第 6 章复习题

1、

(1)

$$\lim_{x\to 0}\frac{\int_0^x (e^{t方}-1)\,dt}{x^2}=\lim_{x\to 0}\frac{e^{x^2}-1}{2x}=\lim_{x\to 0}\frac{e^{x^2*2x}}{2}=0$$

故 a=0

(2)

$$f'(0)=\lim_{x\to 0}\frac{f'(x)-0}{x-0}=\lim_{x\to 0}\frac{\int_0^x (e^{x^2-1})\,dt}{x^3}=\lim_{x\to 0}\frac{e^{x^2-1}}{3x^2}=\frac{1}{3}$$

3、

(1)

$$\int_1^2\frac{1+x^2}{1+x^4}dx=\int_1^2\frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}dx=\int_1^2\frac{d(x-\frac{1}{x})}{2+(x-\frac{1}{x})^2}dx=\frac{1}{\sqrt{2}}arctan\frac{x^2-1}{\sqrt{2}x}|_1^2=\frac{1}{\sqrt{2}}arctan\frac{3\sqrt{2}}{4}$$

(2)

$$\int_0^{\Pi}\frac{sin\theta d\theta}{\sqrt{1-2acos\theta+a^2}}=-\int_0^{\Pi}\frac{dcos\theta}{\sqrt{1-2acos\theta+a^2}}=\frac{\sqrt{1-2acos\theta+a^2}}{a}|_0^{\Pi}=\frac{2}{a}$$

(3)

$$\int_0^1 x\sqrt{\frac{1-x}{1+x}}dx=\int_0^1\frac{x(1-x)}{\sqrt{1-x^2}}dx$$

令 x=sint

$$原式=\int_0^{\frac{\Pi}{2}}\frac{sint(1-sint)}{cost}costdt=\int_0^{\frac{\Pi}{2}}(sint-sin^2 t)dt=(-cost)|_1^{\frac{\Pi}{2}}-\frac{\Pi}{4}=1-\frac{\Pi}{4}$$

(4)

$\int_{\frac{1}{2}}^{2}\frac{|lnx|}{1+x}$dx　令 t=$\frac{1}{x}$

原式=$\int_{2}^{1}-\frac{ln\frac{1}{t}}{1+\frac{1}{t}}$*(- $\frac{1}{t^2}$)dt+$\int_{1}^{2}\frac{lnx}{1+x}$dx

=$\int_{1}^{2}\frac{lnt}{t*(1+t)}$dt+$\int_{1}^{2}\frac{lnt}{1+t}$

=$\int_{1}^{2}lntd(lnt)=\frac{1}{2}(lnt)^2|_1^2$=$\frac{(ln2)^2}{2}$

(5)

$\int_{2}^{e}\frac{1+lnx}{x^2ln^2x}$dx=　$\int_{2}^{e}\frac{d(xlnx)}{(xlnx)^2}$dx=-$\frac{1}{xlnx}$ $|_2^e$=$\frac{1}{2ln2}$-$\frac{1}{e}$

（6）$\int_{0}^{3}arcsin\sqrt{\frac{x}{1+x}}$dx　令 t=$arcsin\sqrt{\frac{x}{1+x}}$　x=tan^2t

原式=$\int_{0}^{\frac{\Pi}{3}}td\,tan^2$t=t$tan^2t|_0^{\frac{\Pi}{3}}$-$\int_{0}^{\frac{\Pi}{3}}tan^2$tdt

=Π-$\int_{0}^{\frac{\Pi}{3}}(sec2-1)dt$

=Π-(tant-t) $|_0^{\frac{\Pi}{3}}$

=$\frac{4\Pi}{3}$-√3

4、（1）

令 x=Π-t,则 dx=-dt

$\int_{0}^{\Pi}xf(sinx)dx=-\int_{\Pi}^{0}(\Pi-t)(\sin(\Pi-t))dt$

=$\int_{0}^{\Pi}(\Pi-t)f(sint)dt$

=Π$\int_{0}^{\Pi}f(sint)dt$-$\int_{0}^{\Pi}tf(sint)dt$

=Π$\int_{0}^{\Pi}f(sinx)dx$-$\int_{0}^{\Pi}xf(sinx)dx$

有$\int_{0}^{\Pi}xf(sinx)dx$=$\frac{\Pi}{2}\int_{0}^{\Pi}f(sinx)dx$

$\int_{0}^{\Pi}\frac{xsinx}{1+cos^2x}$dx=$\frac{\Pi}{2}$ $\int_{0}^{\Pi}\frac{sinx}{1+cos^2x}$dx=-$\frac{\Pi}{2}\int_{0}^{\Pi}\frac{dcosx}{1+cos^2x}$=-$\frac{\Pi}{2}$arctan(cosx) $|_0^{\Pi}$

