习题 2.2

1. 证明：

(1)对于$\forall\varepsilon > 0$，要使$|(2x+1)-3| < \varepsilon$

只需 $2|x-1| < \varepsilon$

即$|x-1| < \frac{\varepsilon}{2}$

令$\delta = \frac{\varepsilon}{2}$，则当$|x-1| < \delta$时

恒有$|(2x+1)-3| < \varepsilon$

$\therefore \lim_{x\to1}(2x+1) = 3$

(2)对于$\forall\varepsilon > 0$，要使$|(3x+1)-7| < \varepsilon$

只需 $3|x-2| < \varepsilon$

即$|x-2| < \frac{\varepsilon}{3}$

令$\delta = \frac{\varepsilon}{3}$，则当$|x-2| < \delta$时

恒有$|(3x+1)-7| < \varepsilon$

$\therefore \lim_{x\to2}(3x+1) = 7$

(3)对于$\forall\varepsilon \geq 0$，要使$|sin\,x - \ sin\,x_0| < \varepsilon$

即 $2\left|\sin\frac{x-x_0}{2}\cos\frac{x+x_0}{2}\right| < \varepsilon$

$\Leftarrow 2\left|\sin\frac{x-x_0}{2}\right| < \varepsilon$

$\Leftarrow |x-x_0| < \varepsilon$

令$\delta = \varepsilon,$则当$|x-x_0| < \delta$时

恒有$|sin\,x - \ sin\,x_0| < \varepsilon$

即$\lim_{x\to0}\sin x = \sin x_0$

2. 证明

(1) $\lim\limits_{x\to 0}[x]=0$

对于$\forall\varepsilon>0$，取δ为ε，则当 $0<x<\delta$时

有$|[x]-0|=0<\delta=\varepsilon$

$\therefore\lim\limits_{x\to 0^+}[x]=0$

(2) $\lim\limits_{x\to\infty}[x]=-1$

$\because$ 当$x\in[-1,0)$时，$[x]=-1$

$\therefore$ 对于$\forall\varepsilon>0$，取δ为ε，则当$-\delta<x<0$ 时

有$|[x]+1|=0<d=$

$\therefore\lim\limits_{x\to\infty}[x]=-1.$

(3) $\lim\limits_{x\to 0^+}x\,sgnx=0$

$\because$ 当$x>0$ 时,$xsgnx=x$

$\therefore$ 对于$\forall\varepsilon>0$，取δ为ε，则当 $0<x<\delta$时

有$|x\operatorname{sgn}x-0|=|x|<\delta=\varepsilon$

$\therefore\lim\limits_{x\to 0^+}x\,sgnx=0$

(4) $\lim\limits_{x\to 0^-}x\,sgnx=0$

$\because$ 当$x<0$ 时,$xsgnx=-x$

$\therefore$ 对于$\forall\varepsilon>0$，取δ为ε，则当$-\delta<x<0$ 时

有$|x\operatorname{sgn}x-0|=-x<\delta=\varepsilon$

$\therefore\lim\limits_{x\to 0^-}x\,sgnx=0$

3.

(1)解：对于$\forall\varepsilon>0$，取$X=\dfrac{1}{\sqrt{\varepsilon}}$，则当$|x|>X$时，

$$\left|\frac{x^2+1}{x^2+2}-1\right| = \left|\frac{1}{x^2+2}\right| = \frac{1}{x^2+2} < \frac{1}{X^2} = \varepsilon.$$

$$\therefore \lim_{x\to\infty}\frac{x^2+1}{x^2+2} = 1.$$

(2)解：对于$\forall\varepsilon > 0$，取$X = \frac{1}{\sqrt{\varepsilon}}$，则当$|x| > X$时，

$$\left|\frac{1}{x^2+1}-0\right| = \left|\frac{1}{x^2+1}\right| = \frac{1}{x^2+1} < \frac{1}{X^2} = \varepsilon.$$

$$\therefore \lim_{x\to\infty}\frac{1}{x^2+1} = 0.$$

(3)解：$\because \left|\left|\sqrt{x^2+1}-x\right|-0\right| = \frac{1}{\sqrt{x^2+1}+x}$

