4. Differential calculus of functions of one real variable

4.3. Power series

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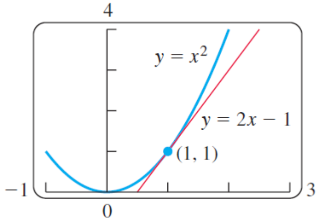
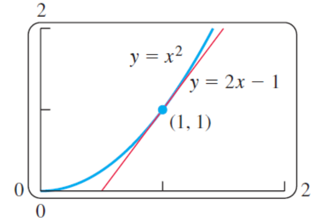
Content:

- Linearization
- Notion of a power series
- Convergence of power series
- Radius and interval of convergence
- Operations on power series

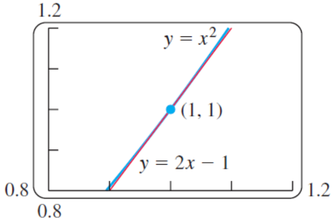
Linearization



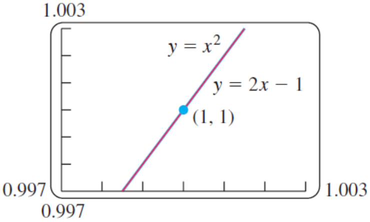
$y = x^2$ and its tangent $y = 2x - 1$ at $(1, 1)$.



Tangent and curve very close near $(1, 1)$.



Tangent and curve very close throughout entire x-interval shown.



Tangent and curve closer still. Computer screen cannot distinguish tangent from

Linearization

DEFINITIONS If f is differentiable at $x = a$, then the approximating function

$$L(x) = f(a) + f'(a)(x - a)$$

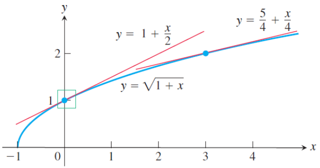
is the **linearization** of f at a. The approximation

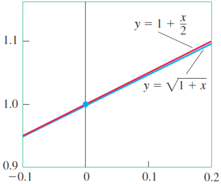
$$f(x) \approx L(x)$$

of f by L is the **standard linear approximation** of f at a. The point $x = a$ is the **center** of the approximation.

(from Thomas' Calculus)

EXAMPLE Find the linearization of $f(x) = \sqrt{1 + x}$ at $x = 0$





Solution Since $f'(x) = \frac{1}{2}(1 + x)^{-1/2}$,
we have $f(0) = 1$ and $f'(0) = 1/2$, giving the linearization

$$L(x) = f(a) + f'(a)(x - a) = 1 + \frac{1}{2}(x - 0) = 1 + \frac{x}{2}.$$

Approximation	True value	$\lvert$True value − approximation$\rvert$
$\sqrt{1.2} \approx 1 + \frac{0.2}{2} = 1.10$	1.095445	$0.004555 < 10^{-2}$
$\sqrt{1.05} \approx 1 + \frac{0.05}{2} = 1.025$	1.024695	$0.000305 < 10^{-3}$
$\sqrt{1.005} \approx 1 + \frac{0.005}{2} = 1.00250$	1.002497	$0.000003 < 10^{-5}$

(from Thomas' Calculus)

EXAMPLE Find the linearization of $f(x) = \sqrt{1 + x}$ at $x = 3$.

Solution We evaluate the equation defining $L(x)$ at $a = 3$. With

$$f(3) = 2, \qquad f'(3) = \frac{1}{2}(1 + x)^{-1/2}\Big|_{x=3} = \frac{1}{4},$$

we have

$$L(x) = 2 + \frac{1}{4}(x - 3) = \frac{5}{4} + \frac{x}{4}.$$

At $x = 3.2$,

$$\sqrt{1 + x} = \sqrt{1 + 3.2} \approx \frac{5}{4} + \frac{3.2}{4} = 1.250 + 0.800 = 2.050,$$

which differs from the true value $\sqrt{4.2} \approx 2.04939$ by less than one one-thousandth. The linearization in Example 1 gives

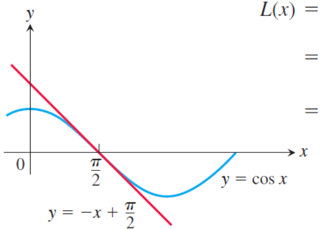
$$\sqrt{1 + x} = \sqrt{1 + 3.2} \approx 1 + \frac{3.2}{2} = 1 + 1.6 = 2.6,$$

a result that is off by more than 25%.

