

第 2 章复习题

1. 证明：反证法：

假设 $\{a_n + b_n\}$ 收敛

因为： $b_n = a_n + b_n - a_n$ 又 $\{a_n\}$ 收敛

则 $\{b_n\}$ 收敛，与 $\{b_n\}$ 发散矛盾

则假设不成立，即 $\{a_n + b_n\}$ 发散

$\{a_n b_n\}$ 不一定发散，如： $a_n = 0$, $b_n = n$, $a_n b_n = 0$, $\lim_{n \rightarrow \infty} a_n b_n = 0$

2. 不能，如： $a_n = n$, $b_n = -n$, $a_n + b_n = 0$, $\{a_n + b_n\}$ 收敛

$a_n = (-1)^n$, $b_n = (-1)^n$, $a_n b_n = 1$, $\{a_n b_n\}$ 收敛

3. 不能，如： $a_n = \frac{1}{\sqrt{n}}$, $b_n = n$, $\lim_{n \rightarrow \infty} a_n = 0$

但 $a_n b_n = \sqrt{n}$ 不收敛

所以 $\lim_{n \rightarrow \infty} a_n b_n$ 不存在

4. 不能，如： $a_n = 2$ (n 为奇), 0 (n 为偶)

$b_n = 0$ (n 为奇), 2 (n 为偶)

$\lim_{n \rightarrow \infty} a_n b_n = 0$, 但 $\lim_{n \rightarrow \infty} a_n$ 和 $\lim_{n \rightarrow \infty} b_n$ 都不存在

5.

(1)

$$a_n \geq \frac{1}{\sqrt{n^2+n}} + \frac{1}{\sqrt{n^2+n}} + \dots + \frac{1}{\sqrt{n^2+n}} = \frac{n}{\sqrt{n^2+n}}$$

$$a_n \leq \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+1}} + \dots + \frac{1}{\sqrt{n^2+1}} = \frac{n}{\sqrt{n^2+1}}$$

$$\text{又} \lim_{n \rightarrow \infty} \frac{n}{\sqrt{n^2+n}} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{\frac{1}{n}+1}} = 1$$

$$\lim_{n \rightarrow \infty} \frac{n}{\sqrt{n^2+1}} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{\frac{1}{n^2}+1}} = 1$$

则由夹逼定理知: $\lim_{n \rightarrow \infty} a_n = 1$

(2) 令 $\max\{A, B, C, D\} = a$ 则:

$$\sqrt[n]{a^n} \leq \sqrt[n]{A^n + B^n + C^n + D^n} \leq \sqrt[n]{4a^n}$$

$$a \leq \sqrt[n]{A^n + B^n + C^n + D^n} \leq a\sqrt[n]{4}$$

又 $\lim_{n \rightarrow \infty} (a\sqrt[n]{4}) = a$, 所以 $\lim_{n \rightarrow \infty} a_n = \max\{A, B, C, D\}$

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(1) 证明: 单调性: $\because 0 < a_1 < 1$

由数学归纳法知 $a_n > 0$

则 $a_{n+1} - a_n = -a_n^2 < 0$

$\therefore 0 < a_{n+1} < a_n$

则 $\{a_n\}$ 单调递减

有界性: $\because a_n > 0$

$\therefore \{a_n\}$ 收敛

$$\text{令} \lim_{n \rightarrow \infty} a_n = a, \text{则} \lim_{n \rightarrow \infty} a_{n+1} = \lim_{n \rightarrow \infty} a_n(1 - a_n)$$

$$\therefore a = a(1 - a)$$

$$\therefore a = 0$$

$$\text{则} \lim_{n \rightarrow \infty} a_n = 0$$

(2) 证明: 单调性: $a_1 = \sqrt{2}, a_2 = \sqrt{3 + 2\sqrt{2}}$, 则 $a_2 > a_1$

$$\text{设 } a_{k+1} > a_k, \text{则 } a_{k+2} = \sqrt{3 + 2a_{k+1}} > \sqrt{3 + 2a_k} = a_{k+1}$$

由数学归纳法知, $\{a_n\}$ 单调递增

有界性: $n=1, a_1=\sqrt{2}<3$
 假设 $n=k, a_k<3$

则 $n=k+1$ 时, $a_{k+1}=\sqrt{3+2a_k}<3$ 成立

$\therefore a_n<3$

$\therefore \{a_n\}$ 收敛

令 $\lim_{n \rightarrow \infty} a_n = a$, 则 $\lim_{n \rightarrow \infty} a_{n+1} = \lim_{n \rightarrow \infty} \sqrt{3+2a_n}$

