4. Differential calculus of functions of one real variable

4.1. Derivatives

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Content:

- Derivative at a point
- Geometric and physical meanings of a derivative
- One-sided derivatives
- Derivative as a function
- Calculating derivatives by definition
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- Differentiation rules
- Derivative formulas
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Derivative at a point

Definition

Let I be an open interval in $\mathbb{R}$, $x_0 \in I$. A function $f : I \to \mathbb{R}$ is said to be **differentiable at x_0** if there exists a finite limit

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} =: f'(x_0).$$

The value $f'(x_0)$ is called the **derivative of f at x_0**.

Notations: $f'(x_0)$, $f'(x)|_{x=x_0}$, $\dfrac{\mathrm{d}f}{\mathrm{d}x}(x_0)$, $\left.\dfrac{\mathrm{d}f(x)}{\mathrm{d}x}\right|_{x=x_0}$, $\left.\dfrac{\mathrm{d}}{\mathrm{d}x}f(x)\right|_{x=x_0}$.

Remark

An equivalent definition: $f'(x_0) = \displaystyle\lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$,

or $f'(x_0) = \displaystyle\lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}$ where $\Delta f = f(x_0 + \Delta x) - f(x_0)$.

Let a function $y = f(x)$ be defined on open interval $I \subset \mathbb{R}$ and continuous at $x_0 \in (a, b)$.

DEFINITION The **average rate of change** of $y = f(x)$ with respect to x over the interval $[x_1, x_2]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + h) - f(x_1)}{h}, \qquad h \neq 0.$$

(from Thomas'Calculus)

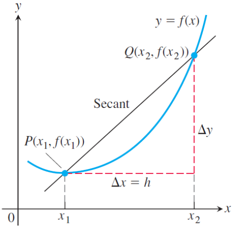


FIGURE A secant to the graph $y = f(x)$. Its slope is $\Delta y/\Delta x$, the average rate of change of f over the interval $[x_1, x_2]$.

Geometrically, the rate of change of f over $[x_1, x_2]$ is the **slope of the line** through the points $P(x_1, f(x_1))$ and $Q(x_2, f(x_2))$. A line joining two points of a curve is a **secant to the curve**. Thus, the average rate of change of f from x_1 to x_2 is identical with the slope of secant PQ.

Geometric meaning of a derivative

Definition

The **slope of the curve** $y = f(x)$ at the point $P(x_0, f(x_0))$ is the number

$$m = \lim_{hx \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

(provided the limit exists). The **tangent line** to the curve at P is the line through P with this slope:

$$y = m(x - x_0) + y_0.$$

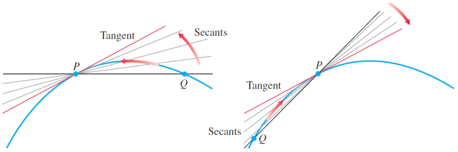


FIGURE The tangent to the curve at P is the line through P whose slope is the limit of the secant slopes as $Q \to P$ from either side.

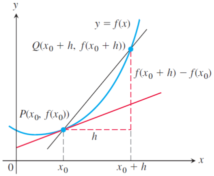


FIGURE The slope of the tangent line at P is $\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$.

(from Thomas'Calculus)

Thus, if the function $f(x)$ is differentiable at x_0, then the tangent line to the curve $f(x)$ at $(x_0, f(x_0))$ is

$$y = f'(x_0)(x - x_0) + y_0.$$

Moreover, $\tan\alpha = f'(x_0)$.

EXAMPLE

(a) Find the slope of the curve $y = 1/x$ at any point $x = a \neq 0$. What is the slope at the point $x = -1$?

(b) Where does the slope equal $-1/4$?

(c) What happens to the tangent to the curve at the point $(a, 1/a)$ as a changes?

Solution

(a) Here $f(x) = 1/x$. The slope at $(a, 1/a)$ is

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = \lim_{h\to 0}\frac{\dfrac{1}{a+h}-\dfrac{1}{a}}{h} = \lim_{h\to 0}\frac{1}{h}\,\frac{a-(a+h)}{a(a+h)}$$

$$= \lim_{h\to 0}\frac{-h}{ha(a+h)} = \lim_{h\to 0}\frac{-1}{a(a+h)} = -\frac{1}{a^2}.$$

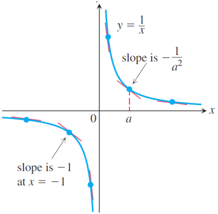
(b) The slope of $y = 1/x$ at the point where $x = a$ is $-1/a^2$. It will be $-1/4$ provided that

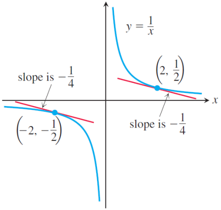
$$-\frac{1}{a^2} = -\frac{1}{4}.$$

This equation is equivalent to $a^2 = 4$, so $a = 2$ or $a = -2$. The curve has slope $-1/4$ at the two points $(2, 1/2)$ and $(-2, -1/2)$ (Figure).

(c) The slope $-1/a^2$ is always negative if $a \neq 0$. As $a \to 0^+$, the slope approaches $-\infty$ and the tangent becomes increasingly steep (Figure). We see this situation again as $a \to 0^-$. As a moves away from the origin in either direction, the slope approaches 0 and the tangent levels off becoming more and more horizontal.

(from Thomas'Calculus)





The **average velocity** of an object moving according to a law $f(x)$ during an interval of time is the ratio between distance traveled (δd) and the time elapsed (δt):

$$V_{av} := \frac{\text{displacement}}{\text{travel time}} = \frac{f(t+\Delta t) - f(t)}{\Delta t}.$$

The **instantaneous velocity at time $t = t_0$** is the limit of the average velocity function when Δt approaches zero:

$$V(t_0) := \lim_{\Delta t \to 0} \frac{f(t_0 + \Delta t) - f(t_0)}{\Delta t}.$$

Thus, $V(t_0) = f'(t_0)$.

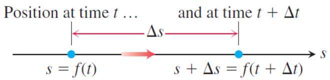


FIGURE The positions of a body moving along a coordinate line at time t and shortly later at time $t + \Delta t$. Here the coordinate line is horizontal.

(from Thomas'Calculus)

EXAMPLE 1 A rock breaks loose from the top of a tall cliff. What is its average speed

(a) during the first 2 sec of fall?

(b) during the 1-sec interval between second 1 and second 2?

Solution The average speed of the rock during a given time interval is the change in distance, Δy, divided by the length of the time interval, Δt. (Increments like Δy and Δt are reviewed in Appendix 3, and pronounced "delta y" and "delta t.") Measuring distance in feet and time in seconds, we have the following calculations:

(a) For the first 2 sec: $$\frac{\Delta y}{\Delta t} = \frac{16(2)^2 - 16(0)^2}{2 - 0} = 32\,\frac{\text{ft}}{\text{sec}}$$

(b) From sec 1 to sec 2: $$\frac{\Delta y}{\Delta t} = \frac{16(2)^2 - 16(1)^2}{2 - 1} = 48\,\frac{\text{ft}}{\text{sec}}$$ ■

(from Thomas'Calculus)

EXAMPLE 2 Find the speed of the falling rock in Example 1 at $t = 1$ and $t = 2$ sec.

Solution We can calculate the average speed of the rock over a time interval $[t_0, t_0 + h]$, having length $\Delta t = h$, as

$$\frac{\Delta y}{\Delta t} = \frac{16(t_0 + h)^2 - 16t_0^2}{h}. \qquad (1)$$

We cannot use this formula to calculate the "instantaneous" speed at the exact moment t_0 by simply substituting $h = 0$, because we cannot divide by zero. But we *can* use it to calculate average speeds over increasingly short time intervals starting at $t_0 = 1$ and $t_0 = 2$.

