习题 6.4

1.（1）解：

$\int_0^1 xe^x dx$

$=xe^x|_0^1-\int_0^1 e^x dx$

$=xe^x|_0^1-e^x|_0^1$

$=1$

（2）解：

$\int_0^{\frac{1}{2}} arcsinx dx$

$=x\cdot \arcsin x|_0^{\frac{1}{2}}-\int_0^{\frac{1}{2}}\frac{x}{\sqrt{1-x^2}}dx$

$=x\cdot \arcsin x|_0^{\frac{1}{2}}+\frac{1}{2}\int_0^{\frac{1}{2}}\frac{d(1-x^2)}{\sqrt{1-x^2}}$

$=x\cdot \arcsin x|_0^{\frac{1}{2}}+\sqrt{1-x^2}|_0^{\frac{1}{2}}$

$=\frac{\pi}{12}+\frac{\sqrt{3}}{2}-1$

（3）解：

由推导结论：

$\int_0^{\frac{\pi}{2}} xdx\cos^2 xdx$

$=\frac{6}{7}\times\frac{4}{5}\times\frac{2}{3}$

$=\frac{16}{35}$

（4）解：

同理：

$\int_0^{\frac{\pi}{2}} sinx^6 dx$

$=\frac{5}{6}\times\frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2}$

$=\frac{5\pi}{32}$

结论推导：

$A_n=\int_0^{\frac{\pi}{2}} sin^n xdx$ (n≥2)

$=-\int_0^{\frac{\pi}{2}}(sinx)^{n-1}d(cosx)$

$=-(sinx)^{n-1}\cos x\Big|_0^{\frac{\pi}{2}}+\int_0^{\frac{\pi}{2}}cosxd(sinx)^{n-1}$

$=(n-1)\int_0^{\frac{\pi}{2}}cosx(sinx)^{n-2}cosxdx$

$=(n-1)(A_{n-2}-A_n)$

$\therefore nA_n=(n-1)A_{n-2}$

$\Rightarrow A_n=\frac{n-1}{n}A_{n-2}$

$\because A_0=\int_0^{\frac{\pi}{2}}(sinx)^0dx=\frac{\pi}{2}$

$A_1=\int_0^{\frac{\pi}{2}}(sinx)^1dx=1$

$$\therefore A_n=\begin{cases}\frac{(n-1)!!}{n!!}\cdot\frac{\pi}{2}\ (n=2k)\\ \frac{(n-1)!!}{n!!}\ (n=2k+1)\end{cases}$$

