习题 6.1

1.（1）✗（$f(x)$在$[a,b]$上可积一定有界，但$f(x)$有界为$f(x)$在$[a,b]$上可积最基本条件）

（2）✗（$\lambda \to 0 \neq n \to \infty$）

（3）√

（4）√

2.（1）C

（2）D（由施瓦茨不等式得$\left(\int_a^b f(x)g(x)\,dx\right)^2 \leq \int_a^b f^2(x)\,dx \int_a^b g^2(x)\,dx$

其中令$f(x) = x, g(x) = 1$ 则$\left(\int_a^b x\,dx\right)^2 \leq \int_a^b x^2\,dx\,(b-a)$

即$\left(\int_0^1 x\,dx\right)^2 \leq \int_0^1 x^2\,dx$）

（3）D

3. $S = \int_0^{10}(10t+1)dt = 510m$

4.（1）解：令$f(x) = x, x \in [0,1]$

将$x \in [0,1], n$等分，$\Delta x_i = \frac{1}{n}$

取$\xi_i = \frac{i}{n}(i = 1,2,\ldots,n)$

则$\sum_{i=1}^n f(\xi_i)\Delta x_i = \frac{1}{n^2}(1 + 2 + \cdots + n)$

$$= \frac{1}{n^2} \cdot \frac{n(n+1)}{2} = \frac{n+1}{2n}$$

当 $\lambda \to 0$ 时，$\pi \to \infty$

故$\int_0^1 x\,dx = \lim_{n \to \infty}\sum_{i=1}^n f(\xi_i)\Delta x_i = \frac{1}{2}$

（2）解：令$g(x) = e^x, x \in [0,1]$

将$x \in [0,1]$ n等分，$\Delta x_i = \frac{1}{n}$

取$\xi_i = \frac{2}{n}(i = 1,2,\ldots,n)$

则$\sum_{i=1}^n f(\xi_i)\Delta x_i = \frac{1}{n}e^{\frac{1}{n}} + \frac{1}{n}e^{\frac{2}{n}} + \cdots + \frac{1}{n}e^{\frac{n}{n}}$

$$= \frac{1}{n}\left(e^{\frac{1}{n}} + e^{\frac{2}{n}} + e^{\frac{3}{n}} + \cdots + e^1\right)$$

$$= \frac{1}{n} \cdot \frac{e^{\frac{1}{n}}(1-e)}{1-e^{\frac{1}{n}}} \text{（等比数列）}$$

由 $\lambda \to 0$ 时，$n \to \infty$ 时

$$\lim_{n \to \infty} \sum_{i=1}^{n} f(\xi_i) \Delta x_i = \lim_{n \to \infty} \frac{1}{n} \cdot \frac{e^{\frac{1}{n}}(1-e)}{-\frac{1}{n}} \quad （等价无穷小）$$

$$= \lim_{n \to \infty} (e-1) e^{\frac{1}{n}} = e - 1$$

5. 由定积分几何意义得 $\frac{1}{2} \cdot 2(k + 8 + k) = 10$

   $\Rightarrow 2k = 2 \Rightarrow k = 1$

6. （1）解：令 $f(x) = \frac{1}{(1+x)^2}$ ，$x \in [0,1]$

   对 $x \in [0,1] n$ 等分，$\Delta x_i = \frac{1}{n}$ ，取 $\xi_i = \frac{2}{n}(i = 1,2,\ldots,n)$

   则 $\sum_{i=1}^{n} f(\xi_i) \Delta x_i = \frac{1}{\left(1+\frac{1}{n}\right)^2} \cdot \frac{1}{n} + \frac{1}{\left(1+\frac{2}{n}\right)^2} \cdot \frac{1}{n} + \cdots + \frac{1}{\left(1+\frac{i}{n}\right)^2} \cdot \frac{1}{n}$

   $= \sum_{i=1}^{n} \frac{n}{(n+i)^2}$

   当 $n \to \infty$ ，$\lambda \to 0$

   则原式 $= \int_0^1 \frac{1}{(1+x)^2} dx$

   （2）解：令 $f(x) = \sin x$

   在 $x \in \left[0, \frac{\pi}{2}\right]$ 分成 n 等份，则 $\Delta x_i = \frac{\pi}{2n}$

   取 $\xi_i = \frac{i\pi}{2n}(i = 0,1,2,\ldots,n-1)$

   $\sum_{i=1}^{n} f(\xi_i) \Delta x_i = \frac{\pi}{2n}\left[\sin 0 + \sin \frac{\pi}{2n} + \cdots + \frac{\sin(n-1)\pi}{2n}\right]$

   当 $n \to \infty$ ，$\lambda \to 0$

   则原式 $= \int_0^{\frac{\pi}{2}} \sin x \, dx$

7. （1）解：原式 $= \int_c^b f(x)\,dx + \int_b^a f(x)\,dx = \int_c^a f(x)\,dx$

   （2）由积分中值定理得

   平均值为 $f(\xi) = \frac{1}{1-0} \int_0^1 f(x)\,dx = \int_0^1 e^x \, dx$

8. （1）由 $f(x) = x$，$g(x) = \sqrt{x}$ 在区间 $[0,1]$ 可积，且在 $[0,1]$ 上 $x \le \sqrt{x}$

   由保序性 $\int_0^1 x \, dx \le \int_0^1 \sqrt{x} \, dx$

   （2）同理 $x \ge \sin x$

   $\int_0^{\frac{\pi}{2}} x \, dx \ge \int_0^{\frac{\pi}{2}} \sin x \, dx$

   （3）因为在 $[-1,0]$ 上 $e^{2x} \le e^x$ 所以 $\int_{-1}^0 e^{2x} \, dx \le \int_{-1}^0 e^x \, dx$

两边加负号即$\int_0^{-1} e^{2x}\, dx \ge \int_0^{-1} e^x\, dx$

（4）由$lnx \ge (lnx)^2$在$[1,2]$上

同理$\int_1^2 lnx\, dx \ge \int_1^2 (lnx)^2\, dx$

（5）由在$\left[0,\frac{\pi}{2}\right]$上 $x \le \tan x$

同理$\int_0^{\frac{\pi}{2}} x\, dx \le \int_0^{\frac{\pi}{2}} \tan x\, dx$

9. （1）由在$x \in [0,1]$内$\frac{1}{1+x^2} \in \left[\frac{1}{2}, 1\right]$

则$\frac{1}{2} \cdot 1 \le \int \frac{dx}{1+x^2} \le 1 \cdot 1$（推论 6.1.1）

即$\frac{1}{2} \le \int \frac{dx}{1+x^2} \le 1$

（2）由于在$x \in [0,2]$，$x^2 - 2x \in [-1,0]$

则$e^{x^2-2x} \in \left[\frac{1}{e}, 1\right]$

$\frac{2}{e} \le \int_0^2 e^{x^2-2x}\, dx \le 2$

（3）在$x \in [0,2\pi]$内 $\frac{1}{1+0.5\cos x} \in \left[\frac{2}{3}, 2\right]$

则$\frac{4\pi}{3} \le \int_0^{2\pi} \frac{dx}{1+0.5\cos x} \le 4\pi$

（4）在$x \in [0,100]$内 则$\frac{e^{-x}}{x+100} \in \left[\frac{\frac{1}{e^{100}}}{200}, \frac{1}{100}\right]$

$\frac{1}{2e^{100}} \le \int_0^{100} \frac{e^{-x}}{x+100} dx \le 1$

10. 证明：（1）$\int_0^2 \frac{1}{\sqrt{1+x^3}} dx = \int_0^1 \frac{1}{\sqrt{1+x^3}} dx + \int_1^2 \frac{1}{\sqrt{1+x^3}} dx$

由积分中值定理得

原式$= f(\xi_1)(1-0) + f(\xi_2)(2-1)$（其中$\xi_1 \in [0,1]$，$\xi_2 \in [1,2]$）

$\qquad = \frac{1}{\sqrt{1+\xi_1^3}} + \frac{1}{\sqrt{1+\xi_2^3}}$

则$\frac{1}{\sqrt{1+0}} + \frac{1}{\sqrt{1+1}} \le \int_0^2 \frac{1}{\sqrt{1+x^3}} dx \le \frac{1}{\sqrt{1+1}} + \frac{1}{\sqrt{1+2^3}}$

即$\frac{1}{3} + \frac{\sqrt{2}}{2} \le \int_0^2 \frac{1}{\sqrt{1+x^3}} dx \le 1 + \frac{\sqrt{2}}{2}$

（2）$\int_0^{\frac{\pi}{2}} \sin^2 x\, dx = \int_0^{\frac{\pi}{6}} \sin^2 x\, dx + \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin^2 x\, dx + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin^2 x\, dx$

积分中值定理

$= \sin^2 \xi_1 \left(\frac{\pi}{6} - 0\right) + \sin^2 \xi_2 \left(\frac{\pi}{3} - \frac{\pi}{6}\right) + \sin^2 \xi_3 \left(\frac{\pi}{2} - \frac{\pi}{3}\right)$（$\xi_1 \in \left[0, \frac{\pi}{6}\right]$, $\xi_2 \in \left[\frac{\pi}{6}, \frac{\pi}{3}\right]$, $\xi_3 \in \left[\frac{\pi}{3}, \frac{\pi}{2}\right]$）

则$\frac{\pi}{6}\sin^2 0+\frac{\pi}{6}\sin^2\frac{\pi}{6}+\frac{\pi}{6}\sin^2\frac{\pi}{3}\leq\int_0^{\frac{\pi}{2}}\sin^2 x\,dx\leq\frac{\pi}{6}\left(\sin^2\frac{\pi}{6}+\sin^2\frac{\pi}{3}+\sin^2\frac{\pi}{2}\right)$

即$\frac{\pi}{6}\leq\int_0^{\frac{\pi}{2}}\sin^2 x\,dx\leq\frac{\pi}{6}\left(\frac{1}{4}+\frac{3}{4}+1\right)=\frac{\pi}{3}$ 得证

11. 解：由题意得 $\int_0^1 f(x)\,dx$为具体值

设$f(x)=x+t$

则$2\int_0^1 f(x)\,dx=2\int_0^1(x+t)\,dx=t$

则$2\int_0^1 x\,dx+2\int_0^1 t\,dx=t$

则$2\int_0^1 x\,dx=-t$

$t=-2\cdot\frac{1}{2}\cdot 1=-1$

故$f(x)=x-1$

习题 6.2

1. （1）$F'(x) = \sqrt{1+x^2}$　$F'(0) = 1$

（2）$F'(x) = 2 - \frac{1}{\sqrt{x}} < 0 \Rightarrow x < \frac{1}{4}$　区间为 $\left(0, \frac{1}{4}\right)$

（3）$F'(x) = f(e^{-x}) \cdot e^{-x}(-1) - f(x) = -f(e^{-x}) \cdot e^{-x} - f(x)$

（4）令 $\int_0^y e^{-t^2}\,dt + \int_0^x \sin^2 t\,dt = F(x)$

$F'(x) = e^{-y^2}y' + \sin^2 x = 0 \Rightarrow y' = -e^{y^2}\sin^2 x$

（5）因为 $[-\pi, \pi]$ 关于原点对称　又 $|\sin x|$ 为偶函数

所以　原式$= 2\int_0^\pi \sin x\,dx = -2\cos x\big|_0^\pi = 4$

2. （1）原式$= \lim\limits_{x \to 0} \frac{\cos x^2}{1} = 1$

（2）原式$= \lim\limits_{x \to 0} \frac{2\int_0^x e^t dt \cdot e^x}{xe^{2x^2}} = \lim\limits_{x \to 0} \frac{2\int_0^x e^t dt}{xe^{2x^2-x}}$

$= \lim\limits_{x \to 0} \frac{2e^x}{e^{2x^2-x} + xe^{2x^2-x} \cdot (4x-1)} = \frac{2 \times 1}{1 + 0 \times 1 \times (-1)} = 2$

