5. Integral calculus of functions of one real variable

5.1. Antiderivative and undefined integral

Content:

- Antiderivative
- Indefinite integral
- Basic integration formulas
- Substitution rule
- Integration by parts
- Trigonometric integrals
- Trigonometric substitution
- Method of partial fractions
- Non-elementary integrals

Definition

Let $f : I \to \mathbb{R}$ be a function, I be an interval in $\mathbb{R}$. A function F is an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all $x \in I$. The process of recovering a function $F(x)$ from its derivative $f(x)$ is called **antidifferentiation**.

EXAMPLE Find an antiderivative for each of the following functions.

(a) $f(x) = 2x$ **(b)** $g(x) = \cos x$ **(c)** $h(x) = \dfrac{1}{x} + 2e^{2x}$

Solution We need to think backward here: What function do we know has a derivative equal to the given function?

(a) $F(x) = x^2$ **(b)** $G(x) = \sin x$ **(c)** $H(x) = \ln|x| + e^{2x}$

Each answer can be checked by differentiating. The derivative of $F(x) = x^2$ is $2x$. The derivative of $G(x) = \sin x$ is $\cos x$, and the derivative of $H(x) = \ln|x| + e^{2x}$ is $(1/x) + 2e^{2x}$. ■

(from Thomas'Calculus)

The function $F(x) = x^2$ is not the only function whose derivative is $2x$. The function $x^2 + 1$ has the same derivative. So does $x^2 + C$ for any constant C. Are there others?

Corollary of the Mean Value Theorem gives the answer: Any two antiderivatives of a function differ by a constant. So the functions $x^2 + C$, where C is an **arbitrary constant**, form *all* the antiderivatives of $f(x) = 2x$. More generally, we have the following result.

THEOREM If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is

$$F(x) + C$$

where C is an arbitrary constant.

(from Thomas'Calculus)

The most general antiderivative of f on I is a family of functions $F(x) + C$ whose graphs are vertical translations of one another. We can select a particular antiderivative from this family by assigning a specific value to C.

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Definition

Let $f : I \to \mathbb{R}$ be a function, I be an interval in $\mathbb{R}$. A function F is an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all $x \in I$. The process of recovering a function $F(x)$ from its derivative $f(x)$ is called **antidifferentiation**.

EXAMPLE Find an antiderivative of $f(x) = 3x^2$ that satisfies $F(1) = -1$.

Solution Since the derivative of x^3 is $3x^2$, the general antiderivative

$$F(x) = x^3 + C$$

gives all the antiderivatives of $f(x)$. The condition $F(1) = -1$ determines a specific value for C. Substituting $x = 1$ into $F(x) = x^3 + C$ gives

$$F(1) = (1)^3 + C = 1 + C.$$

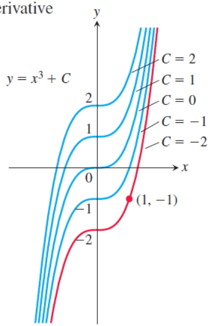
Since $F(1) = -1$, solving $1 + C = -1$ for C gives $C = -2$.

So
$$F(x) = x^3 - 2$$

is the antiderivative satisfying $F(1) = -1$. Notice that this assignment for C selects the particular curve from the family of curves

$$y = x^3 + C$$

that passes through the point $(1, -1)$ in the plane



(from Thomas'Calculus)

Antiderivative

TABLE Antiderivative linearity rules

		Function	General antiderivative
1.	*Constant Multiple Rule*:	$kf(x)$	$kF(x) + C$, k a constant
2.	*Negative Rule*:	$-f(x)$	$-F(x) + C$
3.	*Sum or Difference Rule*:	$f(x) \pm g(x)$	$F(x) \pm G(x) + C$

(from Thomas'Calculus)

Antiderivative

TABLE Antiderivative formulas, k a nonzero constant

Function	General antiderivative	Function	General antiderivative
1. x^n	$\frac{1}{n+1}x^{n+1} + C, \quad n \neq -1$	8. e^{kx}	$\frac{1}{k}e^{kx} + C$
2. $\sin kx$	$-\frac{1}{k}\cos kx + C$	9. $\frac{1}{x}$	$\ln\lvert x\rvert + C, \quad x \neq 0$
3. $\cos kx$	$\frac{1}{k}\sin kx + C$	10. $\frac{1}{\sqrt{1-k^2x^2}}$	$\frac{1}{k}\sin^{-1} kx + C$
4. $\sec^2 kx$	$\frac{1}{k}\tan kx + C$	11. $\frac{1}{1+k^2x^2}$	$\frac{1}{k}\tan^{-1} kx + C$
5. $\csc^2 kx$	$-\frac{1}{k}\cot kx + C$	12. $\frac{1}{x\sqrt{k^2x^2-1}}$	$\sec^{-1} kx + C, kx > 1$
6. $\sec kx \tan kx$	$\frac{1}{k}\sec kx + C$	13. a^{kx}	$\left(\frac{1}{k\ln a}\right)a^{kx} + C, a > 0, a \neq 1$
7. $\csc kx \cot kx$	$-\frac{1}{k}\csc kx + C$		

(from Thomas'Calculus)

Indefinite integral

The collection of all antiderivatives of f is called the **indefinite integral** of f with respect to x, and is denoted by $\int f(x)\,dx$. The symbol $\int$ is an **integral sign**. The function f is the **integrand** of the integral, and x is the **variable of integration**.

TABLE Basic integration formulas

1. $\int k\,dx = kx + C \quad \text{(any number } k)$
2. $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$
3. $\int \frac{dx}{x} = \ln|x| + C$
4. $\int e^x\,dx = e^x + C$
5. $\int a^x\,dx = \frac{a^x}{\ln a} + C \quad (a > 0,\ a \neq 1)$
6. $\int \sin x\,dx = -\cos x + C$
7. $\int \cos x\,dx = \sin x + C$
8. $\int \sec^2 x\,dx = \tan x + C$
9. $\int \csc^2 x\,dx = -\cot x + C$
10. $\int \sec x \tan x\,dx = \sec x + C$
11. $\int \csc x \cot x\,dx = -\csc x + C$
12. $\int \tan x\,dx = \ln|\sec x| + C$
13. $\int \cot x\,dx = \ln|\sin x| + C$
14. $\int \sec x\,dx = \ln|\sec x + \tan x| + C$
15. $\int \csc x\,dx = -\ln|\csc x + \cot x| + C$
16. $\int \sinh x\,dx = \cosh x + C$
17. $\int \cosh x\,dx = \sinh x + C$
18. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$
19. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$
20. $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + C$
21. $\int \frac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1}\left(\frac{x}{a}\right) + C \quad (a > 0)$
22. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\left(\frac{x}{a}\right) + C \quad (x > a > 0)$

Indefinite integral

EXAMPLE Evaluate $\int (x^2 - 2x + 5)\, dx$.

