习题 7.2

1.(1) $y' = e^{x-y}$

$\frac{dy}{dx} = \frac{e^x}{e^y}$

$\Rightarrow\ e^y\,dy = e^x\,dx$

两端积分：$e^y = e^x + C$ （C 为任意常数）

（2）$xy\,dx + \sqrt{1-x^2}\,dy = 0$

$xy\,dx = -\sqrt{1-x^2}\,dy$

$-\frac{x\,dx}{\sqrt{1-x^2}} = \frac{1}{y}dy$

两端积分：$ln|y| = \frac{1}{2}\int \frac{1}{\sqrt{1-x^2}}d(1-x^2)$

$= \sqrt{1-x^2} + C_1$

$\therefore y = e^{\sqrt{1-x^2}+C_1}$

$= e^{C_1} \cdot e^{\sqrt{1-x^2}}$

$= C \cdot e^{\sqrt{1-x^2}}$ (C 为任意常数)

(3) $y' = \sqrt{\frac{1-y^2}{1-x^2}}$

$\frac{dy}{dx} = \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$

$\frac{dy}{\sqrt{1-y^2}} = \frac{dx}{\sqrt{1-x^2}}$

两端积分：$arc\sin y = arc\sin x + C$ （C 为任意常数）

（4）$e^x y\,dx + 2(e^x - 1)\,dy = 0$

$\frac{e^x}{e^x-1}dx = -\frac{2}{y}dy$

两端积分：$ln|e^x - 1| = -2\,ln|y|$ +C

$ln|e^x - 1| + ln\,y^2 = C$

$\therefore$ $(e^x - 1)y^2 = C$ （C 为任意常数)

2.(1) $xy' = y\,ln\frac{y}{x}$

$\frac{dy}{dx} = \frac{y}{x}ln\frac{y}{x}$

令$m=\frac{y}{x}$ ，则 $y=mx$ ，$\frac{dy}{dx}=m+x\frac{dm}{dx}$

$\therefore m+x\frac{dm}{dx}=m\ln m$

$\frac{dm}{m(\ln m-1)}=\frac{dx}{x}$

两端积分：$\ln|\ln m-1|=\ln x+\ln C_1$

$\therefore|\ln m-1|=C_1x$

$\ln\frac{y}{x}=Cx+1$

$y=xe^{cx+1}$（C 为任意常数）

(2) $y'=e^{\frac{y}{x}}+\frac{y}{x}$

$\frac{dy}{dx}=e^{\frac{y}{x}}+\frac{y}{x}$

令$m=\frac{y}{x}$ ，则 $y=mx$ ，$\frac{dy}{dx}=m+x\frac{dm}{dx}$

$m+x\frac{dm}{dx}=e^m+m$

$\frac{dm}{e^m}=\frac{dx}{x}$

两端积分：$\frac{-1}{e^m}=\ln|x|+C$

$\therefore$ -$e^{-\frac{y}{x}}=\ln|x|+C$ （C 为任意常数）

(3) $xy'-y-\sqrt{y^2-x^2}=0$ ($x>0$)

