习题 5.2

1、计算下列不定积分

(1)、$\int \frac{dx}{(3-2x)^2}$

$=-\frac{1}{2}\int \frac{1}{(3-2x)^2}d(3-2x)$

$=\frac{1}{2}\int \left(\frac{1}{3-2x}\right)^1 d(3-2x)$

$=\frac{1}{2}(3-2x)^{-1}+C$

(2)、$\int \tan(5x-3)\,dx$

$=\frac{1}{5}\int \frac{\sin(5x-3)}{\cos(5x-3)}d(5x-3)$

$=-\frac{1}{5}\int \frac{1}{\cos(5x-3)}d[\cos(5x-3)]$

$=-\frac{1}{5}\int \ln|\cos(5x-3)|+C$

(3)、$\int x^3\,e^{-x^4}dx$

$=-\frac{1}{4}\int e^{-x^4}\,d(-x^4)$

$=-\frac{1}{4}e^{-x^4}+C$

(4)、$\int \frac{dx}{x\ln x}$

$=\frac{1}{\ln x}d\ln x$

$=\ln|\ln x|+C$

(5)、$\int \frac{\cos x-\sin x}{\sin x+\cos x}dx$

$=\int \frac{1}{\sin x+\cos x}d(\sin x+\cos x)$

$=\ln|\sin x+\cos x|+C$

(6)、$\int \frac{1}{x^2}a^{\frac{1}{x}}dx$

$=-\int -\frac{1}{x^2}a^{\frac{1}{x}}dx$

$=-\int a^{\frac{1}{x}}d\frac{1}{x}$

$=-\frac{1}{\ln a}\int \ln a\, a^{\frac{1}{x}}d\frac{1}{x}$

$=-\frac{1}{\ln a}a^{\frac{1}{x}}+C$

(7)、$\int \frac{x^3}{\sqrt[3]{x^4+1}}dx$

$=\frac{1}{4}\int \frac{4x^3}{\sqrt[3]{x^4+1}}dx$

$=\frac{1}{4}\int \frac{1}{(x^4+1)^{\frac{1}{3}}}d(x^4+1)$

$=\frac{3}{8}(x^4+1)^{\frac{2}{3}}+C$

(8)、$\int \frac{f\prime(x)}{\sqrt{f(x)}}dx$

$=\int[f(x)]^{\frac{-1}{2}}df(x)$

$=2\sqrt{f(x)}+C$

(9)、$\int \frac{1}{\sqrt{\tan x}\cdot cos^2x}dx$

$=\int \frac{1}{\sqrt{\tan x}}d\tan x$

$=\int(\tan x)^{\frac{-1}{2}}d\tan x$

$=2\int \frac{1}{2}(\tan x)^{\frac{-1}{2}}d\tan x$

$=2\sqrt{\tan x}+C$

(10)、$\int \frac{1}{\sqrt{1-x^2}(arcsinx)^2}dx$

$=\int(arcsinx)^{-2}\,d\,arcsinx$

$=-\frac{1}{arcsinx}+C$

(11)、$\int \frac{\cos\sqrt{x}}{\sqrt{x}}dx$

$=2\int \frac{1}{2}\frac{\cos\sqrt{x}}{\sqrt{x}}dx$

$=2\int \cos\sqrt{x}\,d\sqrt{x}$

$=2\sin\sqrt{x}+C$

(12)、$\int \frac{x}{4+x^4}dx$

$=-\frac{1}{4}\int \frac{-4x}{4+x^4}dx$

$=-\frac{1}{4}\int \frac{1}{x^2+2x+2}-\frac{1}{x^2-2x+2}dx$

$=-\frac{1}{4}\int \frac{1}{x^2+2x+2}dx+\frac{1}{4}\int \frac{1}{x^2-2x+2}dx$

$=-\frac{1}{4}\int \frac{1}{(x+1)^2+1}d(x+1)+\frac{1}{4}\int \frac{1}{(x-1)^2+1}d(x-1)$

$=-\frac{1}{4}arc\tan(x+1)+\frac{1}{4}arc\tan(x-1)+C$

(13)、$\int sin^3x\,dx$

$=-\int sin^2x\,d\cos x$

$=-\int(1-cos^2x)\,d\cos x$

$=-\left(\cos x-\frac{1}{3}cos^3x\right)+C$

$=\frac{1}{3}cos^3x-\cos x+C$

(14)、$\int(x^2-3x+1)^{10}(2x-3)dx$

$=\int(x^2-3x+1)^{10}d(x^2-3x+1)$

$=\frac{1}{11}(x^2-3x+1)^{11}+C$

(15)、$\int \frac{1}{x^2}\cot\frac{1}{x}dx$

$=-\int \frac{\cos\frac{1}{x}}{\sin\frac{1}{x}}d\frac{1}{x}$

$=-\int \frac{1}{\sin\frac{1}{x}}d\frac{1}{\sin\frac{1}{x}}$

$=\ln\left|\sin\frac{1}{x}\right|+C$

(16)、$\int sin^5\,xcos^3xdx$

$=\frac{1}{2}\int 2\sin x\cos x\cdot sin^4xcos^2xdx$

$=\frac{1}{2}\int \sin 2x\left(\frac{1-\cos 2x}{2}\right)^2\frac{1+\cos 2x}{2}dx$

