习题 5.4

(1) $\int \frac{x^3}{1+x}dx$

$= \int (x^2+1-x-\frac{1}{1+x})\,dx$

$= \frac{x^3}{3}-\frac{x^2}{2}+x-ln|x+1|+C$

(2) $\int \frac{x^5+x^4-8}{x^3-x}dx$

$= \int \frac{x^2(x^3-x)+x(x^3-x)+(x^3-x)+x^2+x-8}{x^3-x}dx$

$= \int (x^2+x+1+\frac{1}{x-1}-\frac{8}{x^3-x})\,dx$

$= \frac{x^3}{3}+\frac{x^2}{2}+x+8ln|x|-4\ln|x+1|-3\ln|x-1|+C$

(3) $\int \frac{x^3+1}{x^3-x^2}dx$

$= \int (1-\frac{1}{\mathrm{x}}-\frac{1}{\mathrm{x}^2}+\frac{2}{x-1})\,dx$

$= x+\frac{1}{x}+ln\frac{(\mathrm{x}-1)^2}{|x|}+C$

(4) $\int \frac{x^5}{(x-1)^2(x^2-1)}dx$

$= \int (x+2+\frac{\frac{1}{8}}{1+x}+\frac{\frac{31}{8}}{x-1}+\frac{\frac{9}{4}}{(x-1)^2}+\frac{\frac{1}{2}}{(x-1)^3})\,dx$

$= \frac{x^2}{2}+2x-\frac{1}{4(x-1)^2}-\frac{9}{4(x-1)}+\frac{31}{8}ln|x-1|+\frac{1}{8}\ln|x+1|+C$

(5) $\int \frac{x^4}{1+x^2}dx$

$= \int (x^2-1+\frac{1}{1+x^2})\,dx$

$= \frac{x^3}{3} - x + arctanx + C$

(6) $\int \frac{x^2}{1-x^4} dx$

$= \int (\frac{\frac{1}{4}}{1+x} + \frac{\frac{1}{4}}{1-x} - \frac{\frac{1}{2}}{1+x^2}) dx$

$= \frac{1}{4} ln \left|\frac{1+x}{1-x}\right| - \frac{1}{2} arctanx + C$

(7) $\int \frac{1}{(x+1)^2(x^2+1)} dx$

$= \int \left(\frac{\frac{1}{2}}{x+1} + \frac{\frac{1}{2}}{(x+1)^2} - \frac{\frac{1}{2}x}{1+x^2}\right) dx$

$= \frac{1}{2} ln|x+1| - \frac{1}{4} \ln(x^2+1) - \frac{1}{2(x+1)} + C$

(8) $\int \frac{x^3-x^2-x+3}{x^2-1} dx$

$= \int (x - 1 + \frac{1}{x-1} - \frac{1}{x+1}) dx$

$= \frac{x^2}{2} - x + ln\left|\frac{x-1}{x+1}\right| + C$

(9) $\int \frac{2x+2}{(1+x)^2(x-1)} dx$

$= \int (\frac{1}{x-1} - \frac{1}{x+1}) dx$

$= ln|x-1| - \ln|x+1| + C$

（10）

$$\int \frac{x^3+2x^2+1}{(x-1)(x-2)(x-3)^2} dx$$

$=\int \left(\frac{-1}{x-1} + \frac{17}{x-2} + \frac{-15}{x-3} + \frac{23}{(x-3)^2}\right) dx$

$=-\ln|x-1|+17\ln|x-2|-15\ln|x-3|-\frac{23}{x-3}+c$

（11）

$$\int\frac{x^3}{(x-1)^{100}}dx$$

$=\int\frac{(x-1)^3+3(x-1)^2-3(x-1)-1}{(x-1)^{100}}dx$

$=\int\left(\frac{1}{(x-1)^{97}}+\frac{3}{(x-1)^{98}}+\frac{3}{(x-1)^{99}}+\frac{1}{(x-1)^{100}}\right)dx$

$=-\frac{1}{96(x-1)^{96}}-\frac{3}{97(x-1)^{97}}-\frac{3}{98(x-1)^{98}}-\frac{1}{99(x-1)^{99}}+c$

