Master Internship's report

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General context

The dream of a machine capable of solving mathematical problems is an old dream that takes its roots in the Ancient Greece. The advent of computers in the second part of the XXth century brought back hope to make the dream of the Fathers happen. From the resolution procedure, exhibited by Davis and Putnam in 1962, to modern SAT-solvers, the idea remains the same, take the problem's formula, convert it in a normal form, namely, the conjunctive normal form, and ask the solver whether or not the resulting formula is satisfiable. However, we know, since the 30s that, among the problems we formalize, some are decidable and some are not. Among those that are decidable, some belong to the **NP** class, meaning that they can be solved in polynomial time with a non-deterministic algorithm. Modern SAT-solvers propose, nowaday, a bunch of heuristics to solve that class of problems.

Research problem

The research problem we address is an acceleration of some class of hard problems using a weakening of the classical logic. We divided the research in two different aspect. The first aspect asks whether it is possible or not to improve the preprocessing of the given formula. That is to say, is it possible to apply a specific normal form before the CNF transformation, that would make the formula less redundant ? The second aspect concerns the crafting of an algorithm in a specific proof system of the logic that would solve the class of problems we study faster.

Your contribution

The main contribution is the exhibition of an efficient proof-search algorithm based on a proof system of orthologic. We showed that it is correct and that it admits a cubical worst-case complexity. To evaluate the performance, we studied the specific class of problems of circuit equivalence modulo renaming. This happens to be a class that SAT-solvers struggle to solve. However, we showed that we can easely encode into the framework of orthologic and thus, ask a proof for it in our proof system.

Arguments supporting its validity

Concerning the preprocessing, we mainly tested the procedure on several set of problems more or less hard to solve for SAT-solvers. Concerning the proof-search algorithm, in addition to proofs, we tested our algorithm on a circuit's benchmark, resulting in a great improvement compared to several SAT-solvers we ran in the same set of problems.

Summary and future work

To sum up, we propose a way to improve the current state of the art of SAT-solvers, according to certain class of problems. To achieve this, we exhibit an efficient proof-search algorithm for problems that can be encoded in the orthologic proof system. Next, the immediate step to do is to implement our algorithm within an existing simple SAT-solver, such as Minisat, and test whether the performances are improved, compared to the original version. The question that follows, concerns the behaviour in the framework of first-order orthologic, when dealing with specific theories.

Acknowledgements

1 Introduction

In this section, we shall introduce the key concepts and the notations that will be mainly used in the present report. Few years after the advent of quantum mechanics, a rigorous mathematical foundation has been proposed by von Neumann [vNB18] leading, after some observations [BN36], to a non-classical logical structure that would get rid of the distributivity identities. The associated algebra to such a structure is the class of the so called orthomodular lattices, rather than the classical boolean algebras. This gave birth to a new field of logic called the *quantum logic*. The term of *orthologic* has first been introduced by Goldblatt [Gol74] to refer to the logic that corresponds to the algebraic class of *orthocomplemented lattices* or *ortholattices* for short. Those are a generalization of the boolean algebras and, in particular, give up the axiom of distributivity. For ortholattices are weaker than orthomodular lattices, it is also often called the *minimal quantum logic*. First, we shall give some basic notions about lattices, after what, we will introduce the underlying notion of ortholattices. Then we will take a closer look to a normal form of the orthologic and, finally, we will present a proof system for orthologic.

Definition 1 (Lattice)**.** *A lattice $\langle L, \wedge, \vee \rangle$ is an algebraic structure where L is a set and $\wedge$ and $\vee$ are two binary, commutative and associative operations satisfying the following axiomatic identities, also called absorption laws.*

$$a \vee (a \wedge b) = a$$
$$a \wedge (a \vee b) = a$$

Moreover, the lattice is said to be bounded when there exists two elements $\bot$ and $\top$ such that $x \vee \top = \top$, $x \wedge \bot = \bot$ and $x \vee \bot = x \wedge \top = x$.

Remark. *A lattice can also be seen as a partial order, in which case, we associate it the following order relation.*

$$a \leq b \Leftrightarrow a = (a \wedge b)$$

Or, equivalently, if and only if $b = (a \vee b)$.

Definition 2 (Generated lattice)**.** *A lattice is said to be generated by a family of elements (X_i) if its elements consist in the X_i's and their finite combinations by $\wedge$ and $\vee$.*

Definition 3 (Free lattice)**.** *A free lattice generated by a family of elements (X_i) is a lattice in which there is no other laws of equality than the ones derived by the axiomatic identities of the lattice.*

1.1 Ortholattices

As for classical logic with boolean algebras or, similarly, intuitionnistic logic with Heyting algebras, orthologic also has an underlying algebraic structure that we call *orthocomplemented lattices* or *ortholattices* for short. To introduce it, we need a very last notion that is the *orthocomplement*.

Definition 4 (Orthocomplement)**.** *An orthogonal complementation (or orthocomplement for short) on a bounded lattice L is a function $\neg : L \to L$ that respects, the complement law $\neg a \vee a = \top$ and $\neg a \wedge a = \bot$, the involution law $\neg\neg a = a$ and, finally, the order-reversing law $a \leq b \Rightarrow \neg b \leq \neg a$.*

We now have all the ingredients to properly define the ortholattices. An *ortholattice* is a bounded lattice that comes with an *orthocomplement* $\neg$. We show, in the table 1, an axiomatization of ortholattices that is a sum up of all the axiomatic identities we have seen so far. With that set, one can show that boolean algebras are special cases of ortholattices, in particular, the cases where distributivity holds.

