习题 3.2

本颜色字体均为概念或公式

1.

(1)$(x^n)' = nx^{n-1}$

$\Longrightarrow y = x^3 - 2x^2 + 3x - 4$

$y' = 3x^2 - 4x + 3$

(2)$(uv)' = u'v + uv'$

$y = (x^2 + 3x + 2)(x^2 - 3x + 2)$

方法$a.\, y' = (2x + 3)(x^2 - 3x + 2) + (x^2 + 3x - 2)(2x - 3)$

$\Rightarrow y' = 4x^3 - 10x$

方法$b.\, y = (x^2 + 2)^2 - (3x)^2 = x^4 - 5x^2 + 4$

$\Rightarrow y' = 4x^3 - 10x$

(3) $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$

$y = \frac{\cos x}{x} + \frac{x}{\cos x}$

$\Rightarrow y' = \frac{-\sin x \cdot x - \cos x}{x^2} + \frac{\cos x + x \sin x}{\cos^2 x}$

$\Rightarrow y' = -x^{-1} \cdot \sin x - x^{-2} \cos x + \sec x + x \tan x \cdot \sec x$

Tip: $\frac{1}{\sin x} = \csc x$, $\frac{1}{\cos x} = \sec x$

(4)$(\ln x)' = \frac{1}{x} / (\ln|x|)' = \frac{1}{x}$

$y = x \ln x$

$\Rightarrow y' = \ln x + x \cdot \frac{1}{x}$

$\Rightarrow y' = \ln x + 1$

(5)$y = x^2 + x^{-2}$

$\Rightarrow y' = 2x - 2x^{-3}$

(6)$(e^x)' = e^x$

$y = e^x \cos x$

$\Rightarrow y' = e^x \cos x + e^x(-\sin x)$

$\Rightarrow y' = e^x(\cos x - \sin x)$

(7)$y=exsinx$

$\Rightarrow y' = e^x \sin x + e^x \cos x$

$\Rightarrow y' = e^x(\sin x + \cos x)$

(8)$y = e^x \ln x$

$\Rightarrow y' = e^x \ln x + e^x \cdot \frac{1}{x}$

$\Rightarrow y' = e^x \left(\ln x + \frac{1}{x}\right)$

2.

(1) $\left(f\left(g(x)\right)\right)' = f'\left(g(x)\right) \cdot g'(x)$

$y = e^{x^2 + \sin x}$

$\Rightarrow y' = e^{x^2 + \sin x} \cdot (2x + \cos x)\Big) \cdot g'(x)$

(2)$y = x \ln(x^2 + e^x)$

$\Rightarrow y' = \ln(x^2 + e^x) + x \cdot \frac{1}{x^2 + e^x} \cdot (2x + e^x)$

$$\Rightarrow y' = ln(x^2+e^x) + \frac{2x^2+xe^x}{x^2+e^x}$$

$(3)y = sin\,2x$

$$\Rightarrow y' = 2\cos 2x$$

$(4)y = cos\,2x$

$$\Rightarrow y' = -2\sin 2x$$

$(5)y = \sqrt{x}\,arcsin\sqrt{x}$

$$Tip：(arcsin\,x)' = \frac{1}{\sqrt{1-x^2}};$$

$$(arccos\,x)' = -\frac{1}{\sqrt{1-x^2}};$$

$$(arctan\,x)' = \frac{1}{1+x^2};$$

$$(arccotx)' = -\frac{1}{1+x^2}$$

$$\Rightarrow y' = \frac{1}{2\sqrt{x}} \cdot arcsin\sqrt{x} + \sqrt{x} \cdot \frac{1}{\sqrt{1-x}} \cdot \frac{1}{2\sqrt{x}}$$

$$\Rightarrow y' = \frac{arcsin\sqrt{x}}{2\sqrt{x}} + \frac{1}{2\sqrt{1-x}} \quad (x \neq 0)$$

$(6)y = \sqrt{x}\,arccos\sqrt{x}$

$$\Rightarrow y' = \frac{1}{2\sqrt{x}} \cdot arccos\sqrt{x} + \sqrt{x} \cdot \left(-\frac{1}{\sqrt{1-x}}\right) \cdot \frac{1}{2\sqrt{x}}$$

$$\Rightarrow y' = \frac{arccos\sqrt{x}}{2\sqrt{x}} - \frac{1}{2\sqrt{1-x}} \quad (x \neq 0)$$