易证：

$$\int_0^{\frac{\pi}{2}} sin^n xdx=\int_0^{\frac{\pi}{2}} cos^n xdx$$

2. （1）解：

$\int_0^{\pi} xsin\frac{x}{2}dx$

$=-2\int_0^{\pi} xdcos\frac{x}{2}$

$=-2(x\cos\frac{x}{2}\Big|_0^{\pi}-\int_0^{\pi} xdcos\frac{x}{2}dx)$

$=2(2\sin\frac{x}{2}\Big|_0^{\pi}-x\cos\frac{x}{2}\Big|_0^{\pi})$

$=4$

（2）解：

$$\int_0^e x\,ln^2\,x\,dx$$

$$=\frac{1}{2}\int_0^e ln^2\,x\,dx^2$$

$$=\frac{1}{2}(x^2\,ln^2\,x\,|_0^e-\int_0^e 2x\,ln\,x\,dx)$$

$$=\frac{1}{2}[x^2\,ln^2\,x\,|_0^e-(x^2\,ln\,x-\int_0^e xdx)]$$

$$=\frac{1}{2}[x^2\,ln^2\,x+\frac{1}{2}x^2-x^2\,ln\,x)|_0^e$$

$=\frac{e^2}{4}$（正确答案）

$\int_1^e x\,ln^2\,x\,dx=\frac{1}{4}(e^2-1)$（改正后的题目答案）

（3）

$$\int_0^1 x\,arctan\,x\,dx$$

$$=\frac{1}{2}\int_0^1 arctan\,x\,dx^2$$

$$=\frac{1}{2}(x^2\,arctan\,x\,|_0^1-\int_0^1\frac{x^2}{x^2+1}dx)$$

$$=\frac{1}{2}(x^2\,arctan\,x\,|_0^1-\int_0^1 dx+\int_0^1\frac{dx}{x^2+1})$$

$$=\frac{1}{2}(x^2\,arctan\,x\,|_0^1-x|_0^1+arctan\,x\,||_0^1$$

$$=\frac{\pi}{4}-\frac{1}{2}$$

（4）

$$\int_0^{\frac{\pi}{2}} e^{2x} \cos x\, dx$$

$$= \int_0^{\frac{\pi}{2}} e^{2x} d \sin x$$

$$= e^{2x} \sin x \,|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} 2\mathrm{e}^{2x} \mathrm{sin}x dx$$

$$= e^{2x} \sin x \,|_0^{\frac{\pi}{2}} + 2 \int_0^{\frac{\pi}{2}} e^{2x} d \cos x$$

$$= e^{2x} \sin x \,|_0^{\frac{\pi}{2}} + 2(e^{2x} \cos x \,|_0^{\frac{\pi}{2}} - 2 \int_0^{\frac{\pi}{2}} e^{2x} \cos x\, dx)$$

移项得:

$$原式 = \frac{1}{5}(e^{\pi} - 2)$$

（5）

$$\int_0^1 e^{\sqrt{x}} dx$$

$$令 - \sqrt{x} = t$$

$$\Rightarrow x = t^2$$

$$dx = 2tdt$$

$$原式 = \int_0^{-1} 2te^t dt$$

$$= -2 \int_{-1}^{0} te^t dt$$

$$= -2e^t(t-1)|_{-1}^{0}$$

$$= 2 - 4e^{-1}$$

（6）

$$\int_{\frac{1}{e}}^{e} |\ln x| dx$$

$$= \int_{1}^{e} \ln x \, dx - \int_{\frac{1}{e}}^{1} \ln x \, dx$$

$$= (x \ln x - x)|_1^e - (x \ln x - x)|_{\frac{1}{e}}^{1}$$

$$= 2 - \frac{2}{e}$$

$$\boxed{\int \ln x dx = x \ln x - x + c}$$

（7）

$$\int_{1}^{e} \sin(\ln x) dx$$

$$= [x \sin(\ln x)]|_1^e - \int_{1}^{e} \cos(\ln x) dx$$

$$= [x \sin(\ln x)|_1^e - [x \cos(\ln x)]|_1^e - \int_{1}^{e} \sin(\ln x) dx$$

移项得:

$$原式 = \frac{1}{2}(e \sin 1 - e \cos 1 + 1)$$

（8）

$$\int_0^{\frac{1}{2}} \frac{x \arcsin x}{\sqrt{1-x^2}} dx$$

$$= \int_{0}^{\frac{1}{2}} x \arcsin x \, d \arcsin x$$

令 $\arcsin x = t \Rightarrow x = \sin t$

原式

$$= \int_{0}^{\frac{\pi}{6}} t \sin t \, dt$$

$\frac{\pi}{6}$

$$\boxed{\begin{aligned}&\int x \sin ax dx \\ &= \frac{1}{a^2} \sin ax - \frac{x}{a} \cos ax + C\end{aligned}}$$

$= \frac{1}{2} - \frac{\sqrt{3}\ \pi}{12}$

3. 解：

$\int_0^2 x^2 f^{''}(x)dx$

$= \int_0^2 x^2 d(f^{'}(x))$

$= x^2 f^{'}(x)|_0^2 - 2(\int_0^2 xf^{'}(x)dx)$

$= x^2 f^{'}(x)|_0^2 + 2\int_0^2 \quad f(x) - 2xf(x)|_0^2$

$代入数据得原式 = 0$

4. 证明：

（1）

$\because (f^2(x))' = 2f(x)f'(x)$

$\therefore \int_a^b xf(x)f'(x)dx$

$= \frac{1}{2}\int_a^b xd(f^2(x))$

$= \frac{x}{2}f^2(x)|_a^b - \frac{1}{2}\int_a^b f^2(x)dx$

$= \frac{b}{2}f^2(b)|_a^b - \frac{a}{2}f^2(a) - \frac{1}{2}$

$= -\frac{1}{2}$

（2）

由施瓦茨不等式得$(P169)$

$$(\int_a^b (f(x)\cdot xf'(x))dx)^2 \le \int_a^b f^2(x)dx \cdot \int_a^b (xf'(x))^2 dx$$

