Analysis I

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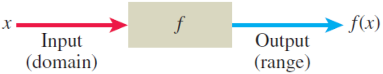
1. Preliminaries

1.4. Functions

Content:

- Definition
- Domain, codomain, range
- Representations of a function
- Injection, surjection, bijection
- Increasing and decreasing functions
- Even and odd functions
- Common functions
- Operations with functions
- Transformations of a graph of a function
- Composite functions
- Inverse functions

A **function** is a rule which assigns, to each of certain real numbers ("inputs") , some other real numbers ("outputs").



(from Thomas' Calculus)

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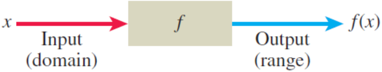

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A **function** from a set $X \subseteq \mathbb{R}$ to a set $Y \subseteq \mathbb{R}$ is a rule that assigns a unique (single) element of Y to each element of X. **Notations**: $f : X \to Y$, $f : x \to y$, $y = f(x)$, $f(x)$, $f(c) := f(x)|_{x=c}$.

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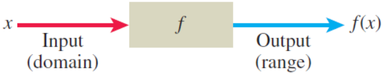

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Example: $y = x^2$, $f(x) = x^2$, $f : x \to x^2$.

Example: this tree grows 20 cm every year, so the height of the tree is ***related*** to its age using the function ***h***:

***h*(age) = age × 20**

So, if the age is 10 years, the height is $h(10) = 200$ cm

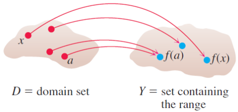
Saying "***h*(10) = 200**" is like saying 10 is related to 200.

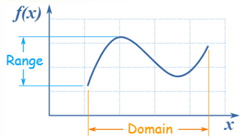
Or 10 → 200

(from www.mathsisfun.com)

Domain, codomain, range

For $f : X \to Y$, X is the **domain** of the function ($D(f)$ or $\mathrm{dom}(f)$), Y is the **codomain** of the function.
The value of f at $x \in X$, $f(x)$, is the **image** of x under f, or the **value** of f applied to the **argument** x. The set of all output values of $f(x)$ as x varies throughout X is the **range** of f ($R(f)$ or $\mathrm{ran}(f)$):
$R(f) := f(X) := \{y \in Y | \exists x \in X : f(x) = y\}$. Obviously, $R(f) \subseteq Y$.





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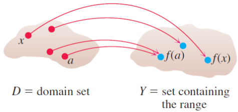
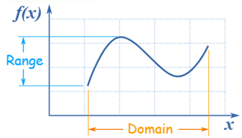


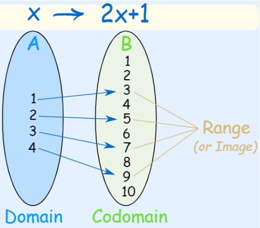


Example

- The set "A" is the **Domain**,
- The set "B" is the **Codomain**,
- And the set of elements that get pointed to in B (the actual values produced by the function) are the **Range**, also called the Image.

And we have:

- Domain: {1, 2, 3, 4}
- Codomain: {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}
- Range: {3, 5, 7, 9}



(from Thomas' Calculus & www.mathsisfun.com)

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Domain, codomain, range

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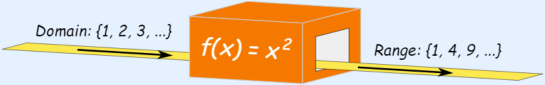
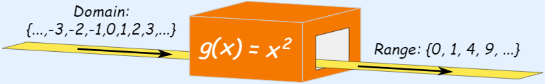
Example: a simple function like $f(x) = x^2$ can have the **domain** (what goes in) of just the counting numbers $\{1,2,3,\ldots\}$, and the **range** will then be the set $\{1,4,9,\ldots\}$



And another function $g(x) = x^2$ can have the domain of integers $\{\ldots,-3,-2,-1,0,1,2,3,\ldots\}$, in which case the range is the set $\{0,1,4,9,\ldots\}$



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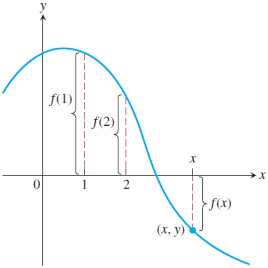
Two functions f and g are **equal**, $f = g$, if $D(f) = D(g)$, $R(f) = R(g)$, and $f(x) = g(x)\ \forall x \in D(f)$.

Graphs of functions

If f is a function with domain D, its **graph**, $\Gamma(f)$, is the set of the points in the Cartesian plane whose coordinates are the input-output pairs for f:

$$\Gamma(f) = \{(x, y) | x \in D, y = f(x)\}.$$

If (x, y) is a point on the graph, then $y = f(x)$ is the height of the graph above (or below) the point x. The height may be positive or negative, depending on the sign of $f(x)$.



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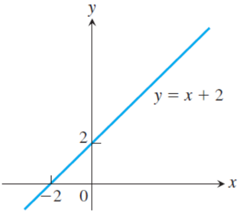

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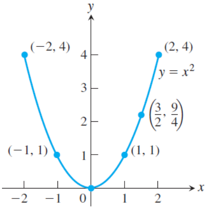
-1	1
0	0
1	1
$\frac{3}{2}$	$\frac{9}{4}$
2	4



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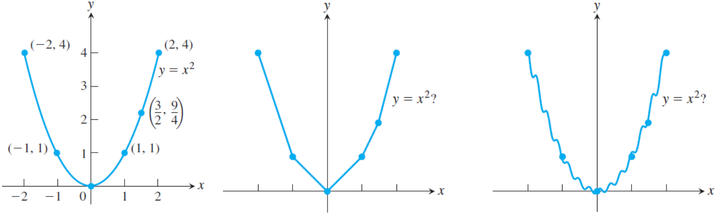
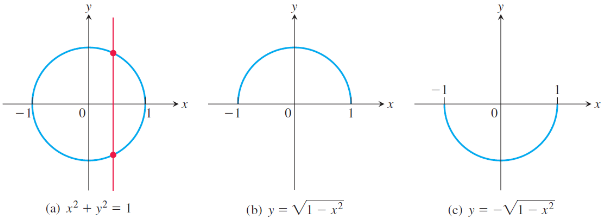


(from Thomas' Calculus)

Not every curve in the coordinate plane can be the graph of a function!

Vertical line test

No vertical line can intersect the graph of a function more than once. If a is in the domain of the function f, then the vertical line $x = a$ will intersect the graph of f at the single point $(a, f(a))$.



