习题6.3

1.

(1) $\int_0^1 (2x-3)^2 dx = \int_0^1 (4x^2-12x+9) dx$

$= \frac{4}{3}x^3 - 6x^2 + 9x \Big|_0^1 = \frac{10}{3}$

(2) 设$f(x)$在$[0,\frac{1}{2}]$上连续可导，则$\int_0^1 f'(\frac{1-x}{2}) dx = \underline{2[f(\frac{1}{2})-f(0)]}$

解：令$t=\frac{1-x}{2}$，$dx=-2dt$　$x\Big|_0^1 \to t\Big|_{\frac{1}{2}}^0$

$\int_0^1 f'(\frac{1-x}{2}) dx = \int_{\frac{1}{2}}^0 f'(t)(-2dt) = 2\int_0^{\frac{1}{2}} f'(t)\, dt = 2f(t)\Big|_0^{\frac{1}{2}}$

$= 2[f(\frac{1}{2}) - f(0)]$

2.

(1) $\int_0^1 x\sqrt{1-x}\, dx$　令$t=\sqrt{1-x}$　$dx=-2t\,dt$　$x\Big|_0^1 \to t\Big|_1^0$

原式$= \int_1^0 t\cdot(1-t^2)(-2t) dt$

$= 2\int_0^1 (t^2-t^4) dt = 2(\frac{1}{3}t^3\Big|_0^1 - \frac{1}{5}t^5\Big|_0^1)$

$= 2\cdot\frac{2}{15} = \frac{4}{15}$

(2) $\int_0^1 x(2-x^2)^5 dx = (-\frac{1}{2})\int_0^1 (2-x^2)^5 d(2-x^2)$

$= (-\frac{1}{2})\,\frac{1}{6}(2-x^2)^6 \Big|_0^1$

$= \frac{21}{4}$

(3) $\int_1^{\sqrt{3}} \frac{dx}{x^2\sqrt{1+x^2}}$　令$x=\tan t$　$x\Big|_1^{\sqrt{3}} \to t\Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}}$

原式$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec^2 t}{\tan^2 t\cdot\sec t}\cdot dt = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\cos t}{\sin^2 t} dt$

$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{d\sin t}{\sin^2 t} = -\frac{1}{\sin t}\Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}}$

$= \sqrt{2} - \frac{2\sqrt{3}}{3}$

(4) $\int_{\frac{\sqrt{2}}{2}}^{1}\frac{\sqrt{1-x^2}}{x^2}dx$　令$t=e^x$　$dx=\frac{dt}{t}$　$x|_0^1\to t|_1^e$

原式$=\int_1^e\frac{t}{t^2+1}\cdot\frac{1}{t}\cdot dt$

$=\int_1^e\frac{dt}{t^2+1}=\arctan t|_1^e=\arctan e-\frac{\pi}{4}$

(5) $\int_0^1\frac{dx}{e^x+1}$　令$t=e^x$　$x=\ln t$　$dx=\frac{dt}{t}$　$x|_0^1\to t|_1^e$

原式$=\int_1^e\frac{1}{t+1}\cdot\frac{1}{t}dt=\int_1^e\left(\frac{1}{t}-\frac{1}{t+1}\right)dt$

$=\ln t|_1^e-\ln(t+1)|_1^e$

$=\ln\frac{2e}{e+1}$

(6) $\int_{\frac{\sqrt{2}}{2}}^{1}\frac{\sqrt{1-x^2}}{x^2}dx$　令$x=\sin t$

原式$=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{\cos t}{\sin^2 t}\cdot\cos t\,dt=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{1-\sin^2 t}{\sin^2 t}dt=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\left(\frac{1}{\sin^2 t}-1\right)dt$

$=-\cot t|_{\frac{\pi}{4}}^{\frac{\pi}{2}}-\left(\frac{\pi}{2}-\frac{\pi}{4}\right)=1-\frac{\pi}{4}$

(7) $\int_0^a\frac{dx}{x+\sqrt{a^2-x^2}}$　令~~θ=si~~ $x=a\sin t$　$dx=a\cos t\,dt$

原式$=\int_0^{\frac{\pi}{2}}\frac{\cos t}{\sin t+\cos t}dt=\frac{1}{2}\int_0^{\frac{\pi}{2}}\frac{\cos t+\sin t-\sin t+\cos t}{\sin t+\cos t}dt$