If we set $t_0 = 1$ and then expand the numerator in Equation (1) and simplify, we find that

$$\frac{\Delta y}{\Delta t} = \frac{16(1 + h)^2 - 16(1)^2}{h} = \frac{16(1 + 2h + h^2) - 16}{h}$$
$$= \frac{32h + 16h^2}{h} = 32 + 16h.$$

Similarly, setting $t_0 = 2$ in Equation (1), the procedure yields

$$\frac{\Delta y}{\Delta t} = 64 + 16h$$

Average and instantaneous rates of change

DEFINITION The **average rate of change** of $y = f(x)$ with respect to x over the interval $[x_1, x_2]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + h) - f(x_1)}{h}, \qquad h \neq 0.$$

(from Thomas'Calculus)

DEFINITION The **instantaneous rate of change** of f with respect to x at x_0 is the derivative

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h},$$

provided the limit exists.

Thus, instantaneous rates are limits of average rates.

(from Thomas'Calculus)

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Remark

The expression $\lim_{h \to 0} \dfrac{f(x_0+h)-f(x_0)}{h}$ is called the **difference quotient of f at x_0 with increment h.**

The following are all interpretations for the limit of the difference quotient,

$$\lim_{h \to 0} \frac{f(x_0+h)-f(x_0)}{h}.$$

1. The slope of the graph of $y = f(x)$ at $x = x_0$

2. The slope of the tangent to the curve $y = f(x)$ at $x = x_0$

3. The rate of change of $f(x)$ with respect to x at $x = x_0$

4. The derivative $f'(x_0)$ at a point

(from Thomas'Calculus)

One-sided derivatives

Definition

Let $x_0 \in \mathbb{R}$, $I = (x_0, b)$. A function $f : I \to \mathbb{R}$ is said to be **right differentiable at x_0** if there exists a finite right-sided limit

$$\lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0} =: f'_+(x_0).$$

The value $f'_+(x_0)$ is called the **right derivative of f at x_0**.

Definition

Let $x_0 \in \mathbb{R}$, $I = (a, x_0)$. A function $f : I \to \mathbb{R}$ is said to be **left differentiable at x_0** if there exists a finite left-sided limit

$$\lim_{x \to x_0^-} \frac{f(x) - f(x_0)}{x - x_0} =: f'_-(x_0).$$

The value $f'_-(x_0)$ is called the **left derivative of f at x_0**.

Derivative as a function

Definition

Let (a, b) be an open interval in $\mathbb{R}$, $x_0 \in (a, b)$.

- A function $f : (a, b) \to \mathbb{R}$ is said to be **differentiable on (a, b)** if it is differentiable at each $x_0 \in (a, b)$.
- A function $f : [a, b] \to \mathbb{R}$ is said to be **differentiable on $[a, b]$** if it is differentiable on (a, b), right differentiable at a and left differentiable at b.

Definition

The **derivative of the function $f(x)$ with respect to the variable x** is the function f' whose value at x is

$$f'(x) := \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

provided the limit exists.

The domain of f' is the set of points in the domain of f for which the limit exists, which means that $D(f') \subseteq D(f)$.

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The process of calculating a derivative is called **differentiation**.

Examples:

1) $y = c$, $c \in \mathbb{R}$. $y' = \lim\limits_{h\to 0} \dfrac{c-c}{h} = 0$.

2) $y = x$. $y' = \lim\limits_{h\to 0} \dfrac{x+h-x}{h} = 1$.

3) $y = x^2$. $y' = \lim\limits_{h\to 0} \dfrac{(x+h)^2 - x^2}{h} = \lim\limits_{h\to 0} \dfrac{2hx + h^2}{h} = \lim\limits_{h\to 0}(2x+h) = 2x$.

4) $y = x^n$, $n \in \mathbb{N}$. $y' = \lim\limits_{h\to 0} \dfrac{(x+h)^n - x^n}{h} = \lim\limits_{h\to 0} \dfrac{nx^{n-1}h + \cdots + h^n}{h} =$
$\lim\limits_{h\to 0}(nx^{n-1} + \cdots + h^{n-1}) = nx^{n-1}$.

5) $y = \dfrac{1}{x}$, $x \neq 0$, $n \in \mathbb{N}$.

$$y' = \lim_{h\to 0} \frac{\dfrac{1}{x+h} - \dfrac{1}{h}}{h} = -\lim_{h\to 0} \frac{1}{x(x+h)} = -\frac{1}{x^2}.$$

6) $y = \dfrac{1}{x^n}$, $x \neq 0$, $n \in \mathbb{N}$. $y' = -\dfrac{n}{x^{n+1}}$.

Examples:

7) $y = \sqrt{x}$, $x > 0$. $y' = \dfrac{1}{2\sqrt{x}}$.

8) $y = \dfrac{x}{x-1}$, $x \neq 1$. $y' = -\dfrac{1}{(x-1)^2}$.

9) $y = \cos x$. $y' = \lim\limits_{h \to 0} \dfrac{\cos(x+h) - \cos x}{h} =$

$$\lim_{h\to 0} \frac{-2\sin\left(x + \dfrac{h}{2}\right)\sin\dfrac{h}{2}}{h} = -\lim_{h\to 0} \sin x \lim_{h \to 0} \frac{\sin\dfrac{h}{2}}{h/2} = -\sin x.$$

10) $y = \sin x$. $y' = \cos x$.

11) $y = a^x$.

$$y' = \lim_{h\to 0} \frac{a^{x+h} - a^x}{h} = a^x \lim_{h\to 0} \frac{a^h - 1}{h} = a^x \ln a.$$

In particular, $(e^x)' = e^x$.

Examples:

1) $y = \sqrt{x}$, $x_0 = 0$. $\lim\limits_{h\to 0+} \dfrac{\sqrt{x_0+h}-\sqrt{x_0}}{h} = \lim\limits_{h\to 0+} \dfrac{1}{\sqrt{h}} = +\infty \notin \mathbb{R}$.
Therefore, the function $y = \sqrt{x}$ is not differentiable at 0.

2) $y = |x|$, $x_0 = 0$. $\lim\limits_{h\to 0+} \dfrac{|x_0+h|-|x_0|}{h} = \lim\limits_{h\to 0+} \dfrac{|h|}{h} = 1$;

$\lim\limits_{h\to 0-} \dfrac{|x_0+h|-|x_0|}{h} = \lim\limits_{h\to 0-} \dfrac{|h|}{h} = -1 \neq 1$.
Therefore, the function $y = |x|$ is not differentiable at 0.

Theorem (Necessary condition of differentiability)

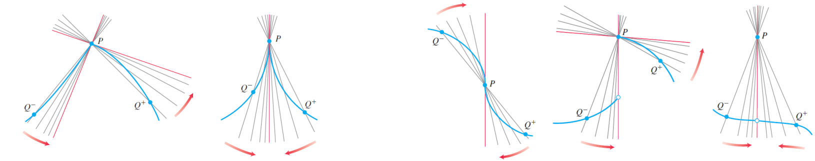
If a function f is differentiable at x_0, then f is continuous at x_0.

Proof: Given $f'(x_0)$ exists, we must show that $\lim\limits_{x\to x_0} f(x) = f(x_0)$, or, equivalently, $\lim\limits_{h\to 0} f(h+x_0) = f(x_0)$. For $h \neq 0$,

$f(x_0+h) = f(x_0+h) - f(x_0) + f(x_0) = h\dfrac{f(x_0+h)-f(x_0)}{h} + f(x_0)$. Then

$\lim\limits_{h\to 0} f(h+x_0) = \lim\limits_{h\to 0} \left(h\dfrac{f(x_0+h)-f(x_0)}{h} + f(x_0)\right) = \lim\limits_{h\to 0} hf'(x_0) + f(x_0) = f(x_0)$.