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Piecewise-defined functions

Sometimes a function is described in pieces by using different formulas on different parts of its domain, for **example**:

$$f(x) = \begin{cases} -x, & x < 0, \\ x^2, & 0 \le x \le 1, \\ 1, & x > 1. \end{cases}$$

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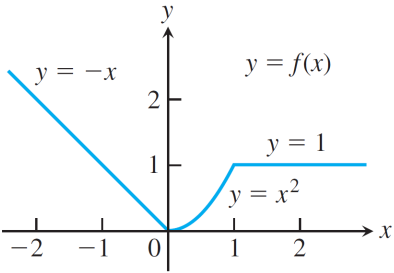

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Sometimes a function is described in pieces by using different formulas on different parts of its domain, for **example**:

Greatest integer function (integer floor function)

The function whose value at any number x is the greatest integer less than or equal to x, $f(x) = \lfloor x \rfloor$.

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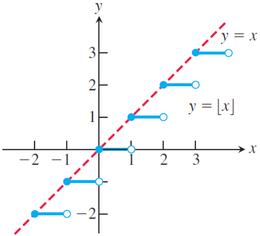

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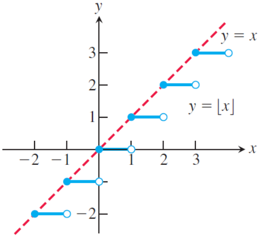

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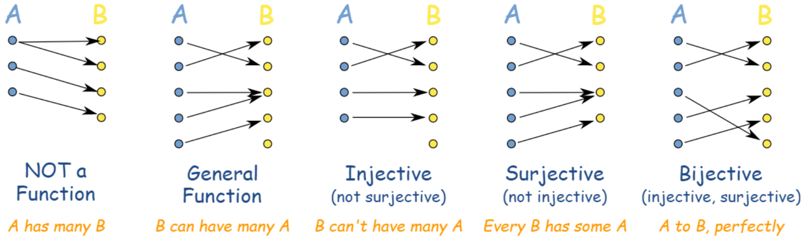
Injection, surjection, bijection

A function $f : X \to Y$ is called

- **injective** (or into, one-to-one, injection) if it maps any two different elements of X to different elements of X: $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ (or $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$),
- **surjective** (or onto, surjection) if for every element of Y, there is at least one element of X that maps to it, i.e. $\forall y \in Y \exists x \in X : f(x) = y$
- **bijective** (or a one-to-one correspondence, bijection) if the function is both injective and surjective.



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Injection, surjection, bijection

Horizontal line test

If every line $y = c$, $c \in Y$

- intersects the graph of f at most once, then f is injective;
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Examples: $f_1 : \mathbb{R} \to \mathbb{R}$, $f_1(x) = x^2$:

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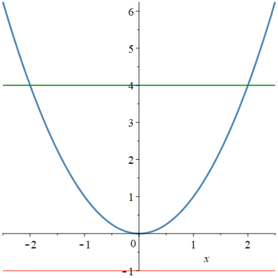
Examples: $f_1 : \mathbb{R} \to \mathbb{R}$, $f_1(x) = x^2$:neither injective (e.g., $f_1(2) = f_1(-2) = 4$), nor surjective (e.g., $\nexists x \in \mathbb{R} : f_1(x) = -1$.)



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Examples: $f_2 : \mathbb{R} \to [0, +\infty)$, $f_2(x) = x^2$:

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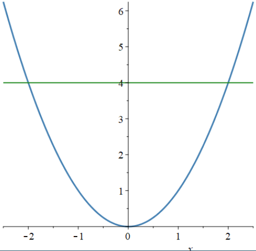
Examples: $f_2 : \mathbb{R} \to [0, +\infty)$, $f_2(x) = x^2$:not injective (e.g., $f_1(2) = f_1(-2) = 4$), but surjective ($\forall y \in [0, +\infty) : \exists x \in \mathbb{R} : y = x^2$.)



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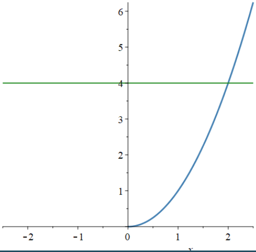
Examples: $f_3 : [0, +\infty) \to [0, +\infty)$, $f_2(x) = x^2$:bijective, i.e. injective ($\forall x, y \in [0, +\infty)$ holds: $x^2 = y^2 \iff x = y$ and surjective ($\forall y \in [0, +\infty) : \exists x \in \mathbb{R} : y = x^2$.)



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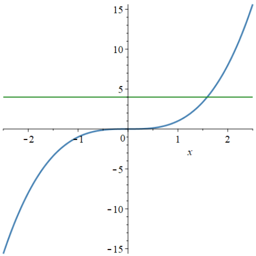
Examples: $f_4 : \mathbb{R} \to \mathbb{R}$, $f_4(x) = x^3$:

Horizontal line test

If every line $y = c$, $c \in Y$

- intersects the graph of f at most once, then f is injective;
- intersects the graph of f at least once, then f is surjective;
- intersects the graph of f exactly once, then f is bijective.

Examples: $f_4 : \mathbb{R} \to \mathbb{R}$, $f_4(x) = x^3$:bijective.



Injection, surjection, bijection

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Injection, surjection, bijection

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Horizontal line test

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Examples: $f_5 : \mathbb{R} \to [-1, 1]$, $f_5(x) = \cos x$:

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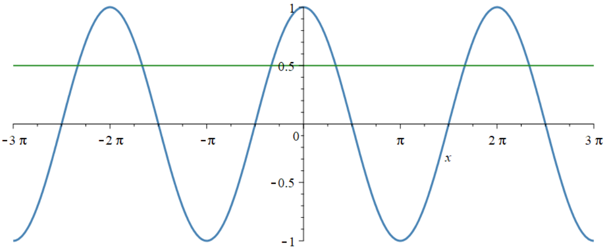
Examples: $f_5 : \mathbb{R} \to [-1, 1]$, $f_5(x) = \cos x$:surjective.



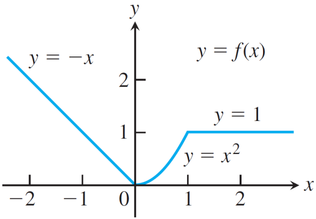
Increasing and decreasing functions

Let $f : D \to \mathbb{R}$ be a function, $D \subseteq \mathbb{R}$, $I \subseteq D$ be an interval, x_1 and x_2 be any two points in I. The function f is

- **increasing on I** if $f(x_2) \geq f(x_1)$ whenever $x_1 < x_2$;
- **decreasing on I** if $f(x_2) \leq f(x_1)$ whenever $x_1 < x_2$.

