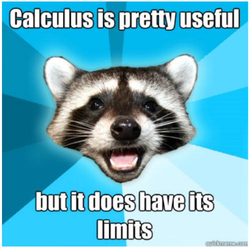
2. Sequences and series

2.2. Limit of a sequence



Content:

Convergent and divergent sequences

Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers.

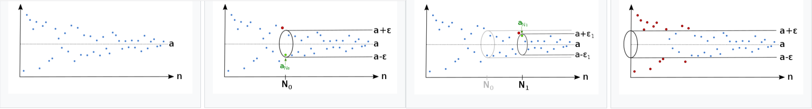
Definition 1 of a limit

A number $a \in \mathbb{R}$ is the **limit of** $\{a_n\}_{n\in\mathbb{N}}$, $a = \lim\limits_{n\to\infty} a_n$, if for any $\varepsilon > 0$ there exists an $n_\varepsilon \in \mathbb{N}$ such that, for all $n > n_\varepsilon$, $|a_n - a| < \varepsilon$.
In this case, we say that $\{a_n\}_{n\in\mathbb{N}}$ **converges to** a as $n \to \infty$, $a_n \overset{n\to\infty}{\to} a$.

Symbolically:

$$a = \lim_{n\to\infty} a_n \iff \forall\varepsilon > 0 \exists n_\varepsilon \in \mathbb{N} : n > n_\varepsilon \Rightarrow |a_n - a| < \varepsilon.$$

If a sequence has a limit, then it called **convergent**. A sequence that does not converge is said to be **divergent**.



Example of a sequence which converges to the limit a.

Regardless which $\varepsilon > 0$ we have, there is an index N_0, so that the sequence lies afterwards completely in the epsilon tube

There is also for a smaller $\varepsilon_1 > 0$ an index N_1, so that the sequence is afterwards inside the epsilon tube

For each $\varepsilon > 0$ there are only finitely many sequence members outside the epsilon tube.

$$a = \lim_{n\to\infty} a_n \iff \forall\varepsilon > 0 \exists n_\varepsilon \in \mathbb{N} : n > n_\varepsilon \Rightarrow |a_n - a| < \varepsilon.$$

Examples:

The sequence $\left\{\frac{1}{n}\right\}_{n\in\mathbb{N}}$ converges to 0.

Proof: Let us take an arbitrary $\varepsilon > 0$ and find n_ε such that $\left|\frac{1}{n}\right| < \varepsilon$ for all $n > n_\varepsilon$. Obviously, if $n_\varepsilon > \frac{1}{\varepsilon}$ then, for all $n < n_\varepsilon$, $\left|\frac{1}{n}\right| < \left|\frac{1}{n_\varepsilon}\right| < \varepsilon$.

$$\lim_{n\to\infty} \frac{n}{n+1} = 1.$$

Proof: Let us take an arbitrary $\varepsilon > 0$ and find n_ε such that $\left|\frac{n}{n+1} - 1\right| < \varepsilon$ for all $n > n_\varepsilon$.
$\left|\frac{n}{n+1} - 1\right| = \left|\frac{n}{n+1} - 1\right| = \left|\frac{1}{n+1}\right| < \left|\frac{1}{n_\varepsilon + 1}\right|$. Solving inequality $\frac{1}{n_\varepsilon + 1} < \varepsilon$ w.r.t. n_ε we conclude that, for any $n_\varepsilon > \frac{1}{\varepsilon} - 1$, $\left|\frac{n}{n+1} - 1\right| < \varepsilon$ for all $n > n_\varepsilon$.

$$\lim_{n\to\infty} (\sqrt{n+1} - \sqrt{n}) = 0.$$

Proof: $\forall\varepsilon > 0$, let $n_\varepsilon > \varepsilon^{-2}$. Observe that $\forall n \in \mathbb{N}$
$|\sqrt{n+1} - \sqrt{n}| = \frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}}$. Then $\forall n > n_\varepsilon$,
$|\sqrt{n+1} - \sqrt{n}| = \frac{1}{\sqrt{n+1} + \sqrt{n}} < \frac{1}{\sqrt{n_\varepsilon}} < \varepsilon$.

Convergent and divergent sequences

Definition 2 of a limit

Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers. A number $a \in \mathbb{R}$ is the limit of $\{a_n\}_{n\in\mathbb{N}}$ if $\forall\varepsilon > 0 \exists n_\varepsilon \in \mathbb{N} : n > n_\varepsilon \Rightarrow a_n \in U(a,\varepsilon)$.

Lemma

Definition 1 and Definition 2 of a limit are equivalent.

Lemma

A sequence of real numbers $\{a_n\}_{n\in\mathbb{N}}$ is divergent iff any $a \in \mathbb{R}$ has a ε-neighborhood $U(a,\varepsilon)$ such that the set $\mathbb{R} \setminus U(a,\varepsilon)$ contains infinitely many elements a_n.

Example: $\{(-1)^n\}_{n\in\mathbb{N}}$ diverges. Indeed, let $\varepsilon = 1/2$. Then $\forall a \in \mathbb{R}$, $U(a,\varepsilon)$ contains at most one of 1 and -1. Then either $a_{2n} \in \mathbb{R} \setminus U(a,\varepsilon) \forall n \in \mathbb{N}$, or $a_{2n-1} \in \mathbb{R} \setminus U(a,\varepsilon) \forall n \in \mathbb{N}$.