$=-\frac{1}{4}\int -2\sin 2x\left(\frac{1-\cos 2x}{2}\right)^2\frac{1+\cos 2x}{2}dx$

$=-\frac{1}{4}\int \frac{1}{8}(1-\cos 2x)^2(1+\cos 2x)d\cos 2x$

$=-\frac{1}{32}\int(1-\cos 2x-cos^2 2x+cos^3 2x)\, d\cos 2x$

$=-\frac{1}{32}cos2x+\frac{1}{64}cos^2 2x+\frac{1}{96}cos^3 2x-\frac{1}{128}cos^4 2x+C$

(17)、$\int\frac{1}{x(x^8+1)}dx$

$=\int(\frac{1}{x}-\frac{x^7}{x^8+1})dx$

$=\int\frac{1}{x}dx-\int\frac{x^7}{x^8+1}dx$

$=\ln|x|-\frac{1}{8}\int\frac{8x^7}{x^8+1}dx$

$=\ln|x|-\frac{1}{8}\int\frac{1}{x^8+1}d(x^8+1)$

$=\ln|x|-\frac{1}{8}\ln(x^8+1)+C$

(18)、$\int\frac{x^2+1}{x^4+1}dx$

$=\frac{1}{2}\int\left(\frac{1}{x^2+\sqrt{2}x+1}+\frac{1}{x^2-\sqrt{2}x+1}\right)dx$

$=\frac{1}{2}\int\left(\frac{1}{(x+\frac{\sqrt{2}}{2})^2+\frac{1}{2}}+\frac{1}{(x-\frac{\sqrt{2}}{2})^2+\frac{1}{2}}\right)dx$

$=\frac{1}{2}\times 2\int\left(\frac{1}{(\sqrt{2}x+1)^2+1}+\frac{1}{(\sqrt{2}x-1)^2+1}\right)dx$

$=\frac{1}{\sqrt{2}}arctan\left(\sqrt{2}x+1\right)+\frac{1}{\sqrt{2}}arctan\left(\sqrt{2}x-1\right)+C$

(19)、$\int\frac{\ln(x+1)-\ln x}{x(x+1)}dx$

$=\int\frac{\ln\frac{x+1}{x}}{x(x+1)}dx$

$=-\int\ln\frac{x+1}{x}\cdot\left[\frac{-1}{x(x+1)}\right]dx$

$=-\int\ln\frac{x+1}{x}\,d\ln\frac{x+1}{x}$

$=-\frac{1}{2}\left(\ln\frac{x+1}{x}\right)^2+C$

2、计算下列定积分

(1)、$\int\frac{1}{x^2(1-x^2)^{\frac{3}{2}}}dx$

令$x=\sin t$,t∈$\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$，$\therefore$ *原式*$=\int\frac{1}{sin^2 t\, cos^3 t}\cdot\cos t\, dt$

$$=\int \frac{1}{sin^2 t\, cos^2 t} dt$$

$$=\int \frac{dt}{sin^2 t} + \int \frac{dt}{cos^2 t}$$

$$=-\cot t + \tan t + C$$

因为$x = sint$,所以$cost = \sqrt{1-x^2}$. 所以原式$= -\frac{\sqrt{1-x^2}}{x} + \frac{x}{\sqrt{1-x^2}} + C$

(2)、$\int \frac{\sqrt{x^2-1}}{x^3} dx$

令$x = \frac{1}{cost}$,$\cos t = \frac{1}{x}$,$\sin t = \sqrt{1-\frac{1}{x^2}}$

原式$= \int \frac{\tan t}{\frac{1}{cos^3 t}} \cdot \frac{\sin t}{\cos t} dt$

$=\int sin^2 tdt = \int \frac{1-\cos 2t}{2} dt = \int 1dt - \int \frac{\cos 2t}{2} dt$

$=t - \frac{1}{4}\sin 2t + C$

$=arccos\frac{1}{x} - \frac{2}{x}\sqrt{1-\frac{1}{x^2}} + C$

(3)、$\int \frac{1}{x\sqrt{x^2-1}} dx$

令$x = \frac{1}{\cos t}$,则原式$= \int \frac{1}{\frac{1}{\cos t}\cdot\frac{\sin t}{\cos t}} \cdot \frac{\sin t}{cos^2 t} dt$

$$=\int 1dt = t + C = arccos\left|\frac{1}{x}\right| + C$$

(4)、$\int \frac{1}{(x^2+a^2)^2} dx$

设$x = a\tan t$,原式$= \int \frac{\frac{a}{cos^2 t}}{\left(\frac{a^2 sin^2 t}{cos^2 t}+a^2\right)^2} dt$

$$=\int \frac{cos^2 t}{a^3} dt = \frac{1}{a^3}\int cos^2 tdt = \frac{1}{a^3}\int \frac{1+\cos 2t}{2} dt$$

$$=\frac{1}{2a^3}\int 1dt + \frac{1}{2a^3}\int \cos 2tdt$$

$$=\frac{t}{2a^3} + \frac{\sin 2t}{4a^3} + C$$

$$=\frac{arctan\frac{x}{a}}{2a^3} + \frac{8xa^4}{x^2+a^2} + C$$

(5)、$\int \frac{dx}{\sqrt{3+2x-x^2}} = \int \frac{dx}{\sqrt{4-(x-1)^2}} = \int \frac{1}{2\sqrt{1-\left(\frac{x-1}{2}\right)^2}} dx$

