5. Integral calculus of functions of one real variable

5.1. Definite integral

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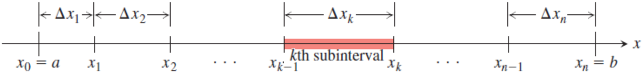
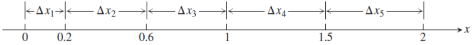
Definition

A **partition** P of an interval $[a, b]$ is a set $P = \{x_0, x_1, \dots, x_n\}$, $n \in \mathbb{N}$, such that
$$a = x_0 < x_1 < x_2 < \cdots < x_n = b.$$
An interval $[x_{k-1}, x_k]$ is called a **k-th subinterval** of P, for $k \in \{1, 2, \dots, n\}$.
The **width** of the k-th interval is $\Delta x_k = x_k - x_{k-1}$.
The **diameter** (or **norm**) of a partition P, $\|P\|$,is the largest of all the subinterval widths.



(from Thomas'Calculus)

EXAMPLE The set $P = \{0, 0.2, 0.6, 1, 1.5, 2\}$ is a partition of $[0, 2]$. There are five subintervals of P: $[0, 0.2]$, $[0.2, 0.6]$, $[0.6, 1]$, $[1, 1.5]$, and $[1.5, 2]$:



The lengths of the subintervals are $\Delta x_1 = 0.2$, $\Delta x_2 = 0.4$, $\Delta x_3 = 0.4$, $\Delta x_4 = 0.5$, and $\Delta x_5 = 0.5$. The longest subinterval length is 0.5, so the norm of the partition is $\|P\| = 0.5$.

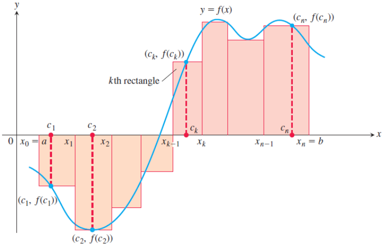
(from Thomas'Calculus)

Definition

Consider an arbitrary bounded function $f : [a, b] \to \mathbb{R}$, a partition $P = \{x_0, x_1, \dots, x_n\}$ of $[a, b]$, and a collection of points $c_k \in [x_{k-1}, x_k]$, $k = \overline{1, n}$. The finite sum

$$S_P(f) = \sum_{k=1}^{n} f(c_k)\Delta x_k$$

is called a **Riemann sum for f on the interval $[a, b]$**.



(from Thomas'Calculus)

Remarks

- There are many such sums, depending on the partition P and the choices of the c_k. For example, let $\Delta x = b - a/n$, $c_k = x_k$, for all $k \in \overline{1,n}$. Then $S_P(f) = \sum_{k=1}^{n} f\left(a + k(b-a)/n\right) b - a/n$.
- When $f(c_k) > 0$, the product $f(c_k)\Delta x_k$ is the area of a rectangle with height $f(c_k)$ and width Δx_k. When $f(c_k) < 0$, $f(c_k)\Delta x_k$ is the negative of the area of a rectangle of width Δx_k that drops from the x-axis to the negative number $f(c_k)$.
- Riemann sums define rectangles approximating the region between the graph of a continuous f and the x-axis. Partitions with norm approaching zero lead to collections of rectangles that approximate this region with increasing accuracy.

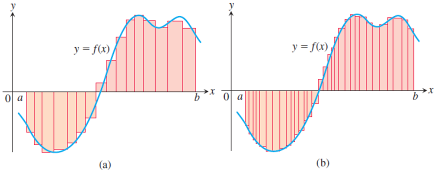


FIGURE The curve with rectangles from finer partitions of $[a, b]$. Finer partitions create collections of rectangles with thinner bases that approximate the region between the graph of f and the x-axis with increasing accuracy.

(from Thomas'Calculus)

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Definition

A function $f : [a, b] \to \mathbb{R}$ is called **integrable** (or more specifically **Riemann–integrable**) **on** $[a, b]$, if there exists a $J \in \mathbb{R}$ such that, for any sequence of partitions of $[a, b]$ $P_j = \{x_0^{(j)}, x_1^{(j)}, \ldots, x_{n_j}^{(j)}\}$, $j \in \mathbb{N}$, with $\lim\limits_{j\to\infty} \|P\| = 0$, and for any collection of points $c_k^{(j)} \in [x_{k-1}^{(j)}, x_k^{(j)}]$, $k = \overline{1, n_j}$, $n \in \mathbb{N}$, the limit of the sequence of the corresponding Riemann sums $S_{P_j}(f; c_1^{(j)}, \ldots, c_{n_j}^{(j)})$ exists and equals J:

$$\lim_{j\to\infty} S_{P_j}(f; c_1^{(j)}, \ldots, c_{n_j}^{(j)}) = \lim_{j\to\infty} \sum_{k=1}^{n_j} f(c_k^{(j)}) \Delta x_k^{(j)} = J.$$

In this case, the number J is called **definite integral** (or **Riemann integral**) **of** f **over** $[a, b]$, $\int\limits_a^b f(x)\,dx$.

Thus, $\int\limits_a^b f(x)\,dx := \lim\limits_{j\to\infty} S_{P_j}(f; c_1^{(j)}, \ldots, c_{n_j}^{(j)})$, where $\lim\limits_{j\to\infty} \|P_j\| = 0$.

Equivalent definition:

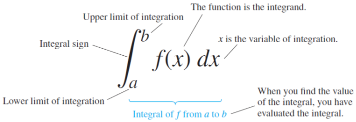
DEFINITION Let $f(x)$ be a function defined on a closed interval $[a, b]$. We say that a number J is the **definite integral of f over $[a, b]$** and that J is the limit of the Riemann sums $\sum_{k=1}^{n} f(c_k)\,\Delta x_k$ if the following condition is satisfied:

Given any number $\epsilon > 0$ there is a corresponding number $\delta > 0$ such that for every partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ with $\|P\| < \delta$ and any choice of c_k in $[x_{k-1}, x_k]$, we have

$$\left| \sum_{k=1}^{n} f(c_k)\,\Delta x_k - J \right| < \epsilon.$$

(from Thomas'Calculus)

That is, $\int\limits_a^b f(x)\,dx := \lim\limits_{\|P\|\to 0} S_P(f) = \lim\limits_{\|P\|\to 0} \sum_{k=1}^{n} f(c_k)\Delta x_k.$



(from Thomas'Calculus)

In the cases where the subintervals all have equal width $\Delta x = (b - a)/n$, we can form each Riemann sum as

$$S_n = \sum_{k=1}^{n} f(c_k)\,\Delta x_k = \sum_{k=1}^{n} f(c_k)\left(\frac{b-a}{n}\right), \qquad \Delta x_k = \Delta x = (b-a)/n \text{ for all } k$$

where c_k is chosen in the kth subinterval. When the limit of these Riemann sums as $n \to \infty$ exists and is equal to J, then J is the definite integral of f over $[a, b]$, so

$$J = \int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\left(\frac{b-a}{n}\right) \qquad \|P\| \to 0 \text{ means } n \to \infty.$$

If we pick the point c_k at the right endpoint of the kth subinterval, so $c_k = a + k\,\Delta x = a + k(b-a)/n$, then the formula for the definite integral becomes

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f\left(a + k\frac{(b-a)}{n}\right)\left(\frac{b-a}{n}\right) \tag{1}$$

Equation (1) gives one explicit formula that can be used to compute definite integrals. Other choices of partitions and locations of points c_k result in the same value for the definite integral when we take the limit as $n \to \infty$ provided that the norm of the partition approaches zero.

