4. Differential calculus of functions of one real variable

4.4. Taylor and Maclaurin Series

Content:

- Definition of Taylor and Maclaurin series
- Taylor polynomials
- Taylor's theorem
- Remainder of Taylor series
- Taylor series of some common functions

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Definition

Let $f : [a,b] \to \mathbb{R}$ be infinitely differentiable on (a,b) and $x_0 \in (a,b)$. Then the **Taylor series generated by f at $x = x_0$** is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k =$$

$$f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2}(x-x_0)^2+\cdots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+\ldots.$$

The **Maclaurin series of f** is the Taylor series generated by f at $x = 0$:

$$\sum_{k=0}^{\infty} \frac{f^{(k)}0)}{k!}x^k = f(0)+f'(0)x+\frac{f''(0)}{2}x^2+\cdots+\frac{f^{(n)}(0)}{n!}x^n+\ldots.$$

EXAMPLE Find the Taylor series generated by $f(x) = 1/x$ at $a = 2$. Where, if anywhere, does the series converge to $1/x$?

Solution We need to find $f(2), f'(2), f''(2), \ldots$. Taking derivatives we get

$$f(x) = x^{-1}, \quad f'(x) = -x^{-2}, \quad f''(x) = 2!x^{-3}, \quad \cdots, \quad f^{(n)}(x) = (-1)^n n! x^{-(n+1)},$$

so that $f(2) = 2^{-1} = \frac{1}{2}, \quad f'(2) = -\frac{1}{2^2}, \quad \frac{f''(2)}{2!} = 2^{-3} = \frac{1}{2^3}, \quad \cdots, \quad \frac{f^{(n)}(2)}{n!} = \frac{(-1)^n}{2^{n+1}}.$

The Taylor series is

$$\begin{aligned} f(2) &+ f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \cdots + \frac{f^{(n)}(2)}{n!}(x-2)^n + \cdots \\ &= \frac{1}{2} - \frac{(x-2)}{2^2} + \frac{(x-2)^2}{2^3} - \cdots + (-1)^n \frac{(x-2)^n}{2^{n+1}} + \cdots. \end{aligned}$$

This is a geometric series with first term $1/2$ and ratio $r = -(x-2)/2$. It converges absolutely for $|x-2| < 2$ and its sum is

$$\frac{1/2}{1 + (x-2)/2} = \frac{1}{2 + (x-2)} = \frac{1}{x}.$$

In this example the Taylor series generated by $f(x) = 1/x$ at $a = 2$ converges to $1/x$ for $|x-2| < 2$ or $0 < x < 4$. ■

(from Thomas'Calculus)

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Definition

The **linearization** of a differentiable function $f : (a, b) \to \mathbb{R}$ at a point x_0 is the polynomial of degree one given by
$P_1(x) = f(x_0) + f'(x_0)(x - x_0)$.

The linearization allows to approximate $f(x)$ at values of x near x_0. If f has derivatives of higher order at x_0, then it has higher-order polynomial approximations as well, one for each available derivative.

Definition

Let $f : (a, b) \to \mathbb{R}$ be N times continuously differentiable on (a, b). Then for any integer $n \in \{0, 1, \dots, N\}$, the **Taylor polynomial of order n** generated by f at $x = x_0$ is the polynomial

$$P_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n.$$

EXAMPLE Find the Taylor series and the Taylor polynomials generated by $f(x) = e^x$ at $x = 0$.

Solution Since $f^{(n)}(x) = e^x$ and $f^{(n)}(0) = 1$ for every $n = 0, 1, 2, \ldots$, the Taylor series generated by f at $x = 0$ is

$$\begin{aligned} f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots \\ = 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!} + \cdots \\ = \sum_{k=0}^{\infty} \frac{x^k}{k!}. \end{aligned}$$

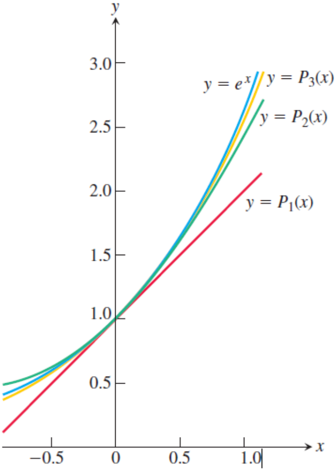

FIGURE The graph of $f(x) = e^x$ and its Taylor polynomials

$P_1(x) = 1 + x$

$P_2(x) = 1 + x + (x^2/2!)$

$P_3(x) = 1 + x + (x^2/2!) + (x^3/3!)$.

