习题 4.7

1. 求下列曲线在指定点处的曲率

（1）曲线 xy=4，点（2, 2）

$$y' = -\frac{4}{x^2}, y'(2) = -1;$$

$$y'' = \frac{8}{x^3}, y''(2) = 1;$$

由曲率公式$k = \frac{|y''|}{(1+(y')^2)^{\frac{3}{2}}}$，带入得 $\mathrm{k} = \frac{\sqrt{2}}{4}$

（2）曲线 $y=4x-x^2$, 点（0, 0）

$$y' = 4-2x，y'(0) = 4$$

$$y'' = -2，y''(0) = -2$$

由曲率公式$k = \frac{|y''|}{(1+(y')^2)^{\frac{3}{2}}}$，带入得 $\mathrm{k} = \frac{2}{\sqrt{17^3}}$

（3）曲线$y = \ln\left(x+\sqrt{1+x^2}\right)$，点（0, 0）

$$y' = \frac{1}{\sqrt{x^2+1}}，y'(0) = 1$$

$$y'' = -x(x^2+1)^{-\frac{3}{2}}，y''(0) = 0$$

由曲率公式$k = \frac{|y''|}{(1+(y')^2)^{\frac{3}{2}}}$，带入得 $\mathrm{k} = 0$

（4）曲线 y=lnx，点（1, 0）

$$y' = \frac{1}{x}，y'(1) = 1$$

$$y' = -\frac{1}{x^2}，y''(1) = -1$$

由曲率公式$k=\frac{|y''|}{(1+(y')^2)^{\frac{3}{2}}}$，带入得 $\mathrm{k}=\frac{\sqrt{2}}{4}$

2. 请证明公式（4.7.4）

证明：$\begin{cases}x=x(t)\\y=y(t)\end{cases}$，$y'=\frac{dy}{dx}=\frac{dy}{dt}\frac{dt}{dx}=\frac{y'(t)}{x'(t)}$

$$y''=\frac{d^2y}{dx^2}=\frac{d\left(\frac{dy}{\alpha x}\right)}{dx}=\frac{d\left(\frac{y'(t)}{x'(t)}\right)}{dt}\frac{dt}{\mathrm{dx}}=\frac{y''(t)x'(t)-x''(t)y'(t)}{[x'(t)]^2}\frac{1}{x'(t)}$$

分别将y'和y''带入 $\mathrm{k}=\frac{|\mathrm{y}''|}{(1+(\mathrm{y}')^2)^{\frac{3}{2}}}$式中得：

$$k=\frac{|x'(t)y''(t)-x''(t)y'(t)|}{[x'(t)^2]^{\frac{3}{2}}\left(1+\left(\frac{y'(t)}{x'(t)}\right)^2\right)^{\frac{3}{2}}}=\frac{|x'(t)y''(t)-x''(t)y'(t)|}{\left[\left(x'(t)\right)^2+\left(y'(t)\right)^2\right]^{\frac{3}{2}}}$$

3. 求由下列参数方程表示的曲线在指定参数处的曲率

(1)曲线$\begin{cases}x=3t^2\\y=3t-t^3\end{cases}$，$\mathrm{t}=1$;

$x'(t)=6t, x'(1)=6; x''(t)=6, x''(1)=6;$

$y'(t)=3-3t^2, y'(1)=0; y''(t)=-6t, y''(1)=-6;$

由 $\mathrm{k}=\frac{|x'(t)y''(t)-x''(t)y'(t)|}{\left[\left(x'(t)\right)^2+\left(y'(t)\right)^2\right]^{\frac{3}{2}}}$,得 $\mathrm{k}=\frac{1}{6}$

(2)曲线$\begin{cases}x=a(cost+tsint)\\y=a(sint-tcost)\end{cases}$, $t=\frac{\pi}{2}$，其中$a>0$.