Infinite limits

Definition

A sequence of real numbers $\{a_n\}_{n\in\mathbb{N}}$

- is **infinitely large**, $\{a_n\}_{n\in\mathbb{N}} \overset{n\to\infty}{\to} \infty$ (or $\lim\limits_{n\to\infty} a_n = \infty$), if for any $\varepsilon > 0$ there is an $n_\varepsilon \in \mathbb{N}$ such that, for all $n > n_\varepsilon$, $|a_n| > \dfrac{1}{\varepsilon}$;
- **tends** (or **diverges**) **to plus infinity**, $\{a_n\}_{n\in\mathbb{N}} \overset{n\to\infty}{\to} +\infty$ (or $\lim\limits_{n\to\infty} a_n = +\infty$) if for any $\varepsilon > 0$ there is an $n_\varepsilon \in \mathbb{N}$ such that, for all $n > n_\varepsilon$, $a_n > \dfrac{1}{\varepsilon}$;
- $\{a_n\}_{n\in\mathbb{N}}$ **tends** (or **diverges**) **to minus infinity**, $\{a_n\}_{n\in\mathbb{N}} \overset{n\to\infty}{\to} -\infty$ (or $\lim\limits_{n\to\infty} a_n = -\infty$) if for any $\varepsilon > 0$ there is an $n_\varepsilon \in \mathbb{N}$ such that, for all $n > n_\varepsilon$, $a_n < -\dfrac{1}{\varepsilon}$.

Not every divergent sequence tends to infinity!

Example: $\lim\limits_{n\to\infty} n = +\infty$ (by the Archimedean property); $\lim\limits_{n\to\infty} \ln n = +\infty$; $\{(-1)^n\}_{n\in\mathbb{N}}$ diverges, but $\lim\limits_{n\to\infty} (-1)^n \neq \infty$.

Infinitesimal sequences

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Definition

A sequence of real numbers $\{a_n\}_{n\in\mathbb{N}}$ is called an **infinitesimal sequence**, if $\lim_{n\to\infty} a_n = 0$.

Examples: $\left\{\frac{1}{n}\right\}_{n\in\mathbb{N}}$, $\left\{\frac{1}{n}\sin\frac{\pi n}{2}\right\}_{n\in\mathbb{N}}$.

Lemma (properties of infinitesimal sequences)

Let $\{a_n\}_{n\in\mathbb{N}}$, $\{b_n\}_{n\in\mathbb{N}}$ be infinitesimal sequences of real numbers.

- A linear combination of infinitesimal sequences is an infinitesimal sequence: $\forall c_1, c_2 \in \mathbb{R}$, the sequence $\{c_1 a_n + c_2 a_n\}_{n\in\mathbb{N}}$ is infinitesimal.
- The product of an infinitesimal sequence and a bounded sequence is an infinitesimal sequence: $\{c_n\}_{n\in\mathbb{N}}$, the sequence $\{a_n c_n\}_{n\in\mathbb{N}}$ is infinitesimal.
- $\lim_{n\to\infty} a_n{=}a \in \mathbb{R} \Leftrightarrow a_n = a + \alpha_n \forall n \in \mathbb{N}$, where $\alpha_n \in \mathbb{R} : \lim_{n\to\infty} \alpha_n = 0$.

Theorem (uniqueness of limits)

A sequence of real numbers $\{a_n\}_{n\in\mathbb{N}}$ can have at most one limit (finite or infinite).

Proof. Let us prove this lemma by contradiction, i.e. assume that there exists a sequence of real numbers $\{a_n\}_{n\in\mathbb{N}}$ and $a, b \in \overline{\mathbb{R}}$, $a \neq b$, such that $\lim\limits_{n\to\infty} a_n = a$, $\lim\limits_{n\to\infty} a_n = b$.

Remainder: Lemma

For any $a, b \in \bar{\mathbb{R}}$ there exist $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that $U(a, \varepsilon_1) \cap U(b, \varepsilon_2) = \emptyset$.

Let us take $\varepsilon_1, \varepsilon_2 > 0$: $U(a, \varepsilon_1) \cap U(b, \varepsilon_2) = \emptyset$. By Def.2, $\exists n_1, n_2 \in \mathbb{N}$:

- $\forall n > n_1, x_n \in U(a, \varepsilon_1)$;
- $\forall n > n_2, x_n \in U(a, \varepsilon_2)$.

Therefore, $\forall n > \max\{n_1, n_2\}, x_n \in U(a, \varepsilon_1) \cap U(a, \varepsilon_2)$, which contradicts to $U(a, \varepsilon_1) \cap U(b, \varepsilon_2) = \emptyset$. ■

Lemma (properties of limits)

Let $\{a_n\}_{n\in\mathbb{N}}$, $\{b_n\}_{n\in\mathbb{N}}$ be sequences of real numbers, $\lim\limits_{n\to\infty} a_n = a \in \mathbb{R}$, $\lim\limits_{n\to\infty} b_n = b \in \mathbb{R}$.