$=\frac{\Pi^2}{4}$

(2)

$\int_0^{\Pi^2} sin^2\sqrt{x}$dx　　令 x=$t^2$

原式=2 $\int_0^{\Pi} tsin^2tdt$=∏$\int_0^{\Pi} sin^2tdt$

=∏$\int_0^{\frac{\Pi}{2}} sin^2tdt$+∏$\int_{\frac{\Pi}{2}}^{\Pi} sin^2tdt$

=2∏$\int_{\frac{\Pi}{2}}^{\Pi} sin^2tdt$

=$\frac{\Pi^2}{2}$

5、

证明：$\int_0^{\frac{\Pi}{2}} sin^nxcos^nxdx$=$\frac{1}{2^n}\int_0^{\frac{\Pi}{2}}(2sinxcosx)^n dx$

=$\frac{1}{2^{n+1}}\int_0^{\frac{\Pi}{2}} sin^n2xd2x$

=$\frac{1}{2^n}\int_0^{\frac{\Pi}{2}} sin^nxdx$

6、

$\int_0^1\frac{dx}{\sqrt{1+x^2}}$=ln|x+$\sqrt{1+x^2}$|　$|_0^1$=$\ln(1+\sqrt{2})$

$\int_0^1\frac{x}{\sqrt{1+x^2}}$dx=$\sqrt{1+x^2}|_0^1$=$\sqrt{2}-1$

又$\int_0^1\frac{dx}{\sqrt{1+x^2}} > \int_0^1\frac{x}{\sqrt{1+x^2}}$dx

所以 $\ln(1+\sqrt{2})>\sqrt{2}-1$

7、

f(x)= $\int_1^x e^{-xt^2}$dt

f'(x)= e^{-x^3}

f'(1)= e^{-1}

8．设$f(x)$在$[a,b]$上连续，$F(x)=\int_a^x(x-t)f(t)\,dt$，$x\epsilon[a,b]$，证明：

⟨1⟩$F''(x)=f(x)$　　　　　　　⟨2⟩$F(x)=\int_a^x[\int_a^u f(t)dt]du$

解：⟨1⟩ $\because$ f(x)在[a, b]连续

$$F(x)=\int_a^x(x-t)f(t)dt=\int_a^x[xf(t)-tf(t)]dt=x\int_a^x f(t)dt-\int_a^x tf(t)dt$$

$$\therefore F'(x)=\int_a^x f(t)dt+xf(x)-xf(x)=\int_a^x f(t)dt$$

$$\therefore F''(x)=f(x)$$

⟨2⟩ $\because$ 由⟨1⟩可知, $\mathrm{F''(x)}=f(x)$

$$\therefore F'(x)=\int_a^u f(t)dt$$

$$\therefore F(x)=\int_a^x[\int_a^u f(t)dt]du$$

9 .设$f(x)$在$(-\infty,+\infty)$内连续可导，当 $x\neq 0$ 时，$f(x)\neq 0$，且$\int_0^{f(x)}t^2dt=\int_0^x f^2(t)e^{-f(t)}\,dt$，求$f(x)$.

解：

$$\because \int_0^{f(x)}t^2dt=\int_0^x f^2(t)e^{-f(t)}\,dt$$

$$\therefore f'(x)f^2(x)=f^2(x)e^{-f(x)}$$

$$\therefore y'=e^{-y}$$

$$\therefore \frac{dy}{dx}=\frac{1}{e^y}$$

$$\therefore e^y dy=dx$$

$$\Rightarrow e^y=x+c$$

$$\Rightarrow y=\ln(x+c)$$

又$\because$ 当 $x=0, f(x)=0$

$$\therefore f(x)=\ln(x+1)$$

10.