Solution If we recognize that $(x^3/3) - x^2 + 5x$ is an antiderivative of $x^2 - 2x + 5$, we can evaluate the integral as

$$\int (x^2 - 2x + 5)\, dx = \overbrace{\frac{x^3}{3} - x^2 + 5x}^{\text{antiderivative}} + \underbrace{C}_{\text{arbitrary constant}}.$$

If we do not recognize the antiderivative right away, we can generate it term-by-term with the Sum, Difference, and Constant Multiple Rules:

$$\begin{aligned}
\int (x^2 - 2x + 5)\, dx &= \int x^2\, dx - \int 2x\, dx + \int 5\, dx \\
&= \int x^2\, dx - 2\int x\, dx + 5\int 1\, dx \\
&= \left(\frac{x^3}{3} + C_1\right) - 2\left(\frac{x^2}{2} + C_2\right) + 5(x + C_3) \\
&= \frac{x^3}{3} + C_1 - x^2 - 2C_2 + 5x + 5C_3.
\end{aligned}$$

(from Thomas'Calculus)

Integration techniques: Substitution rule

Theorem (Substitution rule)

Let $f : I_f \to \mathbb{R}$ has an antiderivative on I_f, $g : I_g \to I_f$ be differentiable on I_g. Then the function $f(g(x))g'(x)$ has an antiderivative on I_f and
$\int f(g(x))g'(x)\,dx = \int f(u)\,du\Big|_{u=g(x)}$.

Proof By the Chain Rule, $F(g(x))$ is an antiderivative of $f(g(x)) \cdot g'(x)$ whenever F is an antiderivative of f:

$$\frac{d}{dx}F(g(x)) = F'(g(x)) \cdot g'(x) \qquad \text{Chain Rule} \qquad = f(g(x)) \cdot g'(x). \qquad F' = f$$

If we make the substitution $u = g(x)$, then

$$\int f(g(x))g'(x)\,dx = \int \frac{d}{dx}F(g(x))\,dx = F(g(x)) + C = F(u) + C = \int F'(u)\,du = \int f(u)\,du.$$

(from Thomas'Calculus)

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The Substitution Method to evaluate $\int f(g(x))g'(x)\,dx$

1. Substitute $u = g(x)$ and $du = (du/dx)\,dx = g'(x)\,dx$ to obtain $\int f(u)\,du$.
2. Integrate with respect to u.
3. Replace u by $g(x)$.

(from Thomas'Calculus)

EXAMPLE Find $\int \sec^2(5x+1)\cdot 5\,dx$

Solution We substitute $u = 5x + 1$ and $du = 5\,dx$. Then,

$$\int \sec^2(5x+1)\cdot 5\,dx = \int \sec^2 u\,du \qquad \text{Let } u = 5x+1,\ du = 5\,dx.$$

$$= \tan u + C \qquad \frac{d}{du}\tan u = \sec^2 u$$

$$= \tan(5x+1) + C. \qquad \text{Substitute } 5x+1 \text{ for } u.$$

(from Thomas'Calculus)

EXAMPLE Find $\int \cos(7\theta + 3)\, d\theta$.

Solution We let $u = 7\theta + 3$ so that $du = 7\, d\theta$. The constant factor 7 is missing from the $d\theta$ term in the integral. We can compensate for it by multiplying and dividing by 7, using the same procedure as in Example 2. Then,

$$\int \cos(7\theta + 3)\, d\theta = \frac{1}{7}\int \cos(7\theta + 3)\cdot 7\, d\theta \qquad \text{Place factor } 1/7 \text{ in front of integral.}$$

$$= \frac{1}{7}\int \cos u\, du \qquad \text{Let } u = 7\theta + 3,\ du = 7\, d\theta.$$

$$= \frac{1}{7}\sin u + C \qquad \text{Integrate.}$$

$$= \frac{1}{7}\sin(7\theta + 3) + C. \qquad \text{Substitute } 7\theta + 3 \text{ for } u.$$

(from Thomas'Calculus)

EXAMPLE Sometimes we observe that a power of x appears in the integrand that is one less than the power of x appearing in the argument of a function we want to integrate. This observation immediately suggests we try a substitution for the higher power of x. This situation occurs in the following integration.

$$\int x^2 e^{x^3}\,dx = \int e^{x^3} \cdot x^2\,dx$$

$$= \int e^u \cdot \frac{1}{3}\,du \qquad \text{Let } u = x^3,\ du = 3x^2\,dx,\ (1/3)\,du = x^2\,dx.$$

$$= \frac{1}{3}\int e^u\,du$$

$$= \frac{1}{3}e^u + C \qquad \text{Integrate with respect to } u.$$

$$= \frac{1}{3}e^{x^3} + C \qquad \text{Replace } u \text{ by } x^3.$$

(from Thomas'Calculus)

EXAMPLE Sometimes we can use trigonometric identities to transform integrals we do not know how to evaluate into ones we can evaluate using the Substitution Rule.

(a) $$\int \sin^2 x\, dx = \int \frac{1 - \cos 2x}{2} dx \qquad \sin^2 x = \frac{1 - \cos 2x}{2}$$

$$= \frac{1}{2}\int (1 - \cos 2x)\, dx$$

$$= \frac{1}{2}x - \frac{1}{2}\frac{\sin 2x}{2} + C = \frac{x}{2} - \frac{\sin 2x}{4} + C$$

(b) $$\int \cos^2 x\, dx = \int \frac{1 + \cos 2x}{2} dx = \frac{x}{2} + \frac{\sin 2x}{4} + C \qquad \cos^2 x = \frac{1 + \cos 2x}{2}$$

(c) $$\int \tan x\, du = \int \frac{\sin x}{\cos x} dx = \int \frac{-du}{u} \qquad u = \cos x,\ du = -\sin x\, dx$$

$$= -\ln|u| + C = -\ln|\cos x| + C$$

$$= \ln \frac{1}{|\cos x|} + C = \ln|\sec x| + C \qquad \text{Reciprocal Rule}$$

■

(from Thomas'Calculus)

Integration techniques: Substitution rule

EXAMPLE An integrand may require some algebraic manipulation before the substitution method can be applied. This example gives two integrals obtained by multiplying the integrand by an algebraic form equal to 1, leading to an appropriate substitution.