（12）

$$\int\frac{1}{x(x^{10}+2)}dx$$

$=\frac{1}{2}\int\left(\frac{1}{x}-\frac{x^9}{x^{10}+2}\right)dx$

$=\frac{1}{2}\ln|x|-\frac{1}{20}\ln(x^{10}+2)+c$

T2（1）

令$\tan\frac{x}{2}=t$，$x=2\arctan t$，$dx=\frac{2}{1+t^2}dt$

$$\int\frac{1}{2\sin x-\cos x+5}dx$$

$=\int\frac{1}{\frac{4t}{1+t^2}-\frac{1-t^2}{1+t^2}+5}\cdot\frac{2}{1+t^2}dt$

$=\int \frac{1}{(3t+1)^2+\left(\sqrt{5}\right)^2} d(3t+1)$

$=\frac{1}{\sqrt{5}} arctan \frac{3t+1}{\sqrt{5}} + c$

$=\frac{1}{\sqrt{5}} arctan \frac{3\, tan\frac{x}{2}+1}{\sqrt{5}} + c$

（2）

令$tan\frac{x}{2} = t$，$x = 2\, arctan\, t$， $dx = \frac{2}{1+t^2} dt$

$\int \frac{1}{(2+cos\, x)\, sin\, x} dx$

$=\frac{1}{3}\int \frac{1}{t^3+3t} d(t^3 + 3t)$

$=\frac{1}{3}\ln|t^3 + 3t| + c$

$=\frac{1}{3} ln \left|tan\frac{x}{2}\left(3 + tan^2\frac{x}{2}\right)\right| + c$

（3）

令$tan\frac{x}{2} = t$，$x = 2\, arctan\, t$， $dx = \frac{2}{1+t^2} dt$

$\int \frac{1+sin\, x}{sin\, x(1+cos\, x)} dx$

$=\int \left(\frac{1}{2}t + 1 + \frac{1}{2t}\right) dt$

$=\frac{1}{4}t^2 + t + \frac{1}{2} ln|t| + c$

$=\frac{1}{4}\tan^2\frac{x}{2} + \tan\frac{x}{2} + \frac{1}{2}\ln\left|\tan\frac{x}{2}\right| + c$