(from Thomas' Calculus)

EXAMPLE Find the linearization of $f(x) = \cos x$ at $x = \pi/2$

Solution Since $f(\pi/2) = \cos(\pi/2) = 0$, $f'(x) = -\sin x$, and $f'(\pi/2) = -\sin(\pi/2) = -1$, we find the linearization at $a = \pi/2$ to be



$$\begin{aligned} L(x) &= f(a) + f'(a)(x - a) \\ &= 0 + (-1)\left(x - \frac{\pi}{2}\right) \\ &= -x + \frac{\pi}{2}. \end{aligned}$$

(from Thomas' Calculus)

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An important linear approximation for roots and powers is

$$(1+x)^k \approx 1+kx \qquad (x \text{ near } 0;\ \text{any number } k)$$

This approximation, good for values of x sufficiently close to zero, has broad application. For example, when x is small,

$$\sqrt{1+x} \approx 1+\frac{1}{2}x \qquad k = 1/2$$

$$\frac{1}{1-x} = (1-x)^{-1} \approx 1+(-1)(-x) = 1+x \qquad k=-1;\ \text{replace } x \text{ by } -x.$$

$$\sqrt[3]{1+5x^4} = (1+5x^4)^{1/3} \approx 1+\frac{1}{3}(5x^4) = 1+\frac{5}{3}x^4 \qquad k = 1/3;\ \text{replace } x \text{ by } 5x^4.$$

$$\frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-1/2} \approx 1+\left(-\frac{1}{2}\right)(-x^2) = 1+\frac{1}{2}x^2 \qquad k=-1/2;\ \text{replace } x \text{ by } -x^2.$$

(from Thomas' Calculus)

Notion of a power series

DEFINITIONS **A power series about $x = 0$** is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots. \tag{1}$$

A power series about $x = a$ is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots \tag{2}$$

in which the **center** a and the **coefficients** $c_0, c_1, c_2, \ldots, c_n, \ldots$ are constants.

(from Thomas'Calculus)

Notion of a power series

EXAMPLE Taking all the coefficients to be 1 in Equation (1) gives the geometric power series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots .$$

This is the geometric series with first term 1 and ratio x. It converges to $1/(1-x)$ for $|x| < 1$. We express this fact by writing

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots , \qquad -1 < x < 1.$$

(from Thomas'Calculus)

Notion of a power series

Two ways of interpreting formula $\dfrac{1}{1-x} = \sum_{j=1}^{\infty} x^j$, $|x| < 1$:

1) formula for the sum of the series on the right;
2) approximation of the function $f(x) = \dfrac{1}{1-x}$ by polynomials $P_n(x) = \sum_{j=1}^{n} x^j$.

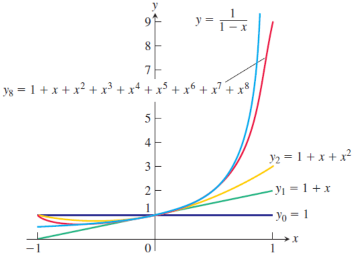


FIGURE The graphs of $f(x) = 1/(1 - x)$ and four of its polynomial approximations.

(from Thomas'Calculus)

EXAMPLE The power series $1 - \frac{1}{2}(x-2) + \frac{1}{4}(x-2)^2 + \cdots + \left(-\frac{1}{2}\right)^n (x-2)^n + \cdots$ is a geometric series with first term 1 and ratio $r = -\frac{x-2}{2}$. The series converges for $\left|\frac{x-2}{2}\right| < 1$ or $0 < x < 4$. The sum is

$$\frac{1}{1-r} = \frac{1}{1 + \frac{x-2}{2}} = \frac{2}{x},$$

so $$\frac{2}{x} = 1 - \frac{(x-2)}{2} + \frac{(x-2)^2}{4} - \cdots + \left(-\frac{1}{2}\right)^n (x-2)^n + \cdots, \qquad 0 < x < 4.$$