If the inequality is strict, then f is **strictly increasing** or **strictly decreasing on I**

Example:



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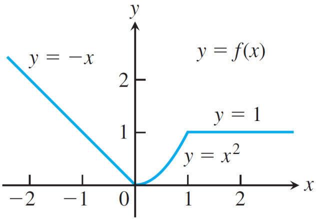

The graphed function is strictly decreasing on $(-\infty, 0]$, strictly increasing on $[0, 1]$, and constant on $[1, +\infty)$.
(from Thomas' Calculus)

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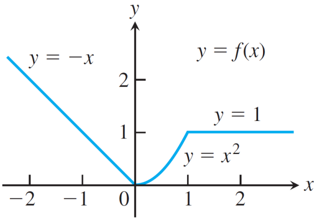


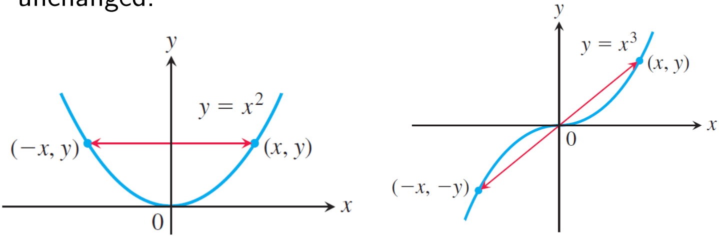
The graphed function is strictly decreasing on $(-\infty, 0]$, strictly increasing on $[0, 1]$, and constant on $[1, +\infty)$.
(from Thomas' Calculus)

A non-decreasing or non-increasing on I function is **monotonic on I**.

A function $f(x)$ with a domain $D \subseteq \mathbb{R}$ is an

- **even function of x** if $f(-x) = f(x)$ for all $x \in D$,
- **odd function of x** if $f(-x) = -f(x)$ for all $x \in D$.

- The graph of an even function is symmetric about the y-axis, i.e. a reflection across the y-axis leaves the graph unchanged.
- The graph of an odd function is symmetric about the origin, i.e. a rotation of 180° about the origin leaves the graph unchanged.

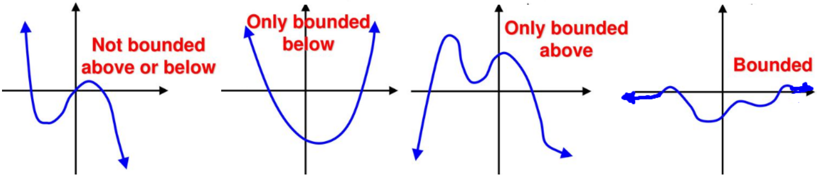


(from Thomas' Calculus)

Bounded and unbounded functions

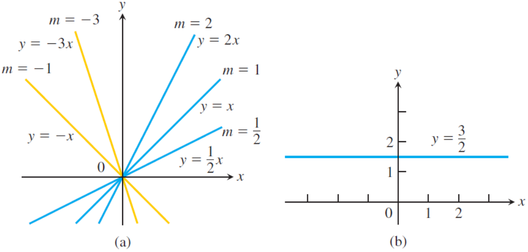
A function $f(x)$ with a domain $D \subseteq \mathbb{R}$ is

- **bounded from below** if there exists an $m \in \mathbb{R}$ such that $f(x) \geq m$ for all $x \in D$; m is called a **lower bound of f**.
- **bounded from above** if there exists an $M \in \mathbb{R}$ such that $f(x) \leq M$ for all $x \in D$; M is called an **upper bound of f**.
- **bounded** if it is bounded both from above and below; equivalently, f is bounded if there exists a $c > 0$ such that $|f(x)| \leq c$ for all $x \in D$;
- **unbounded** if it is not bounded.



(from slideplayer.com/slide/12847291/)

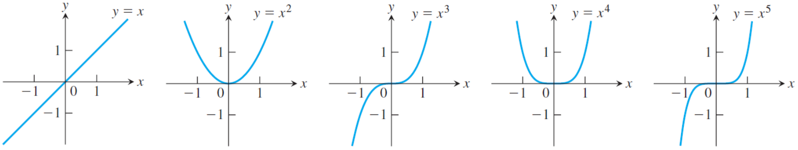
A function of the form $f(x) = mx + b$, for constants m and b, is called a **linear function**.
The function $f(x) = x$ with $m = 1$ and $b = 0$ is called the **identity** function.
The function $f(x) = b$ with $m = 0$ is called a **constant** function.
A linear function with positive slope $m > 0$ whose graph passes through the origin is called a **proportionality relationship**.



(from Thomas' Calculus)

A function of the form $f(x) = x^k$, where $k \in \mathbb{R}$ is constant, is called **a power function**.

$k = n \in \mathbb{N}$:

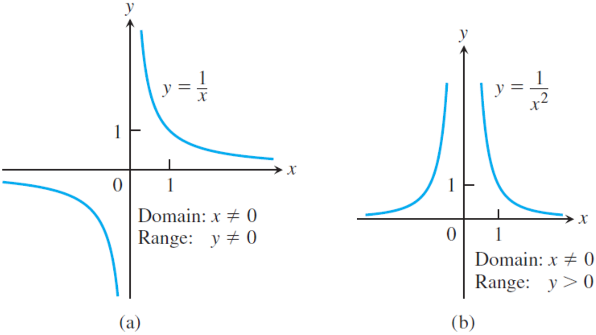


Graphs of $f(x) = x^n$, $n = 1, 2, 3, 4, 5$, defined for $-\infty < x < \infty$.

(from Thomas' Calculus)

A function of the form $f(x) = x^k$, where $k \in \mathbb{R}$ is constant, is called **a power function**.

$k = a \in \mathbb{Z}$:

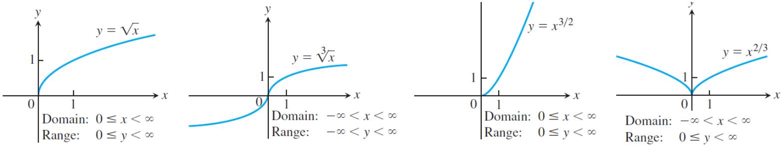


Graphs of the power functions $f(x) = x^a$ for part (a) $a = -1$ and for part (b) $a = -2$.

(from Thomas' Calculus)

A function of the form $f(x) = x^k$, where $k \in \mathbb{R}$ is constant, is called **a power function**.