3. （1）$\int_0^1 \sqrt{x}\left(1 - \sqrt{x}\right)^2 dx = \int_0^1 \sqrt{x}\left(1 + x - 2\sqrt{x}\right)dx$

$= \int_0^1 \left(\sqrt{x} + x^{\frac{3}{2}} - 2x\right)dx = \left(\frac{2}{3}x^{\frac{3}{2}} + \frac{2}{5}x^{\frac{5}{2}} - x^2\right)\Big|_0^1$

$= \frac{2}{3} + \frac{2}{5} - 1 = \frac{1}{15}$

（2）原式$= \int_0^1 \frac{-(x^2+1)+2}{1+x^2}dx = \int_0^1 \left(-1 + \frac{2}{1+x^2}\right)dx$

$= (-x + 2\arctan x)\big|_0^1 = -1 + 2 \times \frac{\pi}{4} = \frac{\pi}{2} - 1$

（3）令　$1 - x = t \Rightarrow x = 1 - t$

$dx = -dt$　$x\big|_0^1 \to t\big|_1^0$

原式$\int_1^0 e^t(-dt) = \int_0^1 e^t dt = e^x\big|_0^1 = e - 1$

（4）原式$= \int_0^1 \frac{\frac{1}{2}}{\sqrt{1-\left(\frac{x}{2}\right)^2}}dx = \int_0^1 \frac{1}{\sqrt{1-\left(\frac{x}{2}\right)^2}}d\left(\frac{1}{2}x\right)$

$= \arcsin\frac{x}{2}\Big|_0^1 = \frac{\pi}{6} - 0 = \frac{\pi}{6}$

（5）原式$= \int_{-1}^2 \sqrt{2+x}\,d(x+2) = \frac{2}{3}(x+2)^{\frac{3}{2}}\Big|_{-1}^2$

$= \frac{2}{3} \times 4^{\frac{3}{2}} - \frac{2}{3} = \frac{2}{3} \times 7 = \frac{14}{3}$

（6）原式= $\int_0^\pi \frac{1-cos2x}{2}dx = \int_0^\pi \frac{1}{2}dx - \frac{1}{4}\int_0^\pi cos2xd(2x)$

$= \frac{x}{2}\Big|_0^\pi - \left(\frac{1}{4}sin2x\right)\Big|_0^\pi = \frac{\pi}{2} - 0 = \frac{\pi}{2}$

（7）原式= $\int_0^{\frac{\pi}{4}} \left(\frac{sinx+cosx}{cosx}\right)^2 dx = \int_0^{\frac{\pi}{4}}(tanx+1)^2\, dx$

$= \int_0^{\frac{\pi}{4}}(tan^2x + 1 + 2tanx)\, dx$

$= \int_0^{\frac{\pi}{4}}(sec^2x + 2tanx)\, dx$

$= (tanx - 2ln|cosx|)\Big|_0^{\frac{\pi}{4}} = 1 - 2ln\frac{\sqrt{2}}{2}$

$= 1 + ln\left(\frac{\sqrt{2}}{2}\right)^{-2} = 1 + ln2$

（8）原式= $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{2sin^2x}\, dx$
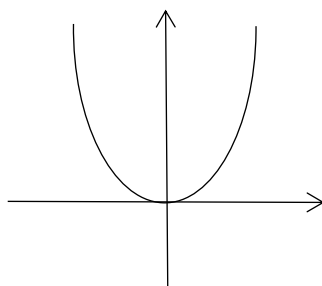
因为 $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ 关于原点对称，又 $\sqrt{2sin^2x}$ 为偶函数

所以原式= $2\int_0^{\frac{\pi}{2}} \sqrt{2}sinx\, dx = -2\sqrt{2}cosx\Big|_0^{\frac{\pi}{2}} = 2\sqrt{2}$

（9）原式= $\int_0^1 x^2\, dx + \int_1^2 1dx$

$= \frac{x^2}{3}\Big|_0^1 + x\Big|_1^2$

$= \frac{1}{3} + 2 - 1 = \frac{4}{3}$

（10）原式= $\int_0^1 0\, dx + \int_1^2 sinxdx + \int_2^3 2sinxdx + \int_3^\pi 3sinxdx$

$= -cosx\Big|_1^2 - 2cosx\Big|_2^3 - 3cosx\Big|_3^\pi$

$= -cos2 + cos1 - 2cos3 + 2cos2 + 3 + 3cos3$

$= cos1 + cos2 + cos3 + 3$

4. 解：$\frac{dy}{dx} = \frac{f^2(t)f'(t)}{f(x)f'(t)} = f(t)$

$\frac{d^2y}{dx^2} = \frac{f'(t)}{f(t)f'(t)} = \frac{1}{f(t)}$

5. 证：$y' = xf(x)$

因为 $f(x)>0$ 当 $x>0$ 时，$y'>0$，$y\uparrow$

当 $x<0$ 时，$y'<0$，$y\downarrow$
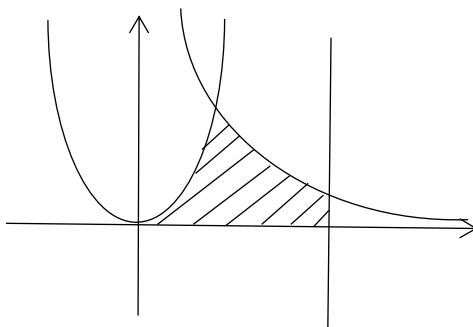
所以当$x = 0$时，y取最小值，得证

6. 解:$S = \int_0^1 x^2 dx + \int_1^2 \frac{1}{x} dx$

$= \frac{x^3}{3}\Big|_0^1 + lnx\Big|_1^2$

$= \frac{1}{3} + ln2$

7. （1）证：$\int_{-\pi}^{\pi} cosnxdx = \int_{-\pi}^{\pi} \frac{1}{n} cosnxd(nx)$

$= \frac{1}{n} sin(nx)\Big|_{-\pi}^{\pi} = \frac{1}{n}[sinn\pi - sin(-n\pi)] = 0$

所以得证

（2）证：$= \int_{-\pi}^{\pi} sinnxdx = \frac{1}{n}\int_{-\pi}^{\pi} \frac{1}{n} sinnxd(nx)$

$= -\frac{1}{n} cosnx\Big|_{-\pi}^{\pi} = -\frac{1}{n}[cosn\pi - cos(-n\pi)]$

$= -\frac{1}{n}(cosn\pi - cosn\pi) = 0$

所以得证

（3）证：$\int_{-\pi}^{\pi} cos^2 nxdx = \int_{-\pi}^{\pi} \frac{1+cos(2nx)}{2} dx$

$= \frac{1}{2}\int_{-\pi}^{\pi} 1dx + \frac{1}{2}\int_{-\pi}^{\pi} \frac{1}{2n} cos(2nx)d(2nx)$

$= \frac{1}{2}x\Big|_{-\pi}^{\pi} + \frac{1}{4n} sin(2nx)\Big|_{-\pi}^{\pi}$

$= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) + 0 = \frac{\pi}{2} + \frac{\pi}{2} = \pi$

所以得证

（4）证：$\int_{-\pi}^{\pi} sin^2 nxdx = \int_{-\pi}^{\pi} \frac{1-cos(2nx)}{2} dx$

$= \int_{-\pi}^{\pi} \frac{1}{2} dx - \int_{-\pi}^{\pi} \frac{1}{4n} cos(2nx)d(2nx)$

$= \frac{x}{2}\Big|_{-\pi}^{\pi} - \frac{1}{4n} sin(2nx)\Big|_{-\pi}^{\pi}$

$= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) - 0 = \frac{\pi}{2} + \frac{\pi}{2} = \pi$

所以得证

8. （1）证：$\int_{-\pi}^{\pi}cosmxcosnxdx = \int_{-\pi}^{\pi}\frac{1}{2}[cos(m+n)x + cos(m-n)x]dx$（积化和差公式）

$\qquad = \frac{1}{2}\int_{-\pi}^{\pi}cos(m+n)xdx + \frac{1}{2}\int_{-\pi}^{\pi}cos(m-n)xdx$

$\qquad = \frac{1}{2(m+n)}\int_{-\pi}^{\pi}cos(m+n)xd(m+n)x + \frac{1}{2(m-n)}\int_{-\pi}^{\pi}cos(m-n)xd(m-n)x$

$\qquad = \frac{1}{2(m+n)}sin(m+nx)|_{-\pi}^{\pi} + \frac{1}{2(m-n)}sin(m-n)x|_{-\pi}^{\pi}$

$\qquad = 0 + 0 = 0$

所以得证

（2）证：$\int_{-\pi}^{\pi}sinmxsinnxdx = \int_{-\pi}^{\pi}\left[-\frac{1}{2}cos(m+n)x + \frac{1}{2}cos(m-n)x\right]dx$

$\qquad = -\frac{1}{2}\int_{-\pi}^{\pi}cos(m+n)xdx + \frac{1}{2}\int_{-\pi}^{\pi}cos(m-n)xdx$

$\qquad = -\frac{1}{2(m+n)}\int_{-\pi}^{\pi}cos(m+n)xd(m+n)x + \frac{1}{2(m-n)}\int_{-\pi}^{\pi}cos(m-n)xd(m-n)x$

$\qquad = -\frac{1}{2(m+n)}sin(m+nx)|_{-\pi}^{\pi} + \frac{1}{2(m-n)}sin(m-n)x|_{-\pi}^{\pi}$

$\qquad = 0 + 0 = 0$

所以得证

（3）证：①$m \neq n$时

$$\int_{-\pi}^{\pi}sinmxcosnxdx = \int_{-\pi}^{\pi}\frac{1}{2}[sin(m+n)x + sin(m-n)x]dx$$
$$= \frac{1}{2}\int_{-\pi}^{\pi}sin(m+n)xdx + \frac{1}{2}\int_{-\pi}^{\pi}sin(m-n)xdx$$
$$= -\frac{1}{2(m+n)}cos(m+n)x\Big|_{-\pi}^{\pi} - \frac{1}{2(m-n)}cos(m-n)x|_{-\pi}^{\pi}$$
$$= -\frac{1}{2(m+n)}[cos(m+n)\pi - cos(-(m+n)\pi)]$$
$$\quad - \frac{1}{2(m-n)}[cos(m-n)\pi - cos(-(m-n)\pi)]$$
$$= 0 - 0 = 0$$

②$m = n$时

$\int_{-\pi}^{\pi}sinmxcosnx\,dx = \frac{1}{2}\int_{-\pi}^{\pi}sin2mxdx = 0$（第 7.（2）的结论）

习题 6.3

1. （1） $\int_0^1 (2x-3)^2 dx$

    令 $2x - 3 = t \Rightarrow x = \frac{3+t}{2}$

    （积分变量变化时，积分区间也要相应变化）

    $\Rightarrow \int_0^1 (2x-3)^2 dx = \int_{-3}^{-1} t^2 d\left(\frac{3+t}{2}\right) = \frac{1}{2} \cdot \frac{t^3}{3}\Big|_{-3}^{-1} = \frac{1}{2}\left(\frac{-1}{3} - \frac{-27}{3}\right) = \frac{13}{3}$

    （2） $f(x)$ 在 $\left[0, \frac{1}{2}\right]$ 上连续可导 $\Rightarrow f(x)$ 在 $\left[0, \frac{1}{2}\right]$ 上可积

    $\int_0^1 f'\left(\frac{1-x}{2}\right) dx$　令 $\frac{1-x}{2} = t \Rightarrow x = 1 - 2t,$　$\frac{1-x}{2}\Big|_0^1 \to t\Big|_{\frac{1}{2}}^0$

    $\Rightarrow \int_0^1 f'\left(\frac{1-x}{2}\right) dx = \int_{\frac{1}{2}}^0 f'(t) d(1-2t) = 2\int_0^{\frac{1}{2}} f'(t) dt = 2f(t)\big|_0^{\frac{1}{2}} = 2\left(f\left(\frac{1}{2}\right) - f(0)\right)$

