习题<6.3>答案

解：

1、(1) $\int_0^1 (2x-3)^2 dx$，

令 $2x-3=t \Rightarrow x=\frac{3+t}{2}$

$(2x-3)\big|_0^1 \rightarrow t\big|_{-3}^{-1}$ 积分变量变化时，积分区间也要相应地变化。

$$\Rightarrow \int_0^1 (2x-3)^2 dx = \int_{-3}^{-1} t^2 d\left(\frac{3+t}{2}\right) = \frac{1}{2}\cdot\frac{t^3}{3}\Big|_{-3}^{-1} = \frac{1}{2}\left(\frac{-1}{3}-\frac{-27}{3}\right) = \frac{13}{3}$$

(2) $f(x)$在$[0,\frac{1}{2}]$上连续可导 $\Rightarrow$ $f(x)$在$[0,\frac{1}{2}]$可积　连续 $\underset{\not\Leftarrow}{\Rightarrow}$ 可积

$\int_0^1 f'\left(\frac{1-x}{2}\right)dx$

令 $\frac{1-x}{2}=t \Rightarrow x=1-2t$，$\frac{1-x}{2}\big|_0^1 \rightarrow t\big|_{\frac{1}{2}}^0$

$$\Rightarrow \int_0^1 f'\left(\frac{1-x}{2}\right)dx = \int_{\frac{1}{2}}^0 f'(t)\,d(1-2t) = 2\int_0^{\frac{1}{2}} f'(t)\,dt = 2f(t)\Big|_0^{\frac{1}{2}}$$

$$=2\left(f\left(\tfrac{1}{2}\right)-f(0)\right)$$

2、解：(1) $\int_0^1 x\sqrt{1-x}\,dx$

$$\xrightarrow[t|_1^0]{令\sqrt{1-x}=t} \int_1^0 t\cdot(1-t^2)\,d(1-t^2)$$

$= 2\int_0^1 (t^2-t^4)\,dt$

$= 2\cdot\frac{t^3}{3}\Big|_0^1 - 2\cdot\frac{t^5}{5}\Big|_0^1$

$= 2\left(\frac{1}{3}-0\right)-2\left(\frac{1}{5}-0\right)$

$= \frac{2}{3}-\frac{2}{5}$

$= \frac{4}{15}$

(2) $\int_0^1 x(2-x^2)^5 dx$

$= -\frac{1}{2}\int_0^1 (2-x^2)^5\,d(2-x^2)$

$= -\frac{(2-x^2)^6}{12}\Big|_0^1$

$= -\left(\frac{1}{12}-\frac{2^6}{12}\right)$

$= \frac{21}{4}$

(3) $\int_1^{\sqrt{3}} \frac{dx}{x^2\sqrt{1+x^2}}$

$$\xrightarrow[t|_{\frac{\pi}{4}}^{\frac{\pi}{3}}]{x=\tan t} \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{\tan^2 t\cdot\sec t}\cdot\sec^2 t\cdot dt$$

$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\cos t}{\sin^2 t}\,dt$

$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{\sin^2 t}\,d\sin t$

$= -\frac{1}{\sin t}\Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}}$

$= -\left(\frac{2\sqrt{3}}{3}-\sqrt{2}\right)$

$= \sqrt{2}-\frac{2\sqrt{3}}{3}$

(4) $\int_0^1 \frac{dx}{e^x+e^{-x}}$

$$\xrightarrow[t|_1^e]{e^x=t} \int_1^e \frac{1}{t+\frac{1}{t}}\cdot\frac{1}{t}\,dt$$

$= \int_1^e \frac{1}{1+t^2}\,dt$

$= \arctan t\Big|_1^e$

$= \arctan e-\frac{\pi}{4}$

15) $\int_0^1 \frac{1}{e^x+1}dx$

$$\xrightarrow[t|_1^e]{e^x=t} \int_1^e \frac{1}{1+t}\cdot\frac{1}{t}dt$$

$$=\int_1^e\left(\frac{1}{t}-\frac{1}{1+t}\right)dt$$

$$=\ln t|_1^e-\ln(1+t)|_1^e$$

$$=1-0-\ln(1+e)+\ln 2$$

$$=1+\ln 2-\ln(1+e)$$

$$=\ln\frac{2e}{1+e}$$

17) $\int_0^a \frac{dx}{x+\sqrt{a^2-x^2}}$

$$\xrightarrow[t|_0^{\frac{\pi}{2}}]{x=a\sin t} \int_0^{\frac{\pi}{2}} \frac{\cos t}{\sin t+\cos t}dt$$