Notice the very close agreement near the center $x = 0$

(from Thomas'Calculus)

Taylor polynomials

EXAMPLE Find the Taylor series and Taylor polynomials generated by $f(x) = \cos x$ at $x = 0$.

Solution The cosine and its derivatives are

$$\begin{aligned} f(x) &= \cos x, & f'(x) &= -\sin x, \\ f''(x) &= -\cos x, & f^{(3)}(x) &= \sin x, \\ &\vdots & &\vdots \\ f^{(2n)}(x) &= (-1)^n \cos x, & f^{(2n+1)}(x) &= (-1)^{n+1} \sin x. \end{aligned}$$

At $x = 0$, the cosines are 1 and the sines are 0, so

$$f^{(2n)}(0) = (-1)^n, \qquad f^{(2n+1)}(0) = 0.$$

The Taylor series generated by f at 0 is

$$\begin{aligned} f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots \\ = 1 + 0\cdot x - \frac{x^2}{2!} + 0\cdot x^3 + \frac{x^4}{4!} + \cdots + (-1)^n\frac{x^{2n}}{(2n)!} + \cdots \\ = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}. \end{aligned}$$

Because $f^{(2n+1)}(0) = 0$, the Taylor polynomials of orders $2n$ and $2n+1$ are identical:

$$P_{2n}(x) = P_{2n+1}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n\frac{x^{2n}}{(2n)!}.$$

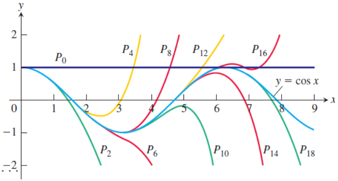

FIGURE : The polynomials

$$P_{2n}(x) = \sum_{k=0}^{n} \frac{(-1)^k x^{2k}}{(2k)!}$$

converge to $\cos x$ as $n \to \infty$. We can deduce the behavior of $\cos x$ arbitrarily far away solely from knowing the values of the cosine and its derivatives at $x = 0$ (Example 3).

(from Thomas'Calculus)

EXAMPLE It can be shown (though not easily) that

$$f(x) = \begin{cases} 0, & x = 0 \\ e^{-1/x^2}, & x \neq 0 \end{cases}$$

has derivatives of all orders at $x = 0$ and that $f^{(n)}(0) = 0$ for all n. This means that the Taylor series generated by f at $x = 0$ is

$$\begin{aligned} f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots \\ &= 0 + 0 \cdot x + 0 \cdot x^2 + \cdots + 0 \cdot x^n + \cdots \\ &= 0 + 0 + \cdots + 0 + \cdots . \end{aligned}$$

The series converges for every x (its sum is 0) but converges to $f(x)$ only at $x = 0$. That is, the Taylor series generated by $f(x)$ in this example is *not* equal to the function $f(x)$ over the entire interval of convergence. ■

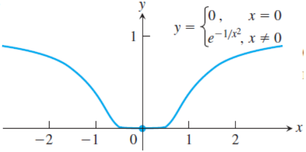

FIGURE The graph of the continuous extension of $y = e^{-1/x^2}$ is so flat at the origin that all of its derivatives there are zero

(from Thomas'Calculus)

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- For what values of x a Taylor series converges to its generating function?
- How accurately do a function's Taylor polynomials approximate the function on a given interval?

Theorem (Taylor's Formula)

Let $f : (a, b) \to \mathbb{R}$ be n times differentiable at $x_0 \in (a, b)$. Then there exists a function $h_n : \mathbb{R} \to \mathbb{R}$ such that

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + h_n(x)(x - x_0)^n \text{ and } \lim_{x \to x_0} h_n(x) = 0$$

$(h_n(x)(x - x_0)^n = o((x - x_0)^n))$.

