# 习题 7.5

**1. 证明$\boldsymbol{y = C_1 e^x + C_2 e^{-x} - 2(\cos x + x \sin x)}$是$\boldsymbol{y'' - y = 4x \sin x}$的通解。**

思路：代入即可

$y'' = C_1 e^x + C_2 e^{-x} - 2\cos x + 2x\sin x$

$\therefore y'' - y = 4x\sin x$

代入即可得

**2. 求下列微分方程的通解**

(1)$y'' - y' + y = 0;$

特征方程：$\lambda^2 - \lambda + 1 = 0$

$\because \Delta < 0$

求共轭副根

$$\lambda_1 = \frac{1 - \sqrt{2}i}{2}, \lambda_2 = \frac{1 + \sqrt{3}i}{2}$$

$$\therefore y = e^{\frac{x}{2}}\left(C_1 \cos\frac{\sqrt{3}}{2}x + C_2 \sin\frac{\sqrt{3}}{2}x\right)$$

(2)$y'' + 2y' - 3y = 0;$

特：$\lambda^2 + 2\lambda - 3 = 0$

解：$\lambda_1 = 1, \lambda_2 = -3$

$\therefore y = C_1 e^{-3x} + C_2 e^x$

(3)$y'' - 8y'' + 16y = 0$

特：$\lambda^2 - 8\lambda + 16 = 0$

解：$\lambda_1 = \lambda_2 = 4$

$y=(C_1+C_2x)e^{4x}$

$(4)y''+y=0$

特：$\lambda^2+1=0$

$\Delta<0$

$\therefore \lambda_1=i,\lambda_2=-i$

$\therefore y=C_1\cos x+C_2\sin x$

$(5)y''-y=\cos x$

对应齐次方程的特征方程为：

$\lambda^2-1=0$

$\lambda_1=1,\lambda_2=-1$

故对应齐次方程的通解：$Y=C_1e^x+C_2e^{-x}$

又 $\because 0$ 不是特征方程的根

故设方程的特解为$y^*=Q_1\cos x+Q_2\sin x$

代入$y''-y=\cos x$，

解得$Q_1=-\dfrac{1}{2},Q_2=0$

$\therefore y^*=-\dfrac{1}{2}\cos x$

故通解：

$y=C_1e^x+C_2\,e^{-x}-\dfrac{1}{2}\ \cos x$

$(6)y''+4y'+4y=e^{-2x}$

对应齐次方程的特征方程为

$\lambda^2+4\lambda+4=0$

$\lambda_1 = \lambda_2 = -2$

$\therefore$ 齐次方程通解：$Y = (C_1 + C_2 x)e^{-2x}$

$\because -2$ 是特征方程的重根

$\therefore$ 设方程的特解为$y^* = x^2 b_0 e^{-2x}$

代入$y'' + 4y' + 4y = e^{-2x}$

解得$b_0 = \frac{1}{2}$

$\therefore y^* = \frac{x^2}{2} e^{-2x}$

$\therefore$ 方程通解：

$y = (C_1 + C_2 x)e^{-2x} + \frac{x^2}{2} e^{-2x}$

(7)$y'' + 2y' + 2y = 2e^{-x} \sin x$;

特征方程：$\lambda^2 + 2\lambda + 2 = 0$

$\therefore \lambda_1 = -1 + i, \lambda_2 = -1 - i$

$\therefore$ 齐次的通解：$Y = e^{-x}(C_1 \cos x + C_2 \sin x)$

$\because -1 + i$是特征方程的根

$\therefore$ 设方程的特解：$y^* = xe^{-x}(Q_1 \cos x + Q_1 \sin x)$

代入$y'' + 2y' + 2y = 2e^{-x} \sin x$

解：$Q_1 = -1, \quad Q_2 = 0$

$\therefore$ 通解：$e^{-x}(C_1 \cos x + C_2 \sin x) - xe^{-x} \cos x$

(8)$y'' - 5y' + 6y = x^2 e^x - xe^{3x}$;

特征方程：$\lambda^2 - 5\lambda + 6 = 0$

$\lambda_1 = 2, \lambda_2 = 3$

$\therefore$ 齐次通解：$Y = C_1e^{2x} + C_2e^{3x}$

设特解$y^* = (b_0 + b_1x + b_2x^2)e^x + x(b_3 + b_4x)e^{3x}$

代入$y'' - 5y' + 6y = x^2e^x - xe^{3x}$

得$b_0 = \frac{7}{4}, b_1 = \frac{3}{2}, b_2 = \frac{1}{2}, b_3 = 1, b_4 = -\frac{1}{2}$

$\therefore$ 通解为：$y = C_1e^{2x} + C_2e^x + e^x\left(\frac{1}{2}x^2 + \frac{3}{2}x + \frac{7}{4}\right) - \left(\frac{x^2}{2} - x\right)e^{3x}$