(a)

$$\int \frac{dx}{e^x + e^{-x}} = \int \frac{e^x\,dx}{e^{2x} + 1} \qquad \text{Multiply by } (e^x/e^x) = 1.$$

$$= \int \frac{du}{u^2 + 1} \qquad \text{Let } u = e^x,\ u^2 = e^{2x},\ du = e^x\,dx.$$

$$= \tan^{-1} u + C \qquad \text{Integrate with respect to } u.$$

$$= \tan^{-1}(e^x) + C \qquad \text{Replace } u \text{ by } e^x.$$

(b)

$$\int \sec x\,dx = \int (\sec x)(1)\,dx = \int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x}\,dx \qquad \frac{\sec x + \tan x}{\sec x + \tan x} \text{ is equal to 1.}$$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\,dx$$

$$= \int \frac{du}{u} \qquad u = \tan x + \sec x,\ du = (\sec^2 + \sec x \tan x)\,dx$$

$$= \ln|u| + C = \ln|\sec x + \tan x| + C.$$

EXAMPLE Evaluate $\int \frac{2z\,dz}{\sqrt[3]{z^2+1}}$.

Solution We can use the substitution method of integration as an exploratory tool: Substitute for the most troublesome part of the integrand and see how things work out. For the integral here, we might try $u = z^2 + 1$ or we might even press our luck and take u to be the entire cube root. Here is what happens in each case, and both substitutions are successful.

Method 1: Substitute $u = z^2 + 1$.

$$\int \frac{2z\,dz}{\sqrt[3]{z^2+1}} = \int \frac{du}{u^{1/3}}$$ Let $u = z^2 + 1$, $du = 2z\,dz$.

$$= \int u^{-1/3}\,du$$ In the form $\int u^n\,du$

$$= \frac{u^{2/3}}{2/3} + C$$ Integrate.

$$= \frac{3}{2}u^{2/3} + C$$

$$= \frac{3}{2}(z^2+1)^{2/3} + C$$ Replace u by $z^2 + 1$.

(from Thomas'Calculus)

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Method 2: Substitute $u = \sqrt[3]{z^2+1}$ instead.

$$\int \frac{2z\,dz}{\sqrt[3]{z^2+1}} = \int \frac{3u^2\,du}{u}$$

Let $u = \sqrt[3]{z^2+1}$,
$u^3 = z^2 + 1$, $3u^2\,du = 2z\,dz$.

$$= 3\int u\,du$$

$$= 3 \cdot \frac{u^2}{2} + C$$

Integrate.

$$= \frac{3}{2}(z^2+1)^{2/3} + C$$

Replace u by $(z^2+1)^{1/3}$. ■

(from Thomas'Calculus)

EXAMPLE

$$\int \frac{\log_2 x}{x} dx = \frac{1}{\ln 2} \int \frac{\ln x}{x} dx \qquad \log_2 x = \frac{\ln x}{\ln 2}$$

$$= \frac{1}{\ln 2} \int u \, du \qquad u = \ln x, \quad du = \frac{1}{x} dx$$

$$= \frac{1}{\ln 2} \frac{u^2}{2} + C = \frac{1}{\ln 2} \frac{(\ln x)^2}{2} + C = \frac{(\ln x)^2}{2 \ln 2} + C$$

(from Thomas'Calculus)

Integration techniques: Substitution rule

EXAMPLE Complete the square to evaluate $\displaystyle\int \frac{dx}{\sqrt{8x - x^2}}$.

Solution We complete the square to simplify the denominator:

$$\begin{aligned} 8x - x^2 &= -(x^2 - 8x) = -(x^2 - 8x + 16 - 16) \\ &= -(x^2 - 8x + 16) + 16 = 16 - (x - 4)^2. \end{aligned}$$

Then

$$\begin{aligned} \int \frac{dx}{\sqrt{8x - x^2}} &= \int \frac{dx}{\sqrt{16 - (x - 4)^2}} \\ &= \int \frac{du}{\sqrt{a^2 - u^2}} && a = 4,\ u = (x - 4),\ du = dx \\ &= \sin^{-1}\left(\frac{u}{a}\right) + C \\ &= \sin^{-1}\left(\frac{x - 4}{4}\right) + C. \end{aligned}$$

■

(from Thomas'Calculus)

EXAMPLE Evaluate the integral

$$\int (\cos x \sin 2x + \sin x \cos 2x)\, dx.$$

Solution Here we can replace the integrand with an equivalent trigonometric expression using the Sine Addition Formula to obtain a simple substitution:

$$\begin{aligned}
\int (\cos x \sin 2x + \sin x \cos 2x)\, dx &= \int (\sin (x + 2x))\, dx \\
&= \int \sin 3x\, dx \\
&= \int \frac{1}{3} \sin u\, du \qquad u = 3x,\ du = 3\, dx \\
&= -\frac{1}{3} \cos 3x + C.
\end{aligned}$$

∎

(from Thomas'Calculus)

EXAMPLE Evaluate

$$\int \frac{3x^2 - 7x}{3x + 2}\, dx.$$

Solution The integrand is an improper fraction since the degree of the numerator is greater than the degree of the denominator. To integrate it, we perform long division to obtain a quotient plus a remainder that is a proper fraction:

$$\frac{3x^2 - 7x}{3x + 2} = x - 3 + \frac{6}{3x + 2}.$$

Therefore,

$$\int \frac{3x^2 - 7x}{3x + 2}\, dx = \int \left(x - 3 + \frac{6}{3x + 2} \right) dx = \frac{x^2}{2} - 3x + 2 \ln |3x + 2| + C. \quad \blacksquare$$

(from Thomas'Calculus)

EXAMPLE Evaluate $\displaystyle\int \frac{3x+2}{\sqrt{1-x^2}}\,dx.$

Solution We first separate the integrand to get

$$\int \frac{3x+2}{\sqrt{1-x^2}}\,dx = 3\int \frac{x\,dx}{\sqrt{1-x^2}} + 2\int \frac{dx}{\sqrt{1-x^2}}.$$