(from Thomas'Calculus)

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Theorem (Boundedness of integrable functions)

If a function $f : [a, b] \to \mathbb{R}$ is integrable over $[a, b]$ then it is bounded on $[a, b]$.

Not every bounded function is integrable!!!

Example: Dirichlet function

$f(x) = \mathbf{1}_{\mathbb{Q}}(x) := \begin{cases} 1 & x \in \mathbb{Q}, \\ 0 & x \notin \mathbb{Q}. \end{cases}$ Indeed, let us take any partition $P = \{x_0, x_1, \dots, x_n\}$ of $[0, 1]$. On the one hand, if we choose $c_k \in [x_{k-1}, x_k]$, $k = \overline{1, n}$, to be rational, then $S_P(f) = \sum_{k=1}^n f(c_k)\Delta x_k = \sum_{k=1}^n 1 \cdot \Delta x_k = 1$. On the other hand, if we choose $c_k \in [x_{k-1}, x_k]$, $k = \overline{1, n}$, to be irrational, then $S_P(f) = \sum_{k=1}^n f(c_k)\Delta x_k = \sum_{k=1}^n 0 \cdot \Delta x_k = 0$. Since P is assumed to be arbitrary, the limit $\lim\limits_{\|P\| \to 0} \sum_{k=1}^n f(c_k)\Delta x_k$ does not exist.

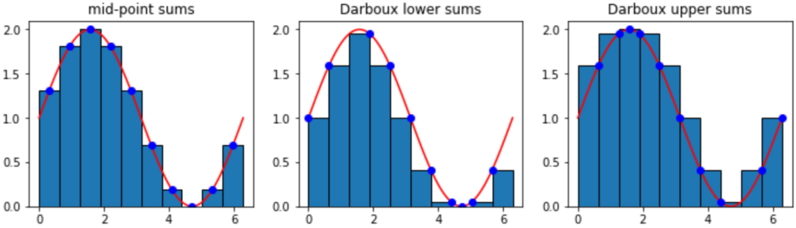
Darboux sums

Definition

Let $f : [a, b] \to \mathbb{R}$ be bounded on $[a, b]$, $P = \{x_0, x_1, \dots, x_n\}$ be a partition of $[a, b]$, and

$$M_k = \sup_{x \in [x_{k-1}, x_k]} f(x), \; m_k = \inf_{x \in [x_{k-1}, x_k]} f(x).$$

Then the **upper Darboux sum** of f with respect to P is $\overline{S}_P(f) = \sum_{i=1}^n (x_i - x_{i-1}) M_i$, and the **lower Darboux sum** of f with respect to P is $\underline{s}_P(f) = \sum_{i=1}^n (x_i - x_{i-1}) m_i$. The lower and upper Darboux sums are often called the **lower** and **upper sums**.



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Properties of Darboux sums:

- For any continuous f such that $f(x) \geq 0$ for all $x \in [a, b]$, for any partition P of $[a, b]$, $\underline{s}_P(f) \leq F \leq \overline{S}_P(f)$, where F is the area between the graph of f and x-axis.
- For any two partitions P and Q of $[a, b]$, $\underline{s}_P(f) \leq \overline{S}_Q(f)$.
- If a partition Q is a **refinement** of partition P (i.e. every subinterval of Q lies entirely in some subinterval of P), then $\underline{s}_P(f) \leq \underline{s}_Q(f) \leq \overline{S}_P(f) \leq \overline{S}_Q(f)$.
- If $S_P(f)$ is a Riemann sum for f on the interval $[a, b]$ corresponding to the partition P, then $\underline{s}_P(f) = \inf\limits_{c_1,\dots,c_n} S_P(f)$ and $\overline{S}_P(f) = \sup\limits_{c_1,\dots,c_n} S_P(f)$.

Definition

The **upper Darboux integral** of f over $[a, b]$ is $J_* := \inf\limits_P \overline{S}_P(f)$.

The **lower Darboux integral** of f over $[a, b]$ is $J^* := \sup\limits_P \underline{s}_P(f)$.

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Theorem (Integrability criterion)

A bounded function $f : [a, b] \to \mathbb{R}$ is integrable over $[a, b]$ if and only if $\lim\limits_{\|P\| \to 0} (\overline{S}_P(f) - \underline{s}_P(f)) = 0$.

Theorem (Darboux's Integrability criterion)

A bounded function $f : [a, b] \to \mathbb{R}$ is integrable over $[a, b]$ if and only if its upper and lower Darboux integrals are equal $J_* = J^* = J$.

In this case, $\int\limits_a^b f(x)\, dx = J$.

Theorem (Riemann's Integrability criterion)

A bounded function $f : [a, b] \to \mathbb{R}$ is integrable over $[a, b]$ if and only if for any $\varepsilon > 0$ there exists a partition P of $[a, b]$ such that $\overline{S}_P(f) - \underline{s}_P(f) < \varepsilon$.

Sufficient integrability conditions

Theorem (Integrability of bounded functions)

Let $f : [a, b] \to \mathbb{R}$ be bounded and monotonic (or piecewise-monotonic) on $[a, b]$. Then f is integrable over $[a, b]$.

Theorem (Integrability of continuous functions)

Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ or has at most finitely many jump discontinuities there. Then f is integrable over $[a, b]$.

Properties of definite integrals

We put by definition $\int\limits_a^a f(x)\,dx = 0$ and $\int\limits_b^a f(x)\,dx = -\int\limits_a^b f(x)\,dx$.

Theorem (Properties of definite integrals)

Let $f, g :\to \mathbb{R}$ be integrable over $[a, b]$. Then the following properties hold.

- $\int\limits_a^b dx = b - a$.
- If $[a^*, b^*] \subset [a, b]$, then f is integrable over $[a^*, b^*]$.
- *Additivity*: for any $a < b < c$, if f is integrable over $[a, c]$ and over $[b, c]$, then it is integrable over $[a, c]$ and
  $\int\limits_a^c f(x)\,dx = \int\limits_a^b f(x)\,dx + \int\limits_b^c f(x)\,dx$.
- *Sum and difference*: $f(x) \pm g(x)$ are integrable over $[a, b]$ and
  $\int\limits_a^b (f(x) + g(x))\,dx = \int\limits_a^b f(x)\,dx + \int\limits_a^b g(x)\,dx$.
- *Linearity*: for any $k \in \mathbb{R}$, $kf(x)$ is integrable over $[a, b]$ and
  $\int\limits_a^b kf(x)\,dx = k\int\limits_a^b f(x)\,dx$.

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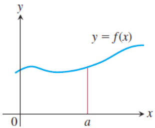
Theorem (Properties of definite integrals, cont'd)

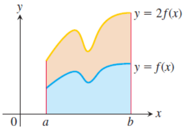
Let $f, g :\to \mathbb{R}$ be integrable over $[a, b]$. Then the following properties hold.

