{\frac{2}{4}}^{\frac{\pi}{2}} \frac{\cos t}{\sin^2 t} \cdot \cos t\, dt$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos^2 t}{\sin^2 t} \mathrm{dt}$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1-\sin^2 t}{\sin^2 t} dt$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\csc^2 t - 1) d\mathrm{t}$$

$$= -\cot t\,\bigg|_{2}^{\pi} - t\bigg|_{4}^{\frac{\pi}{2}}$$

$$= -(0-1) - \left(\frac{\pi}{2} - \frac{\pi}{4}\right)$$

$$= 1 - \frac{\pi}{4}$$

（7）$\int_0^{\frac{\pi}{2}} \frac{cost}{sint+cost} \mathrm{dt}$（令$\mathrm{x} = a \sin t$,则积分上下限改变为$\frac{\pi}{2}$，0）

$$= \int_0^{\frac{\pi}{2}} \frac{\frac{1}{2}(\sin t + \cos t) + \frac{1}{2}(\cos t - \sin t)}{\sin t + \cos t} \mathrm{dt}$$

$=\int_0^{\frac{\pi}{2}}\left(\frac{1}{2}+\frac{1}{2}\cdot\frac{\cos t-\sin t}{\sin t+\cos t}\right)dt$

$=\int_0^{\frac{2}{2}}\left(\frac{1}{2}+\frac{1}{2}\cdot\frac{\cos t-\sin t}{\sin t+cos\,t}\right)d\,t$

$=\frac{\pi}{4}+\frac{1}{2}\ln(sint+cost)\ |_0^{\frac{\pi}{2}}$

$=\frac{\pi}{4}+(|n|-|n|)$

$=\frac{\pi}{4}$

(8) $\mathrm{I}=\int_0^{\frac{\pi}{2}}\frac{\sin\theta}{\sin\theta+\cos\theta}\mathrm{d}\theta$， $J=\int_0^{\frac{\pi}{2}}\frac{\cos\theta}{\sin\theta+\cos\theta}\mathrm{d}$(半余法)

a.$(\mathrm{I}+\mathrm{J}=\int_0^{\frac{\pi}{2}}\frac{\sin\theta+\cos\theta}{\sin\theta+\cos\theta}\mathrm{d}\theta=\int_0^{\frac{\pi}{2}}1\mathrm{d}\theta=\frac{\pi}{2})$

b.$[\ \mathrm{I}-\mathrm{J}=\int_0^{\frac{\pi}{2}}\frac{\sin\theta-\cos\theta}{\sin\theta+\cos\theta}\mathrm{d}\theta=-\int_0^{\frac{\pi}{2}}\frac{1}{\sin\theta+\cos\theta}\mathrm{d}(\sin\theta+\cos\theta)$

$\Rightarrow I-\mathrm{T}=-\ln(\sin\theta+\cos\theta)|_0^{\frac{\pi}{2}}=-(\ln 1-\ln 1)=0\ ]$

$\Rightarrow\ \frac{\mathrm{a+b}}{2}=\mathrm{I}=\frac{\pi}{4}\ \ \frac{\mathrm{a}-b}{2}=J=\frac{\pi}{4}$

(9) $\int_0^{\frac{\pi}{2}}\frac{dx}{a^2\sin^2x+b^2\cos^2x}(a,b>0)$ (令u=tan$x$，则积分上下限改变为$+\infty$， 0)

$=\int_0^{+\infty}\frac{1}{a^2\cdot\frac{u^2}{1+u^2}+b^2\frac{1}{1+u^2}}$ darctanu

$=\int_0^{+\infty}\frac{1+u^2}{a^2u^2+b^2}\cdot\frac{1}{1+u^2}du$

$=\int_0^{+\infty}\frac{1}{au^2+b^2}du$

$=\frac{1}{b^2}\int_0^{+\infty}\frac{1}{1+\left(\frac{a}{b}u\right)^2}\left(d\left(\frac{a}{b}u\right)\right)\cdot\frac{b}{a}$

$=\frac{1}{ab}\arctan\ \frac{a}{b}u\Big|_0^{+\infty}$ （$\lim\limits_{x\to+\infty}\arctan\frac{a}{b}x=\frac{\pi}{2}$,可认为$\arctan\ \frac{a}{b}u=\frac{\pi}{2}$ （$u\to+\infty$)）