In the first of these new integrals, we substitute

$$u = 1 - x^2, \qquad du = -2x\,dx, \qquad \text{so} \qquad x\,dx = -\frac{1}{2}\,du.$$

Then
$$3\int \frac{x\,dx}{\sqrt{1-x^2}} = 3\int \frac{(-1/2)\,du}{\sqrt{u}} = -\frac{3}{2}\int u^{-1/2}\,du$$
$$= -\frac{3}{2}\cdot\frac{u^{1/2}}{1/2} + C_1 = -3\sqrt{1-x^2} + C_1.$$

The second of the new integrals is a standard form,

$$2\int \frac{dx}{\sqrt{1-x^2}} = 2\sin^{-1}x + C_2. \qquad \text{Table 8.1, Formula 18}$$

Combining these results and renaming $C_1 + C_2$ as C gives

$$\int \frac{3x+2}{\sqrt{1-x^2}}\,dx = -3\sqrt{1-x^2} + 2\sin^{-1}x + C.$$

(from Thomas'Calculus)

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EXAMPLE Evaluate $\displaystyle\int \frac{dx}{(1+\sqrt{x})^3}$.

Solution We might try substituting for the term $\sqrt{x}$, but we quickly realize the derivative factor $1/\sqrt{x}$ is missing from the integrand, so this substitution will not help. The other possibility is to substitute for $(1+\sqrt{x})$, and it turns out this works:

$$\int \frac{dx}{(1+\sqrt{x})^3} = \int \frac{2(u-1)\,du}{u^3} \qquad u = 1+\sqrt{x},\ du = \frac{1}{2\sqrt{x}}\,dx; \quad dx = 2\sqrt{x}\,du = 2(u-1)\,du$$

$$= \int\left(\frac{2}{u^2} - \frac{2}{u^3}\right)du = -\frac{2}{u} + \frac{1}{u^2} + C = \frac{1-2u}{u^2} + C$$

$$= \frac{1-2(1+\sqrt{x})}{(1+\sqrt{x})^2} + C$$

$$= C - \frac{1+2\sqrt{x}}{(1+\sqrt{x})^2}.$$

(from Thomas'Calculus)

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Theorem (Integration by parts)

Let $f, g : I \to \mathbb{R}$ be differentiable on I. Then

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx.$$

Proof:

If f and g are differentiable functions of x, the Product Rule says that

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x).$$

In terms of indefinite integrals, this equation becomes

$$\int \frac{d}{dx}[f(x)g(x)]\,dx = \int [f'(x)g(x) + f(x)g'(x)]\,dx$$

or

$$\int \frac{d}{dx}[f(x)g(x)]\,dx = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx.$$

Rearranging the terms of this last equation, we get

$$\int f(x)g'(x)\,dx = \int \frac{d}{dx}[f(x)g(x)]\,dx - \int f'(x)g(x)\,dx,$$

leading to the **integration by parts** formula

(from Thomas'Calculus)

Integration techniques: Integration by parts

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Theorem (Integration by parts)

Let $f, g : I \to \mathbb{R}$ be differentiable. Then
$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx.$

Sometimes it is easier to remember the formula if we write it in differential form. Let $u = f(x)$ and $v = g(x)$. Then $du = f'(x)\,dx$ and $dv = g'(x)\,dx$. Using the Substitution Rule, the integration by parts formula becomes

Integration by Parts Formula

$$\int u\,dv = uv - \int v\,du$$

(from Thomas'Calculus)

EXAMPLE Find

$$\int x \cos x \, dx.$$

Solution We use the formula $\int u \, dv = uv - \int v \, du$ with

$$u = x, \qquad dv = \cos x \, dx,$$
$$du = dx, \qquad v = \sin x. \qquad \text{Simplest antiderivative of } \cos x$$

Then

$$\int x \cos x \, dx = x \sin x - \int \sin x \, dx = x \sin x + \cos x + C.$$

There are four apparent choices available for u and dv in Example 1:

1. Let $u = 1$ and $dv = x \cos x \, dx$.
2. Let $u = x$ and $dv = \cos x \, dx$.
3. Let $u = x \cos x$ and $dv = dx$.
4. Let $u = \cos x$ and $dv = x \, dx$.

(from Thomas'Calculus)

EXAMPLE Find

$$\int \ln x \, dx.$$

Solution Since $\int \ln x \, dx$ can be written as $\int \ln x \cdot 1 \, dx$, we use the formula $\int u \, dv = uv - \int v \, du$ with

$u = \ln x$ Simplifies when differentiated $\qquad dv = dx$ Easy to integrate

$du = \frac{1}{x} dx, \qquad v = x.$ Simplest antiderivative

Then from Equation (2),

$$\int \ln x \, dx = x \ln x - \int x \cdot \frac{1}{x} dx = x \ln x - \int dx = x \ln x - x + C. \qquad \blacksquare$$

(from Thomas'Calculus)

Sometimes we have to use integration by parts more than once.

EXAMPLE Evaluate

$$\int x^2 e^x \, dx.$$

Solution With $u = x^2$, $dv = e^x \, dx$, $du = 2x \, dx$, and $v = e^x$, we have

$$\int x^2 e^x \, dx = x^2 e^x - 2 \int x e^x \, dx.$$

The new integral is less complicated than the original because the exponent on x is reduced by one. To evaluate the integral on the right, we integrate by parts again with $u = x$, $dv = e^x \, dx$. Then $du = dx$, $v = e^x$, and

$$\int x e^x \, dx = x e^x - \int e^x \, dx = x e^x - e^x + C.$$

Using this last evaluation, we then obtain

$$\begin{aligned} \int x^2 e^x \, dx &= x^2 e^x - 2 \int x e^x \, dx \\ &= x^2 e^x - 2x e^x + 2e^x + C, \end{aligned}$$

where the constant of integration is renamed after substituting for the integral on the right. ■

(from Thomas'Calculus)

EXAMPLE Evaluate $\int e^x \cos x\, dx$.

