习题 2.3

1. 证明定理 2.3.1

$\forall \delta > 0$,当 $0 < |x - x_0| < \delta$ 时

$\lim\limits_{x \to x_0} g(x) = A \Rightarrow \quad |g(x) - A| < \varepsilon \Rightarrow A - \varepsilon < g(x) < A + \varepsilon$

同理：$\lim\limits_{x \to x_0} h(x) = A \Rightarrow A - \varepsilon < h(x) < A + \varepsilon$

$\because g(x) \leq f(x) \leq h(x)$

$\therefore A - \varepsilon < g(x) \leq f(x) \leq h(x) < A + \varepsilon$

$\Rightarrow A - \varepsilon < f(x) < A + \varepsilon \Rightarrow \quad |f(x) - A| < \varepsilon \Rightarrow \lim\limits_{x \to x_0} f(x) = A$

2. 利用夹逼定理，求下列函数极限

(1) $\lim\limits_{x \to \infty} \frac{[x]}{x}$

$x - 1 \leq [x] \leq x$

① 对于$x \to +\infty$时，有$\frac{x-1}{x} \leq \frac{[x]}{x} \leq \frac{x}{x} \leq 1$

且 $\lim\limits_{x \to +\infty} \frac{x-1}{x} = \lim\limits_{x \to +\infty} \left(1 - \frac{1}{x}\right) = 1 - 0 = 1$

$\lim\limits_{x \to +\infty} 1 = 1$

$\therefore \lim\limits_{x \to +\infty} \frac{[x]}{x} = 1$

② 对于$x \to -\infty$时，有$\frac{x-1}{x} \geq \frac{[x]}{x} \geq 1$

又 $\lim\limits_{x \to -\infty} \frac{x-1}{x} = \lim\limits_{x \to -\infty} \left(1 - \frac{1}{x}\right) = 1 + 0 = 1$

$\lim\limits_{x \to -\infty} 1 = 1$

$\therefore \lim\limits_{x\to-\infty}\frac{[x]}{x}=1$

综上所述 $\lim\limits_{x\to\infty}\frac{[x]}{x}=1$

(2) $\lim\limits_{x\to+\infty}\sqrt{1+\frac{1}{x^{\alpha}}}(\alpha>0)$

当$x\to+\infty,\alpha>0$ 时，$1+\frac{1}{x^{\alpha}}>1$

故有 $1<\sqrt{1+\frac{1}{x^{\alpha}}}<1+\frac{1}{x^{\alpha}}$

$\lim\limits_{x\to+\infty}1=1,\lim\limits_{x\to\infty}\left(1+\frac{1}{x^{\alpha}}\right)=1+0=1$

$\therefore \lim\limits_{x\to+\infty}\sqrt{1+\frac{1}{x^{\alpha}}}=1$

(3) $\frac{1}{x}-1<\left[\frac{1}{x}\right]\leq\frac{1}{x}$

① 对于$x\to0^{+}$时，有 $1-x<x\left[\frac{1}{x}\right]\leq1$

$\lim\limits_{x\to0^{+}}(1-x)=1-0=1,\lim\limits_{x\to0^{+}}1=1$

由夹逼定理知 $\lim\limits_{x\to0^{+}}x\left[\frac{1}{x}\right]=1$

② 对于$x\to0^{-}$， $1\leq x\left[\frac{1}{x}\right]<1-x$

$\lim\limits_{x\to0^{-}}1=1,\lim\limits_{x\to0^{-}}(1-x)=1-0=1$

由夹逼定理知 $\lim\limits_{x\to0^{-}}x\left[\frac{1}{x}\right]=1$

综上 $\lim\limits_{x\to 0} x\left[\frac{1}{x}\right] = \lim\limits_{x\to 0^+} x\left[\frac{1}{x}\right] = \lim\limits_{x\to 0^-} x\left[\frac{1}{x}\right] = 1$

3. 应用海涅定理，证明下列函数极限不存在

(1) $\lim\limits_{x\to 0} \sin\frac{1}{x}$

设$x_n' = \frac{1}{2n\pi}$，$x_n'' = \frac{1}{2n\pi + \frac{\pi}{2}}$，其中$n$为非 0 整数

显然$x_n' \neq 0, \lim\limits_{n\to\infty} x_n' = 0; x_n'' \neq 0, \lim\limits_{n\to\infty} x_n'' = 0$

$\lim\limits_{n\to\infty} \sin\frac{1}{x_n'} = \lim\limits_{n\to\infty} \sin 2n\pi = 0$

$\lim\limits_{n\to\infty} \sin\frac{1}{x_n''} = \lim\limits_{n\to\infty} \sin\left(2n\pi + \frac{\pi}{2}\right) = 1$

根据海涅定理，$\lim\limits_{x\to 0} \sin\frac{1}{x}$不存在

(2) $\lim\limits_{x\to 0} \cos\frac{1}{x}$

设$f(x) = \frac{1}{\cos x}$, 设$x_n' = \frac{1}{2n\pi}$, $x_n'' = \frac{1}{2n\pi + \frac{\pi}{2}}$, $|n| \in N^*$

显然$x_n' \neq 0, \lim\limits_{n\to\infty} x_n' = 0;\quad x_n'' \neq 0, \lim\limits_{n\to\infty} x_n'' = 0$

$\lim\limits_{n\to\infty} f(x_n') = \lim\limits_{n\to\infty} \cos 2n\pi = 1, \lim\limits_{n\to\infty} f(x_n'') = \lim\limits_{n\to\infty} \cos\left(2n\pi + \frac{\pi}{2}\right) = 0$

