1. $\forall \delta > 0$，当 $0 < |x - x_0| < \delta$ 时

$\lim\limits_{x \to x_0} g(x) = A \Rightarrow |g(x) - A| < \varepsilon \Rightarrow A - \varepsilon < g(x) < A + \varepsilon$

同理 $\lim\limits_{x \to x_0} h(x) = A \Rightarrow A - \varepsilon < h(x) < A + \varepsilon$

$\because g(x) \le f(x) \le h(x)$ $\therefore A - \varepsilon < g(x) \le f(x) \le h(x) < A + \varepsilon$

$\Rightarrow A - \varepsilon < f(x) < A + \varepsilon \Rightarrow |f(x) - A| < \varepsilon \Rightarrow \lim\limits_{x \to x_0} f(x) = A$

2.(1) $\lim\limits_{x \to \infty} \frac{[x]}{x}$

$x - 1 < [x] \le x \Rightarrow \frac{x-1}{x} < \frac{[x]}{x} \le 1$

$\because \lim\limits_{x \to \infty} \frac{x-1}{x} = 1$，$\lim\limits_{x \to \infty} 1 = 1$ $\therefore \lim\limits_{x \to \infty} \frac{[x]}{x} = 1$

(2) $\lim\limits_{x \to +\infty} \sqrt{1 + \frac{1}{x^\alpha}}$ $(\alpha > 0)$

$\sqrt{1} < \sqrt{1 + \frac{1}{x^\alpha}} < 1 + \frac{1}{x^\alpha}$

$\because \lim\limits_{x \to +\infty} \sqrt{1} = 1$ $\lim\limits_{x \to +\infty} 1 + \frac{1}{x^\alpha} = 1$ $\therefore \lim\limits_{x \to +\infty} \sqrt{1 + \frac{1}{x^\alpha}} = 1$

(3) $\lim\limits_{x \to 0} x[\frac{1}{x}]$

$\frac{1}{x} - 1 < [\frac{1}{x}] \le \frac{1}{x} \Rightarrow 1 - x < x[\frac{1}{x}] \le 1$

$\because \lim\limits_{x \to 0} 1 - x = 1$ $\lim\limits_{x \to 0} 1 = 1$ $\therefore \lim\limits_{x \to 0} x[\frac{1}{x}] = 1$

3.(1) $\lim\limits_{x \to 0} \sin\frac{1}{x}$

取 $x_1 = \frac{1}{2n\pi}$ $x_2 = \frac{1}{2n\pi + \frac{\pi}{2}}$ $n \in N^*$

易知 $\lim\limits_{n \to \infty} x_n' = \lim\limits_{n \to \infty} x_2 = 0$，$x_n'' \ne x_2 \ne 0$

$\because \lim_{n\to\infty} \sin\frac{1}{x_n'} = \lim_{n\to\infty} \sin 2n\pi = 0 \qquad \lim_{n\to\infty} \sin\frac{1}{x_n''} = \lim_{n\to\infty} \sin(2n\pi+\frac{\pi}{2}) = 1$

$\therefore \lim_{x\to 0} \sin\frac{1}{x}$ 不存在

(2) $\lim_{x\to 0} \cos\frac{1}{x}$

证明过程同(1)

4. (1) $\lim_{x\to 0} \frac{\sin \alpha x}{\sin \beta x} \ (\beta \neq 0)$

$= \lim_{x\to 0} \frac{\sin\alpha x}{\alpha x} \cdot \frac{\beta x}{\sin\beta x} \cdot \frac{\alpha}{\beta}$

$= \lim_{x\to 0} \frac{\sin\alpha x}{\alpha x} \cdot \lim_{x\to 0} \frac{\beta x}{\sin\beta x} \cdot \lim_{x\to 0} \frac{\alpha}{\beta} = \frac{\alpha}{\beta}$

(2) $\lim_{x\to 0} \frac{\tan\alpha x}{\tan\beta x} \ (\beta\neq 0)$

$= \lim_{x\to 0} \frac{\alpha x}{\beta x} = \frac{\alpha}{\beta}$

(3) $\lim_{x\to 0} \frac{1-\cos x}{x^2}$

$= \lim_{x\to 0} \frac{\frac{1}{2}x^2}{x^2} = \frac{1}{2}$

(4) $\lim_{x\to\frac{\pi}{4}} \frac{\sqrt{2}-2\cos x}{\sin(x-\frac{\pi}{4})}$

$= \lim_{x\to\frac{\pi}{4}} \frac{2\sin x}{\cos(x-\frac{\pi}{4})} = \frac{2\sin x}{\frac{\sqrt{2}}{2}(\cos x+\sin x)} = \frac{\sqrt{2}}{1} = \sqrt{2}$

5. (1) $\lim\limits_{x\to 0}(1-3x)^{\frac{1}{x}}$

$= \lim\limits_{x\to 0}[1+(-3x)]^{\frac{1}{-3x}\cdot(-3)} = e^{-3}$

(2) $\lim\limits_{x\to\infty}\left(\frac{1+x}{2+x}\right)^{\frac{1-x^2}{1-x}}$

$= \lim\limits_{x\to\infty}\left[1+\left(-\frac{1}{x+2}\right)\right]^{-(x+2)\cdot\frac{1+x}{-(x+2)}}$

$= \lim\limits_{x\to\infty} e^{\frac{1+x}{-(x+2)}} = e^{-1}$

(3) $\lim\limits_{x\to 0}(1+\sin x)^{3\csc x}$

$= \lim\limits_{x\to 0}(1+\sin x)^{\frac{1}{\sin x}\cdot 3} = e^{3}$

(4) $\lim\limits_{x\to\infty}\left(\frac{x+1}{x-1}\right)^{x}$

$= \lim\limits_{x\to\infty}\left(1+\frac{2}{x-1}\right)^{\frac{x-1}{2}\cdot\frac{2x}{x-1}}$

$= \lim\limits_{x\to\infty} e^{\frac{2x}{x-1}} = e^{2}$