$R_n(x) = f(x) - P_n(x)$ is the **n-th order remainder** or the **error term**.

Statement

If $R_n(x) = f(x) - P_n(x)$ as $n \to \infty$ for all $x \in (a, b)$, the Taylor series generated by f at $x = x_0$ converges to f on (a, b), that is

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k.$$

Explicit formulas for the remainder

Peano form of the remainder:

$R_n(x) = o(|x - x_0|^n)$.

Mean-value forms of the remainder:

Let $f : (a, b) \to \mathbb{R}$ be $n+1$ times differentiable on (a, b) with $f^{(k)}$ continuous on $a, b]$, for each $k = 0, 1, \ldots, n$. Then for any $x_0, x \in (a, b)$ there exist a $\xi \in (x - 0, x)$ such that
$f(x) = P_n(x) + R_n(x)$, where

- $R_n(x) = \dfrac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)^{n+1}$ – the Lagrange form of the remainder;
- $R_n(x) = \dfrac{f^{(n+1)}(\xi)}{n!}(x - \xi)^n(x - x_0)$ – the Cauchy form of the remainder;
- $R_n(x) = \dfrac{f^{(n+1)}(\xi)}{n!p}(x - \xi)^{n+1}\left(\dfrac{x - x_0}{x - \xi}\right)^p$ with some $p > 0$ – the Schlömilch–Roche form of the remainder.

The number e as a series

EXAMPLE Show that the Taylor series generated by $f(x) = e^x$ at $x = 0$ converges to $f(x)$ for every real value of x.

Solution The function has derivatives of all orders throughout the interval $I = (-\infty, \infty)$. Equations (1) and (2) with $f(x) = e^x$ and $a = 0$ give

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + R_n(x)$$

and $R_n(x) = \dfrac{e^c}{(n+1)!}x^{n+1}$ for some c between 0 and x.

Since e^x is an increasing function of x, e^c lies between $e^0 = 1$ and e^x. When x is negative, so is c, and $e^c < 1$. When x is zero, $e^x = 1$ so that $R_n(x) = 0$. When x is positive, so is c, and $e^c < e^x$. Thus, for $R_n(x)$ given as above,

$$|R_n(x)| \le \frac{|x|^{n+1}}{(n+1)!} \quad \text{when } x \le 0, \qquad e^c < 1$$

and

$$|R_n(x)| < e^x \frac{x^{n+1}}{(n+1)!} \quad \text{when } x > 0. \qquad e^c < e^x$$

Finally, because

$$\lim_{n\to\infty} \frac{x^{n+1}}{(n+1)!} = 0 \quad \text{for every } x,$$

$\lim_{n\to\infty} R_n(x) = 0$, and the series converges to e^x for every x. Thus,

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^k}{k!} + \cdots. \qquad \blacksquare$$

(from Thomas'Calculus)

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We can use the result of Example with $x = 1$ to write

$$e = 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!} + R_n(1),$$

where for some c between 0 and 1,

$$R_n(1) = e^c \frac{1}{(n+1)!} < \frac{3}{(n+1)!}. \qquad e^c < e^1 < 3$$

The Number e as a Series

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}$$

(from Thomas'Calculus)

Estimating the remainder

THEOREM —The Remainder Estimation Theorem If there is a positive constant M such that $|f^{(n+1)}(t)| \leq M$ for all t between x and a, inclusive, then the remainder term $R_n(x)$ in Taylor's Theorem satisfies the inequality

$$|R_n(x)| \leq M \frac{|x - a|^{n+1}}{(n + 1)!}.$$

If this inequality holds for every n and the other conditions of Taylor's Theorem are satisfied by f, then the series converges to $f(x)$.

(from Thomas'Calculus)

EXAMPLE Show that the Taylor series for $\sin x$ at $x = 0$ converges for all x.