- *Product*: $f(x) \cdot g(x)$ is integrable over $[a, b]$.
- *Quotient*: if $\inf\limits_{[a,b]} |g(x)| > 0$, then $\dfrac{f(x)}{g(x)}$ is integrable over $[a, b]$.
- *Domination*: if $f(x) \geq g(x)$ for all $x \in [a, b]$, then $\int\limits_a^b f(x)\,dx \geq \int\limits_a^b g(x)\,dx$. As a consequence, if $f(x) \geq 0$ for all $x \in [a, b]$, then $\int\limits_a^b f(x)\,dx \geq 0$.
- *Max-Min inequality*: if f has maximal and minimal values $\max f = M$ and $\min f = m$ on $[a, b]$, then $m \cdot (b - a) \leq \int\limits_a^b f(x)\,dx \leq M \cdot (b - a)$.
- *Absolute value*: $|f(x)|$ is integrable over $[a, b]$ and $\left|\int\limits_a^b f(x)\,dx\right| \leq \int\limits_a^b |f(x)|\,dx$.



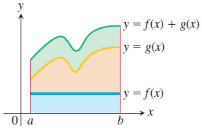
(a) *Zero Width Interval:*

$$\int_a^a f(x)\,dx = 0$$



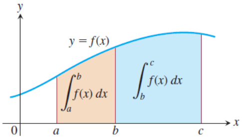
(b) *Constant Multiple:* $(k = 2)$

$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$$



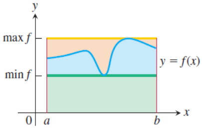
(c) *Sum: (areas add)*

$$\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$



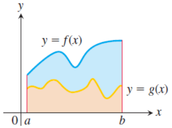
(d) *Additivity for Definite Integrals:*

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$



(e) *Max-Min Inequality:*

$$\min f \cdot (b - a) \le \int_a^b f(x)\,dx \le \max f \cdot (b - a)$$



(f) *Domination:*

$f(x) \ge g(x)$ on $[a, b]$

$$\Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$$

(from Thomas'Calculus)

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EXAMPLE To illustrate some of the rules, we suppose that

$$\int_{-1}^{1} f(x)\,dx = 5, \qquad \int_{1}^{4} f(x)\,dx = -2, \quad \text{and} \quad \int_{-1}^{1} h(x)\,dx = 7.$$

Then

1. $\displaystyle\int_{4}^{1} f(x)\,dx = -\int_{1}^{4} f(x)\,dx = -(-2) = 2$ Rule 1

2. $\displaystyle\int_{-1}^{1} [2f(x) + 3h(x)]\,dx = 2\int_{-1}^{1} f(x)\,dx + 3\int_{-1}^{1} h(x)\,dx$ Rules 3 and 4

$$= 2(5) + 3(7) = 31$$

3. $\displaystyle\int_{-1}^{4} f(x)\,dx = \int_{-1}^{1} f(x)\,dx + \int_{1}^{4} f(x)\,dx = 5 + (-2) = 3$ Rule 5 ■

(from Thomas'Calculus)

EXAMPLE Show that the value of $\int_0^1 \sqrt{1 + \cos x}\, dx$ is less than or equal to $\sqrt{2}$.

Solution The Max-Min Inequality for definite integrals (Rule 6) says that $\min f \cdot (b - a)$ is a *lower bound* for the value of $\int_a^b f(x)\, dx$ and that $\max f \cdot (b - a)$ is an *upper bound*. The maximum value of $\sqrt{1 + \cos x}$ on $[0, 1]$ is $\sqrt{1 + 1} = \sqrt{2}$, so

$$\int_0^1 \sqrt{1 + \cos x}\, dx \leq \sqrt{2} \cdot (1 - 0) = \sqrt{2}. \qquad \blacksquare$$

(from Thomas'Calculus)

Theorem (Mean value theorem for definite integrals)

Let f be continuous on $[a, b]$. Then there exists a point $c \in [a, b]$ such that

$$f(c) = \frac{1}{b-a} \int_a^b f(x)\, dx.$$

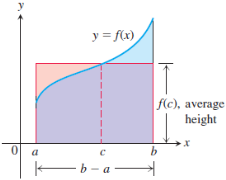

FIGURE The value $f(c)$ in the Mean Value Theorem is, in a sense, the average (or *mean*) height of f on $[a, b]$. When $f \geq 0$, the area of the rectangle is the area under the graph of f from a to b,

$$f(c)(b-a) = \int_a^b f(x)\, dx.$$

(from Thomas'Calculus)

Fundamental theorem of Calculus, Part 1

If $f(t)$ is an integrable function over a finite interval I, then the integral from any fixed number $a \in I$ to another number $x \in I$ defines a new function F (**integral with a variable upper limit**) whose value at x is

$$F(x) = \int_a^x f(t)\,dt.$$

Theorem (Fundamental theorem of Calculus, Part 1)

If a function f is continuous on $[a, b]$, then the function $F(x) = \int_a^x f(t)\,dt$ is continuous on $[a, b]$, differentiable on (a, b), and

$$F'(x) = \frac{d}{dx} \int_a^x f(t)\,dt = f(x).$$

Proof We prove the Fundamental Theorem, Part 1, by applying the definition of the derivative directly to the function $F(x)$, when x and $x + h$ are in (a, b). This means writing out the difference quotient

$$\frac{F(x+h) - F(x)}{h}$$

and showing that its limit as $h \to 0$ is the number $f(x)$ for each x in (a, b). Doing so, we find

$$\begin{aligned} F'(x) &= \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h \to 0} \frac{1}{h}\left[\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt\right] \\ &= \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t)\,dt. \end{aligned}$$

According to the Mean Value Theorem for Definite Integrals, the value before taking the limit in the last expression is one of the values taken on by f in the interval between x and $x + h$. That is, for some number c in this interval,

$$\frac{1}{h} \int_x^{x+h} f(t)\,dt = f(c).$$

As $h \to 0$, $x + h$ approaches x, forcing c to approach x also (because c is trapped between x and $x + h$). Since f is continuous at x, $f(c)$ approaches $f(x)$:

$$\lim_{h \to 0} f(c) = f(x).$$

In conclusion, we have
$$F'(x) = \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t)\,dt = \lim_{h \to 0} f(c) = f(x).$$

If $x = a$ or b, then the limit is interpreted as a one-sided limit with $h \to 0^+$ or $h \to 0^-$, respectively. ∎

(from Thomas'Calculus)

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EXAMPLE Use the Fundamental Theorem to find dy/dx if

(a) $y = \int_a^x (t^3 + 1)\,dt$ **(b)** $y = \int_x^5 3t \sin t\,dt$ **(c)** $y = \int_1^{x^2} \cos t\,dt$ **(d)** $y = \int_{1+3x^2}^4 \frac{1}{2 + e^t}\,dt$

Solution We calculate the derivatives with respect to the independent variable x.

(a) $\frac{dy}{dx} = \frac{d}{dx}\int_a^x (t^3 + 1)\,dt = x^3 + 1$

(b) $\frac{dy}{dx} = \frac{d}{dx}\int_x^5 3t \sin t\,dt = \frac{d}{dx}\left(-\int_5^x 3t \sin t\,dt\right) = -\frac{d}{dx}\int_5^x 3t \sin t\,dt = -3x \sin x$

(c) The upper limit of integration is not x but x^2. This makes y a composite of the two functions,

$$y = \int_1^u \cos t\,dt \qquad \text{and} \qquad u = x^2.$$

We must therefore apply the Chain Rule when finding dy/dx.