Table 1: Axiomatization of ortholattices

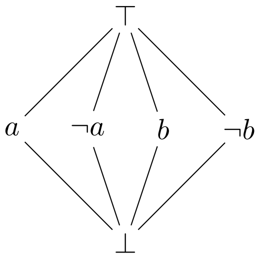
$x \vee y = y \vee x$	$x \wedge y = y \wedge x$
$x \vee (y \vee z) = (x \vee y) \vee z$	$x \wedge (y \wedge z) = (x \wedge y) \wedge z$
$x \vee x = x$	$x \wedge x = x$
$x \vee \top = \top$	$x \wedge \bot = \bot$
$x \vee \bot = x$	$x \wedge \top = x$
$\neg\neg x = x$	
$x \vee \neg x = \top$	$x \wedge \neg x = \bot$
$\neg(x \vee y) = \neg x \wedge \neg y$	$\neg(x \wedge y) = \neg x \vee \neg y$
$x \vee (x \wedge y) = x$	$x \wedge (x \vee y) = x$

Furthermore, note that this set of axiom is not minimal, but it highlights the duality between $\wedge$ and $\vee$, as well as the duality between $\bot$ and $\top$. For the minimal set of axioms of ortholattices, please refer to [McC98]. We shall write $a = b$ if and only if $a \leq b$ and $b \leq a$. Remark also that ortholattices are weaker than orthomodular lattices since they do not admit the following law as an axiom.

$$a \leq b \Rightarrow a \vee (\neg a \wedge b) = b$$

In the following of the report, we will denote orthologic by the acronym OL. Let us illustrate ortholattices through the following example.

Example 5. *The following is an ortholattice, while it is not a boolean algebra.*



This lattice is known as the M_4 lattice. Let us consider the relation $x \wedge (\neg x \vee y) \leq y$. Although it is true in boolean algebras, it does not hold in M_4. Take, as a counterexample, the mapping x to a and y to b.

1.2 Normalization

Given two elements a and b, one would like to know whether $a \leq b$, $a = b$ or $b \leq a$ hold. This is commonly known as the *word problem*. Whitman proposes a procedure to solve it on free lattices [Whi41] and that relies on the following relations.

$$\bigwedge a_i \leq x \Leftrightarrow \exists i, a_i \leq x \tag{1}$$

$$x \leq \bigvee b_i \Leftrightarrow \exists i, x \leq b_i \tag{2}$$

$$\bigvee a_i \leq x \Leftrightarrow \forall i, a_i \leq x \tag{3}$$

$$x \leq \bigwedge b_i \Leftrightarrow \forall i, x \leq b_i \tag{4}$$

Those relations are consequences of the axiomatization of free lattices. The important relation stressed by Whitman is the following ;

$$\bigwedge a_i \leq \bigvee b_j \Leftrightarrow \exists j, \bigwedge a_i \leq b_j \text{ or } \exists i, a_i \leq \bigvee b_j$$

Several decades later, the word problem for *free ortholattices* has been shown to be solvable in quadratic time by Bruns [Bru76] and, more recently, an algorithm has been proposed to efficiently compute such normal forms [GBMK23]. The important property of the normal form for orthologic we are about to expose, is that the computed transformation ain't be larger than the original formula. The following result exhibits the normal form for disjunction, it works dually for conjunction.

Theorem 6 (Normal form [RF95]). *A term $a = a_1 \vee \cdots \vee a_m$, with $m > 1$ is in normal form if and only if ;*

1. *each a_i is in normal form*
2. *if $a_i = \bigwedge a_{ij}$, then for all j, $a_{ij} \not\leq a$*
3. *the family $(a_1, ..., a_n)$ forms an antichain meaning that, if $i \neq j$ then $a_i \not\leq a_j$*

Inductive functions to ensure those properties are explicited in [GBMK23] and are actively used in the proposed algorithm.

Theorem 7. *NF_{OL} is a computable normal form for ortholattices.*

Theorem 8 (Transformation's complexity [GBMK23])**.** *The normal form for OL can be computed by an algorithm with a complexity in time and in space belonging to $\mathcal{O}(n^2)$. Moreover, the resulted form is guaranteed to be the smallest in the equivalence class of the input terms.*

1.3 Proof system

To make orthologic useful at effectively proving statements, we still need a set of rules that will allow us to derive a statement to hypothesis or axioms and hence, to prove it. There has been many proposals of proof systems and, among them, we may highlight [Lau17] or [GK24]. For our purposes, we shall focus on the second proposition, that we will present now.

Definition 9 (Orthologic's sequent)**.** *If ϕ is a formula, we say that ϕ^L and ϕ^R are annotated formulas. A sequent is a set of at most two annotated formulas.*

Therefore, to express the sequent, we will use the notation $\phi^\diamond, \psi^\circ$, where ϕ and ψ are formulas and $\diamond, \circ \in \{L, R\}$. By upper case greek letters Γ and Δ, we denote a set of annotated formula being either empty, or a singleton. Note that, in the sequent calculus Gentzen [Gen35] exhibits for intuitionnistic logic, the right side of the sequent can contain at most one formula. More generally, the proof system presented can be thought of a sequent calculus of classical logic with the restriction that the sequents contain at most two formulas, whatever is the side they belong to. A sequent can be interpreted with ortholattice in the following way. Suppose the sequent ϕ^L, ψ^R, this is exactly $\phi \leq \psi$ in ortholattices. Now, suppose the sequent ϕ^L, ψ^L, this is interpreted by $\phi \leq \neg\psi$. Dually, we have that ϕ^R, ψ^R is interpreted by $\neg\phi \leq \psi$. Finally, ϕ^L is interpreted by $\phi \leq \bot$ and ϕ^R is interpreted by $\top \leq \phi$.

Theorem 10 (Soundness and completeness [GK24])**.** *The orthologic proof system 1 is sound and complete.*

First, let us remark that the law of *excluded middle* is provable in OL, the corresponding sequent being $\phi^R, (\neg\phi)^R$. Furthermore, notice the presence of axioms in this proof system. Indeed, starting with a base knowledge allows to prove more formulas within the OL framework. To see how, let us sketch the following example.

