

习题 5.1

1.C

解析: $F(x)$ 仅为 I 区间内 $f(x)$ 的原函数, 非整个区间 $f(x)$ 的原函数, 故 C 错误。

2.

$$(1) \int f(x) dx = C \Rightarrow C' = (\int f(x) dx)' = 0 = f(x)$$

$$(2) \quad (3) \text{ 区间 } I \text{ 需连续, 并非整个定义域内} \quad \text{例 } f(x) = \frac{1}{x}$$

(3) 定义 5.1.1: 设函数 $f(x)$ 在某区间 I 上有定义, 如果存在可导函数 $F(x)$, 使得对 I 内每一点 x , 都有 $F'(x) = f(x)$ 或 $dF(x) = f(x) dx$, 则称 $F(x)$ 为 $f(x)$ 在区间 I 上的一个原函数。

3.

$$(1) \int (3x^3 - 5x^2 + \frac{3}{x^2}) dx = \int (3x^3) dx - \int (5x^2) dx + \int (\frac{3}{x^2}) dx = \frac{3}{4}x^4 - \frac{5}{3}x^3 - \frac{3}{x} + C$$

$$(2) \int \sqrt{x} \sqrt{x} \sqrt{x} dx = \int \sqrt{x} \sqrt{x \cdot x^{\frac{1}{2}}} dx = \int \sqrt{x} \sqrt{x^{\frac{3}{2}}} dx = \int \sqrt{x \cdot x^{\frac{3}{4}}} dx = \int x^{\frac{7}{8}} dx = \frac{8}{15} x^{\frac{15}{8}} + C$$

$$(3) \int (2 \tan x + 3 \cot x)^2 dx = \int (4 \tan^2 x + 12 \tan x \cdot \cot x + 9 \cot^2 x)^2 dx \\ = \int \left[4 \left(\frac{1 - \cos^2 x}{\cos^2 x} \right) + 12 + 9 \left(\frac{1 - \sin^2 x}{\sin^2 x} \right) \right] dx = \int \left(4 \frac{1}{\cos^2 x} + 9 \frac{1}{\sin^2 x} - 1 \right) dx \\ = 4 \tan x - 9 \cot x - x + C$$

$$(4) \int e^{3x} (3^x - e^{-2x}) dx = \int e^{3x} \cdot e^{x \ln 3} dx - \int e^x dx \\ = \int e^{(ln 3 + 3)x} dx - \int e^x dx = \frac{e^{(ln 3 + 3)x}}{ln 3 + 3} - e^x + C \\ = \frac{e^x \cdot 3^x}{3 + ln 3} - e^x + C$$

$$(5) \int \left(\frac{1}{x} - \frac{3}{\sqrt{1-x^2}} \right) dx = \ln|x| - 3 \arcsin x + C$$

$$(6) \int \frac{\sqrt{x-2} \sqrt[3]{x^2+1}}{4\sqrt{x}} dx = \int \frac{\sqrt{x-2} \sqrt[3]{x^2+1}}{2} d\sqrt{x}$$

$$\text{令 } t = \sqrt{x}$$

$$\text{原式} = \int \frac{t}{2} dt - \int t^{\frac{4}{3}} dt + \frac{1}{2} \int dt$$

$$= \frac{x}{4} - \frac{3}{7} x^{\frac{7}{6}} + \frac{\sqrt{x}}{2} + C$$

$$(7) \int \frac{2^{x-1} - 5^{x-1}}{10^x} dx = \int \frac{1}{2} \left(\frac{1}{5} \right)^x dx - \int \frac{1}{5} \left(\frac{1}{2} \right)^x dx = \frac{1}{5 \cdot 2^x \ln 2} - \frac{1}{2 \cdot 5^x \ln 5} + C$$

$$(8) \int \frac{(1-x)^2}{x(1+x^2)} dx = \int \frac{x^2+1-2x}{x(1+x^2)} dx = \int \frac{1}{x} dx - \int \frac{2}{x^2+1} dx = \ln|x| - 2 \arctan x + C$$

$$(9) \int \frac{x^2}{1+x^2} dx = \int \frac{x^2+1-1}{1+x^2} dx = \int dx - \int \frac{1}{1+x^2} dx = x - \arctan x + C$$

$$(10) \int \frac{1+\cos^2 x}{1+\cos 2x} dx = \int \frac{2-\sin^2 x}{2\sin^2 x} dx = -\frac{1}{2} \int dx + \int \frac{1}{\sin^2 x} dx = \cot x - \frac{1}{2} x + C$$

4. 解：由题意得 $f'(x) = \frac{2}{\sqrt{1-x^2}}$

$$f(x) = \int f'(x) dx = 2 \arcsin x + C$$

$$\text{因为 } f\left(\frac{1}{2}\right) = 0 \text{ 得 } C = -\frac{\pi}{3}$$

$$\text{所以 } f(x) = 2 \arcsin x - \frac{\pi}{3}$$

5. 解：由题意得 $x = \int v dt = t^3 - t + C \text{ m}$

$$\text{因为在 } x(t) \text{ 中, } x(1) = 10 \text{ m}$$

$$\text{所以 } C = 10$$

$$\text{所以当 } t = 3 \text{ 时 } x(3) = 34 \text{ m}$$

6. 证明：因为 $\int f(x) dx = F(x) + C$

$$\text{所以 } F'(x) = f(x) \quad F'(ax+b) = af(ax+b)$$

$$\text{对两边积分得 } F(ax+b) = a \int f(ax+b) dx + C$$

$$\text{因为 } C \in \mathbb{R}$$

$$\text{所以 } \frac{1}{a} F(ax+b) + C = \int f(ax+b) dx$$

习题 5.2

1、计算下列不定积分

$$\begin{aligned}(1)、\int \frac{dx}{(3-2x)^2} \\&= -\frac{1}{2} \int \frac{1}{(3-2x)^2} d(3-2x) \\&= \frac{1}{2} \int \left(\frac{1}{3-2x}\right)^1 d(3-2x) \\&= \frac{1}{2} (3-2x)^{-1} + C\end{aligned}$$

$$\begin{aligned}(2)、\int \tan(5x-3) dx \\&= \frac{1}{5} \int \frac{\sin(5x-3)}{\cos(5x-3)} d(5x-3) \\&= -\frac{1}{5} \int \frac{1}{\cos(5x-3)} d[\cos(5x-3)] \\&= -\frac{1}{5} \int \ln|\cos(5x-3)| + C\end{aligned}$$

$$\begin{aligned}(3)、\int x^3 e^{-x^4} dx \\&= -\frac{1}{4} \int e^{-x^4} d(-x^4) \\&= -\frac{1}{4} e^{-x^4} + C\end{aligned}$$

$$\begin{aligned}(4)、\int \frac{dx}{x \ln x} \\&= \frac{1}{\ln x} d \ln x \\&= \ln|\ln x| + C\end{aligned}$$

$$\begin{aligned}(5)、\int \frac{\cos x - \sin x}{\sin x + \cos x} dx \\&= \int \frac{1}{\sin x + \cos x} d(\sin x + \cos x) \\&= \ln|\sin x + \cos x| + C\end{aligned}$$

$$(6)、\int \frac{1}{x^2} a^{\frac{1}{x}} dx$$

$$\begin{aligned}
&= -\int -\frac{1}{x^2} a^{\frac{1}{x}} dx \\
&= -\int a^{\frac{1}{x}} d\frac{1}{x} \\
&= -\frac{1}{\ln a} \int \ln a a^{\frac{1}{x}} d\frac{1}{x} \\
&= -\frac{1}{\ln a} a^{\frac{1}{x}} + C
\end{aligned}$$