2. （1） $\int_0^1 x\sqrt{1-x}\, dx$　（令 $\sqrt{1-x} = t$）

    $= \int_1^0 t \cdot (1 - t^2) d(1 - t^2)$

    $= 2\int_0^1 (t^2 - t^4) dt$

    $= 2 \cdot \frac{t^3}{3}\Big|_0^1 - 2 \cdot \frac{t^4}{5}\Big|_0^1$

    $= 2\left(\frac{1}{3} - 0\right) - 2\left(\frac{1}{5} - 0\right)$

    $= \frac{2}{3} - \frac{2}{5} = \frac{4}{15}$

    （2） $\int_0^1 x(2 - x^2)^5 dx$

    $= -\frac{1}{2}\int_0^1 (2 - x^2)^5 d(2 - x^2)$

    $= -\left(\frac{1}{12} - \frac{2^6}{12}\right) = \frac{21}{4}$

    （3） $\int_1^{\sqrt{3}} \frac{dx}{x^2\sqrt{1+x^2}}$ （令 $x = \tan t$，积分上下限改变为 $\frac{\pi}{3}$，$\frac{\pi}{4}$）

    $= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{\tan^2 t \cdot \sec t} \cdot \sec^2 t \cdot dt$

    $= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\cos t}{\sin^2 t} dt$

    $= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{\sin^2 t} d\sin t$

$$= -\frac{1}{\sin t}\Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$

$$= -\left(\frac{2\sqrt{3}}{3} - \sqrt{2}\right) = \sqrt{2} - \frac{2\sqrt{3}}{3}$$

(4) $\int_0^1 \frac{dx}{e^x + e^{-x}}$ （令 $e^x = t$，则积分上下限改变为e，1）

$$= \int_1^e \frac{1}{t + \frac{1}{t}} \cdot \frac{1}{t} dt$$

$$= \int_1^e \frac{1}{1 + t^2} dt$$

$$= \arctan t \big|_1^e$$

$$= \arctan e - \frac{\pi}{4}$$

(5) $\int_0^1 \frac{1}{e^x + 1} dx$ （令 $e^x = t$，则积分上下限改变为e，1）

$$= \int_1^e \frac{1}{1+t} \cdot \frac{1}{t} dt$$

$$= \int_1^e \left(\frac{1}{t} - \frac{1}{1+t}\right) dt$$

$$= \ln t \big|_1^e - \ln(1+t)_1^e$$
$$= 1 - 0 - \ln(1+e) + \ln 2$$
$$= 1 + \ln 2 - \ln(1+e)$$

$$= \ln \frac{2e}{1+e}$$

(6) $\int_{\frac{1}{2}}^1 \frac{\sqrt{1-x^2}}{x^2} dx$ （令x $= \sin t$，则积分上下限改变为 $\frac{\pi}{2}$，$\frac{2}{4}$）

$$= \int_{\frac{2}{4}}^{\frac{\pi}{2}} \frac{\cos t}{\sin^2 t} \cdot \cos t \, dt$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos^2 t}{\sin^2 t} dt$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1 - \sin^2 t}{\sin^2 t} dt$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\csc^2 t - 1) dt$$

$$= -\cot t \Big|_{\frac{\pi}{2}} - t \Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= -(0 - 1) - \left(\frac{\pi}{2} - \frac{\pi}{4}\right)$$

$$= 1 - \frac{\pi}{4}$$

(7) $\int_0^{\frac{\pi}{2}} \frac{\cos t}{\sin t + \cos t} dt$ （令x $= a \sin t$，则积分上下限改变为 $\frac{\pi}{2}$，0）

$$= \int_0^{\frac{\pi}{2}} \frac{\frac{1}{2}(\sin t + \cos t) + \frac{1}{2}(\cos t - \sin t)}{\sin t + \cos t} dt$$

$$= \int_0^{\frac{\pi}{2}} \left(\frac{1}{2} + \frac{1}{2} \cdot \frac{\cos t - \sin t}{\sin t + \cos t} \right) dt$$

$$= \int_0^{\frac{2}{2}} \left(\frac{1}{2} + \frac{1}{2} \cdot \frac{\cos t - \sin t}{\sin t + \cos t} \right) dt$$

$$= \frac{\pi}{4} + \frac{1}{2} \ln(\sin t + \cos t) \Big|_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{4} + (|n| - |n|)$$

$$= \frac{\pi}{4}$$

(8) $I = \int_0^{\frac{\pi}{2}} \frac{\sin \theta}{\sin \theta + \cos \theta} d\theta$, $\quad J = \int_0^{\frac{\pi}{2}} \frac{\cos \theta}{\sin \theta + \cos \theta} d$(半余法)

a.$(I + J = \int_0^{\frac{\pi}{2}} \frac{\sin \theta + \cos \theta}{\sin \theta + \cos \theta} d\theta = \int_0^{\frac{\pi}{2}} 1 d\theta = \frac{\pi}{2})$

b.$[\, I - J = \int_0^{\frac{\pi}{2}} \frac{\sin \theta - \cos \theta}{\sin \theta + \cos \theta} d\theta = - \int_0^{\frac{\pi}{2}} \frac{1}{\sin \theta + \cos \theta} d(\sin \theta + \cos \theta)$

$\Rightarrow I - T = -\ln(\sin \theta + \cos \theta)\Big|_0^{\frac{\pi}{2}} = -(\ln 1 - \ln 1) = 0\,]$

$\Rightarrow \frac{a+b}{2} = I = \frac{\pi}{4}\ \frac{a-b}{2} = J = \frac{\pi}{4}$

(9) $\int_0^{\frac{\pi}{2}} \frac{dx}{a^2 \sin^2 x + b^2 \cos^2 x}$ $(a, b > 0)$ (令u=tan$x$，则积分上下限改变为$+\infty$，0)

$$= \int_0^{+\infty} \frac{1}{a^2 \cdot \frac{u^2}{1+u^2} + b^2 \frac{1}{1+u^2}}\, d\arctan u$$

$$= \int_0^{+\infty} \frac{1+u^2}{a^2 u^2 + b^2} \cdot \frac{1}{1+u^2} \, du$$

$$= \int_0^{+\infty} \frac{1}{a u^2 + b^2} \, du$$

$$= \frac{1}{b^2} \int_0^{+\infty} \frac{1}{1 + \left(\frac{a}{b}u\right)^2} \left(d\left(\frac{a}{b}u\right)\right) \cdot \frac{b}{a}$$

$$= \frac{1}{ab} \arctan \frac{a}{b} u \Big|_0^{+\infty} \quad (\lim_{x \to +\infty} \arctan \frac{a}{b} x = \frac{\pi}{2}, \text{可认为} \arctan \frac{a}{b} u = \frac{\pi}{2} \ (u \to +\infty))$$

$$= \frac{1}{ab} \left(\frac{\pi}{2} - 0 \right)$$

(10) $f(x) = \begin{cases} 1 + x^2, & x \le 0 \\ e^{-x}, & x > 0 \end{cases}$

$\int_1^3 f(x-2) dx$ （令$x - 2 = t$，则积分上下限改变为1，-1）

$$= \int_{-1}^1 f(t) dt$$

$$= \int_{-1}^0 f(t) dt + \int_0^1 f(t) dt \quad \text{（分段函数将积分区间相应分段）}$$

$$= \int_{-1}^0 (1 + t^2) dt + \int_0^1 e^{-t} dt$$

$$= t1_{-1}^0 + \frac{t^3}{3}\Big|_{-1}^0 - e^{-t}\big|_0^1$$

$$= 1 + \frac{1}{3} - e^{-1} + 1$$

$$= \frac{7}{3} - \frac{1}{e}$$

3.证明：因为$f(x)$在$[-a, a]$上连续

$\Rightarrow f(x)$在$[-a, a]$上可积

$$\int_{-a}^a x\big(f(x) + f(-x)\big)d\,x = \int_{-a}^a x f(x)dx + \int_{-a}^a x f(-x)dx$$

$\Rightarrow \int_{-a}^a x f(-x)dx$（令$-x = t$，则积分上下限改变为$-a, a$）$\int_a^{-a}(-t)f(t)d(-t) =$

$$\int_a^{-a} t f(t)dt = \int_a^{-a} x f(x)dx$$

$$\Rightarrow \int_{-a}^a x\big(f(x) + f(-x)\big)d\,x = \int_{-a}^a x f(x)dx + \int_a^{-a} x f(x)dx = 0$$

4 证明：

$$\int_0^1 x^m (1-x)^n d\,x \xrightarrow[t|_1^0]{1-x=t} \int_1^0 (1-t)^m t^n d(1-t) = \int_0^1 t^n (1-t)^m d\,t = \int_0^1 x^n (1-x)^m d\,x(等于右式)$$

综上：$\int_0^1 x^m (1-x)^n d\,x = \int_0^1 x^n (1-x)^m d\,x$

5.证明：令$t = \frac{1}{u}$　$(x > 0)$　$(x > 0 \Rightarrow t > 0$即可$\Rightarrow t = \frac{1}{u}(u > 0))$

$$t|_x^1 \Rightarrow u|_{\frac{1}{x}}^1$$

$$\Rightarrow \int_x^1 \frac{1}{1+t^2}dt = \int_{\frac{1}{x}}^1 \frac{1}{1+\left(\frac{1}{u}\right)^2} \cdot \left(-\frac{1}{u^2}\right)du = \int_1^{\frac{1}{x}} \frac{1}{1+u^2}du = \int_1^{\frac{1}{x}} \frac{1}{1+t^2}dt$$

综上：$\int_x^1 \frac{1}{1+t^2}dt = \int_1^{\frac{1}{x}} \frac{1}{1+t^2}dt$　　$(x > 0)$

6.证明：$f(x)$为连续函数$\Rightarrow$在$x \in D$时可积

（1）因为$f(x)$为奇函数

所以$f(x) = -f(-x)$

令$F(x) = \int_0^x f(t)d\,t$，则　$F(-x) = \int_0^{-x} f(t)d\,t$

令$t = -u$，$t|_0^{-x} \to u|_0^x$

$\Rightarrow F(-x) = \int_0^x f(t)d\,t = \int_0^x f_0 f(-u)d(-u) = \int_0^x -f(-u)d\,u = \int_0^x f(u)d\,x =$

$$\int_0^x f(t)d\,t = F(x)$$

故$\int_0^{-x} f(t)\,dt$ 在$f(x)$为奇函数时，为偶函数。

（2）因为$f(x)$为偶函数

所以$f(x) = f(-x)$

令 $G(x) = \int_0^x f(t)dt$, $G(-x) = \int_0^{-x} f(t)dt$

令 $t = -k$, $\quad t|_0^{-x} \to k|_0^x$

$\Rightarrow G(-x) = \int_0^x f(t)dt = \int_0^x f(-k)d(-k) = -\int_0^x f(k)dk = -G(x)$

故当 $f(x)$ 为偶函数时，$\int_0^x f(t)\,dt$ 为奇函数。

习题 6.4

1. （1）解：

$\int_0^1 xe^x dx$

$= xe^x \big|_0^1 - \int_0^1 e^x dx$

$= xe^x \big|_0^1 - e^x \big|_0^1$

$= 1$

（2）解：

$\int_0^{\frac{1}{2}} \arcsin x dx$

$= x \cdot \arcsin x \big|_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} \frac{x}{\sqrt{1-x^2}} dx$

$= x \cdot \arcsin x \big|_0^{\frac{1}{2}} + \frac{1}{2} \int_0^{\frac{1}{2}} \frac{d(1-x^2)}{\sqrt{1-x^2}}$