Example 11. *Suppose, again, the relation $x \wedge (\neg x \vee y) \leq y$. We have seen, in the example 5 that this relation is not valid in ortholattices and, hence, the sequent $(x \wedge (\neg x \vee y))^L, y^R$ is not provable in orthologic without the axiom rule. However, considering the axiom rule, it is possible by using the axiom $x \wedge (\neg x \vee y)$.*

Figure 1: Proof system for propositional orthologic with axioms

$$\frac{\Gamma, \phi^R \qquad \phi^L, \Delta}{\Gamma, \Delta} \ (\textsc{Cut})$$

$$\frac{\Gamma}{\Gamma, \Delta} \ (\textsc{Weakening})$$

$$\frac{}{\phi^L, \phi^R} \ (\textsc{Hyp}) \qquad \frac{\Gamma \cup \Delta \in A}{\Gamma, \Delta} \ (\textsc{Ax})$$

$$\frac{\phi^L, \Delta}{(\phi \wedge \psi)^L, \Delta} \ (\wedge\text{-}L) \qquad \frac{\Gamma, \phi^R \qquad \Gamma, \psi^R}{\Gamma, (\phi \wedge \psi)^R} \ (\wedge\text{-}R)$$

$$\frac{\phi^L, \Delta \qquad \psi^L, \Delta}{(\phi \vee \psi)^L, \Delta} \ (\vee\text{-}L) \qquad \frac{\Gamma, \phi^R}{\Gamma, (\phi \vee \psi)^R} \ (\vee\text{-}R)$$

$$\frac{\Gamma, \phi^R}{\Gamma, (\neg\phi)^L} \ (\neg\text{-}L) \qquad \frac{\phi^L, \Delta}{(\neg\phi)^R, \Delta} \ (\neg\text{-}R)$$

2 Preprocessing

Since the procedure presented in the paper of Davis and Putnam [DP60], the practical methods to decide whether a given formula is satisfiable or not has remained more or less the same. The trick is to normalize the original formula into conjunctive normal form and then, learn knowledge on litterals and propagate it. The Tseitin's transformation [Tse83] allows one to compute the conjunctive normal form of a formula in linear time. However, the resulting formula is usually larger than the original one and carries a lot of redundant information. Nowaday, SAT solvers do some preprocessing on the given CNF formula to speed up the solving. As previously explained, the normal form in OL can be computed in worst-case quadratic time and has the good property of not being larger than the original formula. Therefore, the immediate question to ask is whether a SAT solver answers faster to a formula that has been first normalized into OL normal form and then, into CNF ? The main problem to that question is undeniably the lack of benchmarks that would contain random formulas, that is to say, formulas not in conjunctive normal form that still remain hard for SAT solvers to decide. To test our approach, we mainly used two kind of benchmarks.

2.1 Formula generation

The first way was to generate ourselves such formulas according to a procedure proposed by Navarro and Voronkov [NV05]. The idea is to craft formulas by alternating the disjunction and conjunction operations, given a family of integers, called the *shape*, that specifies the arity of the operators at each level of depth. Formally, suppose a family of integers $(k_1, \cdots, k_n)$ such that $k_i \geq 2$, the sets of formulas $[\![k_1, \cdots, k_n]\!]$ and $\langle k_1, \cdots, k_n \rangle$ are inductively defined such as the following.

1. if $n = 0$ then $[\![]\!]$ and $\langle\rangle$ contain literals only.

2. if $n \geq 1$ then $[\![k_1, \cdots, k_n]\!]$ is the set of conjunctions of arity k_1 of formulas in $\langle k_2, \cdots, k_n \rangle$. Dually, $\langle k_1, \cdots, k_n \rangle$ is the set of disjunctions of arity k_1 of formulas in $[\![k_2, \cdots, k_n]\!]$.

To the purpose of our experimentations, we isolated the specific shape $\langle 6, 3 \rangle$ that generates short – in term of nodes – and hard enough formulas to do proper measures.

2.2 Circuits

The second way to do the benchmarks was to take real-life circuits written in the And-Inverted Graph (AIG) or ISCAS formats. Those have the advantage to be raw and, hence, not in conjunctive normal form. For the AIG format, we used the arithmetic circuits offered by the EPFL combinatorial benchmark suite [AGDM15]. Older, but not easier for the solvers, is the ISCAS benchmark, that contains 21 circuits, all unsatisfiable and generated in the formal verification of correct superscalar microprocessors [Vel01].

2.3 Evaluation

We evaluated both of the benchmarks, the generated one and the AIG/ISCAS one, sticking to the following method. Given a formula F, in one hand we generate its CNF and, on the other hand, we generate its OL normal form that we then transform into a CNF. The two resulting formulas are sent to the same SAT solver and we compare the time it takes to decide. For the generated formula benchmark, we did the evaluation on a set of forty formulas of the shape $\langle 6, 3 \rangle$. Because we lack space, we will, in each benchmarks, show five of them randomly chosen, the remaining of the results can be found in the appendix.

3 The equivalence problem

To try to take advantage of the proof-search algorithm, one needs to find a class of problems that is hard for SAT solver to solve and that is easily encodable in the framework of orthologic. Let us go back to the circuits we used to benchmark the preprocessing method. An interesting question, that is hard for most of the solvers to answer to, is the

Problem	Without NF_{OL}	With NF_{OL}	Speed-up
S1	187.8s	132.9s	0.41
S2	102s	76.4s	0.34
S3	118.8s	88.4s	0.34
S4	172.3s	180s	-0.04
S5	83.5s	98.3s	-0.15

Table 2: Time (in seconds) to decide for $\langle 6,3\rangle$-shaped formulas

Problem	Without NF_{OL}	With NF_{OL}	Speed-up
4pipe3	14.96s	10.41s	0.44
4pipe4	16.4s	11.13s	0.47
5pipe	25.88s	31.56s	-0.18
5pipe1	59.22s	32.77s	0.8
6pipe	354s	200s	0.77

Table 3: Time (in seconds) for CaDiCaL to decide

question of equivalence between two circuits modulo variable renaming. The remaining work is now to prove that this specific problem can effectively be encoded in the OL proof system with axioms.

Theorem 12. *Let C and C' be two equivalent circuits modulo intermediate variable renaming. Furthermore, let z be the output of C and z' be the output of C'. There exists a set of axioms $\mathcal{A}$ such that $z \vdash z'$ in OL.*

Proof. We shall do a syntactic proof by induction on the structure of z.

