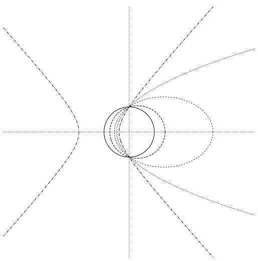
Gravitational slingshot

General solution for the two–body gravitational problem



The expression

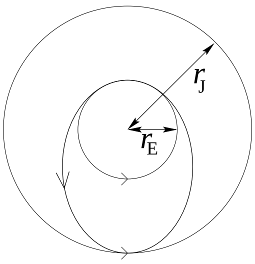
$$r(\theta) = \frac{l}{1 - \epsilon\cos(\theta - \theta_0)}$$

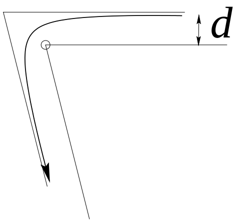
describes all the possible orbits of a tiny mass near a large mass located at the origin. θ_0 determines the orientation of the orbit. The figure shows orbits with fixed θ_0 and l, and different eccentricities ϵ: $\epsilon = 0, 0.3, 0.7, 1, 1.5$.

$\epsilon = 0$	circle
$\epsilon \in (0, 1)$	ellipse
$\epsilon \in (-1, 0)$	ellipse with r_{min} and r_{max} interchanged
$\epsilon = 1$	parabola
$\lvert\epsilon\rvert > 1$	hyperbola

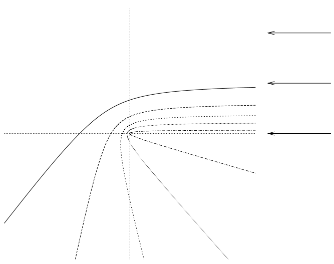
If $|\epsilon| > 1$ then the expression $r(\theta)$ describes two branches, both of which are shown in the above figure. Only one of the branches is a solution of the equation of motion, however. The other branch is the solution of the equation of motion for a *repulsive* $1/r^2$ force.

Slingshot

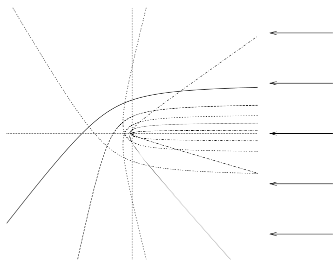




A spacecraft arrives at Jupiter with velocity parallel to Jupiter's, but smaller. As Jupiter overtakes the spacecraft, we transform into Jupiter's frame.

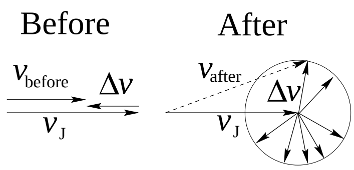


(a)



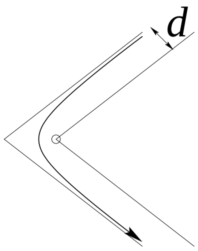
(b)

The outcome depends on the 'impact parameter' d. Figure (a) shows possible trajectories (hyperbolas) if the craft arrives on the upper side of Jupiter. Figure (b) shows possible trajectories for arrivals on both sides.



(c)

The interaction is an elastic collision. The speed of the spacecraft afterwards, relative to Jupiter, is still Δv, but the direction has changed. Figure (c) shows the velocities before and after in the sun frame (v_{after} is the dashed vector). Notice that whatever orientation is chosen, the speed of the spacecraft in the sun frame is always increased by the collision. In this way, the Voyager spacecraft stole energy from Jupiter to speed themselves on



their way to Saturn, Neptune and Uranus. The slingshot manoeuvre was also used by Galileo (Venus, Earth, Earth → Jupiter). On arrival, Galileo slowed itself down relative to Jupiter by a slingshot around the moon Io, and a lengthy thrust of its 400 N engine. It has since adjusted its orbit around Jupiter using further slingshots around other Jovian moons.

Impact parameter relationships

Page 2 gave formulae relating a, b, c, $r_{\min}$, $r_{\max}$, l and ϵ. The impact parameter d is the distance between the trajectory at infinity and the trajectory for direct impact. In this figure, d is shown in the standard orientation such that $r(\theta) = \frac{l}{1-\epsilon\cos\theta}$ and the point of closest approach is at $\theta = \pi$ (in contrast, on the previous page, all particles arrived from a common direction).

We can relate d to the other parameters using the angular momentum at infinity, $J = mv_\infty d$, and the previously derived relationship between J and l,

$$J^2 = GMm^2 l, \tag{1}$$

to obtain

$$l = \frac{v_\infty^2 d^2}{GM} \tag{2}$$

We can eliminate the energy–like terms using

$$E = \frac{1}{2}mv_\infty^2 = -\frac{GMm(1-\epsilon^2)}{2l}, \tag{3}$$

giving the convenient relationship

$$d = \frac{l}{\sqrt{\epsilon^2 - 1}}. \tag{4}$$

The impact parameter d of a hyperbola equals i times the semi-minor axis $b = l/\sqrt{1-\epsilon^2}$. You are not expected to memorize these relationships, but you should understand how they can be derived.

Repulsive inverse–square interactions

If we change the sign on the inverse square force, the only orbits that are possible are hyperbolas. The hyperbola has two branches, one of which is the solution for the attractive force, and the other is the solution for the repulsive force.

The figures below show the trajectories for a variety of impact parameters for incoming velocities equal to 1, 2, 4 and 8 units respectively.