1. $\forall c_1, c_2 \in \mathbb{R}$, $\lim\limits_{n\to\infty}(c_1 a_n + c_2 b_n) = c_1 a + c_2 b$;
2. $\lim\limits_{n\to\infty}(a_n \cdot b_n) = a \cdot b$; $\forall k \in \mathbb{R}$, $\lim\limits_{n\to\infty}(a_n)^k = a^k$;
3. $\lim\limits_{n\to\infty} a_n/b_n = a/b$ provided that $b_n \neq 0$ for all $n \in \mathbb{N}$ and $b \neq 0$;
4. $\lim\limits_{n\to\infty} |a_n| = |a|$.

Remark

True also if $\lim\limits_{n\to\infty} a_n = \pm\infty$ or $\lim\limits_{n\to\infty} b_n = \pm\infty$ under the following rules:

$$a \pm \infty = \pm\infty; a \cdot \pm\infty = \pm\infty, \forall a < 0;\; a \cdot \pm\infty = \mp\infty, \forall a < 0; \frac{a}{\pm\infty} = 0;$$

$$+\infty + \infty = +\infty; -\infty - \infty = -\infty; \pm\infty \cdot \pm\infty = +\infty; \pm\infty \cdot \mp\infty = -\infty;$$

Important: "$\infty - \infty$", "$\pm\infty \cdot 0$", "$\frac{\pm\infty}{\pm\infty}$", "$\infty^0$" are undefined!

Examples:

1)

$$\lim_{n\to\infty} \frac{n^2+4n-5}{2n^2+3n-100} = \lim_{n\to\infty} \frac{n^2\left(1+\frac{4}{n}-\frac{5}{n^2}\right)}{n^2\left(2+\frac{3}{n}-\frac{100}{n^2}\right)} \overset{n\to\infty}{\to} \frac{1+0-0}{2+0-0} = \frac{1}{2}.$$

2) $$\lim_{n\to\infty} \frac{5n^6-1}{10n^4-n^2} = \lim_{n\to\infty} \frac{n^4\left(5n^2-\frac{1}{n^4}\right)}{n^4\left(10-\frac{1}{n^2}\right)} \overset{n\to\infty}{\to} \frac{5n^2-\frac{1}{n^4}}{10-\frac{1}{n^2}} \overset{"\frac{+\infty}{10}"}{=} +\infty.$$

3) $a_n = n+1,\ b_n = n,\ \forall n \in \mathbb{N}$; $\lim\limits_{n\to\infty} a_n = +\infty,\ \lim\limits_{n\to\infty} b_n = +\infty,$

$\lim\limits_{n\to\infty}(a_n - b_n) = \lim\limits_{n\to\infty} 1 = 1.$

4) $a_n = 2n,\ b_n = n,\ \forall n \in \mathbb{N}$; $\lim\limits_{n\to\infty} a_n = +\infty,\ \lim\limits_{n\to\infty} b_n = +\infty,$

$\lim\limits_{n\to\infty}(a_n - b_n) = \lim\limits_{n\to\infty} n = +\infty.$

5) $a_n = n + \sin\frac{\pi n}{2},\ b_n = n,\ \forall n \in \mathbb{N}$; $\lim\limits_{n\to\infty} a_n = +\infty,\ \lim\limits_{n\to\infty} b_n = +\infty,$

$\lim\limits_{n\to\infty}(a_n - b_n) = \lim\limits_{n\to\infty} \sin\frac{\pi n}{2}$ – does not exist.

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Lemma (Limit passage in inequalities)

Let $\{a_n\}_{n\in\mathbb{N}}$, $\{b_n\}_{n\in\mathbb{N}}$ be sequences of real numbers.

- If $a_n = a$ for all $n \in \mathbb{N}$ (i.e. $\{a_n\}_{n\in\mathbb{N}}$ is a constant sequence), then $\lim\limits_{n\to\infty} a_n = a$.
- If $\lim\limits_{n\to\infty} a_n = a \in \overline{\mathbb{R}}$, $\lim\limits_{n\to\infty} b_n = b \in \overline{\mathbb{R}}$, and $a < b$ (resp., $a > b$), then there exists an $N \in \mathbb{N}$ such that, for all $n < N$, $a_n < b_n$ (resp., $a_n > b_n$).
  Corollary: if $a, b \in \overline{\mathbb{R}}$, $a < b$ (resp., $a > b$), and $\lim\limits_{n\to\infty} a_n = a$, then there exists an $N \in \mathbb{N}$ such that, for all $n < N$, $a_n < b$ (resp.,, $a_n > b$).
- If $\lim\limits_{n\to\infty} a_n = a \in \overline{\mathbb{R}}$, $\lim\limits_{n\to\infty} b_n = b \in \overline{\mathbb{R}}$, and $a_n \leq b_n$ for all $n \in \mathbb{N}$, then $a \leq b$.

Important: $a_n < b_n \not\Rightarrow \lim\limits_{n\to\infty} a_n < \lim\limits_{n\to\infty} b_n$! E.g., $a_n = \frac{1}{n}$, $b_n = 0$, $\forall n \in \mathbb{N}$.

Sandwich theorem

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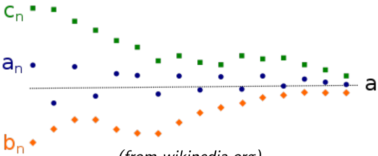
Theorem (Sandwich or (squeeze) theorem)

If $a_n \in \mathbb{R}$, $b_n \in \mathbb{R}$, $c_n \in \mathbb{R}$, $a_n \leq b_n \leq c_n$ for all $n \in \mathbb{N}$, and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = a \in \overline{\mathbb{R}}$$

then $\lim\limits_{n\to\infty} b_n = a$.