- if z is an axiom, then it is straightforward.
- if $z = z_1 \wedge z_2$, we designate as axioms $(z, z_1 \wedge z_2)$ and $(z_1' \wedge z_2', z')$. Then, we first apply the cut rule,

  $$\frac{z \vdash z_1 \wedge z_2 \qquad z_1 \wedge z_2 \vdash z'}{z \vdash z'} \ (\textsc{Cut})$$

  The first sequent is an axiom. To prove the second one, we apply again the cut rule, leading us to,

  $$\frac{z_1 \wedge z_2 \vdash z_1' \wedge z_2' \qquad z_1' \wedge z_2' \vdash z'}{z_1 \wedge z_2 \vdash z'} \ (\textsc{Cut})$$

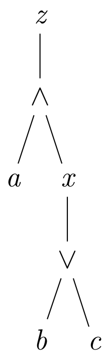
  Here, the second sequent is an axiom by hypothesis whereas the first one can be proven by applying first ($\wedge$-R), then successively ($\wedge$-L), giving us the two sequents $z_1 \vdash z_1'$ and $z_2 \vdash z_2'$ that admit a proof by induction hypothesis.

- if $z = z_1 \lor z_2$, we designate as axioms $(z, z_1 \lor z_2)$ and $(z'_1 \lor z'_2, z')$. Then, the reasonning is exactly the same until the second application of the cut rule, where it remains to prove the sequent $z_1 \lor z_2 \vdash z'_1 \lor z'_2$. Here, we apply first ($\lor$-$L$) and then successively ($\lor$-R), giving us the two sequents $z_1 \vdash z'_1$ and $z_2 \vdash z'_2$, both admitting a proof by induction hypothesis.

- if $z = \neg z_1$, we designate as axioms $(z, \neg z_1)$ and $(\neg z'_1, z')$. Then, we first apply the cut rule to introduce $\neg z_1$, producing the two sequents $z \vdash \neg z_1$ and $\neg z_1 \vdash z'$. The first one is an axiom. To prove the second one, we apply, once again, the cut rule to introduce $\neg z'_1$, giving us the two sequents $\neg z_1 \vdash \neg z'_1$ and $\neg z'_1 \vdash z'$. The second one is an axiom. To prove the remaining one, we successively apply ($\neg$-L) and ($\neg$-R), leading to the sequent $z'_1 \vdash z_1$, that admits a proof by induction hypothesis.

Finally, the set of axioms $\mathcal{A}$ we choose to prove a sequent is the union of the axioms described in each case, for each subsequent. □

Example 13. *Let us now illustrate the equivalence problem on a simple example. Suppose the following circuit.*



The formula represented is, in fact, $a \land (b \lor c)$. Therefore, the variables we rename to build the new circuit are essentially z and x. One wants to prove either the sequent z^L, z'^R or z'^L, z^R in OL. Since we have the equalities ; $x = b \lor c$ (resp. $x' = b \lor c$) and $z = a \land x$ (resp. $z' = a \land x'$), we give the following axioms to our system ; $x^L, (b \lor c)^R$ and $(b \lor c)^L, x^R$ (resp. for x') and $z^L, (a \land x)^R$ and $(a \land x)^L, z^R$ (resp. for z'). According to theorem 12, the OL proof system with axioms is able to find a proof for the equivalence statement.

4 Proof search procedure for OL with axioms

Bound to the proof system we have presented in the introduction, researchers have proposed a proof-search algorithm for orthologic with axioms that has a cubical complexity in time [GK24]. The idea is to start from the goal sequent and to compute a backward search to eventually reach the hypothesis and axioms by applying the rules of 1. To be efficient, some *memoization* would be used to keep in memory, for each sequent, whether they have been proved or not. However, the algorithm is not correct since, in the framework of orthologic proof system with axioms, it may lead to cycles and, hence, not terminate. The following example may enlight the problem we encounter.

Example 14. *The problem lies with the cut rule, for it is the one rule that may introduce formulas that are not subformulas of the sequent one seeks to prove. According to the backward algorithm, an instance of the cut rule is generated for each subformulas of the set of axioms. Suppose now that we want to prove the sequent* $\Gamma, (A_1 \wedge A_2)^R$ *with the axiom* $\Delta, (A_1 \wedge A_2)^R$*. By the rule* $(\wedge\text{-}R)$*, we now have to prove* Γ, A_1^R *and* Γ, A_2^R*. Let us focus on the first one. At the moment, note that the initial sequent* $\Gamma, (A_1 \wedge A_2)^R$ *is neither proved nor refuted. Since* Γ, A_1^R *is not an axiom, nor an hypothesis, let us craft the instances of the cut rule with subformulas of the only axiom we have. In particular, with the subformula* $A_1 \wedge A_2$*, this leads us to prove, in particular,* $\Gamma, (A_1 \wedge A_2)^R$*, which is the sequent we are trying to prove. Therefore, there is a cycle in the proof procedure, leading to non-termination!*

To remedy this, the immediate idea is to divide the algorithm in two distinct parts. First, we build, backwardly, the structure of every sequent we might have to prove in order to prove the goal sequent, according to the rules of our proof system. Then, we explore the resulted structure forwardly, until we reach the goal sequent. This was our first approach, in the following, we shall present the algorithm in more detail and we shall also show that it still runs in worst-case cubical time.

4.1 Backward-forward approach

As previously sketched, the idea of the algorithm we introduce can be decomposed in two parts such as what follows ;

1. First, it constructs backwardly the hypergraph of the proof space. The nodes of that hypergraph are the sequents and the hyperedges are the relations between the premisses and the conclusion.
2. Second, it explores the resulted hypergraph forwardly *i.e.* starting from the axioms and targetting the goal. The way it does this is a variant of the classical breadth-first search algorithm.

To be clear on what data structure we are using, let us define the one that composes the core of the algorithm, that is the *directed hypergraph.*

Definition 15. *A directed hypergraph is a tuple* $\langle V, E\rangle$ *where* V *is a set of vertices and* $E \subseteq \mathcal{P}(V) \times \mathcal{P}(V)$ *is a set of subsets of pairs whose elements are subsets of* V*. The first component of such pair being the source nodes and the second component being the target nodes.*

In fact, in our specific case, the elements of E are essentially in $\mathcal{P}(V) \times V$ since there is only one sequent to conclude with. With that set, we propose the proof-search algorithm 1 for orthologic. We shall now prove the correction and study carefully the complexity of that algorithm. To that last, we will prove several intermediate lemmas that will help to prove the complexity theorem.

