

习题 6.1

1. (1) \times ($f(x)$ 在 $[a, b]$ 上可积一定有界, 但 $f(x)$ 有界为 $f(x)$ 在 $[a, b]$ 上可积最基本条件)

(2) \times ($\lambda \rightarrow 0 \neq n \rightarrow \infty$)

(3) \checkmark

(4) \checkmark

2. (1) C

(2) D (由施瓦茨不等式得 $\left(\int_a^b f(x)g(x) dx\right)^2 \leq \int_a^b f^2(x) dx \int_a^b g^2(x) dx$

其中令 $f(x) = x, g(x) = 1$ 则 $\left(\int_a^b x dx\right)^2 \leq \int_a^b x^2 dx (b-a)$

即 $\left(\int_0^1 x dx\right)^2 \leq \int_0^1 x^2 dx$)

(3) D

3. $S = \int_0^{10} (10t + 1) dt = 510m$

4. (1) 解: 令 $f(x) = x, x \in [0, 1]$

将 $x \in [0, 1], n$ 等分, $\Delta x_i = \frac{1}{n}$

取 $\xi_i = \frac{i}{n} (i = 1, 2, \dots, n)$

则 $\sum_{i=1}^n f(\xi_i) \Delta x_i = \frac{1}{n^2} (1 + 2 + \dots + n)$

$$= \frac{1}{n^2} \cdot \frac{n(n+1)}{2} = \frac{n+1}{2n}$$

当 $\lambda \rightarrow 0$ 时, $\pi \rightarrow \infty$

故 $\int_0^1 x dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(\xi_i) \Delta x_i = \frac{1}{2}$

(2) 解: 令 $g(x) = e^x, x \in [0, 1]$

将 $x \in [0, 1] n$ 等分, $\Delta x_i = \frac{1}{n}$

取 $\xi_i = \frac{i}{n} (i = 1, 2, \dots, n)$

则 $\sum_{i=1}^n f(\xi_i) \Delta x_i = \frac{1}{n} e^{\frac{1}{n}} + \frac{1}{n} e^{\frac{2}{n}} + \dots + \frac{1}{n} e^{\frac{n}{n}}$

$$= \frac{1}{n} \left(e^{\frac{1}{n}} + e^{\frac{2}{n}} + e^{\frac{3}{n}} + \dots + e^1 \right)$$

$$= \frac{1}{n} \cdot \frac{e^{\frac{1}{n}}(1-e)}{1-e^{\frac{1}{n}}} \quad (\text{等比数列})$$

由 $\lambda \rightarrow 0$ 时, $n \rightarrow \infty$ 时

$$\begin{aligned}\lim_{n \rightarrow \infty} \sum_{i=1}^n f(\xi_i) \Delta x_i &= \lim_{n \rightarrow \infty} \frac{1}{n} \cdot \frac{e^{\frac{1}{n}}(1-e)}{-\frac{1}{n}} \quad (\text{等价无穷小}) \\ &= \lim_{n \rightarrow \infty} (e-1)e^{\frac{1}{n}} = e-1\end{aligned}$$

5. 由定积分几何意义得 $\frac{1}{2} \cdot 2(k+8+k) = 10$

$$\Rightarrow 2k = 2 \Rightarrow k = 1$$

6. (1) 解: 令 $f(x) = \frac{1}{(1+x)^2}$, $x \in [0,1]$

对 $x \in [0,1]$ n 等分, $\Delta x_i = \frac{1}{n}$, 取 $\xi_i = \frac{2}{n} (i = 1, 2, \dots, n)$

$$\text{则 } \sum_{i=1}^n f(\xi_i) \Delta x_i = \frac{1}{\left(1+\frac{1}{n}\right)^2} \cdot \frac{1}{n} + \frac{1}{\left(1+\frac{2}{n}\right)^2} \cdot \frac{1}{n} + \dots + \frac{1}{\left(1+\frac{n}{n}\right)^2} \cdot \frac{1}{n}$$

$$= \sum_{i=1}^n \frac{n}{(n+i)^2}$$

当 $n \rightarrow \infty$, $\lambda \rightarrow 0$

$$\text{则原式} = \int_0^1 \frac{1}{(1+x)^2} dx$$

(2) 解: 令 $f(x) = \sin x$

在 $x \in \left[0, \frac{\pi}{2}\right]$ 分成 n 等份, 则 $\Delta x_i = \frac{\pi}{2n}$

取 $\xi_i = \frac{i\pi}{2n} (i = 0, 1, 2, \dots, n-1)$

$$\sum_{i=1}^n f(\xi_i) \Delta x_i = \frac{\pi}{2n} \left[\sin 0 + \sin \frac{\pi}{2n} + \dots + \frac{\sin(n-1)\pi}{2n} \right]$$

当 $n \rightarrow \infty$, $\lambda \rightarrow 0$

$$\text{则原式} = \int_0^{\frac{\pi}{2}} \sin x dx$$

7. (1) 解: 原式 $= \int_c^b f(x) dx + \int_b^a f(x) dx = \int_c^a f(x) dx$

(2) 由积分中值定理得

$$\text{平均值为 } f(\xi) = \frac{1}{1-0} \int_0^1 f(x) dx = \int_0^1 e^x dx$$

8. (1) 由 $f(x) = x$, $g(x) = \sqrt{x}$ 在区间 $[0,1]$ 可积, 且在 $[0,1]$ 上 $x \leq \sqrt{x}$

$$\text{由保序性 } \int_0^1 x dx \leq \int_0^1 \sqrt{x} dx$$

(2) 同理 $x \geq \sin x$

$$\int_0^{\frac{\pi}{2}} x dx \geq \int_0^{\frac{\pi}{2}} \sin x dx$$

(3) 因为在 $[-1,0]$ 上 $e^{2x} \leq e^x$ 所以 $\int_{-1}^0 e^{2x} dx \leq \int_{-1}^0 e^x dx$