Solution The function and its derivatives are

$$\begin{aligned} f(x) &= \sin x, & f'(x) &= \cos x, \\ f''(x) &= -\sin x, & f'''(x) &= -\cos x, \\ &\vdots & &\vdots \\ f^{(2k)}(x) &= (-1)^k \sin x, & f^{(2k+1)}(x) &= (-1)^k \cos x, \end{aligned}$$

so

$$f^{(2k)}(0) = 0 \quad \text{and} \quad f^{(2k+1)}(0) = (-1)^k.$$

The series has only odd-powered terms and, for $n = 2k + 1$, Taylor's Theorem gives

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^k x^{2k+1}}{(2k+1)!} + R_{2k+1}(x).$$

All the derivatives of $\sin x$ have absolute values less than or equal to 1, so we can apply the Remainder Estimation Theorem with $M = 1$ to obtain

$$|R_{2k+1}(x)| \le 1 \cdot \frac{|x|^{2k+2}}{(2k+2)!}.$$

$(|x|^{2k+2}/(2k+2)!) \to 0$ as $k \to \infty$, whatever the value of x, so $R_{2k+1}(x) \to 0$ and the Maclaurin series for $\sin x$ converges to $\sin x$ for every x. Thus,

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots.$$

(from Thomas'Calculus)

EXAMPLE Show that the Taylor series for $\cos x$ at $x = 0$ converges to $\cos x$ for every value of x.

Solution We add the remainder term to the Taylor polynomial for $\cos x$ to obtain Taylor's formula for $\cos x$ with $n = 2k$:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^k \frac{x^{2k}}{(2k)!} + R_{2k}(x).$$

Because the derivatives of the cosine have absolute value less than or equal to 1, the Remainder Estimation Theorem with $M = 1$ gives

$$|R_{2k}(x)| \leq 1 \cdot \frac{|x|^{2k+1}}{(2k+1)!}.$$

For every value of x, $R_{2k}(x) \to 0$ as $k \to \infty$. Therefore, the series converges to $\cos x$ for every value of x. Thus,

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots.$$

(from Thomas'Calculus)

Since every Taylor series is a power series, the operations of adding, subtracting, and multiplying Taylor series are all valid on the intersection of their intervals of convergence.

EXAMPLE Using known series, find the first few terms of the Taylor series for the given function using power series operations.

(a) $\frac{1}{3}(2x + x\cos x)$ **(b)** $e^x \cos x$

Solution

(a) $$\frac{1}{3}(2x + x\cos x) = \frac{2}{3}x + \frac{1}{3}x\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^k\frac{x^{2k}}{(2k)!} + \cdots\right)$$

$$= \frac{2}{3}x + \frac{1}{3}x - \frac{x^3}{3!} + \frac{x^5}{3\cdot 4!} - \cdots = x - \frac{x^3}{6} + \frac{x^5}{72} - \cdots$$

(b) $$e^x\cos x = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right)\cdot\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)$$

Multiply the first series by each term of the second series.

$$= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right) - \left(\frac{x^2}{2!} + \frac{x^3}{2!} + \frac{x^4}{2!2!} + \frac{x^5}{2!3!} + \cdots\right)$$

$$+ \left(\frac{x^4}{4!} + \frac{x^5}{4!} + \frac{x^6}{2!4!} + \cdots\right) + \cdots$$

$$= 1 + x - \frac{x^3}{3} - \frac{x^4}{6} + \cdots$$

(from Thomas'Calculus)

Examples

The Binomial Series

For $-1 < x < 1$,

$$(1+x)^m = 1 + \sum_{k=1}^{\infty} \binom{m}{k} x^k,$$

where we define

$$\binom{m}{1} = m, \qquad \binom{m}{2} = \frac{m(m-1)}{2!},$$

and

$$\binom{m}{k} = \frac{m(m-1)(m-2)\cdots(m-k+1)}{k!} \qquad \text{for } k \geq 3.$$

(from Thomas'Calculus)