Continuity does not imply differentiability!



1. a *corner*, where the one-sided derivatives differ.

2. a *cusp*, where the slope of *PQ* approaches ∞ from one side and $-\infty$ from the other.

3. a *vertical tangent*, where the slope of *PQ* approaches ∞ from both sides or approaches $-\infty$ from both sides (here, $-\infty$).

4. a *discontinuity* (two examples shown).

(from Thomas'Calculus)

Derivative of a Constant Function

If f has the constant value $f(x) = c$, then

$$\frac{df}{dx} = \frac{d}{dx}(c) = 0.$$

(from Thomas'Calculus)

Derivative of a Positive Integer Power

If n is a positive integer, then

$$\frac{d}{dx}x^n = nx^{n-1}.$$

(from Thomas'Calculus)

Power Rule (General Version)

If n is any real number, then

$$\frac{d}{dx}x^n = nx^{n-1},$$

for all x where the powers x^n and x^{n-1} are defined.

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EXAMPLE Differentiate the following powers of x.

(a) x^3 **(b)** $x^{2/3}$ **(c)** $x^{\sqrt{2}}$ **(d)** $\dfrac{1}{x^4}$ **(e)** $x^{-4/3}$ **(f)** $\sqrt{x^{2+\pi}}$

Solution

(a) $\dfrac{d}{dx}(x^3) = 3x^{3-1} = 3x^2$ **(b)** $\dfrac{d}{dx}(x^{2/3}) = \dfrac{2}{3}x^{(2/3)-1} = \dfrac{2}{3}x^{-1/3}$

(c) $\dfrac{d}{dx}\left(x^{\sqrt{2}}\right) = \sqrt{2}x^{\sqrt{2}-1}$

(d) $\dfrac{d}{dx}\left(\dfrac{1}{x^4}\right) = \dfrac{d}{dx}(x^{-4}) = -4x^{-4-1} = -4x^{-5} = -\dfrac{4}{x^5}$

(e) $\dfrac{d}{dx}(x^{-4/3}) = -\dfrac{4}{3}x^{-(4/3)-1} = -\dfrac{4}{3}x^{-7/3}$

(f) $\dfrac{d}{dx}\left(\sqrt{x^{2+\pi}}\right) = \dfrac{d}{dx}\left(x^{1+(\pi/2)}\right) = \left(1 + \dfrac{\pi}{2}\right)x^{1+(\pi/2)-1} = \dfrac{1}{2}(2 + \pi)\sqrt{x^{\pi}}$

(from Thomas'Calculus)

Differentiation rules: arithmetic rules

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Derivative Constant Multiple Rule

If u is a differentiable function of x, and c is a constant, then

$$\frac{d}{dx}(cu) = c\frac{du}{dx}.$$

(from Thomas'Calculus)

Derivative Sum Rule

If u and v are differentiable functions of x, then their sum $u + v$ is differentiable at every point where u and v are both differentiable. At such points,

$$\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}.$$

(from Thomas'Calculus)

EXAMPLE

(a) The derivative formula

$$\frac{d}{dx}(3x^2) = 3 \cdot 2x = 6x$$

says that if we rescale the graph of $y = x^2$ by multiplying each y-coordinate by 3, then we multiply the slope at each point by 3 (Figure 3.10).

(b) Negative of a function

The derivative of the negative of a differentiable function u is the negative of the function's derivative. The Constant Multiple Rule with $c = -1$ gives

$$\frac{d}{dx}(-u) = \frac{d}{dx}(-1 \cdot u) = -1 \cdot \frac{d}{dx}(u) = -\frac{du}{dx}.$$

The next rule says that the derivative of the sum of two differentiable functions is the sum of their derivatives.

(from Thomas'Calculus)

EXAMPLE Find the derivative of the polynomial $y = x^3 + \frac{4}{3}x^2 - 5x + 1$.

Solution

$$\frac{dy}{dx} = \frac{d}{dx}x^3 + \frac{d}{dx}\left(\frac{4}{3}x^2\right) - \frac{d}{dx}(5x) + \frac{d}{dx}(1) \qquad \text{Sum and Difference Rules}$$

$$= 3x^2 + \frac{4}{3}\cdot 2x - 5 + 0 = 3x^2 + \frac{8}{3}x - 5$$

We can differentiate any polynomial term by term, the way we differentiated the polynomial in Example 3. All polynomials are differentiable at all values of x.

(from Thomas'Calculus)

EXAMPLE Does the curve $y = x^4 - 2x^2 + 2$ have any horizontal tangents? If so, where?

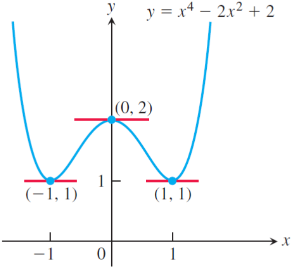
Solution The horizontal tangents, if any, occur where the slope dy/dx is zero. We have

$$\frac{dy}{dx} = \frac{d}{dx}(x^4 - 2x^2 + 2) = 4x^3 - 4x.$$

Now solve the equation $\frac{dy}{dx} = 0$ for x:

$$\begin{aligned} 4x^3 - 4x &= 0 \\ 4x(x^2 - 1) &= 0 \\ x &= 0, 1, -1. \end{aligned}$$

The curve $y = x^4 - 2x^2 + 2$ has horizontal tangents at $x = 0$, 1, and -1. The corresponding points on the curve are (0, 2), (1, 1), and $(-1, 1)$.



(from Thomas'Calculus)

Derivative Product Rule

If u and v are differentiable at x, then so is their product uv, and

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x).$$

(from Thomas'Calculus)

Derivative Quotient Rule

If u and v are differentiable at x and if $v(x) \neq 0$, then the quotient u/v is differentiable at x, and

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$$

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}.$$

(from Thomas'Calculus)

EXAMPLE Find the derivative of $y = (x^2 + 1)(x^3 + 3)$.

Solution

(a) From the Product Rule with $u = x^2 + 1$ and $v = x^3 + 3$, we find

$$\frac{d}{dx}\left[(x^2 + 1)(x^3 + 3)\right] = (x^2 + 1)(3x^2) + (x^3 + 3)(2x) \qquad \frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$= 3x^4 + 3x^2 + 2x^4 + 6x$$

$$= 5x^4 + 3x^2 + 6x.$$

(b) This particular product can be differentiated as well (perhaps better) by multiplying out the original expression for y and differentiating the resulting polynomial:

$$y = (x^2 + 1)(x^3 + 3) = x^5 + x^3 + 3x^2 + 3$$

$$\frac{dy}{dx} = 5x^4 + 3x^2 + 6x.$$

This is in agreement with our first calculation. ∎

(from Thomas'Calculus)

EXAMPLE Find the derivative of **(a)** $y = \dfrac{t^2 - 1}{t^3 + 1}$, **(b)** $y = e^{-x}$.