根据海涅定理，$\lim\limits_{x\to 0} \cos\frac{1}{x}$不存在

4. 求下列函数极限

(1) $\lim\limits_{x\to 0} \frac{\sin\alpha x}{\sin\beta x} (\beta \neq 0)$

$= \lim\limits_{x\to 0} \frac{\sin\alpha x}{\alpha x} \cdot \frac{\beta x}{\sin\beta x} \cdot \frac{\alpha x}{\beta x}$

$$=\lim_{x\to 0}\frac{\sin\alpha x}{\alpha x}\cdot\lim_{x\to 0}\frac{\beta x}{\sin\beta x}\cdot\lim_{x\to 0}\frac{\alpha x}{\beta x}$$

$$=1\cdot 1\cdot\frac{\alpha}{\beta}=\frac{\alpha}{\beta}$$

(2) $\lim_{x\to 0}\frac{\tan\alpha x}{\tan\beta x}(\beta\neq 0)=\lim_{x\to 0}\frac{\sin\alpha x}{\sin\beta x}\cdot\frac{\cos\beta x}{\cos\alpha x}=\lim_{x\to 0}\frac{\sin\alpha x}{\sin\beta x}\cdot\lim_{x\to 0}\frac{\cos\beta x}{\cos\alpha x}$

$$=\frac{\alpha}{\beta}\cdot\frac{\cos 0}{\cos 0}=\frac{\alpha}{\beta}$$

(3) $\lim_{x\to 0}\frac{1-\cos x}{x^2}=\lim_{x\to 0}\frac{1-\left(2\cos^2\frac{x}{2}-1\right)}{x^2}=\lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{x^2}$

$$=\frac{1}{2}\left(\lim_{x\to 0}\frac{\sin\frac{x}{2}}{x}\right)^2=\frac{1}{2}\cdot 1^2=\frac{1}{2}$$

(4)设$y=x-\frac{\pi}{4}$

$$\lim_{x\to\frac{\pi}{4}}\frac{\sqrt{2}-2\cos x}{\sin\left(x-\frac{\pi}{4}\right)}=\lim_{y\to 0}\frac{\sqrt{2}-2\cos\left(y+\frac{\pi}{4}\right)}{\sin y}$$

$$=\lim_{y\to 0}\frac{\sqrt{2}-\sqrt{2}\cos y+\sqrt{2}\sin y}{\sin y}$$

$$=\sqrt{2}\lim_{y\to 0}\frac{1-\cos y}{\sin y}+\sqrt{2}=\sqrt{2}\lim_{y\to 0}\frac{2-2\cos^2\frac{y}{2}}{2\sin\frac{y}{2}\cos\frac{y}{2}}+\sqrt{2}$$

$$=\sqrt{2}\lim_{y\to 0}\frac{2\sin^2\frac{y}{2}}{2\sin\frac{y}{2}\cos\frac{y}{2}}+\sqrt{2}$$

$$=\sqrt{2}\lim_{y\to 0}\tan\frac{y}{2}+\sqrt{2}=\sqrt{2}\cdot 0+\sqrt{2}=\sqrt{2}$$

5. 求下列函数极限

(1) $\lim_{x\to 0}(1-3x)^{\frac{1}{x}}$

$$= \lim_{x\to 0}\left(1+\frac{1}{\frac{1}{3x}}\right)^{-\frac{1}{3x}\cdot(-3)}$$

$$= \left[\lim_{x\to 0}\left(1+\frac{1}{-\frac{1}{3x}}\right)^{-\frac{1}{3x}}\right]^{-3}$$

$$= e^{-3}$$

(2) $\lim\limits_{x\to\infty}\left(\frac{1+x}{2+x}\right)^{\frac{1-x^2}{1-x}}$

$$= \lim_{x\to\infty}\left(1-\frac{1}{2+x}\right)^{1+x}$$

$$= \lim_{x\to\infty}\left(1+\frac{1}{-(2+x)}\right)^{-(2+x)\cdot\frac{x+1}{-(2+x)}}$$

$$= \left[\lim_{x\to\infty}\left(1+\frac{1}{-(2+x)}\right)^{-(2+x)}\right]^{\lim\limits_{x\to\infty}\left(\frac{1}{2+x}-1\right)}$$

$$= e^{-1}$$

(3) $\lim\limits_{x\to 0}(1+\sin x)^{3\csc x}$

$$= \left[\lim_{x\to 0}\left(1+\frac{1}{\frac{1}{\sin x}}\right)^{\frac{1}{\sin x}}\right]^{3}$$

$$= e^{3}$$

(4) $\lim\limits_{x\to\infty}\left(\frac{x+1}{x-1}\right)^{x}$

$$= \lim_{x\to\infty}\left(1+\frac{2}{x-1}\right)^{x}$$

$$= \lim_{x\to\infty}\left(1+\frac{1}{\frac{x-1}{2}}\right)^{\frac{x-1}{2}\cdot\frac{2x}{x-1}}$$

$$= \left[\lim_{x\to\infty}\left(1+\frac{1}{\frac{x-1}{2}}\right)^{\frac{x-1}{2}}\right]^{\lim\limits_{x\to\infty}\frac{2x}{x-1}}$$

$$= e^{\lim\limits_{x\to\infty}\frac{2}{1-\frac{1}{x}}}$$

$$= e^{\frac{2}{1-0}}$$

$$= e^{2}$$