(from wikipedia.org)

Corollary

Let $a_n \in \mathbb{R}$, $b_n \in \mathbb{R}$, $a_n \leq b_n$ for all $n \in \mathbb{N}$.

- If $\lim\limits_{n\to\infty} a_n = +\infty$ then $\lim\limits_{n\to\infty} b_n = +\infty$.
- If $\lim\limits_{n\to\infty} a_n = -\infty$ then $\lim\limits_{n\to\infty} a_n = -\infty$.

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Theorem (Sandwich or (squeeze) theorem)

If $a_n \in \mathbb{R}$, $b_n \in \mathbb{R}$, $c_n \in \mathbb{R}$, $a_n \leq b_n \leq c_n$ for all $n \in \mathbb{N}$, and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = a \in \overline{\mathbb{R}}$$

then $\lim\limits_{n\to\infty} b_n = a$.

Proof:

Let us take any $\varepsilon > 0$. By Def. 1 of the limit,

$a = \lim\limits_{n\to\infty} a_n \Rightarrow \exists n_{\varepsilon,1} \in \mathbb{N} : \forall n > n_{\varepsilon,1},\ |a_n - a| < \varepsilon;$

$a = \lim\limits_{n\to\infty} c_n \Rightarrow \exists n_{\varepsilon,2} \in \mathbb{N} : \forall n > n_{\varepsilon,2},\ |c_n - a| < \varepsilon.$

Let $n_\varepsilon = \max\{n_{\varepsilon/2,1}, n_{\varepsilon/2,2}\}$. Then $\forall n > n_\varepsilon$ $b_n - a \leq c_n - a \leq |c_n - a| < \varepsilon;$

$b_n - a \geq a_n - a \geq -|a_n - a| > -\varepsilon.$

Therefore, $-\varepsilon < b_n - a < \varepsilon \iff |b_n - a| \leq \varepsilon$.

Example:

1) $\lim\limits_{n\to\infty} \dfrac{\cos n}{n} = 0$ because $-\dfrac{1}{n} \leq \dfrac{\cos n}{n} \leq \dfrac{1}{n}$.

Some remarks

Lemma

Changing a finite number of terms in a sequence has no effect on the convergence, divergence or the value of limit if it exists.

Corollary

Sandwich theorem is also valid if $a_n \leq b_n \leq c_n$ for all $n \geq N$ with some $N \in \mathbb{N}$.

Common limits

For any $c > 0$,

$$\lim_{n\to\infty} \sqrt[n]{c} = 1.$$

Proof: For $c = 1$, the sequence is constant and converges to 1.
Let $c > 1$. Let us define $b_n := \sqrt[n]{c} - 1$ and prove that $\lim_{n\to\infty} b_n = 0$. For all $n \in \mathbb{N}$, $b_n > 0$ and $c = (1 + b_n)^n$. From the binomial expansion,

$$\begin{aligned} c = (1+b_n)^n &= \binom{n}{0} + \binom{n}{1} b_n + \binom{n}{2} b_n^2 + \cdots + \binom{n}{n} b_n^n \\ &= 1 + nb_n + \underbrace{\binom{n}{2} b_n^2 + \cdots + \binom{n}{n} b_n^n}_{>0} . \end{aligned}$$

Thus, $c > nb_n > 0$, and

$$0 < b_n < \frac{c}{n} \underset{n\to\infty}{\to} 0.$$

By the Sandwich theorem, $\lim_{n\to\infty} b_n = 0$. Therefore, $\lim_{n\to\infty} \sqrt[n]{c} = \lim_{n\to\infty} b_n + 1 = 1$.
In case $0 < c < 1$, we have $1/c > 1 \Rightarrow \lim_{n\to\infty} \sqrt[n]{c} = \lim_{n\to\infty} \frac{1}{\sqrt[n]{c}} = \frac{1}{\lim_{n\to\infty} \sqrt[n]{c}} = 1.$

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1. For any $p > 1$, $\lim\limits_{n\to\infty} p^n = +\infty$.
2. For any $p \in (-1, 1)$, $\lim\limits_{n\to\infty} p^n = 0$.

Proof:

1) From the Bernoulli inequality,

$$p^n = (1 + (p-1))^n \geq 1 + n(p-1) \overset{n\to\infty}{\to} +\infty \Rightarrow \lim_{n\to\infty} p^n = +\infty.$$

2) For $0 < |p| < 1$, denote $q = \dfrac{1}{p} > 1$. Then

$$\lim_{n\to\infty} p^n = \lim_{n\to\infty} \frac{1}{q^n} = \frac{1}{\lim\limits_{n\to\infty} q^n} = 0.$$

For any $p < -1$, the limit does not exists.