当$x\to\infty$时，不妨设$x > 1$，有$\sqrt{x^2+1}+x > x$

$$\therefore \left|\left|\sqrt{x^2+1}-x\right|-0\right| < \frac{1}{x}$$

对于$\forall\varepsilon > 0$，可取$X = \max\left\{1, \frac{1}{\varepsilon}\right\}$

只要$x > X$时，就有$\left|\left|\sqrt{x^2+1}-x\right|-0\right| < \frac{1}{x} < \frac{1}{X} = \varepsilon$

$$\therefore \lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right) = 0$$

(4) $\because \left|\frac{\sqrt{x+2}-\sqrt{3}}{x-1}\right| = \left|\frac{x+2-3}{(x-1)\left(\sqrt{x}+\sqrt{3}\right)}\right| = \frac{1}{\sqrt{x+2}+\sqrt{3}}$

$$< \frac{1}{\sqrt{x+2}} < \frac{1}{\sqrt{x}}$$

对于$\forall\varepsilon > 0$，可取$X = \frac{1}{\varepsilon^2}$

只要$x > X$时，就有$\left|\frac{\sqrt{x+2}-\sqrt{3}}{x-1}\right| < \frac{1}{\sqrt{x}} < \frac{1}{\sqrt{X}} = \varepsilon$

$\therefore \lim_{x \to \infty} \frac{\sqrt{x+2}-\sqrt{3}}{x-1} = 0$

4.

解：由题意$f(x) = \begin{cases} 2, & x > 0 \\ 0, & x < 0 \end{cases}$

$\therefore \lim_{x \to 0^+} f(x) = 2,$

$\lim_{x \to 0^-} f(x) = 0$

$\because \lim_{x \to 0^+} f(x) \neq \lim_{x \to 0^-} f(x)$

$\therefore \lim_{x \to 0} f(x)$不存在

$\lim_{x \to +\infty} f(x) = 2,$

$\lim_{x \to -\infty} f(x) = 0$

$\because \lim_{x \to -\infty} f(x) \neq \lim_{x \to +\infty} f(x)$

$\therefore \lim_{x \to \infty} f(x)$不存在

5.

(1)解：$\frac{x+1}{x^2+2} = \frac{1+1}{1+2} = \frac{2}{3}$

(2)解：$\lim_{x \to -1} \frac{x^3+1}{x^2+2} = \frac{-1+1}{1+2} = 0$

(3)解：$\lim_{x \to 2} \frac{x^2-4}{x-2} = \lim_{x \to 2}(x+2) = 4$

(4)解：$\lim_{x \to -2} \frac{x^2-4}{x+2} = \lim_{x \to -2}(x-2) = -4$

(5)解：$\lim_{x \to \infty} \frac{x^3+x+1}{x^3+2x+1} = \lim_{x \to \infty} \frac{1+\frac{1}{x^2}+\frac{1}{x^3}}{1+\frac{2}{x^2}+\frac{1}{x^3}} = \frac{\lim_{x \to \infty}\left(1+\frac{1}{x^2}+\frac{1}{x^3}\right)}{\lim_{x \to \infty}\left(1+\frac{2}{x^2}+\frac{1}{x^3}\right)} = 1$

(6)解： $\lim\limits_{x\to\infty}\dfrac{x^3+1}{x^4+1}=\lim\limits_{x\to\infty}\dfrac{\frac{1}{x}+\frac{1}{x^4}}{1+\frac{1}{x^4}}=0$

(7)解： $\lim\limits_{x\to+\infty}\dfrac{x+2}{x+1}=1$

(8)解： $\lim\limits_{x\to-\infty}\dfrac{x^2+2}{x^2+1}=1$

(9)解： $\lim\limits_{x\to+\infty}\left(\dfrac{2x+1}{x+2}\right)^{\frac{\sin x}{x}}$

$\because \lim\limits_{x\to+\infty}\left(\dfrac{2x+1}{x+2}\right)=2$， $\lim\limits_{x\to\infty}\dfrac{\sin x}{x}=0$

($\sin x$是有界变量，$\dfrac{1}{x}$ 是无穷小量，无穷小量与有界变量的乘积是无穷小量)

$\therefore \lim_{x\to+\infty}\left(\dfrac{2x+1}{x+2}\right)^{\frac{\sin x}{x}}=2^0=1$

(10) $\lim\limits_{x\to a}\dfrac{\sqrt[3]{x}-\sqrt[3]{a}}{\sqrt[3]{x-a}}=\lim\limits_{x\to a}\dfrac{\dfrac{x-a}{x^{\frac{2}{3}}+(ax)^{\frac{1}{3}}+a^{\frac{2}{3}}}}{(x-a)^{\frac{1}{3}}}=\lim\limits_{x\to a}\dfrac{(x-a)^{\frac{2}{3}}}{x^{\frac{2}{3}}+(ax)^{\frac{1}{3}}+a^{\frac{2}{3}}}=0$

$x^3-y^3=(x-y)(x^2+xy+y^2)$

$\Rightarrow x-y=\dfrac{x^3-y^3}{x^2+xy+y^2}$

$\therefore x^{\frac{1}{3}}-y^{\frac{1}{3}}=\dfrac{x-y}{x^{\frac{2}{3}}+(xy)^{\frac{1}{3}}+y^{\frac{2}{3}}}$

6.