$=\frac{1}{ab}\left(\frac{\pi}{2}-0\right)$

(10) $f(x)=\begin{cases}1+x^2, & x\le 0\\ e^{-x}, & x>0\end{cases}$

$\int_1^3 f(x-2)d\,x$（令$x-2=t$，则积分上下限改变为 1，-1）

$=\int_{-1}^{1}f(t)d\,t$

$=\int_{-1}^{0}f(t)d\,t+\int_0^1 f(t)d\,t$（分段函数将积分区间相应分段）

$=\int_{-1}^{0}(1+t^2)d\,t+\int_0^1 e^{-t}d\,t$

$= t1_{-1}^{0} + \frac{t^3}{3}\Big|_{-1}^{0} - e^{-t}|_0^1$

$= 1 + \frac{1}{3} - e^{-1} + 1$

$= \frac{7}{3} - \frac{1}{e}$

3.证明：因为$f(x)$在$[-a,a]$上连续

$\Rightarrow f(x)$在$[-a,a]$上可积

$\int_{-a}^{a} x\big(f(x)+f(-x)\big)d\,x = \int_{-a}^{a} x\,f(x)dx + \int_{-a}^{a} x\,f(-x)dx$

$\Rightarrow \int_{-a}^{a} x\,f(-x)dx$（令$-x=t$，则积分上下限改变为$-a,a$）$\int_{a}^{-a}(-t)f(t)d(-t) =$

$\int_{a}^{-a} t\,f(t)dt = \int_{a}^{-a} x\,f(x)dx$

$\Rightarrow \int_{-a}^{a} x\big(f(x)+f(-x)\big)d\,x = \int_{-a}^{a} x\,f(x)dx + \int_{a}^{-a} x\,f(x)dx = 0$

4 证明：

$\int_0^1 x^m(1-x)^n d\,x \overset{1-x=t}{\underset{t|_1^0}{=\!=}} \int_1^0 (1-t)^m t^n d(1-t) = \int_0^1 t^n(1-t)^m d\,t = \int_0^1 x^n(1-$

$x)^m d\,x$(等于右式)

综上：$\int_0^1 x^m(1-x)^n d\,x = \int_0^1 x^n(1-x)^m d\,x$

5.证明：令$t=\frac{1}{u}$　$(x>0)$（$x>0 \Rightarrow t>0$即可$\Rightarrow t=\frac{1}{u}(u>0)$）

$t|_x^1 \Rightarrow u|_{\frac{1}{x}}^{1}$

$\Rightarrow \int_x^1 \frac{1}{1+t^2}dt = \int_{\frac{1}{x}}^{1} \frac{1}{1+\left(\frac{1}{u}\right)^2}\cdot\left(-\frac{1}{u^2}\right)du = \int_1^{\frac{1}{x}} \frac{1}{1+u^2}du = \int_1^{\frac{1}{x}} \frac{1}{1+t^2}dt$

综上：$\int_x^1 \frac{1}{1+t^2}dt = \int_1^{\frac{1}{x}} \frac{1}{1+t^2}dt$　　$(x>0)$

6.证明：$f(x)$为连续函数⇒在$x\in D$时可积

（1）因为$f(x)$为奇函数

所以$f(x) = -f(-x)$

令$F(x) = \int_0^x f(t)d\,t$，则　$F(-x) = \int_0^{-x} f(t)d\,t$

令$t=-u$，$t|_0^{-x} \to u|_0^x$

$\Rightarrow F(-x) = \int_0^x f(t)d\,t = \int_0^x f_0 f(-u)d(-u) = \int_0^x -f(-u)d\,u = \int_0^x f(u)d\,x =$

$\int_0^x f(t)d\,t = F(x)$

故$\int_0^{-x} f(t)\,dt$ 在$f(x)$为奇函数时，为偶函数。

（2）因为$f(x)$为偶函数

所以$f(x) = f(-x)$

令$G(x)=\int_0^x f(t)d\,t$，$G(-x)=\int_0^{-x} f(t)d\,t$

令$t=-k$，　$t|_0^{-x}\to k|_0^x$

$\Rightarrow G(-x)=\int_0^x f(t)d\,t=\int_0^x f(-k)d(-k)=-\int_0^x f(k)d\,k=-G(x)$

故当$f(x)$为偶函数时，$\int_0^x f(t)\,dt$为奇函数。