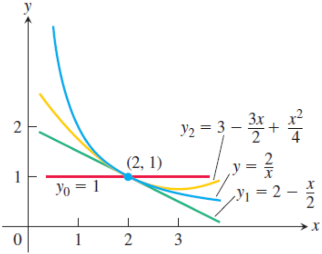
Series generates useful polynomial approximations of $f(x) = 2/x$ for values of x near 2:

$P_0(x) = 1$

$P_1(x) = 1 - \frac{1}{2}(x-2) = 2 - \frac{x}{2}$

$P_2(x) = 1 - \frac{1}{2}(x-2) + \frac{1}{4}(x-2)^2 = 3 - \frac{3x}{2} + \frac{x^2}{4},$

and so on



(from Thomas'Calculus)

Notion of a power series

EXAMPLE For what values of x do the following power series converge?

(a) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$

(b) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{2n-1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$

(c) $\displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

(d) $\displaystyle\sum_{n=0}^{\infty}n!x^n = 1 + x + 2!x^2 + 3!x^3 + \cdots$

Solution Apply the Ratio Test to the series $\sum |u_n|$, where u_n is the nth term of the power series in question.

(a) $\displaystyle\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{x^{n+1}}{n+1}\cdot\frac{n}{x}\right| = \frac{n}{n+1}|x| \to |x|.$

The series converges absolutely for $|x| < 1$. It diverges if $|x| > 1$ because the nth term does not converge to zero. At $x = 1$, we get the alternating harmonic series $1 - 1/2 + 1/3 - 1/4 + \cdots$, which converges. At $x = -1$, we get $-1 - 1/2 - 1/3 - 1/4 - \cdots$, the negative of the harmonic series; it diverges. Series (a) converges for $-1 < x \le 1$ and diverges elsewhere.

(from Thomas'Calculus)

Notion of a power series

EXAMPLE For what values of x do the following power series converge?

(a) $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$

(b) $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{2n-1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$

(c) $\sum_{n=0}^{\infty}\frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

(d) $\sum_{n=0}^{\infty} n!x^n = 1 + x + 2!x^2 + 3!x^3 + \cdots$

Solution Apply the Ratio Test to the series $\sum |u_n|$, where u_n is the nth term of the power series in question.

(b) $\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{x^{2n+1}}{2n+1}\cdot\frac{2n-1}{x^{2n-1}}\right| = \frac{2n-1}{2n+1}x^2 \to x^2.$ $\quad 2(n+1) - 1 = 2n + 1$

The series converges absolutely for $x^2 < 1$. It diverges for $x^2 > 1$ because the nth term does not converge to zero. At $x = 1$ the series becomes $1 - 1/3 + 1/5 - 1/7 + \cdots$, which converges by the Alternating Series Theorem. It also converges at $x = -1$ because it is again an alternating series that satisfies the conditions for convergence. The value at $x = -1$ is the negative of the value at $x = 1$. Series (b) converges for $-1 \le x \le 1$ and diverges elsewhere.

(from Thomas'Calculus)

Notion of a power series

EXAMPLE For what values of x do the following power series converge?

(a) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$

(b) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{2n-1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$

(c) $\displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

(d) $\displaystyle\sum_{n=0}^{\infty}n!x^n = 1 + x + 2!x^2 + 3!x^3 + \cdots$

Solution Apply the Ratio Test to the series $\sum|u_n|$, where u_n is the nth term of the power series in question.

(c) $\displaystyle\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{x^{n+1}}{(n+1)!}\cdot\frac{n!}{x^n}\right| = \frac{|x|}{n+1} \to 0$ for every x. $\quad\displaystyle\frac{n!}{(n+1)!} = \frac{1\cdot 2\cdot 3\cdots n}{1\cdot 2\cdot 3\cdots n\cdot(n+1)}$

The series converges absolutely for all x.

(d) $\displaystyle\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{(n+1)!x^{n+1}}{n!x^n}\right| = (n+1)|x| \to \infty$ unless $x = 0$.