$a=\sqrt{3+2a}$

有极限的保号性知 $a=3$

则 $\lim_{n \rightarrow \infty} a_n = 3$

7、求下列数列极限

$$(1) \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n-2}\right)^{n+1}$$

解: 原式 $= \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n-2}\right)^{-(n-2) \cdot \frac{n+1}{-(n-2)}} = e^{-1} \cdot 1^\infty$

$$(2) \lim_{n \rightarrow \infty} \left(\frac{1+n}{2+n}\right)^n$$

解: 原式 $= \lim_{n \rightarrow \infty} \left(1 + \frac{-1}{n+2}\right)^{-(n+2) \cdot \frac{n}{-(n+2)}} = e^{-1} \cdot 1^\infty$

$$(3) \lim_{n \rightarrow \infty} n \sin \frac{1}{n}$$

解: 原式 $= \lim_{n \rightarrow \infty} \frac{\sin \frac{1}{n}}{\frac{1}{n}} = 1$

$$(4) \lim_{n \rightarrow \infty} (\sqrt{n+2} - 2\sqrt{n+1} + \sqrt{n}) * \sqrt{n}$$

解: 原式 $= \lim_{n \rightarrow \infty} [(\sqrt{n+2} - \sqrt{n+1}) - (\sqrt{n+1} - \sqrt{n})] * \sqrt{n}$

$$= \lim_{n \rightarrow \infty} \left(\frac{\sqrt{n}}{\sqrt{n+2} + \sqrt{n+1}} - \frac{\sqrt{n}}{\sqrt{n+1} + \sqrt{n}} \right)$$

$$= \frac{1}{2} - \frac{1}{2}$$

$$= 0$$

$$(5) \lim_{n \rightarrow \infty} \tan^n \left(\frac{\pi}{4} + \frac{2}{n} \right)$$

解: 原式 $= \lim_{n \rightarrow \infty} \left(\frac{1 + \tan^2 \frac{2}{n}}{1 - \tan^2 \frac{2}{n}} \right)^n = \lim_{n \rightarrow \infty} \left(\frac{1 - \tan^2 \frac{2}{n} + 2 \tan^2 \frac{2}{n}}{1 - \tan^2 \frac{2}{n}} \right)^n$

$$\begin{aligned}
&= \lim_{n \rightarrow \infty} \left(1 + \frac{2 \tan^2 \frac{2}{n}}{1 - \tan^2 \frac{2}{n}} \right)^{\frac{1 - \tan^2 \frac{2}{n}}{2 \tan^2 \frac{2}{n}} \cdot \frac{2 \tan^2 \frac{2}{n}}{1 - \tan^2 \frac{2}{n}}} \\
&= e^{\lim_{n \rightarrow \infty} \left(\frac{2n \tan^2 \frac{2}{n}}{1 - \tan^2 \frac{2}{n}} \right)} = e^{\lim_{n \rightarrow \infty} \frac{2n \cdot \frac{2}{n}}{1 - \frac{2}{n}}} \\
&= e^{\frac{4}{1-0}} = e^4
\end{aligned}$$

$$(6) \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{n+k}{n^2+k}$$

解：令 $a_n = \sum_{k=1}^n \frac{n+k}{n^2+k}$ 由于 $\forall 1 \leq k \leq n$ 有 $\frac{n+k}{n^2+n} \leq \frac{n+k}{n^2+k} \leq \frac{n+k}{n^2+1}$

$$\text{则 } \sum_{k=1}^n \frac{n+k}{n^2+n} \leq a_n \leq \sum_{k=1}^n \frac{n+k}{n^2+1}$$

$$\text{即 } \frac{n^2 + \frac{n(n+1)}{2}}{n^2+n} \leq a_n \leq \frac{n^2 + \frac{n(n+1)}{2}}{n^2+1}$$

$$\text{因为 } \lim_{n \rightarrow \infty} \frac{n^2 + \frac{n(n+1)}{2}}{n^2+n} = \lim_{n \rightarrow \infty} \frac{n^2 + \frac{n(n+1)}{2}}{n^2+1}$$

$$\text{由夹逼定理知 } \lim_{n \rightarrow \infty} a_n = \frac{3}{2} \text{ 即 } \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{n+k}{n^2+k} = \frac{3}{2}$$