$k = a \in \mathbb{Q}$:



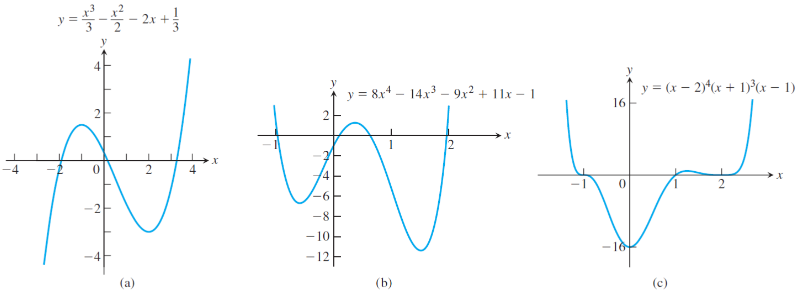
Graphs of the power functions $f(x) = x^a$ for $a = \frac{1}{2}, \frac{1}{3}, \frac{3}{2}$, and $\frac{2}{3}$.

(from Thomas' Calculus)

A function f is a **polynomial** if

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

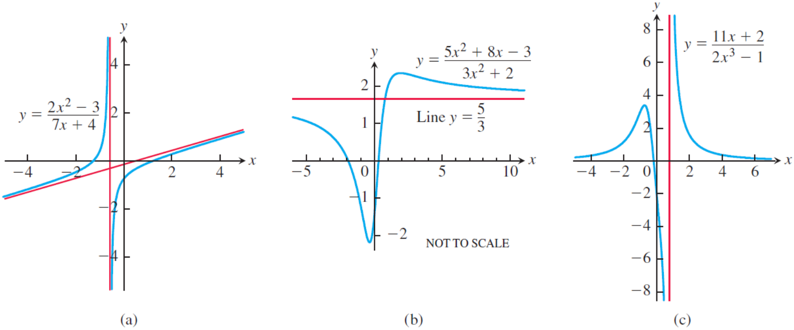
where n is a nonnegative integer and the numbers $a_0, a_1, a_2, \ldots,$ are real constants (called the **coefficients** of the polynomial).



Graphs of three polynomial functions.

(from Thomas' Calculus)

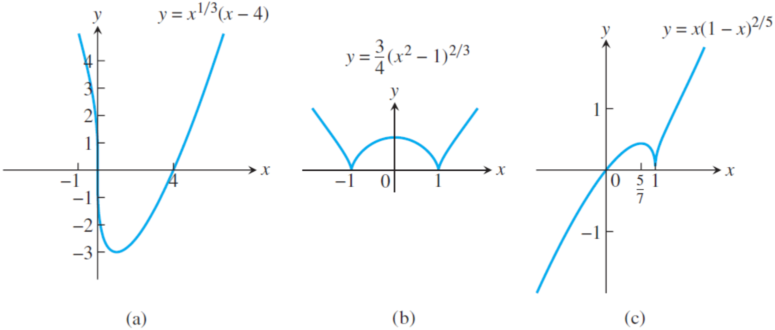
A **rational function** is a quotient or ratio $f(x) = p(x)/q(x)$, where p and q are polynomials.



(from Thomas' Calculus)

Algebraic and transcendental functions

Any function constructed from polynomials using algebraic operations (addition, subtraction, multiplication, division, and taking roots) lies within the class of an **algebraic functions**.

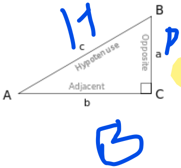


(from Thomas' Calculus)

The functions that are not algebraic are **transcendental**.

Trigonometric functions

The **trigonometric functions** (also called circular functions, angle functions or goniometric functions) are real functions which relate an angle of a right-angled triangle to ratios of two side lengths.



sine

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

cosine

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

tangent

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

cosecant

$$\csc\theta = \frac{\text{hypotenuse}}{\text{opposite}}$$

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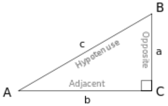

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Whenever the quotients are defined,

$$\tan\theta = \frac{\sin\theta}{\cos\theta}, \ \cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}, \ \sec\theta = \frac{1}{\cos\theta}, \ \csc\theta = \frac{1}{\sin\theta}$$

Function $f(x)$ is **periodic** if there is a positive number p such that $f(x+p) = f(x)$ for every value of x. The smallest such value of p is the **period** of f.

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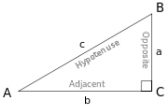

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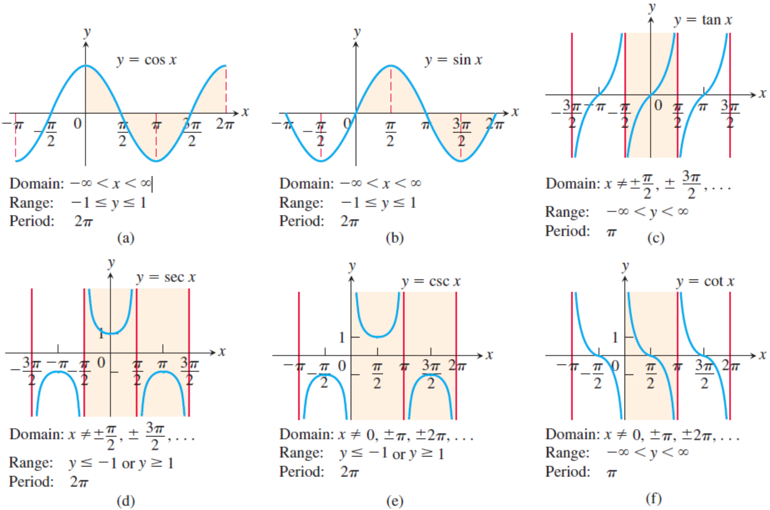


(from Thomas' Calculus)

Trigonometric functions

Values of $\sin\theta$, $\cos\theta$, and $\tan\theta$ for selected values of θ

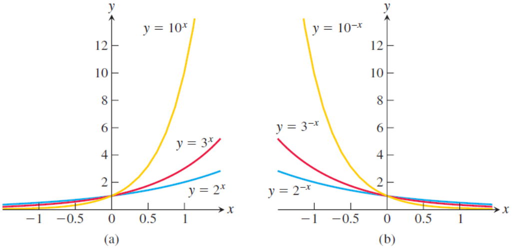
Degrees	-180	-135	-90	-45	0	30	45	60	90	120	135	150	180	270	360
θ (radians)	$-\pi$	$\frac{-3\pi}{4}$	$\frac{-\pi}{2}$	$\frac{-\pi}{4}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π
$\sin\theta$	0	$\frac{-\sqrt{2}}{2}$	-1	$\frac{-\sqrt{2}}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1	0
$\cos\theta$	-1	$\frac{-\sqrt{2}}{2}$	0	$\frac{\sqrt{2}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{-\sqrt{2}}{2}$	$\frac{-\sqrt{3}}{2}$	-1	0	1
$\tan\theta$	0	1		-1	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$		$-\sqrt{3}$	-1	$\frac{-\sqrt{3}}{3}$	0		0