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \left(\frac{d}{du}\int_1^u \cos t\,dt\right) \cdot \frac{du}{dx} = \cos u \cdot \frac{du}{dx} = \cos(x^2) \cdot 2x = 2x \cos x^2$$

(d) $\frac{d}{dx}\int_{1+3x^2}^4 \frac{1}{2 + e^t}\,dt = \frac{d}{dx}\left(-\int_4^{1+3x^2} \frac{1}{2 + e^t}\,dt\right) = -\frac{d}{dx}\int_4^{1+3x^2} \frac{1}{2 + e^t}\,dt$

$$= -\frac{1}{2 + e^{(1+3x^2)}} \frac{d}{dx}(1 + 3x^2) = -\frac{6x}{2 + e^{(1+3x^2)}}$$

(from Thomas'Calculus)

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Theorem (Fundamental theorem of Calculus, Part 2 (a.k.a. the Newton–Leibniz formula or Evaluation theorem))

If a function f is continuous on $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Proof Part 1 of the Fundamental Theorem tells us that an antiderivative of f exists, namely

$$G(x) = \int_a^x f(t)\,dt.$$

Thus, if F is *any* antiderivative of f, then $F(x) = G(x) + C$ for some constant C for $a < x < b$

Since both F and G are continuous on $[a, b]$, we see that $F(x) = G(x) + C$ also holds when $x = a$ and $x = b$ by taking one-sided limits (as $x \to a^+$ and $x \to b^-$).

Evaluating $F(b) - F(a)$, we have

$$\begin{aligned} F(b) - F(a) &= [G(b) + C] - [G(a) + C] \\ &= G(b) - G(a) \\ &= \int_a^b f(t)\,dt - \int_a^a f(t)\,dt = \int_a^b f(t)\,dt - 0 = \int_a^b f(t)\,dt. \ \blacksquare \end{aligned}$$

(from Thomas'Calculus)

The Evaluation Theorem is important because it says that to calculate the definite integral of f over an interval $[a, b]$ we need do only two things:

1. Find an antiderivative F of f, and
2. Calculate the number $F(b) - F(a)$, which is equal to $\int_a^b f(x)\,dx$.

This process is much easier than using a Riemann sum computation. The power of the theorem follows from the realization that the definite integral, which is defined by a complicated process involving all of the values of the function f over $[a, b]$, can be found by knowing the values of *any* antiderivative F at only the two endpoints a and b. The usual notation for the difference $F(b) - F(a)$ is

$$F(x)\Big]_a^b \qquad \text{or} \qquad \Big[F(x)\Big]_a^b,$$

depending on whether F has one or more terms.

(from Thomas'Calculus)

EXAMPLE We calculate several definite integrals using the Evaluation Theorem, rather than by taking limits of Riemann sums.

(a) $$\int_0^{\pi} \cos x\, dx = \sin x \Big]_0^{\pi} = \sin \pi - \sin 0 = 0 - 0 = 0$$

(b) $$\int_{-\pi/4}^{0} \sec x \tan x\, dx = \sec x \Big]_{-\pi/4}^{0} = \sec 0 - \sec\left(-\frac{\pi}{4}\right) = 1 - \sqrt{2}$$

(c) $$\int_1^4 \left(\frac{3}{2}\sqrt{x} - \frac{4}{x^2}\right) dx = \left[x^{3/2} + \frac{4}{x}\right]_1^4 = \left[(4)^{3/2} + \frac{4}{4}\right] - \left[(1)^{3/2} + \frac{4}{1}\right]$$
$$= [8 + 1] - [5] = 4$$

(d) $$\int_0^1 \frac{dx}{x+1} = \ln |x + 1| \Big]_0^1 = \ln 2 - \ln 1 = \ln 2$$

(e) $$\int_0^1 \frac{dx}{x^2+1} = \tan^{-1} x \Big]_0^1 = \tan^{-1} 1 - \tan^{-1} 0 = \frac{\pi}{4} - 0 = \frac{\pi}{4}. \quad \blacksquare$$

(from Thomas'Calculus)

The Relationship Between Integration and Differentiation

The conclusions of the Fundamental Theorem tell us several things. Equation (2) can be rewritten as

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x),$$

which says that if you first integrate the function f and then differentiate the result, you get the function f back again. Likewise, replacing b by x and x by t in Equation (6) gives

$$\int_a^x F'(t)\,dt = F(x) - F(a),$$

so that if you first differentiate the function F and then integrate the result, you get the function F back (adjusted by an integration constant). In a sense, the processes of integration and differentiation are "inverses" of each other. The Fundamental Theorem also says that every continuous function f has an antiderivative F. It shows the importance of finding antiderivatives in order to evaluate definite integrals easily. Furthermore, it says that the differential equation $dy/dx = f(x)$ has a solution (namely, any of the functions $y = F(x) + C$) for every continuous function f.

(from Thomas'Calculus)

Definite integral substitutions

THEOREM –Substitution in Definite Integrals If g' is continuous on the interval $[a, b]$ and f is continuous on the range of $g(x) = u$, then

$$\int_a^b f(g(x)) \cdot g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du.$$

Proof Let F denote any antiderivative of f. Then,

$$\begin{aligned}
\int_a^b f(g(x)) \cdot g'(x)\, dx &= F(g(x)) \Big]_{x=a}^{x=b} && \frac{d}{dx} F(g(x)) = F'(g(x))g'(x) = f(g(x))g'(x) \\
&= F(g(b)) - F(g(a)) \\
&= F(u) \Big]_{u=g(a)}^{u=g(b)} \\
&= \int_{g(a)}^{g(b)} f(u)\, du. && \text{Fundamental Theorem, Part 2}
\end{aligned}$$

■

To use the formula, make the same u-substitution $u = g(x)$ and $du = g'(x)\, dx$ you would use to evaluate the corresponding indefinite integral. Then integrate the transformed integral with respect to u from the value $g(a)$ (the value of u at $x = a$) to the value $g(b)$ (the value of u at $x = b$).

(from Thomas'Calculus)

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EXAMPLE Evaluate $\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx$.

Solution We have two choices.

Method 1: Transform the integral and evaluate the transformed integral with the transformed limits given in Theorem 7.

$$\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx$$

Let $u = x^3 + 1$, $du = 3x^2\,dx$.
When $x = -1$, $u = (-1)^3 + 1 = 0$.
When $x = 1$, $u = (1)^3 + 1 = 2$.

$$= \int_0^2 \sqrt{u}\,du = \frac{2}{3}u^{3/2}\Big]_0^2 = \frac{2}{3}\left[2^{3/2} - 0^{3/2}\right] = \frac{2}{3}\left[2\sqrt{2}\right] = \frac{4\sqrt{2}}{3}$$

Method 2: Transform the integral as an indefinite integral, integrate, change back to x, and use the original x-limits.

$$\int 3x^2\sqrt{x^3+1}\,dx = \int \sqrt{u}\,du$$

Let $u = x^3 + 1$, $du = 3x^2\,dx$.

$$= \frac{2}{3}u^{3/2} + C = \frac{2}{3}(x^3+1)^{3/2} + C$$

Replace u by $x^3 + 1$.

$$\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx = \frac{2}{3}(x^3+1)^{3/2}\Big]_{-1}^{1}$$

Use the integral just found, with limits of integration for x.

$$= \frac{2}{3}\left[((1)^3+1)^{3/2} - ((-1)^3+1)^{3/2}\right] = \frac{2}{3}\left[2^{3/2} - 0^{3/2}\right] = \frac{2}{3}\left[2\sqrt{2}\right] = \frac{4\sqrt{2}}{3}$$

(from Thomas'Calculus)

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EXAMPLE We use the method of transforming the limits of integration.