EXAMPLE

$$(1+x)^{1/2} = 1 + \frac{x}{2} + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}x^2 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}x^3$$

$$+ \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{4!}x^4 + \cdots$$

$$= 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + \cdots .$$

Substitution for x gives still other approximations. For example,

$$\sqrt{1-x^2} \approx 1 - \frac{x^2}{2} - \frac{x^4}{8} \qquad \text{for } |x^2| \text{ small}$$

$$\sqrt{1-\frac{1}{x}} \approx 1 - \frac{1}{2x} - \frac{1}{8x^2} \qquad \text{for } \left|\frac{1}{x}\right| \text{ small, that is, } |x| \text{ large.}$$

(from Thomas'Calculus)

Evaluating Indeterminate Forms

We can sometimes evaluate indeterminate forms by expressing the functions involved as Taylor series.

EXAMPLE Evaluate $\lim_{x \to 1} \dfrac{\ln x}{x - 1}$.

Solution We represent $\ln x$ as a Taylor series in powers of $x - 1$. This can be accomplished by calculating the Taylor series generated by $\ln x$ at $x = 1$ directly or by replacing x by $x - 1$ in the series for $\ln(1 + x)$ Either way, we obtain

$$\ln x = (x - 1) - \frac{1}{2}(x - 1)^2 + \cdots,$$

from which we find that

$$\lim_{x \to 1} \frac{\ln x}{x - 1} = \lim_{x \to 1} \left(1 - \frac{1}{2}(x - 1) + \cdots\right) = 1.$$

Of course, this particular limit can be evaluated using l'Hôpital's Rule just as well. ■

(from Thomas'Calculus)

EXAMPLE Evaluate $\lim_{x\to 0} \dfrac{\sin x - \tan x}{x^3}$.

Solution The Taylor series for $\sin x$ and $\tan x$, to terms in x^5, are

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots, \qquad \tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots.$$

Subtracting the series term by term, it follows that

$$\sin x - \tan x = -\frac{x^3}{2} - \frac{x^5}{8} - \cdots = x^3\left(-\frac{1}{2} - \frac{x^2}{8} - \cdots\right).$$

Division of both sides by x^3 and taking limits then gives

$$\begin{aligned}\lim_{x\to 0} \frac{\sin x - \tan x}{x^3} &= \lim_{x\to 0}\left(-\frac{1}{2} - \frac{x^2}{8} - \cdots\right)\\ &= -\frac{1}{2}.\end{aligned}$$

(from Thomas'Calculus)

EXAMPLE Find $\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right)$.

Solution Using algebra and the Taylor series for $\sin x$, we have

$$\frac{1}{\sin x}-\frac{1}{x}=\frac{x-\sin x}{x\sin x}=\frac{x-\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots\right)}{x\cdot\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots\right)}$$

$$=\frac{x^3\left(\frac{1}{3!}-\frac{x^2}{5!}+\cdots\right)}{x^2\left(1-\frac{x^2}{3!}+\cdots\right)}=x\cdot\frac{\frac{1}{3!}-\frac{x^2}{5!}+\cdots}{1-\frac{x^2}{3!}+\cdots}.$$

Therefore,

$$\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right)=\lim_{x\to 0}\left(x\cdot\frac{\frac{1}{3!}-\frac{x^2}{5!}+\cdots}{1-\frac{x^2}{3!}+\cdots}\right)=0.$$

From the quotient on the right, we can see that if $|x|$ is small, then

$$\frac{1}{\sin x}-\frac{1}{x}\approx x\cdot\frac{1}{3!}=\frac{x}{6}\quad\text{or}\quad\csc x\approx\frac{1}{x}+\frac{x}{6}.$$

(from Thomas'Calculus)

Some common Taylor series

TABLE Frequently used Taylor series

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n, \qquad |x| < 1$$

$$\frac{1}{1+x} = 1 - x + x^2 - \cdots + (-x)^n + \cdots = \sum_{n=0}^{\infty} (-1)^n x^n, \qquad |x| < 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \qquad |x| < \infty$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \qquad |x| < \infty$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, \qquad |x| < \infty$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1} \frac{x^n}{n} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}, \qquad -1 < x \le 1$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}, \qquad |x| \le 1$$

(from Thomas'Calculus)