Common limits

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For any $c \in \mathbb{R}$,

$$\lim_{n\to\infty} \frac{c^n}{n!} = 0.$$

Proof:

Let us fix $n_c \in \mathbb{N}$: $|c| < n_c$. Then $\left|\frac{c}{2n_c}\right| < \frac{n_c}{2n_c} = \frac{1}{2}$. Then for any $n > 2n_c$,

$$\begin{aligned}
0 \le \left|\frac{c^n}{n!}\right| &= \left|\frac{c^{2n_c}}{(2n_c)!} \cdot \frac{c}{2n_c+1} \cdot \frac{c}{2n_c+2} \cdot \ldots \cdot \frac{c}{n}\right| \\
&< \left|\frac{c^{2n_c}}{(2n_c)!} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \ldots \cdot \frac{1}{2}\right| = \left|\frac{c^{2n_c}}{(2n_c)!} \cdot \frac{1}{2^{n-2n_c}}\right| \\
&= \underbrace{\frac{2^{n_c}|c|^{2n_c}}{(2n_c)!}}_{\text{finite number}} \cdot \underbrace{\frac{1}{2^n}}_{\overset{n\to\infty}{\to} 0} \overset{n\to\infty}{\to} 0.
\end{aligned}$$

By the Sandwich theorem, $\lim_{n\to\infty} \frac{|c|^n}{n!} = 0$. Since $-x \le |x| \le x$, the Sandwich theorem implies $\lim_{n\to\infty} \frac{c^n}{n!} = 0$.

$$\lim_{n\to\infty} \sqrt[n]{n!} = +\infty.$$

Proof:

From the previous example, $\lim_{n\to\infty} \dfrac{c^n}{n!} = 0$. Therefore, for any $c > 0$ there exists an $n_c \in \mathbb{N}$ such that, for all $n > n_a$,

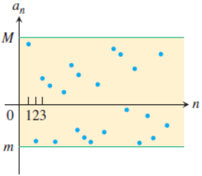
$$\frac{c^n}{n!} < 1.$$

Therefore, for all $n > n_c$, $\sqrt[n]{n!} > c$. Since $c > 0$ is arbitrary, this means that $\lim_{n\to\infty} \sqrt[n]{n!} = +\infty$.

Theorem

If a sequence of real numbers has a finite limit, then it is bounded.

In general, the converse is not true!

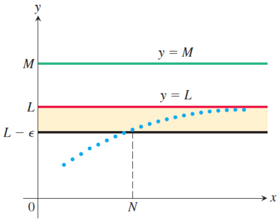


Some bounded sequences bounce around between their bounds and fail to converge to any limiting value.

(from Thomas' Calculus)

Theorem (Weierstrass theorem)

- Any monotonically increasing sequence of real numbers $\{a_n\}_{n\in\mathbb{N}}$ has a finite limit if it is bounded from above, and tends to plus infinity, if it is unbounded; $\lim\limits_{n\to\infty} a_n = \sup\limits_{n\in\mathbb{N}}\{a_n\}$.
- Any monotonically decreasing sequence of real numbers $\{a_n\}_{n\in\mathbb{N}}$ has a finite limit if it is bounded from below, and tends to minus infinity, if it is unbounded; $\lim\limits_{n\to\infty} a_n = \inf\limits_{n\in\mathbb{N}}\{a_n\}$.



If the terms of a nondecreasing sequence have an upper bound M, they have a limit $L \leq M$.

Monotonic sequence theorems

Theorem (Weierstrass theorem)

- Any monotonically increasing sequence of real numbers $\{a_n\}_{n\in\mathbb{N}}$ has a finite limit if it is bounded from above, and tends to plus infinity, if it is unbounded; $\lim\limits_{n\to\infty} a_n = \sup\limits_{n\in\mathbb{N}}\{a_n\}$.
- Any monotonically decreasing sequence of real numbers $\{a_n\}_{n\in\mathbb{N}}$ has a finite limit if it is bounded from below, and tends to minus infinity, if it is unbounded; $\lim\limits_{n\to\infty} a_n = \inf\limits_{n\in\mathbb{N}}\{a_n\}$.

Remark

Thus, any monotone sequence of real numbers has a finite limit if the sequence is bounded, and an infinite limit, if the sequence is unbounded.

Corollary

A monotonically increasing (resp., decreasing) sequence of real numbers is convergent if and only if it is bounded from above (resp., from below).

Examples:

1) The sequence $\{\arctan n\}_{n\in\mathbb{N}}$ is monotone and bounded and hence converges.

2) Let $k>0$, $a_0>0$, and $a_n=\frac{1}{2}\left(a_{n-1}+\frac{k}{a_{n-1}}\right)$, $\forall n\in\mathbb{N}$. Prove that $\lim\limits_{n\to\infty} a_n=\sqrt{k}$.

