

习题 6.6

1.

$$\begin{aligned}(1) S &= \int_0^4 \sqrt{1 + (y')^2} dx = \int_0^4 \sqrt{1 + \frac{9}{4}x} dx = \left[\frac{4}{9} \times \frac{2}{3} \times \left(1 + \frac{9}{4}x\right)^{\frac{3}{2}} \right]_0^4 \\ &= \frac{4}{9} \times \frac{2}{3} \times \left[\left(1 + \frac{9}{4} \times 4\right)^{\frac{3}{2}} - 1 \right] = \frac{8}{27} (10^{\frac{3}{2}} - 1)\end{aligned}$$

$$(2) x' = \frac{y}{2} - \frac{1}{2y}$$

$$\begin{aligned}S &= \int_1^e \sqrt{1 + (x')^2} dy = \int_1^e \sqrt{1 + \left(\frac{y}{2} - \frac{1}{2y}\right)^2} dy \\ &= \int_1^e \frac{1}{2} \left(y + \frac{1}{y}\right) dy = \frac{1}{2} \left[\frac{1}{2} y^2 + \ln y \right]_1^e \\ &= \frac{1}{2} \times \left(\frac{1}{2} e^2 + 1 - \frac{1}{2} \right) = \frac{1}{4} (e^2 + 1)\end{aligned}$$

(3) 由题目可知, $x \geq 0, y \geq 0$

$$y = (1 - \sqrt{x})^2 \quad y' = \frac{dy}{dx} = 2(1 - \sqrt{x}) \left(-\frac{1}{2\sqrt{x}}\right) = 1 - \frac{1}{\sqrt{x}}.$$

$$\begin{aligned}S &= \int_0^1 \sqrt{1 + (y')^2} dx = \int_0^1 \sqrt{1 + \left(1 - \frac{1}{\sqrt{x}}\right)^2} dx \\ &= 2 \int_0^1 \sqrt{2x - 2\sqrt{x} + 1} d\sqrt{x} = 1 + \frac{\sqrt{2}}{2} \ln(1 + \sqrt{2}).\end{aligned}$$

(4) 设 $x = a \cos^3 t$ $y = a \sin^3 t$ ($0 \leq t \leq 2\pi$)

$$S = 4 \int_0^{\frac{\pi}{2}} \sqrt{[(a \cos^3 t)']^2 + [(a \sin^3 t)']^2} dt$$

$$= 4a \int_0^{\frac{\pi}{2}} \sqrt{(-3 \cos^2 t \sin t)^2 + (3 \sin^2 t \cos t)^2} dt$$

$$= 12a \int_0^{\frac{\pi}{2}} \sin t \cos t dt = 6a.$$

$$(5) S = \int_0^{2\pi} \sqrt{[(a(\cos t + t \sin t))']^2 + [(a(\sin t - t \cos t))']^2} dt$$

$$= |a| \int_0^{2\pi} \sqrt{(t \cos t)^2 + (t \sin t)^2} dt$$

$$= |a| \int_0^{2\pi} t dt = 2\pi^2 |a|$$

$$(6) S = \int_0^{2\pi} \sqrt{r^2 + (r')^2} d\theta = \int_0^{2\pi} \sqrt{a^2(1 + \cos \theta)^2 + a^2 \sin^2 \theta} d\theta$$

$$= 4a \int_0^{\pi} \sqrt{\frac{1 + \cos \theta}{2}} d\theta = 4a \int_0^{\pi} \cos \frac{\theta}{2} d\theta = 8a$$

2.

$$(1) \int_a^b |f(x) - g(x)| dx.$$

面积 $A \geq 0$, 且 $f(x)$ 、 $g(x)$ 的大小无法确定

$$\text{故面积为 } \int_a^b |f(x) - g(x)| dx$$

$$(2) \pi \int_a^b |f^2(x) - g^2(x)| dx$$

在区间 $[a, b]$ 上, 由曲线 $y = f(x)$, y

$= g(x)$ 所围成的平面绕 x 轴旋转

一周所成的旋转体的体积微元为

$$dV = \pi |f^2(x) - g^2(x)| dx$$

$$\therefore V = \pi \int_a^b |f^2(x) - g^2(x)| dx$$

3.

