习题 5.2

1、计算下列不定积分

(1)、$\int \frac{dx}{(3-2x)^2}$

$=-\frac{1}{2}\int \frac{1}{(3-2x)^2}\,d(3-2x)$

$=-\frac{1}{2}\int \left(\frac{1}{3-2x}\right)^2 d(3-2x)$

$=\frac{1}{2}(3-2x)^{-1}+C$

(2)、$\int \tan(5x-3)\,dx$

$=\frac{1}{5}\int \frac{\sin(5x-3)}{\cos(5x-3)}\,d(5x-3)$

$=-\frac{1}{5}\int \frac{1}{\cos(5x-3)}\,d[\cos(5x-3)]$

$=-\frac{1}{5}\int \ln|\cos(5x-3)|+C$

(3)、$\int x^3 e^{-x^4}dx$

$=-\frac{1}{4}\int e^{-x^4}\,d(-x^4)$

$=-\frac{1}{4}e^{-x^4}+C$

(4)、$\int \frac{dx}{x\ln x}$

$=\frac{1}{\ln x}\,d\ln x$

$=\ln|\ln x|+C$

(5)、$\int \frac{\cos x-\sin x}{\sin x+\cos x}\,dx$

$=\int \frac{1}{\sin x+\cos x}\,d(\sin x+\cos x)$

$=\ln|\sin x+\cos x|+C$

(6)、$\int \frac{1}{x^2}a^{\frac{1}{x}}dx$

$$=-\int -\frac{1}{x^2}a^{\frac{1}{x}}dx$$

$$=-\int a^{\frac{1}{x}}d\frac{1}{x}$$

$$=-\frac{1}{\ln a}\int \ln a \ a^{\frac{1}{x}}d\frac{1}{x}$$

$$=-\frac{1}{\ln a}a^{\frac{1}{x}}+C$$

(7)、 $\int \frac{x^3}{\sqrt[3]{x^4+1}}dx$

$$=\frac{1}{4}\int \frac{4x^3}{\sqrt[3]{x^4+1}}dx$$

$$=\frac{1}{4}\int \frac{1}{(x^4+1)^{\frac{1}{3}}}d(x^4+1)$$

$$=\frac{3}{8}(x^4+1)^{\frac{2}{3}}+C$$

(8)、 $\int \frac{f'(x)}{\sqrt{f(x)}}dx$

$$=\int [f(x)]^{\frac{-1}{2}}df(x)$$

$$=2\sqrt{f(x)}+C$$

(9)、 $\int \frac{1}{\sqrt{\tan x}\cdot \cos^2 x}dx$

$$=\int \frac{1}{\sqrt{\tan x}}d\tan x$$

$$=\int (\tan x)^{\frac{-1}{2}}d\tan x$$

$$=2\int \frac{1}{2}(\tan x)^{\frac{-1}{2}}d\tan x$$

$$=2\sqrt{\tan x}+C$$

(10)、 $\int \frac{1}{\sqrt{1-x^2}(arcsinx)^2}dx$

$$=\int (arcsinx)^{-2}\ d\ arcsinx$$

$$=-\frac{1}{arcsinx}+C$$

(11)、 $\int \frac{\cos \sqrt{x}}{\sqrt{x}}dx$

$$=2\int \frac{1}{2}\frac{\cos \sqrt{x}}{\sqrt{x}}dx$$

$$=2\int \cos \sqrt{x}\ d\sqrt{x}$$
$$=2\sin \sqrt{x}+C$$

(12)、 $\int \frac{x}{4+x^4} dx$

$= -\frac{1}{4} \int \frac{-4x}{4+x^4} dx$

$= -\frac{1}{4} \int \frac{1}{x^2+2x+2} - \frac{1}{x^2-2x+2} dx$

$= -\frac{1}{4} \int \frac{1}{x^2+2x+2} dx + \frac{1}{4} \int \frac{1}{x^2-2x+2} dx$