设$f(x)$在$[2,4]$上连续可导，且$f(2)=f(4)=0$.证明：$|\int_2^4 f(x)dx| \le \max\limits_{2\le x\le 4}|f'(x)|$

解：取$x\in[2,4]$,在$[2,x]$和$[x,4]$上分别对$f(x)$使用

拉格朗日中值定理，则$\exists\varepsilon_1\in[2,x],\varepsilon_2\in[x,4]$,使得

$f(x)-f(2)=f'(\varepsilon_1)(x-2)\Rightarrow f(x)=f'(\varepsilon_1)(x-2)$

$f(4)-f(x)=f'(\varepsilon_2)(4-x)\Rightarrow f(x)=f'(\varepsilon_2)(x-4)$

令$M=\max|f'(x)|(x\in[2,4])$

$|f(x)|\le M(x-2)$

$|f(x)|\le M(4-x)$

又$\because |\int_2^4 f(x)dx| \le \int_2^4 |f(x)|dx \le \int_2^3 M(x-2)dx+\int_3^4 M(4-x)dx=M$

$\therefore \max\limits_{2\le x\le 4}|f'(x)| \ge |\int_2^4 f(x)dx|$

11.

设$f(x)$在$[0,1]$上连续，在$(0,1)$内可导，且$3\int_{\frac{2}{3}}^{1} f(x)dx=f(0)$.试证：在（0,1）内至少存在一点$\xi$,使$f'(\xi$

$\because 3\int_{\frac{2}{3}}^{1} f(x)dx=f(0)$

由积分中值定理可知：$\exists\xi_1\in(\frac{2}{3},1)$

$f(\xi_1)=f(0)$

由罗尔中值定理可知,$\exists\xi\in(0,\xi_1)\in(0,1)$

使得$f'(\xi)=0$

12.设 f(x)在[0,1]上可导，且 $2\int_0^{\frac{1}{2}} xf(x)\,dx=f(1)$.证明：在(0,1)内至少存在一点ξ，使$f'(\xi)=-\frac{f(\xi)}{\xi}$.

解：令$\mathrm{F(x)=xf(x)}$

$$F'(x)=f(x)+xf'(x)$$

$$f(1)-2\int_0^{\frac{1}{2}} xf(x)\,dx=0$$

$$\therefore \int_0^{\frac{1}{2}}[f(1)-xf(x)]\,dx=0$$

由积分中值定理$\exists x_1\in\left[0,\frac{1}{2}\right]$，$x_1f(x_1)=f(1)$

$\therefore F(x_1)=F(1)$

$\therefore \exists\xi\in(x_1,1)\quad F'(\xi)$=0

即$f'(\xi)=-\frac{f(\xi)}{\xi}$

13. 曲线 y=ax^2+bx 在[0,1]上的一段位于 x 轴上方，且与直线 x=1 及 x 轴所围成图形的面积为$\frac{1}{3}$，确定 a、b 的值，使得该图形绕 x 轴一周所得旋转体的体积最小.

解：$\mathrm{f(x)=ax^2+bx}$

$\int_0^1(ax^2+bx)\,dx=\frac{1}{3}$

$\therefore\left(\frac{a}{3}x^3+\frac{b}{2}x^2\right)\Big|_0^1=\frac{a}{3}+\frac{b}{2}=\frac{1}{3}$

$\therefore \mathrm{2a+3b=2}$

$b=\frac{2-2a}{3}$

$V=\int_0^1\pi(ax^2+bx)^2\,dx$

$=\pi\int_0^1(a^2x^4+b^2x^2+2abx^3)\,dx$

$=\pi\left(\frac{1}{5}a^2x^5+\frac{1}{3}b^2x^3+\frac{1}{2}abx^4\right)\Big|_0^1$

$=\frac{a^2}{5}\pi+\frac{b^2}{3}\pi+\frac{ab}{2}\pi$

$\mathrm{V'(a)}=\frac{2}{5}\pi\mathrm{a}+\frac{2\pi}{3}\cdot\frac{2-2\mathrm{a}}{3}\cdot\left(-\frac{2}{3}\right)+(\frac{\pi}{3}-\frac{2}{3}\pi\mathrm{a})$

$=\frac{2}{5}\pi a-\frac{8}{27}\pi+\frac{8}{27}\pi a+\frac{\pi}{3}-\frac{18}{27}\pi a$