Solution Let $u = e^x$ and $dv = \cos x\, dx$. Then $du = e^x\, dx$, $v = \sin x$, and

$$\int e^x \cos x\, dx = e^x \sin x - \int e^x \sin x\, dx.$$

The second integral is like the first except that it has $\sin x$ in place of $\cos x$. To evaluate it, we use integration by parts with

$$u = e^x, \qquad dv = \sin x\, dx, \qquad v = -\cos x, \qquad du = e^x\, dx.$$

Then

$$\begin{aligned}\int e^x \cos x\, dx &= e^x \sin x - \left(-e^x \cos x - \int (-\cos x)(e^x\, dx)\right) \\ &= e^x \sin x + e^x \cos x - \int e^x \cos x\, dx.\end{aligned}$$

The unknown integral now appears on both sides of the equation. Adding the integral to both sides and adding the constant of integration give

$$2\int e^x \cos x\, dx = e^x \sin x + e^x \cos x + C_1.$$

Dividing by 2 and renaming the constant of integration give

$$\int e^x \cos x\, dx = \frac{e^x \sin x + e^x \cos x}{2} + C. \qquad \blacksquare$$

(from Thomas'Calculus)

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EXAMPLE Obtain a formula that expresses the integral $\int \cos^n x\, dx$ in terms of an integral of a lower power of $\cos x$.

Solution We may think of $\cos^n x$ as $\cos^{n-1} x \cdot \cos x$. Then we let

$$u = \cos^{n-1} x \qquad \text{and} \qquad dv = \cos x\, dx,$$

so that $du = (n-1)\cos^{n-2} x\,(-\sin x\, dx) \qquad \text{and} \qquad v = \sin x.$

Integration by parts then gives

$$\begin{aligned}\int \cos^n x\, dx &= \cos^{n-1} x \sin x + (n-1)\int \sin^2 x \cos^{n-2} x\, dx \\ &= \cos^{n-1} x \sin x + (n-1)\int (1-\cos^2 x)\cos^{n-2} x\, dx \\ &= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\, dx - (n-1)\int \cos^n x\, dx.\end{aligned}$$

If we add $(n-1)\int \cos^n x\, dx$ to both sides of this equation, we obtain

$$n\int \cos^n x\, dx = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\, dx.$$

We then divide through by n, and the final result is

$$\int \cos^n x\, dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n}\int \cos^{n-2} x\, dx.$$

The formula found in Example is called a **reduction formula** because it replaces an integral containing some power of a function with an integral of the same form having the power reduced. When n is a positive integer, we may apply the formula repeatedly until the remaining integral is easy to evaluate. For example, the result in Example tells us that

$$\int \cos^3 x\, dx = \frac{\cos^2 x \sin x}{3} + \frac{2}{3}\int \cos x\, dx = \frac{1}{3}\cos^2 x \sin x + \frac{2}{3}\sin x + C.$$

Products of Powers of Sines and Cosines

$$\int \sin^m x \cos^n x \, dx,$$

where m and n are nonnegative integers (positive or zero).

Case 1 If ***m* is odd**, we write m as $2k + 1$ and use the identity $\sin^2 x = 1 - \cos^2 x$ to obtain

$$\sin^m x = \sin^{2k+1} x = (\sin^2 x)^k \sin x = (1 - \cos^2 x)^k \sin x. \tag{1}$$

Then we combine the single $\sin x$ with dx in the integral and set $\sin x \, dx$ equal to $-d(\cos x)$.

Case 2 If ***m* is even and *n* is odd** in $\int \sin^m x \cos^n x \, dx$, we write n as $2k + 1$ and use the identity $\cos^2 x = 1 - \sin^2 x$ to obtain

$$\cos^n x = \cos^{2k+1} x = (\cos^2 x)^k \cos x = (1 - \sin^2 x)^k \cos x.$$

We then combine the single $\cos x$ with dx and set $\cos x \, dx$ equal to $d(\sin x)$.

Case 3 If **both *m* and *n* are even** in $\int \sin^m x \cos^n x \, dx$, we substitute

$$\sin^2 x = \frac{1 - \cos 2x}{2}, \qquad \cos^2 x = \frac{1 + \cos 2x}{2} \tag{2}$$

to reduce the integrand to one in lower powers of $\cos 2x$.

(from Thomas'Calculus)

EXAMPLE Evaluate $\int \sin^3 x \cos^2 x \, dx$.

Solution This is an example of Case 1.

$$\begin{aligned}
\int \sin^3 x \cos^2 x \, dx &= \int \sin^2 x \cos^2 x \sin x \, dx && m \text{ is odd.} \\
&= \int (1 - \cos^2 x)(\cos^2 x)(-d(\cos x)) && \sin x \, dx = -d(\cos x) \\
&= \int (1 - u^2)(u^2)(-du) && u = \cos x \\
&= \int (u^4 - u^2) \, du && \text{Multiply terms.} \\
&= \frac{u^5}{5} - \frac{u^3}{3} + C = \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C
\end{aligned}$$

(from Thomas'Calculus)

EXAMPLE Evaluate $\displaystyle\int \sin^2 x \cos^4 x \, dx.$

Solution This is an example of Case 3.

$$\int \sin^2 x \cos^4 x \, dx = \int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right)^2 dx \qquad m \text{ and } n \text{ both even}$$

$$= \frac{1}{8}\int (1-\cos 2x)(1+2\cos 2x+\cos^2 2x)\, dx$$

$$= \frac{1}{8}\int (1+\cos 2x-\cos^2 2x-\cos^3 2x)\, dx$$

$$= \frac{1}{8}\left[x+\frac{1}{2}\sin 2x-\int (\cos^2 2x+\cos^3 2x)\, dx\right]$$

For the term involving $\cos^2 2x$, we use

$$\int \cos^2 2x \, dx = \frac{1}{2}\int (1+\cos 4x)\, dx = \frac{1}{2}\left(x+\frac{1}{4}\sin 4x\right). \qquad \text{Omitting the constant of integration until the final result}$$

For the $\cos^3 2x$ term, we have

$$\int \cos^3 2x \, dx = \int (1-\sin^2 2x)\cos 2x \, dx \qquad \begin{aligned} u &= \sin 2x, \\ du &= 2\cos 2x \, dx \end{aligned}$$

$$= \frac{1}{2}\int (1-u^2)\, du = \frac{1}{2}\left(\sin 2x-\frac{1}{3}\sin^3 2x\right). \qquad \text{Again omitting } C$$

Combining everything and simplifying, we get

$$\int \sin^2 x \cos^4 x \, dx = \frac{1}{16}\left(x-\frac{1}{4}\sin 4x+\frac{1}{3}\sin^3 2x\right)+C. \qquad \blacksquare$$

(from Thomas'Calculus)

EXAMPLE Evaluate $\int \sin^2 x \cos^4 x \, dx$.