The series diverges for all values of x except $x = 0$.

(from Thomas'Calculus)

Convergence of power series

THEOREM —The Convergence Theorem for Power Series If the power series $\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$ converges at $x = c \neq 0$, then it converges absolutely for all x with $|x| < |c|$. If the series diverges at $x = d$, then it diverges for all x with $|x| > |d|$.

(from Thomas'Calculus)

COROLLARY TO THEOREM The convergence of the series $\sum c_n(x-a)^n$ is described by one of the following three cases:

1. There is a positive number R such that the series diverges for x with $|x-a| > R$ but converges absolutely for x with $|x-a| < R$. The series may or may not converge at either of the endpoints $x = a - R$ and $x = a + R$.
2. The series converges absolutely for every x ($R = \infty$).
3. The series converges at $x = a$ and diverges elsewhere ($R = 0$).

(from Thomas'Calculus)

Definition

The value R from the above theorem is called the **radius of convergence** of the series $\sum_{n=0}^{\infty} c_n(x-a)^n$, and $(a-R, a+R)$ is its **interval of convergence**.

Radius of convergence

Remarks

- The interval of convergence may be open, closed, or half-open, depending on the particular series.
- At points x with $|x - a| < R$, the series converges absolutely.
- If the series converges for all values of x, we say its radius of convergence is infinite. If it converges only at $x = a$, we say its radius of convergence is zero.

How to Test a Power Series for Convergence

1. *Use the Ratio Test (or Root Test) to find the interval where the series converges absolutely*. Ordinarily, this is an open interval

$$|x - a| < R \qquad \text{or} \qquad a - R < x < a + R.$$

2. *If the interval of absolute convergence is finite, test for convergence or divergence at each endpoint*.
3. *If the interval of absolute convergence is $a - R < x < a + R$, the series diverges for $|x - a| > R$* (it does not even converge conditionally) because the nth term does not approach zero for those values of x.

(from Thomas'Calculus)

Let $\sum_{n=0}^{\infty} c_n(x-a)^n$ has the radius of convergence R and

$$\lim_{n\to\infty} \sqrt[n]{|c_n|} = c \text{ or } \lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right| = c.$$

- If $c = 0$ then $R = +\infty$.
- If $c = +\infty$ then $R = 0$.
- If $c \in (0, \infty)$ then $R = \dfrac{1}{c}$.

Examples:

1) $\sum_{j=1}^{\infty} \frac{(x-1)^j}{j2^j}$, $a=1$, $c_j = \frac{1}{j2^j}$.

$\lim_{n\to\infty} \sqrt[n]{|c_n|} = \lim_{n\to\infty} \frac{1}{2\sqrt[j]{j}} = \frac{1}{2} = c$. Therefore, the radius of convergence is $R = \frac{1}{c} = 2$ and the series converges absolutely for $x \in (-1,3)$, diverges for $x \in (-\infty,-1) \cup (3,+\infty)$.

$x=-1: \sum_{j=1}^{\infty} \frac{(x-1)^j}{j2^j} = \sum_{j=1}^{\infty} \frac{(-2)^j}{j2^j} = \sum_{j=1}^{\infty} \frac{(-1)^j}{j}$ – converges conditionally.

$x=3: \sum_{j=1}^{\infty} \frac{(x-1)^j}{j2^j} = \sum_{j=1}^{\infty} \frac{(2)^j}{j2^j} = \sum_{j=1}^{\infty} \frac{1}{j}$ – harmonic series, diverges.

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Examples:

2) $\sum_{j=1}^{\infty} \frac{(j+1)^5 x^{2j}}{2j+1}$, $a = 0$, $c_j = \frac{(j+1)^5}{2j+1}$.