$= -\frac{1}{4} \int \frac{1}{(x+1)^2+1} d(x+1) + \frac{1}{4} \int \frac{1}{(x-1)^2+1} d(x-1)$

$= -\frac{1}{4} arc \tan(x+1) + \frac{1}{4} arc \tan(x-1) + C$

(13)、 $\int \sin^3 x \, dx$
$= - \int \sin^2 x \, d \cos x$
$= - \int (1 - \cos^2 x) \, d \cos x$
$= - \left(\cos x - \frac{1}{3} \cos^3 x \right) + C$

$= \frac{1}{3} \cos^3 x - \cos x + C$

(14)、 $\int (x^2 - 3x + 1)^{10} (2x - 3) dx$
$= \int (x^2 - 3x + 1)^{10} d(x^2 - 3x + 1)$

$= \frac{1}{11} (x^2 - 3x + 1)^{11} + C$

(15)、 $\int \frac{1}{x^2} \cot \frac{1}{x} dx$

$= - \int \frac{\cos \frac{1}{x}}{\sin \frac{1}{x}} d \frac{1}{x}$

$= - \int \frac{1}{\sin \frac{1}{x}} d \sin \frac{1}{x}$

$= \ln \left| \sin \frac{1}{x} \right| + C$

(16)、 $\int \sin^5 x \cos^3 x dx$

$= \frac{1}{2} \int 2 \sin x \cos x \cdot \sin^4 x \cos^2 x dx$

$= \frac{1}{2} \int \sin 2x \left(\frac{1-\cos 2x}{2} \right)^2 \frac{1+\cos 2x}{2} dx$

$= -\frac{1}{4} \int -2 \sin 2x \left(\frac{1-\cos 2x}{2} \right)^2 \frac{1+\cos 2x}{2} dx$

$= -\frac{1}{4} \int \frac{1}{8} (1 - \cos 2x)^2 (1 + \cos 2x) d \cos 2x$

$$=-\frac{1}{32}\int(1-\cos 2x-\cos^2 2x+\cos^3 2x)\,d\cos 2x$$

$$=-\frac{1}{32}\cos 2x+\frac{1}{64}\cos^2 2x+\frac{1}{96}\cos^3 2x-\frac{1}{128}\cos^4 2x+C$$

(17)、$\int\frac{1}{x(x^8+1)}dx$

$$=\int\left(\frac{1}{x}-\frac{x^7}{x^8+1}\right)dx$$

$$=\int\frac{1}{x}dx-\int\frac{x^7}{x^8+1}dx$$

$$=\ln|x|-\frac{1}{8}\int\frac{8x^7}{x^8+1}dx$$

$$=\ln|x|-\frac{1}{8}\int\frac{1}{x^8+1}d(x^8+1)$$

$$=\ln|x|-\frac{1}{8}\ln(x^8+1)+C$$

(18)、$\int\frac{x^2+1}{x^4+1}dx$

$$=\frac{1}{2}\int\left(\frac{1}{x^2+\sqrt{2}x+1}+\frac{1}{x^2-\sqrt{2}x+1}\right)dx$$

$$=\frac{1}{2}\int\left(\frac{1}{(x+\frac{\sqrt{2}}{2})^2+\frac{1}{2}}+\frac{1}{(x-\frac{\sqrt{2}}{2})^2+\frac{1}{2}}\right)dx$$

$$=\frac{1}{2}\times 2\int\left(\frac{1}{(\sqrt{2}x+1)^2+1}+\frac{1}{(\sqrt{2}x-1)^2+1}\right)dx$$

$$=\frac{1}{\sqrt{2}}arctan(\sqrt{2}x+1)+\frac{1}{\sqrt{2}}arctan(\sqrt{2}x-1)+C$$