By mathematical induction, $a_n>0\ \forall n\in\mathbb{N}$. Moreover, $a_n\geq\sqrt{k}\ \forall n\in\mathbb{N}$. Indeed, note that, for $t>0$, $(t-1)^2\geq 0\Rightarrow t+\frac{1}{t}\geq 2$. Taking $t=\frac{a_n}{\sqrt{k}}$, we get

$$a_{n+1}=\frac{1}{2}\left(a_n+\frac{k}{a_n}\right)=\frac{\sqrt{k}}{2}\left(\frac{a_n}{\sqrt{k}}+\frac{\sqrt{k}}{a_n}\right)\geq\frac{\sqrt{k}}{2}2=\sqrt{k},\forall n\in\mathbb{N}.$$

Let us prove that $\{a_n\}_{n\in\mathbb{N}}$ monotonically decreases. Since $a_n\geq\sqrt{k}\ \forall n\in\mathbb{N}$,

$a_{n+1}=\frac{1}{2}\left(a_n+\frac{k}{a_n}\right)=\frac{a_n}{2}\left(1+\frac{k}{a_n^2}\right)\leq\frac{a_n}{2}\left(1+\frac{k}{k}\right)=a_n,\ \forall n\in\mathbb{N}.$

Thus, the sequence $\{a_n\}_{n\in\mathbb{N}}$ is bounded from below and monotonically decreasing, therefore, it has a finite limit (by the Weierstrass theorem). How to calculate it? Denote $\lim\limits_{n\to\infty} a_n=x$. Then $\lim\limits_{n\to\infty} a_{n-1}=x$, and

$$x=\lim_{n\to\infty} a_n=\lim_{n\to\infty}\left(\frac{1}{2}\left(a_{n-1}+\frac{k}{a_{n-1}}\right)\right)=\frac{1}{2}\left(x+\frac{k}{x}\right).$$

Solving the above equation for $x\geq 0$ (because of $a_n\geq 0\forall n\in\mathbb{N}$), we get $x=\sqrt{k}$.

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Examples: Let $a_n = \left(1+\frac{1}{n}\right)^n$, $n \in \mathbb{N}$. Does $\{a_n\}_{n\in\mathbb{N}}$ converge?

From the binomial expansion,

$$a_n = \left(1+\frac{1}{n}\right)^n = 1 + n\cdot\frac{1}{n} + \frac{n(n-1)}{2!}\frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!}\frac{1}{n^3}$$
$$+\cdots+\frac{n(n-1)(n-2)\ldots(n-k+1)}{k!}\frac{1}{n^k}+\cdots+\frac{n(n-1)\ldots 1}{n!}\frac{1}{n^n}$$
$$=1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+\cdots+\frac{1}{k!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{k-1}{n}\right)$$
$$+\cdots+\frac{1}{n!}\left(1-\frac{1}{n}\right)\ldots\left(1-\frac{n-1}{n}\right).$$ Then

$$a_n < 1+1+\frac{1}{2!}\left(1-\frac{1}{n+1}\right)+\frac{1}{3!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)+\cdots+\frac{1}{k!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{k-1}{n}\right)$$
$$+\cdots+\frac{1}{n!}\left(1-\frac{1}{n+1}\right)\ldots\left(1-\frac{n-1}{n+1}\right)+\frac{1}{(n+1)!}\left(1-\frac{1}{n+1}\right)\ldots\left(1-\frac{n}{n+1}\right)=a_{n+1}.$$

Thus, the sequence $\{a_n\}_{n\in\mathbb{N}}$ strictly monotonically increases. Furthermore, observe that $1-\frac{k}{n}<1$ and $2^{n-1} = \underbrace{1\cdot 2\cdot 2\cdot\ldots\cdot 2}_{n \text{ factors}} \le 1\cdot 2\cdot 3\cdot\ldots\cdot n = n!$. Therefore, $\frac{1}{n!}\le\frac{1}{2^{n-1}}$ for all $n\in\mathbb{N}$. This yields

$$a_n < 2+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!}\le 2+\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^{n-1}} = 3+1-\frac{1}{2^{n-1}} < 3.$$

Hence, $2\le a_n < a_{n+1} < 3$ for all $n\in\mathbb{N} \Rightarrow$ the sequence $\{a_n\}_{n\in\mathbb{N}}$ is strictly monotonically increasing and bounded and thus convergent; $2 \le \lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n \le 3$.

Euler's number as a limit

Definition

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n.$$

$(e \approx 2,718281828459045)$

Corollary: alternative representations

- $e = \lim_{n\to\infty}\left(1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!}\right);$
- $e = \lim_{n\to\infty}\frac{n}{\sqrt[n]{n!}}.$

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Definition

Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers, and $\{n_k\}_{k\in\mathbb{N}}$ be a strictly monotonically increasing sequence of natural numbers. Then $\{a_{n_k}\}_{k\in\mathbb{N}}$ is a **subsequence** of $\{a_n\}_{n\in\mathbb{N}}$.

Thus, a subsequence of $\{a_n\}_{n\in\mathbb{N}}$ is a sequence that can be derived from $\{a_n\}_{n\in\mathbb{N}}$ by deleting some or no elements without changing the order of the remaining elements.

Examples:

For the sequence $\{n\}_{n\in\mathbb{N}}$, $\{2k\}_{k\in\mathbb{N}}$ is its subsequence, $\{k^2\}_{k\in\mathbb{N}}$ is its subsequence, $\left\{\frac{1}{k}\right\}$ is not its subsequence.

Lemma (properties of subsequences)

- If a sequence is bounded from above (resp., below), then any its subsequence is also bounded from above (resp., below).
- If a sequence of real numbers $\{a_n\}_{n\in\mathbb{N}}$ converges to $a \in \overline{\mathbb{R}}$, then any its subsequence also converges to a.
- If a sequence is monotonically increasing (resp., decreasing), then any its subsequence is also monotonically increasing (resp., decreasing), and, therefore, has the same limit.
- A sequence is convergent iff every its subsequence is convergent.