$$\begin{aligned}
(7)、\int \frac{x^3}{\sqrt[3]{x^4+1}} dx \\
&= \frac{1}{4} \int \frac{4x^3}{\sqrt[3]{x^4+1}} dx \\
&= \frac{1}{4} \int \frac{1}{(x^4+1)^{\frac{1}{3}}} d(x^4+1) \\
&= \frac{3}{8} (x^4+1)^{\frac{2}{3}} + C
\end{aligned}$$

$$\begin{aligned}
(8)、\int \frac{f'(x)}{\sqrt{f(x)}} dx \\
&= \int [f(x)]^{-\frac{1}{2}} df(x) \\
&= 2\sqrt{f(x)} + C
\end{aligned}$$

$$\begin{aligned}
(9)、\int \frac{1}{\sqrt{\tan x} \cos^2 x} dx \\
&= \int \frac{1}{\sqrt{\tan x}} d \tan x \\
&= \int (\tan x)^{-\frac{1}{2}} d \tan x \\
&= 2 \int \frac{1}{2} (\tan x)^{-\frac{1}{2}} d \tan x \\
&= 2\sqrt{\tan x} + C
\end{aligned}$$

$$\begin{aligned}
(10)、\int \frac{1}{\sqrt{1-x^2} (\arcsin x)^2} dx \\
&= \int (\arcsin x)^{-2} d \arcsin x \\
&= -\frac{1}{\arcsin x} + C
\end{aligned}$$

$$\begin{aligned}
(11)、\int \frac{\cos \sqrt{x}}{\sqrt{x}} dx \\
&= 2 \int \frac{1}{2} \frac{\cos \sqrt{x}}{\sqrt{x}} dx \\
&= 2 \int \cos \sqrt{x} d\sqrt{x} \\
&= 2 \sin \sqrt{x} + C
\end{aligned}$$

$$\begin{aligned}
(12)、\int \frac{x}{4+x^4} dx \\
&= -\frac{1}{4} \int \frac{-4x}{4+x^4} dx \\
&= -\frac{1}{4} \int \frac{1}{x^2+2x+2} - \frac{1}{x^2-2x+2} dx \\
&= -\frac{1}{4} \int \frac{1}{x^2+2x+2} dx + \frac{1}{4} \int \frac{1}{x^2-2x+2} dx \\
&= -\frac{1}{4} \int \frac{1}{(x+1)^2+1} d(x+1) + \frac{1}{4} \int \frac{1}{(x-1)^2+1} d(x-1) \\
&= -\frac{1}{4} \arctan(x+1) + \frac{1}{4} \arctan(x-1) + C
\end{aligned}$$

$$\begin{aligned}
(13)、\int \sin^3 x dx \\
&= -\int \sin^2 x d \cos x \\
&= -\int (1 - \cos^2 x) d \cos x \\
&= -\left(\cos x - \frac{1}{3} \cos^3 x \right) + C \\
&= \frac{1}{3} \cos^3 x - \cos x + C
\end{aligned}$$

$$\begin{aligned}
(14)、\int (x^2 - 3x + 1)^{10} (2x - 3) dx \\
&= \int (x^2 - 3x + 1)^{10} d(x^2 - 3x + 1) \\
&= \frac{1}{11} (x^2 - 3x + 1)^{11} + C
\end{aligned}$$

$$\begin{aligned}
(15)、\int \frac{1}{x^2} \cot \frac{1}{x} dx \\
&= -\int \frac{\cos \frac{1}{x}}{\sin \frac{1}{x}} d \frac{1}{x} \\
&= -\int \frac{1}{\sin \frac{1}{x}} d \frac{1}{\sin \frac{1}{x}} \\
&= \ln \left| \sin \frac{1}{x} \right| + C
\end{aligned}$$

$$\begin{aligned}
(16)、\int \sin^5 x \cos^3 x dx \\
&= \frac{1}{2} \int 2 \sin x \cos x \cdot \sin^4 x \cos^2 x dx \\
&= \frac{1}{2} \int \sin 2x \left(\frac{1 - \cos 2x}{2} \right)^2 \frac{1 + \cos 2x}{2} dx \\
&= -\frac{1}{4} \int -2 \sin 2x \left(\frac{1 - \cos 2x}{2} \right)^2 \frac{1 + \cos 2x}{2} dx \\
&= -\frac{1}{4} \int \frac{1}{8} (1 - \cos 2x)^2 (1 + \cos 2x) d \cos 2x
\end{aligned}$$

$$\begin{aligned}
&= -\frac{1}{32} \int (1 - \cos 2x - \cos^2 2x + \cos^3 2x) d \cos 2x \\
&= -\frac{1}{32} \cos 2x + \frac{1}{64} \cos^2 2x + \frac{1}{96} \cos^3 2x - \frac{1}{128} \cos^4 2x + C
\end{aligned}$$

$$\begin{aligned}
(17)、\int \frac{1}{x(x^8+1)} dx \\
&= \int \left(\frac{1}{x} - \frac{x^7}{x^8+1} \right) dx \\
&= \int \frac{1}{x} dx - \int \frac{x^7}{x^8+1} dx \\
&= \ln|x| - \frac{1}{8} \int \frac{8x^7}{x^8+1} dx \\
&= \ln|x| - \frac{1}{8} \int \frac{1}{x^8+1} d(x^8+1) \\
&= \ln|x| - \frac{1}{8} \ln(x^8+1) + C
\end{aligned}$$

$$\begin{aligned}
(18)、\int \frac{x^2+1}{x^4+1} dx \\
&= \frac{1}{2} \int \left(\frac{1}{x^2+\sqrt{2}x+1} + \frac{1}{x^2-\sqrt{2}x+1} \right) dx \\
&= \frac{1}{2} \int \left(\frac{1}{(x+\frac{\sqrt{2}}{2})^2+\frac{1}{2}} + \frac{1}{(x-\frac{\sqrt{2}}{2})^2+\frac{1}{2}} \right) dx \\
&= \frac{1}{2} \times 2 \int \left(\frac{1}{(\sqrt{2}x+1)^2+1} + \frac{1}{(\sqrt{2}x-1)^2+1} \right) dx \\
&= \frac{1}{\sqrt{2}} \arctan(\sqrt{2}x+1) + \frac{1}{\sqrt{2}} \arctan(\sqrt{2}x-1) + C
\end{aligned}$$

$$\begin{aligned}
(19)、\int \frac{\ln(x+1)-\ln x}{x(x+1)} dx \\
&= \int \frac{\ln \frac{x+1}{x}}{x(x+1)} dx \\
&= - \int \ln \frac{x+1}{x} \cdot \left[\frac{-1}{x(x+1)} \right] dx \\
&= - \int \ln \frac{x+1}{x} d \ln \frac{x+1}{x} \\
&= -\frac{1}{2} \left(\ln \frac{x+1}{x} \right)^2 + C
\end{aligned}$$

2、计算下列定积分

$$(1)、\int \frac{1}{x^2(1-x^2)^{\frac{3}{2}}} dx$$

$$\text{令 } x = \sin t, t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \therefore \text{原式} = \int \frac{1}{\sin^2 t \cos^3 t} \cdot \cos t dt$$

$$\begin{aligned}
&= \int \frac{1}{\sin^2 t \cos^2 t} dt \\
&= \int \frac{dt}{\sin^2 t} + \int \frac{dt}{\cos^2 t} \\
&= -\cot t + \tan t + C
\end{aligned}$$

因为 $x = \sin t$, 所以 $\cos t = \sqrt{1-x^2}$. 所以原式 $= -\frac{\sqrt{1-x^2}}{x} + \frac{x}{\sqrt{1-x^2}} + C$