同除x并移项$\frac{dy}{dx}=\frac{y}{x}+\sqrt{\left(\frac{y}{x}\right)^2-1}$

令$m=\frac{y}{x}$ ，则 $y=mx$ ，$\frac{dy}{dx}=m+x\frac{dm}{dx}$

$\therefore m+x\frac{dm}{dx}=m+\sqrt{m^2-1}$

$\frac{dm}{\sqrt{m^2-1}}=\frac{dx}{x}$

两端积分：$\ln\left|m+\sqrt{m^2-1}\right|=\ln|x+1|+\ln c_1$

$\therefore m+\sqrt{m^2-1}=cx$

$\frac{y}{x}\frac{+\sqrt{y^2-x^2}}{x}=cx$

$y=cx^2-\sqrt{y^2-x^2}$（c 为任意常数）

(4) $\frac{dy}{dx}=\frac{2x-y+5}{2x-y-4}$

令$m=2x-y$ 则 $y=2x-m$ $\frac{dy}{dx}=2-\frac{dm}{dx}$

$\therefore 2-\frac{dm}{dx}=1+\frac{9}{m-4}$

$\frac{m-4}{m-13}dm=dx$

两端积分：$\int\left(1+\frac{9}{m-13}\right)dm=x$

$\Rightarrow m+9\ln|m-13|=x+c_1$

$\Rightarrow \ln|m-13|=\frac{1}{9}(x-m+c_1)$

$\Rightarrow m-13=e^{\frac{x-m}{9}}\cdot e^{\frac{c_1}{9}}$

$\therefore 2x-y-13=ce^{\frac{y-x}{9}}$（$c$ 为任意常数）

(5) $(2x-y+1)dx+(2y-x-1)dy=0$

$\frac{dy}{dx}=\frac{2x-y+1}{x-2y+1}$

显然$\begin{vmatrix}2 & -1\\ 1 & -2\end{vmatrix}$=-3≠0

设$\begin{cases}x=X+s\\ y=Y+t\end{cases}$ 则$dx=dX,dy=dY$

解方程组：$\begin{cases}2s-t+1=0\\ s-2t+1=0\end{cases}\Rightarrow\begin{cases}s=-\frac{1}{3}\\ t=\frac{1}{3}\end{cases}$

$\therefore$原方程可化为$\frac{dY}{dX}=\frac{2X-Y}{X-2Y}=\frac{2-\frac{Y}{X}}{1-\frac{2Y}{X}}$

设$m=\frac{Y}{X}$ $\therefore\frac{dY}{dX}=m+X\frac{dm}{dX}$

$\therefore m+x\frac{dm}{dx}=\frac{2-m}{1-2m}$

$-\frac{1}{2}\cdot\frac{2m-1}{1-m+m^2}dm=\frac{dX}{X}$

$\ln|1-m+m^2|=-2\ln|X|+\ln c$

$\Rightarrow 1-\frac{Y}{X}+\frac{Y^2}{X^2}=\frac{c}{X^2}$

$\Rightarrow X^2-XY+Y^2=c$

$\left(x+\frac{1}{3}\right)^2-\left(x+\frac{1}{3}\right)\left(y-\frac{1}{3}\right)+\left(y-\frac{1}{3}\right)^2=c$

$x^2-xy+y^2+x-y=c$（c 为任意常数）

(6)$y(1+x^2y^2)\,dx = x\,dy$

设$z=xy$ $\therefore \frac{dz}{dx}=y+x\frac{dy}{dx}$ ①

$\therefore y(1+z^2)\,dx = x\,dy$

$1+z^2=\frac{x}{y}\frac{dy}{dx}$

由①式可知: $\frac{dz}{y\,dx}=1+\frac{x\,dy}{y\,dx}$

$\Rightarrow \frac{x\,dz}{z\,dx}=1+\frac{x\,dy}{y\,dx}$

$\therefore 1+z^2=\frac{x\,dz}{z\,dx}+1$

$\frac{dx}{x}=\frac{dz}{z(1+z^2)}$

$4\,ln|x| = 2\,ln|z| - ln|z+z^2| + ln\,c$

$x^4=\frac{z^2\cdot c}{2+z^2}$

$y=cx\sqrt{x^2y^2+2}$ (c 为任意常数)

3.(1) $xy'+y=cos\,x$

解: $\frac{dy}{dx}+\frac{y}{x}=\frac{cos\,x}{x}$ ①

常数变易法: $\frac{dy}{dx}+\frac{y}{x}=0$

$\frac{dy}{y}=-\frac{dx}{x}$

积分: $ln|y|=-\,ln|x|+c_1$

$y=Cx$ $(C=\pm e^{c_1})$

$y=\frac{u}{x}$ ②

$\Rightarrow \frac{dy}{dx}=\frac{du}{dx}\cdot\frac{1}{x}-\frac{u}{x^2}$ ③

将②③带入①中 $\frac{du}{dx}\cdot\frac{1}{x}-\frac{u}{x^2}+\frac{u}{x^2}=\frac{cos\,x}{x}$

$\Rightarrow du=\cos x\,dx$

积分 $u=sin\,x+c$

代入②中 通解为$y=(sin\,x+c)\frac{1}{x}$ （c 为任意常数）

（2）$y'-\frac{2y}{x}=x^2\,sin\,3x$

解：$\frac{dy}{dx}-\frac{2}{x}y=x^2\,sin\,3x$ ①

$\frac{dy}{dx}-\frac{2}{x}y=0$

$\frac{dy}{y}=2\frac{dx}{x}$

积分:

$ln|y|=2\,ln|x|+c_1$

$y=cx^2(c=\pm e^{c_1})$

$y=ux^2$ ②

$\frac{dy}{dx}=\frac{du}{dx}x^2+2ux$ ③

将②③代入①中

$\frac{du}{dx}x^2+2ux-2ux=x^2\sin 3x$

$\Rightarrow du=\sin 3x\,dx$

积分 $u=-\frac{1}{3}cos\,3x+c$

代入②中 通解 $y=\left(-\frac{1}{3}cos\,3x+c\right)x^2$ (c 为任意常数)

(3)$(y^2-6x)y'+2y=0$②

解 $\frac{dx}{dy}-\frac{3x}{y}=-\frac{y}{2}$ ①

$\frac{dx}{dy}=\frac{3x}{y}$

$\frac{dx}{x}=\frac{3\,dy}{y}$

积分 $ln|x|=3\,ln|y|+c_1$

$x=cy^3$ $(c=\pm e^{c_1})$

$x=uy^3$ ②

$\frac{dx}{dy}=\frac{du}{dy}y^3+3uy^2$ ③

将②③代入①中 $\frac{du}{dy}y^3+3uy^2-3uy^2=-\frac{y}{2}$

$\Rightarrow du=-\frac{1}{2y^2}dy$

积分 $u=\frac{1}{2y}+c$

代入②中 $x=\left(\frac{1}{2y}+c\right)y^3=cy^3+\frac{y^2}{2}$ （c 为任意常数)

（4）$y'\cos x+y\sin x=1$

解：$\frac{dy}{dx}+y\tan x=\frac{1}{\cos x}$①

$\frac{dy}{dx}+y\tan x=0$

$\frac{dy}{y}=-\tan x\,dx$

积分 $\ln|y|=-\ln|\sec x|+c_1$

$y=c\cos x$ $(c=\pm e^{c_1})$

$y=u\cos x$ ②

$\frac{dy}{dx}=\frac{du}{dx}\cos x-u\sin x$③

将②③代入①中

$\frac{du}{dx}\cos x-u\sin x+n\sin x=\frac{1}{\cos x}$

$\Rightarrow\ du=\frac{1}{\cos^2 x}dx$

积分：$u=\tan x+c$

代入②通解：$y=(\tan x+c)\cos x=c\cos x+\sin x$ （c 为任意常数）

4（1）$y'+2\frac{y}{x}=x^2y^{\frac{4}{3}}$

解 $y^{-\frac{4}{3}}\frac{dy}{dx}+2\frac{1}{x}\cdot y^{-\frac{1}{3}}=x^2$①

$z=y^{-\frac{1}{3}}$

$\frac{dz}{dx}=-\frac{1}{3}y^{-\frac{4}{3}}\frac{dy}{dx}$

代入①中

$\frac{dz}{dx}-\frac{2}{3}\frac{z}{x}=-\frac{1}{3}x^2$②

$\frac{dz}{dx}=\frac{2}{3}\frac{z}{x}$

$\frac{dz}{z}=\frac{2}{3}\frac{dx}{x}$

积分$\ln|z|=\frac{2}{3}\ln|x|+c_1$

$z=cx^{\frac{2}{3}}(c=\pm e^{c_1})$

$z = ux^{\frac{2}{3}}$③

$\frac{dz}{dx} = \frac{du}{dx}x^{\frac{2}{3}} + \frac{2}{3}ux^{-\frac{1}{3}}$④

将③④代入②中

$\frac{du}{dx}x^{\frac{2}{3}} + \frac{2}{3}ux^{-\frac{1}{3}} - \frac{2}{3}ux^{-\frac{1}{3}} = -\frac{1}{3}x^2$

$\Rightarrow du = -\frac{1}{3}x^{\frac{4}{3}}dx$

积分 $u = -\frac{1}{7}x^{\frac{7}{3}} + c$

代入③中$z = \left(-\frac{1}{7}x^{\frac{7}{3}} + c\right)x^{\frac{2}{3}} = -\frac{1}{7}x^3 + cx^{\frac{2}{3}}$

$y = \left(-\frac{1}{7}x^3 + c'x^{\frac{2}{3}}\right)^{-3}$ (c'为任意常数)