8、求下列函数极限

$$(1) \text{ 解：原式} = \lim_{n \rightarrow \infty} \frac{1 + \frac{1}{x}}{x + \sqrt{1 + \frac{1}{x^2}}} = \frac{1}{2}$$

$$(2) \text{ 解：原式} = \lim_{n \rightarrow \infty} \frac{\frac{(2x+3)^{20} \cdot (3x+2)^{30}}{(3x)^{20} \cdot (3x)^{30}}}{\frac{(2x+1)^{50}}{(3x)^{50}}} = \lim_{n \rightarrow \infty} \frac{\left(\frac{2}{3} + \frac{1}{x}\right)^{20} \cdot \left(1 + \frac{2}{3x}\right)^{30}}{\left(\frac{2}{3} + \frac{1}{3x}\right)^{50}} = \lim_{n \rightarrow \infty} \frac{\left(\frac{2}{3}\right)^{20} \cdot 1^{30}}{\left(\frac{2}{3}\right)^{50}} = \left(\frac{2}{3}\right)^{30}$$

(3)解：当 $m=n$ 时，原式=1

$$\text{当 } m > n \text{ 时，原式} = \lim_{x \rightarrow \infty} \frac{x^{\frac{1}{m} - \frac{1}{n}} - \frac{1}{\sqrt[n]{x}}}{1 - \frac{1}{\sqrt[n]{x}}} = \frac{0-0}{1-0} = 0$$

当 $m < n$ 时，原式=+∞

$$(4) \text{ 解：原式} = \lim_{x \rightarrow 0} \frac{\cos x - 1 + 1 \cos 3x}{x^2} = \lim_{x \rightarrow 0} \frac{\cos x - 1}{x^2} + \lim_{x \rightarrow 0} \frac{\cos 3x}{x^2} = \lim_{x \rightarrow 0} \frac{-\frac{1}{2}x^2}{x^2} + \lim_{x \rightarrow 0} \frac{9x^2}{x^2} = 4$$

(5)解：令 $t=1-x$ ，则当 $x \rightarrow 1$ 时， $t \rightarrow 0$

$$\text{原式} = \lim_{t \rightarrow 0} t \tan \frac{\pi}{2} (1-t) = \lim_{t \rightarrow 0} t \cos \frac{t}{2} = \lim_{t \rightarrow 0} t \frac{\cos \frac{\pi}{2} t}{\sin \frac{\pi}{2} t} = \lim_{t \rightarrow 0} t \frac{\cos \frac{\pi}{2} t}{\frac{\pi}{2} t} = \frac{2}{\pi}$$

$$(6) \text{ 解：原式} = \lim_{x \rightarrow 0} \frac{2+e^{\frac{1}{x}}}{1+e^{\frac{1}{x}}} - \lim_{x \rightarrow 0} \frac{\sin x}{x} = \frac{2+0}{1+0} - 1 = 1$$

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解: $\lim_{x \rightarrow 1} f(x) = \lim_{x \rightarrow 1} [3x^2 + 2x \lim_{x \rightarrow 1} f(x)] = 3 + 2 \lim_{x \rightarrow 1} f(x)$

可以解得 $\lim_{x \rightarrow 1} f(x) = -3$

代入原式得 $f(x) = 3x^2 - 6x$

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(1) 解: 显然 $3x$ 和 $\sqrt{ax^2 + bx + 1}$ 是同阶无穷大量

$$\therefore a = 9$$

$$\lim_{x \rightarrow \infty} \frac{-b - \frac{1}{x}}{3 + \sqrt{a + \frac{b}{x} + \frac{1}{x^2}}} = 2$$

$$\therefore b = -12$$

(2) 解: $x^2 + ax + b$ 和 $x - 1$ 是同阶无穷小

$$\text{当 } x \rightarrow 1, x^2 + ax + b \rightarrow 1 + a + b \rightarrow 0$$

$$\text{再洛必达 } 2x + a = 5, \therefore a = 3, b = -4$$

11.(1) 解: 因为 $\lim_{x \rightarrow 0} \frac{f(x)}{1 - \cos x} = \lim_{x \rightarrow 0} \frac{f(x)}{\frac{1}{2}x^2}$

$$\text{则 } \lim_{x \rightarrow 0} \frac{f(x)}{x^2} = 2$$

$$\text{所以 } \lim_{x \rightarrow 0} \left(1 + \frac{f(x)}{x}\right)^{\frac{1}{x}} = \lim_{x \rightarrow 0} \left(1 + \frac{f(x)}{x}\right)^{\frac{x}{f(x)} \times \frac{f(x)}{x^2}} = \lim_{x \rightarrow 0} e^{\frac{f(x)}{x^2}} = e^2$$

