1. Preliminaries

1.1. Elements of logic and theory of sets

1.1.1. Propositions and logical operations

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Content:

- Propositions: basic notions
- Logical operations
- Tautology, contradiction, equivalence
- Laws of propositional logic
- Predicates and quantified statements
- Negation of statements
- Proof techniques

Proposition is a declarative sentence that is true or false.

- Every proposition has a **truth** value (T or F).
- The value may be: known/widely accepted as true or false; unknown; a matter of opinion (true for some people) or even a false belief.

Examples:

Proposition: Math is fun. (*opinion*)
Proposition: $2 + 1 = 3$. (*truth*)
Proposition: All cats are red. (*false*)
Proposition: If it is raining, then I take an umbrella. (*opinion*)
Proposition: The Sun orbits around the Earth. (*false belief*)
Not a proposition: Never stop learning.
Not a proposition: Are you going dancing?

Constructing propositions

Operations to construct compound propositions:

- conjunction $\wedge$ AND;
- disjunction $\vee$ OR;
- negation $\neg$ NOT;
- implication $\Rightarrow$ IF–THEN;
- bijunction $\iff$ IFF.

Any proposition can be represented by a truth table.

Conjunction

Let A and B be propositions. Then "*A and B*", written symbolically $A \wedge B$, is the **conjunction** of A and B.

$A \wedge B$ is true if and only if A is true and B is true.

Truth table:

A	B	$A \wedge B$
True	True	True
True	False	False
False	True	False
False	False	False

Example:

A: I am at the university.

B: Today is the first lecture on Analysis I.

$A \wedge B$: I am at the university and today is the first lecture on Analysis I.

Disjunction

Let A and B be propositions. Then "*A or B*", written symbolically $A \vee B$, is the **disjunction** of A and B.

$A \vee B$ is true if and only if A is true or B is true.

Truth table:

A	B	$A \vee B$
True	True	True
True	False	True
False	True	True
False	False	False

Example:

A: I am at the university.

B: Today is the first lecture on Analysis I.

$A \vee B$: I am at the university or today is the first lecture on Analysis I.

Negation

Let A be a proposition. Then "*not A*", written symbolically $\neg A$, is the **negation** of A.

$\neg A$ is true if and only if A is false.

Truth table:

A	$\neg A$
True	False
False	True

Example:

A: The Earth is round.

$\neg A$: It is not true that the Earth is round.

$\sim$ The Earth is not round.

Conditional statements (implication)

Let A and B be propositions. Then "If A then B", written symbolically $A \Rightarrow B$, is a **conditional statement** or **implication**.

A is the *hypothesis* (premise) and B is the *conclusion* (consequence).

Truth table:

A	B	$A \rightarrow B$
True	True	True
True	False	False
False	True	True
False	False	True

Example:

A: It is raining.

B: Streets are wet.

$A \Rightarrow B$: *If* it is raining *then* streets are wet.

From a false premise anything can be implied!

$\neg A \Rightarrow B$: *If* it is not raining *then* streets are wet. **T**

$\neg A \Rightarrow \neg B$: *If* it is not raining *then* streets are not wet. **T**

$A \Rightarrow \neg B$: *If* it is raining *then* streets are not wet. **F**

Biconditional statements (bijunction)

Let A and B be propositions. Then " A if and only if B", written symbolically $A \Leftrightarrow B$, is a **biconditional statement** or **bijunction**.

Other ways to say this: *A iff B*; *if A then B, and conversely (vise verse)*; *A is necessary and sufficient for B*.

Truth table:

A	B	$A \leftrightarrow$ B
True	True	True
True	False	False
False	True	False
False	False	True

Example:

A: You buy an airline ticket.

B: You can take a flight.

$A \Leftrightarrow B$: You buy an airline ticket *iff* you can take a flight.

True only if you do **both** or **neither**.
Doing **only one** or the other makes the proposition **false**.

A **tautology** is a proposition that is always true.

A **contradiction** is a proposition that is always false.

Two propositions A and B are **logically equivalent**, $A \equiv B$, if they always have the same truth value.
$A \equiv B$ iff $A \Leftrightarrow B$ is a tautology.

Showing equivalence:
Use truth tables or laws of logic.
Showing non-equivalence:
Find at least one row of the truth table where values differ.