(2)、 $\int \frac{\sqrt{x^2-1}}{x^3} dx$

令 $x = \frac{1}{\cos t}$, $\cos t = \frac{1}{x}$, $\sin t = \sqrt{1 - \frac{1}{x^2}}$

$$\begin{aligned}
\text{原式} &= \int \frac{\tan t}{\frac{1}{\cos^3 t}} \cdot \frac{\sin t}{\cos t} dt \\
&= \int \sin^2 t dt = \int \frac{1-\cos 2t}{2} dt = \int 1 dt - \int \frac{\cos 2t}{2} dt \\
&= t - \frac{1}{4} \sin 2t + C \\
&= \arccos \frac{1}{x} - \frac{2}{x} \sqrt{1 - \frac{1}{x^2}} + C
\end{aligned}$$

(3)、 $\int \frac{1}{x\sqrt{x^2-1}} dx$

$$\begin{aligned}
\text{令 } x &= \frac{1}{\cos t}, \text{ 则原式} = \int \frac{1}{\frac{1}{\cos t} \cdot \frac{\sin t}{\cos^2 t}} \cdot \frac{\sin t}{\cos^2 t} dt \\
&= \int 1 dt = t + C = \arccos \left| \frac{1}{x} \right| + C
\end{aligned}$$

(4)、 $\int \frac{1}{(x^2+a^2)^2} dx$

$$\begin{aligned}
\text{设 } x &= a \tan t, \text{ 原式} = \int \frac{\frac{a}{\cos^2 t}}{\left(\frac{a^2 \sin^2 t}{\cos^2 t} + a^2 \right)^2} dt \\
&= \int \frac{\cos^2 t}{a^3} dt = \frac{1}{a^3} \int \cos^2 t dt = \frac{1}{a^3} \int \frac{1+\cos 2t}{2} dt \\
&= \frac{1}{2a^3} \int 1 dt + \frac{1}{2a^3} \int \cos 2t dt \\
&= \frac{t}{2a^3} + \frac{\sin 2t}{4a^3} + C \\
&= \frac{\arctan \frac{x}{a}}{2a^3} + \frac{8xa^4}{x^2+a^2} + C
\end{aligned}$$

$$\begin{aligned}
(5)、\int \frac{dx}{\sqrt{3+2x-x^2}} &= \int \frac{dx}{\sqrt{4-(x-1)^2}} = \int \frac{1}{2\sqrt{1-\left(\frac{x-1}{2}\right)^2}} dx \\
&= \arcsin \left(\frac{x-1}{2} \right) + C
\end{aligned}$$

(6)、 $\int \frac{x^2}{\sqrt{9-x^2}} dx$

$$\begin{aligned}
 \text{设 } x = 3\sin t, \text{ 则原式} &= \int \frac{9\sin^2 t}{\sqrt{9(1-\sin^2 t)}} \cdot 3\cos t dt \\
 &= \int \frac{9\sin^2 t \cdot 3\cos t}{3\cos t} dt = \int 9\sin^2 t dt \\
 &= 9 \int \frac{1-\cos 2t}{2} dt = \frac{9}{2} (\int 1 dt - \int \cos 2t dt) \\
 &= \frac{9}{2} t - \frac{9}{4} \sin 2t + C = \frac{9}{2} t - \frac{x\sqrt{9-x^2}}{2} + C
 \end{aligned}$$

$$\text{因为 } \sin t = \frac{x}{3}, \text{ 所以 } \cos t = \frac{\sqrt{9-x^2}}{3}$$

$$\text{原式} = \frac{9}{2} \arcsin \frac{x}{3} - \frac{x\sqrt{9-x^2}}{2} + C$$

$$(7)、\int \frac{\sqrt{x^2-4}}{x} dx$$

$$\begin{aligned}
 \text{令 } x = \frac{2}{\cos t}, \text{ 则原式} &= \int \sqrt{\frac{4\left(\frac{1}{\cos^2 t}-1\right)}{\frac{2}{\cos t}}} \cdot \frac{2\sin t}{\cos^2 t} dt \\
 &= \int 2\tan^2 t dt = \int 2 \frac{\sin^2 t}{\cos^2 t} dt = 2 \int \frac{1}{\cos^2 t} dt - 2 \int 1 dt \\
 &= 2\tan t - 2t + C \\
 \text{因为 } \cos t = \frac{2}{x}, \text{ 所以 } \sin t &= \frac{\sqrt{x^2-2}}{x}, \text{ 所以原式} = \frac{2\sqrt{x^2-2}}{x} \times \frac{x}{2} - 2\arccos \left| \frac{2}{x} \right| + C \\
 &= \sqrt{x^2-2} - 2\arccos \left| \frac{2}{x} \right| + C
 \end{aligned}$$

$$(8)、\int \frac{x^3}{(1+x^2)^{\frac{3}{2}}} dx$$

$$\begin{aligned}
 \text{令 } x = \tan t, t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \text{ 则原式} &= \int \frac{\tan^3 t}{\frac{1}{\cos^3 t}} \cdot \frac{1}{\cos^2 t} dt = \int \frac{\sin^3 t}{\cos^3 t} dt \\
 &= \int \frac{\sin t(1-\cos^2 t)}{\cos^3 t} dt = \int \frac{\sin t}{\cos t} dt - \int \sin t dt \\
 &= -\frac{1}{\cos t} + \cos t + C
 \end{aligned}$$

$$\text{因为 } \tan x = x, \text{ 所以 } \cos t = \frac{1}{\sqrt{x^2+1}}, \text{ 所以原式} = \sqrt{x^2+1} + \frac{1}{\sqrt{x^2+1}} + C$$

$$(9)、\int \frac{1}{(1-x^2)^{\frac{3}{2}}} dx$$

$$\text{令 } x = \sin t, t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \text{ 则原式} = \int \frac{1}{\cos^3 t} \cdot \cos t dt = \int \frac{1}{\cos^2 t} dt = \tan t + C$$

$$\text{因为 } \sin t = x, \text{ 所以 } \cos t = \sqrt{1-x^2}, \text{ 所以原式} = \frac{x}{\sqrt{1-x^2}} + C$$

$$(10)、\int \frac{1}{(a^2+x^2)^{\frac{3}{2}}} dx$$

$$\text{设 } x = atant, t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \text{ 所以原式} = \int \frac{1}{\left(a^2+a^2\frac{\sin^2 t}{\cos^2 t}\right)^{\frac{3}{2}}} \cdot \frac{a}{\cos^2 t} dt$$

$$= \frac{1}{a^2} \int \cos t dt = \frac{1}{a^2} \sin t + C$$

因为 $\tan t = \frac{x}{a}$, 所以 $\sin t = \frac{x}{\sqrt{x^2+a^2}}$, 原式 $= \frac{1}{a^2} \frac{x}{\sqrt{x^2+a^2}} + C$

(11)、 $\int \frac{1}{x^2\sqrt{x^2+9}} dx$

令 $x = 3\tan t$, $t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, 则原式 $= \int \frac{1}{9\tan^2 t \cdot \frac{3}{\cos t}} \cdot 3 \frac{1}{\cos^2 t} dt$

$$= \int \frac{1}{9\tan^2 t \cdot \cos t} dt$$

$$= \frac{1}{9} \int \frac{\cos t}{\sin^2 t} dt = -\frac{1}{9} \frac{1}{\sin t} + C$$