(2) 解: 因为 $\sqrt{1 + f(x) \sin x^2} - 1 \rightarrow 0 \quad (x \rightarrow 0)$

$$f(x) \sin x^2 \rightarrow 0 \quad (x \rightarrow 0)$$

$$\text{所以 } \lim_{x \rightarrow 0} \frac{\sqrt{1 + f(x) \sin x^2} - 1}{1 - \cos x} = \lim_{x \rightarrow 0} \frac{\frac{1}{2} f(x) \sin x^2}{\frac{1}{2} x^2} = \lim_{x \rightarrow 0} \frac{f(x) x^2}{x^2} = 3$$

$$\text{即 } \lim_{x \rightarrow 0} f(x) = 3$$

12.(1) 解: $\lim_{x \rightarrow 0^+} f(x) = \frac{1}{x+1} = 1$

$$\lim_{x \rightarrow 0^-} f(x) = -\frac{1}{x+1} = -1$$

$x=0$ 为第一类间断点中的跳跃间断点

$$\lim_{x \rightarrow 1^+} f(x) = \frac{1}{x+1} = \frac{1}{2}$$

$$\lim_{x \rightarrow 1^-} f(x) = \frac{1}{x+1} = \frac{1}{2}$$

而 $f(1) = 0$, $x=1$ 为第一类间断点中的可去间断点
同理, $x = -1$ 为第一类间断点中的可去间断点

(2)解: 当 $x=k\pi$ 时, $\sin x = 0$

$$\lim_{x \rightarrow 2k\pi^+} f(x) = 1$$

$$\lim_{x \rightarrow 2k\pi^-} f(x) = -1$$

$x=2k\pi$ 为第一类间断点中的跳跃间断点
同理, $x = (2k+1)\pi$ 为第一类间断点中的跳跃间断点

13.证: 反证法: 假设 $f(x)$ 在 R 上无界

① $f(x)$ 在 $x=x_0$ 时, 有 $\lim_{x \rightarrow x_0} f(x) = \infty$

则 $x=x_0$ 是 $f(x)$ 的无穷间断点
不满足连续条件

② $f(x)$ 在 $x \rightarrow \infty$ 时, 有 $\lim_{x \rightarrow \infty} f(x) = \infty$

则 $f(x)$ 不满足周期条件
故 $f(x)$ 有界

14.把分段点找到, 令其左右相等即可

① 当 $|x| < 1$ 时, $\lim_{x \rightarrow \infty} |x^n| = 0$

$$f(x) = ax^2 + bx$$

② 当 $|x|=1$ 时, $f(x) = a+b|x|$

③ 当 $|x| > 1$ 时, $f(x) = \frac{1}{x}$

$$\text{即 } f(x) = \begin{cases} \frac{1}{x}, & x < -1 \\ a-b, & x = -1 \\ ax^2 + bx, & -1 < x < 1 \\ a+b, & x = 1 \\ \frac{1}{x}, & x > 1 \end{cases}$$

任意取分段点左右极限相等, 联立方程
这里取 -1 和 1

$$\begin{cases} \lim_{x \rightarrow -1^-} f(x) = f(-1) \\ \lim_{x \rightarrow 1^+} f(x) = f(1) \end{cases}$$

$$\text{则} \begin{cases} a - b = -1 \\ a + b = 1 \end{cases} \text{解得: } \begin{cases} a = 0 \\ b = 1 \end{cases}$$

15.

证：因为 $f(x+y) = f(x) + f(y)$ ，令 $y = \Delta x$ 且 $\Delta x \rightarrow 0$

$$\text{原式} = f(x + \Delta x) = f(x) + f(\Delta x)$$

$$\text{两边同时取极限: } \lim_{\Delta x \rightarrow 0} f(x + \Delta x) = \lim_{\Delta x \rightarrow 0} f(x) + \lim_{\Delta x \rightarrow 0} f(\Delta x)$$

$$\text{又因为 } \lim_{\Delta x \rightarrow 0} f(\Delta x) = 0, \lim_{\Delta x \rightarrow 0} f(x) = f(x)$$

$$\text{所以 } \lim_{\Delta x \rightarrow 0} f(x + \Delta x) = f(x)$$

则 $f(x)$ 连续

16.

