习题 4.7

1. 求下列曲线在指定点处的曲率

（1）曲线 xy=4，点（2, 2）

$$y' = \frac{4}{x^2}, y'(2) = -1;$$

$$y'' = \frac{8}{x^3}\ , y''(2) = 1;$$

由曲率公式$k = \frac{|y''|}{(1+(y')^2)^{\frac{3}{2}}}$，带入得 $\mathrm{k} = \frac{\sqrt{2}}{4}$

（2）曲线 y=4x−x^2, 点（0, 0）

$$y' = 4 - 2x，y'(0) = 4$$

$$y'' = -2，y''(0) = -2$$

由曲率公式$k = \frac{|y''|}{(1+(y')^2)^{\frac{3}{2}}}$，带入得 $\mathrm{k} = \frac{2}{\sqrt{17^3}}$

（3）曲线$y = \ln\left(x + \sqrt{1+x^2}\right)$，点（0, 0）

$$y' = \frac{1}{\sqrt{x^2+1}}，y'(0) = 1$$

$$y'' = -x(x^2+1)^{-\frac{3}{2}}，y''(0) = 0$$

由曲率公式$k = \frac{|y''|}{(1+(y')^2)^{\frac{3}{2}}}$，带入得 $\mathrm{k} = 0$

（4）曲线 y=lnx，点（1, 0）

$$y' = \frac{1}{x}，y'(1) = 1$$

$$y' = -\frac{1}{x^2}，y''(1) = -1$$

由曲率公式$k=\frac{|y''|}{(1+(y')^2)^{\frac{3}{2}}}$，带入得 $\mathrm{k}=\frac{\sqrt{2}}{4}$

2. 请证明公式（4.7.4）

证明：$\begin{cases}x=x(t)\\y=y(t)\end{cases}$，$y'=\frac{dy}{dx}=\frac{dy}{dt}\frac{dt}{dx}=\frac{y'(t)}{x'(t)}$

$$y''=\frac{d^2y}{dx^2}=\frac{d\left(\frac{dy}{\alpha x}\right)}{dx}=\frac{d\left(\frac{y'(t)}{x'(t)}\right)}{dt}\frac{dt}{\mathrm{dx}}=\frac{y''(t)x'(t)-x''(t)y'(t)}{[x'(t)]^2}\frac{1}{x'(t)}$$

分别将y'和y''带入 $\mathrm{k}=\frac{|\mathrm{y}''|}{(1+(\mathrm{y}')^2)^{\frac{3}{2}}}$式中得：

$$k=\frac{|x'(t)y''(t)-x''(t)y'(t)|}{[x'(t)^2]^{\frac{3}{2}}\left(1+\left(\frac{y'(t)}{x'(t)}\right)^2\right)^{\frac{3}{2}}}=\frac{|x'(t)y''(t)-x''(t)y'(t)|}{\left[\left(x'(t)\right)^2+\left(y'(t)\right)^2\right]^{\frac{3}{2}}}$$

3. 求由下列参数方程表示的曲线在指定参数处的曲率

(1)曲线$\begin{cases}x=3t^2\\y=3t-t^3\end{cases}$，$\mathrm{t}=1$;

$x'(t)=6t,x'(1)=6;x''(t)=6,x''(1)=6;$

$y'(t)=3-3t^2,y'(1)=0;y''(t)=-6t,y''(1)=-6;$

由 $\mathrm{k}=\frac{|x'(t)y''(t)-x''(t)y'(t)|}{\left[\left(x'(t)\right)^2+\left(y'(t)\right)^2\right]^{\frac{3}{2}}}$,得 $\mathrm{k}=\frac{1}{6}$

(2)曲线$\begin{cases}x=a(cost+tsint)\\y=a(sint-tcost)\end{cases}$,$t=\frac{\pi}{2}$，其中$a>0$.