因为 $\tan t = \frac{x}{3}$, 所以 $\sin t = \frac{x}{\sqrt{x^2+9}}$

所以原式 $= -\frac{1}{9} \frac{\sqrt{x^2+9}}{x} + C$

习题 5.3

1. 求下列不定积分

$$(1) \int x \cos x \, dx = \int x \, d \sin x = x \sin x - \int \sin x \, d x$$

$$= x \sin x + \cos x + C$$

$$(2) \int \ln x \, d x = \int x \ln x \, d x$$

$$= x \ln x - \int x \, d \ln x$$

$$= x \ln x - \int x \cdot \frac{1}{x} \, dx$$

$$= x \ln x - x + C$$

$$(3) \int x^2 e^x \, d x = \int x^2 (e^x)' \, d x = x^2 e^x - \int e^x \cdot 2x \, dx$$

$$= x^2 e^x - \left(e^x \cdot 2x - \int e^x \cdot 2 \, dx \right)$$

$$= e^x (x^2 - 2x + 2) + C$$

$$(4) \int \arcsin x \, d x = \int x' \arcsin x \, d x$$

$$= x \arcsin x - \int x \cdot \frac{1}{\sqrt{1-x^2}} \, dx$$

$$= x \arcsin x + \sqrt{1-x^2} + C$$

$$(5) \int \frac{\ln(\ln x)}{x} \, dx = \int (\ln x)' \ln(\ln x) \, dx$$

$$= \ln x \cdot \ln(\ln x) - \int \ln x \cdot \frac{\frac{1}{x}}{\ln x} \, dx$$

$$= \ln x \cdot \ln(\ln x) - \ln x + C$$

$$(6) \int e^{2x} \cos x \, dx = \int \frac{1}{2} (e^{2x})' \cos x \, dx$$

$$= \frac{1}{2} \left(e^{2x} \cos x + \int e^{2x} \sin x \, dx \right)$$

$$= \frac{1}{2} \left[e^{2x} \cos x + \frac{1}{2} \left(e^{2x} \sin x - \int e^{2x} \cos x \, dx \right) \right]$$

$$\text{移项可得} \int e^{2x} \cos x \, dx = \frac{1}{5} e^{2x} (\sin x + 2 \cos 2x) + C$$

$$(7) \int x \sin x \cos x \, dx = \frac{1}{2} \int x \sin 2x \, dx = -\frac{1}{4} \int x (\cos 2x)' \, dx$$

$$= -\frac{1}{4} \left(x \cos 2x - \int \cos 2x \, dx \right)$$

$$= -\frac{1}{4} \left(x \cos 2x - \frac{1}{2} \sin 2x \right) + C$$

$$(8) \int x f''(x) \, dx = \int x (f'(x))' \, dx$$

$$= x f'(x) - \int f'(x) \, dx$$

$$= x f'(x) - f(x) + C$$

$$(9) \int x \sin^2 x \, dx = \int x \frac{1 - \cos 2x}{2} \, dx$$

$$= \frac{1}{2} \int x \left(x - \frac{1}{2} \sin 2x \right)' \, dx$$

$$= \frac{1}{2} \left[x^2 - \frac{1}{2} x \sin 2x - \int \left(x - \frac{1}{2} \sin 2x \right) \, dx \right]$$

$$= \frac{1}{2} \left[x^2 - \frac{1}{2} x \sin 2x - \frac{1}{2} x^2 + \left(-\frac{1}{4} \cos 2x \right) \right]$$

$$= \frac{1}{4}x^2 - \frac{x}{4}\sin 2x - \frac{1}{8}\cos 2x + C$$

$$(10) \int x(\arctan x)^2 dx$$

$$= \int \frac{1}{2}(x^2)'(\arctan x)^2 dx$$

$$= \frac{1}{2}[x^2 \arctan x]^2 - \int 2 \arctan x \left(1 - \frac{1}{1+x^2}\right) dx$$

$$= \frac{1}{2}x^2(\arctan x)^2 - \int x' \arctan x dx + \frac{1}{2}[(\arctan x)^2]' dx$$

$$= \frac{1}{2}x^2(\arctan x)^2 - \left(x \arctan x - \int \frac{x}{1+x^2} dx\right) + \frac{1}{2}(\arctan x)^2$$

$$= \frac{1+x^2}{2}(\arctan x)^2 - x \arctan x + \frac{\ln(1+x^2)}{2} + C$$

$$(11) \int \ln(x + \sqrt{1+x^2}) dx$$

$$= \int x' \ln(x + \sqrt{1+x^2}) dx$$

$$= x \ln(x + \sqrt{1+x^2}) - \int \frac{1 + \frac{2x}{2\sqrt{1+x^2}}}{x + \sqrt{1+x^2}} \cdot x dx$$

$$= x \ln(x + \sqrt{1+x^2}) - \int \frac{x}{\sqrt{1+x^2}} dx$$

$$= x \ln(x + \sqrt{1+x^2}) - \sqrt{1+x^2} + C$$

$$(12) \int \frac{x \cos x}{\sin^3 x} dx = \int -x \cdot \frac{1}{2} \left(\frac{1}{\sin^2 x} \right)' dx$$

$$= - \left(\frac{x}{\sin^2 x} - \int \frac{1}{\sin^2 x} dx \right)$$

$$= - \frac{x}{2 \sin^2 x} - \frac{1}{2} \cot x + C$$

$$(13) \int \sec^5 x dx = \int (\tan x)' \sec^3 x dx$$

$$= \tan x \sec^3 x - 3 \int \tan x \sec^4 \sin x dx$$

$$\text{得 ① } 4 \int \sec^5 x dx = \tan x \sec^3 x + 3 \int \sec^3 x dx$$

$$\text{由 } \int \sec^3 x dx = \tan x \sec x - \int \tan x \frac{\sin x}{\cos^2 x} dx$$

$$\text{得 ② } 2 \int \sec^3 x dx = \tan x \sec x + \int \frac{1}{\cos x} dx$$

$$= \tan x \sec x + \ln|\sec x + \tan x| + C$$

②代入①得

$$\begin{aligned} \int \sec^5 x dx &= \frac{1}{4} \tan x \sec^3 x + \frac{3}{8} \tan x \sec x + \frac{3}{8} \ln|\sec x + \tan x| \\ &\quad + C \end{aligned}$$

$$(14) \int \frac{x^2 \arctan x}{1+x^2} dx$$

$$= \int \left(1 - \frac{1}{1+x^2}\right) \arctan x dx$$

$$= (x - \arctan x) \arctan x - \int (x - \arctan x) \frac{1}{1+x^2} dx$$

$$= (x - \arctan x) \arctan x - \frac{1}{2} \ln(1+x^2) + \int \frac{\arctan x}{1+x^2} dx$$

$$\text{由 } \int \frac{\arctan x}{1+x^2} dx = (\arctan x)^2 - \int \frac{\arctan x}{1+x^2} dx$$

$$\text{得 } \int \frac{\arctan x}{1+x^2} dx = \frac{1}{2} (\arctan x)^2 + C$$

$$\int \frac{x^2 \arctan x}{1+x^2} dx = x \arctan x - \frac{1}{2} \ln(1+x^2) - \frac{1}{2} (\arctan x)^2 + C$$

2.对于正整数 $n \geq 2$, 建立 $I_n = \int \sin^n x dx$ 的递推公式

$$\begin{aligned}
I_n &= \int \sin^{n-1} x \cdot \sin x \, dx = - \int \sin^{n-1} x (\cos x)' \, dx \\
&= - \sin^{n-1} x \cos x + \int \cos x (n-1) \sin^{n-2} x \cos x \, dx \\
&= - \sin^{n-1} x \cos x + \int (1 - \sin^2 x)(n-1) \sin^{n-2} x \, dx \\
&= - \sin^{n-1} x \cos x + (n-1) \left(\int \sin^{n-2} x \, dx - \int \sin^n x \, dx \right) \\
\text{整理可得 } I_n &= -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} I_{n-2}
\end{aligned}$$