Solution

(a) We apply the Quotient Rule with $u = t^2 - 1$ and $v = t^3 + 1$:

$$\frac{dy}{dt} = \frac{(t^3 + 1) \cdot 2t - (t^2 - 1) \cdot 3t^2}{(t^3 + 1)^2} \qquad \frac{d}{dt}\left(\frac{u}{v}\right) = \frac{v(du/dt) - u(dv/dt)}{v^2}$$

$$= \frac{2t^4 + 2t - 3t^4 + 3t^2}{(t^3 + 1)^2}$$

$$= \frac{-t^4 + 3t^2 + 2t}{(t^3 + 1)^2}.$$

(b) $$\frac{d}{dx}(e^{-x}) = \frac{d}{dx}\left(\frac{1}{e^x}\right) = \frac{e^x \cdot 0 - 1 \cdot e^x}{(e^x)^2} = \frac{-1}{e^x} = -e^{-x}$$

(from Thomas'Calculus)

EXAMPLE Find the derivative of

$$y = \frac{(x-1)(x^2-2x)}{x^4}.$$

Solution Using the Quotient Rule here will result in a complicated expression with many terms. Instead, use some algebra to simplify the expression. First expand the numerator and divide by x^4:

$$y = \frac{(x-1)(x^2-2x)}{x^4} = \frac{x^3-3x^2+2x}{x^4} = x^{-1} - 3x^{-2} + 2x^{-3}.$$

Then use the Sum and Power Rules:

$$\begin{aligned}\frac{dy}{dx} &= -x^{-2} - 3(-2)x^{-3} + 2(-3)x^{-4} \\ &= -\frac{1}{x^2} + \frac{6}{x^3} - \frac{6}{x^4}.\end{aligned}$$

■

(from Thomas'Calculus)

Differentiation rules: Chain rule

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Theorem (Chain rule)

Let $g : (a, b) \to \mathbb{R}$ be differentiable at $x_0 \in (a, b)$, and let $f : R(g) \to \mathbb{R}$ be differentiable at $g(x_0)$. Then $f \circ g : (a, b) \to \mathbb{R}$ is differentiable at x_0 and

$$(f \circ g)'(x_0) = f'(g(x_0))g'(x_0).$$

Equivalently,

$$\frac{\mathrm{d}}{\mathrm{d}x} f(g(x))\Big|_{x=x_0} = \left(\frac{\mathrm{d}}{\mathrm{d}y} f(y)\Big|_{y=g(x_0)}\right)\left(\frac{\mathrm{d}}{\mathrm{d}x} g(x)\Big|_{x=x_0}\right).$$

Ways to Write the Chain Rule

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\frac{dy}{dx} = f'(g(x)) \cdot g'(x)$$

$$\frac{d}{dx} f(u) = f'(u) \frac{du}{dx}$$

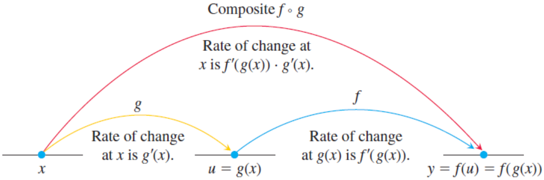

FIGURE Rates of change multiply: The derivative of $f \circ g$ at x is the derivative of f at $g(x)$ times the derivative of g at x.

(from Thomas' Calculus)

EXAMPLE Differentiate $\sin(x^2 + e^x)$ with respect to x.

Solution We apply the Chain Rule directly and find

$$\frac{d}{dx}\sin\underbrace{(x^2 + e^x)}_{\text{inside}} = \cos\underbrace{(x^2 + e^x)}_{\substack{\text{inside}\\ \text{left alone}}} \cdot \underbrace{(2x + e^x)}_{\substack{\text{derivative of}\\ \text{the inside}}}.$$

EXAMPLE Differentiate $y = e^{\cos x}$.

Solution Here the inside function is $u = g(x) = \cos x$ and the outside function is the exponential function $f(x) = e^x$. Applying the Chain Rule, we get

$$\frac{dy}{dx} = \frac{d}{dx}(e^{\cos x}) = e^{\cos x}\frac{d}{dx}(\cos x) = e^{\cos x}(-\sin x) = -e^{\cos x}\sin x.$$

Generalizing Example we see that the Chain Rule gives the formula

$$\boxed{\frac{d}{dx}e^u = e^u\frac{du}{dx}.}$$

(from Thomas'Calculus)

EXAMPLE Find the derivative of $g(t) = \tan(5 - \sin 2t)$.

Solution Notice here that the tangent is a function of $5 - \sin 2t$, whereas the sine is a function of $2t$, which is itself a function of t. Therefore, by the Chain Rule,

$$\begin{aligned}
g'(t) &= \frac{d}{dt}(\tan(5 - \sin 2t)) \\
&= \sec^2(5 - \sin 2t) \cdot \frac{d}{dt}(5 - \sin 2t) && \text{Derivative of } \tan u \text{ with } u = 5 - \sin 2t \\
&= \sec^2(5 - \sin 2t) \cdot \left(0 - \cos 2t \cdot \frac{d}{dt}(2t)\right) && \text{Derivative of } 5 - \sin u \text{ with } u = 2t \\
&= \sec^2(5 - \sin 2t) \cdot (-\cos 2t) \cdot 2 \\
&= -2(\cos 2t)\sec^2(5 - \sin 2t).
\end{aligned}$$

(from Thomas'Calculus)

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EXAMPLE The Power Chain Rule simplifies computing the derivative of a power of an expression.

(a) $$\frac{d}{dx}(5x^3 - x^4)^7 = 7(5x^3 - x^4)^6 \frac{d}{dx}(5x^3 - x^4)$$

Power Chain Rule with $u = 5x^3 - x^4, n = 7$

$$= 7(5x^3 - x^4)^6(5 \cdot 3x^2 - 4x^3) = 7(5x^3 - x^4)^6(15x^2 - 4x^3)$$

(b) $$\frac{d}{dx}\left(\frac{1}{3x-2}\right) = \frac{d}{dx}(3x - 2)^{-1}$$

$$= -1(3x - 2)^{-2}\frac{d}{dx}(3x - 2)$$

Power Chain Rule with $u = 3x - 2, n = -1$

$$= -1(3x - 2)^{-2}(3) = -\frac{3}{(3x - 2)^2}$$

In part (b) we could also find the derivative with the Derivative Quotient Rule.

(c) $$\frac{d}{dx}(\sin^5 x) = 5\sin^4 x \cdot \frac{d}{dx}\sin x$$

Power Chain Rule with $u = \sin x, n = 5$, because $\sin^n x$ means $(\sin x)^n$, $n \neq -1$.

$$= 5\sin^4 x \cos x$$

(d) $$\frac{d}{dx}\left(e^{\sqrt{3x+1}}\right) = e^{\sqrt{3x+1}} \cdot \frac{d}{dx}\left(\sqrt{3x + 1}\right)$$

$$= e^{\sqrt{3x+1}} \cdot \frac{1}{2}(3x + 1)^{-1/2} \cdot 3$$

Power Chain Rule with $u = 3x + 1, n = 1/2$

$$= \frac{3}{2\sqrt{3x + 1}}e^{\sqrt{3x+1}}$$

■

EXAMPLE we saw that the absolute value function $y = |x|$ is not differentiable at $x = 0$. However, the function is differentiable at all other real numbers, as we now show. Since $|x| = \sqrt{x^2}$, we can derive the following formula:

$$\begin{aligned}
\frac{d}{dx}(|x|) &= \frac{d}{dx}\sqrt{x^2} \\
&= \frac{1}{2\sqrt{x^2}} \cdot \frac{d}{dx}(x^2) && \text{Power Chain Rule with } u = x^2, n = 1/2, x \neq 0 \\
&= \frac{1}{2|x|} \cdot 2x && \sqrt{x^2} = |x| \\
&= \frac{x}{|x|}, \quad x \neq 0. && \blacksquare
\end{aligned}$$

(from Thomas'Calculus)

Differentiation rules: implicit functions

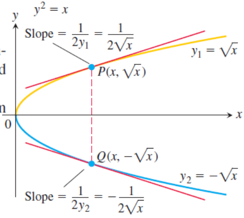
Implicit Differentiation

1. Differentiate both sides of the equation with respect to x, treating y as a differentiable function of x.
2. Collect the terms with dy/dx on one side of the equation and solve for dy/dx.

(from Thomas'Calculus)

EXAMPLE Find dy/dx if $y^2 = x$.

