4.3

1\. $f(x)=x^4-5x^3+x^2-3x+4$ $\qquad f(4)=-56$

$f'(x)=4x^3-15x^2+2x-3$ $\qquad f'(4)=21$

$f''(x)=12x^2-30x+2$ $\qquad f''(4)=74$

$f^{(3)}(x)=24x-30$ $\qquad f^{(3)}(4)=66$

$f^{(4)}(x)=24$ $\qquad f^{(4)}(4)=24$

$f(x)$在$x=4$的泰勒公式为

$$f(4)+f'(4)(x-4)+\frac{f''(4)}{2!}(x-4)^2+\frac{f^{(3)}(4)}{3!}(x-4)^3+\frac{f^{(4)}(4)}{4!}(x-4)^4$$

$$=(x-4)^4+11(x-4)^3+37(x-4)^2+21(x-4)-56$$

(2) 设$f(x)=\ln(1-x)$ 定义域$(-\infty,1)$

$f^{(k)}(x)=-\frac{(k-1)!}{(1-x)^k}$, $k=1,2,\cdots,n$ $\qquad f^{(k)}(\frac{1}{2})=-\frac{(k-1)!}{(\frac{1}{2})^k}=-(k-1)!\cdot 2^k$, $k=1,2,\cdots,n$

$f(x)$在$x=\frac{1}{2}$的n阶泰勒公式

$$f(\tfrac{1}{2})+f'(\tfrac{1}{2})(x-\tfrac{1}{2})+\frac{f''(\frac{1}{2})}{2!}(x-\tfrac{1}{2})^2+\cdots+\frac{f^{(n)}(\frac{1}{2})}{n!}(x-\tfrac{1}{2})^n+o((x-\tfrac{1}{2})^n)$$

$$=-\ln 2-2(x-\tfrac{1}{2})-2(x-\tfrac{1}{2})^2-\frac{8}{3}(x-\tfrac{1}{2})^3-\cdots-\frac{2^n}{n}(x-\tfrac{1}{2})^n+o((x-\tfrac{1}{2})^n)$$

2\.

(1) $f^{(k)}(x)=\frac{(-1)^k\cdot k!}{x^{k+1}}$ $\qquad f^{(k)}(-1)=\frac{(-1)^k\cdot k!}{(-1)^{k+1}}=-k!$ $\quad (k=0,1,2,\cdots,n)$

则$f(x)$在$x=-1$的n阶泰勒公式为

$$f(x)+f'(x)(x+1)+\frac{f''(x)}{2!}(x+1)^2+\cdots+\frac{f^{(n)}(x)}{n!}(x+1)^n+o((x+1)^n)$$

$$=f(-1)+f'(-1)(x+1)+\frac{f''(-1)}{2!}(x+1)^2+\cdots+\frac{f^{(n)}(-1)}{n!}(x+1)^n+o((x+1)^n)$$

$$=-1-(x+1)-(x+1)^2-(x+1)^3-\cdots-(x+1)^n+o((x+1)^n)$$

(3) 设$f(x)=\frac{1}{2}(e^x+e^{-x})$

$f^{(k)}(x)=\begin{cases}\frac{1}{2}(e^x+e^{-x}), & k\text{为偶数}\\ \frac{1}{2}(e^x-e^{-x}), & k\text{为奇数}\end{cases}$ $\qquad f^{(k)}(0)=\begin{cases}1, & k\text{为偶数}\\ 0, & k\text{为奇数}\end{cases}$ $\quad (k=0,1,2,\cdots,r$

则$f(x)$在$x=0$的20阶泰勒公式

$$f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\cdots+\frac{f^{(20)}(0)}{20!}x^{20}+o(x^{20})$$

$$=1+\frac{1}{2!}x^2+\frac{1}{4!}x^4+\cdots+\frac{1}{20!}x^{20}+o(x^{20})$$

(4) 设 $f(x)=x\cdot e^x$

$$f^{(k)}(x)=(x+k)\cdot e^x \qquad f^{(k)}(0)=k \quad , k=0,1,2,\cdots,n$$

$f(x)$ 在 $x=0$ 的 n 阶泰勒公式为

$$f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\cdots+\frac{f^{(n)}(0)}{n!}x^n+o(x^n)$$

$$=x+x^2+\frac{x^3}{2!}+\frac{x^4}{3!}+\cdots+\frac{x^n}{(n-1)!}+o(x^n)$$

3. (1) 解：由泰勒公式知：

$$(1+x^2)^{\frac{1}{4}}=1+\frac{1}{4}x^2+o(x^2)$$

$$(1-x^2)^{\frac{1}{4}}=1-\frac{1}{4}x^2+o(x^2)$$

则原式 $=\lim\limits_{x\to 0}\dfrac{\left[1+\frac{1}{4}x^2+o(x^2)\right]-\left[1-\frac{1}{4}x^2+o(x^2)\right]}{x^2}=\lim\limits_{x\to 0}\left(\dfrac{1}{2}+\dfrac{o(x^2)}{x^2}\right)=\dfrac{1}{2}$

(2) 解：由泰勒公式知：

$$\cos x^2=1-\frac{1}{2!}x^4+o(x^4)$$

$$x^2\cos x=x^2-\frac{1}{2!}x^4+o(x^4)$$

$$\sin x^2=x^2-\frac{1}{3!}x^6+o(x^6)$$

则原式 $=\lim\limits_{x\to 0}\dfrac{-x^2+o(x^4)}{x^2-\frac{1}{3!}x^6+o(x^6)}=\lim\limits_{x\to 0}\dfrac{-1+\frac{o(x^4)}{x^2}}{1-\frac{1}{3!}x^6+\frac{o(x^6)}{x^2}}=\dfrac{-1}{1}=-1$