线性代换

$$=\int_0^{\frac{\pi}{2}} \frac{\frac{1}{2}(\sin t+\cos t)+\frac{1}{2}(\cos t-\sin t)}{\sin t+\cos t}dt$$

$$=\int_0^{\frac{\pi}{2}}\left(\frac{1}{2}+\frac{1}{2}\cdot\frac{\cos t-\sin t}{\sin t+\cos t}\right)dt$$

$$=\frac{t}{2}\Big|_0^{\frac{\pi}{2}}+\frac{1}{2}\int_0^{\frac{\pi}{2}}\frac{1}{\sin t+\cos t}d(\sin t+\cos t)$$

$$=\frac{\pi}{4}+\frac{1}{2}\ln(\sin t+\cos t)\Big|_0^{\frac{\pi}{2}}$$

$$=\frac{\pi}{4}+\frac{1}{2}(\ln 1-\ln 1)$$

$$=\frac{\pi}{4}$$

16) $\int_{\frac{\sqrt{2}}{2}}^1 \frac{\sqrt{1-x^2}}{x^2}dx$

$$\xrightarrow[t|_{\frac{\pi}{4}}^{\frac{\pi}{2}}]{x=\sin t} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos t}{\sin^2 t}\cdot\cos t\,dt$$

$$=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{\cos^2 t}{\sin^2 t}dt$$

$$=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{1-\sin^2 t}{\sin^2 t}dt$$

$$=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}(\csc^2 t-1)dt$$

$$=-\cot t\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}}-t\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$=-(0-1)-\left(\frac{\pi}{2}-\frac{\pi}{4}\right)$$

$$=1-\frac{\pi}{4}$$

18) $I=\int_0^{\frac{\pi}{2}}\frac{\sin\theta}{\sin\theta+\cos\theta}d\theta$, $J=\int_0^{\frac{\pi}{2}}\frac{\cos\theta}{\sin\theta+\cos\theta}d\theta$

半余法

$$\Rightarrow\begin{cases} I+J=\int_0^{\frac{\pi}{2}}\frac{\sin\theta+\cos\theta}{\sin\theta+\cos\theta}d\theta=\int_0^{\frac{\pi}{2}}1\,d\theta=\frac{\pi}{2} & ① \\ I-J=\int_0^{\frac{\pi}{2}}\frac{\sin\theta-\cos\theta}{\sin\theta+\cos\theta}d\theta=-\int_0^{\frac{\pi}{2}}\frac{1}{\sin\theta+\cos\theta}d(\sin\theta+\cos\theta) & \end{cases}$$

$$\Rightarrow I-J=-\ln(\sin\theta+\cos\theta)\Big|_0^{\frac{\pi}{2}}=-(\ln 1-\ln 1)=0 \quad ②$$

$$\Rightarrow \frac{①+②}{2}=I=\frac{\pi}{4}$$

$$\frac{①-②}{2}=J=\frac{\pi}{4}$$

▲(9) $\int_0^{\frac{\pi}{2}} \frac{dx}{a^2\sin^2 x+b^2\cos^2 x}$ $(a,b>0)$

$\xlongequal[u|_0^{+\infty}]{令u=\tan x} \int_0^{+\infty} \frac{1}{a^2\cdot\frac{u^2}{1+u^2}+b^2\cdot\frac{1}{1+u^2}}\, d\arctan u$

$= \int_0^{+\infty} \frac{1+u^2}{a^2u^2+b^2}\cdot\frac{1}{1+u^2}\, du$

$= \int_0^{+\infty} \frac{1}{a^2u^2+b^2}\, du$

$= \frac{1}{b^2}\int_0^{+\infty} \frac{1}{1+(\frac{a}{b}u)^2}\left(d\left(\frac{a}{b}u\right)\right)\cdot\frac{b}{a}$

$= \frac{1}{ab}\arctan\frac{a}{b}u\Big|_0^{+\infty}$ $\quad \lim\limits_{x\to+\infty}\arctan\frac{a}{b}x=\frac{\pi}{2}$

$\downarrow$ 可认为 $\arctan\frac{a}{b}u=\frac{\pi}{2}$ $(u\to+\infty)$

$= \frac{1}{ab}\left(\frac{\pi}{2}-0\right)$

$= \frac{\pi}{2ab}$

(10) $f(x)=\begin{cases}1+x^2, & x\le 0\\ e^{-x}, & x>0\end{cases}$

$\int_1^3 f(x-2)\,dx$

$\xlongequal[t|_{-1}^{1}]{x-2=t} \int_{-1}^{1} f(t)\,dt$

$= \int_{-1}^{0} f(t)\,dt + \int_0^1 f(t)\,dt$ 分段函数将积分区间相应分段.