Solution The equation $y^2 = x$ defines two differentiable functions of x that we can actually find, namely $y_1 = \sqrt{x}$ and $y_2 = -\sqrt{x}$ We know how to calculate the derivative of each of these f for $x > 0$: $\dfrac{dy_1}{dx} = \dfrac{1}{2\sqrt{x}}$ and $\dfrac{dy_2}{dx} = -\dfrac{1}{2\sqrt{x}}$.



But suppose that we knew only that the equation $y^2 = x$ defined y as one or more differentiable functions of x for $x > 0$ without knowing exactly what these functions were. Could we still find dy/dx?

The answer is yes. To find dy/dx, we simply differentiate both sides of the equation $y^2 = x$ with respect to x, treating $y = f(x)$ as a differentiable function of x:

$$y^2 = x$$

$$2y\frac{dy}{dx} = 1 \qquad \text{The Chain Rule gives } \frac{d}{dx}(y^2) = \frac{d}{dx}[f(x)]^2 = 2f(x)f'(x) = 2y\frac{dy}{dx}.$$

$$\frac{dy}{dx} = \frac{1}{2y}.$$

This one formula gives the derivatives we calculated for *both* explicit solutions $y_1 = \sqrt{x}$ and $y_2 = -\sqrt{x}$:

$$\frac{dy_1}{dx} = \frac{1}{2y_1} = \frac{1}{2\sqrt{x}} \quad \text{and} \quad \frac{dy_2}{dx} = \frac{1}{2y_2} = \frac{1}{2(-\sqrt{x})} = -\frac{1}{2\sqrt{x}}.$$

(from Thomas'Calculus)

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EXAMPLE Find dy/dx if $y^2 = x^2 + \sin xy$

Solution We differentiate the equation implicitly.

$$y^2 = x^2 + \sin xy$$

$$\frac{d}{dx}(y^2) = \frac{d}{dx}(x^2) + \frac{d}{dx}(\sin xy)$$ Differentiate both sides with respect to x . . .

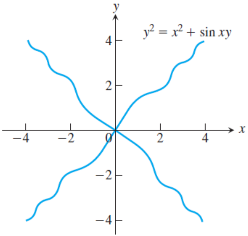
$$2y\frac{dy}{dx} = 2x + (\cos xy)\frac{d}{dx}(xy)$$ . . . treating y as a function of x and using the Chain Rule.

$$2y\frac{dy}{dx} = 2x + (\cos xy)\left(y + x\frac{dy}{dx}\right)$$ Treat xy as a product.

$$2y\frac{dy}{dx} - (\cos xy)\left(x\frac{dy}{dx}\right) = 2x + (\cos xy)y$$ Collect terms with dy/dx.

$$(2y - x\cos xy)\frac{dy}{dx} = 2x + y\cos xy$$

$$\frac{dy}{dx} = \frac{2x + y\cos xy}{2y - x\cos xy}$$ Solve for dy/dx.



Notice that the formula for dy/dx applies everywhere that the implicitly defined curve has a slope. Notice again that the derivative involves *both* variables x and y, not just the independent variable x.

(from Thomas'Calculus)

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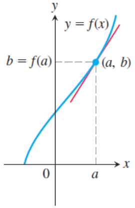
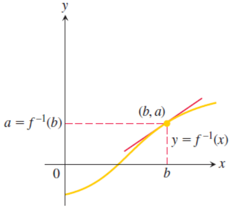
Theorem (Derivative of inverse function)

Let $f : [a, b] \to \mathbb{R}$ be monotonic and continuous, i.e. there exists an inverse function $f^{-1} : R(f) \to [a, b]$. If f is differentiable at $x_0 \in [a, b]$ and $f'(x_0) \neq 0$, then the inverse function $x = f^{-1}(y)$ is differentiable at $y_0 = f(x_0)$ and

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)}.$$

Equivalently, $\left.\frac{\mathrm{d}f^{-1}(x)}{\mathrm{d}x}\right|_{x=y_0} = \frac{1}{\left.\frac{\mathrm{d}f(x)}{\mathrm{d}x}\right|_{x=f^{-1}(y_0)}}$; $(f^{-1})(x) = \frac{1}{f'\left(f^{-1}(x)\right)}$.





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EXAMPLE 1 The function $f(x) = x^2, x > 0$ and its inverse $f^{-1}(x) = \sqrt{x}$ have derivatives $f'(x) = 2x$ and $(f^{-1})'(x) = 1/(2\sqrt{x})$.

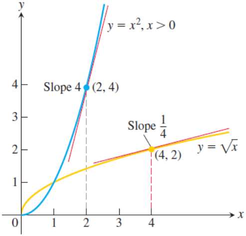
Let's verify that Theorem gives the same formula for the derivative of $f^{-1}(x)$:

$$
\begin{aligned}
(f^{-1})'(x) &= \frac{1}{f'(f^{-1}(x))} \\
&= \frac{1}{2(f^{-1}(x))} \qquad f'(x) = 2x \text{ with } x \text{ replaced by } f^{-1}(x) \\
&= \frac{1}{2(\sqrt{x})}.
\end{aligned}
$$

Theorem gives a derivative that agrees with the known derivative of the square root function.

Let's examine Theorem 3 at a specific point. We pick $x = 2$ (the number a) and $f(2) = 4$ (the value b). Theorem 3 says that the derivative of f at 2, which is $f'(2) = 4$, and the derivative of f^{-1} at $f(2)$, which is $(f^{-1})'(4)$, are reciprocals. It states that

$$(f^{-1})'(4) = \frac{1}{f'(f^{-1}(4))} = \frac{1}{f'(2)} = \left.\frac{1}{2x}\right|_{x=2} = \frac{1}{4}.$$



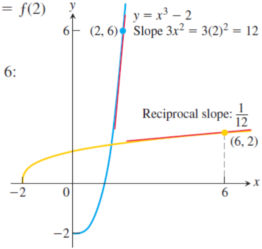
(from Thomas'Calculus)

EXAMPLE Let $f(x) = x^3 - 2, x > 0$. Find the value of df^{-1}/dx at $x = 6 = f(2)$ without finding a formula for $f^{-1}(x)$.

Solution We apply Theorem 3 to obtain the value of the derivative of f^{-1} at $x = 6$:

$$\left.\frac{df}{dx}\right|_{x=2} = \left.3x^2\right|_{x=2} = 12$$

$$\left.\frac{df^{-1}}{dx}\right|_{x=f(2)} = \frac{1}{\left.\frac{df}{dx}\right|_{x=2}} = \frac{1}{12}. \qquad \text{Eq. (1)}$$



(from Thomas'Calculus)

Differentiation rules: inverse functions

$$
\begin{aligned}
y &= \ln x && x > 0 \\
e^y &= x && \text{Inverse function relationship} \\
\frac{d}{dx}(e^y) &= \frac{d}{dx}(x) && \text{Differentiate implicitly.} \\
e^y \frac{dy}{dx} &= 1 && \text{Chain Rule} \\
\frac{dy}{dx} &= \frac{1}{e^y} = \frac{1}{x}. && e^y = x
\end{aligned}
$$

No matter which derivation we use, the derivative of $y = \ln x$ with respect to x is

$$\frac{d}{dx}(\ln x) = \frac{1}{x}, \quad x > 0.$$

The Chain Rule extends this formula to positive functions $u(x)$:

$$\boxed{\frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}, \qquad u > 0.}$$

(from Thomas'Calculus)

The Derivatives of a^u

We start with the equation $a^x = e^{\ln(a^x)} = e^{x \ln a}$, $a > 0$:

$$\frac{d}{dx}a^x = \frac{d}{dx}e^{x\ln a} = e^{x\ln a} \cdot \frac{d}{dx}(x \ln a) \qquad \frac{d}{dx}e^u = e^u\frac{du}{dx}$$
$$= a^x \ln a.$$

That is, if $a > 0$, then a^x is differentiable and

$$\frac{d}{dx}a^x = a^x \ln a.$$

$$\frac{d}{dx}e^x = e^x \ln e = e^x.$$

If $a > 0$ and u is a differentiable function of x, then a^u is a differentiable function of x and

$$\frac{d}{dx}a^u = a^u \ln a \frac{du}{dx}.$$

(from Thomas'Calculus)

Differentiation rules: $\log_a f(x)$

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The Derivatives of $\log_a u$

Taking derivatives, we have

$$\frac{d}{dx}\log_a x = \frac{d}{dx}\left(\frac{\ln x}{\ln a}\right)$$

$$= \frac{1}{\ln a}\cdot\frac{d}{dx}\ln x \qquad \text{$\ln a$ is a constant.}$$

$$= \frac{1}{\ln a}\cdot\frac{1}{x} = \frac{1}{x\ln a}.$$

If u is a differentiable function of x and $u > 0$, the Chain Rule gives a more general formula.