$=\frac{1}{2}\left[\frac{\pi}{2}+\ln(\sin t+\cos t)|_0^{\frac{\pi}{2}}\right]=\frac{\pi}{4}$

(8) $I=\int_0^{\frac{\pi}{2}}\frac{\sin\theta}{\sin\theta+\cos\theta}d\theta$，$J=\int_0^{\frac{\pi}{2}}\frac{\cos\theta}{\sin\theta+\cos\theta}d\theta$

$I+J=\int_0^{\frac{\pi}{2}}1\,d\theta=\frac{\pi}{2}$ ①

$I-J=\int_0^{\frac{\pi}{2}}\frac{\sin\theta-\cos\theta}{\sin\theta+\cos\theta}d\theta=-\int_0^{\frac{\pi}{2}}\frac{}{\sin\theta+\cos\theta}d(\sin\theta+\cos\theta)=0$ ②

联立①②，得$I=J=\frac{\pi}{4}$

(9) $\int_0^{\frac{\pi}{2}}\frac{dx}{a^2\sin^2x+b^2\cos^2x}=\frac{1}{|ab|}\int_0^{\frac{\pi}{2}}\frac{d(|\frac{a}{b}|\tan x)}{(|\frac{a}{b}|\tan x)^2+1}$

$=\frac{1}{|ab|}\arctan(|\frac{a}{b}|\tan x)\Big|_0^{\frac{\pi}{2}}$

$=\frac{1}{|ab|}\cdot\frac{\pi}{2}=\frac{\pi}{2ab}$

(10) $\int_1^3 f(x-2)dx \quad f(x)=\begin{cases}1+x^2, & x\le 0\\ e^{-x}, & x>0\end{cases}$

令$t=x-2$，$x\big|_1^3\to t\big|_{-1}^{1}$ $dx=dt$

原式$=\int_{-1}^{1}f(t)dt=\int_{-1}^{1}f(x)dx=\int_{-1}^{0}(1+x^2)dx+\int_0^1 e^{-x}dx$

$=(x+\frac{1}{3}x^3)\Big|_{-1}^{0}+(-e^{-x})\Big|_0^1$

$=\frac{7}{3}-\frac{1}{e}$

3. 证明：要证 $\int_{-a}^{a}x(f(x)+f(-x))dx=0$，即证$\int_{-a}^{a}xf(x)=-\int_{-a}^{a}xf(-x)dx$

令$-x=t$，$dx=-dt$

$-\int_{-a}^{a}xf(-x)dx=-\int_{a}^{-a}(-t)f(t)\cdot(-dt)=\int_{-a}^{a}tf(t)dt=\int_{-a}^{a}xf(x)dx$

得证

4. 证明：令$t=1-x$

$\int_1^0 x^m(1-x)^n dx=\int_0^1(1-t)^m t^n(-dt)$

$=\int_0^1(1-t)^m t^n dt=\int_0^1 x^n(1-x)^m dx$

5. 证明：令$\frac{1}{x}=t$，$dt=-\frac{1}{x^2}dx$ $\quad t\big|_x^1\to x\big|_{\frac{1}{x}}^{1}$

原式$=\int_x^1\frac{1}{1+(\frac{1}{x})^2}\cdot(-\frac{1}{x^2})dx=\int_1^{\frac{1}{x}}\frac{dx}{x^2+1}=\int_1^{\frac{1}{x}}\frac{1}{t^2+1}dt$

得证

6、(1) $g(-x) = \int_0^{-x} f(t)\,dt$

$= \int_0^{x} f(-m)\,d(-m) = \int_0^{x} f(m)\,d(m) = g(x)$

$\therefore$ $g(x)$ 为偶函数.

即 $f(x)$ 为奇函数时，$\int_0^{x} f(t)\,dt$ 为偶函数

(2) $g(-x) = \int_0^{-x} f(t)\,dt = \int_0^{x} f(-m)\,d(-m) = -\int_0^{x} f(m)\,dm = -g(x)$

$\therefore$ $f(x)$ 为偶函数时，$\int_0^{x} f(t)\,dt$ 为奇函数.