$x'(t)=atcost,x'\left(\frac{\pi}{2}\right)=0;$

$x''(t)=a(cost-tsint),x''\left(\frac{\pi}{2}\right)=-\frac{\pi a}{2};$

$y'(t) = atsint, y'\left(\frac{\pi}{2}\right) = \frac{\pi a}{2};$

$y''(t) = a(sint + tcost), y''\left(\frac{\pi}{2}\right) = a;$

由$k = \frac{|x'(t)y''(t) - x''(t)y'(t)|}{\left[\left(x'(t)\right)^2 + \left(y'(t)\right)^2\right]^{\frac{3}{2}}}$,得$k = \frac{2}{\pi a}$

4.求曲线 $y=x^2$ 上任一点处的曲率，并问哪一点处曲率最大？

由 $y' = 2x, y'' = 2$

带入公式 $k = \frac{|y''|}{(1+(y')^2)^{\frac{3}{2}}}$，得；$k = \frac{2}{(1+4x^2)^{\frac{3}{2}}}$

所以当 x=0 时，k 取最大值，

即曲线 $y=x^2$ 在 x=0 处曲率最大。

5.求椭圆周$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$上任一点处的曲率，并问哪一点处曲率最大？其中 a>b>0.

设$\begin{cases} x = acost \\ y = bsint \end{cases}$ (a>b>0)

则$x'(t) = -asint$；$x''(t) = -acost$；

$y'(t) = bcost; y''(t) = -bsint$；

$$\Rightarrow k = \frac{|x'(t)y''(t) - x''(t)y'(t)|}{\left[\left(x'(t)\right)^2 + \left(y'(t)\right)^2\right]^{\frac{3}{2}}}$$

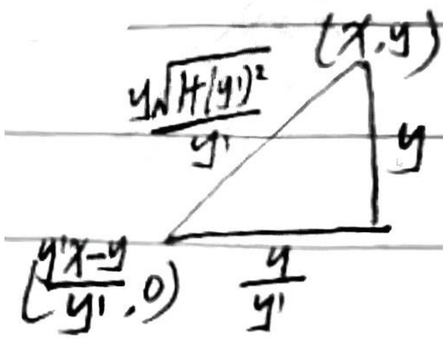
$$\Rightarrow k = \frac{ab}{(a^2\sin^2 t + b^2\cos^2 t)^{\frac{3}{2}}}$$

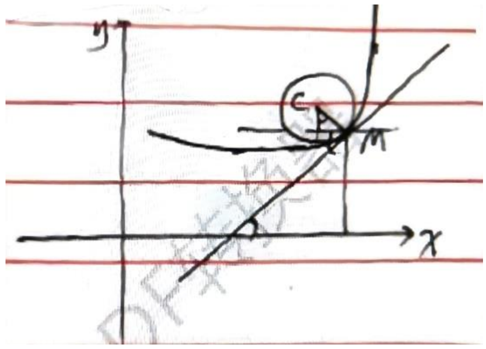
$$\Rightarrow k = \frac{ab}{(a^2(1-cos^2t) + b^2\cos^2 t)^{\frac{3}{2}}}$$

$$\Rightarrow k = \frac{ab}{\left(a^2 + (b^2 - a^2) \cos^2 t\right)^{\frac{3}{2}}}$$

所以当 cost=0，即 t=$\pm\frac{\pi}{2}$时，k 取最大值，此时 x=$\pm a$

6.





M点处曲率为$k=\frac{|y''|}{(H+(y)^2)^{\frac{2}{2}}}=\frac{|y''(x)|}{\left(1+(y'(x)^2)^{\frac{2}{2}}\right)}$

M点处切线为$\alpha=y't-y'x+y$

$C(\alpha,\beta)$

$\alpha=x-r\, sin\, arctan\, y'$

$\beta=y+r\, cos\, arctan\, y'$

$r=\frac{1}{k}$

$sin\, arctan\, y'=\frac{1}{\sqrt{1+(y)^2}}$

$cos\, arctan\, y'=\frac{y'}{\sqrt{1+(y')^2}}$

则$\begin{cases}\alpha=x-\frac{(1+(y'')^2)y'}{y''}\\ \beta=y+\frac{1+(y')^2}{y''}\end{cases}$

7.

解：$y=ln\, x$ 与x轴交点为$(1,0)$

$$y' = \frac{1}{x}, y'' = \frac{-1}{x^2}, k = \frac{1}{(2)^{\frac{3}{2}}} = \frac{\sqrt{2}}{4}$$

则$\rho = \dfrac{1}{k} = 2\sqrt{2}$

设圆心为(α, β)

$$a = x - \frac{(1 + (y')^2)y'}{y''} = 1 - \frac{2}{-1} = 3$$

$$\beta = y + \frac{1 + (y')^2}{y''} = \frac{2}{1} = -2$$

则方程为$(x - 3)^2 + (y + 2)^2 = 8$