$= x \cdot \arcsin x \big|_0^{\frac{1}{2}} + \sqrt{1-x^2} \big|_0^{\frac{1}{2}}$

$= \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1$

（3）解：

由推导结论：

$\int_0^{\frac{\pi}{2}} x dx \cos^2 x dx$

$= \frac{6}{7} \times \frac{4}{5} \times \frac{2}{3}$

$= \frac{16}{35}$

（4）解：

同理：

$\int_0^{\frac{\pi}{2}} \sin x^6 dx$

$= \frac{5}{6} \times \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2}$

$= \frac{5\pi}{32}$

结论推导：

$A_n = \int_0^{\frac{\pi}{2}} sin^n x \, dx$ （n≥2）

$= -\int_0^{\frac{\pi}{2}} (sinx)^{n-1} d(cosx)$

$= -(sinx)^{n-1} \cos x \Big|_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} cosx \, d(sinx)^{n-1}$

$= (n-1) \int_0^{\frac{\pi}{2}} cosx (sinx)^{n-2} cosx \, dx$

$= (n-1)(A_{n-2} - A_n)$

$\therefore nA_n = (n-1)A_{n-2}$

$\Rightarrow A_n = \frac{n-1}{n} A_{n-2}$

$\because A_0 = \int_0^{\frac{\pi}{2}} (sinx)^0 dx = \frac{\pi}{2}$

　　$A_1 = \int_0^{\frac{\pi}{2}} (sinx)^1 dx = 1$

$$\therefore A_n = \begin{cases} \dfrac{(n-1)!!}{n!!} \cdot \dfrac{\pi}{2} & (n = 2k) \\[2mm] \dfrac{(n-1)!!}{n!!} & (n = 2k+1) \end{cases}$$

易证：

$$\int_0^{\frac{\pi}{2}} sin^n x \, dx = \int_0^{\frac{\pi}{2}} cos^n x \, dx$$

2. （1）解：

$\displaystyle\int_0^{\pi} xsin\frac{x}{2} dx$

$= -2\int_0^{\pi} x \, dcos\frac{x}{2}$

$= -2\left(xcos\frac{x}{2} \Big|_0^{\pi} - \int_0^{\pi} x \, dcos\frac{x}{2} dx \right)$

$= 2\left(2sin\frac{x}{2} \Big|_0^{\pi} - xcos\frac{x}{2} \Big|_0^{\pi} \right)$

$= 4$

（2）解：

$$\int_0^e x\,ln^2\,x\,dx$$

$$=\frac{1}{2}\int_0^e ln^2\,x\,dx^2$$

$$=\frac{1}{2}(x^2\,ln^2\,x\,|_0^e - \int_0^e 2x\,ln\,x\,dx)$$

$$=\frac{1}{2}[x^2\,ln^2\,x\,|_0^e - (x^2\,ln\,x - \int_0^e x dx)]$$

$$=\frac{1}{2}[x^2\,ln^2\,x + \frac{1}{2}x^2 - x^2\,ln\,x)|_0^e$$

$$=\frac{e^2}{4}\text{（正确答案）}$$

$\int_1^e x\,ln^2\,x\,dx = \frac{1}{4}(e^2-1)$ （改正后的题目答案）

（3）

$$\int_0^1 x\,arctan\,x\,dx$$

$$=\frac{1}{2}\int_0^1 arctan\,x\,dx^2$$

$$=\frac{1}{2}(x^2\,arctan\,x\,|_0^1 - \int_0^1 \frac{x^2}{x^2+1}dx)$$

$$=\frac{1}{2}(x^2\,arctan\,x\,|_0^1 - \int_0^1 dx + \int_0^1 \frac{dx}{x^2+1})$$

$$=\frac{1}{2}(x^2\,arctan\,x\,|_0^1 - x|_0^1 + arctan\,x\,||_0^1$$

$$=\frac{\pi}{4} - \frac{1}{2}$$

（4）

$$\int_0^{\frac{\pi}{2}} e^{2x} \cos x \, dx$$

$$= \int_0^{\frac{\pi}{2}} e^{2x} d \sin x$$

$$= e^{2x} \sin x \,|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} 2e^{2x} \sin x dx$$

$$= e^{2x} \sin x \,|_0^{\frac{\pi}{2}} + 2\int_0^{\frac{\pi}{2}} e^{2x} d \cos x$$

$$= e^{2x} \sin x \,|_0^{\frac{\pi}{2}} + 2(e^{2x} \cos x \,|_0^{\frac{\pi}{2}} - 2\int_0^{\frac{\pi}{2}} e^{2x} \cos x \, dx)$$

移项得：

$$原式 = \frac{1}{5}(e^{\pi} - 2)$$

（5）

$$\int_0^1 e^{\sqrt{x}} \, dx$$

$$令 -\sqrt{x} = t$$

$$\Rightarrow x = t^2$$

$$dx = 2tdt$$

$$原式 = \int_0^{-1} 2te^t dt$$

$$= -2\int_{-1}^0 te^t dt$$

$$= -2e^t(t-1)|_{-1}^0$$

$$= 2 - 4e^{-1}$$

（6）

$$\int_{\frac{1}{e}}^{e} |\ln x| dx$$

$$= \int_{1}^{e} \ln x \, dx - \int_{\frac{1}{e}}^{e} \ln x \, dx$$

$$= (x \ln x - x)|_{1}^{e} - (x \ln x - x)|_{\frac{1}{e}}^{1}$$

$$= 2 - \frac{2}{e}$$

$$\boxed{\int \ln x \, dx = x \ln x - x + c}$$

（7）

$$\int_{1}^{e} \sin(\ln x) dx$$

$$= [x \sin(\ln x)]|_{1}^{e} - \int_{1}^{e} \cos(\ln x) dx$$

$$= [x \sin(\ln x)|_{1}^{e} - [x \cos(\ln x)]|_{1}^{e} - \int_{1}^{e} \sin(\ln x) dx$$

移项得：

$$原式 = \frac{1}{2}(e \sin 1 - e \cos 1 + 1)$$

（8）

$$\int_{0}^{\frac{1}{2}} \frac{x \arcsin x}{\sqrt{1-x^2}} dx$$

$$= \int_{0}^{\frac{1}{2}} x \arcsin x \, d \arcsin x$$

令 $\arcsin x = t \Rightarrow x = \sin t$

原式

$$= \int_{0}^{\frac{\pi}{6}} t \sin t \, dt$$

$$\boxed{\int x \sin ax \, dx = \frac{1}{a^2} \sin ax - \frac{x}{a} \cos ax + C}$$

$$= \frac{1}{2} - \frac{\sqrt{3}}{12} \pi$$

3. 解：

$$\int_0^2 x^2 f''(x)dx$$

$$= \int_0^2 x^2 d(f'(x))$$

$$= x^2 f'(x)|_0^2 - 2(\int_0^2 xf'(x)dx)$$

$$= x^2 f'(x)|_0^2 + 2\int_0^2 f(x) - 2xf(x)|_0^2$$

代入数据得原式 = 0

4. 证明：

（1）

$$\because (f^2(x))' = 2f(x)f'(x)$$

$$\therefore \int_a^b xf(x)f'(x)dx$$

$$= \frac{1}{2}\int_a^b xd(f^2(x))$$

$$= \frac{x}{2}f^2(x)|_a^b - \frac{1}{2}\int_a^b f^2(x)dx$$

$$= \frac{b}{2}f^2(b)|_a^b - \frac{a}{2}f^2(a) - \frac{1}{2}$$

$$= -\frac{1}{2}$$

（2）

由施瓦茨不等式得(P169)

$$(\int_a^b (f(x) \cdot xf'(x))dx)^2 \le \int_a^b f^2(x)dx \cdot \int_a^b (xf'(x))^2 dx$$

$$\Rightarrow \frac{1}{4} \le \int_a^b x^2 (f'(x))^2 \, dx \, \text{得证}$$

5. 证明:

（1）

$$\int_0^{\frac{\pi}{2}} [f(x) + f''(x)] \sin x \, dx$$

$$= \int_0^{\frac{\pi}{2}} f(x) \sin x \, dx + \int_0^{\frac{\pi}{2}} f''(x) \sin x \, dx$$

$$= \int_0^{\frac{\pi}{2}} f(x) \sin x \, dx + \sin x \cdot f'(x)|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} f'(x) \cos x \, dx$$

$$= f'(\frac{\pi}{2}) + \int_0^{\frac{\pi}{2}} f(x) \sin x \, dx - \int_0^{\frac{\pi}{2}} \cos x \, df(x)$$

$$= f'(\frac{\pi}{2}) + \int_0^{\frac{\pi}{2}} f(x) \sin x \, dx - (\cos x \cdot f(x)|_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} f(x) \sin x \, dx)$$

$$= f(0) + f'(\frac{\pi}{2})$$

（2）

$$\int_0^{\frac{\pi}{2}} [f(x) + f''(x)] \cos x \, dx$$

$$= \int_0^{\frac{\pi}{2}} f(x) \cos x \, dx + \int_0^{\frac{\pi}{2}} \cos x \, df'(x)$$

$$= \int_0^{\frac{\pi}{2}} f(x) \cos x \, dx + \cos x f'(x)|_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} f'(x) \sin x \, dx$$

$$= -f'(0) + \int_0^{\frac{\pi}{2}} f(x) \cos x \, dx + \int_0^{\frac{\pi}{2}} \sin x \, df(x)$$

$$= -f'(0) + \int_0^{\frac{\pi}{2}} f(x) \cos x \, dx + \sin x \, f(x) \Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} f(x) \cos x \, dx$$

$$= f\left(\frac{\pi}{2}\right) - f'(0)$$

6. 解（1）

$$f(x) = x^2, f'(x) = 2x, f(0) = 0, f'\left(\frac{\pi}{2}\right) = \pi$$

$$\int_0^{\frac{\pi}{2}} x^2 \sin x \, dx$$

$$= \int_0^{\frac{\pi}{2}} (x^2 + 2) \sin x \, dx - 2 \int_0^{\frac{\pi}{2}} \sin x \, dx$$

$$= \pi + 2 \cos x \Big|_0^{\frac{\pi}{2}}$$

$$= \pi - 2$$

（2）

$$f(x) = x^4, f'(x) = 4x^3, f\left(\frac{\pi}{2}\right) = \frac{\pi^4}{16}, f'(0) = 0$$

$$g(x) = x^2, g'(x) = 2x, g\left(\frac{\pi}{2}\right) = \frac{\pi^2}{4}, g'(0) = 0$$

$$\int_0^{\frac{\pi}{2}} x^4 \cos x \, dx$$

$$= \int_0^{\frac{\pi}{2}} (x^4 + 12x^2) \cos x \, dx - 12 \int_0^{\frac{\pi}{2}} x^2 \cos x \, dx$$

$$= \int_0^{\frac{\pi}{2}} (x^4 + 12x^2) \cos x \, dx - 12 \left(\int_0^{\frac{\pi}{2}} (x^2 + 2) \cos x \, dx - \int_0^{\frac{\pi}{2}} 2 \cos x \, dx \right)$$

$$= \frac{\pi^4}{16} - 3\pi^2 + 24 \quad （课本后答案错误）$$

习题 6.5

1. （1）×　$x = 0$为奇点

$$\int_{-1}^{1} \frac{1}{x} dx = \int_{-1}^{0} \frac{1}{x} dx + \int_{0}^{1} \frac{1}{x} dx = ln|x|\Big|_{-1}^{0+\varepsilon} ① + ln|x|\Big|_{0+\varepsilon}^{1} ② \quad (\varepsilon \to 0)$$

①$\to +\infty$　②$\to -\infty$

①②左右两边极限均不存在，所以原式极限不存在（P189）

（2）×　$x = 0$为$\int_{-\infty}^{+\infty} \frac{1}{x^2} dx$奇点

$$\int_{-\infty}^{+\infty} \frac{1}{x^2} dx = \int_{-\infty}^{0} \frac{1}{x^2} dx + \int_{0}^{+\infty} \frac{1}{x^2} dx = -\left(\frac{1}{x}\Big|_{-\infty}^{0} + \frac{1}{x}\Big|_{0}^{+\infty} \right)$$

$$① \qquad ②$$

①$\to +\infty$　②$\to -\infty$

左右两边极限不存在，原式极限不存在（P189 课本）

（3）×　$\int sindx = -cosx$ 当$x \to \infty$时，$-cosx$无极限，发散

（4）×　反例：令$p = 0$ $\int_{0}^{2} \frac{1}{1-x} dx = \int_{0}^{1} \frac{1}{1-x} dx + \int_{1}^{2} \frac{1}{1-x} dx$