Lemma 16. *Let A be a set of axioms and S be a sequent. The set of generated sequents from subformulas of sequents in A and S is of cardinal at most $4(||A|| + ||S||)^2$.*

Proof. Considering A and S, we have in total, and in the worst case, $||A|| + ||S||$ subformulas. To build sequents, we consider annotated formulas, that is to say, whether they are in the left part or in the right part of the sequent. Hence, we end up with $2(||A|| + ||S||)$ annotated subformulas. Sequents in OL are, by definition, the data of a pair of such annotated formula. Hence $4(||A|| + ||S||)^2$. □

Lemma 17. *Let S be a sequent. In the OL sequent calculus, assuming that axioms are never hypothesis, there exists at most $7 + 4|A|$ instances of rules whose conclusion is S.*

Proof. Let us enumerate every possible cases to derive the sequent S.

1. If S is of the form (s^L, s^R), it is an hypothesis and we therefore apply the *hypothesis* rule. If S is an axiom, we apply the *axiom* rule. Since we supposed that axioms never are hypothesis, those two cases are disjoint.

2. At any time, it is possible to apply the weakening rule. Suppose $S = (\Gamma, \Delta)$, there is two ways to derive it, either by dropping Γ or by dropping Δ.

3. Depending on the structure of Γ being either Left or Right, we can derive S in at most two ways. For the Left case, by using the ($\wedge$-L) rule, which let us derive either ϕ^L or ψ^L, the remaining rules derive only one sequent deterministically. For the Right case, by using the ($\vee$-R) rule, which let us derive either ϕ^R or ψ^R.

4. The same reasonning is done on the structure of Δ, which gives us at most two additional ways to derive S.

5. Finally, to craft instances of the cut rule, the algoritm 1 chooses terms among A^*. Remark that $|A^*| = 2|A|$ since it flattens the pairs of formulas. Therefore, for each term $a \in A^*$, there is two ways to build an instance of the cut rule either by deriving the sequents Γ, a^R and a^L, Δ or by deriving the sequents Δ, a^R and a^L, Γ. We end-up with $4|A|$ ways to instantiate the cut rule.

In the end, the first case gives us one way, the second case gives us two ways, the third case gives us two ways, the fourth case gives us two ways and the fifth case gives us $4|A|$ ways. In total, we get at most $7 + 4|A|$ ways to derive a sequent. □

Theorem 18. *The algorithm 1 is correct.*

Algorithm 1 Cubic time proof-search algorithm for OL with axioms

```
$(A, S)$ *the given problem*
$A^* \leftarrow \{a, b \mid (a^\circ, b^\diamond) \in A\}$
$V \leftarrow \{S\}$
$E \leftarrow \varnothing$
$W \leftarrow V$ *the set of vertices to visit*
**while** $W \neq \varnothing$ **do**
    $s = (\Gamma, \Delta) \leftarrow \mathrm{choose}(W)$
    $W \leftarrow W \setminus \{s\}$
    **if** $(\Gamma, \Delta) = (\phi^L, \phi^R) \mid\mid (\Gamma, \Delta) \in A$ **then**
        $E \leftarrow E \cup \{\langle \varnothing, s \rangle\}$ *hyperedge for either an hypothesis or an axiom*
        **continue**
    **end if**
    $E \leftarrow E \cup \{\langle \{(\Gamma, \Gamma)\}, s \rangle\} \cup \{\langle \{(\Delta, \Delta)\}, s \rangle\}$ *hyperedges for the weakening rule*
    $W \leftarrow W \cup \{(\Gamma, \Gamma), (\Delta, \Delta)\}$
    **if** $\Delta = \neg\phi^\circ$ **then**
        $E \leftarrow E \cup \{\langle \{(\Gamma, \phi^\circ)\}, s \rangle\}$
    **else if** $\Delta = (\phi \vee \psi)^L$ **then**
        $E \leftarrow E \cup \{\langle \{(\Gamma, \phi^L), (\Gamma, \psi^L)\}, s \rangle\}$
    **else if** $\Delta = (\phi \wedge \psi)^R$ **then**
        $E \leftarrow E \cup \{\langle \{(\Gamma, \phi^R), (\Gamma, \psi^R)\}, s \rangle\}$
    **else if** $\Delta = (\phi \vee \psi)^R$ **then**
        $E \leftarrow E \cup \{\langle \{(\Gamma, \phi^R)\}, s \rangle\} \cup \{\langle \{(\Gamma, \psi^R)\}, s \rangle\}$
    **end if**
    *the dual matching is applied to* $\Gamma$
    $E \leftarrow E \cup \bigcup_{a \in A^*} \{\langle \{(\Gamma, a^R), (a^L, \Delta)\}, s \rangle, \langle \{(\Delta, a^R), (a^L, \Gamma)\}, s \rangle\}$
    $W \leftarrow W \cup \bigcup_{a \in A^*} \{(\Gamma, a^R), (a^L, \Delta), (\Delta, a^R), (a^L, \Gamma)\}$
    $V \leftarrow V \cup W$
**end while**
$H \leftarrow \langle V, E \rangle$ *the resulting hypergraph*
$P \leftarrow A$ *the set of proven sequents*
$R \leftarrow \varnothing$
**while** $P \neq \varnothing$ **do**
    $s \leftarrow \mathrm{choose}(P)$
    $P \leftarrow P \setminus \{s\}$
    $R \leftarrow R \cup \{s\}$
    $E \leftarrow E[(K, d) \mapsto (K \setminus \{s\}, d)]$
    $P \leftarrow \{t \in V \mid (\varnothing, t) \in E\}$
**end while**
**return** $S \in R$ *checks whether the goal S is proven or not*
```

