习题 4.3

1.

$f(x)=x^4-5x^3+x^2-3x+4 \quad f(4)=-56$

$f'(x)=4x^3-15x^2+2x-3 \quad f'(4)=21$

$f''(x)=12x^2-30x+2 \quad f''(4)=74$

$f^{(3)}(x)=24x-30 \quad f^{(3)}(4)=66$

$f^{(4)}(x)=24 \quad f^{(4)}(4)=24$

$f(x)$在$x=4$ 的泰勒公式为

$$f(4)+f'(4)(x-4)+\frac{f''(4)}{2!}(x-4)^2+\frac{f^{(3)}(4)}{3!}(x-4)^3+\frac{f^{(4)}(4)}{4!}(x-4)^4$$

$$=(x-4)^4+11(x-4)^3+37(x-4)^2+21(x-4)-56$$

2.

(1)$f^{(k)}(x)=\frac{(-1)^k\cdot k!}{x^{k+1}} \quad f^{(k)}(-1)=\frac{(-1)^k\cdot k!}{(-1)^{k+1}}=-k! \quad (k=0,1,2,\cdots,n)$

则$f(x)$在$x=-1$ 的n阶泰勒公式为

$$f(x)+f'(x)(x+1)+\frac{f''(x)}{2!}(x+1)^2+\cdots+\frac{f^{(n)}(x)}{n!}(x+1)^n+o((x+1)^n)$$

$$=f(-1)+f'(-1)(x+1)+\frac{f''(-1)}{2!}(x+1)^2+\cdots$$

$$+\frac{f^{(n)}(-1)}{n!}(x+1)^n+o((x+1)^n)$$

$$=-1-(x+1)-(x+1)^2-(x+1)^3-\cdots-(x+1)^n+o((x+1)^n)$$

(2)设$f(x)=\ln(1-x)$ 定义域$(-\infty,1)$

$$f^{(k)}(x)=-\frac{(k-1)!}{(1-x)^{k}},k=1,2,\cdots,n$$

$$f^{(k)}\left(\frac{1}{2}\right)=-\frac{(k-1)!}{\left(\frac{1}{2}\right)^{k}}=-(k-1)!\cdot 2^{k},k=1,2,\cdots,n$$

$f(x)$在$x=\frac{1}{2}$的n阶泰勒公式为

$$f\left(\frac{1}{2}\right)+f'\left(\frac{1}{2}\right)\left(x-\frac{1}{2}\right)+\frac{f''\left(\frac{1}{2}\right)}{2!}\left(x-\frac{1}{2}\right)^{2}+\cdots+\frac{f^{(n)}\left(\frac{1}{2}\right)}{n!}\left(x-\frac{1}{2}\right)^{n}+o\left(\left(x-\frac{1}{2}\right)^{n}\right)$$

$$=-\ln 2-2\left(x-\frac{1}{2}\right)-2\left(x-\frac{1}{2}\right)^{2}-\frac{8}{3}\left(x-\frac{1}{2}\right)^{3}-\cdots-\frac{2^{n}}{n}\left(x-\frac{1}{2}\right)^{n}+o\left(\left(x-\frac{1}{2}\right)^{n}\right)$$

(3)设$f(x)=\frac{1}{2}(e^{x}+e^{-x})$

$$f^{(k)}(x)=\begin{cases}\frac{1}{2}(e^{x}+e^{-x}), & k\text{为偶数}\\ \frac{1}{2}(e^{x}-e^{-x}), & k\text{为奇数}\end{cases}$$

$$f^{(k)}(0)=\begin{cases}1, & k\text{为偶数}\\ 0, & k\text{为奇数}\end{cases}(k=0,1,2,\cdots,n)$$