习题 5.4

$$\begin{aligned}(1) \quad & \int \frac{x^3}{1+x} dx \\&= \int \left(x^2 + 1 - x - \frac{1}{1+x}\right) dx \\&= \frac{x^3}{3} - \frac{x^2}{2} + x - \ln|x+1| + C\end{aligned}$$

$$\begin{aligned}(2) \quad & \int \frac{x^5+x^4-8}{x^3-x} dx \\&= \int \frac{x^2(x^3-x)+x(x^3-x)+(x^3-x)+x^2+x-8}{x^3-x} dx \\&= \int \left(x^2 + x + 1 + \frac{1}{x-1} - \frac{8}{x^3-x}\right) dx \\&= \frac{x^3}{3} + \frac{x^2}{2} + x + 8\ln|x| - 4\ln|x+1| - 3\ln|x-1| + C\end{aligned}$$

$$\begin{aligned}(3) \quad & \int \frac{x^3+1}{x^3-x^2} dx \\&= \int \left(1 - \frac{1}{x} - \frac{1}{x^2} + \frac{2}{x-1}\right) dx \\&= x + \frac{1}{x} + \ln \frac{(x-1)^2}{|x|} + C\end{aligned}$$

$$\begin{aligned}(4) \quad & \int \frac{x^5}{(x-1)^2(x^2-1)} dx \\&= \int \left(x + 2 + \frac{\frac{1}{8}}{1+x} + \frac{\frac{31}{8}}{x-1} + \frac{\frac{9}{4}}{(x-1)^2} + \frac{\frac{1}{2}}{(x-1)^3}\right) dx \\&= \frac{x^2}{2} + 2x - \frac{1}{4(x-1)^2} - \frac{9}{4(x-1)} + \frac{31}{8}\ln|x-1| + \frac{1}{8}\ln|x+1| + C\end{aligned}$$

$$\begin{aligned}(5) \quad & \int \frac{x^4}{1+x^2} dx \\&= \int \left(x^2 - 1 + \frac{1}{1+x^2}\right) dx\end{aligned}$$

$$= \frac{x^3}{3} - x + \arctan x + C$$

$$\begin{aligned} (6) \quad & \int \frac{x^2}{1-x^4} dx \\ &= \int \left(\frac{\frac{1}{4}}{1+x} + \frac{\frac{1}{4}}{1-x} - \frac{\frac{1}{2}}{1+x^2} \right) dx \\ &= \frac{1}{4} \ln \left| \frac{1+x}{1-x} \right| - \frac{1}{2} \arctan x + C \end{aligned}$$

$$\begin{aligned} (7) \quad & \int \frac{1}{(x+1)^2(x^2+1)} dx \\ &= \int \left(\frac{\frac{1}{2}}{x+1} + \frac{\frac{1}{2}}{(x+1)^2} - \frac{\frac{1}{2}x}{1+x^2} \right) dx \\ &= \frac{1}{2} \ln|x+1| - \frac{1}{4} \ln(x^2+1) - \frac{1}{2(x+1)} + C \end{aligned}$$

$$\begin{aligned} (8) \quad & \int \frac{x^3-x^2-x+3}{x^2-1} dx \\ &= \int \left(x-1 + \frac{1}{x-1} - \frac{1}{x+1} \right) dx \\ &= \frac{x^2}{2} - x + \ln \left| \frac{x-1}{x+1} \right| + C \end{aligned}$$

$$\begin{aligned} (9) \quad & \int \frac{2x+2}{(1+x)^2(x-1)} dx \\ &= \int \left(\frac{1}{x-1} - \frac{1}{x+1} \right) dx \\ &= \ln|x-1| - \ln|x+1| + C \end{aligned}$$

(10)

$$\begin{aligned} & \int \frac{x^3 + 2x^2 + 1}{(x-1)(x-2)(x-3)^2} dx \\ &= \int \left(\frac{-1}{x-1} + \frac{17}{x-2} + \frac{-15}{x-3} + \frac{23}{(x-3)^2} \right) dx \end{aligned}$$

$$= -\ln|x-1| + 17\ln|x-2| - 15\ln|x-3| - \frac{23}{x-3} + c$$

(11)

$$\begin{aligned} & \int \frac{x^3}{(x-1)^{100}} dx \\ &= \int \frac{(x-1)^3 + 3(x-1)^2 - 3(x-1) - 1}{(x-1)^{100}} dx \\ &= \int \left(\frac{1}{(x-1)^{97}} + \frac{3}{(x-1)^{98}} + \frac{3}{(x-1)^{99}} + \frac{1}{(x-1)^{100}} \right) dx \\ &= -\frac{1}{96(x-1)^{96}} - \frac{3}{97(x-1)^{97}} - \frac{3}{98(x-1)^{98}} - \frac{1}{99(x-1)^{99}} + c \end{aligned}$$

(12)

$$\begin{aligned} & \int \frac{1}{x(x^{10}+2)} dx \\ &= \frac{1}{2} \int \left(\frac{1}{x} - \frac{x^9}{x^{10}+2} \right) dx \\ &= \frac{1}{2} \ln|x| - \frac{1}{20} \ln(x^{10}+2) + c \end{aligned}$$

T2 (1)

$$\text{令 } \tan \frac{x}{2} = t, \quad x = 2 \arctan t, \quad dx = \frac{2}{1+t^2} dt$$

$$\begin{aligned} & \int \frac{1}{2 \sin x - \cos x + 5} dx \\ &= \int \frac{1}{\frac{4t}{1+t^2} - \frac{1-t^2}{1+t^2} + 5} \cdot \frac{2}{1+t^2} dt \end{aligned}$$

$$\begin{aligned}
&= \int \frac{1}{(3t+1)^2 + (\sqrt{5})^2} d(3t+1) \\
&= \frac{1}{\sqrt{5}} \arctan \frac{3t+1}{\sqrt{5}} + c \\
&= \frac{1}{\sqrt{5}} \arctan \frac{3 \tan \frac{x}{2} + 1}{\sqrt{5}} + c
\end{aligned}$$

(2)

$$\begin{aligned}
&\text{令 } \tan \frac{x}{2} = t, \quad x = 2 \arctan t, \quad dx = \frac{2}{1+t^2} dt \\
&\int \frac{1}{(2+\cos x) \sin x} dx \\
&= \frac{1}{3} \int \frac{1}{t^3+3t} d(t^3+3t) \\
&= \frac{1}{3} \ln |t^3+3t| + c \\
&= \frac{1}{3} \ln \left| \tan \frac{x}{2} \left(3 + \tan^2 \frac{x}{2} \right) \right| + c
\end{aligned}$$

(3)

$$\begin{aligned}
&\text{令 } \tan \frac{x}{2} = t, \quad x = 2 \arctan t, \quad dx = \frac{2}{1+t^2} dt \\
&\int \frac{1+\sin x}{\sin x(1+\cos x)} dx \\
&= \int \left(\frac{1}{2}t + 1 + \frac{1}{2t} \right) dt \\
&= \frac{1}{4}t^2 + t + \frac{1}{2} \ln |t| + c \\
&= \frac{1}{4} \tan^2 \frac{x}{2} + \tan \frac{x}{2} + \frac{1}{2} \ln \left| \tan \frac{x}{2} \right| + c
\end{aligned}$$

(4)

$$\text{令 } \tan \frac{x}{2} = t, \quad x = 2 \arctan t, \quad dx = \frac{2}{1+t^2} dt$$

$$\begin{aligned} & \int \frac{1}{\sin x + \tan x} dx \\ &= \int \left(-\frac{1}{2}t + \frac{1}{2t} \right) dt \\ &= -\frac{1}{4}t^2 + \frac{1}{2} \ln|t| + c \\ &= -\frac{1}{4} \tan^2 \frac{x}{2} + \frac{1}{2} \ln \left| \tan \frac{x}{2} \right| + c \end{aligned}$$

(5)

$$\text{令 } \tan x = t, \quad x = \arctan t, \quad dx = \frac{1}{1+t^2}$$

$$\begin{aligned} & \int \frac{1}{(\sin x + \cos x)^2} dx \\ &= \int \frac{1}{1 + \sin 2x} dx \\ &= \int \frac{1}{1 + \frac{2t}{1+t^2}} \cdot \frac{1}{1+t^2} dt \\ &= \int \frac{1}{(t+1)^2} dt \\ &= -\frac{1}{t+1} + c \\ &= -\frac{1}{\tan x + 1} + c \end{aligned}$$