(19)、$\int\frac{\ln(x+1)-\ln x}{x(x+1)}dx$

$$=\int\frac{\ln\frac{x+1}{x}}{x(x+1)}dx$$

$$=-\int\ln\frac{x+1}{x}\cdot\left[\frac{-1}{x(x+1)}\right]dx$$

$$=-\int\ln\frac{x+1}{x}\,d\ln\frac{x+1}{x}$$

$$=-\frac{1}{2}\left(\ln\frac{x+1}{x}\right)^2+C$$

2、计算下列定积分

(1)、$\int\frac{1}{x^2(1-x^2)^{\frac{3}{2}}}dx$

令$x=\sin t,$ t$\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$, ∴原式$=\int\frac{1}{\sin^2 t\,\cos^3 t}\cdot\cos t\,dt$

$$= \int \frac{1}{sin^2 t\, cos^2 t} dt$$

$$= \int \frac{dt}{sin^2 t} + \int \frac{dt}{cos^2 t}$$

$$= -\cot t + \tan t + C$$

因为$x = sint$,所以$cost = \sqrt{1-x^2}$. 所以原式$= -\frac{\sqrt{1-x^2}}{x} + \frac{x}{\sqrt{1-x^2}} + C$

(2)、$\int \frac{\sqrt{x^2-1}}{x^3} dx$

令$x = \frac{1}{cost}$,$\cos t = \frac{1}{x}$,$\sin t = \sqrt{1 - \frac{1}{x^2}}$

原式$= \int \frac{\tan t}{\frac{1}{cos^3 t}} \cdot \frac{\sin t}{\cos t} dt$

$= \int sin^2 t\, dt = \int \frac{1-\cos 2t}{2} dt = \int 1 dt - \int \frac{\cos 2t}{2} dt$

$= t - \frac{1}{4} \sin 2t + C$

$= arccos\frac{1}{x} - \frac{2}{x}\sqrt{1 - \frac{1}{x^2}} + C$

(3)、$\int \frac{1}{x\sqrt{x^2-1}} dx$

令$x = \frac{1}{\cos t}$,则原式$= \int \frac{1}{\frac{1}{\cos t}\frac{\sin t}{\cos t}} \cdot \frac{\sin t}{cos^2 t} dt$

$$= \int 1 dt = t + C = arccos\left|\frac{1}{x}\right| + C$$

(4)、$\int \frac{1}{(x^2+a^2)^2} dx$

设$x = a\tan t$,原式$= \int \frac{\frac{a}{cos^2 t}}{\left(\frac{a^2 sin^2 t}{cos^2 t} + a^2\right)^2} dt$

$$= \int \frac{cos^2 t}{a^3} dt = \frac{1}{a^3}\int cos^2 t\, dt = \frac{1}{a^3}\int \frac{1+\cos 2t}{2} dt$$

$$= \frac{1}{2a^3}\int 1 dt + \frac{1}{2a^3}\int \cos 2t\, dt$$

$$= \frac{t}{2a^3} + \frac{\sin 2t}{4a^3} + C$$

$$= \frac{arctan\frac{x}{a}}{2a^3} + \frac{8xa^4}{x^2+a^2} + C$$

(5)、$\int \frac{dx}{\sqrt{3+2x-x^2}} = \int \frac{dx}{\sqrt{4-(x-1)^2}} = \int \frac{1}{2\sqrt{1-\left(\frac{x-1}{2}\right)^2}} dx$

$= arcsin\left(\frac{x-1}{2}\right) + C$

(6)、$\int \frac{x^2}{\sqrt{9-x^2}} dx$

$$设 x = 3sint, 则原式 = \int \frac{9sin^2t}{\sqrt{9(1-sin^2t)}} \cdot 3costdt$$

$$=\int \frac{9sin^2t \cdot 3cost}{3cost} dt = \int 9 sin^2tdt$$

$$=9 \int \frac{1-cos2t}{2} dt = \frac{9}{2}(\int 1dt - \int cos2tdt)$$

$$=\frac{9}{2}t - \frac{9}{4}sin2t + C = \frac{9}{2}t - \frac{x\sqrt{9-x^2}}{2} + C$$