则$f(x)$在$x=0$ 的 20 阶泰勒公式为

$$f(0)+f'(0)x+\frac{f''(0)}{2!}x^{2}+\frac{f'''(0)}{3!}x^{3}+\cdots+\frac{f^{(20)}(0)}{20!}x^{20}+o(x^{20})$$

$$=1+\frac{1}{2!}x^{2}+\frac{1}{4!}x^{4}+\cdots+\frac{1}{20!}x^{20}+o(x^{20})$$

(4)设$f(x)=xe^{x}$

$f^{(k)}(x)=(x+k)\cdot e^{x}\quad f^{(k)}(0)=k\quad ,k=0,1,2,\cdots,n$

$f(x)$在$x=0$ 的n阶泰勒公式为

$$f(0)+f'(0)x+\frac{f''(0)}{2!}x^{2}+\cdots+\frac{f^{(n)}(0)}{n!}x^{n}+o(x^{n})$$

$$= x + x^2 + \frac{x^3}{2!} + \frac{x^4}{3!} + \cdots + \frac{x^n}{(n-1)!} + o(x^n)$$

3.

(1)解：由泰勒公式知

$$(1+x^2)^{\frac{1}{4}} = 1 + \frac{1}{4}x^2 + o(x^2)$$

$$(1-x^2)^{\frac{1}{4}} = 1 - \frac{1}{4}x^2 + o(x^2)$$

$$\text{则原式} = \lim_{x\to 0}\frac{\left[1+\frac{1}{4}x^2+o(x^2)\right]-\left[1-\frac{1}{4}x^2+o(x^2)\right]}{x^2}$$

$$= \lim_{x\to 0}\left(\frac{1}{2}+\frac{o(x^2)}{x^2}\right) = \frac{1}{2}$$

(2)解：由泰勒公式知

$$\cos x^2 = 1 - \frac{1}{2!}x^4 + o(x^4)$$

$$x^2\cos x = x^2 - \frac{1}{2!}x^4 + o(x^4)$$

$$\sin x^2 = x^2 - \frac{1}{3!}x^6 + o(x^6)$$

$$\text{则原式} = \lim_{x\to 0}\frac{-x^2+o(x^4)}{x^2-\frac{1}{3!}x^6+o(x^6)} = \lim_{x\to 0}\frac{-1+\frac{o(x^4)}{x^2}}{1-\frac{1}{3!}x^6+\frac{o(x^6)}{x^2}} = \frac{-1}{1} = -1$$

(3)解：由泰勒公式知

$$e^{x^2} = 1 + x^2 + o(x^2)$$

$$\sin^2 2x = \frac{1-\cos 4x}{2} \quad \cos 4x = 1 - \frac{1}{2!}(4x^2) + 0(x^2)$$

则原式 $= \frac{1}{2}\lim_{x\to 0}\frac{o(x^2)}{8x^2-o(x^2)} = \frac{1}{2}\lim_{x\to 0}\frac{\frac{o(x^2)}{x^2}}{8-\frac{o(x^2)}{x^2}} = 0$

(4)解：原式 $= \lim_{x\to 0}\frac{\tan x-\sin x}{[x\ln(1+x)-x^2]\left(\sqrt{\tan x+1}+\sqrt{1+\sin x}\right)}$

$$= \lim_{x\to 0}\frac{\sin x-\frac{1}{2}\sin 2x}{2(x\ln(1+x)-x^2)}$$

$$= \lim_{x\to 0}\frac{\left[x-\frac{1}{3!}x^3+o(x^3)\right]-\frac{1}{2}\left[2x-\frac{8}{3!}x^3+o(x^3)\right]}{-x^3+o(x^3)}$$

$$= -\frac{1}{2}$$

4.

解：(1)*因为*$f(x)=\sqrt[3]{1+x}=(1+x)^{\frac{1}{3}}$

$$\approx 1+\frac{1}{3}x+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}x^2+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)}{3!}x^3$$

$$= 1+\frac{1}{3}x-\frac{1}{9}x^2+\frac{5}{81}x^3,$$

$$R_3(x)=\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)\left(\frac{1}{3}-3\right)}{4!}(1+\xi)^{\frac{1}{3}-4}x^4,$$