(6)

$$\text{令 } \cos x = t, \quad \sin^2 x = 1 - t^2$$

$$\int \frac{1}{\sin x \cos^3 x} dx$$

$$\begin{aligned}
&= -\int \frac{1}{\sin^2 x \cos^3 x} d(\cos x) \\
&= -\int \left(\frac{1}{t} + \frac{1}{t^3} + \frac{t}{1-t^2} \right) dt \\
&= -\ln|\cos x| + \frac{1}{2\cos^2 x} + \ln|\sin x| + c \\
(7) \quad &\int \frac{\cos x}{1+\sin x} dx \\
&= \int \frac{1}{1+\sin x} d(\sin x + 1) \\
&= \ln(1 + \sin x) + C
\end{aligned}$$

$$\begin{aligned}
(8) \quad &\text{令 } \tan \frac{x}{2} = t \\
&\int \frac{1}{3+5\cos x} dx \\
&= \frac{1}{4} \int \left(\frac{1}{2-t} + \frac{1}{2+t} \right) dt \\
&= \frac{1}{4} \ln \left| \frac{2+\tan \frac{x}{2}}{2-\tan \frac{x}{2}} \right| + C
\end{aligned}$$

$$\begin{aligned}
(9) \quad &\text{令 } \tan \frac{x}{2} = t \\
&\int \frac{1}{\sin 2x - 2\sin x} dx \\
&= -\frac{1}{4} \int \left(\frac{1}{t^3} + \frac{1}{t} \right) dt \\
&= -\frac{1}{4} \left(-\frac{1}{2t^2} + \ln|t| \right) + C \\
&= \frac{1}{8} \cos^2 \frac{x}{2} - \frac{1}{4} \ln \left| \tan \frac{x}{2} \right| + C
\end{aligned}$$

$$\begin{aligned}
 (10) & \int \frac{1}{\sin^4 x + \cos^4 x} dx \\
 &= \int \frac{\sec^4 x}{(\tan^4 x) + 1} d \tan x \quad \text{Let } \tan x = t \\
 &= \int \frac{1+t^2}{1+t^4} dt \\
 &= \frac{\sqrt{2}}{2} \arctan \left(\frac{t - \frac{1}{t}}{\sqrt{2}} \right) + C \\
 &= \frac{\sqrt{2}}{2} \arctan \left(\frac{\tan^2 x - 1}{\sqrt{2} \tan x} \right) + C
 \end{aligned}$$

$$3 (1) \text{ Let } t = \sqrt{\frac{x}{1-x}}, x = \frac{t^2}{1+t^2}$$

$$\begin{aligned}
 & \int \frac{1}{x} \sqrt{\frac{x}{1-x}} dx \\
 &= \int \frac{2}{1+t^2} dt \\
 &= 2 \arctan t + C \\
 &= 2 \arctan \sqrt{\frac{x}{1-x}} + C
 \end{aligned}$$

$$\begin{aligned}
 (2) & \int \frac{\sqrt{x}}{\sqrt[3]{x^2-4}\sqrt{x}} dx, \text{ Let } t = \sqrt[12]{x}, x = t^{12} \\
 &= \int \frac{t^6}{t^8-t^3} dt^{12} \\
 &= 12 \int \left(\frac{t^4}{t^5-1} + t^4 + t^9 \right) dt \\
 &= \frac{6}{5} x^{\frac{5}{6}} + \frac{12}{5} x^{\frac{5}{12}} + \frac{12}{5} \ln \left| x^{\frac{5}{12}} - 1 \right| + C
 \end{aligned}$$

$$(3) \int \frac{1+\sqrt{1-x^2}}{1-\sqrt{1-x^2}} dx$$

$$\begin{aligned}
&= \int \frac{(1+\sqrt{1-x^2})(1+\sqrt{1-x^2})}{(1-\sqrt{1-x^2})(1+\sqrt{1-x^2})} dx \\
&= \int \frac{2-x^2+2\sqrt{1-x^2}}{x^2} dx \\
&= -\frac{2}{x} - x - 2 \int \sqrt{1-x^2} d\left(\frac{1}{x}\right) \\
&= -\frac{2}{x} - x - \frac{2}{x} \sqrt{1-x^2} - 2 \int \frac{dx}{\sqrt{1-x^2}} \\
&= -\frac{2+x^2}{x} - \frac{2}{x} \sqrt{1-x^2} - 2 \arcsin x + C
\end{aligned}$$

$$\begin{aligned}
(4) \int \sqrt{\frac{e^x-1}{e^x+1}} dx, \quad \text{let } t = \sqrt{\frac{e^x-1}{e^x+1}}, \quad dx = \frac{4t}{1-t^4} dt \\
= \int t \frac{4t}{1-t^4} dt \\
= 2 \int \left(\frac{1}{1-t^2} + \frac{1}{1+t^2} \right) dt \\
= \ln \left| \frac{1-t}{1+t} \right| + \arctan t + C \\
= \ln \left| \frac{1-\sqrt{\frac{e^x-1}{e^x+1}}}{1+\sqrt{\frac{e^x-1}{e^x+1}}} \right| + \arctan \sqrt{\frac{e^x-1}{e^x+1}} + C
\end{aligned}$$

$$\begin{aligned}
(5) \int \frac{1}{1+\sqrt[3]{x+1}} dx, \quad \text{let } t = \sqrt[3]{x+1}, \quad x = t^3 - 1 \\
= 3 \int \left(t - 1 + \frac{1}{1+t} \right) dt \\
= \frac{3}{2} t^2 - 3t + 3 \ln|1+t| + C \\
= \frac{3}{2} \sqrt[3]{(x+1)^2} - 3\sqrt[3]{x+1} + 3 \ln|1+\sqrt[3]{x+1}| + C
\end{aligned}$$

$$\begin{aligned}
(6) \int \frac{x}{\sqrt{5+x-x^2}} dx \\
= \frac{2}{\sqrt{21}} \int \frac{x}{\sqrt{1-\left[\frac{2}{\sqrt{21}}\left(x-\frac{1}{2}\right)\right]^2}} dx
\end{aligned}$$

$$= \int x d \arcsin \frac{2x-1}{\sqrt{21}}, \quad \Leftrightarrow t = \arcsin \frac{2x-1}{\sqrt{21}}$$

$$= \left(\int \left(\frac{1}{2} + \frac{\sqrt{21}}{2} \sin t \right) dt \right)$$

$$= \frac{1}{2} t - \frac{\sqrt{21}}{2} \cos t + C$$

$$= -\sqrt{5+x-x^2} + \frac{1}{2} \arcsin \frac{2x-1}{\sqrt{21}} + C$$

第 5 章复习题

1. (1) $\int (\cos \frac{x}{2} - \sin \frac{x}{2})^2 dx =$ _____

$$\begin{aligned}\text{原式} &= \int (1 - \sin x) dx \\ &= \int 1 dx - \int \sin x dx \\ &= x + \cos x + C\end{aligned}$$

(2) 若 $a \neq 0$, 则 $\int \frac{1}{x^2 \sqrt{x^2 + a^2}} dx =$ _____

令 $x = a \tan t$

$$\begin{aligned}\text{原式} &= \int \frac{1}{a^3 (\tan t)^2 \sec t} \cdot a (\sec t)^2 dt \\ &= \frac{1}{a^2} \int \frac{\sec t}{(\tan t)^2} dt \\ &= \frac{1}{a^2} \int \frac{1}{(\sin t)^2} d \sin t \\ &= -\frac{x}{a^2 \sqrt{x^2 + a^2}} + C\end{aligned}$$