(a) $$\int_{\pi/4}^{\pi/2} \cot\theta \csc^2\theta \, d\theta = \int_1^0 u \cdot (-du)$$

Let $u = \cot\theta$, $du = -\csc^2\theta \, d\theta$,
$-du = \csc^2\theta \, d\theta$.
When $\theta = \pi/4$, $u = \cot(\pi/4) = 1$.
When $\theta = \pi/2$, $u = \cot(\pi/2) = 0$.

$$= -\int_1^0 u \, du = -\left[\frac{u^2}{2}\right]_1^0 = -\left[\frac{(0)^2}{2} - \frac{(1)^2}{2}\right] = \frac{1}{2}$$

(b) $$\int_{-\pi/4}^{\pi/4} \tan x \, dx = \int_{-\pi/4}^{\pi/4} \frac{\sin x}{\cos x} \, dx$$

Let $u = \cos x$, $du = -\sin x \, dx$.
When $x = -\pi/4$, $u = \sqrt{2}/2$.
When $x = \pi/4$, $u = \sqrt{2}/2$.

$$= -\int_{\sqrt{2}/2}^{\sqrt{2}/2} \frac{du}{u} = -\ln|u| \Big]_{\sqrt{2}/2}^{\sqrt{2}/2} = 0$$

Integrate, zero width interval

(from Thomas'Calculus)

Evaluating definite integrals by parts

Theorem (Integration by parts)

Let $f, g : [a, b] \to \mathbb{R}$ be continuously differentiable on $[a, b]$. Then

$$\int f(x)g'(x)\,dx = f(x)g(x)\big|_a^b - \int_a^b f'(x)g(x)\,dx.$$

EXAMPLE $\displaystyle\int_0^4 xe^{-x}\,dx.$

Solution Let $u = x$, $dv = e^{-x}\,dx$, $v = -e^{-x}$, and $du = dx$. Then,

$$\int_0^4 xe^{-x}\,dx = -xe^{-x}\Big]_0^4 - \int_0^4 (-e^{-x})\,dx = \left[-4e^{-4} - (-0e^{-0})\right] + \int_0^4 e^{-x}\,dx$$

$$= -4e^{-4} - e^{-x}\Big]_0^4 = -4e^{-4} - (e^{-4} - e^{-0}) = 1 - 5e^{-4} \approx 0.91. \quad \blacksquare$$

(from Thomas'Calculus)

Definite integrals of symmetric functions

THEOREM Let f be continuous on the symmetric interval $[-a, a]$.

(a) If f is even, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.

(b) If f is odd, then $\int_{-a}^{a} f(x)\,dx = 0$.

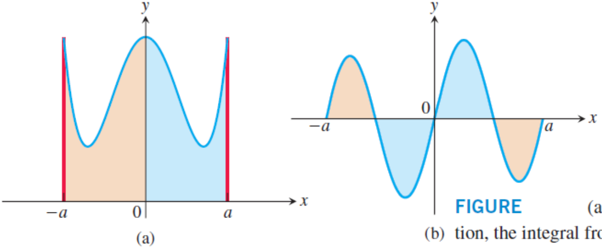


FIGURE (a) For f an even function, the integral from $-a$ to a is twice the integral from 0 to a. (b) For f an odd function, the integral from $-a$ to a equals 0.

(from Thomas'Calculus)

EXAMPLE Evaluate $\int_{-2}^{2}(x^4 - 4x^2 + 6)\,dx$.

Solution Since $f(x) = x^4 - 4x^2 + 6$ satisfies $f(-x) = f(x)$, it is even on the symmetric interval $[-2, 2]$, so

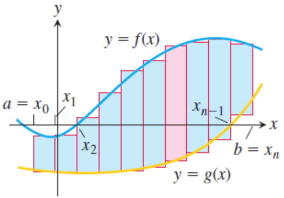
$$\begin{aligned}\int_{-2}^{2}(x^4 - 4x^2 + 6)\,dx &= 2\int_{0}^{2}(x^4 - 4x^2 + 6)\,dx \\ &= 2\left[\frac{x^5}{5} - \frac{4}{3}x^3 + 6x\right]_0^2 \\ &= 2\left(\frac{32}{5} - \frac{32}{3} + 12\right) = \frac{232}{15}.\end{aligned}$$

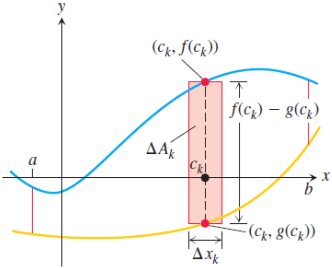
(from Thomas'Calculus)

DEFINITION If f and g are continuous with $f(x) \geq g(x)$ throughout $[a, b]$, then the **area of the region between the curves $y = f(x)$ and $y = g(x)$ from a to b** is the integral of $(f - g)$ from a to b:

$$A = \int_a^b [f(x) - g(x)]\, dx.$$

When applying this definition it is helpful to graph the curves. The graph reveals which curve is the upper curve f and which is the lower curve g. It also helps you find the limits of integration if they are not given. You may need to find where the curves intersect to determine the limits of integration, and this may involve solving the equation $f(x) = g(x)$ for values of x. Then you can integrate the function $f - g$ for the area between the intersections.





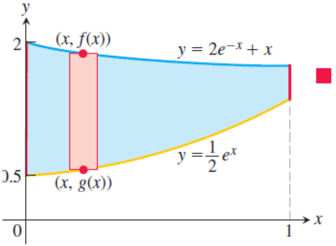
EXAMPLE Find the area of the region bounded above by the curve $y = 2e^{-x} + x$, below by the curve $y = e^x/2$, on the left by $x = 0$, and on the right by $x = 1$.

Solution Figure displays the graphs of the curves and the region whose area we want to find. The area between the curves over the interval $0 \le x \le 1$ is given by

$$A = \int_0^1 \left[(2e^{-x} + x) - \frac{1}{2}e^x\right]dx = \left[-2e^{-x} + \frac{1}{2}x^2 - \frac{1}{2}e^x\right]_0^1$$

$$= \left(-2e^{-1} + \frac{1}{2} - \frac{1}{2}e\right) - \left(-2 + 0 - \frac{1}{2}\right)$$

$$= 3 - \frac{2}{e} - \frac{e}{2} \approx 0.9051.$$



(from Thomas'Calculus)

Areas between curves

EXAMPLE Find the area of the region enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$.

Solution First we sketch the two curves (Figure 5.28). The limits of integration are found by solving $y = 2 - x^2$ and $y = -x$ simultaneously for x.

$$2 - x^2 = -x \qquad \text{Equate } f(x) \text{ and } g(x).$$
$$x = -1, \qquad x = 2. \qquad \text{Solve.}$$

The region runs from $x = -1$ to $x = 2$. The limits of integration are $a = -1$, $b = 2$.