（2）$\frac{dy}{dx} = \frac{1}{xy + x^3y^3}$

解 $\frac{dx}{dy} = xy + x^3y^3$

$\Rightarrow x^{-3}\frac{dx}{dy} - yx^{-2} = y^3$

$z = x^{-2}$

$\frac{dz}{dy} = -2x^{-3}\frac{dx}{dy}$

$\frac{dz}{dy} + 2yz = -2y^3$ ①

$\frac{dz}{dy} + 2yz = 0$

$\frac{dz}{z} = -2y\,dy$

积分:$ln|z| = -y^2 + c_1$

$z = ce^{-y^2}$

$z = ue^{-y^2}$②

$\frac{dz}{dy} = \frac{du}{dy}e^{-y^2} - 2ue^{-y^2}y$③

将②③代入①中

$\frac{du}{dy}e^{-y^2} - 2ue^{-y^2}y + 2ue^{-y^2}y = -2y^3$

$\Rightarrow du = -2y^3e^{y^2}\,dy$

积分:$u = (1 - y^2)e^{y^2} + c$

代入②中:$z = 1 - y^2 + ce^{-y^2} = x^{-2}$

$\therefore -x^{-2} - y^2 + 1 + ce^{-y^2} = 0$（$c$ 为任意常数）

（3）$\frac{dy}{dx} = \frac{1}{x-y} + 1$

解:　设$x - y = z$,则$\frac{dz}{dx} = -\frac{dy}{dx} + 1$

代入原方程:$-\frac{dz}{dx} = \frac{1}{z}$

$-z\,dz = dx$

$z^2 = -2(x - c_1)$

$(x - y)^2 = -2x + c$（c 为任意常数）

（4）$(1 - xy + x^2y^2)\,dx + (x^3y - x^2)\,dy = 0$

解:令 $z = xy$，则$dz = x\,dy + y\,dx$

$\therefore dy = \frac{x\,dz - z\,dx}{x^2}$

$\therefore$代入原方程:$(1 - z + z^2)\,dx + x^2(z - 1)\frac{x\,dz - z\,dx}{x} = 0$

$\Rightarrow (1 - z + z^2)\,dx + (z - 1)x\,dz - (z - 1)z\,dx = 0$

$\Rightarrow (z - 1)x\,dz + dx = 0$

$\therefore (z - 1)\,dz = -\frac{dx}{x}$

两端积分:$\frac{1}{2}z^2 - z = -\,ln|x| + c$

$\therefore ln|x| + \frac{1}{2}x^2y^2 - xy = c$（$c$ 为任意常数）

5（1）$y' + 3y = 8$，　$y(0) = 2$

$\frac{dy}{dx} = 8 - 3y$

$\frac{dy}{8-3y} = dx$

两端积分：$-\frac{1}{3}ln|8 - 3y| = x + c$

$\therefore 8 - 3y = ce^{-3x}$

代入$y(0) = 2$　　$c = 2$

$\therefore$特解为：$y = \frac{8-2e^{-3x}}{3}$

（2）$xyy' = x^2 + y^2$，$y(1) = 1$

$$\frac{dy}{dx}=\frac{x}{y}+\frac{y}{x}$$

令$m=\frac{y}{x}$，则$y=mx$，$\frac{dy}{dx}=m+x\frac{dm}{dx}$

$$\therefore m+x\frac{dm}{dx}=m+\frac{1}{m}$$

$$m\,dm=\frac{dx}{x}$$

两端积分：$\frac{1}{2}m^2=ln|x|+ln\,c$

$$\therefore \frac{y^2}{x^2}=ln\,x^2+c$$

代入$y(1)=1$ $\therefore c=1$

$\therefore$特解为：$\frac{y^2}{x^2}=2\,ln\,x+1$

（3）$(y-x^2y)\,dy+x\,dx=0$，$y\left(\sqrt{2}\right)=0$

$$\frac{x}{x^2-1}dx=y\,dy$$

两端积分：$ln|x^2-1|=y^2+c$

代入$y\left(\sqrt{2}\right)=0$ $\therefore c=0$

$$\therefore y^2=ln(x^2-1)$$

（4）$xy'=y+x\,cos^2\left(\frac{y}{x}\right)$，$y(1)=\frac{\pi}{4}$