等式右侧两极限均不存在

（5）√　$\int \frac{1}{x^p} dx = \frac{1}{1-p} x^{1-p}$

$$\int_{0}^{+\infty} \frac{1}{x^p} dx = \frac{1}{1-p} x^{1-p}\Big|_{0+\varepsilon}^{+\infty} \quad (\varepsilon \to 0)$$

①$1 - p < 0$时，$x \to +\infty$时x^{1-p}为0，

$x \to 0$，$x^{1-p} \to +\infty$，发散

②$1 - p > 0$且$1 - p < 1$时 $x \to +\infty$，$x^{1-p} \to +\infty$，

$x \to 0$，$x^{1-p} \to 0$，发散

2. （1）$\int_{1}^{+\infty} \frac{lnx}{x^2} dx = \int_{1}^{+\infty} \frac{lnx-1}{x^2} dx + \int_{1}^{+\infty} \frac{1}{x^2} dx$

$$= -\frac{lnx}{x}\Big|_{1}^{+\infty} - \frac{1}{x}\Big|_{1}^{+\infty}$$

$$= -(0 - 0) - (0 - 1) = 1$$

（2）$\int_{0}^{1} \frac{1}{\sqrt{1-x^2}} dx = arcsinx\Big|_{0}^{1} = \frac{\pi}{2}$

（3）$\int_{0}^{1} \frac{e^{\sqrt{x}}}{\sqrt{x}} dx = 2 \int_{0}^{1} e^{\sqrt{x}} d\sqrt{x} = 2e^{\sqrt{x}}\Big|_{0}^{1} = 2(e - 1)$

（4）$\int_{-\infty}^{+\infty} e^{-a|x|} dx = \int_{-\infty}^{0} e^{ax} dx + \int_{0}^{+\infty} e^{-ax} dx$

$$= \frac{1}{a} e^{ax}\Big|_{-\infty}^{0} - \frac{1}{a} e^{-ax}\Big|_{0}^{+\infty}$$

$$= \frac{1}{a} - \lim_{x \to -\infty} \frac{1}{a} e^{ax} - \lim_{x \to +\infty} \frac{1}{a} e^{-ax} + \frac{1}{a}$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$= 0 \qquad = 0$$

$$= \frac{2}{a}$$

3. （1）$\int_{-\infty}^{+\infty} \frac{dx}{x^2+x+1} = \int_{-\infty}^{+\infty} \frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}} dx$ （配方）

$$= \frac{4}{3}\int_{-\infty}^{+\infty} \frac{1}{\left[\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right]^2+1} dx$$

$$= \frac{2}{\sqrt{3}}\int_{-\infty}^{+\infty} \frac{1}{\left[\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right]^2+1} d\left[\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right]$$

$$= \frac{2}{\sqrt{3}} arctan\left[\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right]\Big|_{-\infty}^{+\infty}$$

$$= \frac{2}{\sqrt{3}}\left[\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right] = \frac{2\sqrt{3}}{3}\pi$$

（2）$\int_{2}^{+\infty} \frac{dx}{x^2+x-2} = \int_{2}^{+\infty} \frac{dx}{(x-1)(x+2)}$

$$= \frac{1}{3}\int_{2}^{+\infty}\left(\frac{1}{x-1}-\frac{1}{x+2}\right)dx \quad （裂项）$$

$$= \frac{1}{3}\left(ln|x-1|\Big|_{2}^{+\infty}-ln|x+2|\Big|_{2}^{+\infty}\right)$$

$$= \frac{2}{3}ln2 \quad （注：\lim_{x\to+\infty}ln|x-1|=\lim_{x\to+\infty}ln|x+2|）$$

（3）$\int_{0}^{1}x^2lnxdx = \frac{1}{3}\int_{0}^{1}lnxdx^3$

$$= \frac{1}{3}\left(x^2lnx\Big|_{0}^{1}-\int_{0}^{1}x^3dlnx\right)$$

$$= \frac{1}{3}x^3\left(lnx-\frac{1}{3}\right)\Big|_{0}^{1} = -\frac{1}{9}$$

（4）令$\sqrt{x}=t$

$$\int_{0}^{+\infty}e^{-t}dt^2 = \int_{0}^{+\infty}2t\cdot e^{-t}dt$$

$$= -2(1+t)e^{-t}\Big|_{0}^{+\infty}$$
$$= -2\cdot(-1) = 2$$

（5）题目错误，原题应为：$\int_{1}^{2}\frac{x}{\sqrt{x-1}}dx$

令$\sqrt{x-1}=t, x=t^2+1$

原式$= \int_{0}^{1}(1+t^2)dt = \left(t+\frac{t^3}{3}\right)\Big|_{0}^{1} = \frac{8}{3}$

（6）$\int_{0}^{1}\sqrt{\frac{x}{1-x}}dx$

令$\sqrt{1-x}=t, \sqrt{x}=\sqrt{1-t^2}$

原式$= \int_{0}^{1}\frac{\sqrt{1-t^2}}{t}d(1-t^2)$

$$= 2\int_{0}^{1}\sqrt{1-t^2}dt$$

$$= 2 \times \frac{\pi}{4} = \frac{\pi}{2}$$

（令 $f(t) = \sqrt{1-t^2} \Rightarrow f^2(t) + t^2 = 1$ ， $(x^2 + y^2) = 1$ 为圆方程而 $\sqrt{1-t^2} \geq 0$，所以

为半圆，那么从-1积到1的面积为$\frac{1}{2} \times \pi \times 1^2 = \frac{\pi}{2}$）

（7） $\int_a^b \frac{1}{\sqrt{(x-a)(b-x)}} dx$

$(x-a)(b-x) = bx - x^2 - ab + ax = -x^2 + (a+b)x - ab$

$= -\left(x - \frac{a+b}{2}\right)^2 + \left(\frac{a-b}{2}\right)^2$ （这一步就是配方）

$= \frac{|b-a|}{2}\left[1 - \left(\frac{x - \frac{a+b}{2}}{\frac{|a-b|}{2}}\right)^2\right]$

所以原式$= \int_a^b \frac{1}{\sqrt{1 - \left(\frac{x - \frac{a+b}{2}}{\frac{|a-b|}{2}}\right)^2}} dx \cdot \frac{2}{|b-a|}$

令$\left(\frac{x - \frac{a+b}{2}}{\frac{|a-b|}{2}}\right)^2$为①，把①看成整体

发现$\frac{2}{|a-b|} dx = d\left(\frac{2}{b-a}x - \frac{a+b}{b-a}\right)$ （任意常数，根据整体配）

所以原式$\int_a^b \frac{1}{\sqrt{1 - ①^2}} d①$

$= arcsin① \Big|_a^b$ $x = b \to ① = +1$

$= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi$ $x = a \to ① = -1$

注：3.（7）看似复杂，实则为对关于x的一元二次多项式配方，化成$\frac{1}{\sqrt{1-a^2}}$或$\frac{1}{\sqrt{1+a^2}}$，

达到求积分的目的

（8） $\int_0^1 (lnx)^2 dx = x(lnx)^2 \Big|_0^1 - \int_0^1 x \, d[(lnx)^2]$

$$= x(lnx)^2 \Big|_0^1 - 2\int_0^1 lnx \, dx$$

$$= [(lnx)^2 - 2lnx + 2] \Big|_0^1 = 2$$

4. $\int_2^{+\infty} \frac{1}{x(lnx)^k} dx = \int_2^{+\infty} \frac{1}{(lnx)^k} dlnx$

令$lnx = m$

$\int_{ln2}^{+\infty} \frac{1}{m^k} dm = \frac{1}{1-k} \cdot m^{1-k} \Big|_{ln2}^{+\infty}$

$= \lim_{m \to +\infty} \frac{m^{1-k}}{1-k} \cdot \frac{(ln2)^{1-k}}{1-k}$ （$k \neq 1$时，该项为常值）

若收敛，则$1 - k < 0$，才能使$m^{1-k} \to 0, k > 1$

5. （1）因为 $\lim\limits_{x \to +\infty} x^2 \cdot \frac{x^2}{x^4-x^2+1} = \lim\limits_{x \to +\infty} \frac{1}{1-\frac{1}{x^2}+\frac{1}{x^4}} = 1$

所以 $\int_0^{+\infty} \frac{x^2}{x^4-x^2+1}$ 收敛

注：用来判断敛散性的一种方法：极限审敛法（与书上不太相同）

1. 对于无穷限广义积分 $\int_a^{+\infty} f(x)\,dx$ ： $f(x)$ 在 $[a,+\infty)$ 连续且非负

①若 $\exists P > 1, \lim\limits_{x \to +\infty} x^P f(x) = C < +\infty$,则 $\int_a^{+\infty} f(x)\,dx$ 收敛

②若 $\lim\limits_{x \to +\infty} xf(x) = d > 0$（或 $= +\infty$）则 $\int_a^{+\infty} f(x)dx$ 发散

2. 对于非负积分 $\int_a^b f(x)dx, a$ 为瑕点，$f(x)$ 在 $[a,b]$ 连续且非负

①若 $\exists 0 < q < 1, \lim\limits_{x \to a^+} (x-a)^q f(x)$ 存在，则 $\int_a^b f(x)\,dx$ 收敛

②若 $\lim\limits_{x \to a^+} (x-a)f(x) = d > 0$（或 $= +\infty$），则 $\int_a^b f(x)dx$ 发散

（2）因为 $x \cdot \sqrt[3]{x^2+1} > x \cdot x^{\frac{2}{3}} = x^{\frac{5}{3}}$

所以 $\int_1^{+\infty} \frac{1}{x\sqrt[3]{x^2+1}}dx < \int_1^{+\infty} \frac{1}{x^{\frac{5}{3}}}dx$

因为 $\int_1^{+\infty} \frac{1}{x^{\frac{5}{3}}}dx$ 收敛（比较判别法）

所以 $\int_1^{+\infty} \frac{1}{x\sqrt[3]{x^2+1}}dx$ 收敛

（3）$\int_0^2 \frac{dx}{lnx}$

$x = 0, x = 1$ 为可能瑕点

原式 $= \int_0^1 \frac{dx}{lnx} + \int_1^2 \frac{dx}{lnx}$

主要上述两式任意一式不收敛，原式不收敛，否则收敛

$\int_1^2 \frac{dx}{lnx}$ 利用极限审敛法

$\lim\limits_{x \to 1^+} \frac{(x-1)}{lnx} = \lim\limits_{x \to 1^+} \frac{x-1}{\underset{\downarrow}{x-1}} = 1$ 存在极限

等价无穷小

$\int_1^2 \frac{dx}{lnx}$ 发散，所以原式发散

（4）$\int_0^{+\infty} \frac{x^m}{x^{n+1}}dx \le \int_0^{+\infty} \frac{x^m}{x^n}dx$

$\int_0^{+\infty} x^{m+n}dx = \frac{1}{m-n+1}x^{m-n+1}\Big|_0^{+\infty}$

所以当 $m - n + 1 < 0$ 时，原式收敛，否则不收敛

（5）利用极限审敛法：

若 $n = p > 1, x^p \cdot \dfrac{arctanx}{x^p} = arctanx$

$x \to +\infty, arctanx \to \dfrac{\pi}{2}$ 存在极限为满足原式收敛的一个必要条件

另：$\lim\limits_{x \to 0}(x - 0)^q \cdot \dfrac{arctanx}{x^n} = \dfrac{arctanx}{x^{n-q}}$（0为可能奇点）