其中 ξ 介于 $0,x$之间. 故

$$\sqrt[3]{30}=\sqrt[3]{27+3}=3\sqrt[3]{1+\frac{1}{9}}\approx 3\left[1+\frac{1}{3}\cdot\frac{1}{9}-\frac{1}{9}\left(\frac{1}{9}\right)^2+\frac{5}{81}\left(\frac{1}{9}\right)^3\right]$$

$$\approx 3.10724.$$

误差　$|R_3| = 3\cdot\left|\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)\left(\frac{1}{3}-3\right)}{4!}(1+\xi)^{\frac{1}{3}-4}\left(\frac{1}{9}\right)^4\right|,$

ξ介于 0 与$\dfrac{1}{9}$之间, 即 $0<\xi<\dfrac{1}{9}$, 因此

$$|R_3|=\left|\frac{80}{4!\cdot 3^{11}}\right|\approx 1.88\times 10^{-5}.$$

5.

解：设$f(x)=2^x \quad f^{(k)}(x)=2^x(\ln 2)^k \quad (k=0,1,2,\cdots,n)$

$f(x)$在$x=0$ 的n阶泰勒公式为

$$1+\ln 2\cdot x+\frac{(\ln 2)^2}{2!}x^2+\frac{(\ln 2)^3}{3!}x^3+\cdots+\frac{(\ln 2)^n}{n!}x^n$$

则$2^{\frac{1}{5}}\approx 1+\ln 2\times\dfrac{1}{5}+\dfrac{(\ln 2)^2}{2!}\times\left(\dfrac{1}{5}\right)^2+\dfrac{(\ln 2)^3}{3!}\times\left(\dfrac{1}{5}\right)^3\approx 1.149$

6.

$$\lim_{x\to 0}\left(1+x+\frac{f(x)}{x}\right)^{\frac{1}{x}}=\lim_{x\to 0}\left(1+x+\frac{f(x)}{x}\right)^{\frac{1}{x+\frac{f(x)}{x}}\cdot\frac{x+\frac{f(x)}{x}}{x}}=e^3$$

$$\Rightarrow \lim_{x\to 0}\left(x+\frac{f(x)}{x}\right)=0 \quad \lim_{x\to 0}\frac{x+\frac{f(x)}{x}}{x}=3$$

$$\Rightarrow \quad \lim_{x\to 0}\frac{f(x)}{x}=0 \quad \lim_{x\to 0}\frac{f(x)}{x^2}=2$$

$$\lim_{x\to 0}\left(1+\frac{f(x)}{x}\right)^{\frac{1}{x}}=\lim_{x\to 0}\left(1+\frac{f(x)}{x}\right)^{\frac{x}{f(x)}\cdot\frac{1}{x}\cdot\frac{f(x)}{x}}=e^{\lim\limits_{x\to 0}\frac{f(x)}{x^2}}=e^2$$

7.

由待定系数法

构造函数$P(x)=\frac{x^3}{2}+\left(\frac{1}{2}-f(0)\right)x^2+f(0)$

设$F(x)=f(x)-P(x)$，显然$F(x)$在$[-1,1]$上有连续的三阶导数

且$F(-1)=F(1)=F(0)=F'(0)=0$

对$F(x)$在$[-1,0],[0,1]$用罗尔定理得

存在$-1<\theta_1<0\quad 0<\theta_2<1$

使$F'(\theta_1)=F'(\theta_2)=0$

对$F'(x)$在$[\theta_1,0],[0,\theta_2]$上用罗尔定理得

存在$-1<\theta_1<\eta_1<0\quad 0<\eta_2<\theta_2<1$

$$F''(\eta_1)=F''(\eta_2)=0$$

对$F''(x)$在$[\eta_1,\eta_2]$上用罗尔定理得

存在 $\xi\in(\eta_1,\eta_2)\subsetneq(-1,1)$

$F'''(\xi)=0$ 即 $f'''(\xi)=3$

8. 由泰勒公式得

$f(x+h)=f(x)+f'(x)h+\frac{1}{2}f''(x)h^2+o(h^2)$

$f(x-h)=f(x)-f'(x)h+\frac{1}{2}f''(x)h^2+o(h^2)$

$f(x)\le\frac{1}{2}[f(x-h)+f(x+h)]$

$\Rightarrow f''(x)+o(h^2)\ge 0$

令 $h\to 0$ 则 $f''(x)\ge 0$