$$\frac{dy}{dx}=\frac{y}{x}+cos^2\left(\frac{y}{x}\right)$$

令$m=\frac{y}{x}$，则$y=mx$,$\frac{dy}{dx}=m+x\frac{dy}{dx}$

$$\therefore m+x\frac{dm}{dx}=m+cos^2\,m$$

$$\frac{dm}{cos^2\,m}=\frac{dx}{x}$$

两端积分:$tan\,m=ln|x|+c$

$$tan\frac{y}{x}=ln|x|+c$$

代入$y(1)=\frac{\pi}{4}$ $c=1$

$$\therefore tan\frac{y}{x}=ln\,x+1$$

$e^x\left(c-2\left(xe^{-x}-\int e^{-x}\,dx\right)\right)$ （5） $y'-\frac{4xx}{x^2+1}=x\sqrt{y}$, $y(0)=0$

令$z=y^{\frac{1}{2}}$ ， $z'=\frac{1}{2\sqrt{y}}y'$

$\therefore z'-\frac{2x}{x^2+1}z=\frac{1}{2}x$

$z=e^{-\int P(x)\,dx}\left(\int Q(x)e^{\int P(x)\,dx}\,dx+c\right)$

$=\frac{1}{4}(x^2+1)(ln(x^2+1)+c)$

由$y(0)=0$ $\therefore c=0$

$\therefore$将$z=y^{\frac{1}{2}}$代入上式$y=\frac{1}{16}(x^2+1)^2\,ln^2(x^2+1)$

6、$\frac{dy}{dx}=2x+y$且$y(0)=0$

$y=e^{\int dx}\left(c+\int 2xe^{-x}\,dx\right)$

$=e^x\left(c-2\int x\,de^{-x}\right)$

$=e^x\left(c-2\left(xe^{-x}-\int e^{-x}\,dx\right)\right)$

$=e^x(c-2xe^{-x}-2e^{-x})$

$=ce^x-2x-2$

代入$y(0)=0$ $\therefore c=2$

$\therefore$所求曲线方程为$y=2e^x-2x-2$

7. 解: $y'+\frac{y}{\arcsin x\sqrt{1-x^2}}=\frac{1}{arc\sin x}$

$\frac{dy}{y}=-\frac{dx}{\arcsin x\sqrt{1-x^2}}$

积分 $ln|y|=-ln|\arcsin x|+c_1$

$y=c\frac{1}{arc\sin x}$

$y=u\frac{1}{arc\sin x}$

$\frac{dy}{dx}=\frac{du}{dx}\frac{1}{\arcsin x}+u\frac{1}{\sqrt{1-x^2}(\arcsin x)^2}$

$\frac{du}{dx}\frac{1}{\arcsin x}=\frac{1}{arc\sin x}$

$du=dx$

积分　　$u = x + c$

$$y = \frac{x+c}{\arcsin x}$$

代入$\left(\frac{1}{2}, 0\right)$ $\frac{1}{2} + c = 0$

$$c = -\frac{1}{2}$$

$$y = \frac{x-\frac{1}{2}}{\arcsin x}$$

8、$\frac{dy(x)}{dx} = y(x) + e^x$

$y(x) = e^x(x + c)$

$y(0) = 1$

$\therefore c = 1$

$\therefore y(x) = e^x(x + 1)$

9、证（1）$\phi_1'(x) + P(x)\phi_1(x) = 0$

$\phi_2'(x) + P(x)\phi_2(x) = 0$

$\phi_1'(x) + \phi_2'(x) + P(x)[\phi_1(x) + \phi_2(x)] = 0$

$[\phi_1(x) + \phi_2(x)]' + P(x)[\phi_1(x) + \phi_2(x)] = 0$

故$\phi_1(x) + \phi_2(x)$ 为$y' + P(x)y = 0$的解

（2）同（1）

（3）$\phi_1'(x) + P(x)\phi_1(x) = 0$

$\psi_1'(x) + P(x)\psi_1(x) = Q(x)$

$[\phi_1'(x) + \psi_1'(x)] + P(x)[\phi_1(x) + \psi_1(x)] = Q(x)$

$[\phi_1(x) + \psi_1(x)]' + P(x)[\phi_1(x) + \psi_1(x)] = Q(x)$

故$\phi_1(x) + \psi_1(x)$为$y' + P(x)y = Q(x)$的解