Theorem (Bolzano–Weierstrass theorem)

- Any bounded sequence of real numbers has a convergent subsequence.
- Any unbounded sequence of real numbers has an infinitely large subsequence. Namely, any unbounded from below sequence has a diverging to $-\infty$ subsequence, and any unbounded from above sequence has a diverging to $+\infty$ subsequence.

Subsequential limits

Definition

An element $a \in \overline{\mathbb{R}}$ is a subsequential limit of a sequence of real numbers, if there exists a subsequence converging to a.

Corollaries from Bolzano–Weierstrass theorem

Any sequence of real numbers $\{a_n\}_{n\in\mathbb{N}}$

- has at least one finite or infinite subsequential limit. If the sequence is bounded then it has at least one finite subsequential limit;
- converges if and only if it is bounded and has only one subsequential limit;
- diverges to $+\infty$ (resp., $-\infty$) if and only if it has a subsequence converging to $+\infty$ (resp., $-\infty$);
- has an accumulation point $a \in \overline{\mathbb{R}}$ iff it has a subsequence $\{a_{n_k}\}_{k\in\mathbb{N}}$ which converges to a.

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Examples: 1) $\{1/n\}_{n\in\mathbb{N}}$, unique accumulation point $a=0$, every subsequence converges to 0.

2) $\{a_n\}_{n\in\mathbb{N}}$ with $a_n=(-1)^n+\dfrac{1}{n}$ has two accumulation points: $a=-1$ and $b=1$.

There exist subsequences converging to a and to b:

$$\lim_{k\to\infty} a_{2k}=\lim_{k\to\infty}\left((-1)^{2k}+\frac{1}{2k}\right)=\lim_{k\to\infty}\left(1+\frac{1}{2k}\right)=1,$$

$$\lim_{k\to\infty} a_{2k-1}=\lim_{k\to\infty}\left((-1)^{2k-1}+\frac{1}{2k-1}\right)=\lim_{k\to\infty}\left(-1+\frac{1}{2k-1}\right)=-1.$$

3) $\{n\}_{n\in\mathbb{N}}\to\infty$, no accumulation points, no convergent subsequences.

4) $\{(-n)^n\}_{n\in\mathbb{N}}\to\infty$, no convergent subsequences;

$\{(-2k)^{2k}\}_{k\in\mathbb{N}}\to+\infty$, $\{(-(2k+1))^{2k+1}\}_{k\in\mathbb{N}}\to-\infty$.

Diverging sequences may have both subsequences converging to $+\infty$ or $-\infty$.

5) $\{(1-(-n)^n)\,n\}_{n\in\mathbb{N}}\to\infty$, no convergent subsequences;

$\{\left(1-(-2k)^{2k}\right)2k\}_{k\in\mathbb{N}}\to-\infty$,

$\{\left(1-(-(2k+1))^{2k+1}\right)(2k+1)\}_{k\in\mathbb{N}}\to+\infty$.

Limit inferior and limit superior

Definition

The least subsequential limit of a sequence $\{a_n\}_{n\in\mathbb{N}}$ is called the **limit inferior of** $\{a_n\}_{n\in\mathbb{N}}$, $\underline{\lim}_{n\to\infty} a_n$ (or $\liminf_{n\to\infty} a_n$), and the greatest subsequential limit of $\{a_n\}_{n\in\mathbb{N}}$ is called the **limit superior of** $\{a_n\}_{n\in\mathbb{N}}$, $\overline{\lim}_{n\to\infty} a_n$ (or $\limsup_{n\to\infty} a_n$).

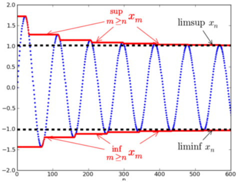
$$\underline{\lim}_{n\to\infty} a_n \neq \inf_{n\in\mathbb{N}} a_n, \quad \overline{\lim}_{n\to\infty} a_n \neq \sup_{n\in\mathbb{N}} a_n!!!$$

$$\underline{\lim}_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = \lim_{n\to\infty}\left(\inf_{m\geq n} a_m\right),$$

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty}\left(\sup_{m\geq n} a_m\right);$$

$$\overline{\lim}_{n\to\infty} a_n = \liminf_{n\to\infty} x_n = \sup_{n\geqslant 0}\inf_{m\geqslant n} x_m = \sup\{\inf\{x_m : m\geqslant n\} : n\geqslant 0\},$$

$$\limsup_{n\to\infty} x_n = \inf_{n\geqslant 0}\sup_{m\geqslant n} x_m = \inf\{\sup\{x_m : m\geqslant n\} : n\geqslant 0\}.$$



(from wikipedia.org)

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Examples:

1) $\{1/n\}_{n\in\mathbb{N}}$, $\underline{\lim}_{n\to\infty} a_n = 0$, $\overline{\lim}_{n\to\infty} a_n = 0$.

2) $\{a_n\}_{n\in\mathbb{N}}$ with $a_n = (-1)^n + \dfrac{1}{n}$ $\underline{\lim}_{n\to\infty} a_n = -1$, $\overline{\lim}_{n\to\infty} a_n = 1$.

3) $\{n\}_{n\in\mathbb{N}} \to \infty$, $\underline{\lim}_{n\to\infty} a_n = +\infty$, $\overline{\lim}_{n\to\infty} a_n = +\infty$.