Solution This is an example of Case 3.

$$\int \sin^2 x \cos^4 x \, dx = \int \left(\frac{1 - \cos 2x}{2}\right)\left(\frac{1 + \cos 2x}{2}\right)^2 dx$$ *m and n both even*

$$= \frac{1}{8}\int (1 - \cos 2x)(1 + 2\cos 2x + \cos^2 2x)\, dx$$

$$= \frac{1}{8}\int (1 + \cos 2x - \cos^2 2x - \cos^3 2x)\, dx$$

$$= \frac{1}{8}\left[x + \frac{1}{2}\sin 2x - \int (\cos^2 2x + \cos^3 2x)\, dx\right]$$

For the term involving $\cos^2 2x$, we use

$$\int \cos^2 2x \, dx = \frac{1}{2}\int (1 + \cos 4x)\, dx = \frac{1}{2}\left(x + \frac{1}{4}\sin 4x\right).$$ *Omitting the constant of integration until the final result*

For the $\cos^3 2x$ term, we have

$$\int \cos^3 2x \, dx = \int (1 - \sin^2 2x)\cos 2x \, dx$$ $u = \sin 2x$, $du = 2\cos 2x \, dx$

$$= \frac{1}{2}\int (1 - u^2)\, du = \frac{1}{2}\left(\sin 2x - \frac{1}{3}\sin^3 2x\right).$$ *Again omitting C*

Combining everything and simplifying, we get

$$\int \sin^2 x \cos^4 x \, dx = \frac{1}{16}\left(x - \frac{1}{4}\sin 4x + \frac{1}{3}\sin^3 2x\right) + C.$$ ■

Eliminating Square Roots

In the next example, we use the identity $\cos^2\theta = (1 + \cos 2\theta)/2$ to eliminate a square root.

EXAMPLE Evaluate $$\int_0^{\pi/4} \sqrt{1 + \cos 4x}\, dx.$$

Solution To eliminate the square root, we use the identity

$$\cos^2\theta = \frac{1 + \cos 2\theta}{2} \qquad \text{or} \qquad 1 + \cos 2\theta = 2\cos^2\theta.$$

With $\theta = 2x$, this becomes

$$1 + \cos 4x = 2\cos^2 2x.$$

Therefore,

$$\int_0^{\pi/4} \sqrt{1 + \cos 4x}\, dx = \int_0^{\pi/4} \sqrt{2\cos^2 2x}\, dx = \int_0^{\pi/4} \sqrt{2}\sqrt{\cos^2 2x}\, dx$$

$$= \sqrt{2}\int_0^{\pi/4} |\cos 2x|\, dx = \sqrt{2}\int_0^{\pi/4} \cos 2x\, dx \qquad \cos 2x \geq 0 \text{ on } [0, \pi/4]$$

$$= \sqrt{2}\left[\frac{\sin 2x}{2}\right]_0^{\pi/4} = \frac{\sqrt{2}}{2}[1 - 0] = \frac{\sqrt{2}}{2}. \qquad \blacksquare$$

(from Thomas'Calculus)

EXAMPLE Evaluate $\int \sec^3 x\, dx$.

Solution We integrate by parts using

$$u = \sec x, \qquad dv = \sec^2 x\, dx, \qquad v = \tan x, \qquad du = \sec x \tan x\, dx.$$

Then

$$\begin{aligned}\int \sec^3 x\, dx &= \sec x \tan x - \int (\tan x)(\sec x \tan x\, dx) \\ &= \sec x \tan x - \int (\sec^2 x - 1) \sec x\, dx \qquad \tan^2 x = \sec^2 x - 1 \\ &= \sec x \tan x + \int \sec x\, dx - \int \sec^3 x\, dx.\end{aligned}$$

Combining the two secant-cubed integrals gives

$$2\int \sec^3 x\, dx = \sec x \tan x + \int \sec x\, dx$$

and

$$\int \sec^3 x\, dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln |\sec x + \tan x| + C.$$

(from Thomas'Calculus)

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Integrals of Powers of tan x and sec x

We know how to integrate the tangent and secant and their squares. To integrate higher powers, we use the identities $\tan^2 x = \sec^2 x - 1$ and $\sec^2 x = \tan^2 x + 1$, and integrate by parts when necessary to reduce the higher powers to lower powers.

EXAMPLE Evaluate $\int \tan^4 x\, dx$.

Solution

$$\int \tan^4 x\, dx = \int \tan^2 x \cdot \tan^2 x\, dx = \int \tan^2 x \cdot (\sec^2 x - 1)\, dx$$

$$= \int \tan^2 x \sec^2 x\, dx - \int \tan^2 x\, dx = \int \tan^2 x \sec^2 x\, dx - \int (\sec^2 x - 1)\, dx$$

$$= \int \tan^2 x \sec^2 x\, dx - \int \sec^2 x\, dx + \int dx$$

In the first integral, we let $u = \tan x, \quad du = \sec^2 x\, dx$ and have

$$\int u^2\, du = \frac{1}{3} u^3 + C_1.$$

The remaining integrals are standard forms, so

$$\int \tan^4 x\, dx = \frac{1}{3} \tan^3 x - \tan x + x + C.$$

(from Thomas'Calculus)

EXAMPLE Evaluate

$$\int \tan^4 x \sec^4 x \, dx.$$

Solution

$$\begin{aligned}
\int (\tan^4 x)(\sec^4 x)\, dx &= \int (\tan^4 x)(1 + \tan^2 x)(\sec^2 x)\, dx && \sec^2 x = 1 + \tan^2 x \\
&= \int (\tan^4 x + \tan^6 x)(\sec^2 x)\, dx \\
&= \int (\tan^4 x)(\sec^2 x)\, dx + \int (\tan^6 x)(\sec^2 x)\, dx \\
&= \int u^4\, du + \int u^6\, du = \frac{u^5}{5} + \frac{u^7}{7} + C && u = \tan x, \; du = \sec^2 x\, dx \\
&= \frac{\tan^5 x}{5} + \frac{\tan^7 x}{7} + C
\end{aligned}$$

(from Thomas'Calculus)