（4）

令$tan\frac{x}{2}=t$，$x=2\,arctan\,t$， $dx=\frac{2}{1+t^2}dt$

$$\int\frac{1}{sin\,x+tan\,x}dx$$

$=\int\left(-\frac{1}{2}t+\frac{1}{2t}\right)dt$

$=-\frac{1}{4}t^2+\frac{1}{2}ln|t|+c$

$=-\frac{1}{4}tan^2\frac{x}{2}+\frac{1}{2}ln\left|tan\frac{x}{2}\right|+c$

（5）

令$tan\,x=t$，$x=arc\,tan\,t$， $dx=\frac{1}{1+x^2}$

$$\int\frac{1}{(sin\,x+cos\,x)^2}dx$$

$=\int\frac{1}{1+sin\,2x}dx$

$=\int\frac{1}{1+\frac{2t}{1+t^2}}\cdot\frac{1}{1+t^2}dt$

$=\int\frac{1}{(t+1)^2}dt$

$=-\frac{1}{t+1}+c$

$=-\frac{1}{tan\,x+1}+c$

（6）

令$cos\,x=t$，$sin^2x=1-t^2$

$$\int\frac{1}{sin\,x\,cos^3x}dx$$

$=-\int \frac{1}{sin^2 x\, cos^3 x} d(cos\, x)$

$=-\int \left(\frac{1}{t}+\frac{1}{t^3}+\frac{t}{1-t^2}\right) dt$

$=-\, ln|cos\, x|+\frac{1}{2\, cos^2 x}+ln|sin\, x|+c$

（7） $\int \frac{\cos x}{1+\sin x} dx$

$=\int \frac{1}{1+\sin x} d(\sin x+1)$

$=\ln(1+\sin x)+C$

(8) 令$tan\frac{x}{2}=t$

$\int \frac{1}{3+5\cos x} dx$

$=\frac{1}{4}\int \left(\frac{1}{2-t}+\frac{1}{2+t}\right) dt$

$=\frac{1}{4}\ln\left|\frac{2+\tan\frac{x}{2}}{2-\tan\frac{x}{2}}\right|+C$

(9) 令$tan\frac{x}{2}=t$

$\int \frac{1}{\sin 2x-2\sin x} dx$

$=-\frac{1}{4}\int \left(\frac{1}{t^3}+\frac{1}{t}\right) dt$

$=-\frac{1}{4}\left(-\frac{1}{2t^2}+\ln|t|\right)+C$

$=\frac{1}{8}cos^2\frac{x}{2}-\frac{1}{4}\ln\left|\tan\frac{x}{2}\right|+C$

(10)$\int \frac{1}{sin^4x+cos^4x}dx$

$=\int \frac{sec^4x}{(tan^4x)+1}d\tan x$ 　令$tan\,x=t$

$=\int \frac{1+t^2}{1+t^4}dt$

$=\frac{\sqrt{2}}{2}\arctan\left(\frac{t-\frac{1}{t}}{\sqrt{2}}\right)+C$

$=\frac{\sqrt{2}}{2}\arctan\left(\frac{tan^2x-1}{\sqrt{2}\tan x}\right)+C$

3 (1)令 t=$\sqrt{\frac{x}{1-x}}, x=\frac{t^2}{1+t^2}$

$\int \frac{1}{x}\sqrt{\frac{x}{1-x}}dx$

$=\int \frac{2}{1+t^2}dt$

$=2arctant+C$

$=2arctan\sqrt{\frac{x}{1-x}}+C$

(2)$\int \frac{\sqrt{x}}{\sqrt[3]{x^2}-\sqrt[4]{x}}dx$, 令$t=\sqrt[12]{x}, x=t^{12}$

$=\int \frac{t^6}{t^8-t^3}dt^{12}$

$=12\int(\frac{t^4}{t^5-1}+t^4+t^9)$dt

$=\frac{6}{5}x^{\frac{5}{6}}+\frac{12}{5}x^{\frac{5}{12}}+\frac{12}{5}\ln\left|x^{\frac{5}{12}}-1\right|+C$

(3)$\int \frac{1+\sqrt{1-x^2}}{1-\sqrt{1-x^2}}dx$

$$=\int \frac{\left(1+\sqrt{1-x^2}\right)\left(1+\sqrt{1-x^2}\right)}{\left(1-\sqrt{1-x^2}\right)\left(1+\sqrt{1-x^2}\right)}dx$$

$$=\int \frac{2-x^2+2\sqrt{1-x^2}}{x^2}dx$$

$$=-\frac{2}{x}-x-2\int \sqrt{1-x^2}d\left(\frac{1}{x}\right)$$

$$=-\frac{2}{x}-x-\frac{2}{x}\sqrt{1-x^2}-2\int \frac{dx}{\sqrt{1-x^2}}$$

$$=-\frac{2+x^2}{x}-\frac{2}{x}\sqrt{1-x^2}-2arcsinx+C$$

(4)$\int \sqrt{\frac{e^x-1}{e^x+1}}dx$，令$t=\sqrt{\frac{e^x-1}{e^x+1}}, dx=\frac{4t}{1-t^4}dt$

$$=\int t\frac{4t}{1-t^4}dt$$

$$=2\int \left(\frac{1}{1-t^2}+\frac{1}{1+t^2}\right)dt$$

$$=\ln\left|\frac{1-t}{1+t}\right|+arctant+C$$

$$=\ln\left|\frac{1-\sqrt{\frac{e^x-1}{e^x+1}}}{1+\sqrt{\frac{e^x-1}{e^x+1}}}\right|+arctan\sqrt{\frac{e^x-1}{e^x+1}}+C$$

(5)$\int \frac{1}{1+\sqrt[3]{x+1}}dx$，令$t=\sqrt[3]{x+1}, x=t^3-1$

$$=3\int \left(t-1+\frac{1}{1+t}\right)dt$$

$$=\frac{3}{2}t^2-3t+3\ln|1+t|+C$$

$$=\frac{3}{2}\sqrt[3]{(x+1)^2}-3\sqrt[3]{x+1}+3\ln\left|1+\sqrt[3]{x+1}\right|+C$$

(6)$\int \frac{x}{\sqrt{5+x-x^2}}dx$

$$=\frac{2}{\sqrt{21}}\int \frac{x}{\sqrt{1-\left[\frac{2}{\sqrt{21}}\left(x-\frac{1}{2}\right)\right]^2}}dx$$

$= \int xdarcsin\frac{2x-1}{\sqrt{21}}$, 令$t = arcsin\frac{2x-1}{\sqrt{21}}$

$= \left(\int (\frac{1}{2} + \frac{\sqrt{21}}{2}\sin t)\right)$dt

$=\frac{1}{2}t - \frac{\sqrt{21}}{2}\cos t + C$

$=-\sqrt{5 + x - x^2} + \frac{1}{2}arcsin\frac{2x-1}{\sqrt{21}} + C$