(3) $\int \frac{1 + \cos x}{x + \sin x} dx =$ _____

$$\begin{aligned}\text{原式} &= \int \frac{1}{x + \sin x} d(x + \sin x) \\ &= \ln |x + \sin x| + C\end{aligned}$$

(4) $\int \frac{\sqrt{\ln x}}{x} dx =$ _____

$$\begin{aligned}\text{原式} &= \int \sqrt{\ln x} d \ln x \\ &= \frac{2}{3} (\ln x)^{\frac{3}{2}} + C\end{aligned}$$

2. (1) $\int \frac{\arctan x}{x^2(1+x^2)} dx$

$$\begin{aligned}&= \int \left[\arctan x \left(\frac{1}{x^2} - \frac{1}{x^2+1} \right) \right] dx \\ &= \int \frac{\arctan x}{x^2} dx - \int \frac{\arctan x}{x^2+1} dx\end{aligned}$$

$$\begin{aligned}
&= - \int \arctan x \, d\frac{1}{x} - \int \arctan x \, d(\arctan x) \\
&= -\frac{\arctan x}{x} + \int \frac{1}{x(x^2+1)} dx - \frac{(\arctan x)^2}{2} \\
&= -\frac{\arctan x}{x} + \frac{1}{2} \int \frac{1}{x^2(x^2+1)} dx^2 - \frac{(\arctan x)^2}{2} \\
&= -\frac{\arctan x}{x} - \frac{(\arctan x)^2}{2} + \frac{1}{2} \int \frac{1}{x^2} dx^2 - \frac{1}{2} \int \frac{1}{x^2+1} d(x^2+1) \\
&= \frac{1}{2} \ln \frac{x^2}{x^2+1} - \frac{\arctan x}{x} - \frac{(\arctan x)^2}{2} + C
\end{aligned}$$

$$(2) \int \frac{1}{(1-x)\sqrt{1-x^2}} dx$$

$$\text{令 } x = \sin t, \quad t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$\begin{aligned}
\text{原式} &= \int \frac{1}{1-\sin t} dt \\
&= \int \frac{1+\sin t}{(1-\sin t)(1+\sin t)} dt \\
&= \int \frac{1+\sin t}{(\cos t)^2} dt \\
&= \int \frac{1}{(\cos t)^2} dt + \int \frac{\sin t}{(\cos t)^2} dt \\
&= \tan t + \frac{1}{\cos t} + C \\
&= \frac{x+1}{\sqrt{1-x^2}} + C
\end{aligned}$$

$$(3) \int \frac{e^{\arctan x}}{(1+x^2)^{\frac{3}{2}}} dx$$

$$\text{令 } x = \tan t, \quad t = \arctan x$$

$$\begin{aligned}
\text{原式} &= \int e^t \cos t \, dt \\
&= e^t \cos t + \int e^t \sin t \, dt \\
&= e^t \cos t + e^t \sin t - \int e^t \cos t \, dt \\
\text{原式} &= \frac{e^t \cos t + e^t \sin t}{2} + C \\
&= \frac{1+x}{2\sqrt{1+x^2}} e^{\arctan x} + C
\end{aligned}$$

$$(4) \int \frac{x^2-1}{x\sqrt{x^4+3x^2+1}} dx$$

$$= \int \frac{1 - \frac{1}{x^2}}{\sqrt{x^2 + \frac{1}{x^2} + 3}} dx$$

$$= \int \frac{1}{\sqrt{(x + \frac{1}{x})^2 + 1}} d(x + \frac{1}{x})$$

$$\text{令 } x + \frac{1}{x} = \tan t, \quad \sec t = \sqrt{(x + \frac{1}{x})^2 + 1}$$

$$\text{原式} = \int \frac{1}{\sqrt{(x + \frac{1}{x})^2 + 1}} d(x + \frac{1}{x})$$

$$= \int \sec t \, dt$$

$$= \ln |\sec t + \tan t| + C$$

$$= \ln \left| \sqrt{(x + \frac{1}{x})^2 + 1} + x + \frac{1}{x} \right| + C$$

$$(5) \int \frac{1}{(\sin x)^2 + 3} dx$$

$$= \int \frac{(\sec x)^2}{(\tan x)^2 + 3(\sec x)^2} dx$$

$$= \int \frac{1}{(\tan x)^2 + 3(\sec x)^2} d \tan x$$

$$= \frac{1}{x} \int \frac{1}{4(\tan x)^2 + 3} d(2 \tan x)$$

$$= \frac{\sqrt{3}}{6} \arctan\left(\frac{2 \tan x}{\sqrt{3}}\right) + C$$

$$(6) \int \frac{x e^x}{\sqrt{e^x - 1}} dx$$

$$\text{令 } \sqrt{e^x - 1} = t, \quad x = \ln(t^2 + 1)$$

$$\text{原式} = 2 \int \ln(t^2 + 1) dt$$

$$= 2t \ln(t^2 + 1) - 4 \int \frac{t^2}{t^2 + 1} dt$$

$$= 2t \ln(t^2 + 1) - 4 \int 1 \, dt + 4 \int \frac{1}{t^2 + 1} dt$$

$$= 2t \ln(t^2 + 1) - 4t + 4 \arctan t + C$$

$$= 2x\sqrt{e^x - 1} - 4\sqrt{e^x - 1} + 4\arctan\sqrt{e^x - 1} + C$$

$$\begin{aligned} (7) \quad \text{原式} &= \int \frac{\sin^4 x}{\cos^4 x} dx \\ &= \int \frac{\cos^4 x - 2\cos^2 x + 1}{\cos^4 x} dx \\ &= \int 1 dx - \int \frac{2}{\cos^2 x} dx + \int \frac{1}{\cos^4 x} dx \\ &= x - 2\tan x + \int (\tan^2 x + 1) d\tan x \\ &= x - 2\tan x + \frac{1}{3}\tan^3 x + \tan x + C \\ &= x - \tan x + \frac{1}{3}\tan^3 x + C \end{aligned}$$

$$(8) \quad \text{原式} = \int \frac{\arcsin x}{x^2 \sqrt{1-x^2}} dx + \int \frac{\arcsin x}{\sqrt{1-x^2}} dx$$

$$\text{令 } x = \sin t, \quad t = \arcsin x, \quad dx = \cos t dt \quad \left(-\frac{\pi}{2} < t < \frac{\pi}{2}\right)$$

$$\begin{aligned} \text{则 } \int \frac{\arcsin x}{x^2 \sqrt{1-x^2}} dx &= \int \frac{t}{\sin^2 t} dt \\ &= -\int t d\cot t \\ &= -t\cot t + \int \cot t dt \\ &= -t\cot t + \int \frac{\cos t}{\sin t} dt \\ &= -t\cot t + \int \frac{1}{\sin t} d\sin t \\ &= -t\cot t + \ln|\sin t| \\ &= -\arcsin x \cdot \cot(\arcsin x) + \ln|x| + C \\ \int \frac{\arcsin x}{\sqrt{1-x^2}} dx &= \int t dt = \frac{1}{2}t^2 = \frac{1}{2}(\arcsin x)^2 \end{aligned}$$

$$\text{综上, 原式} = -\arcsin x \cdot \cot(\arcsin x) + \ln|x| + \frac{1}{2}(\arcsin x)^2 + C$$

$$\begin{aligned} (9) \quad \text{原式} &= \int \frac{x e^x + e^x - e^x}{(1+x)^2} dx \\ &= \int \frac{(x+1)e^x}{(1+x)^2} dx - \int \frac{e^x}{(1+x)^2} dx \end{aligned}$$

$$\begin{aligned}
&= \int \frac{e^x}{x+1} dx + \int e^x d\frac{1}{x+1} \\
&= \int \frac{e^x}{x+1} dx + \frac{e^x}{x+1} - \int \frac{1}{x+1} de^x \\
&= \frac{e^x}{x+1} + C
\end{aligned}$$