$x'(t)=atcost, x'\left(\frac{\pi}{2}\right)=0;$

$x''(t)=a(cost-tsint), x''\left(\frac{\pi}{2}\right)=-\frac{\pi a}{2};$

$y'(t)=atsint, y'\left(\frac{\pi}{2}\right)=\frac{\pi a}{2}$;

$y''(t)=a(sint+tcost), y''\left(\frac{\pi}{2}\right)=a;$

由$k=\frac{|x'(t)y''(t)-x''(t)y'(t)|}{\left[\left(x'(t)\right)^2+\left(y'(t)\right)^2\right]^{\frac{3}{2}}}$,得$k=\frac{2}{\pi a}$

4.求曲线 $y=x^2$ 上任一点处的曲率，并问哪一点处曲率最大？

由 $\mathrm{y}'=2\mathrm{x}, \mathrm{y}''=2$

带入公式 $\mathrm{k}=\frac{|\mathrm{y}''|}{(1+(\mathrm{y}')^2)^{\frac{3}{2}}}$，得；$\mathrm{k}=\frac{2}{(1+4x^2)^{\frac{3}{2}}}$

所以当 x=0 时，k 取最大值，

即曲线 $y=x^2$ 在 x=0 处曲率最大。

5.求椭圆周$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$上任一点处的曲率，并问哪一点处曲率最大？其中 a>b>0.

设$\begin{cases}x=acost\\y=bsint\end{cases}$(a>b>0)

则$x'(t)=-asint$;$x''(t)=-acost$;

$y'(t)=bcost; y''(t)=-bsint$;

$\Rightarrow \mathrm{k}=\frac{|x'(t)y''(t)-x''(t)y'(t)|}{\left[\left(x'(t)\right)^2+\left(y'(t)\right)^2\right]^{\frac{3}{2}}}$

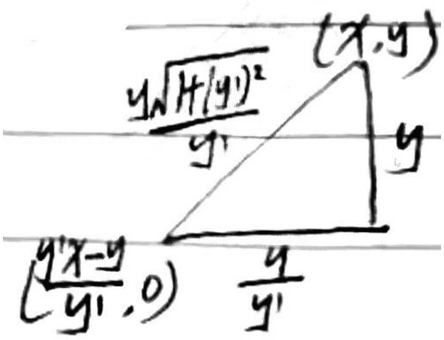
$\Rightarrow \mathrm{k}=\frac{ab}{(a^2\sin^2 t+b^2\cos^2 t)^{\frac{3}{2}}}$

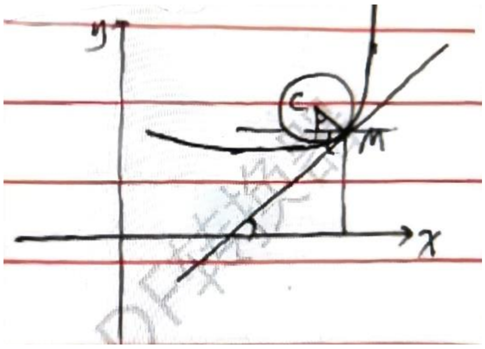
$\Rightarrow \mathrm{k}=\frac{ab}{(a^2(1-cos^2t)+b^2\cos^2 t)^{\frac{3}{2}}}$

$\Rightarrow \mathrm{k}=\frac{ab}{\left(a^2+(b^2-a^2)\cos^2 t\right)^{\frac{3}{2}}}$

所以当 cost=0，即 t=$\pm\frac{\pi}{2}$时，k 取最大值，此时 x=$\pm a$

6.





M点处曲率为$k=\frac{|y''|}{(H+(y)^2)^{\frac{2}{2}}}=\frac{|y''(x)|}{\left(1+(y'(x)^2)^{\frac{2}{2}}\right)}$

M点处切线为$\alpha=y't-y'x+y$

$C(\alpha,\beta)$

$\alpha=x-r\,sin\,arctan\,y'$

$\beta=y+r\,cos\,arctan\,y'$

$r=\frac{1}{k}$

$sin\,arctan\,y'=\frac{1}{\sqrt{1+(y)^2}}$

$cos\,arctan\,y'=\frac{y'}{\sqrt{1+(y')^2}}$

则$\begin{cases}\alpha=x-\frac{(1+(y'')^2)y'}{y''}\\ \beta=y+\frac{1+(y')^2}{y''}\end{cases}$

7.

解：$y=ln\,x$与x轴交点为$(1,0)$

$$y' = \frac{1}{x}, y'' = \frac{-1}{x^2}, k = \frac{1}{(2)^{\frac{3}{2}}} = \frac{\sqrt{2}}{4}$$

则$\rho = \frac{1}{k} = 2\sqrt{2}$

设圆心为(α, β)

$$a = x - \frac{(1 + (y')^2)y'}{y''} = 1 - \frac{2}{-1} = 3$$

$$\beta = y + \frac{1 + (y')^2}{y''} = \frac{2}{1} = -2$$

则方程为$(x-3)^2 + (y+2)^2 = 8$