因为 $n > 1, n > q\left(q \in (0,1)\right)$

所以该极限为 $\dfrac{0}{0}$ 型，用洛必达：

$\lim\limits_{x \to 0} \dfrac{\frac{1}{1+x^2}}{\frac{x^{n-q-1}}{n-q}} = \lim\limits_{x \to 0} \dfrac{n-q}{x^{n-q-1}}$

要使该极限存在，则 $n - q - 1 \Rightarrow n < q + 1$

所以 $n < 2$

综上，$1 < n < 2$ 时，原式收敛，否则不收敛

（6）用比较判别法的极限形式：

$\int_0^{+\infty} \dfrac{sin^2x}{x} dx \le \int_0^{+\infty} \dfrac{1}{x} dx = lnx \Big|_0^{+\infty}$

　　　　　① 　　　　　　②

因为 $\tau = 1$ 所以①与②有相同的敛散性（P189~P191）

因为 $lnx \Big|_0^{+\infty}$ 极限不存在

所以原式发散

（8）因为 $\lim\limits_{x \to 1^-} \dfrac{lnx}{1-x^2} = \lim\limits_{x \to 1^-} \dfrac{x-1}{1-x^2} = \lim\limits_{x \to 1^-} \dfrac{-1}{1+x} = -\dfrac{1}{2}$

所以只有 $x = 0$ 为 $\int_0^1 \dfrac{lnx}{1-x^2} dx$ 的可能奇点、瑕点

利用极限审敛法：

$\lim\limits_{x \to 0} \dfrac{(x-0)^q \cdot lnx}{1-x^2} = \lim\limits_{x \to 0} \dfrac{lnx}{x^{-q}-x^{2-q}}$ 属于 $\dfrac{\infty}{\infty}$ 型

洛必达法则：$\exists q \in (0,1)$ 则 $\lim\limits_{x \to 0} \dfrac{\frac{1}{x}}{-q \cdot x^{-q-1}-(2-q)x^{1-q}} = \lim\limits_{x \to 0} \dfrac{\frac{1}{x}}{-qx^{-q-1}} = -\dfrac{1}{q}x^q = 0$

$\underset{q\in(0,1)\,则该项为\,0}{\downarrow}$

所以原式收敛

注：对于本题判断敛散性，用比较判别法和极限审敛法，若用极限审敛法，则找到所用瑕点，再去判断

6. （1）$\int_x^1 \dfrac{cost}{t^2} dt \le \int_x^1 \dfrac{1}{t^2} dt$

由等价无穷小 $x \to 0$ 时 $1 - cosx \sim \dfrac{1}{2}x^2$

所以 $cost \sim 1 - \dfrac{1}{2}t^2$

所以 $\int_x^1 \dfrac{1-\frac{t^2}{2}}{t^2} dt = \int_x^1 \dfrac{cost}{t^2} dt \le \int_x^1 \dfrac{1}{t^2} dt$

由夹逼定理：$-\dfrac{3}{2} + \dfrac{x}{2} + \dfrac{1}{x} \le \int_x^1 \dfrac{cost}{t^2} dt \le -1 + \dfrac{1}{x}$（$x \to 0$）

所以 $\lim\limits_{x \to 0} \int_x^1 \frac{\cos t}{t^2} dt = 1$

（2）$\int_0^x \sqrt{1+t^4} dt > \int_0^x \sqrt{t^4} dt = \int_0^x t^2 dt = \frac{1}{3} t^3 \Big|_0^x$

因为 $x \to \infty$

所以 $\lim\limits_{x \to +\infty} \frac{1}{3} t^3 \Big|_0^x \to +\infty$

所以原式为 $\frac{\infty}{\infty}$ 型，用洛必达法则

$\lim\limits_{x \to +\infty} \frac{\sqrt{1+t^4}}{3x^2} = \frac{1}{3} \lim\limits_{x \to +\infty} \frac{\sqrt{1+x^4}}{\sqrt{x^4}} = \frac{1}{3}$

7. $\int_1^{+\infty} \frac{1}{x^2} dx = -\frac{1}{x} \Big|_1^{+\infty} = -0 - (-1)] = 1$ 收敛

所以 $\int_1^{+\infty} \left[f^2(x) + \frac{1}{x^2} \right] dx$ 收敛

$f^2(x) \to a^2, \frac{1}{x^2} \to b^2$

因为 $ab < \frac{1}{2}(a^2 + b^2)$ 基本不等式

所以 $\left| f(x) \cdot \frac{1}{x} \right| < \frac{1}{2} \left[f^2(x) + \frac{1}{x^2} \right]$

所以 $\dfrac{\left| \frac{f(x)}{x} \right|}{f^2(x) + \frac{1}{x^2}} < \frac{1}{2} = \tau$

由比较判别法 $\tau \in \left(0, \frac{1}{2} \right)$

又因为 $\int_1^{+\infty} \left[f^2(x) + \frac{1}{x^2} \right] dx$ 收敛

所以 $\int_1^{+\infty} \left| \frac{f(x)}{x} \right| dx$ 收敛

即 $\int_1^{+\infty} \frac{f(x)}{x} dx$ 绝对收敛

习题 6.6

1.

$(1) S = \int_0^4 \sqrt{1 + (y')^2} \, dx = \int_0^4 \sqrt{1 + \frac{9}{4}x} \, dx = \left[\frac{4}{9} \times \frac{2}{3} \times \left(1 + \frac{9}{4}x\right)^{\frac{3}{2}}\right]_0^4$

$= \frac{4}{9} \times \frac{2}{3} \times \left[\left(1 + \frac{9}{4} \times 4\right)^{\frac{3}{2}} - 1\right] = \frac{8}{27}\left(10^{\frac{3}{2}} - 1\right)$

$(2) x' = \frac{y}{2} - \frac{1}{2y}$

$S = \int_1^e \sqrt{1 + (x')^2} \, dy = \int_1^e \sqrt{1 + \left(\frac{y}{2} - \frac{1}{2y}\right)^2} \, dy$

$= \int_1^e \frac{1}{2}\left(y + \frac{1}{y}\right) dy = \frac{1}{2}\left[\frac{1}{2}y^2 + \ln y\right]_1^e$

$= \frac{1}{2} \times \left(\frac{1}{2}e^2 + 1 - \frac{1}{2}\right) = \frac{1}{4}(e^2 + 1)$

(3)由题目可知，$x \geq 0$，$y \geq 0$

$y = \left(1 - \sqrt{x}\right)^2 \quad y' = \frac{dy}{dx} = 2(1 - \sqrt{x})\left(-\frac{1}{2\sqrt{x}}\right) = 1 - \frac{1}{\sqrt{x}}.$

$S = \int_0^1 \sqrt{1 + (y')^2} \, dx = \int_0^1 \sqrt{1 + \left(1 - \frac{1}{\sqrt{x}}\right)^2} \, dx$

$= 2\int_0^1 \sqrt{2x - 2\sqrt{x} + 1} \, d\sqrt{x} = 1 + \frac{\sqrt{2}}{2}\ln(1 + \sqrt{2}).$

(4)设$x = a\cos^3 t \quad y = a\sin^3 t \, (0 \leq t \leq 2\pi)$

$S = 4\int_0^{\frac{\pi}{2}} \sqrt{[(a\cos^3 t)']^2 + [(a\sin^3 t)']^2} \, dt$

$$= 4a \int_0^{\frac{\pi}{2}} \sqrt{(-3\cos^2 t \sin t)^2 + (3\sin^2 t \cos t)^2} \, dt$$

$$= 12a \int_0^{\frac{\pi}{2}} \sin t \cos t \, dt = 6a.$$

$$(5) S = \int_0^{2\pi} \sqrt{\left[\left(a(\cos t + t \sin t) \right)' \right]^2 + \left[\left(a(\sin t - t \cos t) \right)' \right]^2} \, dt$$

$$= |a| \int_0^{2\pi} \sqrt{(t \cos t)^2 + (t \sin t)^2} \, dt$$

$$= |a| \int_0^{2\pi} t \, dt = 2\pi^2 |a|$$

$$(6) S = \int_0^{2\pi} \sqrt{r^2 + (r')^2} \, d\theta = \int_0^{2\pi} \sqrt{a^2(1 + \cos\theta)^2 + a^2 \sin^2\theta} \, d\theta$$

$$= 4a \int_0^{\pi} \sqrt{\frac{1 + \cos\theta}{2}} \, d\theta = 4a \int_0^{\pi} \cos\frac{\theta}{2} \, d\theta = 8a$$

2.

$(1) \int_a^b |f(x) - g(x)| dx.$

面积$A \geq 0$，且$f(x)$、$g(x)$的大小无法确定

故面积为$\int_a^b |f(x) - g(x)| dx$

$(2) \pi \int_a^b |f^2(x) - g^2(x)| dx$

在区间$[a, b]$上，由曲线$y = f(x)$，y

$\qquad = g(x)$所围成的平面绕 x 轴旋转

一周所成的旋转体的体积微元为

$dV = \pi |f^2(x) - g^2(x)| dx$

$$\therefore V = \pi \int_a^b |f^2(x) - g^2(x)|dx$$

3.

(1)由 $\begin{cases} y = x^2 \\ x + y = 2 \end{cases}$ 得$x = 1$ 或$x = -2$

$$S = \int_{-2}^1 (2 - x - x^2)dx = \left(2x - \frac{1}{2}x^2 - \frac{1}{3}x^3\right)\Big|_{-2}^1 = \frac{9}{2}$$

$(2)S = \int_{\frac{1}{e}}^1 |ln\, x|dx + \int_1^e ln\, x \cdot dx$

$$= \int_1^{\frac{1}{e}} ln\, x\, dx + \int_1^e ln\, x\, dx$$

$$= (x\, ln\, x - x)|_1^{\frac{1}{e}} + (x\, ln\, x - x)|_1^e$$

$$= 2 - \frac{2}{e}$$

(3)令$x = a\, sin\, t$，$y = b\, cos\, t$

$$则S = 4\int_0^a y\, dx = \frac{4b}{a}\int_0^a \sqrt{a^2 - x^2}\, dx = \int_a^{4b} \int_0^{\frac{\pi}{2}} a^2\, cos^2\, t\, dt$$

$$= 4ab\int_0^{\frac{\pi}{2}} \frac{1 + cos\, 2t}{2}\, dt = 2ab\left[t + \frac{1}{2}sin\, 2t\right]_0^{\frac{\pi}{2}} = ab\pi$$

$(4)S = \int_0^1 (e^x - e^{-x})dx = [e^x - (-e^{-x})]_0^1$

$$= e + e^{-1} - 2$$

$(5)S = \int_0^{\frac{\pi}{4}} (cos\, x - sin\, x)dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (cos\, x - sin\, x)dx$

$$= (sin\, x + cos\, x)|_0^{\frac{\pi}{4}} + (-cos\, x - sin\, x)|_{\frac{\pi}{4}}^{\frac{\pi}{2}} = 2\sqrt{2} - 2$$

(6) 由 $\begin{cases} y = \dfrac{1}{2}x^2 \\ x^2 + y^2 = 8 \end{cases}$ 　得两曲线的交点为 $(-2,2),(2,2)$

则 $S_1 = \displaystyle\int_{-2}^{2} \left(\sqrt{8-x^2} - \dfrac{1}{2}x^2 \right) dx = 2\int_{0}^{2} \left(\sqrt{8-x^2} - \dfrac{1}{2}x^2 \right) dx$

$= 2 \left[4\arcsin\dfrac{x}{\sqrt{8}} + \dfrac{1}{2}x\sqrt{8-x^2} - \dfrac{1}{6}x^3 \right] \Bigg|_{0}^{2}$

$= 2\pi + \dfrac{4}{3}$

$S_2 = S - S_1 = \pi \left(2\sqrt{2} \right)^2 - 2\pi - \dfrac{4}{3} = 6\pi - \dfrac{4}{3}$

4.