(from Thomas' Calculus)

- **Trigonometric identities**: $\cos^2\theta + \sin^2 = 1$, $1 + \tan^2\theta = \sec^2\theta$, $1 + \cot^2\theta = \csc^2\theta$;
- **Addition formulas**: $\cos(A+B) = \cos A\cos B - \sin A \sin B$, $\sin(A+B) = \sin A\cos B + \cos A \sin B$;
- **Double-angle formulas**: $\cos 2\theta = \cos^2\theta - \sin^2\theta$, $\sin 2\theta = 2\sin\theta\cos\theta$;
- **Half-angle formulas**: $\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$, $\sin^2\theta = \dfrac{1-\cos 2\theta}{2}$;
- **The law of cosines**: if a, b, and c are sides of a triangle and if θ is the angle opposite c, then $c^2 = a^2 + b^2 - 2ab\cos\theta$;
- $-|\theta| \le \sin\theta \le |\theta|$, $-|\theta| \le 1 - \cos\theta \le |\theta|$.

Exponential functions

Functions of the form $f(x) = a^x$, where the base $a > 0$ is a positive constant and $a \neq 1$, are called **exponential functions**.



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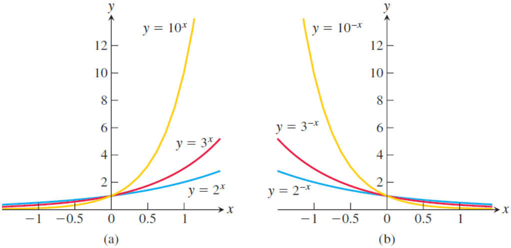
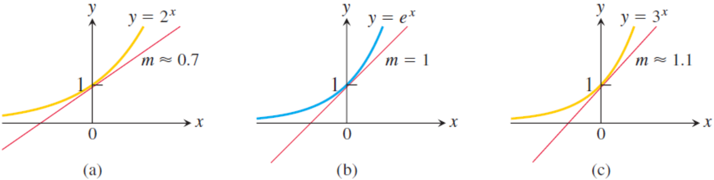


(from Thomas' Calculus)

Properties: $\forall a > 0, b > 0, x \in \mathbb{R}, y \in \mathbb{R}$:

- $a^0 = 1,\ a^{-x} = 1/a^x$;
- $a^x \cdot a^y = a^{x+y},\ \dfrac{a^x}{a^y} = a^{x-y}$;
- $(a^x)^y = (a^y)^x = a^{xy},\ a^x \cdot b^x = (ab)^x,\ \dfrac{a^x}{b^x} = \left(\dfrac{a}{b}\right)^x$.

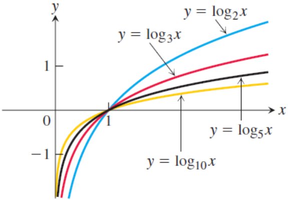
The most important exponential function used for modeling natural, physical, and economic phenomena is the **natural exponential function** $f(x) = e^x$, $e \approx 2.718281828\ldots$.
The function $y = y_0 e^{kx}$, $y_0 > 0, k \neq 0$ is a model for **exponential growth** if $k > 0$ and a model for **exponential decay** if $k < 0$.
Here y_0 represents a constant.



Among the exponential functions, the graph of $y = e^x$ has the property that the slope m of the tangent line to the graph is exactly 1 when it crosses the y-axis. The slope is smaller for a base less than e, such as 2^x, and larger for a base greater than e, such as 3^x.

(from Thomas' Calculus)

Logarithmic functions are the functions $f(x) = \log_a x$, where the base $a \neq 1$ is a positive constant.



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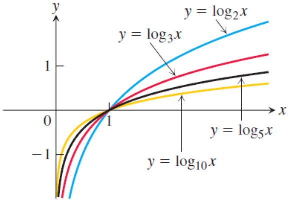
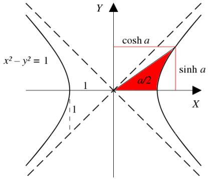


(from Thomas' Calculus)

Properties: $\forall a > 0, a \neq 1, b > 0, x, y > 0,\ p \in \mathbb{R}$:

- $\log_a(xy) \log_a x + \log_b y$, $\log_a \dfrac{x}{y} = \log_a x - \log_a y$;
- $\log_a(x^p) = p \log_a x$, for $p \neq 0$ $\log_a(\sqrt[p]{x}) = \frac{1}{p} \log_a x$;
- $a^{\log_a x} = x$, $\log_a a^x = x$;
- $a^x = e^{x \ln a}$; for $b \neq 1$, $\log_a x = \frac{\log_b x}{\log_b a} = \frac{\ln x}{\ln a}$.

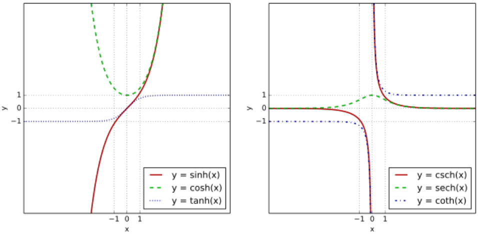
Hyperbolic functions are analogues of the ordinary trigonometric functions, but defined using the hyperbola rather than the circle.



(from wikipedia.org)

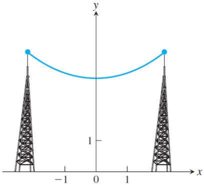
$$\sinh x = \frac{e^x - e^{-x}}{2} = -i\sin(ix), \cosh x = \frac{e^x + e^{-x}}{2} = \cos(ix),$$
$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} = -i\tan(ix),$$
$$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}} = i\cot(ix),$$
$$\mathrm{sech} x = \frac{1}{\cosh x} = \sec(ix), \ \mathrm{csch} x = \frac{1}{\sinh x} = i\csc(ix).$$

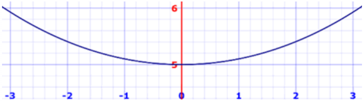
Hyperbolic functions



(from wikipedia.org)

A hanging cable forms a curve called a **catenary** defined using the cosh function: $f(x) = a\cosh(x/a)$





(from Thomas' Calculus & mathsisfun.com)

Operations with functions

Let $f : D(f) \to R(f)$, $g : D(g) \to R(g)$. Then $\forall x \in D_f \cap D_g$, the functions $f+g$, $f-g$, fg, f/g are defined as:

- $(f+g)(x) = f(x) + g(x)$
- $(f-g)(x) = f(x) - g(x)$
- $(cf)(x) = cf(x)$ for any $c \in \mathbb{R}$
- $(f \cdot g)(x) = f(x) \cdot g(x)$
- $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ for $x : g(x) \neq 0$.