Denote $y_j = x^2$ and consider the series $\sum_{j=1}^{\infty} \frac{(j+1)^5 y^j}{2j+1}$, $a = 0$.

$$\lim_{n\to\infty} \sqrt[n]{|c_n|} = \lim_{n\to\infty} \sqrt[j]{\frac{(j+1)^5}{2j+1}} = 1.$$

Therefore, the radius of convergence is $R = \frac{1}{c} = 1$ and the series converges absolutely for $|y| < 1$, diverges for $|y| > 1$.
Hence, the series converges absolutely for $|x| < 1$, diverges for $|x| > 1$.
$|x| = 1$: $\sum_{j=1}^{\infty} \frac{(j+1)^5 x^{2j}}{2j+1} = \sum_{j=1}^{\infty} \frac{(j+1)^5}{2j+1}$ – diverges because $c_n \nrightarrow 0$.

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THEOREM —The Series Multiplication Theorem for Power Series If $A(x) = \sum_{n=0}^{\infty} a_n x^n$ and $B(x) = \sum_{n=0}^{\infty} b_n x^n$ converge absolutely for $|x| < R$, and

$$c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_{n-1} b_1 + a_n b_0 = \sum_{k=0}^{n} a_k b_{n-k},$$

then $\sum_{n=0}^{\infty} c_n x^n$ converges absolutely to $A(x)B(x)$ for $|x| < R$:

$$\left(\sum_{n=0}^{\infty} a_n x^n\right) \cdot \left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} c_n x^n.$$

(from Thomas'Calculus)

$$\left(\sum_{n=0}^{\infty} x^n\right) \cdot \left(\sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}\right)$$

$$= (1 + x + x^2 + \cdots)\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots\right) \qquad \text{Multiply second series } \ldots$$

$$= \underbrace{\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots\right)}_{\text{by } 1} + \underbrace{\left(x^2 - \frac{x^3}{2} + \frac{x^4}{3} - \cdots\right)}_{\text{by } x} + \underbrace{\left(x^3 - \frac{x^4}{2} + \frac{x^5}{3} - \cdots\right)}_{\text{by } x^2} + \cdots$$

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THEOREM If $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for $|x| < R$, then $\sum_{n=0}^{\infty} a_n (f(x))^n$ converges absolutely for any continuous function f on $|f(x)| < R$.

(from Thomas'Calculus)

Since $1/(1-x) = \sum_{n=0}^{\infty} x^n$ converges absolutely for $|x| < 1$, it follows from Theorem that $1/(1-4x^2) = \sum_{n=0}^{\infty} (4x^2)^n$ converges absolutely for $|4x^2| < 1$ or $|x| < 1/2$.

(from Thomas'Calculus)

Differentiation of power series

THEOREM —The Term-by-Term Differentiation Theorem If $\sum c_n(x-a)^n$ has radius of convergence $R > 0$, it defines a function

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n \qquad \text{on the interval} \qquad a - R < x < a + R.$$

This function f has derivatives of all orders inside the interval, and we obtain the derivatives by differentiating the original series term by term:

$$f'(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1},$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1)c_n(x-a)^{n-2},$$

and so on. Each of these derived series converges at every point of the interval $a - R < x < a + R$.

(from Thomas'Calculus)

EXAMPLE Find series for $f'(x)$ and $f''(x)$ if

$$f(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots + x^n + \cdots$$

$$= \sum_{n=0}^{\infty} x^n, \qquad -1 < x < 1.$$

Solution We differentiate the power series on the right term by term:

$$f'(x) = \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots + nx^{n-1} + \cdots$$

$$= \sum_{n=1}^{\infty} nx^{n-1}, \qquad -1 < x < 1;$$

$$f''(x) = \frac{2}{(1-x)^3} = 2 + 6x + 12x^2 + \cdots + n(n-1)x^{n-2} + \cdots$$

$$= \sum_{n=2}^{\infty} n(n-1)x^{n-2}, \qquad -1 < x < 1. \qquad \blacksquare$$

(from Thomas'Calculus)

Caution Term-by-term differentiation might not work for other kinds of series. For example, the trigonometric series

$$\sum_{n=1}^{\infty} \frac{\sin(n!x)}{n^2}$$

converges for all x. But if we differentiate term by term we get the series

$$\sum_{n=1}^{\infty} \frac{n! \cos(n!x)}{n^2},$$

which diverges for all x. This is not a power series since it is not a sum of positive integer powers of x.

(from Thomas'Calculus)