The area between the curves is

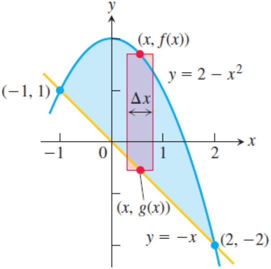
$$A = \int_a^b [f(x) - g(x)]\, dx = \int_{-1}^{2} [(2 - x^2) - (-x)]\, dx$$

The area between the curves is

$$A = \int_a^b [f(x) - g(x)]\, dx = \int_{-1}^{2} [(2 - x^2) - (-x)]\, dx$$

$$= \int_{-1}^{2} (2 + x - x^2)\, dx = \left[2x + \frac{x^2}{2} - \frac{x^3}{3}\right]_{-1}^{2}$$

$$= \left(4 + \frac{4}{2} - \frac{8}{3}\right) - \left(-2 + \frac{1}{2} + \frac{1}{3}\right) = \frac{9}{2}.$$



(from Thomas'Calculus)

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If the formula for a bounding curve changes at one or more points, we subdivide the region into subregions that correspond to the formula changes and apply the formula for the area between curves to each subregion.

EXAMPLE Find the area of the region in the first quadrant that is bounded above by $y = \sqrt{x}$ and below by the x-axis and the line $y = x - 2$.

Solution The sketch (Figure) shows that the region's upper boundary is the graph of $f(x) = \sqrt{x}$. The lower boundary changes from $g(x) = 0$ for $0 \le x \le 2$ to $g(x) = x - 2$ for $2 \le x \le 4$ (both formulas agree at $x = 2$). We subdivide the region at $x = 2$ into subregions A and B, shown in Figure

The limits of integration for region A are $a = 0$ and $b = 2$. The left-hand limit for region B is $a = 2$. To find the right-hand limit, we solve the equations $y = \sqrt{x}$ and $y = x - 2$ simultaneously for x:

$$\sqrt{x} = x - 2$$
$$x = 1, \quad x = 4.$$

Only the value $x = 4$ satisfies the equation $\sqrt{x} = x - 2$. The value $x = 1$ is an extraneous root introduced by squaring. The right-hand limit is $b = 4$.

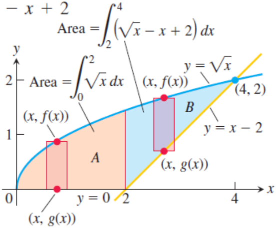
$$\text{For } 0 \le x \le 2: \qquad f(x) - g(x) = \sqrt{x} - 0 = \sqrt{x}$$

$$\text{For } 2 \le x \le 4: \qquad f(x) - g(x) = \sqrt{x} - (x - 2) = \sqrt{x} - x + 2$$

We add the areas of subregions A and B to find the total area:

$$\text{Total area} = \underbrace{\int_0^2 \sqrt{x}\,dx}_{\text{area of } A} + \underbrace{\int_2^4 \left(\sqrt{x} - x + 2\right) dx}_{\text{area of } B}$$

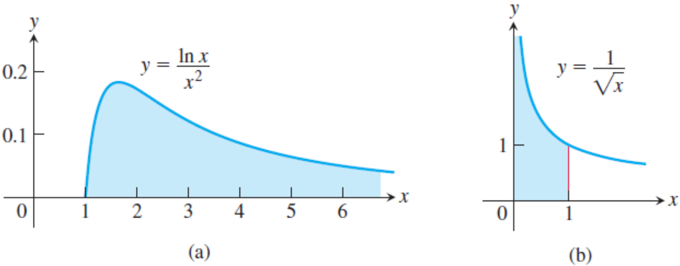
$$= \left[\frac{2}{3}x^{3/2}\right]_0^2 + \left[\frac{2}{3}x^{3/2} - \frac{x^2}{2} + 2x\right]_2^4 = \frac{2}{3}(8) - 2 = \frac{10}{3}.$$



Up to now, we have required definite integrals to have two properties:

- the finite domain of integration $[a, b]$;
- the finite range of the integrand on this domain.

What if one or both of these conditions are not met?



Are the areas under these infinite curves finite? We will see that the answer is yes for both curves.

(from Thomas'Calculus)

Infinite Limits of Integration

Consider the infinite region (unbounded on the right) that lies under the curve $y = e^{-x/2}$ in the first quadrant (Figure). You might think this region has infinite area, but we will see that the value is finite. We assign a value to the area in the following way. First find the area $A(b)$ of the portion of the region that is bounded on the right by $x = b$

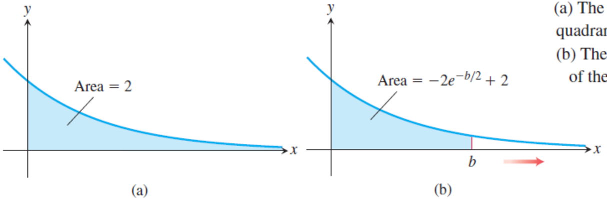
$$A(b) = \int_0^b e^{-x/2}\,dx = -2e^{-x/2}\Big]_0^b = -2e^{-b/2} + 2$$

Then find the limit of $A(b)$ as $b \to \infty$

$$\lim_{b\to\infty} A(b) = \lim_{b\to\infty} \left(-2e^{-b/2} + 2\right) = 2.$$

The value we assign to the area under the curve from 0 to ∞ is

$$\int_0^{\infty} e^{-x/2}\,dx = \lim_{b\to\infty} \int_0^b e^{-x/2}\,dx = 2.$$



(a) The area in the first quadrant under the curve $y = e^{-x/2}$. (b) The area is an improper integral of the first type.

(from Thomas'Calculus)

DEFINITION Integrals with infinite limits of integration are **improper integrals of Type I**.

1. If $f(x)$ is continuous on $[a, \infty)$, then
$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx.$$
2. If $f(x)$ is continuous on $(-\infty, b]$, then
$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx.$$
3. If $f(x)$ is continuous on $(-\infty, \infty)$, then
$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx,$$
where c is any real number.

In each case, if the limit is finite we say that the improper integral **converges** and that the limit is the **value** of the improper integral. If the limit fails to exist, the improper integral **diverges**.

(from Thomas'Calculus)

The choice of c in Part 3 is unimportant. We can evaluate or determine the convergence or divergence of $\int\limits_{-\infty}^{+\infty} f(x)dx$ with any convenient choice.

EXAMPLE Is the area under the curve $y = (\ln x)/x^2$ from $x = 1$ to $x = \infty$ finite? If so, what is its value?

Solution We find the area under the curve from $x = 1$ to $x = b$ and examine the limit as $b \to \infty$. If the limit is finite, we take it to be the area under the curve . The area from 1 to b is

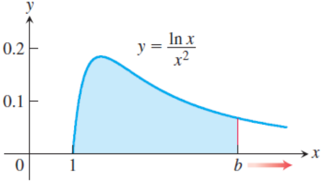
$$\int_1^b \frac{\ln x}{x^2}\,dx = \left[(\ln x)\left(-\frac{1}{x}\right)\right]_1^b - \int_1^b \left(-\frac{1}{x}\right)\left(\frac{1}{x}\right)dx = -\frac{\ln b}{b} - \left[\frac{1}{x}\right]_1^b = -\frac{\ln b}{b} - \frac{1}{b} + 1.$$

The limit of the area as $b \to \infty$ is

$$\int_1^\infty \frac{\ln x}{x^2}\,dx = \lim_{b\to\infty}\int_1^b \frac{\ln x}{x^2}\,dx = \lim_{b\to\infty}\left[-\frac{\ln b}{b} - \frac{1}{b} + 1\right] = -\left[\lim_{b\to\infty}\frac{\ln b}{b}\right] - 0 + 1$$

$$= -\left[\lim_{b\to\infty}\frac{1/b}{1}\right] + 1 = 0 + 1 = 1.$$

Thus, the improper integral converges and the area has finite value 1.