$$\Rightarrow \frac{1}{4} \le \int_a^b x^2 (f'(x))^2\, dx \text{得证}$$

5. 证明:

(1)

$$\int_0^{\frac{\pi}{2}} [f(x)+f''(x)]\sin x\, dx$$

$$= \int_0^{\frac{\pi}{2}} f(x)\sin x\, dx + \int_0^{\frac{\pi}{2}} f''(x)\sin x\, dx$$

$$= \int_0^{\frac{\pi}{2}} f(x)\sin x\, dx + \sin x\cdot f'(x)|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} f'(x)\cos x\, dx$$

$$= f'(\frac{\pi}{2}) + \int_0^{\frac{\pi}{2}} f(x)\sin x\, dx - \int_0^{\frac{\pi}{2}} \cos x\, df(x)$$

$$= f'(\frac{\pi}{2}) + \int_0^{\frac{\pi}{2}} f(x)\sin x\, dx - (\cos x\cdot f(x)|_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} f(x)\sin x\, dx)$$

$$= f(0) + f'(\frac{\pi}{2})$$

（2）

$$\int_0^{\frac{\pi}{2}} [f(x)+f''(x)]\cos x\, dx$$

$$= \int_0^{\frac{\pi}{2}} f(x)\cos x\, dx + \int_0^{\frac{\pi}{2}} \cos x\, df'(x)$$

$$= \int_0^{\frac{\pi}{2}} f(x)\cos x\, dx + \cos x\, f'(x)|_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} f'(x)\sin x\, dx$$

$$= -f'(0) + \int_0^{\frac{\pi}{2}} f(x)\cos x\, dx + \int_0^{\frac{\pi}{2}} \sin x\, df(x)$$

$$= -f'(0) + \int_0^{\frac{\pi}{2}} f(x)\cos x\,dx + \sin x\, f(x)\Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} f(x)\cos x\,dx$$

$$= f(\frac{\pi}{2}) - f'(0)$$

6. 解（1）

$$f(x) = x^2, f'(x) = 2x, f(0) = 0, f'(\frac{\pi}{2}) = \pi$$

$$\int_0^{\frac{\pi}{2}} x^2 \sin x\,dx$$

$$= \int_0^{\frac{\pi}{2}} (x^2+2)\sin x\,dx - 2\int_0^{\frac{\pi}{2}} \sin x\,dx$$

$$= \pi + 2\cos x\Big|_0^{\frac{\pi}{2}}$$

$$= \pi - 2$$

（2）

$$f(x) = x^4, f'(x) = 4x^3, f(\frac{\pi}{2}) = \frac{\pi^4}{16}, f'(0) = 0$$

$$g(x) = x^2, g'(x) = 2x, g(\frac{\pi}{2}) = \frac{\pi^2}{4}, g'(0) = 0$$

$$\int_0^{\frac{\pi}{2}} x^4 \cos x\,dx$$

$$= \int_0^{\frac{\pi}{2}} (x^4+12x^2)\cos x\,dx - 12\int_0^{\frac{\pi}{2}} x^2\cos x\,dx$$

$$= \int_0^{\frac{\pi}{2}} (x^4+12x^2)\cos x\,dx - 12(\int_0^{\frac{\pi}{2}} (x^2+2)\cos x\,dx - \int_0^{\frac{\pi}{2}} 2\cos x\,dx)$$

$= \frac{\pi^4}{16} - 3\pi^2 + 24$ *（课本后答案错误）*