Proof. In the beginning of the first loop, the sets W and V only contain the goal sequent and the edge set E is empty. At each iteration, we pop a sequent s from W and, for each rule, we add the hypothesis sequents to W and the hyperedges of those sequents to $s = (\Gamma, \Delta)$. For the hypothesis rule, we check whether s is equal to a sequent of the form ϕ^L, ϕ^R. If it is the case, then we add the hyperedge $\langle \varnothing, s \rangle$. For the remaining rules, one has to explicitely do the comparison with Γ and Δ. Let us focus on Δ with the ($\wedge$-R) rule ; the proof is the same for Γ and the other rules. If Γ is equal to $(\phi, \psi)^R$ then, according to the rule, on needs to prove Γ, ϕ^R and Γ, ψ^R, therefore, the algorithm adds the hyperedge $\langle \{(\Gamma, \phi^R), (\Gamma, \psi^R)\}, s \rangle$. Finally, to handle the cut rule, it enumerates every formula a in the set of formulas that belong to axioms A^* and adds the hyperedges going from Γ, a^R and a^L, Δ to s and reciprocally, from Δ, a^R and a^L, Γ to s. All those new sequents are added to W for the following iteration. According to lemma 16, we explore a bounded set of sequents and, since we visit them once, W is, at some point, decreasing until reaching emptiness. Let us now prove the correctness for the forward search. At the beginning of the loop, we have H the hypergraph structure and P the set of proven sequents to be explored, composed only of the axioms at the beginning. Furthermore, R is the set of all proven sequents so far. At each iteration, the algorithm pops a sequent s from P, adds it to R and alters the set of hyperedges E in the following way, for each hyperedge, it removes s from the source. If, after the operation, the source of a given hyperedge is empty, it means that the destination has been proven, therefore, it adds it to P. Since H is a finite structure, P is, at some point, decrease until emptiness. By the end of the loop, R contains every provable sequents of H. Hence, the algorithm returns whether or not S, the goal sequent, belongs to R. □

Theorem 19. *The algorithm 1 runs in* $\mathcal{O}((1 + |A|)n^2)$.

Proof. To begin, let us analyze the complexity of the first part of the algorithm. Let H be the hypergraph constructed by algorithm 1. By lemma 16, we assert that the cardinal of V is at most $4(||A|| + ||S||)^2$, that belongs to $\mathcal{O}(n^2)$. By lemma 17, we know that the number of hyperedges we need to create for each sequent is at most $7 + 4|A|$, that belongs to $\mathcal{O}(1 + |A|)$. Hence, the algorithm builds the hypergraph H in $\mathcal{O}((1 + |A|)n^2)$. Let us now analyze the complexity of the forward part. The set of proven sequents is of cardinal at most the cardinal of V. Therefore, it belongs to $\mathcal{O}(n^2)$. Hence the final complexity.

□

There is several optimizations to do and that are done in the OCaml implementation. We can split them in two distinct parts. The reduction of the search space and the acceleration of forward exploration of the hypergraph. Let us start with the former. When building the hypergraph backward, there is no need to add remaining hyperedges once the axiom or hypothese hyperedge has been added. Also, we can give the priority to revertible rules that is to say, once we found such, no need to add other hyperedges. In the OL sequent calculus, invertible rules are ($\vee$-L) and ($\wedge$-R). This considerably reduces the size of the hypergraph but is that always useful to do ? In fact, although it accelerates the backward constuction and the search, we could wonder whether there exists shorter proofs passing through those erased derivations ? The second optimization is about the exploration of the hypergraph. To improve that, we mainly use memoization and the average constant time access provided by hash tables in average. Still, this is not enough to beat the Minisat solver. The main problem lies in the use of the cut rule that generates at most $7 + 4|A|$ hyperedges.

4.2 Forward approach

Although the complexity is still cubical, in practice, the main issue with the backward-forward approach is the construction of the hypergraph. Indeed, the algorithm builds a huge subpart (according to some of our optimizations) of it, always leading to the worst case in time *and* in space. Furthermore, most of the sequents belonging to that hypergraph won't be of any relevance in the proof. Let us refine this strategy by using the hypergraph *implicitely* to find a path between the goal sequent and the axioms. To achieve that, we completely give up the backward approach to only proceed forwardly. The idea of the algorithm is the following, we shall emphasize its efficiency. Each time we add a sequent to the set of proven sequents, we want to deduce every new sequent according to the set of rules of the OL proof system. To do this efficiently, we are not allowed to do any kind of test while going through the set of proven sequents. Therefore, the data structure we were using until now is not sufficient and we shall introduce a new one. We can split the algorithm in three important parts, namely ; the cut rule, the rules with one premisse and the rules with two premisses.

Remark. *For the sake of legibility in what follows, we will write (a, b) for the sequent a^L, b^R. Furthermore, we will write SF, the set of subformulas of our problem and AF, the set of formulas that compose the axioms. By P, we denote the set of proven sequents.*

Suppose we just proved the sequent (a, b). For the cut rule, it is possible to efficiently deduce the new sequent quite easily since we only have to lookup among the already proven sequents. Therefore, we need two dual maps $\overrightarrow{P} = x \mapsto \{y \in SF \mid x \vdash y\}$ and $\overleftarrow{P} = x \mapsto \{y \in SF \mid y \vdash x\}$ that are a partition of P. Hence, to deduce the new sequent from (a, b), we can just add forall $x \in \overrightarrow{P}(b)$, (a, x) and, dually, forall $y \in \overleftarrow{P}(a)$, (y, b). Doing that, we nevertheless have an important issue that is that the algorithm is more likely to loop since it will, at some point, add a sequent that has already been proved. The first thing to remediate to this is to put a check at the beginning of the function that will test whether the sequent we're trying to add hasn't been proved previously. Although, we can achieve that in constant time with a hash table, we still *check* the belonging for every sequent we encounter. Let us now have a look to the rules with only one premisse, that is ($\wedge$-L) and ($\vee$-R). We analyze only the former, since the strategy is dual. We know (a, b) is proven, from that, we can deduce that $a \vdash b \vee c$ and $a \wedge c \vdash b$ for a given c and such that $b \vee c$ and $a \wedge c$ still belong to the set of subformulas SF of our problem. To achieve that efficiently, we use two hash maps $A^{\diamond}$, with $\diamond \in \{\wedge, \vee\}$, that bind a formula x to the set of $\phi \in SF$ that are of the shape $\phi = x \diamond y$ or $\phi = y \diamond x$. Those two dual maps are computable in linear time with respect to the cardinal of the set of subformulas of the given problem. Finally, it remains to handle the case of the rules admitting two premisses, namely, ($\vee$-L) and ($\wedge$-R). Let us analyze only the left disjunction case, since the right conjunction is entirely symmetric. Roughly speaking, the difficulty here is to maintain the intersection between the set of subformulas SF and the set of formulas appearing in the left part of the proven sequents. For instance, suppose we have proved (a, b) we would like to deduce every $a \vee c \vdash b$ such that $a \vee c \in SF$. To achieve that, we obviously need $(c, b) \in P$ which, with the naive method, forces us to go through either SF or P and doing some useless testing. To do this efficiently, we need to ensure that every element we encounter in our datastructure will allow to produce a new proven sequent. The data structure we introduce takes advantage of the common element the three sequents of the rule have in common. In our case, it is the right part, that is to say b. First, let us consider B a hash map that binds to every formula x another hash map that itself binds to every formula y a set $\{x \vee y, y \vee x\} \subseteq SF$. It is clear that those sets have cardinal at most 2 depending on whether the formulas are in SF or not. We now need the data structure that will keep in memory that *intersection* between the proven sequents and the subformulas we sketched before. We shall call it D ; it is again a hash map that binds every formula x to another hash map that itself binds every formula y to a set of formulas that is of the shape $\{y \vee z, z \vee y \in SF \mid \forall z, (z, x) \in P\}$. To make things just said clearer, let us sum up the types of each structure.