因为 $sint = \frac{x}{3}$ ，所以 $cost = \frac{\sqrt{9-x^2}}{3}$

原式 $= \frac{9}{2}arcsin\frac{x}{3} - \frac{x\sqrt{9-x^2}}{2} + C$

（7）、 $\int \frac{\sqrt{x^2-4}}{x}dx$

令 $x = \frac{2}{cost}$ ，则原式 $= \int \frac{\sqrt{4(\frac{1}{cos^2t}-1)}}{\frac{2}{cost}} \cdot \frac{2sint}{cos^2t} dt$

$$=\int 2tan^2tdt = \int 2\frac{sin^2t}{cos^2t}dt = 2\int \frac{1}{cos^2t}dt - 2\int 1dt$$

$$=2tant - 2t + C$$

因为 $cost = \frac{2}{x}$ ，所以 $sint = \frac{\sqrt{x^2-2}}{x}$ ，所以原式 $= \frac{2\sqrt{x^2-2}}{x} \times \frac{x}{2} - 2arccos\left|\frac{2}{x}\right| + C$

$$=\sqrt{x^2-2} - 2arccos\left|\frac{2}{x}\right| + C$$

（8）、 $\int \frac{x^3}{(1+x^2)^{\frac{3}{2}}}dx$

令 $x = tanx$ ， $t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ ，则原式 $= \int \frac{tan^3t}{\frac{1}{cos^3t}} \cdot \frac{1}{cos^2t} dt = \int \frac{sin^3t}{cos^3t} dt$

$$=\int \frac{sint(1-cos^2t)}{cos^3t} dt = \int \frac{sint}{cos^3t} dt - \int sintdt$$

$$=\frac{1}{cost} + cost + C$$

因为 $tanx = x$ ，所以 $cost = \frac{1}{\sqrt{x^2+1}}$ ，所以原式 $= \sqrt{x^2+1} + \frac{1}{\sqrt{x^2+1}} + C$

（9）、 $\int \frac{1}{(1-x^2)^{\frac{3}{2}}}dx$

令 $x = sint$ ， $t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ ，则原式 $= \int \frac{1}{cos^3t} \cdot costdt = \int \frac{1}{cos^2t} dt = tant + C$

因为 $sint = x$ ，所以 $cost = \sqrt{1-x^2}$ ，所以原式 $= \frac{x}{\sqrt{1-x^2}} + C$

（10）、 $\int \frac{1}{(a^2+x^2)^{\frac{3}{2}}}dx$

设 $x = atant$ ， $t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ ，所以原式 $= \int \frac{1}{\left(a^2+a^2\frac{sin^2t}{cos^2t}\right)^{\frac{3}{2}}} \cdot \frac{a}{cos^2t} dt$

$$=\frac{1}{a^2}\int cost dt = \frac{1}{a^2}sint + C$$

因为$tant = \frac{x}{a}$ ，所以$sint = \frac{x}{\sqrt{x^2+a^2}}$ ，原式$= \frac{1}{a^2}\frac{x}{\sqrt{x^2+a^2}} + C$

(11)、$\int \frac{1}{x^2\sqrt{x^2+9}}dx$

令$x = 3tant$ ，$t \in \left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ ，则原式$= \int \frac{1}{9tan^2t\cdot\frac{3}{cost}} \cdot 3\frac{1}{cos^2t}dt$

$$=\int \frac{1}{9tan^2t\cdot cost}dt$$
$$=\frac{1}{9}\int \frac{cost}{sin^2t}dt = -\frac{1}{9}\frac{1}{sint} + C$$

因为$tant = \frac{x}{3}$ ，所以$sint = \frac{x}{\sqrt{x^2+9}}$

所以原式$=-\frac{1}{9}\frac{\sqrt{x^2+9}}{x} + C$