$$(1) \text{ 由 } \begin{cases} y = x^2 \\ x + y = 2 \end{cases} \text{ 得 } x = 1 \text{ 或 } x = -2$$

$$S = \int_{-2}^1 (2 - x - x^2) dx = \left(2x - \frac{1}{2}x^2 - \frac{1}{3}x^3 \right) \Big|_{-2}^1 = \frac{9}{2}$$

$$(2) S = \int_{\frac{1}{e}}^1 |\ln x| dx + \int_1^e \ln x \cdot dx$$

$$= \int_{\frac{1}{e}}^1 \ln x dx + \int_1^e \ln x dx$$

$$= (x \ln x - x) \Big|_{\frac{1}{e}}^1 + (x \ln x - x) \Big|_1^e$$

$$= 2 - \frac{2}{e}$$

$$(3) \text{ 令 } x = a \sin t, \quad y = b \cos t$$

$$\text{则 } S = 4 \int_0^a y dx = \frac{4b}{a} \int_0^a \sqrt{a^2 - x^2} dx = \int_a^{4b} \int_0^{\frac{\pi}{2}} a^2 \cos^2 t dt$$

$$= 4ab \int_0^{\frac{\pi}{2}} \frac{1 + \cos 2t}{2} dt = 2ab \left[t + \frac{1}{2} \sin 2t \right]_0^{\frac{\pi}{2}} = ab\pi$$

$$(4) S = \int_0^1 (e^x - e^{-x}) dx = [e^x - (-e^{-x})]_0^1$$

$$= e + e^{-1} - 2$$

$$(5) S = \int_0^{\frac{\pi}{4}} (\cos x - \sin x) dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\cos x - \sin x) dx$$

$$= (\sin x + \cos x) \Big|_0^{\frac{\pi}{4}} + (-\cos x - \sin x) \Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} = 2\sqrt{2} - 2$$

$$(6) \text{ 由 } \begin{cases} y = \frac{1}{2}x^2 \\ x^2 + y^2 = 8 \end{cases} \text{ 得两曲线的交点为 } (-2, 2), (2, 2)$$

$$\begin{aligned} \text{则 } S_1 &= \int_{-2}^2 \left(\sqrt{8-x^2} - \frac{1}{2}x^2 \right) dx = 2 \int_0^2 \left(\sqrt{8-x^2} - \frac{1}{2}x^2 \right) dx \\ &= 2 \left[4 \arcsin \frac{x}{\sqrt{8}} + \frac{1}{2}x\sqrt{8-x^2} - \frac{1}{6}x^3 \right] \Big|_0^2 \\ &= 2\pi + \frac{4}{3} \end{aligned}$$

$$S_2 = S - S_1 = \pi(2\sqrt{2})^2 - 2\pi - \frac{4}{3} = 6\pi - \frac{4}{3}$$

4.

$$(1) S = \frac{1}{2} \int_0^{\frac{\pi}{2}} r^2 d\theta$$

$$\begin{aligned} x_2 &= 4 \int_0^{\frac{\pi}{2}} \frac{\cos 2\theta + 1}{2} d\theta \\ &= (\sin 2\theta + 2x) \Big|_0^{\frac{\pi}{2}} = \pi \end{aligned}$$

$$(2) S = \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} r^2 d\theta = \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2a^2 \cos 2\theta d\theta$$

$$= \left(\frac{1}{2} a^2 \sin 2\theta \right) \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 4 \times \frac{a^2}{2} \sin 2\theta \Big|_0^{\frac{\pi}{4}} = 2a^2$$

(3) $0 \leq t \leq 2\pi$, 该图形关于x轴与y轴都对称

$$x' = -3a \cos^2 t \sin t$$

$$\begin{aligned} S &= 4 \int_0^{\frac{\pi}{2}} |a \sin^3 t (-3a \cos^2 t \sin t)| dt \\ &= 12a^2 \int_0^{\frac{\pi}{2}} \sin^4 t \cos^2 t dt = \frac{3}{8} \pi a^2 \end{aligned}$$

5.