$=\frac{1}{27}\pi+\frac{2}{5}\pi a-\frac{10}{27}\pi a=0$时得$a=-\frac{5}{4}$

$\therefore a=-\frac{5}{4}\qquad b=\frac{3}{2}$

14.设在（$-\infty$， $+\infty$）内 f(x)>0.f’(x)连续，设 F(x)=$\begin{cases} \dfrac{\int_0^x tf(t)dt}{\int_0^x f(t)dt} & x \neq 0 \\ 0 & x = 0 \end{cases}$

<1>求 F’(x)

<2>证明 F’(x)在（$-\infty$， $+\infty$）连续

<3>证明 F(x)在（$-\infty$， $+\infty$）内单调递增

<1>当 $x \neq 0$时

$$F'(x)= \frac{xf(x)\int_0^x f(t)dt-f(x)\int_0^x tf(t)dt}{[\int_0^x f(t)dt]^2} = \frac{f(x)\int_0^x (x-t)f(t)dt}{[\int_0^x f(t)dt]^2}$$

当 $x = 0$时

$$F'(0)= \lim_{x\to 0}\frac{F(x)-F(0)}{x-0} = \frac{\int_0^x tf(t)dt}{x\int_0^x f(t)dt} = \lim_{x\to 0}\frac{xf(x)}{\int_0^x f(t)dt+xf(x)} =\lim_{x\to 0}\frac{f(x)+xf'(x)}{2f(x)+xf'(x)}$$

又因为在（$-\infty$， $+\infty$）f(x)>0

所以 F’(0)=$\frac{1}{2}$

综上所诉 F’(x)=$\begin{cases} \dfrac{f(x)\int_0^x (x-t)f(t)dt}{[\int_0^x f(t)dt]^2} & x \neq 0 \\ \dfrac{1}{2} & x = 0 \end{cases}$

<2>当 $x \neq 0$时 $\lim\limits_{x\to x_0} F'(x)$=F’($x_0$)

当 $x = 0$时$\lim\limits_{x\to 0} F'(x) = \lim\limits_{x\to 0}\dfrac{xf(x)\int_0^x f(t)dt-f(x)\int_0^x tf(t)dt}{[\int_0^x f(t)dt]^2}$

$$=\lim_{x\to 0}\frac{f(x)\int_0^x f(t)dt+xf'(x)\int_0^x f(t)dt+xf^2(x)-f'(x)\int_0^x tf(t)dt-xf^2(x)}{2f(x)\int_0^x f(t)dt}$$

$$=\lim_{x\to 0}\frac{f(x)\int_0^x f(t)dt+f'(x)\int_0^x (t-1)f(t)dt}{2f(x)\int_0^x f(t)dt}$$

$$=\lim_{x\to 0}\frac{f'(x)\int_0^x f(t)dt+f^2(x)+f''(x)\int_0^x (t-1)f(t)dt+f'(x)(x-1)f(x)}{2f'(x)\int_0^x f(t)dt+2f^2(x)}$$

$$=\lim_{x\to 0}\frac{f^2(x)}{2f^2(x)} = \frac{1}{2}$$

（3） $x = 0$时，$F^{'}(x) = \frac{1}{2} > 0$

$x \neq 0$时

$$F^{'}(x) = \frac{xf(x)\int_0^x f(t)d\,t - f(x)\int_0^x t\,f(t)dt}{\left[\int_0^x f(t)d\,t\right]^2}$$

设$g(x) = xf(x)\int_0^x f(t)d\,t - f(x)\int_0^x t\,f(t)dt.$

$$= f(x)\left[x\int_0^x f(t)d\,t - \int_0^x t\,f(t)dt\right]$$

$$h(x) = x\int_0^x f(t)d\,t - \int_0^x t\,f(t)dt.$$

$$h^{'}(x) = \int_0^x f(t)d\,t + xf(x) - xf(x)$$

$$= \int_0^x f(t)d\,t$$

$h^{''}(x) = f(x) > 0 \quad \therefore h(x)$递增

又因为$h^{'}(0) = 0$

$x < 0$时　$h^{'}(x) < 0$　$x > 0$时$h^{'}(x) > 0$ $\therefore h(x)$在$\left(-\infty, 0\right) \downarrow \left(0, +\infty\right) \uparrow$

又$\because h(0) = 0 \quad \therefore h(x) > 0 \quad (x \neq 0$时)

即$x \neq 0$ 时，$g(x) > 0$即$F^{'}(x) > 0$

综上所述$F(x)$在$\left(-\infty, +\infty\right)$内单调递增。