设 $f(x)$ 在 $(0, +\infty)$ 上连续，且满足 $f(x^2) = f(x), x \in (0, +\infty)$. 证明 $f(x)$ 在 $(0, +\infty)$ 上为常值函数.

$$\text{证明: } \because f(x^2) = f(x)$$

$$\therefore f(x) = f(x^{\frac{1}{2}}) = f(x^{\frac{1}{4}}) = f(x^{\frac{1}{8}}) = \dots = f(x^{\frac{1}{2^n}})$$

$$\because n \rightarrow \infty \text{ 时, } \lim_{n \rightarrow \infty} x^{\frac{1}{2^n}} = 1$$

$$\therefore \text{当 } n \rightarrow \infty \text{ 时, } f(x) = f(x^{\frac{1}{2}}) = f(x^{\frac{1}{4}}) = f(x^{\frac{1}{8}}) = \dots = f(x^{\frac{1}{2^n}}) = f(\lim_{n \rightarrow \infty} x^{\frac{1}{2^n}}) = f(1)$$

$$\therefore f(x) = f(1) \text{ 为常值函数}$$

17.

设 $f(x)$ 在 $[a, b]$ 上有定义，满足 $a \leq f(x) \leq b, x \in [a, b]$ ，假设存在常数 $L \in [0, 1]$ ，使得任意 $x', x'' \in [a, b], |f(x') - f(x'')| \leq L|x' - x''|$.

试证明：(1) $f(x)$ 在 $[a, b]$ 上连续。

(2) 存在唯一 $\xi \in [a, b]$ ，使得 $f(\xi) = \xi$

(3) 对于任意的 $x_1 \in [a, b]$ ，定义迭代序列 $x_{n+1} = f(x_n), n = 1, 2, 3, \dots, \lim_{n \rightarrow \infty} x_n = \xi$

证明:

$$(1) \because |f(x') - f(x'')| \leq L|x' - x''|, \text{不妨令 } x'' = x_0, x' \rightarrow x_0 \in [a, b]$$

$$\therefore L|x' - x_0| \rightarrow 0$$

$$\text{又} \because |f(x') - f(x_0)| \geq 0, \lim_{x' \rightarrow x_0} |f(x') - f(x_0)| \leq \lim_{x' \rightarrow x_0} L|x' - x_0| = 0$$

$$\therefore \lim_{x' \rightarrow x_0} |f(x') - f(x_0)| = 0, \therefore \lim_{x' \rightarrow x_0} f(x') = f(x_0)$$

故连续.

$$(2) \text{根据题意设 } F(x) = f(x) - x,$$

$$\because a \leq f(x) \leq b, \text{所以 } F(a) \geq 0, F(b) \leq 0$$

并且 $F(a) \cdot F(b) \leq 0$, 由零点存在定理: 必存在唯一 $\xi \in [a, b]$, 使得 $F(\xi) = 0$,

$$\therefore f(\xi) = \xi$$

$$(3) \text{由题意: } |f(x_n) - f(\xi)| \leq L|x_n - \xi|$$

$$\text{由 (2): } f(\xi) = \xi, \therefore |f(x_n) - \xi| \leq L|x_n - \xi|$$

$$\therefore |x_{n+1} - \xi| \leq L|x_n - \xi|, \text{设数列 } x_n \text{ 存在且为 } A, \text{则有 } \lim_{n \rightarrow \infty} x_n = A$$

$$\therefore \text{当 } n \rightarrow \infty \text{ 时, } |A - \xi| \leq L|A - \xi|, \text{又因为 } L \neq 1, \text{则 } A = \xi$$

$$\text{故假设成立, } \lim_{n \rightarrow \infty} x_n = A = \xi$$

证毕.

18.

函数 $f(x)$ 在 $[0, 1]$ 上连续, $f(0) = f(1)$, 证明: 对于任意的自然数 $n \geq 2$, 存在 ξ_n , 使得 $f(\xi_n) = f(\frac{1}{n} + \xi_n)$.