$= \int_{-1}^{0}(1+t^2)\,dt + \int_0^1 e^{-t}\,dt$

$= t|_{-1}^{0} + \frac{t^3}{3}\Big|_{-1}^{0} - e^{-t}\Big|_0^1$

$= 1+\frac{1}{3}-e^{-1}+1$

$= \frac{7}{3}-\frac{1}{e}$

△定积分的值与积分变量无关

3. 证：∵ $f(x)$在$[-a,a]$上连续

$\Rightarrow f(x)$在$[-a,a]$可积

$\int_{-a}^{a} x(f(x)+f(-x))\,dx = \int_{-a}^{a} xf(x)\,dx + \int_{-a}^{a} xf(-x)\,dx$

$\Rightarrow \int_{-a}^{a} xf(-x)\,dx \xlongequal[t|_{a}^{-a}]{-x=t} \int_{a}^{-a}(-t)f(t)\,d(-t) = \int_{a}^{-a} tf(t)\,dt = \int_{a}^{-a} xf(x)\,dx$

$\Rightarrow \int_{-a}^{a} x(f(x)+f(-x))\,dx = \int_{-a}^{a} xf(x)\,dx + \int_{a}^{-a} xf(x)\,dx = 0$

4. 证：$\int_0^1 x^m(1-x)^n\,dx \xlongequal[t|_1^0]{1-x=t} \int_1^0 (1-t)^m t^n\,d(1-t) = \int_0^1 t^n(1-t)^m\,dt = \int_0^1 x^n(1-x)^m\,dx$

$=$右式

综上：$\int_0^1 x^m(1-x)^n\,dx = \int_0^1 x^n(1-x)^m\,dx$

5. 证：令 $t=\frac{1}{u}$ $(x>0)$ $\quad x>0 \Rightarrow t>0$即可 $\Rightarrow$ 可以令 $t=\frac{1}{u}$ $(u>0)$

$t|_x^1 \Rightarrow u|_{\frac{1}{x}}^{1}$

$\Rightarrow \int_x^1 \frac{1}{1+t^2}\,dt = \int_{\frac{1}{x}}^{1} \frac{1}{1+(\frac{1}{u})^2}\cdot\left(-\frac{1}{u^2}\right)du = \int_1^{\frac{1}{x}} \frac{1}{1+u^2}\,du = \int_1^{\frac{1}{x}} \frac{1}{1+t^2}\,dt$ =右式

综上：$\int_x^1 \frac{1}{1+t^2}\,dt = \int_1^{\frac{1}{x}} \frac{1}{1+t^2}\,dt$ $(x>0)$

6. 证：$f(x)$ 为连续函数 $\Rightarrow$ $f(x)$ 在 $x\in D$ 时可积

(1) ∵ $f(x)$ 为奇函数

∴ $f(x) = -f(-x)$

令 $F(x) = \int_0^x f(t)\,dt$，则 $F(-x) = \int_0^{-x} f(t)\,dt$

令 $t=-u$，$t|_0^{-x} \to u|_0^x$

$\Rightarrow F(-x) = \int_0^{-x} f(t)\,dt = \int_0^x f(-u)\,d(-u) = \int_0^x -f(-u)\,du = \int_0^x f(u)\,dx = \int_0^x f(t)\,dt = F(x)$

故 $\int_0^x f(t)\,dt$ 在 $f(x)$ 为奇函数时，为偶函数

(2) ∵ $f(x)$ 为偶函数

∴ $f(x) = f(-x)$

令 $G(x) = \int_0^x f(t)\,dt$，则 $G(-x) = \int_0^{-x} f(t)\,dt$

令 $t=-k$，$t|_0^{-x} \to k|_0^x$

$\Rightarrow G(-x) = \int_0^{-x} f(t)\,dt = \int_0^x f(-k)\,d(-k) = -\int_0^x f(k)\,dk = -G(x)$

故当 $f(x)$ 为偶函数时，$\int_0^x f(t)\,dt$ 为奇函数.