For $a > 0$ and $a \neq 1$,

$$\frac{d}{dx}\log_a u = \frac{1}{u\ln a}\frac{du}{dx}.$$

(from Thomas'Calculus)

Logarithmic Differentiation

The derivatives of positive functions given by formulas that involve products, quotients, and powers can often be found more quickly if we take the natural logarithm of both sides before differentiating. This enables us to use the laws of logarithms to simplify the formulas before differentiating. The process, called **logarithmic differentiation**, is illustrated in the next example.

EXAMPLE Find dy/dx if $y = \dfrac{(x^2+1)(x+3)^{1/2}}{x-1}, \qquad x > 1.$

Solution We take the natural logarithm of both sides and simplify the result with the algebraic properties of logarithms:

$$\begin{aligned}
\ln y &= \ln \frac{(x^2+1)(x+3)^{1/2}}{x-1} \\
&= \ln\left((x^2+1)(x+3)^{1/2}\right) - \ln(x-1) && \text{Rule 2} \\
&= \ln(x^2+1) + \ln(x+3)^{1/2} - \ln(x-1) && \text{Rule 1} \\
&= \ln(x^2+1) + \frac{1}{2}\ln(x+3) - \ln(x-1). && \text{Rule 4}
\end{aligned}$$

We then take derivatives of both sides with respect to x

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x^2+1}\cdot 2x + \frac{1}{2}\cdot\frac{1}{x+3} - \frac{1}{x-1}.$$

Next we solve for dy/dx:

$$\frac{dy}{dx} = y\left(\frac{2x}{x^2+1} + \frac{1}{2x+6} - \frac{1}{x-1}\right).$$

Finally, we substitute for y:

$$\frac{dy}{dx} = \frac{(x^2+1)(x+3)^{1/2}}{x-1}\left(\frac{2x}{x^2+1} + \frac{1}{2x+6} - \frac{1}{x-1}\right).$$

(from Thomas'Calculus)

General power rule

The definition of the general exponential function enables us to raise any positive number to any real power n, rational or irrational. That is, we can define the power function $y = x^n$ for any exponent n.

DEFINITION For any $x > 0$ and for any real number n,

$$x^n = e^{n \ln x}.$$

Because the logarithm and exponential functions are inverses of each other, the definition gives

$$\ln x^n = n \ln x, \quad \text{for all real numbers } n.$$

That is, the rule for taking the natural logarithm of any power holds for *all* real exponents n, not just for rational exponents.

General Power Rule for Derivatives

For $x > 0$ and any real number n,

$$\frac{d}{dx} x^n = n x^{n-1}.$$

If $x \leq 0$, then the formula holds whenever the derivative, x^n, and x^{n-1} all exist.

(from Thomas'Calculus)

EXAMPLE Differentiate $f(x) = x^x$, $x > 0$.

Solution We note that $f(x) = x^x = e^{x \ln x}$, so differentiation gives

$$\begin{aligned} f'(x) &= \frac{d}{dx}(e^{x \ln x}) \\ &= e^{x \ln x} \frac{d}{dx}(x \ln x) && \frac{d}{dx} e^u,\ u = x \ln x \\ &= e^{x \ln x}\left(\ln x + x \cdot \frac{1}{x}\right) \\ &= x^x(\ln x + 1). && x > 0 \end{aligned}$$

We can also find the derivative of $y = x^x$ using logarithmic differentiation, assuming y' exists. ■

(from Thomas'Calculus)

Differentiation rules: summary

Rules	Function	Derivative
Multiplication by constant	cf	cf'
Power Rule	x^n	nx^{n-1}
Sum Rule	$f + g$	$f' + g'$
Difference Rule	$f - g$	$f' - g'$
Product Rule	fg	$f g' + f' g$
Quotient Rule	f/g	$\frac{f' g - g' f}{g^2}$
Reciprocal Rule	$1/f$	$-f'/f^2$
Chain Rule (as "Composition of Functions")	$f \circ g$	$(f' \circ g) \times g'$
Chain Rule (using $'$)	$f(g(x))$	$f'(g(x))g'(x)$
Chain Rule (using $\frac{d}{dx}$)	$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$	

(from mathisfun.com)

Derivative formulas

Derivative Formulas of Elementary Functions

- $\frac{d}{dx}.x^n = n.\ x^{n-1}$
- $\frac{d}{dx}.\ k = 0$, where k is a constant
- $\frac{d}{dx}.e^x = e^x$
- $\frac{d}{dx}.a^x = a^x.\ \log_e .a$, where $a > 0$, $a \neq 1$
- $\frac{d}{dx}.\log x = 1/x$, $x > 0$
- $\frac{d}{dx}.\ \log_a e = 1/x \log_a e$
- $\frac{d}{dx}.\sqrt{x} = 1/(2\sqrt{x})$

Derivative Formulas of Trigonometric Functions

- $\frac{d}{dx}.\sin x = \cos x$
- $\frac{d}{dx}.\cos x = -\sin x$
- $\frac{d}{dx}.\tan x = \sec^2 x$, $x \neq (2n+1)\ \pi/2$, $n \in I$
- $\frac{d}{dx}.\ \cot x = -\operatorname{cosec}^2 x$, $x \neq n\pi$, $n \in I$
- $\frac{d}{dx}.\ \sec x = \sec x \tan x$, $x \neq (2n+1)\ \pi/2$, $n \in I$
- $\frac{d}{dx}.\operatorname{cosec} x = -\operatorname{cosec} x \cot x$, $x \neq n\pi$, $n \in I$

(from mathisfun.com)

Derivative formulas

Derivative Formulas of Hyperbolic Functions

- $\frac{d}{dx}.\ \mathrm{sinhx} = \mathrm{coshx}$
- $\frac{d}{dx}.\ \mathrm{coshx} = \sin \mathrm{hx}$
- $\frac{d}{dx}.\ \tan \mathrm{hx} = \sec \mathrm{h}^2 x$
- $\frac{d}{dx}.\ \cot \mathrm{hx} = -\mathrm{cosec}\ \mathrm{h}^2 x$
- $\frac{d}{dx}.\ \sec \mathrm{hx} = -\mathrm{sech}\ \mathrm{hx} \tan \mathrm{hx}$
- $\frac{d}{dx}.\ \mathrm{cosec}\ \mathrm{hx} = -\mathrm{cosec}\ \mathrm{hx} \cot \mathrm{hx}$

Differentiation of Inverse Trigonometric Functions

- $\frac{d}{dx}.\ \sin^{-1} x = \frac{1}{\sqrt{(1-x^2}}, -1 < x < 1$
- $\frac{d}{dx}.\cos^{-1} x = -\frac{1}{\sqrt{(1-x^2}}, -1 < x < 1$
- $\frac{d}{dx}.\ \tan^{-1} x = \frac{1}{(1+x^2)}$
- $\frac{d}{dx}.\ \cot^{-1} x = -\frac{1}{(1+x^2)}$
- $\frac{d}{dx}.\ \mathrm{cosec}^{-1} x = -\frac{1}{|x|\sqrt{x^2-1}}, |x| > 1$