4) $\{(-n)^n\}_{n\in\mathbb{N}} \to \infty$, $\underline{\lim}_{n\to\infty} a_n = -\infty$, $\overline{\lim}_{n\to\infty} a_n = +\infty$.

5) $\{(1-(-n)^n)\, n\}_{n\in\mathbb{N}} \to \infty$, $\underline{\lim}_{n\to\infty} a_n = 0$, $\overline{\lim}_{n\to\infty} a_n = +\infty$.

Statement

Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers. Then

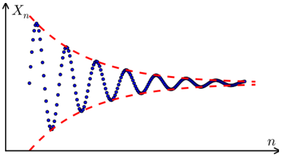
- $\lim_{n\to\infty} a_n = a \in \overline{\mathbb{R}} \Leftrightarrow \underline{\lim}_{n\to\infty} a_n = \overline{\lim}_{n\to\infty} a_n = a$;
- $\forall N \in \mathbb{N}$, $\{a_n\}_{n\in\mathbb{N}}$ converges iff $\{a_{N+k}\}_{k\in\mathbb{N}}$ converges, and $\lim_{n\to\infty} a_n = \lim_{k\to\infty} a_{N+k}$.

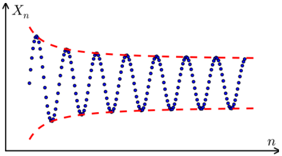
Cauchy sequence

Definition

A sequence of real numbers $\{a_n\}_{n\in\mathbb{N}}$ is called a **Cauchy sequence** if for every $\varepsilon > 0$, there is an $N_\varepsilon \in \mathbb{N}$ such that, for all $m, n > N_\varepsilon$, $|a_m - a_n| < \varepsilon$.

Thus, a Cauchy sequence is a sequence whose elements become arbitrarily close to each other as the sequence progresses.





Left: a Cauchy sequence; right: a sequence that is not Cauchy. *(from wikipedia.org)*

Remark

It is not sufficient for each term to become arbitrarily close to the preceding term! Example: $\{\sqrt{n}\}_{n\in\mathbb{N}}$. $a_{n+1} - a_n = \sqrt{n+1} - \sqrt{n} = \dfrac{1}{\sqrt{n+1}+\sqrt{n}} \overset{n\to\infty}{\to} 0$,
but $\exists\varepsilon > 0 : \forall N \in \mathbb{N}\exists n, m > N$: $|a_n - a_m| = |\sqrt{n} - \sqrt{m}| \geq \varepsilon$; e.g., $\forall\varepsilon > 0$, $n \geq (\varepsilon + \sqrt{m})^2$.

Cauchy convergence criterium

Theorem (Cauchy convergence criterium)

A sequence of real numbers converges if and only if it is a Cauchy sequence.

Remark

Cauchy convergence criterium

- states the existence of the limit, no information about its value;
- sufficiency (every Cauchy sequence converges to an element of $\mathbb{R}$) means the completeness of $\mathbb{R}$. It is possible to construct $\mathbb{R}$ just by using Cauchy sequences of rational numbers.

Corollary

A sequence of real numbers $\{a_n\}_{n\in\mathbb{N}}$ converges if and only if for any $\varepsilon > 0$ there exists an $n_\varepsilon \in \mathbb{N}$ such that, for all $n > n_\varepsilon$, $|x_n - x_{n_\varepsilon}| < \varepsilon$.

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Examples:

1) $\{a_n\}_{n\in\mathbb{N}}$, $a_1 = 1$, $a_{n+1} = 1 + \dfrac{1}{a_n}$ is a Cauchy sequence.

Indeed, note that $a_n > 1 \forall n \in \mathbb{N}$ and $a_n a_{n-1} = a_{n-1} + 1 > 2$. Then

$$|a_{n+1} - a_n| = |\frac{a_{n-1} - a_n}{a_n a_{n-1}}| \leq \frac{1}{2}|a_n - a_{n-1}| \leq \frac{1}{2^{n-1}}|a_2 - a_1|,\ \forall n \geq 2.$$

Hence

$$|a_m - a_n| \leq |a_m - a_{m-1}| + |a_{m-1} - a_{m-2}| + ... + |a_{n+1} - a_n| \leq |a_2 - a_1|\frac{\alpha^{n-1}}{1-\alpha}, \alpha = \frac{1}{2}$$

So given, $\epsilon > 0$, we can choose N such that $\frac{1}{2^{N-1}} < \frac{\epsilon}{2}$.

2) $\{a_n\}_{n\in\mathbb{N}}$, $a_1 = 1$, $a_{n+1} = \dfrac{1}{2}\left(a_n + \dfrac{2}{a_n}\right)$ is a Cauchy sequence of rational numbers which converges $\sqrt{2}$ (discussed earlier).

3) $a_1 = 0.9$, $a_2 = 0.99$, $a_3 = 0.999$, $\ldots$, i.e. $\left\{1 - \dfrac{1}{10^n}\right\}_{n\in\mathbb{N}}$.

$$\lim_{n\to\infty}\left(1 - \frac{1}{10^n}\right) = 1 - \lim_{n\to\infty}\frac{1}{10^n} = 1 - 0 = 1.$$