$(7)y = x^2\,arctan\frac{1}{x}$

$$\Rightarrow y' = 2x \cdot arctan\frac{1}{x} + x^2\frac{1}{1+\left(\frac{1}{x}\right)^2} \cdot \left(-\frac{1}{x^2}\right)$$

$$\Rightarrow y' = 2x\, arctan\frac{1}{x} - \frac{x^2}{x^2+1}$$

$$(8) y = x^2 arccot\frac{1}{x}$$

$$\Rightarrow y' = 2x \cdot arccot\frac{1}{x} + x^2\left(-\frac{1}{1+\left(\frac{1}{x}\right)^2}\right) \cdot \left(\frac{1}{x^2}\right)$$

$$\Rightarrow y' = 2x arccot\frac{1}{x} - \frac{x^2}{x^2+1}$$

(9) $(sec\, x)' = tan\, x \cdot sec\, x$

$(csc\, x)' = -cot\, x\, csc\, x$

$(tan\, x)' = sec^2\, x$

$(cot\, x)' = -csc^2\, x$

$y = sec\, x^2$

$\Rightarrow y' = tan\, x^2 - sec\, x^2 \cdot (2x)$

$\Rightarrow y' = 2x\, tan\, x^2\, sec\, x^2$

$(10) y = csc\sqrt{x}$

$$\Rightarrow y' = -cot\sqrt{x}\, csc\sqrt{x} \cdot \frac{1}{2\sqrt{x}}$$

$$\Rightarrow y' = -\frac{cot\sqrt{x}\, csc\, cos\, x}{2\sqrt{x}}$$

$(11) y = x^{\frac{1}{2}} + x^{-\frac{1}{2}}$

$$\Rightarrow y' = \frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{2}x^{-\frac{3}{2}}$$

$$\Rightarrow y' = \frac{1}{2}\left(x^{-\frac{1}{2}} - x^{-\frac{3}{2}}\right)$$

$$(12) y = \frac{e^{\sqrt{x}} - e^{-\sqrt{x}}}{e^{\sqrt{x}} + e^{-\sqrt{x}}}$$

$$\Rightarrow y = \frac{e^{\sqrt{x}} + e^{-\sqrt{x}} - 2e^{-\sqrt{x}}}{e^{\sqrt{x}} + e^{-\sqrt{x}}}$$

$$y = 1 - 2\frac{1}{e^{2\sqrt{x}} + 1}$$

$$\Rightarrow y' = -2\frac{-e^{2\sqrt{x}} \cdot \frac{1}{\sqrt{x}}}{\left(e^{2\sqrt{x}} + 1\right)^2}$$

$$\Rightarrow y' = \frac{2e^{2\sqrt{x}}}{\sqrt{x}\left(e^{2\sqrt{x}} + 1\right)^2} \quad (x \neq 0)$$

3.

$$(1) \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

对两边关于x求导

$$\Rightarrow \frac{1}{a^2} \cdot 2x + \frac{1}{b^2} \cdot 2y \cdot y' = 0$$

$$\Rightarrow y' = -\frac{b^2 x}{a^2 y} \quad (y \neq 0)$$

$$(2) x^2 + 2xy - y^2 = 2x$$

对两边关于x求导

$$\Rightarrow 2x + 2y + 2xy' - 2y \cdot y' = 2$$

$$\Rightarrow (x - y)y' = 1 - x - y$$

$$\Rightarrow y' = \frac{1 - x - y}{x - y} \quad (x \neq y)$$

$$(3) \sqrt{x} + \sqrt{y} = \sqrt{a}$$

对两边关于x求导

$$\Rightarrow \frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}} \cdot y' = 0$$

$$\Rightarrow y' = -\frac{\sqrt{xy}}{x} \quad (x > 0, y > 0)$$

(4)$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$

对两边关于x求导

$$\Rightarrow \frac{2}{3}x^{-\frac{1}{3}} + \frac{2}{3}y^{-\frac{1}{3}} - y' = 0$$

$$\Rightarrow y' = -\left(\frac{y}{x}\right)^{\frac{1}{3}} \quad (x \neq 0)$$

(5) $arctan\frac{y}{x} = ln\sqrt{x^2 + y^2}$

对两边关于x求导

$$\Rightarrow \frac{1}{1 + \left(\frac{y}{x}\right)^2} \cdot \frac{y'x - y}{x^2} = \frac{1}{\sqrt{x^2 + y^2}} \cdot \frac{2x + 2y \cdot y'}{2\sqrt{x^2 + y^2}}$$