(3) 解：由泰勒公式知：

$$e^{x^2}=1+x^2+o(x^2)$$

$$\sin^2 2x=\frac{1-\cos 4x}{2} \qquad \cos 4x=1-\frac{1}{2!}(4x^2)+o(x^2)$$

则原式 $=\dfrac{1}{2}\lim\limits_{x\to 0}\dfrac{o(x^2)}{8x^2-o(x^2)}=\dfrac{1}{2}\lim\limits_{x\to 0}\dfrac{\frac{o(x^2)}{x^2}}{8-\frac{o(x^2)}{x^2}}=0$

(4) 解：原式 $=\lim\limits_{x\to 0}\dfrac{\tan x-\sin x}{[x\ln(1+x)-x^2](\sqrt{\tan x+1}+\sqrt{1+\sin x})}=\lim\limits_{x\to 0}\dfrac{\sin x-\frac{1}{2}\sin 2x}{2(x\ln(1+x)-x^2)}$

$$=\lim_{x\to 0}\frac{\left[x-\frac{1}{3!}x^3+o(x^3)\right]-\frac{1}{2}\left[2x-\frac{8}{3!}x^3+o(x^3)\right]}{-x^3+o(x^3)}=-\frac{1}{2}$$

5. 设 $f(x)=2^x$ $\qquad f^{(k)}(x)=2^x(\ln 2)^k \quad (k=0,1,2,\cdots,n)$

$f(x)$ 在 $x=0$ 的 n 阶泰勒公式为

$$1+\ln 2\cdot x+\frac{(\ln 2)^2}{2!}x^2+\frac{(\ln 2)^3}{3!}x^3+\cdots+\frac{(\ln 2)^n}{n!}x^n$$

则 $2^{\frac{1}{5}}\approx 1+\ln 2\times\frac{1}{5}+\frac{(\ln 2)^2}{2!}\times\left(\frac{1}{5}\right)^2+\frac{(\ln 2)^3}{3!}\times\left(\frac{1}{5}\right)^3\approx 1.149$

6. $\lim\limits_{x\to 0}\left(1+x+\frac{f(x)}{x}\right)^{\frac{1}{x}}=\lim\limits_{x\to 0}\left(1+x+\frac{f(x)}{x}\right)^{\frac{1}{x+\frac{f(x)}{x}}\cdot\frac{x+\frac{f(x)}{x}}{x}}=e^3$

$\Longrightarrow \lim\limits_{x\to 0}\left(x+\frac{f(x)}{x}\right)=0 \qquad \lim\limits_{x\to 0}\frac{x+\frac{f(x)}{x}}{x}=3$

$\Longrightarrow \lim\limits_{x\to 0}\frac{f(x)}{x}=0 \qquad \lim\limits_{x\to 0}\frac{f(x)}{x^2}=2 \quad \cancel{\Longrightarrow \lim\limits_{x\to 0}f(x)=0}$

~~由洛必达法则知：~~ $\cancel{\lim\limits_{x\to 0}\frac{f'(x)}{2x}=2 \Longrightarrow f'(0)=0}$

$\lim\limits_{x\to 0}\left(1+\frac{f(x)}{x}\right)^{\frac{1}{x}}=\lim\limits_{x\to 0}\left(1+\frac{f(x)}{x}\right)^{\frac{x}{f(x)}\cdot\frac{1}{x}\cdot\frac{f(x)}{x}}=e^{\lim\limits_{x\to 0}\frac{f(x)}{x^2}}=e^2$

7、用待定系数法，

构造函数 $P(x)=\frac{x^3}{2}+\left(\frac{1}{2}-f(0)\right)x^2+f(0)$

设 $F(x)=f(x)-P(x)$　显然 $F(x)$ 在 $[-1,1]$ 上有连续的三阶导数

且

$$F(-1)=F(1)=F(0)=F'(0)=0$$

对 $F(x)$ 在 $[-1,0]$，$[0,1]$ 用罗尔定理得.

存在　$-1<\theta_1<0$　$0<\theta_2<1$

使　$F'(\theta_1)=F'(\theta_2)=0$

对 $F'(x)$ 在 $[\theta_1,0]$，$[0,\theta_2]$ 上用罗尔定理得

存在　$-1<\theta_1<\eta_1<0$　$0<\eta_2<\theta_2<1$

$$F''(\eta_1)=F''(\eta_2)=0$$

对 $F''(x)$ 在 $[\eta_1,\eta_2]$ 上用罗尔定理得.

存在　$\xi\in(\eta_1,\eta_2)\subset(-1,1)$

使得：　$F'''(\xi)=0$　即 $f'''(\xi)=3$

8. 由泰勒公式得:

$$f(x+h) = f(x) + f'(x)h + \frac{1}{2} f''(x) h^2 + o(h^2)$$

$$f(x-h) = f(x) - f'(x)h + \frac{1}{2} f''(x) h^2 + o(h^2)$$

$$f(x) \leq \frac{1}{2} [f(x-h) + f(x+h)]$$

$$\Rightarrow \quad f''(x) + o(h^2) \geq 0$$

令 $h \to 0$ 则 $f''(x) \geq 0$