(1) $S = \dfrac{1}{2} \displaystyle\int_{0}^{\frac{\pi}{2}} r^2 \, d\theta$

$x_2 = 4 \displaystyle\int_{0}^{\frac{\pi}{2}} \dfrac{\cos 2\theta + 1}{2} \, d\theta$

$= (\sin 2\theta + 2x) \big|_{0}^{\frac{\pi}{2}} = \pi$

(2) $S = \dfrac{1}{2} \displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} r^2 \, d\theta = \dfrac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2a^2 \cos 2\theta \, d\theta$

$= \left(\dfrac{1}{2} a^2 \sin 2\theta \right) \Bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 4 \times \dfrac{a^2}{2} \sin 2\theta \Bigg|_{0}^{\frac{\pi}{4}} = 2a^2$

(3) $0 \le t \le 2\pi$，该图形关于 x 轴与 y 轴都对称

$x' = -3a \cos^2 t \sin t$

$S = 4 \displaystyle\int_{0}^{\frac{\pi}{2}} |a \sin^3 t \, (-3a \cos^2 t \sin t)| \, dt$

$= 12a^2 \displaystyle\int_{0}^{\frac{\pi}{2}} \sin^4 t \cos^2 t \, dt = \dfrac{3}{8}\pi a^2$

5.

(1)两曲线交点为(0,0)与(1,1)的旋转体体积

$$V = \pi \int_0^1 \left[\left(\sqrt{x} \right)^2 - (x^2)^2 \right] d\,x = \pi \int_0^1 (x - x^4) d\,x = \frac{3}{10}$$

$$(2) V = \pi \int_{-a}^a \left[\left(b + \sqrt{a^2 - x^2} \right)^2 - \left(b - \sqrt{a^2 - x^2} \right)^2 \right] dx$$

$$= 4\pi b \int_{-a}^a \sqrt{a^2 - x^2}\, dx = 2\pi^2 a^2 b$$

$$(3) V = \pi \int_{-a}^a y^2 dx = 3\pi a^3 \int_0^\pi \sin^7 t \cos^2 t\, dt$$

$$= 6\pi a^3 \int_0^{\frac{\pi}{2}} (\sin^7 t - \sin^9 t) dt = \frac{32}{105} \pi a^3$$

6. 证明：旋转曲面的方程为$\pm\sqrt{y^2 + z^2} = f(x)$，由旋转曲面的对称性，取次曲面的上半部分$\Sigma: z = \sqrt{f^2(x) - y^2}$

Σ在 $x\text{-}O\text{-}y$ 面上的投影区域为

$$D = \{(x,y) \mid -f(x) \le y \le f(x) \quad a \le x \le b\}$$

$$S = 2 \iint_D \sqrt{1 + z_x^2 + z_y^2}\, dxdy$$

$$= 2 \iint_D \sqrt{1 + \left[\frac{f(x)f'(x)}{\sqrt{f^2(x) - y^2}} \right]^2 + \left[\frac{-y}{\sqrt{f^2(x) - y^2}} \right]^2}\, dxdy$$

$$= 2 \int_a^b f(x)\sqrt{1 + [f'(x)]^2}\, dx \int_{-f(x)}^{f(x)} \frac{1}{\sqrt{f^2(x) - y^2}}\, dy$$

$$= 2 \int_a^b f(x)\sqrt{1 + [f'(x)]^2} \cdot \left[arcsin \frac{y}{f(x)} \right]_{-f(x)}^{f(x)} dx$$

$$= 2\pi \int_a^b f(x)\sqrt{1 + [f'(x)]^2}\, dx$$

$(1) y' = 2x^{-\frac{1}{2}}$

$$S = 2\pi \int_0^1 2\, x^{\frac{1}{2}} \sqrt{1 + \left(2x^{-\frac{1}{2}}\right)^2}\, dx$$

$$= 4\pi \int_0^1 \sqrt{x+4}\, dx = 4\pi \times \frac{2}{3}(x+4)^{\frac{3}{2}}\Big|_0^1$$

$$= \frac{8\pi}{3}\left(5\sqrt{5} - 8\right)$$

$(2) y' = \dfrac{dy}{dx} = \dfrac{3a\sin^2 t \cos t}{-3a\cos^2 t \sin t} = -\tan t$

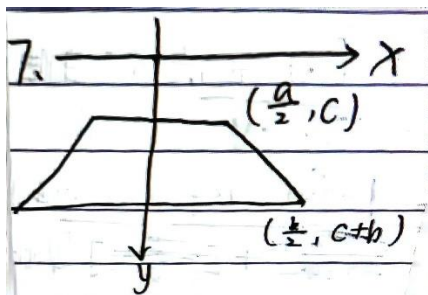由对称性知

$$S = 2\int_0^a 2\pi y \cdot \sqrt{1+(y')^2}\, dx$$

$$= -4\pi \int_0^{\frac{\pi}{2}} a\sin^3 t \sqrt{1+\tan^2 t}\,(-3a\cos^2 t \sin t)\, dt$$

$$= 12a^2\pi \int_0^{\frac{\pi}{2}} \sin^4 t \sec t \cos^2 t\, dt$$

$$= 12a^2\pi \int_0^{\frac{\pi}{2}} \sin^4 t\, d\sin t$$

$$= \frac{12}{5}a^2\pi$$

7.

如图，AB 的方程为 $y = \dfrac{2h}{b-a}\left(x - \dfrac{a}{2}\right) + c$ 对于薄板上每一点 (x, y) 的

压力 $dF = \rho g y \cdot x\, dy$

由对称性可知

$$P = 2 \int_c^{c+h} dF = \int_c^{c+h} \left[a + \frac{b-a}{h}(y-c) \right] \rho g y \, dy$$

$$= \frac{1}{6} \rho g h (3ac + 3bc + ab + 2bh)$$

8. 球的密度与水相同$\Longrightarrow$球在水中移动时不做功，x 为积分变量，

x$\in$[0,2r]。把球体分为很多薄层，将相应于$[x, x+dx]$的那一层球体

抬到水面时不做功，从离开水面时开始做功且 x-0-y 面上方圆的方

程为$(x-r)^2 + y^2 = r^2$，可知，将相应于$[x, x+dx]$的那一薄层球体

提升到$[x-2r, x+dx-2r]$位置时所做的功微元为(ρ为水密度)

$$dW = \rho g (2r-x) \pi y^2 dx = \rho g \pi (2r-x)[r^2 - (x-r)^2] dx$$

$$= \rho g \pi (2r-x)(2rx - x^2) dx = \rho g \pi (x^3 - 4rx^2 + 4r^2 x) dx$$

$$\text{故} W = \int_0^{2r} dw = \rho g \pi \int_0^{2r} (x^3 - 4rx^2 + 4r^2 x) dx$$

$$= \rho g \pi \left(\frac{x^4}{4} - \frac{4rx^3}{3} + 2r^2 x^2 \right) \Big|_0^{2r} = \frac{4}{3} \rho g \pi r^4$$

第 6 章复习题

1、

(1)

$$\lim_{x\to0}\frac{\int_0^x (e^{t方}-1) \, dt}{x^2}=\lim_{x\to0}\frac{e^{x^2}-1}{2x}=\lim_{x\to0}\frac{e^{x^2}*2x}{2}=0$$

故 a=0

(2)

$$f'(0)= \lim_{x\to0}\frac{f'(x)-0}{x-0}= \lim_{x\to0}\frac{\int_0^x (e^{x^2}-1) \, dt}{x^3}=\lim_{x\to0}\frac{e^{x^2}-1}{3x^2}=\frac{1}{3}$$

3、

(1)

$$\int_1^2\frac{1+x^2}{1+x^4}dx=\int_1^2\frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}dx=\int_1^2\frac{d(x-\frac{1}{x})}{2+(x-\frac{1}{x})^2}dx=\frac{1}{\sqrt{2}}arctan\frac{x^2-1}{\sqrt{2}x}\Big|_1^2=\frac{1}{\sqrt{2}}arctan\frac{3\sqrt{2}}{4}$$

(2)

$$\int_0^\Pi\frac{sin\theta d\theta}{\sqrt{1-2acos\theta+a^2}}=-\int_0^\Pi\frac{dcos\theta}{\sqrt{1-2acos\theta+a^2}}=\frac{\sqrt{1-2acos\theta+a^2}}{a}\Big|_0^\Pi=\frac{2}{a}$$

(3)

$$\int_0^1 x\sqrt{\frac{1-x}{1+x}}dx=\int_0^1\frac{x(1-x)}{\sqrt{1-x^2}}dx$$

令 x=sint

$$原式=\int_0^{\frac{\Pi}{2}}\frac{sint(1-sint)}{cost}costdt=\int_0^{\frac{\Pi}{2}}(sint-sin^2 t)dt = (-cost)\Big|_1^{\frac{\Pi}{2}}-\frac{\Pi}{4}=1-\frac{\Pi}{4}$$

(4)

$\int_{\frac{1}{2}}^{2} \frac{|lnx|}{1+x} dx$ 令 $t=\frac{1}{x}$

原式 $=\int_{2}^{1} -\frac{ln\frac{1}{t}}{1+\frac{1}{t}}*(-\frac{1}{t^2})dt + \int_{1}^{2} \frac{lnx}{1+x} dx$

$=\int_{1}^{2} \frac{lnt}{t*(1+t)}dt + \int_{1}^{2} \frac{lnt}{1+t}$

$=\int_{1}^{2} lntd(lnt) = \frac{1}{2}(lnt)^2|_{1}^{2} = \frac{(ln2)^2}{2}$

(5)

$\int_{2}^{e} \frac{1+lnx}{x^2 ln^2 x}dx = \int_{2}^{e} \frac{d(xlnx)}{(xlnx)^2}dx = -\frac{1}{xlnx}|_{2}^{e} = \frac{1}{2ln2} - \frac{1}{e}$

(6) $\int_{0}^{3} arcsin\sqrt{\frac{x}{1+x}}dx$ 令 $t=arcsin\sqrt{\frac{x}{1+x}}$ $x=tan^2t$

原式 $=\int_{0}^{\frac{\Pi}{3}} td\ tan^2t = ttan^2t|_{0}^{\frac{\Pi}{3}} - \int_{0}^{\frac{\Pi}{3}} tan^2tdt$

$=\Pi - \int_{0}^{\frac{\Pi}{3}}(sec2-1)dt$

$=\Pi - (tant-t)\ |_{0}^{\frac{\Pi}{3}}$

$=\frac{4\Pi}{3} - \sqrt{3}$

4、（1）

令 $x=\Pi-t$,则 $dx=-dt$

$\int_{0}^{\Pi} xf(sinx)dx = -\int_{\Pi}^{0}(\Pi-t)(sin(\Pi-t))dt$

$=\int_{0}^{\Pi}(\Pi-t)f(sint)dt$

$=\Pi\int_{0}^{\Pi} f(sint)dt - \int_{0}^{\Pi} tf(sint)dt$

$=\Pi\int_{0}^{\Pi} f(sinx)dx - \int_{0}^{\Pi} xf(sinx)dx$

有 $\int_{0}^{\Pi} xf(sinx)dx = \frac{\Pi}{2}\int_{0}^{\Pi} f(sinx)dx$

$\int_{0}^{\Pi} \frac{xsinx}{1+cos^2x}dx = \frac{\Pi}{2}\int_{0}^{\Pi} \frac{sinx}{1+cos^2x}dx = -\frac{\Pi}{2}\int_{0}^{\Pi} \frac{dcosx}{1+cos^2x} = -\frac{\Pi}{2}arctan(cosx)\ |_{0}^{\Pi}$

$$=\frac{\Pi^2}{4}$$

(2)

$$\int_0^{\Pi^2} sin^2\sqrt{x}dx \quad 令 x=t^2$$

原式$=2\int_0^{\Pi} tsin^2tdt=\Pi\int_0^{\Pi} sin^2tdt$

$$=\Pi\int_0^{\frac{\Pi}{2}} sin^2tdt+\Pi\int_{\frac{\Pi}{2}}^{\Pi} sin^2tdt$$

$$=2\Pi\int_{\frac{\Pi}{2}}^{\Pi} sin^2tdt$$

$$=\frac{\Pi^2}{2}$$

5、

证明：$\int_0^{\frac{\Pi}{2}} sin^nxcos^nxdx=\frac{1}{2^n}\int_0^{\frac{\Pi}{2}}(2sinxcosx)^ndx$

$$=\frac{1}{2^{n+1}}\int_0^{\frac{\Pi}{2}} sin^n2xd2x$$

$$=\frac{1}{2^n}\int_0^{\frac{\Pi}{2}} sin^nxdx$$

6、

$\int_0^1 \frac{dx}{\sqrt{1+x^2}}=ln|x+\sqrt{1+x^2}| \ |_0^1=ln(1+\sqrt{2})$

$\int_0^1 \frac{x}{\sqrt{1+x^2}}dx=\sqrt{1+x^2}|_0^1=\sqrt{2}-1$

又$\int_0^1 \frac{dx}{\sqrt{1+x^2}}> \int_0^1 \frac{x}{\sqrt{1+x^2}}dx$

所以 $ln(1+\sqrt{2})> \sqrt{2}-1$

7、

f(x)= $\int_1^x e^{-xt^2}dt$

f'(x)= e^{-x^3}

f'(1)= e^{-1}

8．设$f(x)$在$[a,b]$上连续，$F(x) = \int_a^x (x-t)f(t)\,dt$，x∈[a,b]，证明:

⟨1⟩$F''(x) = f(x)$　　　　　　　　⟨2⟩$F(x) = \int_a^x [\int_a^u f(t)dt]du$

解：⟨1⟩ ∵ f(x)在[a,b]连续

$$F(x) = \int_a^x (x-t)f(t)dt = \int_a^x [xf(t) - tf(t)]dt = x\int_a^x f(t)dt - \int_a^x tf(t)dt$$

$$\therefore F'(x) = \int_a^x f(t)dt + xf(x) - xf(x) = \int_a^x f(t)dt$$

$$\therefore F''(x) = f(x)$$

⟨2⟩ ∵ 由⟨1⟩可知，$F''(x) = f(x)$

$$\therefore F'(x) = \int_a^u f(t)dt$$

$$\therefore F(x) = \int_a^x [\int_a^u f(t)dt]du$$

9．设$f(x)$在$(-\infty, +\infty)$内连续可导，当 x ≠ 0 时，f(x) ≠ 0，且$\int_0^{f(x)} t^2 dt = \int_0^x f^2(t)e^{-f(t)}\,dt$，求$f(x)$.

解：　　　∵ $\int_0^{f(x)} t^2 dt = \int_0^x f^2(t)e^{-f(t)}\,dt$

$$\therefore f'(x)f^2(x) = f^2(x)e^{-f(x)}$$
$$\therefore y' = e^{-y}$$
$$\therefore \frac{dy}{dx} = \frac{1}{e^y}$$
$$\therefore e^y dy = dx$$
$$\Rightarrow e^y = x + c$$
$$\Rightarrow y = \ln(x + c)$$

又∵ 当 x = 0, f(x) = 0

$$\therefore f(x) = \ln(x + 1)$$

10.

设$f(x)$在[2,4]上连续可导，且$f(2)=f(4)=0$.证明：$|\int_2^4 f(x)dx|\le \max_{2\le x\le 4}|f'(x)|$

解：取$x\in[2,4]$,在$[2,x]$和$[x,4]$上分别对$f(x)$使用

拉格朗日中值定理，则$\exists \varepsilon_1\in[2,x],\varepsilon_2\in[x,4]$,使得

$f(x)-f(2)=f'(\varepsilon_1)(x-2)\Rightarrow f(x)=f'(\varepsilon_1)(x-2)$

$f(4)-f(x)=f'(\varepsilon_2)(4-x)\Rightarrow f(x)=f'(\varepsilon_2)(x-4)$

令$M=\max|f'(x)|(x\in[2,4])$

$|f(x)|\le M(x-2)$

$|f(x)|\le M(4-x)$

又$\because |\int_2^4 f(x)dx|\le \int_2^4|f(x)|dx\le \int_2^3 M(x-2)dx+\int_3^4 M(4-x)dx=M$

$\therefore \max_{2\le x\le 4}|f'(x)|\ge|\int_2^4 f(x)dx|$

11.

设$f(x)$在[0,1]上连续，在(0,1)内可导，且$3\int_{\frac{2}{3}}^1 f(x)dx=f(0)$.试证：在（0,1）内至少存在一点$\xi$,使$f'(\xi)$

$\because 3\int_{\frac{2}{3}}^1 f(x)dx=f(0)$

由积分中值定理可知：$\exists \xi_1\in(\frac{2}{3},1)$

$f(\xi_1)=f(0)$

由罗尔中值定理可知,$\exists \xi\in(0,\xi_1)\in(0,1)$

使得$f'(\xi)=0$

12.设 f(x)在[0,1]上可导，且 $2\int_0^{\frac{1}{2}}xf(x)\,dx=f(1)$.证明：在

(0,1)内至少存在一点 ξ ，使$f'(\xi)=-\frac{f(\xi)}{\xi}$.

解：令$F(x)=xf(x)$

$F'(x)=f(x)+xf'(x)$

$f(1)-2\int_0^{\frac{1}{2}}xf(x)\,dx=0$

$\therefore \int_0^{\frac{1}{2}}[f(1)-xf(x)]\,dx=0$

由积分中值定理$\exists x_1\in[0,\frac{1}{2}]$，$x_1f(x_1)=f(1)$

$$\therefore F(x_1) = F(1)$$

$$\therefore \exists \xi \in (x_1, 1) \quad F'(\xi)=0$$

即$f'(\xi) = -\dfrac{f(\xi)}{\xi}$

13. 曲线 y=ax²+bx 在[0,1]上的一段位于 x 轴上方，且与直线 x=1 及 x 轴所围成图形的面积为 $\frac{1}{3}$，确定 a、b 的值，使得该图形绕 x 轴一周所得旋转体的体积最小.

解：$f(x) = ax^2 + bx$

$$\int_0^1 (ax^2 + bx)\, dx = \frac{1}{3}$$

$$\therefore \left(\frac{a}{3} x^3 + \frac{b}{2} x^2 \right)\Big|_0^1 = \frac{a}{3} + \frac{b}{2} = \frac{1}{3}$$

$$\therefore 2a + 3b = 2$$

$$b = \frac{2-2a}{3}$$

$$V = \int_0^1 \pi (ax^2 + bx)^2\, dx$$

$$= \pi \int_0^1 (a^2 x^4 + b^2 x^2 + 2abx^3)\, dx$$

$$= \pi \left(\frac{1}{5} a^2 x^5 + \frac{1}{3} b^2 x^3 + \frac{1}{2} abx^4 \right)\Big|_0^1$$

$$= \frac{a^2}{5} \pi + \frac{b^2}{3} \pi + \frac{ab}{2} \pi$$

$$V'(a) = \frac{2}{5} \pi a + \frac{2\pi}{3} \cdot \frac{2-2a}{3} \cdot \left(-\frac{2}{3} \right) + \left(\frac{\pi}{3} - \frac{2}{3} \pi a \right)$$

$$= \frac{2}{5} \pi a - \frac{8}{27} \pi + \frac{8}{27} \pi a + \frac{\pi}{3} - \frac{18}{27} \pi a$$

$$= \frac{1}{27} \pi + \frac{2}{5} \pi a - \frac{10}{27} \pi a = 0 \text{时得} a = -\frac{5}{4}$$

$$\therefore a = -\frac{5}{4} \qquad b = \frac{3}{2}$$

14.设在 $(-\infty, +\infty)$ 内 f(x)>0.f'(x)连续，设 F(x)=$\begin{cases} \dfrac{\int_0^x tf(t)dt}{\int_0^x f(t)dt} & x \neq 0 \\ 0 & x = 0 \end{cases}$

<1>求 F'(x)

<2>证明 F'(x)在 $(-\infty, +\infty)$ 连续

<3>证明 F(x)在 $(-\infty, +\infty)$ 内单调递增

<1>当 $x \neq 0$ 时

$$F'(x)= \frac{xf(x)\int_0^x f(t)dt - f(x)\int_0^x tf(t)dt}{[\int_0^x f(t)dt]^2} = \frac{f(x)\int_0^x (x-t)f(t)dt}{[\int_0^x f(t)dt]^2}$$

当 $x = 0$ 时

$$F'(0)= \lim_{x \to 0} \frac{F(x)-F(0)}{x-0} = \frac{\int_0^x tf(t)dt}{x\int_0^x f(t)dt} = \lim_{x \to 0} \frac{xf(x)}{\int_0^x f(t)dt + xf(x)} = \lim_{x \to 0} \frac{f(x)+xf'(x)}{2f(x)+xf'(x)}$$

又因为在 $(-\infty, +\infty)$ f(x)>0

所以 F'(0)=$\frac{1}{2}$

综上所诉 F'(x)=$\begin{cases} \dfrac{f(x)\int_0^x (x-t)f(t)dt}{[\int_0^x f(t)dt]^2} & x \neq 0 \\ \dfrac{1}{2} & x = 0 \end{cases}$

<2>当 $x \neq 0$ 时 $\quad \lim_{x \to x_0} F'(x)$=F'$(x_0)$

当 $x = 0$ 时 $\lim_{x \to 0} F'(x) = \lim_{x \to 0} \frac{xf(x)\int_0^x f(t)dt - f(x)\int_0^x tf(t)dt}{[\int_0^x f(t)dt]^2}$

$$=\lim_{x \to 0} \frac{f(x)\int_0^x f(t)dt + xf'(x)\int_0^x f(t)dt + xf^2(x) - f'(x)\int_0^x tf(t)dt - xf^2(x)}{2f(x)\int_0^x f(t)dt}$$

$$=\lim_{x \to 0} \frac{f(x)\int_0^x f(t)dt + f'(x)\int_0^x (t-1)f(t)dt}{2f(x)\int_0^x f(t)dt}$$

$$=\lim_{x \to 0} \frac{f'(x)\int_0^x f(t)dt + f^2(x) + f''(x)\int_0^x (t-1)f(t)dt + f'(x)(x-1)f(x)}{2f'(x)\int_0^x f(t)dt + 2f^2(x)}$$

$$=\lim_{x \to 0} \frac{f^2(x)}{2f^2(x)} = \frac{1}{2}$$

（3） $x = 0$ 时， $F'(x) = \frac{1}{2} > 0$

$x \neq 0$ 时

$$F'(x) = \frac{xf(x)\int_0^x f(t)d\,t - f(x)\int_0^x t f(t)dt}{\left[\int_0^x f(t)d\,t\right]^2}$$

设 $g(x) = xf(x)\int_0^x f(t)d\,t - f(x)\int_0^x t f(t)dt.$

$$= f(x)\left[x \int_0^x f(t)dt - \int_0^x t f(t)dt\right]$$

$$h(x) = x \int_0^x f(t)dt - \int_0^x t f(t)dt.$$

$$h'(x) = \int_0^x f(t)dt + xf(x) - xf(x)$$

$$= \int_0^x f(t)dt$$

$$h''(x) = f(x) > 0 \quad \therefore h(x)递增$$

又因为$h'(0) = 0$

$x < 0$时 $\quad h'(x) < 0 \quad x > 0$时$h'(x) > 0 \therefore h(x)$在$(-\infty, 0) \downarrow (0, +\infty) \uparrow$

又$\because h(0) = 0 \quad \therefore h(x) > 0 \quad (x \neq 0$时$)$

即$x \neq 0$ 时，$g(x) > 0$即$F'(x) > 0$

综上所述$F(x)$在$(-\infty, +\infty)$内单调递增。