$$\Rightarrow \frac{y'x - y}{x^2 + y^2} = \frac{x + y \cdot y'}{x^2 + y^2}$$

$$\Rightarrow y'x - y = x + y - y'$$

$$\Rightarrow y' = \frac{x + y}{x - y} \quad (x \neq y, x \neq 0)$$

(6)$x^y = y^x \quad (x > 0, y > 0)$

$$\Rightarrow y\,ln\,x = x\,ln\,y$$

对两边关于x求导

$$\Rightarrow y'\,ln\,x + y\frac{1}{x} = ln\,y + x\frac{1}{y} \cdot y'$$

1°由$y\,ln\,x = x\,ln\,y$变形得

$$\Rightarrow y'\left(\ln x-\frac{x}{y}\right)=y'(\ln y-1)\cdot\frac{x}{y}$$

$$\ln y-\frac{y}{x}=\frac{y}{x}(\ln x-1)$$

$$\Rightarrow y'=\frac{y^2(\ln x-1)}{x^2(\ln y-1)}\quad (x>0,y>0)$$

$$2^\circ\Rightarrow y'=\frac{xy\ln y-y^2}{xy\ln x-x^2}\quad (x>0,y>0)$$

$(7) x-y+\xi\sin y=0\ \ (\xi\text{为参数})$

对两边关于x求导

$$\Rightarrow 1-y'+\xi\cos y\cdot y'=0$$

$$\Rightarrow y'=\frac{1}{1-\xi\cos y}$$

4.

$(1) y=x^{\sin x}$

对两边取对数

$$\Rightarrow \ln y=\sin x\ln x$$

$$\Rightarrow \frac{1}{y}\cdot y'=\cos x\ln x+\sin x\frac{1}{x}$$

$$\Rightarrow y'=y\cos x\ln x+\frac{y\sin x}{x}$$

$$\Rightarrow y'=x^{\sin x}\left(\cos x\ln x+\frac{\sin x}{x}\right)$$

$(2) y=x^{\ln x}$

对两边取对数

$$\Rightarrow \ln y=\ln x\cdot\ln x=\ln^2 x$$

$$\Rightarrow \frac{1}{y} \cdot y' = 2\ln x \cdot \frac{1}{x}$$

$$\Rightarrow y' = \frac{2y\ln x}{x}$$

$$\Rightarrow y' = 2x^{(\ln x)-1} \ln x$$

$$(3) y = \sqrt[3]{\frac{(x-1)(x-2)}{(x-3)(x-4)}}$$

对两边取对数

$$\Rightarrow \ln y = \frac{1}{3} \ln \frac{(x-1)(x-2)}{(x-3)(x-4)}$$

$$= \frac{1}{3}\ln(x-1) + \frac{1}{3}\ln(x-2) - \frac{1}{3}\ln(x-3) - \frac{1}{3}\ln(x-4)$$

$$\Rightarrow \frac{1}{y} \cdot y' = \frac{1}{3}\frac{1}{x-1} + \frac{1}{3} \cdot \frac{1}{x-2} - \frac{1}{3} \cdot \frac{1}{x-3} - \frac{1}{3} \cdot \frac{1}{x-4}$$

$$\Rightarrow y' = \frac{-4x^2 + 20x - 22}{3(x-1)(x-2)(x-3)(x-4)} \cdot y$$

$$\Rightarrow y' = \frac{-4x^2 + 20x - 22}{3(x-1)(x-2)(x-2)(x-4)} \sqrt[3]{\frac{(x-1)(x-2)}{(x-3)(x-4)}}$$

5.

