NYU | COURANT INSTITUTE OF MATHEMATICAL SCIENCES

MATH-GA 2791.001 Financial Securities and Markets
Fall 2024, Homework 1

Prof. Bernhard Hientzsch
Due: See deadline on Brighspace

General Guidelines For All Homework In This Course

Student collaboration: Work should be done individually. While you can discuss the course content to develop a better understanding, homework solutions should be worked out alone and written up by yourself. Questions are sometimes asked in a loose way so that you have more freedom in the way to answer them. At the same time, it is less likely that two answers are very similar without indicating too much collaboration between student that went far beyond these guidelines.

Important note: While parts of homework assignments in this course may be similar to homework given in previous years, please do not refer to or copy past homework submissions. First, it defeats the purpose of learning. Second, it violates academic integrity and may lead to failing the course.[1]

Use of ChatGPT: You can use ChatGPT or other AI engines. Note however that you have to demonstrate a complete understanding of your submission. Code generated by ChatGPT can be used but most likely will need additional re-factoring and commenting work (although ChatGPT does write some comments).

Questions About Homework: Please post any (clarification) questions about homework assignments on the NYU Brightspace course discussion board so that all students can benefit, without presenting (parts of) solutions.

Homework Deliverables: Please follow these guidelines:

1. Your write-up should be typeset and written in clean and error-free English. Poor writings will receive deductions. Explain clearly and fully your answer in a way that demonstrate your good understanding of the mathematical and financial principles you are using. If needed, make sure you describe and motivate all assumptions and any other issues you may encounter in your work.

2. Final submission should be in a Jupyter notebook. Use cells in Markdown mode to write your answers and explanation of the scientific methods used.

[1] Please refer to the course syllabus as well as the Graduate School of Arts and Science Policies and Procedures Manual for the full policy on issues of academic misconduct.

Grading:

1. Homework is scored out of 100 points even if the total number of points attainable might be higher.
2. Extra credit: homework can have you earn more then 100 points which can be used for other homework or quizzes.

Objective

The purpose of this homework is to develop a better understanding of the Black-Scholes model and how option pricing and hedging can work in practice.

When a question asks you to write a function, if not explicitly, it is implicitly also asking that you provide example of tests for that function, usually by testing some outputs, plotting some graphs or tables of some testing results.

Technical Guidelines

The simulations we do will assume a daily time step, i.e. $\delta t = 1/360$ (count all days and assume exchanges are always open and assuming a year is 360 days). When relevant, time to maturity will be computed as actual360 i.e. number $t_2 - t_1$ will be equal to the number of days between the date t_2 and t_1 (while excluding t_1, think of when there is just one day between two dates) divided by 360.

Use python standard packages (numpy, scipy, pandas, datetime, matplotlib, seaborn etc.) to simulate standard Gaussian variables, plot your graph and histogram, etc. write your code in general.

Homework Questions

A standard way to simulate paths of stochastic processes over an interval $[0, T]$ is to consider a number of time step N, then set $\delta t = T/N$ and then simulate W_t at discrete times $t_i = i\delta t$.

To simulate a Brownian motion ($\mathcal{BM}$) from t_i to t_{i+1} we use that $W_{t_{i+1}} = W_{t_i} + (W_{t_{i+1}} - W_{t_i})$, with the property that $W_{t_{i+1}} - W_{t_i}$ is independent of W_{t_i} together with $W_{t_{i+1}} - W_{t_i} \sim \sqrt{t_{i+1} - t_i} Z_i = \sqrt{\delta t} Z_i$ with $Z_i \sim \mathcal{N}(0, 1)$. As such, we can simulate $Z_1, ..., Z_N$ and set $W_{t_i} = W_{t_{i-1}} + \sqrt{\delta t} Z_i$.

1. Simulation of Brownian Motion Paths (5 pts)
   (a) Write a function that simulate paths of Brownian Motion.
   (b) For 20k paths, estimate the average $E(W_t)$ at each time step, as well as the standard deviation $\sqrt{Var(W_t)}$ from the Brownian paths.
   (c) Plot the results along with theoretical values for $E(W_t)$ and $\sqrt{Var(W_t)}$.