Products of Sines and Cosines

The integrals

$$\int \sin mx \sin nx \, dx, \qquad \int \sin mx \cos nx \, dx, \qquad \text{and} \qquad \int \cos mx \cos nx \, dx$$

arise in many applications involving periodic functions. We can evaluate these integrals through integration by parts, but two such integrations are required in each case. It is simpler to use the identities

$$\sin mx \sin nx = \frac{1}{2}\left[\cos(m-n)x - \cos(m+n)x\right], \tag{3}$$

$$\sin mx \cos nx = \frac{1}{2}\left[\sin(m-n)x + \sin(m+n)x\right], \tag{4}$$

$$\cos mx \cos nx = \frac{1}{2}\left[\cos(m-n)x + \cos(m+n)x\right]. \tag{5}$$

(from Thomas'Calculus)

EXAMPLE Evaluate

$$\int \sin 3x \cos 5x \, dx.$$

Solution From Equation (4) with $m = 3$ and $n = 5$, we get

$$\begin{aligned}\int \sin 3x \cos 5x \, dx &= \frac{1}{2}\int [\sin(-2x) + \sin 8x]\, dx \\ &= \frac{1}{2}\int (\sin 8x - \sin 2x)\, dx \\ &= -\frac{\cos 8x}{16} + \frac{\cos 2x}{4} + C.\end{aligned}$$

(from Thomas'Calculus)

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Procedure for a Trigonometric Substitution

1. Write down the substitution for x, calculate the differential dx, and specify the selected values of θ for the substitution.
2. Substitute the trigonometric expression and the calculated differential into the integrand, and then simplify the results algebraically.
3. Integrate the trigonometric integral, keeping in mind the restrictions on the angle θ for reversibility.
4. Draw an appropriate reference triangle to reverse the substitution in the integration result and convert it back to the original variable x.

(from Thomas'Calculus)

EXAMPLE Evaluate $\displaystyle\int \frac{x^2\,dx}{\sqrt{9-x^2}}$.

Solution We set $x = 3\sin\theta, \quad dx = 3\cos\theta\, d\theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$$9 - x^2 = 9 - 9\sin^2\theta = 9(1-\sin^2\theta) = 9\cos^2\theta.$$

Then
$$\int \frac{x^2\,dx}{\sqrt{9-x^2}} = \int \frac{9\sin^2\theta \cdot 3\cos\theta\,d\theta}{|3\cos\theta|} = 9\int \sin^2\theta\,d\theta = 9\int \frac{1-\cos 2\theta}{2}\,d\theta$$

$\cos\theta > 0$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$$= \frac{9}{2}\left(\theta - \frac{\sin 2\theta}{2}\right) + C = \frac{9}{2}(\theta - \sin\theta\cos\theta) + C$$

$\sin 2\theta = 2\sin\theta\cos\theta$

$$= \frac{9}{2}\left(\sin^{-1}\frac{x}{3} - \frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3}\right) + C = \frac{9}{2}\sin^{-1}\frac{x}{3} - \frac{x}{2}\sqrt{9-x^2} + C.$$

(from Thomas'Calculus)

The method for rewriting rational functions as a sum of simpler fractions is called the **method of partial fractions**.

General Description of the Method

Success in writing a rational function $f(x)/g(x)$ as a sum of partial fractions depends on two things:

- *The degree of $f(x)$ must be less than the degree of $g(x)$*. That is, the fraction must be proper. If it isn't, divide $f(x)$ by $g(x)$ and work with the remainder term.

- *We must know the factors of $g(x)$*. In theory, any polynomial with real coefficients can be written as a product of real linear factors and real quadratic factors. In practice, the factors may be hard to find.

Here is how we find the partial fractions of a proper fraction $f(x)/g(x)$ when the factors of g are known. A quadratic polynomial (or factor) is **irreducible** if it cannot be written as the product of two linear factors with real coefficients. That is, the polynomial has no real roots.

(from Thomas'Calculus)

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Method of Partial Fractions when $f(x)/g(x)$ is Proper

1. Let $x - r$ be a linear factor of $g(x)$. Suppose that $(x - r)^m$ is the highest power of $x - r$ that divides $g(x)$. Then, to this factor, assign the sum of the m partial fractions:

$$\frac{A_1}{(x - r)} + \frac{A_2}{(x - r)^2} + \cdots + \frac{A_m}{(x - r)^m}.$$

   Do this for each distinct linear factor of $g(x)$.

2. Let $x^2 + px + q$ be an irreducible quadratic factor of $g(x)$ so that $x^2 + px + q$ has no real roots. Suppose that $(x^2 + px + q)^n$ is the highest power of this factor that divides $g(x)$. Then, to this factor, assign the sum of the n partial fractions:

$$\frac{B_1x + C_1}{(x^2 + px + q)} + \frac{B_2x + C_2}{(x^2 + px + q)^2} + \cdots + \frac{B_nx + C_n}{(x^2 + px + q)^n}.$$

   Do this for each distinct quadratic factor of $g(x)$.

3. Set the original fraction $f(x)/g(x)$ equal to the sum of all these partial fractions. Clear the resulting equation of fractions and arrange the terms in decreasing powers of x.

4. Equate the coefficients of corresponding powers of x and solve the resulting equations for the undetermined coefficients.

(from Thomas'Calculus)

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EXAMPLE Use partial fractions to evaluate

$$\int \frac{x^2 + 4x + 1}{(x-1)(x+1)(x+3)}\, dx.$$

Solution The partial fraction decomposition has the form

$$\frac{x^2 + 4x + 1}{(x-1)(x+1)(x+3)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x+3}.$$

To find the values of the undetermined coefficients A, B, and C, we clear fractions and get

$$\begin{aligned} x^2 + 4x + 1 &= A(x+1)(x+3) + B(x-1)(x+3) + C(x-1)(x+1) \\ &= A(x^2 + 4x + 3) + B(x^2 + 2x - 3) + C(x^2 - 1) \\ &= (A + B + C)x^2 + (4A + 2B)x + (3A - 3B - C). \end{aligned}$$

The polynomials on both sides of the above equation are identical, so we equate coefficients of like powers of x, obtaining

$$\begin{aligned} &\text{Coefficient of } x^2: && A + B + C = 1 \\ &\text{Coefficient of } x^1: && 4A + 2B \phantom{{}-C} = 4 \\ &\text{Coefficient of } x^0: && 3A - 3B - C = 1 \end{aligned}$$