(1) 两曲线交点为(0,0)与(1,1)的旋转体体积

$$V = \pi \int_0^1 [(\sqrt{x})^2 - (x^2)^2] dx = \pi \int_0^1 (x - x^4) dx = \frac{3}{10}$$

$$(2) V = \pi \int_{-a}^a \left[(b + \sqrt{a^2 - x^2})^2 - (b - \sqrt{a^2 - x^2})^2 \right] dx$$

$$= 4\pi b \int_{-a}^a \sqrt{a^2 - x^2} dx = 2\pi^2 a^2 b$$

$$(3) V = \pi \int_{-a}^a y^2 dx = 3\pi a^3 \int_0^\pi \sin^7 t \cos^2 t dt$$

$$= 6\pi a^3 \int_0^{\frac{\pi}{2}} (\sin^7 t - \sin^9 t) dt = \frac{32}{105} \pi a^3$$

详细解释如下：

$$= 4\pi b \cdot \frac{1}{2} \pi a^2 = 2\pi^2 a^2 b$$

13) $\begin{cases} x = a \cos^2 t \\ y = a \sin^2 t \end{cases}$ 星形线, 绕x轴

解: $V = \pi \int_{-a}^a y^2 dx$

$$= \pi \int_0^\pi a^2 \sin^4 t d(a \cos^2 t)$$

$$= \pi \int_0^\pi a^2 \sin^4 t \cdot a \cdot 3 \cos^2 t \cdot \sin t dt$$

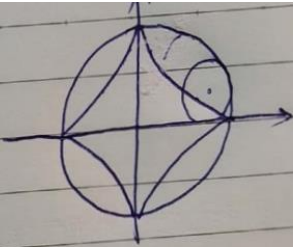
$$= \pi \int_0^\pi a^3 \sin^5 t \cos^2 t dt$$

$$= \frac{32}{105} \pi a^3$$

由补元/Th2 以B 对称性

$$2 \cdot \frac{(1+1)!!(2-1)!!}{(1+2)!!} = 2 \cdot \frac{2 \times 1 \times 1}{3 \times 1 \times 1} = \frac{4}{3}$$

$$= \frac{16}{105} \times 2 = \frac{32}{105}$$



例2: 令 $I(m, n) = \int_0^{\frac{\pi}{2}} \cos^m x \sin^n x dx$
 (这里根据不定积分那里: $\int \cos^m x \sin^n x dx = \int \cos^{m-1} x \sin^n x d(\sin x) = \frac{1}{n+1} \int \cos^{m-1} x d(\sin^{n+1} x)$
 找到递推式)

$$\Rightarrow I(m, n) = \frac{m-1}{m+n} I(m-2, n) = \frac{n-1}{m+n} I(m, n-2)$$

$$\Rightarrow I(m, n) = \begin{cases} \frac{(m-1)!!(n-1)!!}{(m+n)!!} \cdot \frac{2}{2}, & m, n \text{ 都为偶数时} \\ \frac{(m-1)!!(n-1)!!}{(m+n)!!} \cdot 1, & m, n \text{ 一奇一偶时} \\ & \text{或 } m, n \text{ 都为奇数时} \end{cases}$$

 特别地

$$\int_0^{\frac{\pi}{2}} \sin^n x dx = \int_0^{\frac{\pi}{2}} \cos^n x dx = I(n, 0) = I(0, n) = \begin{cases} \frac{(n-1)!!}{n!!} \cdot \frac{2}{2}, & n \text{ 为偶数时} \\ \frac{(n-1)!!}{n!!}, & n \text{ 为奇数时} \end{cases}$$

 注: $0!! = 1, 1!! = 1, 2!! = 2$

6. 证明: 旋转曲面的方程为 $\pm\sqrt{y^2 + z^2} = f(x)$, 由旋转曲面的对称性, 取该曲面的上半部分 $\Sigma: z = \sqrt{f^2(x) - y^2}$