(from mathisfun.com)

Derivative formulas

Differentiation of Inverse Hyperbolic Functions

- $\dfrac{d}{dx}.\sinh^{-1} x = \dfrac{1}{\sqrt{(x^2+1)}}$
- $\dfrac{d}{dx}.\cosh^{-1} x = -\dfrac{1}{\sqrt{(x^2-1)}}$
- $\dfrac{d}{dx}.\tanh^{-1} x = \dfrac{1}{(1-x^2)}$
- $\dfrac{d}{dx}.\coth^{-1} x = -\dfrac{1}{x(1-x^2)}$
- $\dfrac{d}{dx}.\operatorname{cosech}^{-1} x = -\dfrac{1}{x\sqrt{1+x^2}}$

(from mathisfun.com)

Definition

Let a function $f : (a, b) \to \mathbb{R}$ be differentiable on (a, b), and suppose that the function $f'(x)$ is differentiable at $x_0 \in (a, b)$. Then $f''(x_0) := (f'(x))'|_{x=x_0}$ is the **second-order derivative of f at x_0.**

If $y = f(x)$ is a differentiable function, then its derivative $f'(x)$ is also a function. If f' is also differentiable, then we can differentiate f' to get a new function of x denoted by f''. So $f'' = (f')'$. The function f'' is called the second (or second-order) derivative of f.

Notations: $f''(x)$, y'', $\dfrac{d^2 f(x)}{dx^2}$, $\dfrac{d^2 y}{dx^2}$, $\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)$, $\dfrac{dy'}{dx}$, $D^2(f)(x)$, $D_x^2 f(x)$.

If $f''(x)$ is differentiable, then $f'''(x) = \dfrac{d}{dx}\left(\dfrac{df''(x)}{dx}\right) = \dfrac{d^3 f(x)}{dx^3}$ is the **third** (or **third-order**) **derivative of f with respect to x**.

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n-th derivative of f, $n \in \mathbb{N}$:

If the $(n-1)$-th derivative $f^{(n-1)}(x)$ of the function $f(x)$ is differentiable, then

$$f^{(n)}(x) = \frac{d}{dx}\left(\frac{df^{(n-1)}(x)}{dx}\right) = \frac{d^n f(x)}{dx^n}$$

is the **n-th order derivative of f with respect to x**.
With 0-th derivative of the function we mean the function itself: $f^{(0)} := f$.

Higher-order one-sided derivatives of f are defined in similar way.

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EXAMPLE The first four derivatives of $y = x^3 - 3x^2 + 2$ are

$$\begin{aligned}
&\text{First derivative:} && y' = 3x^2 - 6x \\
&\text{Second derivative:} && y'' = 6x - 6 \\
&\text{Third derivative:} && y''' = 6 \\
&\text{Fourth derivative:} && y^{(4)} = 0.
\end{aligned}$$

All polynomial functions have derivatives of all orders. In this example, the fifth and later derivatives are all zero.

(from Thomas' Calculus)

Examples:

1) $y = a^x$, $y' = a^x \ln a$, $y^{(n)} = a^x \ln^n a$.
2) $y = e^x$, $y' = e^x$, $y^{(n)} = e^x$.
3) $y = \sin x$, $y' = \cos x$, $y^{(n)} = \sin\left(x + \frac{\pi n}{2}\right)$.
4) $y = \cos x$, $y' = -\sin x$, $y^{(n)} = \cos\left(x + \frac{\pi n}{2}\right)$.

Definition

A function $f : D = (a, b) \to \mathbb{R}$ is called to be ***n*-th time continuously differentiable on *D*** $f \in C^n(D)$, if all its derivatives up to n-th order exist at each point of D and continuous on D $(n = 0, 1, 2, \ldots.)$

Differentiability classes:

- $C^0(D)$ (or $C(D)$) – set of all continuous on D functions;
- $C^1(D)$ – set of all continuously differentiable on D functions;
- $\ldots$
- $C^\infty(D)$ – set of all functions, which are infinitely many times continuously differentiable on D (smooth functions).

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Examples:

1) $y = |x|$ – continuous but not differentiable at 0;

2) $y = |x|^{k+1}$, k is an even integer – k times continuously differentiable at 0;

3) $y = \begin{cases} x^2 \sin\dfrac{1}{x} \text{ if } x \neq 0 \\ 0 \text{ if } x = 0 \end{cases}$ – differentiable but not continuously differentiable at 0;

4) polynomial, trigonometric, exponential functions – smooth whenever defined.

Theorem (Sum and product of higher-order derivatives)

Let functions $y_1 = f_1(x)$ and $y_2 = f_2(x)$ be n-th time differentiable at x_0, $n \in \mathbb{N} \cup \{0\}$. Then the functions $y_1 + y_2 = f_1(x) + f_2(x)$ and $y_1 \cdot y_2 = f_1(x) \cdot f_2(x)$ are n-th time differentiable at x_0 as well. Moreover,

- $(y_1 + y_2)^{(n)} = y_1^{(n)} + y_2^{(n)}$;
- $(y_1 \cdot y_2)^{(n)} = \sum_{k=0}^{n} C_n^k y_1^{(n-k)} y_2^{(k)}$, where $C_n^k = \left(\begin{array}{c} n \\ k \end{array} \right) = \dfrac{n!}{k!(n-k)!}$.

$$(x^3 \sin x)^{(10)} = x^3 \sin\left(x + 10\frac{\pi}{2}\right) + 10 \cdot 3x^2 \sin\left(x + 9\frac{\pi}{2}\right) +$$

$$+ 10 \cdot 9 \cdot 3x \sin\left(x + 8\frac{\pi}{2}\right) + 10 \cdot 9 \cdot 8 \sin\left(x + 7\frac{\pi}{2}\right) =$$

$$= -x^3 \sin x + 30\, x^2 \cos x + 270\, x \sin x - 720 \cos x.$$

Higher-order derivatives of implicit functions

Implicit differentiation can also be used to find higher derivatives.

EXAMPLE Find d^2y/dx^2 if $2x^3 - 3y^2 = 8$.

Solution To start, we differentiate both sides of the equation with respect to x in order to find $y' = dy/dx$.

$$\frac{d}{dx}(2x^3 - 3y^2) = \frac{d}{dx}(8)$$

$$6x^2 - 6yy' = 0 \qquad \text{Treat } y \text{ as a function of } x.$$

$$y' = \frac{x^2}{y}, \qquad \text{when } y \neq 0 \qquad \text{Solve for } y'.$$

We now apply the Quotient Rule to find y''.

$$y'' = \frac{d}{dx}\left(\frac{x^2}{y}\right) = \frac{2xy - x^2y'}{y^2} = \frac{2x}{y} - \frac{x^2}{y^2} \cdot y'$$

Finally, we substitute $y' = x^2/y$ to express y'' in terms of x and y.

$$y'' = \frac{2x}{y} - \frac{x^2}{y^2}\left(\frac{x^2}{y}\right) = \frac{2x}{y} - \frac{x^4}{y^3}, \qquad \text{when } y \neq 0$$

(from Thomas' Calculus)

Motion along a line

Suppose that an object (or body, considered as a whole mass) is moving along a coordinate line (an s-axis), usually horizontal or vertical, so that we know its position s on that line as a function of time t: $s = f(t)$.

