4. Differential calculus of functions of one real variable

4.2. Applications of derivatives: Mean value theorems

Content:

- Rolle's Lemma
- Lagrange's Mean value theorem and corollaries
- Cauchy's Mean value theorem

Lemma (Rolle's Lemma)

Suppose that $y = f(x)$ is continuous over the closed interval $[a, b]$ and differentiable at every point of its interior (a, b). If $f(a) = f(b)$, then there exists a point c in (a, b) at which $f'(c) = 0$.

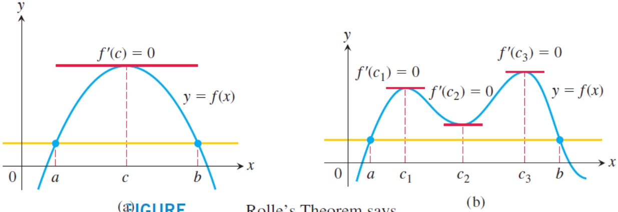
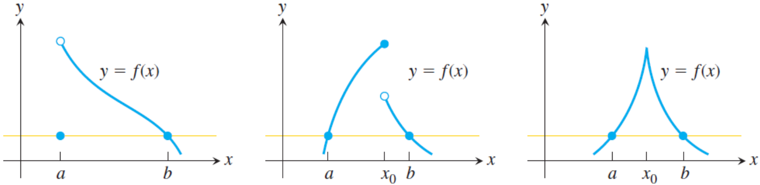


FIGURE Rolle's Theorem says that a differentiable curve has at least one horizontal tangent between any two points where it crosses a horizontal line. It may have just one (a), or it may have more (b).

(from Thomas'Calculus)

Lemma (Rolle's Lemma)

Suppose that $y = f(x)$ is continuous over the closed interval $[a, b]$ and differentiable at every point of its interior (a, b). If $f(a) = f(b)$, then there is at least one number c in (a, b) at which $f'(c) = 0$.



(a) Discontinuous at an endpoint of $[a, b]$

(b) Discontinuous at an interior point of $[a, b]$

(c) Continuous on $[a, b]$ but not differentiable at an interior point

FIGURE There may be no horizontal tangent if the hypotheses of Rolle's Theorem do not hold.

(from Thomas'Calculus)

Rolle's Theorem may be combined with the Intermediate Value Theorem to show when there is only one real solution of an equation $f(x) = 0$, as we illustrate in the next example.

EXAMPLE Show that the equation

$$x^3 + 3x + 1 = 0$$

has exactly one real solution.

Solution We define the continuous function

$$f(x) = x^3 + 3x + 1.$$

Since $f(-1) = -3$ and $f(0) = 1$, the Intermediate Value Theorem tells us that the graph of f crosses the x-axis somewhere in the open interval $(-1, 0)$. Now, if there were even two points $x = a$ and $x = b$ where $f(x)$ was zero, Rolle's Theorem would guarantee the existence of a point $x = c$ in between them where f' was zero. However, the derivative

$$f'(x) = 3x^2 + 3$$

is never zero (because it is always positive). Therefore, f has no more than one zero. ■

(from Thomas'Calculus)

Theorem (Lagrange's Mean value theorem)

Suppose that $y = f(x)$ is continuous over the closed interval $[a, b]$ and differentiable at every point of its interior (a, b). Then there exists a point c in (a, b) at which $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

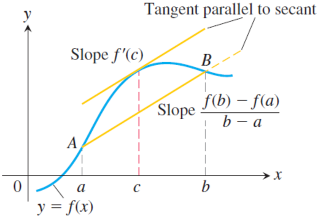


FIGURE Geometrically, the Mean Value Theorem says that somewhere between a and b the curve has at least one tangent parallel to the secant joining A and B.

(from Thomas'Calculus)

Theorem (Lagrange's Mean value theorem)

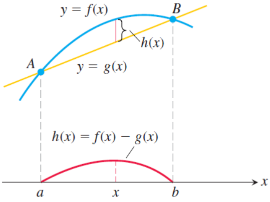
Suppose that $y = f(x)$ is continuous over the closed interval $[a, b]$ and differentiable at every point of its interior (a, b). Then there exists a point c in (a, b) at which $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

Proof We picture the graph of f and draw a line through the points $A(a, f(a))$ and $B(b, f(b))$. (See Figure) The secant line is the graph of the function

$$g(x) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a) \qquad (2)$$

(point-slope equation). The vertical difference between the graphs of f and g at x is

$$h(x) = f(x) - g(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a). \qquad (3)$$



(from Thomas'Calculus)

Theorem (Lagrange's Mean value theorem)

Suppose that $y = f(x)$ is continuous over the closed interval $[a, b]$ and differentiable at every point of its interior (a, b). Then there exists a point c in (a, b) at which $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

The function h satisfies the hypotheses of Rolle's Theorem on $[a, b]$. It is continuous on $[a, b]$ and differentiable on (a, b) because both f and g are. Also, $h(a) = h(b) = 0$ because the graphs of f and g both pass through A and B. Therefore $h'(c) = 0$ at some point $c \in (a, b)$. This is the point we want for Equation (1) in the theorem.