两边加负号即 $\int_0^{-1} e^{2x} dx \geq \int_0^{-1} e^x dx$

(4) 由 $\ln x \geq (\ln x)^2$ 在 $[1, 2]$ 上

$$\text{同理 } \int_1^2 \ln x dx \geq \int_1^2 (\ln x)^2 dx$$

(5) 由在 $\left[0, \frac{\pi}{2}\right]$ 上 $x \leq \tan x$

$$\text{同理 } \int_0^{\frac{\pi}{2}} x dx \leq \int_0^{\frac{\pi}{2}} \tan x dx$$

9. (1) 由在 $x \in [0, 1]$ 内 $\frac{1}{1+x^2} \in \left[\frac{1}{2}, 1\right]$

$$\text{则 } \frac{1}{2} \cdot 1 \leq \int \frac{dx}{1+x^2} \leq 1 \cdot 1 \quad (\text{推论 6.1.1})$$

$$\text{即 } \frac{1}{2} \leq \int \frac{dx}{1+x^2} \leq 1$$

(2) 由于在 $x \in [0, 2]$, $x^2 - 2x \in [-1, 0]$

$$\text{则 } e^{x^2-2x} \in \left[\frac{1}{e}, 1\right]$$

$$\frac{2}{e} \leq \int_0^2 e^{x^2-2x} dx \leq 2$$

(3) 在 $x \in [0, 2\pi]$ 内 $\frac{1}{1+0.5 \cos x} \in \left[\frac{2}{3}, 2\right]$

$$\text{则 } \frac{4\pi}{3} \leq \int_0^{2\pi} \frac{dx}{1+0.5 \cos x} \leq 4\pi$$

(4) 在 $x \in [0, 100]$ 内 则 $\frac{e^{-x}}{x+100} \in \left[\frac{\frac{1}{e^{100}}}{200}, \frac{1}{100}\right]$

$$\frac{1}{2e^{100}} \leq \int_0^{100} \frac{e^{-x}}{x+100} dx \leq 1$$

10. 证明: (1) $\int_0^2 \frac{1}{\sqrt{1+x^3}} dx = \int_0^1 \frac{1}{\sqrt{1+x^3}} dx + \int_1^2 \frac{1}{\sqrt{1+x^3}} dx$

由积分中值定理得

$$\text{原式} = f(\xi_1)(1-0) + f(\xi_2)(2-1) \quad (\text{其中 } \xi_1 \in [0, 1], \xi_2 \in [1, 2])$$

$$= \frac{1}{\sqrt{1+\xi_1^3}} + \frac{1}{\sqrt{1+\xi_2^3}}$$

$$\text{则 } \frac{1}{\sqrt{1+0}} + \frac{1}{\sqrt{1+1}} \leq \int_0^2 \frac{1}{\sqrt{1+x^3}} dx \leq \frac{1}{\sqrt{1+1}} + \frac{1}{\sqrt{1+2^3}}$$

$$\text{即 } \frac{1}{3} + \frac{\sqrt{2}}{2} \leq \int_0^2 \frac{1}{\sqrt{1+x^3}} dx \leq 1 + \frac{\sqrt{2}}{2}$$

(2) $\int_0^{\frac{\pi}{2}} \sin^2 x dx = \int_0^{\frac{\pi}{6}} \sin^2 x dx + \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin^2 x dx + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin^2 x dx$

积分中值定理

$$= \sin^2 \xi_1 \left(\frac{\pi}{6} - 0\right) + \sin^2 \xi_2 \left(\frac{\pi}{3} - \frac{\pi}{6}\right) + \sin^2 \xi_3 \left(\frac{\pi}{2} - \frac{\pi}{3}\right) \quad (\xi_1 \in \left[0, \frac{\pi}{6}\right], \xi_2 \in \left[\frac{\pi}{6}, \frac{\pi}{3}\right], \xi_3 \in \left[\frac{\pi}{3}, \frac{\pi}{2}\right])$$

$$\text{则} \frac{\pi}{6} \sin^2 0 + \frac{\pi}{6} \sin^2 \frac{\pi}{6} + \frac{\pi}{6} \sin^2 \frac{\pi}{3} \leq \int_0^{\frac{\pi}{2}} \sin^2 x \, dx \leq \frac{\pi}{6} \left(\sin^2 \frac{\pi}{6} + \sin^2 \frac{\pi}{3} + \sin^2 \frac{\pi}{2} \right)$$

$$\text{即} \frac{\pi}{6} \leq \int_0^{\frac{\pi}{2}} \sin^2 x \, dx \leq \frac{\pi}{6} \left(\frac{1}{4} + \frac{3}{4} + 1 \right) = \frac{\pi}{3} \text{ 得证}$$

11. 解：由题意得 $\int_0^1 f(x) \, dx$ 为具体值

$$\text{设} f(x) = x + t$$

$$\text{则} 2 \int_0^1 f(x) \, dx = 2 \int_0^1 (x + t) \, dx = t$$

$$\text{则} 2 \int_0^1 x \, dx + 2 \int_0^1 t \, dx = t$$

$$\text{则} 2 \int_0^1 x \, dx = -t$$

$$t = -2 \cdot \frac{1}{2} \cdot 1 = -1$$

$$\text{故} f(x) = x - 1$$