There are several ways of solving such a system of linear equations for the unknowns A, B, and C, including elimination of variables or the use of a calculator or computer. Whatever method is used, the solution is $A = 3/4$, $B = 1/2$, and $C = -1/4$. Hence we have

$$\begin{aligned} \int \frac{x^2 + 4x + 1}{(x-1)(x+1)(x+3)}\, dx &= \int \left[\frac{3}{4}\frac{1}{x-1} + \frac{1}{2}\frac{1}{x+1} - \frac{1}{4}\frac{1}{x+3} \right] dx \\ &= \frac{3}{4}\ln|x-1| + \frac{1}{2}\ln|x+1| - \frac{1}{4}\ln|x+3| + K, \end{aligned}$$

where K is the arbitrary constant of integration (to avoid confusion with the undetermined coefficient we labeled as C). ■

(from Thomas'Calculus)

EXAMPLE Use partial fractions to evaluate

$$\int \frac{6x+7}{(x+2)^2}\,dx.$$

Solution First we express the integrand as a sum of partial fractions with undetermined coefficients.

$$\begin{aligned} \frac{6x+7}{(x+2)^2} &= \frac{A}{x+2} + \frac{B}{(x+2)^2} \\ 6x+7 &= A(x+2) + B \qquad \text{Multiply both sides by } (x+2)^2. \\ &= Ax + (2A+B) \end{aligned}$$

Equating coefficients of corresponding powers of x gives

$$A = 6 \quad \text{and} \quad 2A + B = 12 + B = 7, \quad \text{or} \quad A = 6 \quad \text{and} \quad B = -5.$$

Therefore,

$$\begin{aligned} \int \frac{6x+7}{(x+2)^2}\,dx &= \int \left(\frac{6}{x+2} - \frac{5}{(x+2)^2} \right) dx \\ &= 6\int \frac{dx}{x+2} - 5\int (x+2)^{-2}\,dx \\ &= 6\ln|x+2| + 5(x+2)^{-1} + C \end{aligned}$$

The next example shows how to handle the case when $f(x)/g(x)$ is an improper fraction. It is a case where the degree of f is larger than the degree of g.

EXAMPLE Use partial fractions to evaluate

$$\int \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3}\, dx.$$

Solution First we divide the denominator into the numerator to get a polynomial plus a proper fraction.

$$\begin{array}{r@{}l} & 2x \\ x^2 - 2x - 3 & \overline{)\,2x^3 - 4x^2 - x - 3} \\ & \underline{2x^3 - 4x^2 - 6x} - 3 \\ & 5x - 3 \end{array}$$

Then we write the improper fraction as a polynomial plus a proper fraction.

$$\frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} = 2x + \frac{5x - 3}{x^2 - 2x - 3}$$

We found the partial fraction decomposition of the fraction on the right in the opening example, so

$$\begin{aligned} \int \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3}\, dx &= \int 2x\, dx + \int \frac{5x - 3}{x^2 - 2x - 3}\, dx \\ &= \int 2x\, dx + \int \frac{2}{x + 1}\, dx + \int \frac{3}{x - 3}\, dx \\ &= x^2 + 2 \ln |x + 1| + 3 \ln |x - 3| + C. \end{aligned}$$

EXAMPLE Use partial fractions to evaluate $\displaystyle\int \frac{-2x+4}{(x^2+1)(x-1)^2}\,dx$.

Solution The denominator has an irreducible quadratic factor as well as a repeated linear factor, so we write

$$\frac{-2x+4}{(x^2+1)(x-1)^2} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1} + \frac{D}{(x-1)^2}.$$

Clearing the equation of fractions gives

$$\begin{aligned} -2x+4 &= (Ax+B)(x-1)^2 + C(x-1)(x^2+1) + D(x^2+1) \\ &= (A+C)x^3 + (-2A+B-C+D)x^2 \\ &\quad + (A-2B+C)x + (B-C+D). \end{aligned}$$

Equating coefficients of like terms gives

Coefficients of x^3: $\quad 0 = A + C$

Coefficients of x^2: $\quad 0 = -2A + B - C + D$

Coefficients of x^1: $\quad -2 = A - 2B + C$

Coefficients of x^0: $\quad 4 = B - C + D$

We solve these equations simultaneously to find the values of A, B, C, and D:

$-4 = -2A, \quad A = 2$ — Subtract fourth equation from second.

$C = -A = -2$ — From the first equation

$B = (A + C + 2)/2 = 1$ — From the third equation and $C = -A$

$D = 4 - B + C = 1.$ — From the fourth equation.

We substitute these values into Equation , obtaining

$$\frac{-2x+4}{(x^2+1)(x-1)^2} = \frac{2x+1}{x^2+1} - \frac{2}{x-1} + \frac{1}{(x-1)^2}.$$

Finally, using the expansion above we can integrate:

$$\begin{aligned} \int \frac{-2x+4}{(x^2+1)(x-1)^2}\,dx &= \int \left(\frac{2x+1}{x^2+1} - \frac{2}{x-1} + \frac{1}{(x-1)^2}\right) dx \\ &= \int \left(\frac{2x}{x^2+1} + \frac{1}{x^2+1} - \frac{2}{x-1} + \frac{1}{(x-1)^2}\right) dx \end{aligned}$$

Non-elementary integrals

Integrals of functions that do not have elementary antiderivatives are called **non-elementary integrals**.
These integrals can sometimes be expressed with infinite series or approximated using numerical methods for their evaluation

Examples of nonelementary integrals include the error function (which measures the probability of random errors)

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$$

and integrals such as

$$\int \sin x^2\, dx \qquad \text{and} \qquad \int \sqrt{1 + x^4}\, dx$$

that arise in engineering and physics. These and a number of others, such as

$$\int \frac{e^x}{x}\, dx, \qquad \int e^{(e^x)}\, dx, \qquad \int \frac{1}{\ln x}\, dx, \qquad \int \ln(\ln x)\, dx, \qquad \int \frac{\sin x}{x}\, dx,$$

$$\int \sqrt{1 - k^2 \sin^2 x}\, dx, \qquad 0 < k < 1,$$

look so easy they tempt us to try them just to see how they turn out. It can be proved, however, that there is no way to express these integrals as finite combinations of elementary functions. The same applies to integrals that can be changed into these by substitution.

(from Thomas'Calculus)