$$y' = \frac{dy}{dx}, y = y(t), x = x(t) \Rightarrow y' = \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$\Rightarrow x = x(t)$ 在 $t \in D$ 时单调

$$(1) \begin{cases} x = 1 - t^2 \\ y = 1 - t^3 \end{cases} \Rightarrow \frac{dx}{dt} = -2t, \quad \frac{dy}{dt} = -3t^2 \Rightarrow y' = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3}{2}t$$

$$(2) \begin{cases} x = \ln(1 + t^2) \\ y = t - \arctan t \end{cases}$$

$$\Rightarrow \frac{dx}{dt} = \frac{2t}{1+t^2}, \frac{dy}{dt} = 1 - \frac{1}{1+t^2} = \frac{t^2}{1+t^2} \Rightarrow y' = \frac{dy}{dx} = \frac{\frac{dy}{dx}}{\frac{dx}{dt}} = \frac{t}{2}$$

$$(3)\begin{cases} x = a\cos^3 t \\ y = a\sin^3 t \end{cases} \Rightarrow \frac{dx}{dt} = 3a\cos^2 t\,(-\sin t), \quad \frac{dy}{dt} = 3a\sin^2 t \cdot \cos t$$

$$\Rightarrow y' = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = -\tan t$$

6.

证明：$\because \sqrt{x} + \sqrt{y} = \sqrt{a}(a > 0) \quad (x \geq 0, y \geq 0)$

$\therefore$ 抛物线与x轴交点为$P_1(a,0)$，与y轴交点为$P_2(0,a)$

对$\sqrt{x} + \sqrt{y} = \sqrt{a}(a > 0)$两边关于$x$求导

$$\Rightarrow \frac{1}{2\sqrt{x}} + \frac{y'}{2\sqrt{y}} = 0$$

$$\Rightarrow y' = -\frac{\sqrt{xy}}{x} = k_{切}$$

$\Rightarrow l_{切}: y - y_0 = -\dfrac{\sqrt{x_0 y_0}}{x_0}(x - x_0) \Rightarrow$ 抛物线在x_0点处的切线

在x_0点处$\sqrt{x_0} + \sqrt{y_0} = \sqrt{a} \Rightarrow x_0 + 2\sqrt{x_0 y_0} + y_0 = a$

$\Rightarrow l_{切}$与x轴交点为$P_1\left(x_0 + \sqrt{x_0 y_0}, 0\right)$，与$y$轴交点为$P_2\left(0, y_0 + \sqrt{x_0 y_0}\right)$

$x_0\, y_0 = \sqrt{x_0\, y_0}(x - x_0)$，$y - y_0 = \sqrt{x_0 y_0}$

$$\Rightarrow x_0 + \sqrt{x_0 y_0} + y_0 + \sqrt{x_0 y_0} = x_0 + 2\sqrt{x_0 y_0} + y_0 = a$$

故抛物线$\sqrt{x} + \sqrt{y} = \sqrt{a}(a > 0)$上任一点的切线截两个坐标轴的截距之和为$a$

7.

证明：$\begin{cases} x = a(\cos t + t\sin t) \\ y = a(\sin t + t\cos t) \end{cases}$

$$\Rightarrow \begin{matrix} \dfrac{dx}{dt} = a(-\sin t + \sin t + t\cos t) = at\cos t \\ \dfrac{dy}{dt} = a(\cos t - \cos t + t\sin t) = at\sin t \end{matrix}$$

$$\Rightarrow y' = \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{\sin t}{\cos t} = \tan t$$

$$\Rightarrow k_{切} = -\frac{1}{y'} = -\cot t$$

$\Rightarrow$ 曲线上点x_0的切线为$y - y_0 = -\cot t_0 (x - x_0)$

$l_{法}$： $\cos t_0 x + \sin t_0 y - x_0 \cos t_0 - y_0 \sin t_0 = 0$

$$\Rightarrow d = \frac{|\cos t_0 \cdot x_1 + \sin t_0 \cdot y_1 - x_0 \cos t_0 - y_0 \sin t_0|}{\sqrt{\cos^2 t_0 + \sin^2 t_0}} \left((x_1, y_1)\text{为原点}\right)$$