$$\begin{aligned}
(10) \text{ 原式} &= \int \frac{x^4 - x^2 + 1}{x^6 + 1} dx + \int \frac{x^2}{x^6 + 1} dx \\
&= \int \frac{(x^2 - 1)^2}{x^6 + 1} dx + \frac{1}{3} \int \frac{1}{x^6 + 1} dx^3 \\
&= \arctan x + \frac{1}{3} \arctan x^3 + C
\end{aligned}$$

$$\begin{aligned}
(11) \text{ 原式} &= \int \frac{e^x}{\cos x + 1} dx + \int \frac{\sin x}{\cos x + 1} dx \\
&= \int \frac{e^x}{2\cos^2 \frac{x}{2}} dx + \int \frac{2\sin \frac{x}{2} \cos \frac{x}{2}}{2\cos^2 \frac{x}{2}} e^x dx \\
&= \int e^x d(\tan \frac{x}{2}) + \int e^x \tan \frac{x}{2} dx \\
&= e^x \tan \frac{x}{2} - \int \tan \frac{x}{2} e^x dx + \int e^x \tan \frac{x}{2} dx \\
&= e^x \tan \frac{x}{2} + C
\end{aligned}$$

$$\begin{aligned}
(12) \text{ 原式} &= \int e^{x \ln x} (\ln x + 1) dx \\
&= \int e^{x \ln x} d(x \ln x) \\
&= e^{x \ln x} + C + \\
&= x + C
\end{aligned}$$

$$\begin{aligned}
(13) \text{ 原式} &= \int \frac{(x+a) \ln(x+a) + (x+b) \ln(x+b)}{(x+a)(x+b)} dx \\
&= \int \frac{\ln(x+a)}{x+b} dx + \int \frac{\ln(x+b)}{x+a} dx \\
&= \int \frac{\ln(x+a)}{x+b} dx + \ln(x+b) d\ln(x+a) \\
&= \int \frac{\ln(x+a)}{x+b} dx + \ln(x+b) \cdot \ln(x+a) - \int \frac{\ln(x+a)}{x+b} dx \\
&= \ln(x+a) \cdot \ln(x+b) + C
\end{aligned}$$

$$3.(1) \text{由题得: } f(x) = \left(\frac{\cos x}{x}\right)' = \frac{-x \sin x - \cos x}{x^2}$$

$$\begin{aligned} \int x f'(x) dx &= \int x df(x) \\ &= x f(x) - \int f(x) dx \\ &= \frac{-x \sin x - \cos x}{x} - \frac{\cos x}{x} + C \\ &= -\frac{x \sin x + 2 \cos x}{x} + C \end{aligned}$$

$$(2) \text{因 } \int x f'(x) dx = \arcsin x + C,$$

$$\text{则 } f(x) = \frac{1}{x \sqrt{1-x^2}}$$

$$\begin{aligned} \text{则 } \int \frac{1}{f(x)} dx &= \int x \sqrt{1-x^2} dx \\ &= -\frac{1}{2} \sqrt{1-x^2} d(1-x^2) \\ &= -\frac{1}{3} \sqrt{(1-x^2)^3} \end{aligned}$$

$$(3) \text{设 } f^{-1}(x) = x, \text{ 则 } x = f(y)$$

$$\begin{aligned} \int x f^{-1}(x) dx &= \int y df(y) \\ &= y f(y) - F(y) + C \\ &= x f^{-1}(x) - F(f^{-1}(x)) + C \end{aligned}$$

$$4. \text{令 } u = \sin^2 x, |u| \leq 1$$

$$\text{所以原式即为 } f'(u) = 1 - u$$

$$\text{两边取积分得 } f(u) = u - \frac{1}{2} u^2 + C$$

$$\text{所以 } f(x) = x - \frac{1}{2} x^2 + C$$

$$5. \text{令 } t = x^2 - 1, \text{ 则原式即为 } f(t) = \ln \frac{t+1}{t-1}$$

$$\text{所以 } f[\varphi(x)] = \ln \frac{\varphi(x)+1}{\varphi(x)-1} = \ln x$$

$$\varphi(x) = \frac{x+1}{x-1}$$

$$\int \varphi(x) dx = \int \frac{x+1}{x-1} dx = \int 1 + \frac{2}{x-1} dx = x + \ln(x-1)^2 + C$$

$$6. (1) \int \min\{|x|, x^2\} dx = \begin{cases} \int x dx & \begin{cases} -\frac{1}{2} x^2 + C_1, x < -1 \\ \frac{1}{3} x^3 + C_2, |x| \leq 1 \\ \frac{1}{2} x^2 + C_3, x > 1 \end{cases} \\ \int x^2 dx & \end{cases}$$

$\therefore \min\{|x|, x^2\}$ 在定义域上连续, $\therefore \int \min\{|x|, x^2\} dx$ 在定义域上也连续

$$\therefore \lim_{x \rightarrow -1^-} \left(-\frac{1}{2} x^2 + C_1\right) = \lim_{x \rightarrow -1^+} \left(\frac{1}{3} x^3 + C_2\right) \text{ 得 } C_1 = C_2 + \frac{1}{6}$$

$$\text{同理 } C_3 = C_2 + \frac{1}{6}, \text{ 令 } C_2 = C$$

$$\therefore \int \min\{|x|, x^2\} dx = \begin{cases} -\frac{1}{2} x^2 + C, x < -1 \\ \frac{1}{3} x^3 + C, |x| \leq 1 \\ \frac{1}{2} x^2 + C, x > 1 \end{cases}$$

$$(2). \int \max\{1, x^2, x^3\} dx = \begin{cases} \int x^2 dx & \begin{cases} \frac{1}{3} x^3 + C_1, x < -1 \\ x + C_2, |x| \leq 1 \\ \frac{1}{4} x^4 + C_3, x > 1 \end{cases} \\ \int 1 dx & \\ \int x^3 dx & \end{cases}$$

$$\text{同 (1) } C_1 = C_2 - \frac{2}{3}, C_3 = C_2 + \frac{3}{4}, \text{ 令 } C_2 = C$$

$$\therefore \int \max\{1, x^2, x^3\} dx = \begin{cases} \frac{1}{3} x^3 - \frac{2}{3} + C, x < -1 \\ x + C, |x| \leq 1 \\ \frac{1}{4} x^4 + \frac{3}{4} + C, x > 1 \end{cases}$$

$$7. (1) \text{ 由题 } y' = 2x-1$$

$$\text{两边取积分得 } y = x^2 - x + C$$

又 $x=1$ 时, $y=0$

\therefore 带入得 $C=0$

$\therefore y = x^2 - x$

(2) 由题 $y' = \frac{1}{x}$

两边取积分得 $y = \ln|x| + C$;

又 $x=e^2$ 时, $y=4$

\therefore 带入得 $C=2$

$\therefore y = \ln|x| + 2$

8. 由题取 $x=0$, $\xi=1$;

则有 $f(0+1)=f(1)=f(0)f(1)$

$\therefore f(1) \neq 0$, $\therefore f(0)=1$;

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

又 $\therefore f(x + \Delta x) = f(x)f(\Delta x)$

$$\begin{aligned} \therefore f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{f(x)[f(\Delta x) - 1]}{\Delta x} \\ &= f(x) \lim_{\Delta x \rightarrow 0} \frac{f(0+\Delta x) - f(0)}{\Delta x} \\ &= f(x) f'(0) \end{aligned}$$

综上 $f'(x) = f(x) f'(0)$

令 $y=f(x)$, 则有 $\frac{dy}{dx} = y f'(0)$

当 $y=0$ 时显然成立;

当 $y \neq 0$ 时有 $\frac{dy}{y} = f'(0) dx$

两边取积分得 $\ln|y| = f(0)x$

$$\therefore f(x) = y = Ce^{f'(0)x}$$