Σ 在 xOy 面上的投影区域为

$$D = \{(x, y) \mid -f(x) \leq y \leq f(x) \quad a \leq x \leq b\}$$

$$S = 2 \iint_D \sqrt{1 + z_x^2 + z_y^2} dx dy$$

$$= 2 \iint_D \sqrt{1 + \left[\frac{f(x)f'(x)}{\sqrt{f^2(x) - y^2}} \right]^2 + \left[\frac{-y}{\sqrt{f^2(x) - y^2}} \right]^2} dx dy$$

$$= 2 \int_a^b f(x) \sqrt{1 + [f'(x)]^2} dx \int_{-f(x)}^{f(x)} \frac{1}{\sqrt{f^2(x) - y^2}} dy$$

$$= 2 \int_a^b f(x) \sqrt{1 + [f'(x)]^2} \cdot \left[\arcsin \frac{y}{f(x)} \right]_{-f(x)}^{f(x)} dx$$

$$= 2\pi \int_a^b f(x) \sqrt{1 + [f'(x)]^2} dx$$

$$(1)y' = 2x^{-\frac{1}{2}}$$

$$S = 2\pi \int_0^1 2x^{\frac{1}{2}} \sqrt{1 + \left(2x^{-\frac{1}{2}}\right)^2} dx$$

$$= 4\pi \int_0^1 \sqrt{x+4} dx = 4\pi \times \frac{2}{3} (x+4)^{\frac{3}{2}} \Big|_0^1$$

$$= \frac{8\pi}{3} (5\sqrt{5} - 8)$$

$$(2)y' = \frac{dy}{dx} = \frac{3a \sin^2 t \cos t}{-3a \cos^2 t \sin t} = -\tan t$$

由对称性知

$$S = 2 \int_0^a 2\pi y \cdot \sqrt{1 + (y')^2} dx$$

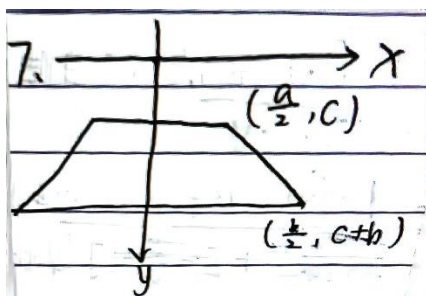
$$= -4\pi \int_0^{\frac{\pi}{2}} a \sin^3 t \sqrt{1 + \tan^2 t} (-3a \cos^2 t \sin t) dt$$

$$= 12a^2\pi \int_0^{\frac{\pi}{2}} \sin^4 t \sec t \cos^2 t dt$$

$$= 12a^2\pi \int_0^{\frac{\pi}{2}} \sin^4 t d \sin t$$

$$= \frac{12}{5} a^2\pi$$

7.



如图，AB 的方程为 $y = \frac{2h}{b-a} \left(x - \frac{a}{2}\right) + c$ 对于薄板上每一点 (x, y) 的

压力 $dF = \rho g y \cdot x dy$

由对称性可知

$$\begin{aligned}P &= 2 \int_c^{c+h} dF = \int_c^{c+h} \left[a + \frac{b-a}{h}(y-c) \right] \rho g y dy \\&= \frac{1}{6} \rho g h (3ac + 3bc + ab + 2bh)\end{aligned}$$

8. 球的密度与水相同 \Rightarrow 球在水中移动时不做功, x 为积分变量,

$x \in [0, 2r]$ 。把球体分为很多薄层, 将相应于 $[x, x+dx]$ 的那一层球体

抬到水面时不做功, 从离开水面时开始做功且 $x=0$ 面上方圆的方

程为 $(x-r)^2 + y^2 = r^2$, 可知, 将相应于 $[x, x+dx]$ 的那一薄层球体

提升到 $[x-2r, x+dx-2r]$ 位置时所做的功微元为 (ρ 为水密度)

$$dW = \rho g (2r-x) \pi y^2 dx = \rho g \pi (2r-x) [r^2 - (x-r)^2] dx$$

$$= \rho g \pi (2r-x) (2rx - x^2) dx = \rho g \pi (x^3 - 4rx^2 + 4r^2x) dx$$

$$\text{故 } W = \int_0^{2r} dW = \rho g \pi \int_0^{2r} (x^3 - 4rx^2 + 4r^2x) dx$$

$$= \rho g \pi \left(\frac{x^4}{4} - \frac{4rx^3}{3} + 2r^2x^2 \right) \Big|_0^{2r} = \frac{4}{3} \rho g \pi r^4$$