$$\begin{aligned} P &: \mathcal{P}(SF \times SF) \\ A^{\diamond}, \overrightarrow{P}, \overleftarrow{P} &: SF \to \mathcal{P}(SF) \\ B^{\diamond}, C, D &: SF \to SF \to \mathcal{P}(SF) \end{aligned}$$

The algorithm 2 exhibits the pseudo-code of everything that have been said.

Theorem 20. *The algorithm 2 is correct.*

Proof. Before the first call to the function [add], the maps $A^{\diamond}$ and $B^{\diamond}$ are built and are not meant to be changed afterwards. The map C (resp. D) is empty before the first call to [add]. Each time a call to [add] is done with the sequent (a, b) as argument, if the sequent is the goal we seek, then the algorithm stops, returning true. Otherwise, it first updates C (resp . D) by making the union of conjunctive (resp. disjunctive) formulas of the form $b \wedge k$ or $k \wedge b$ (resp. $b \vee k$ or $k \vee b$) that may be proven if, at some point, (a, k) is proven. Here, the k lie in the set of keys of $B^{\wedge}(b)$, therefore there is all of them. After having updated C (resp. D), the algorithm will deduce new sequents. Remark that the order we do this, meaning the update, then the deduction, is of no importance

Algorithm 2 Cubical forward proof-search algorithm for OL with axioms

```
Let $SF$ be the set of the subformulas of a given problem $(\mathcal{A}, S)$
Let $AF \subseteq SF$ be the set of subformulas of the axioms
$P, \overrightarrow{P}, \overleftarrow{P} \leftarrow \varnothing$
$A^\diamond \leftarrow x \mapsto \{\phi \in SF \mid \forall y, \phi = x \diamond y \text{ or } \phi = y \diamond x\}$
$B^\diamond \leftarrow x \mapsto y \mapsto \{x \diamond y, y \diamond x\} \subseteq SF$
$C \leftarrow \varnothing$
$D \leftarrow \varnothing$
procedure ADD$(a, b)$
    if $(a, b) \in P$ then
        return false
    end if
    if $(a, b) = S$ then
        return true
    end if
    $P \leftarrow P \cup \{(a, b)\}$
    $\overrightarrow{P}(a) \leftarrow \overrightarrow{P}(a) \cup \{b\}$
    $\overleftarrow{P}(b) \leftarrow \overleftarrow{P}(b) \cup \{a\}$
    for $k \in keys(B^\vee(a))$ do
        $D(b)(k) \leftarrow D(b)(k) \cup B^\vee(a)(k)$
    end for
    for $k \in keys(B^\wedge(b))$ do
        $C(a)(k) \leftarrow C(a)(k) \cup B^\wedge(b)(k)$
    end for
    $res \leftarrow$ false
    for $\phi \in A^\wedge(a) \cup D(b)(a) \cup \overleftarrow{P}(a)$ do
        $res \leftarrow res \mid\mid ADD(\phi, b)$
    end for
    for $\phi \in A^\vee(b) \cup C(a)(b) \cup \overrightarrow{P}(b)$ do
        $res \leftarrow res \mid\mid ADD(a, \phi)$
    end for
    return res
end procedure
for $(a_1, a_2) \in \mathcal{A}$ do
    $ADD(a_2, a_2)$
end for
```

for the proof. However it becomes interesting for optimizations, for instance, to do a tail-recursive program. There is two kind of sequents we can deduce, (ϕ, b) and (a, ϕ) where $\phi \in SF$. Let us handle both of the cases.

1. for the case (ϕ, b), the ϕ formulas belong to the union of $A^{\wedge}(a)$, $D(b)(a)$ and $\overleftarrow{P}(a)$. At this state of the algorithm, $A^{\wedge}(a)$, as we said before, is not altered and therefore, it is still of the shape $\{\phi \in SF \mid \forall y, \phi = a \wedge y \text{ or } \phi = y \wedge a\}$. Hence, adding the sequent (ϕ, b) to the set of proven ones is correct. Furthermore, at that point of the algorithm, $D(b)(a)$ contains exactly the following intersection set, $\{a \vee x \in SF \mid (x, b) \in P\}$. Therefore, since (a, b) and (x, b) are proven, adding $(a \vee x, b)$ is correct. Finally, $\overleftarrow{P}(a)$ contains every ϕ such that (ϕ, a) is proven and, since we just added (a, b), adding (ϕ, b) is correct. However, (ϕ, b) might have already been proven, the check, to avoid cycles, is done in the begining of the function [add]. Hence, there is no cycles. Therefore, adding (ϕ, b) doesn't create any cycle and is correct.

2. for the case (a, ϕ), the reasonning is completely dual.