Example: For $f(x) = \sqrt{x}$, $g(x) = \sqrt{1-x}$, find $f+g$, $f-g$, $g-f$, $f \cdot g$, $\dfrac{f}{g}$, $\frac{g}{f}$ and their domains.

Example: For $f(x) = \sqrt{x}$, $g(x) = \sqrt{1-x}$, find $f+g$, $f-g$, $g-f$, $f \cdot g$, $\dfrac{f}{g}$, $\frac{g}{f}$ and their domains.

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Example: For $f(x) = \sqrt{x}$, $g(x) = \sqrt{1-x}$, find $f+g$, $f-g$, $g-f$, $f \cdot g$, $\frac{f}{g}$, $\frac{g}{f}$ and their domains.

$D(f) = [0, +\infty)$, $D(g) = (-\infty, 1]$,

$D(f) \cap D(g) = [0, +\infty) \cap (-\infty, 1] = [0, 1]$.

Function	Formula	Domain
$f+g$	$(f+g)(x) = \sqrt{x} + \sqrt{1-x}$	$[0, 1] = D(f) \cap D(g)$
$f-g$	$(f-g)(x) = \sqrt{x} - \sqrt{1-x}$	$[0, 1]$
$g-f$	$(g-f)(x) = \sqrt{1-x} - \sqrt{x}$	$[0, 1]$
$f \cdot g$	$(f \cdot g)(x) = f(x)g(x) = \sqrt{x(1-x)}$	$[0, 1]$
f/g	$\frac{f}{g}(x) = \frac{f(x)}{g(x)} = \sqrt{\frac{x}{1-x}}$	$[0, 1)$ ($x = 1$ excluded)
g/f	$\frac{g}{f}(x) = \frac{g(x)}{f(x)} = \sqrt{\frac{1-x}{x}}$	$(0, 1]$ ($x = 0$ excluded)

Example: For $f(x) = \sqrt{x}$, $g(x) = \sqrt{1-x}$, find $f+g$, $f-g$, $g-f$, $f \cdot g$, $\frac{f}{g}$, $\frac{g}{f}$ and their domains.

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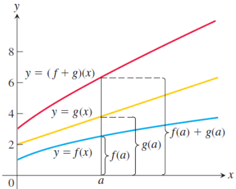
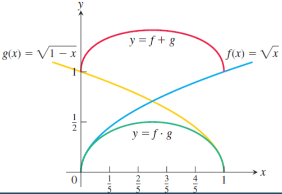




Transformations of a graph of a function

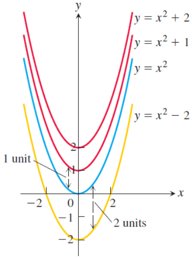
Shift Formulas

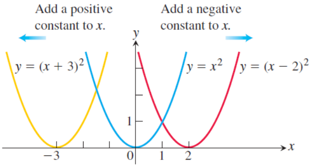
Vertical Shifts

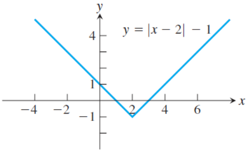
$y = f(x) + k$	Shifts the graph of f *up* k units if $k > 0$ Shifts it *down* $\lvert k \rvert$ units if $k < 0$

Horizontal Shifts

$y = f(x + h)$	Shifts the graph of f *left* h units if $h > 0$ Shifts it *right* $\lvert h \rvert$ units if $h < 0$







(from Thomas' Calculus)

Transformations of a graph of a function

Vertical and Horizontal Scaling and Reflecting Formulas

For $c > 1$, the graph is scaled:

$y = cf(x)$	Stretches the graph of f vertically by a factor of c.
$y = \frac{1}{c}f(x)$	Compresses the graph of f vertically by a factor of c.
$y = f(cx)$	Compresses the graph of f horizontally by a factor of c.
$y = f(x/c)$	Stretches the graph of f horizontally by a factor of c.

For $c = -1$, the graph is reflected:

$y = -f(x)$	Reflects the graph of f across the x-axis.
$y = f(-x)$	Reflects the graph of f across the y-axis.

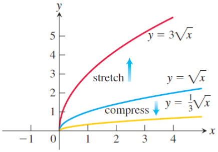


FIGURE 1.32 Vertically stretching and compressing the graph $y = \sqrt{x}$ by a factor of 3 (Example 4a).

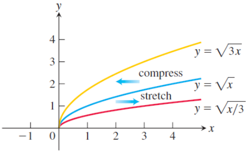


FIGURE 1.33 Horizontally stretching and compressing the graph $y = \sqrt{x}$ by a factor of 3 (Example 4b).

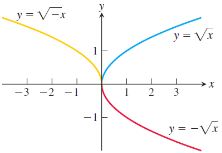


FIGURE 1.34 Reflections of the graph $y = \sqrt{x}$ across the coordinate axes (Example 4c).

(from Thomas' Calculus)

Transformations of a graph of a function

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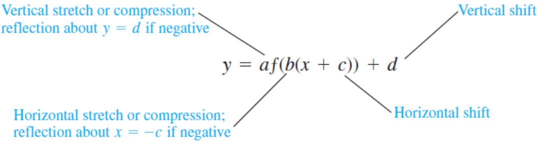
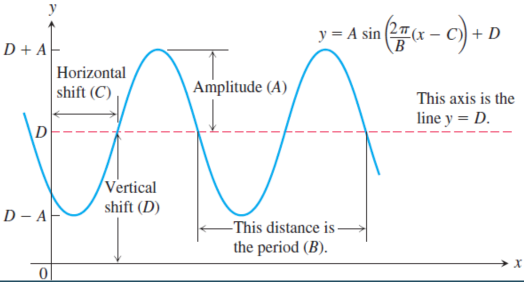
Transformations of trigonometric graphs:



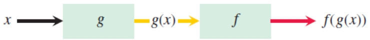
Let $f(x) = A \sin\left(\frac{2\pi}{B}(x - C) + D\right)$, ($|A|$ is the amplitude, $|B|$ is the period, C is the horizontal shift, D is the vertical shift):



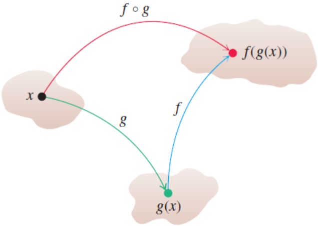
If f and g are functions, the **composite function** $f \circ g$ ("f composed with g") is defined by

$$(f \circ g)(x) = f(g(x)).$$

$$D(f \circ g) = \{x \in \mathbb{R} : x \in D(g),\, g(x) \in D(f)\}.$$



A composite function $f \circ g$ uses the output $g(x)$ of the first function g as the input for the second function f.