DEFINITION **Velocity (instantaneous velocity)** is the derivative of position with respect to time. If a body's position at time t is $s = f(t)$, then the body's velocity at time t is

$$v(t) = \frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}.$$

Besides telling how fast an object is moving along the horizontal line, its velocity tells the direction of motion. When the object is moving forward (s increasing), the velocity is positive; when the object is moving backward (s decreasing), the velocity is negative. If the coordinate line is vertical, the object moves upward for positive velocity and downward for negative velocity.

DEFINITION **Speed** is the absolute value of velocity.

$$\text{Speed} = |v(t)| = \left|\frac{ds}{dt}\right|$$

The rate at which a body's velocity changes is the *body's acceleration.* The acceleration measures how quickly the body picks up or loses speed. An acceleration of an object may also lead to a change in direction.
A sudden change in acceleration is called a *jerk*.

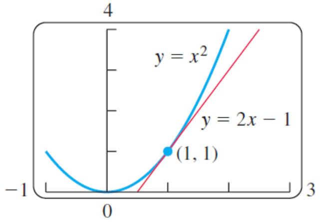
DEFINITIONS **Acceleration** is the derivative of velocity with respect to time. If a body's position at time t is $s = f(t)$, then the body's acceleration at time t is

$$a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}.$$

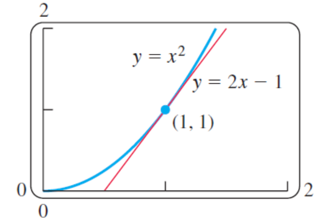
Jerk is the derivative of acceleration with respect to time:

$$j(t) = \frac{da}{dt} = \frac{d^3s}{dt^3}.$$

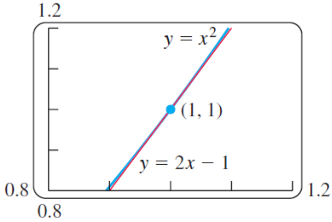
(from Thomas' Calculus)



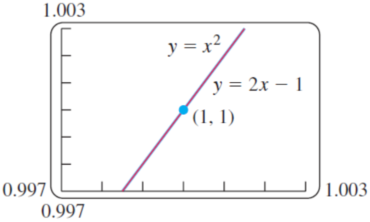
$y = x^2$ and its tangent $y = 2x - 1$ at $(1, 1)$.



Tangent and curve very close near $(1, 1)$.



Tangent and curve very close throughout entire x-interval shown.



Tangent and curve closer still. Computer screen cannot distinguish tangent from

Linearization

DEFINITIONS If f is differentiable at $x = a$, then the approximating function

$$L(x) = f(a) + f'(a)(x - a)$$

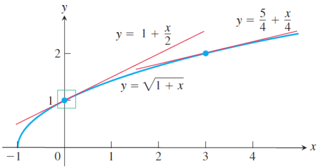
is the **linearization** of f at a. The approximation

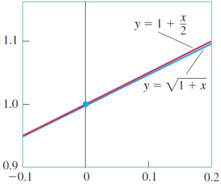
$$f(x) \approx L(x)$$

of f by L is the **standard linear approximation** of f at a. The point $x = a$ is the **center** of the approximation.

(from Thomas' Calculus)

EXAMPLE Find the linearization of $f(x) = \sqrt{1+x}$ at $x = 0$





Solution Since $f'(x) = \frac{1}{2}(1+x)^{-1/2}$,
we have $f(0) = 1$ and $f'(0) = 1/2$, giving the linearization

$$L(x) = f(a) + f'(a)(x-a) = 1 + \frac{1}{2}(x-0) = 1 + \frac{x}{2}.$$

Approximation	True value	|True value − approximation|
$\sqrt{1.2} \approx 1 + \frac{0.2}{2} = 1.10$	1.095445	$0.004555 < 10^{-2}$
$\sqrt{1.05} \approx 1 + \frac{0.05}{2} = 1.025$	1.024695	$0.000305 < 10^{-3}$
$\sqrt{1.005} \approx 1 + \frac{0.005}{2} = 1.00250$	1.002497	$0.000003 < 10^{-5}$

(from Thomas' Calculus)

EXAMPLE Find the linearization of $f(x) = \sqrt{1 + x}$ at $x = 3$.

Solution We evaluate the equation defining $L(x)$ at $a = 3$. With

$$f(3) = 2, \qquad f'(3) = \frac{1}{2}(1 + x)^{-1/2}\Big|_{x=3} = \frac{1}{4},$$

we have

$$L(x) = 2 + \frac{1}{4}(x - 3) = \frac{5}{4} + \frac{x}{4}.$$

At $x = 3.2$,

$$\sqrt{1 + x} = \sqrt{1 + 3.2} \approx \frac{5}{4} + \frac{3.2}{4} = 1.250 + 0.800 = 2.050,$$

which differs from the true value $\sqrt{4.2} \approx 2.04939$ by less than one one-thousandth. The linearization in Example 1 gives

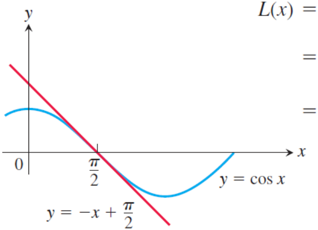
$$\sqrt{1 + x} = \sqrt{1 + 3.2} \approx 1 + \frac{3.2}{2} = 1 + 1.6 = 2.6,$$

a result that is off by more than 25%.

(from Thomas' Calculus)

EXAMPLE Find the linearization of $f(x) = \cos x$ at $x = \pi/2$

Solution Since $f(\pi/2) = \cos(\pi/2) = 0$, $f'(x) = -\sin x$, and $f'(\pi/2) = -\sin(\pi/2) = -1$, we find the linearization at $a = \pi/2$ to be



$$\begin{aligned} L(x) &= f(a) + f'(a)(x - a) \\ &= 0 + (-1)\left(x - \frac{\pi}{2}\right) \\ &= -x + \frac{\pi}{2}. \end{aligned}$$

(from Thomas' Calculus)

An important linear approximation for roots and powers is

$$(1 + x)^k \approx 1 + kx \qquad (x \text{ near } 0; \text{ any number } k)$$

This approximation, good for values of x sufficiently close to zero, has broad application. For example, when x is small,

$$\sqrt{1 + x} \approx 1 + \frac{1}{2}x \qquad k = 1/2$$

$$\frac{1}{1 - x} = (1 - x)^{-1} \approx 1 + (-1)(-x) = 1 + x \qquad k = -1; \text{ replace } x \text{ by } -x.$$

$$\sqrt[3]{1 + 5x^4} = (1 + 5x^4)^{1/3} \approx 1 + \frac{1}{3}(5x^4) = 1 + \frac{5}{3}x^4 \qquad k = 1/3; \text{ replace } x \text{ by } 5x^4.$$

$$\frac{1}{\sqrt{1 - x^2}} = (1 - x^2)^{-1/2} \approx 1 + \left(-\frac{1}{2}\right)(-x^2) = 1 + \frac{1}{2}x^2 \qquad k = -1/2; \text{ replace } x \text{ by } -x^2.$$

(from Thomas' Calculus)

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DEFINITION Let $y = f(x)$ be a differentiable function. The **differential dx** is an independent variable. The **differential dy** is

$$dy = f'(x)\,dx.$$

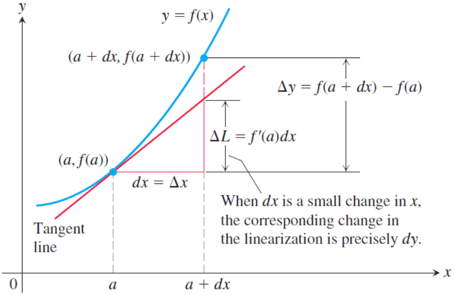

FIGURE Geometrically, the differential dy is the change ΔL in the linearization of f when $x = a$ changes by an amount $dx = \Delta x$.

(from Thomas' Calculus)