$(9)x^2y'' + 4xy' + 2y = 0(x > 0)$

设$x = e^t$，则原方程转化为

$D(D-1)y + 4Dy + 2y = 0$

$D^2 + 3Dy + 2y = 0$

特征方程：$\lambda^2 + 3\lambda + 2 = 0$

$\lambda_1 = -1, \lambda_2 = -2$

$\therefore y = C_1e^{-t} + C_2e^{-2t}$

$\quad = \frac{c_1}{x} + \frac{c_2}{x_2}$

$(10)x^3y''' + x^2y'' - 4xy' = 3x^2$

齐次方程：$x = e^t, t = ln\,x$

$\therefore D(D-1)(D-2)y + D(D-1)y - 4Dy = 0$

特征方程：$\lambda^3 - 2\lambda^2 - 2\lambda = 0$

$\lambda_1 = 0, \lambda_2 = -1, x_3 = 3$

$\therefore Y = C_1 + \frac{C_2}{x} + C_3x^3$

设特解$y^* = b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4$

代入$x^3y''' + x^2y'' - 4xy' = 3x^2$

得$b_0 = 0, b_1 = 0, b_2 = -\frac{1}{2}, b_3 = b_4 = 0$

$\therefore y^* = -\frac{x^2}{2}$

$\therefore y = C_1 + \frac{C_2}{x} + C_3x^3 - \frac{x^2}{2}$

思路：先求齐次欧拉方程的通解，随后求特解

**3. 求下列微分方程的特解**

(1) $\because y'' + 3y' + 2y = \sin x, y(0) = 0, y'(0) = 0$

$\because$ 特征方程$\lambda^2 + 3\lambda + 2 = 0$ 的根为

$\lambda_1 = -1, \lambda_2 = -2$

$\therefore$ 对应齐次方程的通解$Y = C_1e^{-x} + C_2e^{-2x}$

$\because 0 + i$不是特征方程的根

设方程的特解$y^* = Q_1 \cos x + Q_2 \sin x$

代入$y'' + 3y' + 2y = \sin x$

解得$Q_1 = -\frac{3}{10} Q_2 = \frac{1}{10}$

$\therefore$ 通解$y = C_1e^{-x} + C_2e^{-2x} - \frac{3}{10}\cos x + \frac{1}{10}\sin x$

$y(0) = 0, y'(0) = 0$

$C_1 = \frac{1}{2}, C_2 = -\frac{1}{5}$

$\therefore$ 特解：$y = \frac{1}{2}e^{-x} - \frac{1}{5}e^{-2x} - \frac{3}{10}\cos x + \frac{1}{10}\sin x$

$(2) y'' + 2y' + 2y = xe^x, y(0) = 0, y'(0) = 0$

特征方程：$\lambda^2 + 2\lambda + 2 = 0$

$\lambda_1 = -1 + i, \lambda_2 = -1 - i$

$\therefore$ 对应齐次方程通解$Y = e^{-x}(C_1 \cos x + C_2 \sin x)$

$\because -1$ 不是特征方程的根

设方程的特解$y^* = (b_0 + b_1 x)e^{-x}$

代入$y' + 2y' + 2y = xe^{-x}$

解得$b_0 = 0 \quad b_1 = 1$

$\therefore$ 通解：$y = e^{-x}(C_1 \cos x + C_2 \sin x) + xe^{-x}$

代入$y(0) = 0, y'(0) = 0$

$C_1 = 0$，$C_2 = -1$

$\therefore$ 特解$y = e^{-x}(x - \sin x)$

**4. 设二阶常系数线性微分方程$y'' + ay' + by = Ce^x$的一个特解为$y = e^{3x} + \left(1 + \frac{x}{4}\right)e^x$，试确定 $a, b, c$ ，并求通解。**

①代入特解$y' = -3e^{-3x} + e^x + \dfrac{e^x + xe^x}{4}, y'' = 9e^{-3x} + \dfrac{3}{2}e^x + \dfrac{xe^x}{4}$

得：

$$9e^{-3x} + \frac{3}{2}e^x + \frac{xe^x}{4} + a(-3)e^{-3x} + \frac{5ae^x}{4} + \frac{axe^x}{4} + be^{-3x} + be^x + \frac{bxe^x}{4} = Ce^x$$

$$\begin{cases} 9 - 3a + b = 0 \\ \dfrac{3}{2} + \dfrac{5}{4}a + b = c \\ \dfrac{1}{4} + \dfrac{a}{4} + \dfrac{b}{4} = 0 \end{cases} \Longrightarrow \begin{cases} a = 2 \\ b = -3 \\ c = 1 \end{cases}$$

$\therefore$ 原方程为$y'' + 2y' - 3y = e^x$

特征方程：$\lambda^2 + 2\lambda - 3 = 0 \quad \lambda_1 = -3 \lambda_2 = 1$

$\therefore$ 对应齐次方程的通解：$Y = C_1e^{-3x} + C_2e^x$

$\because$ 1 是特征方程的解

$\therefore$ 设特解$\mathrm{y}^* = \mathrm{x}b_0\mathrm{e}^{\mathrm{x}}$

代入$y'' + 2y' - 3y = e^x$得$b_0 = \frac{1}{4}$

$\therefore$ 通解：$y = C_1e^{-3x} + C_2e^x + \frac{x}{4}e^x$