Arrow diagram for $f \circ g$. If x lies in the domain of g and $g(x)$ lies in the domain of f, then the functions f and g can be composed to form $(f \circ g)(x)$.

(from Thomas' Calculus)

Composite functions

Example: For $f(x) = \sqrt{x}$, $g(x) = x + 1$, find $f \circ g$, $g \circ f$, $f \circ f$, $g \circ f$, and their domains.

Composite functions

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	Composite	**Domain**
(a)	$(f \circ g)(x) = f(g(x)) = \sqrt{g(x)} = \sqrt{x+1}$	$[-1, \infty)$
(b)	$(g \circ f)(x) = g(f(x)) = f(x) + 1 = \sqrt{x} + 1$	$[0, \infty)$
(c)	$(f \circ f)(x) = f(f(x)) = \sqrt{f(x)} = \sqrt{\sqrt{x}} = x^{1/4}$	$[0, \infty)$
(d)	$(g \circ g)(x) = g(g(x)) = g(x) + 1 = (x + 1) + 1 = x + 2$	$(-\infty, \infty)$

(from Thomas' Calculus)

Inverse functions

Suppose that f is an injective function on a domain $D(f)$ with range $R(f)$. The **inverse function** f^{-1} is defined by

$$f^{-1}(b) = a \text{ if } f(a) = b.$$

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$$f^{-1}(b) = a \text{ if } f(a) = b.$$

When the function **f** turns the apple into a banana,

Then the **inverse** function $\mathbf{f^{-1}}$ turns the banana back to the apple

(from www.mathsisfun.com)

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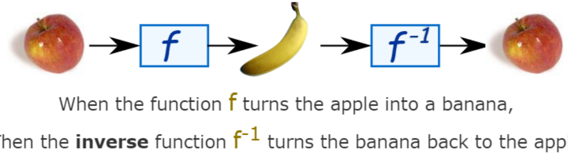
$$f^{-1}(b) = a \text{ if } f(a) = b.$$



(from www.mathsisfun.com)

$$f^{-1} \neq \frac{1}{f}!$$

Examples:

Suppose a one-to-one function $y = f(x)$ is given by a table of values

x	1	2	3	4	5	6	7	8
$f(x)$	3	4.5	7	10.5	15	20.5	27	34.5

A table for the values of $x = f^{-1}(y)$ can then be obtained by simply interchanging the values in the columns (or rows) of the table for f:

y	3	4.5	7	10.5	15	20.5	27	34.5
$f^{-1}(y)$	1	2	3	4	5	6	7	8

(from Thomas' Calculus)

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Examples:

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y	3	4.5	7	10.5	15	20.5	27	34.5
$f^{-1}(y)$	1	2	3	4	5	6	7	8

(from Thomas' Calculus)

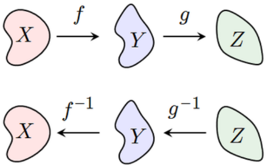
To convert Fahrenheit to Celsius: $\mathbf{f(F) = (F - 32) \times \frac{5}{9}}$

The **Inverse Function** (Celsius back to Fahrenheit): $\mathbf{f^{-1}(C) = (C \times \frac{9}{5}) + 32}$

°C to °F	Divide by 5, then multiply by 9, then add 32
°F to °C	Deduct 32, then multiply by 5, then divide by 9

(from www.mathsisfun.com)

- if f is invertible, then $D(f^{-1}) = R(f)$, $R(f^{-1}) = D(f)$;
- $f \circ f^{-1} = \mathrm{id}_x$ and $f^{-1} \circ f = \mathrm{id}_y$, i.e.
  $(f \circ f^{-1})(x) = x \ \forall x \in D(f)$,
  $(f \circ f^{-1})(y) = y \ \forall y \in D(f^{-1})($ or in $)R(f)$;
- if f and g are invertible, then $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$;



(from wikipedia.org)

- if an inverse function exists for a given function f, then it is unique;
- any monotonic on an interval I function f is invertible on I.

To find then inverse of $y = f(x)$,

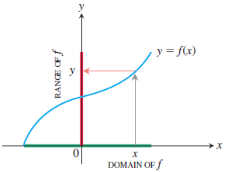
1. solve the equation $y = f(x)$ for x. This gives a formula $x = f^{-1}(y)$, where x is expressed as a function of y;
2. interchange x and y, obtaining a formula $y = f^{-1}(x)$, where f^{-1} is expressed in the conventional format with x as the independent variable and y as the dependent variable.

Put "y" for "f(x)" and solve for x:

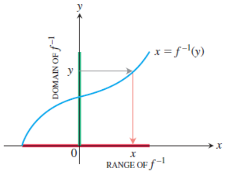
The function:	**f(x) = 2x+3**
Put "y" for "f(x)":	**y = 2x+3**
Subtract 3 from both sides:	**y-3 = 2x**
Divide both sides by 2:	**(y-3)/2 = x**
Swap sides:	**x = (y-3)/2**
Solution (put "f^{-1}(y)" for "x") :	**f^{-1}(y) = (y-3)/2**

(from www.mathsisfun.com)

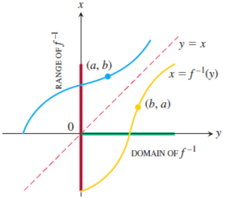
Graphes of inverses functions



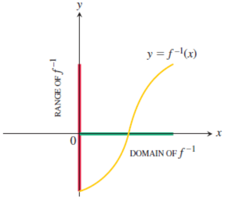
(a) To find the value of f at x, we start at x, go up to the curve, and then over to the y-axis.



(b) The graph of f^{-1} is the graph of f, but with x and y interchanged. To find the x that gave y, we start at y and go over to the curve and down to the x-axis. The domain of f^{-1} is the range of f. The range of f^{-1} is the domain of f.



(c) To draw the graph of f^{-1} in the more usual way, we reflect the system across the line $y = x$.



(d) Then we interchange the letters x and y. We now have a normal-looking graph of f^{-1} as a function of x.

The graph of $y = f^{-1}(x)$ is obtained by reflecting the graph of $y = f(x)$ about the line $y = x$.