解： $\lim\limits_{x\to1^-}\left(\dfrac{x+5}{x^2+1}+5\right)=\dfrac{1+5}{1+1}+5=8$

$\lim\limits_{x\to1^+}\left(6+\dfrac{x^2-1}{x-1}\right)=\lim\limits_{x\to1^+}(6+x+1)=7+1=8$

$\because \lim\limits_{x\to1^-}f(x)=\lim\limits_{x\to1^+}f(x)=8$

$\therefore \lim\limits_{x\to1}f(x)=8$

7.

(1)要证 $\lim\limits_{x\to 0}\frac{x+1}{x}=\infty$

即证 $\lim\limits_{x\to 0}\left(1+\frac{1}{x}\right)=\infty$

只需证 $\lim\limits_{x\to 0}\frac{1}{x}=\infty$

只需证 $\lim\limits_{x\to 0}x=0$

对于$\forall\varepsilon>0$，取δ为ε，则当 $0<|x-0|<\delta$时

有$|x-0|=|x|<\delta=\varepsilon$

$\therefore \lim\limits_{x\to 0}x=0$, 即 $\lim\limits_{x\to 0}\frac{x+1}{x}=\infty$

(2)要证 $\lim\limits_{x\to 0^+}e^{\frac{1}{x}}=+\infty$

即证 $\lim\limits_{x\to 0^+}e^{-\frac{1}{x}}=0$

对于$\forall\varepsilon>0$，取$\delta=-\frac{1}{\ln\varepsilon}$, $0<|x-0|<\delta$

有 $\left|e^{-\frac{1}{x}}-0\right|=e^{-\frac{1}{x}}<\delta=\varepsilon$

$\therefore \lim\limits_{x\to 0^+}e^{-\frac{1}{x}}=0$, 即 $\lim\limits_{x\to +\infty}e^{\frac{1}{x}}=+\infty$

(3)要证 $\lim\limits_{x\to\infty}x^2=+\infty$

即证 $\lim\limits_{x\to\infty}\frac{1}{x^2}=0$

对于$\forall\varepsilon>0$，取$X=\frac{1}{\sqrt{\varepsilon}}$，则当$|x|>X$时，

有 $\left|\frac{1}{x^2}-0\right|=\frac{1}{x^2}<\frac{1}{X^2}=\varepsilon$

$\therefore \lim\limits_{x\to\infty}\frac{1}{x^2}=0, \lim\limits_{x\to\infty}x^2=+\infty$

(4)要证 $\lim\limits_{x\to-\infty} x^3 = -\infty$

即证 $\lim\limits_{x\to-\infty} \frac{1}{x^3} = 0$

对于$\forall\varepsilon > 0$，取$X = \frac{1}{\sqrt[3]{\varepsilon}}$，则当$|x| > X$时，

有 $\left|\frac{1}{x^3} - 0\right| = \left|\frac{1}{x^3}\right| < \frac{1}{X^3} = \varepsilon$

$\therefore \lim\limits_{x\to-\infty} \frac{1}{x^3} = 0$，即 $\lim\limits_{x\to-\infty} x^3 = -\infty$

8.

解：①当$m = n$时

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{a_m + a_{m-1} + \frac{1}{x} + \cdots + a_1 \frac{1}{x^{m-1}} + a_0 \frac{1}{x^m}}{b_n + b_n - \frac{1}{x} + \cdots + b_1 \frac{1}{x^{n-1}} + b_0 \frac{1}{x^n}} = \frac{a_m}{b_n}$$

②当 $m<n$ 时

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{a_m \frac{1}{x^{n-m}} + a_{m-1} \frac{1}{x^{n-m+1}} + \cdots + a_0 \frac{1}{x^n}}{b_n + b_{n=1} \frac{1}{x} + \cdots + b_0 \frac{1}{x^n}} = 0$$

③当 $m>n$ 时

令$g(x) = a_m x^m + a_{m-1} x^{m-1} + \cdots + a_1 x + a_0$

$h(x) = b_n x^n + b_{n-1} x^{n-1} + \cdots + b_1 x + b_0$

由②得 $\lim\limits_{x\to\infty} \frac{h(x)}{g(x)} = 0$

$\therefore \lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} \frac{g(x)}{h(x)} = \infty.$