2. Simulation of Black-Scholes Paths (5 pts)
   We assume a Black-Scholes underlying with $dS_t = \mu S_t dt + \sigma S_t dW_t$. As a baseline, we assume $S_0 = 100$, $\mu = 5\%$, $r = 3\%$, $\sigma = 30\%$

   To simulate the underlying S_t we notice that

$$S_{t_{i+1}} = S_{t_i} e^{(\mu - \frac{1}{2}\sigma^2)(t_{i+1} - t_i) + \sigma(W_{t_{i+1}} - W_{t_i})} = S_{t_i} e^{(\mu - \frac{1}{2}\sigma^2)\delta t + \sigma\sqrt{\delta t} Z_{i+1}}$$

   Repeat the previous question but on S_t this time.

3. Pricing by Simulation (20 pts)
   We want to price a call option with strike $K = S_0$ with expiry $T = 1Y$ by simulation
   (a) After having simulated underlying prices as above, compute the discounted payoff for each of the trajectories. Perform statistics to compute mean and standard deviation (and standard error) to obtain price estimates, together with "standard error bars", for the call option.
   (b) Repeat but for two other strike prices.
   (c) How does price and standard error behave for different number of paths and those different strikes?

4. Simulation of Hedging (30 pts)
   We consider to be short a call option with strike $K = S_0$ with expiry $T = 0.5Y$.
   (a) Simulate (daily) values for the hedging portfolio $P_t = \Delta_t S_t + \varphi_t B_t$. Explain in details how you perform the simulation.
   (b) Plot paths (daily time steps) of the portfolio compared to the theoretical value $C_{bs}(t, S_t, K, r, \sigma, T)$ for a few paths. What do you observe?
   (c) Plot histogram of the hedging error $\varepsilon_T = P_T - (S_T - K)^+$. Display $E(\varepsilon_T)$ and $\sqrt{Var(\varepsilon_T)}$ in the title of the each plot.

5. Behavior of Hedging error (30 pts)
   We want to observe how ε_T behave as we increase μ gradually from 5% to 20% and decreases the number of time step to about weekly $\delta t = 1/60$ or biweekly $\delta t = 1/30$. We also want to understand the effect of the volatility σ and we increases it gradually to $\sigma = 80\%$.
   (a) Show how the histogram of ε_T changes as dt, μ and σ increase.
   (b) Comment on your findings.

6. Estimation of Volatility (30 pts)

   Recall that for a diffusion process $dX_t = a_t dt + b_t dW_t$, the quadratic variation is $\langle X \rangle_T = \lim_N \sum_{i=1}^{N} (X_{t_i} - X_{t_{i-1}})^2 = \int_0^T b_t^2 dt$ where the convergence is a.s.

   (a) In the Black-Scholes model, set $X_t = \ln(S_t)$. What is $\langle X \rangle_T$?

   (b) For a given path of S_t, show distinct plots (for a few paths) of $\frac{360}{n} \sum_{i=1}^{n} (X_{t_i} - X_{t_{i-1}})^2$ as n increases from 1 to $N = 360$. What do you observe? This is called the **historical** volatility. (Often done on business day basis, but we here look at every day.)

   (c) For a given path of S_t, compute the value σ that makes the hedging of an ATM call perfect. It is called the **Break/Even** (B/E) volatility. Note that σ affects both initial price and the value of the delta. Repeat on 10k paths and show the distribution of that volatility. Comment on your findings.

7. Holding Value by Simulation (30 pts)

   We want to model the holding value of a call option with strike $K = S_0$ with maturity T at time $t = 0.5Y$ as a function of the value of the underlying at time t, S_t.

   (a) Simulate Black-Scholes paths as in previous question, evaluate the payoff, and collect the value of the underlying at time t. Visualize the data that you collected.

   (b) Pick some set of basis functions (Longstaff-Schwartz article mentions a number of possible choices), pick the first 2, 3, or 4 of them. Set up the ordinary least square problem to find the best approximation of the conditional expectation of the payoff as a linear combination of those basis functions given the value of the underlying at time t and solve it. Plot the approximation(s) of the conditional expectation against the data that you collected.

   (c) How do those approximations differ for different number of basis functions? How do they compare against the Black-Scholes formula (do not forget to include discounting in the comparison or in the OLS regression)? Comment on your findings.

   (d) How could you use this to decide whether to exercise a Bermudan call option with an early exercise opportunity at $t = 0.5Y$?