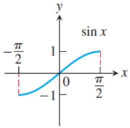
(from Thomas' Calculus)

Inverses of common functions

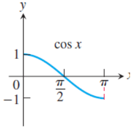
Inverses			Careful!
$+$	<=>	$-$	
$\times$	<=>	$\div$	Don't divide by zero
$\frac{1}{x}$	<=>	$\frac{1}{y}$	x and y not zero
x^2	<=>	$\sqrt{y}$	x and y $\geq$ 0
x^n	<=>	$\sqrt[n]{y}$ **or** $y^{\frac{1}{n}}$	n not zero *(different rules when n is odd, even, negative or positive)*
e^x	<=>	$\ln(y)$	y > 0
a^x	<=>	$\log_a(y)$	y and a > 0
$\sin(x)$	<=>	$\sin^{-1}(y)$	$-\pi/2$ to $+\pi/2$
$\cos(x)$	<=>	$\cos^{-1}(y)$	0 to π
$\tan(x)$	<=>	$\tan^{-1}(y)$	$-\pi/2$ to $+\pi/2$

(from www.mathsisfun.com)

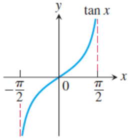
Domain restrictions that make the trigonometric functions one-to-one



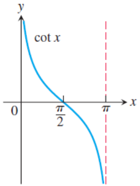
$y = \sin x$
Domain: $[-\pi/2, \pi/2]$
Range: $[-1, 1]$



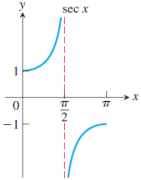
$y = \cos x$
Domain: $[0, \pi]$
Range: $[-1, 1]$



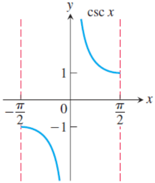
$y = \tan x$
Domain: $(-\pi/2, \pi/2)$
Range: $(-\infty, \infty)$



$y = \cot x$
Domain: $(0, \pi)$
Range: $(-\infty, \infty)$



$y = \sec x$
Domain: $[0, \pi/2) \cup (\pi/2, \pi]$
Range: $(-\infty, -1] \cup [1, \infty)$

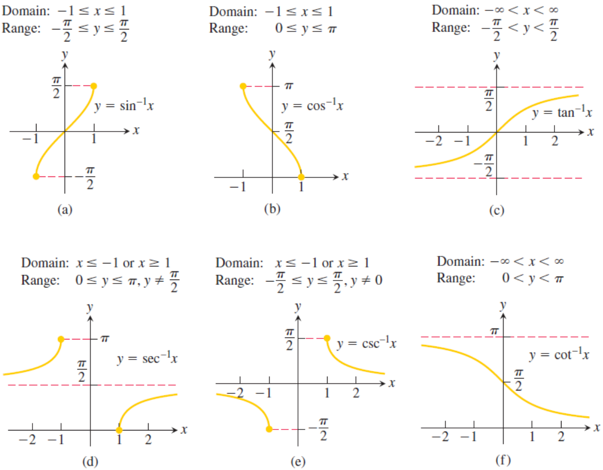


$y = \csc x$
Domain: $[-\pi/2, 0) \cup (0, \pi/2]$
Range: $(-\infty, -1] \cup [1, \infty)$

(from Thomas' Calculus)

Inverse trigonometric functions

$$\sin^{-1} x = \arcsin x,\ \cos^{-1} x = \arccos x,\ \tan^{-1} x = \arctan x,$$
$$\cot^{-1} x = \mathrm{arccot} x,\ \sec^{-1} x = \mathrm{arcsec} x,\ \csc^{-1} x = \mathrm{arccsc} x$$

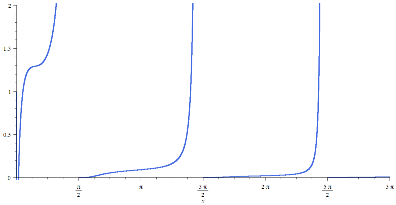


An **elementary function** is a function defined as taking sums, products, and compositions of finitely many polynomial, rational, trigonometric, hyperbolic, and exponential functions, including possibly their inverse functions.
Any function that is not elementary is **non-elementary**.

Examples:

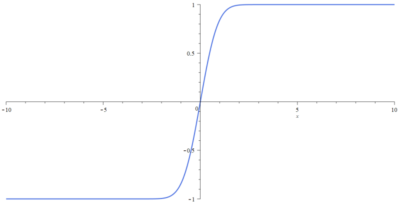
Elementary function:

$$f(x) = \frac{e^{\tan x}}{1+x^2} \sin\left(\sqrt{1+(\ln x)^2}\right)$$



Non-elementary function:

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$$



1. Preliminaries

1.6. Some useful inequalities

Some useful inequalities

The arithmetic and geometric means

For any $a_1, \dots, a_n \geq 0$,

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdot \cdots \cdot a_n}.$$

$\frac{a_1+a_2+\cdots+a_n}{n}$ is called the **arithmetic mean** (or average), and $\sqrt[n]{a_1 a_2 \cdot \cdots \cdot a_n}$ is the **geometric mean** (average).

The Bernoulli inequality

For any $a \in (-1, +\infty)$ and $n \in \mathbb{N} \cup \{0\}$,

$$1 + na \leq (1 + a)^n.$$

The Cauchy–Schwarz inequality

For any $a_j, b_j \in \mathbb{R}$, $j = 1, 2, \dots, n$, $n \in \mathbb{N}$,

$$(a_1 b_1 + a_2 b_2 + \cdots + a_n b_n)^2 \leq \left(a_1^2 + a_2^2 + \cdots + a_n^2\right)\left(b_1^2 + b_2^2 + \cdots + b_n^2\right).$$

Some useful inequalities

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The Hölder inequality

For any $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$, $a_j, b_j \in \mathbb{R}$, $j = 1, 2, \dots, n$, $n \in \mathbb{N}$,

$$|a_1 b_1| + |a_2 b_2| + \cdots + |a_n b_n| \le \big(|a_1|^p + |a_2|^p + \cdots + |a_n|^p\big)^{1/p} \big(|b_1|^q + |b_2|^q + \cdots + |b_n|^q\big)^{1/q}.$$

The Young inequality

For any $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$, $a, b \in [0, +\infty)$,

$$ab \le \frac{a^p}{p} + \frac{b^q}{q}.$$

The Minkowski inequality

For any $p \in (1, \infty)$, $a_j, b_j \in \mathbb{R}$, $j = 1, 2, \dots, n$, $n \in \mathbb{N}$,

$$(|a_1 + b_1|^p + \cdots + |a_n + b_n|^p)^{1/p} \le (|a_1|^p + |a_2|^p + \cdots + |a_n|^p)^{1/p} + (|b_1|^p + |b_2|^p + \cdots + |b_n|^p)^{1/p}.$$

Thank you for your attention!!!