$=arcsin\left(\frac{x-1}{2}\right) + C$

(6)、$\int \frac{x^2}{\sqrt{9-x^2}} dx$

设$x=3sint$,则原式$=\int\frac{9sin^2t}{\sqrt{9(1-sin^2t)}}\cdot 3costdt$

$$=\int\frac{9sin^2t\cdot 3cost}{3cost}dt=\int 9\,sin^2tdt$$

$$=9\int\frac{1-cos2t}{2}dt=\frac{9}{2}(\int 1dt-\int cos2tdt)$$

$$=\frac{9}{2}t-\frac{9}{4}sin2t+C=\frac{9}{2}t-\frac{x\sqrt{9-x^2}}{2}+C$$

因为$sint=\frac{x}{3}$,所以$cost=\frac{\sqrt{9-x^2}}{3}$

原式$=\frac{9}{2}arcsin\frac{x}{3}-\frac{x\sqrt{9-x^2}}{2}+C$

(7)、$\int\frac{\sqrt{x^2-4}}{x}dx$

令$x=\frac{2}{cost}$,则原式$=\int\frac{\sqrt{4\left(\frac{1}{cos^2t}-1\right)}}{\frac{2}{cost}}\cdot\frac{2sint}{cos^2t}dt$

$$=\int 2tan^2tdt=\int 2\frac{sin^2t}{cos^2t}dt=2\int\frac{1}{cos^2t}dt-2\int 1dt$$

$$=2tant-2t+C$$

因为$cost=\frac{2}{x}$,所以$sint=\frac{\sqrt{x^2-2}}{x}$,所以原式$=\frac{2\sqrt{x^2-2}}{x}\times\frac{x}{2}-2arccos\left|\frac{2}{x}\right|+C$

$$=\sqrt{x^2-2}-2arccos\left|\frac{2}{x}\right|+C$$

(8)、$\int\frac{x^3}{(1+x^2)^{\frac{3}{2}}}dx$

令$x=tanx$,$t\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$,则原式$=\int\frac{tan^3t}{\frac{1}{cos^3t}}\cdot\frac{1}{cos^2t}dt=\int\frac{sin^3t}{cos^3t}dt$

$$=\int\frac{sint(1-cos^2t)}{cos^3t}dt=\int\frac{sint}{cost}dt-\int sintdt$$

$$=\frac{1}{cost}+cost+C$$

因为$tanx=x$,所以$cost=\frac{1}{\sqrt{x^2+1}}$,所以原式$=\sqrt{x^2+1}+\frac{1}{\sqrt{x^2+1}}+C$

(9)、$\int\frac{1}{(1-x^2)^{\frac{3}{2}}}dx$

令$x=sint$,$t\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$,则原式$=\int\frac{1}{cos^3t}\cdot costdt=\int\frac{1}{cos^2t}dt=tant+C$

因为$sint=x$,所以$cost=\sqrt{1-x^2}$,所以原式$=\frac{x}{\sqrt{1-x^2}}+C$

(10)、$\int\frac{1}{(a^2+x^2)^{\frac{3}{2}}}dx$

设$x=atant$,$t\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$,所以原式$=\int\frac{1}{\left(a^2+a^2\frac{sin^2t}{cos^2t}\right)^{\frac{3}{2}}}\cdot\frac{a}{cos^2t}dt$

$$=\frac{1}{a^2}\int costdt=\frac{1}{a^2}sint+C$$

因为$tant=\frac{x}{a}$,$所以sint=\frac{x}{\sqrt{x^2+a^2}}$,$原式=\frac{1}{a^2}\frac{x}{\sqrt{x^2+a^2}}+C$

(11)、$\int\frac{1}{x^2\sqrt{x^2+9}}dx$

令$x=3tant$,$t\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$,$则原式=\int\frac{1}{9tan^2t\cdot\frac{3}{cost}}\cdot 3\frac{1}{cos^2t}dt$

$$=\int\frac{1}{9tan^2t\cdot cost}dt$$

$$=\frac{1}{9}\int\frac{cost}{sin^2t}dt=-\frac{1}{9}\frac{1}{sint}+C$$

因为$tant=\frac{x}{3}$,$所以sint=\frac{x}{\sqrt{x^2+9}}$

所以原式=$-\frac{1}{9}\frac{\sqrt{x^2+9}}{x}+C$