To verify Equation (1), we differentiate both sides of Equation (3) with respect to x and then set $x = c$:

$$h'(x) = f'(x) - \frac{f(b) - f(a)}{b - a} \qquad \text{Derivative of Eq. (3) . . .}$$

$$h'(c) = f'(c) - \frac{f(b) - f(a)}{b - a} \qquad \text{. . . with } x = c$$

$$0 = f'(c) - \frac{f(b) - f(a)}{b - a} \qquad h'(c) = 0$$

$$f'(c) = \frac{f(b) - f(a)}{b - a}, \qquad \text{Rearranged}$$

which is what we set out to prove. ∎

(from Thomas'Calculus)

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EXAMPLE The function $f(x) = x^2$ is continuous for $0 \le x \le 2$ and differentiable for $0 < x < 2$. Since $f(0) = 0$ and $f(2) = 4$, the Mean Value Theorem says that at some point c in the interval, the derivative $f'(x) = 2x$ must have the value $(4 - 0)/(2 - 0) = 2$. In this case we can identify c by solving the equation $2c = 2$ to get $c = 1$. However, it is not always easy to find c algebraically, even though we know it always exists. ■

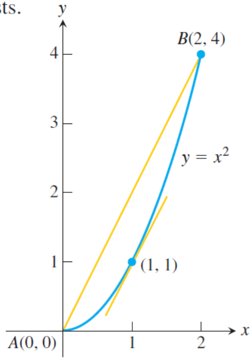

FIGURE As we find in Example 2, $c = 1$ is where the tangent is parallel to the secant line.

(from Thomas'Calculus)

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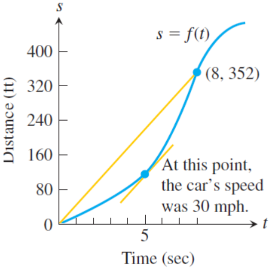
A Physical Interpretation

We can think of the number $(f(b) - f(a))/(b - a)$ as the average change in f over $[a, b]$ and $f'(c)$ as an instantaneous change. Then the Mean Value Theorem says that at some interior point the instantaneous change must equal the average change over the entire interval.

EXAMPLE If a car accelerating from zero takes 8 sec to go 352 ft, its average velocity for the 8-sec interval is $352/8 = 44$ ft/sec. The Mean Value Theorem says that at some point during the acceleration the speedometer must read exactly 30 mph (44 ft/sec) ■



(from Thomas'Calculus)

Corollary 1 from Lagrange's Mean value theorem)

If $f'(x) = 0$ at each $x \in (a, b)$, then $f(x) = C$ for all $x \in (a, b)$, where C is a constant.

Corollary 2 from Lagrange's Mean value theorem)

If $f'(x) = g'(x)$ at each $x \in (a, b)$, then there exists a constant C such that $f(x) = g(x) + C$ for all $x \in (a, b)$.

Corollary 3 from Lagrange's Mean value theorem)

Taking $x = a$, $\Delta x = b - a$, we obtain
$f(x + \Delta x) - f(x) = f'(x + \theta \Delta x)\Delta x$, $0 < \theta < 1$, or

$$f(x + \Delta x) - f(x) = f'(x)\Delta x.$$

Example

Proof that $\ln bx = \ln b + \ln x$ The argument starts by observing that $\ln bx$ and $\ln x$ have the same derivative:

$$\frac{d}{dx}\ln(bx) = \frac{b}{bx} = \frac{1}{x} = \frac{d}{dx}\ln x.$$

According to Corollary 2 of the Mean Value Theorem, then, the functions must differ by a constant, which means that

$$\ln bx = \ln x + C$$

for some C.

Since this last equation holds for all positive values of x, it must hold for $x = 1$. Hence,

$$\begin{aligned} \ln(b \cdot 1) &= \ln 1 + C \\ \ln b &= 0 + C \qquad \ln 1 = 0 \\ C &= \ln b. \end{aligned}$$

By substituting, we conclude

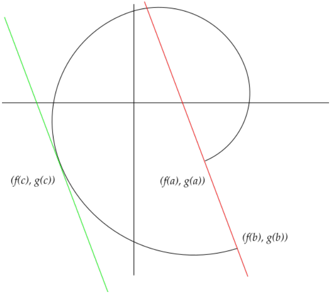
$$\ln bx = \ln b + \ln x. \qquad \blacksquare$$

(from Thomas'Calculus)

Theorem (Cauchy's Mean value theorem)

Suppose that the functions f and g are continuous over the closed interval $[a, b]$ and differentiable at every point of its interior (a, b). Then there exists a point c in (a, b) at which

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$



(from wikipedia.org)