$= a\cos^2 t_0 + at_0 \sin t_0 \cos t_0 + a\sin^2 t_0 - at_0 \sin t_0 \cos t_0$

$= a(\cos^2 t_0 + \sin^2 t_0)$

$= a$

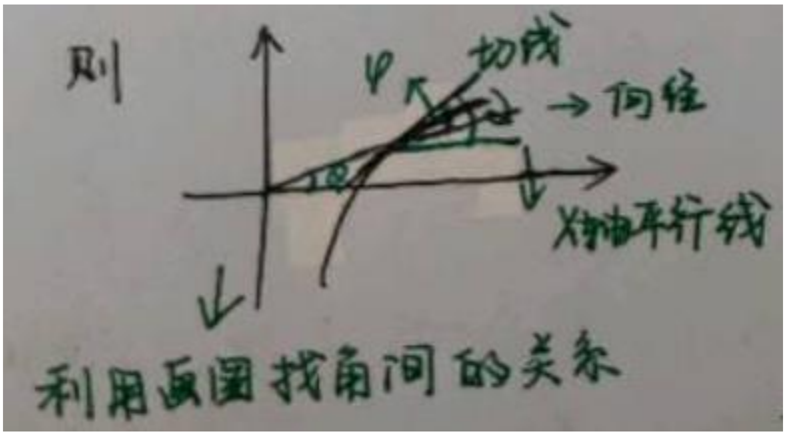
8.

(1)由题得l：$r = r(\theta). \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$ ，$r = r(\theta)$关于θ可导

$$\Rightarrow \begin{matrix} \dfrac{dx}{d\theta} = r'\cos\theta - r\sin\theta = r'(\theta)\cos\theta - r(\theta)\sin\theta \\ \dfrac{dy}{d\theta} = r'\sin\theta + r\cos\theta = r'(\theta)\sin\theta + r(\theta)\cos\theta \end{matrix}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{r'(\theta)\sin\theta + r(\theta)\cos\theta}{r'(\theta)\cos\theta - r(\theta)\sin\theta} = \frac{r'(\theta)\tan\theta + r(\theta)}{r'(\theta) - r(\theta)\tan\theta}$$

(2)设点$P(r,\theta)$处切线与x轴夹角为α



$\Rightarrow$ 同位角：$\theta = \alpha - \varphi$

即$\varphi = \alpha - \theta$

$$\Rightarrow \begin{matrix} \tan\varphi = \tan(\alpha - \theta) = \dfrac{\tan\alpha - \tan\theta}{1 + \tan\alpha\tan\theta} \\ \tan\alpha = k_{切} = \dfrac{dy}{dx} = \dfrac{r'(\theta)\tan\theta + r(\theta)}{r'(\theta) - r(\theta)\tan\theta} \end{matrix}$$

上下同乘$[r'(\theta) - r(\theta)\tan\theta]$

$$\Rightarrow \tan\varphi = \frac{\dfrac{r'(\theta)\tan\theta + r(\theta)}{r'(\theta) - r(\theta)\tan\theta} - \tan\theta}{1 + \dfrac{r'(\theta)\tan\theta + r(\theta)}{r'(\theta) - r(\theta)\tan\theta}\tan\theta}$$

$$\Rightarrow \tan\varphi = \frac{r'(\theta)\tan\theta + r(\theta) - r'(\theta)\tan\theta + r(\theta)\tan^2\theta}{r'(\theta) - r(\theta)\tan\theta + r'(\theta)\tan^2\theta + r(\theta)\tan\theta}$$

$$\Rightarrow \tan\varphi = \frac{r(\theta)(1 + \tan^2\theta)}{r'(\theta)(1 + \tan^2\theta)} = \frac{r(\theta)}{r'(\theta)}$$

9.

解： $\because$ 极坐标曲线$r = e^\theta$

$\therefore$ 其可用极角θ作为参数

表示如下

$$\begin{cases} x = r\cos\theta = e^\theta\cos\theta \\ y = r\sin\theta = e^\theta\sin\theta \end{cases}$$

$$\frac{dy}{d\theta} = e^{\theta} si\, n\, \theta + e^{\theta} \cos\theta$$

$$\frac{dx}{d\theta} = e^{\theta} \cos\theta - e^{\theta} \sin\theta$$

$$\Rightarrow y' = \frac{\sin\theta + \cos\theta}{\cos\theta - \sin\theta}$$

点$\left(e^{\frac{\pi}{2}}, \frac{\pi}{2}\right)$处的极角为$\frac{\pi}{2}$

$$\Rightarrow y_1' = \frac{1+0}{0-1} = -1 \quad y_1 = e^{\frac{\pi}{2}}, x_1 = 0$$

$$\Rightarrow l_{切} = y - y_1 = y_1'(x - x_1)$$

$$\Rightarrow l_{切}: y = -x + e^{\frac{\pi}{2}}$$

$$\Rightarrow l_{切} = x + y - e^{\frac{\pi}{2}} = 0$$