$$\text{证: 令 } F(x) = f(x) - f\left(\frac{1}{n} + x\right) \because f(x) \text{ 在 } x \in [0, 1] \text{ 上连续, } \therefore \frac{1}{n} + x \in [0, 1]$$

$$\therefore F(x) = f(x) - f\left(\frac{1}{n} + x\right) \text{ 在 } [0, 1 - \frac{1}{n}] \text{ 上连续, 即 } F(x) \text{ 在 } [0, \frac{n-1}{n}] \text{ 上连续}$$

$$\because \text{函数 } F(x) \text{ 连续, } \therefore \text{由连续函数性质: 存在 } m, M \text{ 满足 } m \leq F(x) \leq M,$$

$$\text{令 } x = \frac{k}{n}, k = 0, 1, 2, 3, \dots, n-1 \text{ (注意自变量范围),}$$

$$\therefore m \leq F\left(\frac{0}{n}\right) \leq M$$

$$m \leq F\left(\frac{1}{n}\right) \leq M$$

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$$m \leq F\left(\frac{n-1}{n}\right) \leq M$$

将n个式子相加, 得 $nm \leq F\left(\frac{0}{n}\right) + F\left(\frac{1}{n}\right) + \cdots + F\left(\frac{n-1}{n}\right) \leq nM$

$$\therefore m \leq \frac{\sum_{k=0}^{n-1} F\left(\frac{k}{n}\right)}{n} \leq M, \text{ 由介值定理, 存在 } \xi_n \in [0, 1], \text{ 使得 } \frac{\sum_{k=0}^{n-1} F\left(\frac{k}{n}\right)}{n} = F(\xi_n)$$

展开:

$$nF(\xi_n) = F\left(\frac{0}{n}\right) + F\left(\frac{1}{n}\right) + \cdots + F\left(\frac{n-1}{n}\right)$$

$$= f(0) - f\left(\frac{1}{n} + 0\right) + f\left(\frac{1}{n}\right) - f\left(\frac{1}{n} + \frac{1}{n}\right) + f\left(\frac{2}{n}\right) - f\left(\frac{1}{n} + \frac{2}{n}\right) \cdots + f\left(\frac{n-1}{n}\right) - f\left(\frac{1}{n} + \frac{n-1}{n}\right)$$

$$= f(0) - f\left(\frac{1}{n}\right) + f\left(\frac{1}{n}\right) - f\left(\frac{2}{n}\right) + f\left(\frac{2}{n}\right) - f\left(\frac{3}{n}\right) \cdots + f\left(\frac{n-1}{n}\right) - f\left(\frac{n}{n}\right)$$

$$= f(0) - f(1) = 0$$

$$\therefore nF(\xi_n) = 0, \quad F(\xi_n) = 0, \quad \text{即 } F(\xi_n) = f(\xi_n) - f\left(\frac{1}{n} + \xi_n\right) = 0,$$

$$\therefore f(\xi_n) = f\left(\frac{1}{n} + \xi_n\right).$$

证毕.

19.

对于任意的x, 函数满足 $f(x) = f(2x)$, 且 $f(x)$ 在 $x = 0$ 处连续, 证明: $f(x)$ 为常值函数

证明: $\because f(x) = f(2x)$

$$\therefore f(x) = f\left(\frac{x}{2}\right) = f\left(\frac{x}{4}\right) = f\left(\frac{x}{8}\right) = \cdots = f\left(\frac{x}{2^n}\right)$$

$$\text{当 } n \rightarrow \infty \text{ 时, } \lim_{n \rightarrow \infty} \frac{x}{2^n} = 0$$

$$\therefore \text{当 } n \rightarrow \infty \text{ 时, } f(x) = f\left(\frac{x}{2}\right) = f\left(\frac{x}{4}\right) = f\left(\frac{x}{8}\right) = \cdots = f\left(\frac{x}{2^n}\right) = f\left(\lim_{n \rightarrow \infty} \frac{x}{2^n}\right) = f(0)$$

$\because f(x)$ 在 $x = 0$ 处连续, $\therefore f(0)$ 存在

故 $f(x) = f(0)$ 恒为常数.

20.

对于任意的 x , 总有 $\varphi(x) \leq f(x) \leq \psi(x)$, 且 $\lim_{x \rightarrow \infty} (\varphi(x) - \psi(x)) = 0$. 问: 极限 $\lim_{x \rightarrow \infty} f(x)$ 是否存在, 给出理由.

解: $\lim_{x \rightarrow \infty} f(x)$ 不一定存在, 理由如下

$\because \lim_{x \rightarrow \infty} (\varphi(x) - \psi(x)) = 0$ 不等价于 $\varphi(x)$, $\psi(x)$ 极限存在,

\therefore 不满足夹逼准则, $\lim_{x \rightarrow \infty} f(x)$ 不一定存在

反例如: $\varphi(x) = \sin x - \frac{1}{x}$, $f(x) = \sin x$, $\psi(x) = \sin x + \frac{1}{x}$