This was the main proof for the [add] procedure. The main procedure calls [add] passing it, as arguments, the sequents in $\mathcal{A}$, that is to say, the axioms of the given problem. Hence, the algorithm is correct with respect to the OL proof system. □

Theorem 21. *The algorithm 2 has a time complexity belonging to $\mathcal{O}(n^3)$.*

Proof. Let us start the complexity analysis by the one-premisse rules, that is to say, the right disjunction and the left conjunction. To achieve that, we make use of the maps $A^{\wedge}$ and $A^{\vee}$ that map every formula $s \in SF$ to a set of formulas in SF containing formulas that admit s as subformula. This set is built at the very beginning in linear time with respect to the size n of the set of subformulas SF. Then, each time we make a call to the function [add] with the sequent (a, b), the algorithm goes through the sets given by $A^{\wedge}(a)$ and $A^{\vee}(b)$. The cardinal of both of these is bounded by the cardinal of SF because a given formula can appear in at most every subformulas of the problem, that is n. Therefore, the complexity for that subset of rules in overall is $\mathcal{O}(n^3)$. Let us now analyze the rules with two premisses that are the right conjunction and the left disjunction. For the sake of brevity of the proof, let us focus on the disjunction case only, since the conjunction case is completely dual. As roughly explained before, we handle it with the map D. At each [add] function call, we first update the D map. To do so, we gather every formulas in SF of the shape $a \vee x$ where x belongs to SF. Through the map $B^{\vee}$, we access those in constant time. Therefore, the updating is done in k, where k is the cardinal of the set of x such that $a \vee x$ or $x \vee a$ are in SF and is, hence, bounded by n. In the overall, we therefore lie in $\mathcal{O}(n^3)$. Finally, to analyzse the cut rule case, let us remark that, given the sequent (a, b), if a and b is not in the set of the axiom formulas AF, then the complexity to add the sequent is constant. Otherwise, if either a or b is in AF, then the complexity to add the sequent is linear. Let's build a partition of the set of our sequents such as, $I = \{(a, b) \mid a \in AF \text{ or } b \in AF\}$ and $J = \{(a, b) \mid a \notin AF \text{ and } b \notin AF\}$. Splitting the sum of operations done, depending on the case we are in, we get,

$$\sum_{(a,b)\in I} n + \sum_{(a,b)\in J} 1 \leq n^2(|AF| + 1)$$

Summing up every cases, we finally get that the overall complexity of the algorithm 2 lies, in the worst case, in $\mathcal{O}(n^3)$. □

Remark. *Let us take time to write some remarks on the OCaml implementation. First of all, to avoid stack overflowing, the algorithm has been written tail-recursively. Then, pay a special attention to the data structure B. Indeed, it supposes that the operators $\wedge$ and $\vee$ are of arity exactly two. Therefore, one cannot encode the formulas in an ADT that, for instance, would type the conjunction constructor such as $And : formula\ list \rightarrow formula$. Doing that, one would need to filter the formulas having arity exactly 2 to add them to the datastructure B, which results in an unsound algorithm, according to the OL proof system.*

4.3 Evaluation

Now that we have a working efficient algorithm, we are ready to compare it to SAT solvers on the class of problems of circuit equivalence checking, modulo variable renaming, presented in section 3. Among the benchmarks we dispose, a circuit that appears to be particularly hard is the circuit of multiplication operation. Its hardness is one of the reason why one cannot easily break the RSA cipher protocol with a computer. The testing procedure is the following. First, we parse the given AIGER file and we store the data in a convenient algebraic data type. Then, from this data, we build the circuit equivalence by renaming the intermediate variables (the ones that name the operations). From the resulting formula, we compute, on one hand, the conjunctive normal form of its negation, applying the usual Tseitin's transformation and, on the other hand, we encode it into OL with axioms. Finally, we can give that encoding to our algorithm, so it tries to prove it. On the other side, we run a SAT solver on the produced CNF. Note that, if it terminates, it must yield unsatisfiability, because the equivalence formula is valid. Otherwise, it means that the encoding is wrong. In our case, we expect the solver to yield unsat and our algorithm to find a proof. The table 4 shows the time taken by our algorithm (OLSC) *versus* the time taken by the CryptoMinisat [SNC09] and CaDiCaL [BFF$^+$24] solvers to decide the equivalence problem on several instances of the multiplier circuit.

Depending on the instances of the multiplier circuits, the proof-search algorithm 2 (OLSC) performs 20 to 30 times faster than CryptoMinisat and 5 to 9 time faster than CaDiCaL.

	OLSC	CryptoMinisat	CaDiCaL
mult 12 bits	1.6s	37.3s	5.7s
mult 16 bits	3.5s	97.5s	18.2s
mult 20 bits	5.4s	165.7s	30.9s
mult 24 bits	8.6s	294.4s	56.5s
mult 32 bits	20.3s	> 500s	117.3s
mult 40 bits	37.3s	> 500s	327.2s

Table 4: Time (in seconds) to solve the equivalence problem, comparing our algorithm with SAT solvers

5 Going further

The obtained results lead us to consider two main improvements on SAT solvers in order to speed-up the decision procedure. The first improvement concerns the preprocessing. Before converting any formula to conjunctive normal form, it might be useful to transform it first to the OL normal form. Depending on the shape of the formula, the decision procedure is faster. The question is now to identify the species on which the OL normalization will certainly lead to a net speed-up of the SAT solver. The second improvement naturally concerns the proof algorithm itself. For some class of hard problems, one can show that it is encodable in the OL framework and, hence, provable by our proof-search algorithm. Therefore, it would be interesting for SAT solvers to provide an interface that allows one to specify the encoding of a given class of problems into the OL framework. Our algorithm would then be called when the solver has to deal with such class of problems. There is also improvements to do on the algorithm 2 itself. The main problem is the datastructure $B^\diamond$ because it enforces us not to use an ADT able to flatten the formula, leading to stack overflows of the datastructure when the formula is too big. Therefore, we would rather need to type $B^\diamond$ with SF list $\to \mathcal{P}(SF)$. This maps an arbitrary list of subformulas to the set of every existing permutation that lies in SF, according to the operator $\diamond$.

6 Conclusion

Along this work, we investigated several ways to improve the performance of SAT solvers using some properties of *orthologics*. On one hand, we have shown that preprocessing the formulas using the OL normal form could, in average, speed up the solving. On the other hand, we provided an efficient proof-search algorithm to prove statement within the OL proof system.

7 Appendix

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