(from Thomas'Calculus)

Improper integrals

EXAMPLE Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$.

Solution According to the definition (Part 3), we can choose $c = 0$ and write

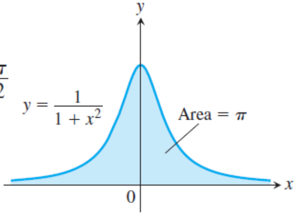
$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{\infty} \frac{dx}{1+x^2}.$$

Next we evaluate each improper integral on the right side of the equation above.

$$\begin{aligned}
\int_{-\infty}^{0} \frac{dx}{1+x^2} &= \lim_{a\to-\infty} \int_{a}^{0} \frac{dx}{1+x^2} = \lim_{a\to-\infty} \tan^{-1} x \Big]_a^0 \\
&= \lim_{a\to-\infty} (\tan^{-1} 0 - \tan^{-1} a) = 0 - \left(-\frac{\pi}{2}\right) = \frac{\pi}{2}
\end{aligned}$$

$$\begin{aligned}
\int_{0}^{\infty} \frac{dx}{1+x^2} &= \lim_{b\to\infty} \int_{0}^{b} \frac{dx}{1+x^2} = \lim_{b\to\infty} \tan^{-1} x \Big]_0^b \\
&= \lim_{b\to\infty} (\tan^{-1} b - \tan^{-1} 0) = \frac{\pi}{2} - 0 = \frac{\pi}{2}
\end{aligned}$$

Thus, $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \frac{\pi}{2} + \frac{\pi}{2} = \pi.$



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The Integral $\int_1^\infty \frac{dx}{x^p}$

The function $y = 1/x$ is the boundary between the convergent and divergent improper integrals with integrands of the form $y = 1/x^p$. As the next example shows, the improper integral converges if $p > 1$ and diverges if $p \le 1$.

Solution If $p \neq 1$,

$$\int_1^b \frac{dx}{x^p} = \frac{x^{-p+1}}{-p+1}\bigg]_1^b = \frac{1}{1-p}(b^{-p+1} - 1) = \frac{1}{1-p}\left(\frac{1}{b^{p-1}} - 1\right).$$

Thus,
$$\int_1^\infty \frac{dx}{x^p} = \lim_{b\to\infty}\int_1^b \frac{dx}{x^p} = \lim_{b\to\infty}\left[\frac{1}{1-p}\left(\frac{1}{b^{p-1}} - 1\right)\right] = \begin{cases} \frac{1}{p-1}, & p > 1 \\ \infty, & p < 1 \end{cases}$$

because
$$\lim_{b\to\infty} \frac{1}{b^{p-1}} = \begin{cases} 0, & p > 1 \\ \infty, & p < 1. \end{cases}$$

Therefore, the integral converges to the value $1/(p-1)$ if $p > 1$ and it diverges if $p < 1$. If $p = 1$, the integral also diverges:

$$\int_1^\infty \frac{dx}{x^p} = \int_1^\infty \frac{dx}{x} = \lim_{b\to\infty}\int_1^b \frac{dx}{x} = \lim_{b\to\infty} \ln x\bigg]_1^b = \lim_{b\to\infty}(\ln b - \ln 1) = \infty.$$

(from Thomas'Calculus)

Integrands with Vertical Asymptotes

Another type of improper integral arises when the integrand has a vertical asymptote—an infinite discontinuity—at a limit of integration or at some point between the limits of integration. If the integrand f is positive over the interval of integration, we can again interpret the improper integral as the area under the graph of f and above the x-axis between the limits of integration.

Consider the region in the first quadrant that lies under the curve $y = 1/\sqrt{x}$ from $x = 0$ to $x = 1$ First we find the area of the portion from a to 1

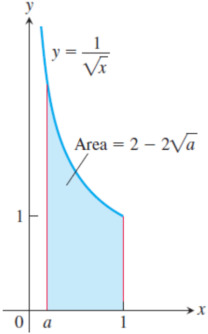
$$\int_a^1 \frac{dx}{\sqrt{x}} = 2\sqrt{x}\Big]_a^1 = 2 - 2\sqrt{a}.$$

Then we find the limit of this area as $a \to 0^+$:

$$\lim_{a\to 0^+} \int_a^1 \frac{dx}{\sqrt{x}} = \lim_{a\to 0^+} \left(2 - 2\sqrt{a}\right) = 2.$$

Therefore the area under the curve from 0 to 1 is finite and is defined to be

$$\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{a\to 0^+} \int_a^1 \frac{dx}{\sqrt{x}} = 2.$$



(from Thomas'Calculus)

DEFINITION Integrals of functions that become infinite at a point within the interval of integration are **improper integrals of Type II**.

1. If $f(x)$ is continuous on $(a, b]$ and discontinuous at a, then

$$\int_a^b f(x)\,dx = \lim_{c \to a^+} \int_c^b f(x)\,dx.$$

2. If $f(x)$ is continuous on $[a, b)$ and discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{c \to b^-} \int_a^c f(x)\,dx.$$

3. If $f(x)$ is discontinuous at c, where $a < c < b$, and continuous on $[a, c) \cup (c, b]$, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

In each case, if the limit is finite we say the improper integral **converges** and that the limit is the **value** of the improper integral. If the limit does not exist, the integral **diverges**.

EXAMPLE Investigate the convergence of

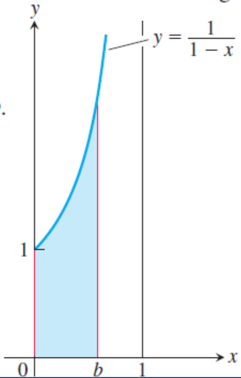
$$\int_0^1 \frac{1}{1-x}\,dx.$$

Solution The integrand $f(x) = 1/(1-x)$ is continuous on $[0, 1)$ but discontinuous at $x = 1$ and becomes infinite as $x \to 1^-$. We evaluate the integral as

$$\lim_{b\to 1^-} \int_0^b \frac{1}{1-x}\,dx = \lim_{b\to 1^-} \Big[-\ln|1-x|\Big]_0^b$$

$$= \lim_{b\to 1^-} [-\ln(1-b) + 0] = \infty.$$

The limit is infinite, so the integral diverges.



EXAMPLE Evaluate $\displaystyle\int_0^3 \frac{dx}{(x-1)^{2/3}}$.

Solution The integrand has a vertical asymptote at $x = 1$ and is continuous on $[0, 1)$ and $(1, 3]$. Thus, by Part 3 of the definition above,

$$\int_0^3 \frac{dx}{(x-1)^{2/3}} = \int_0^1 \frac{dx}{(x-1)^{2/3}} + \int_1^3 \frac{dx}{(x-1)^{2/3}}.$$

Next, we evaluate each improper integral on the right-hand side of this equation.

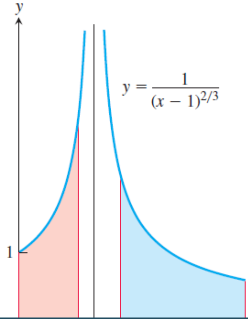
$$\int_0^1 \frac{dx}{(x-1)^{2/3}} = \lim_{b\to 1^-}\int_0^b \frac{dx}{(x-1)^{2/3}} = \lim_{b\to 1^-} 3(x-1)^{1/3}\Big]_0^b = \lim_{b\to 1^-}\left[3(b-1)^{1/3} + 3\right] = 3$$

$$\int_1^3 \frac{dx}{(x-1)^{2/3}} = \lim_{c\to 1^+}\int_c^3 \frac{dx}{(x-1)^{2/3}} = \lim_{c\to 1^+} 3(x-1)^{1/3}\Big]_c^3$$

$$= \lim_{c\to 1^+}\left[3(3-1)^{1/3} - 3(c-1)^{1/3}\right] = 3\sqrt[3]{2}$$

We conclude that

$$\int_0^3 \frac{dx}{(x-1)^{2/3}} = 3 + 3\sqrt[3]{2}.$$



Tests for convergence and divergence

THEOREM —Direct Comparison Test Let f and g be continuous on $[a, \infty)$ with $0 \leq f(x) \leq g(x)$ for all $x \geq a$. Then

1. $\displaystyle\int_a^\infty f(x)\,dx$ converges if $\displaystyle\int_a^\infty g(x)\,dx$ converges.
2. $\displaystyle\int_a^\infty g(x)\,dx$ diverges if $\displaystyle\int_a^\infty f(x)\,dx$ diverges.

THEOREM —Limit Comparison Test If the positive functions f and g are continuous on $[a, \infty)$, and if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L, \qquad 0 < L < \infty,$$

then

$$\int_a^\infty f(x)\,dx \quad \text{and} \quad \int_a^\infty g(x)\,dx$$

both converge or both diverge.

(from Thomas'Calculus)

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EXAMPLE Does the integral $\int_1^\infty e^{-x^2}\,dx$ converge?

Solution By definition,

$$\int_1^\infty e^{-x^2}\,dx = \lim_{b\to\infty}\int_1^b e^{-x^2}\,dx.$$

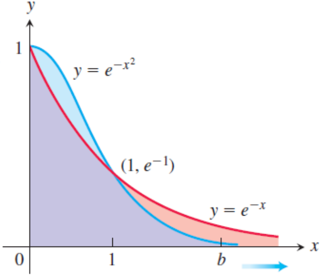
We cannot evaluate this integral directly because it is nonelementary. But we *can* show that its limit as $b\to\infty$ is finite. We know that $\int_1^b e^{-x^2}\,dx$ is an increasing function of b. Therefore either it becomes infinite as $b\to\infty$ or it has a finite limit as $b\to\infty$. It does not become infinite: For every value of $x \geq 1$, we have $e^{-x^2} \leq e^{-x}$ so that

$$\int_1^b e^{-x^2}\,dx \leq \int_1^b e^{-x}\,dx = -e^{-b} + e^{-1} < e^{-1} \approx 0.36788.$$

Hence,

$$\int_1^\infty e^{-x^2}\,dx = \lim_{b\to\infty}\int_1^b e^{-x^2}\,dx$$

converges to some definite finite value. We do not know exactly what the value is except that it is something positive and less than 0.37. Here we are relying on the completeness property of the real numbers



(from Thomas'Calculus)

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EXAMPLE

(a) $\displaystyle\int_1^\infty \frac{\sin^2 x}{x^2}\,dx$ converges because

$$0 \le \frac{\sin^2 x}{x^2} \le \frac{1}{x^2} \quad \text{on} \quad [1, \infty) \quad \text{and} \quad \int_1^\infty \frac{1}{x^2}\,dx \quad \text{converges.}$$

(b) $\displaystyle\int_1^\infty \frac{1}{\sqrt{x^2 - 0.1}}\,dx$ diverges because

$$\frac{1}{\sqrt{x^2 - 0.1}} \ge \frac{1}{x} \quad \text{on} \quad [1, \infty) \quad \text{and} \quad \int_1^\infty \frac{1}{x}\,dx \quad \text{diverges.}$$

(c) $\displaystyle\int_0^{\pi/2} \frac{\cos x}{\sqrt{x}}\,dx$ converges because $0 \le \dfrac{\cos x}{\sqrt{x}} \le \dfrac{1}{\sqrt{x}}$ on $\left[0, \dfrac{\pi}{2}\right]$,

and $$\int_0^{\pi/2} \frac{dx}{\sqrt{x}} = \lim_{a \to 0^+} \int_a^{\pi/2} \frac{dx}{\sqrt{x}} = \lim_{a \to 0^+} \sqrt{4x}\,\Big]_a^{\pi/2} = \lim_{a \to 0^+} \left(\sqrt{2\pi} - \sqrt{4a}\right) = \sqrt{2\pi}$$

converges.

(from Thomas'Calculus)

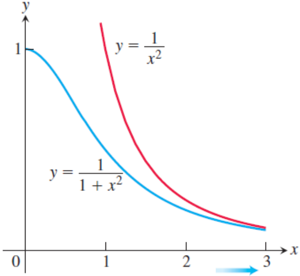
EXAMPLE Show that $\displaystyle\int_1^\infty \frac{dx}{1+x^2}$

converges by comparison with $\int_1^\infty (1/x^2)\,dx$. Find and compare the two integral values.

Solution The functions $f(x) = 1/x^2$ and $g(x) = 1/(1+x^2)$ are positive and continuous on $[1, \infty)$. Also,

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{1/x^2}{1/(1+x^2)} = \lim_{x\to\infty} \frac{1+x^2}{x^2}$$

$$= \lim_{x\to\infty} \left(\frac{1}{x^2} + 1\right) = 0 + 1 = 1,$$



a positive finite limit. Therefore,

$$\int_1^\infty \frac{dx}{1+x^2} \text{ converges because } \int_1^\infty \frac{dx}{x^2} \text{ converges.}$$

The integrals converge to different values, however:

$$\int_1^\infty \frac{dx}{x^2} = \frac{1}{2-1} = 1$$

and $$\int_1^\infty \frac{dx}{1+x^2} = \lim_{b\to\infty} \int_1^b \frac{dx}{1+x^2} = \lim_{b\to\infty} [\tan^{-1} b - \tan^{-1} 1] = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}.$$

(from Thomas'Calculus)

EXAMPLE Investigate the convergence of $\int_1^\infty \frac{1-e^{-x}}{x}\,dx$.

Solution The integrand suggests a comparison of $f(x) = (1 - e^{-x})/x$ with $g(x) = 1/x$. However, we cannot use the Direct Comparison Test because $f(x) \leq g(x)$ and the integral of $g(x)$ *diverges*. On the other hand, using the Limit Comparison Test we find that

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \left(\frac{1-e^{-x}}{x}\right)\left(\frac{x}{1}\right) = \lim_{x\to\infty} (1 - e^{-x}) = 1,$$

which is a positive finite limit.

Therefore, $\int_1^\infty \frac{1-e^{-x}}{x}\,dx$ diverges because $\int_1^\infty \frac{dx}{x}$ diverges. Approximations to the improper integral are given in Table Note that the values do not appear to approach any fixed limiting value as $b \to \infty$.

TABLE

b	$\int_1^b \frac{1-e^{-x}}{x}\,dx$
2	0.5226637569
5	1.3912002736
10	2.0832053156
100	4.3857862516
1000	6.6883713446
10000	8.9909564376
100000	11.2935415306

(from Thomas'Calculus)