Different ways of expressing $A \Rightarrow B$

if A **then** B	If it is raining then streets are wet.	**T**
if A, B	If it is raining, streets are wet.	**T**
A implies B	Rain implies that streets are wet.	**T**
A only if B	It is raining only if streets are wet.	**T**
B if A	Streets are wet if it is raining.	**T**
B when A	Streets are wet when it is raining.	**T**
B whenever A	Streets are wet whenever it is raining.	**T**
B follows from A	Streets being wet follows from being a rain.	**T**
A is sufficient for B	Being a rain is sufficient for streets being wet.	**T**
B is necessary for A	Streets being wet is necessary for being a rain.	**T**

$A \Rightarrow B$: If it is raining then streets are wet.

B **if** A	Streets are wet if it is raining.	**T**
B **only if** A	Streets are wet only if it is raining.	**F**
A **if** B	It is raining if streets are wet.	**F**
A **only if** B	It is raining only if streets are wet.	**T**

Which statements are equivalent?

Remember:

$$\begin{array}{ccc}
(A \Rightarrow B) & \not\Leftrightarrow & (B \Rightarrow A) \\
\Updownarrow & & \Updownarrow \\
(B \text{ if } A) & \not\Leftrightarrow & (A \text{ if } B) \\
\Updownarrow & & \Updownarrow \\
(A \text{ only if } B) & \not\Leftrightarrow & (B \text{ only if } A)
\end{array}$$

A **is sufficient for** B	Rain is **sufficient** for streets being wet.
B **is necessary for** A	Wet streets are **necessary** for there being rain.
A **is necessary and sufficient for** A	Rain is **necessary and sufficient** for streets being wet.(**F**)

Converse, inverse, contrapositive

Implication	$A \Rightarrow B$	If it is raining then streets are wet. **T**
Converse	$B \Rightarrow A$	If streets are wet then it is raining. **F**
Inverse	$\neg A \Rightarrow \neg B$	If it is not raining then streets are not wet. **F**
Contrapositive	$\neg B \Rightarrow \neg A$	If streets are not wet then it is not raining. **T**

Which statements are equivalent?

$$(A \Rightarrow B) \iff (\neg B \Rightarrow \neg A)$$

$$(B \Rightarrow A) \iff (\neg A \Rightarrow \neg B)$$

Precedence rules

All logical operations can be applied to build up arbitrarily complex **compound** propositions.
Any proposition can become a term inside another proposition.

Examples: let A, B, C, D be propositions.

- $A \vee B$, $C \Rightarrow D$ – combination of simple propositions;
- $(A \vee B) \Rightarrow C$, $(A \vee B) \wedge (C \vee D)$ – combination of simple and compound propositions.

Parenthesis indicate the order of evaluation.

Operation	Precedence
$\neg$	1
$\wedge$	2
$\vee$	3
$\rightarrow$	4
$\leftrightarrow$	5

Examples:

- $(A \vee B) \vee C \Leftrightarrow A \vee B \vee C$
- $(A \vee B) \wedge C \not\Leftrightarrow A \vee B \wedge C$

Laws of propositional logic

Let A, B, C be propositions.

Idempotent laws	$A \vee A \equiv A,\ A \wedge A \equiv A$
Associative laws	$A \wedge (B \wedge C) \equiv (A \wedge B) \wedge C$ $A \vee (B \vee C) \equiv (A \vee B) \vee C$
Commutative laws	$A \wedge B \equiv B \wedge A,\ A \vee B \equiv B \vee A$
Distributive laws	$(A \vee (B \wedge C)) \equiv ((A \vee B) \wedge (A \vee C))$ $(A \wedge (B \vee C)) \equiv ((A \wedge B) \vee (A \wedge C))$
Identity laws	$A \vee F \equiv A,\ A \wedge T \equiv A$
Domination laws	$A \wedge F \equiv F,\ A \vee T \equiv T$
Double negation	$\neg\neg A \equiv A$

Laws of propositional logic (cont'd)

Complement laws	$A \wedge \neg A \equiv F$, $A \vee \neg A \equiv T$, $\neg T \equiv F$, $\neg F \equiv T$
De Morgan's laws:	$\neg(A \wedge B) \equiv (\neg A) \vee (\neg B)$ $\neg(A \vee B) \equiv (\neg A) \wedge (\neg B)$
Absorption laws	$A \vee (A \wedge B) \equiv A$ $A \wedge (A \vee B) \equiv A$
Conditional identities	$A \Rightarrow B \equiv \neg A \vee B$ $A \Leftrightarrow B \equiv (A \Rightarrow B) \wedge (B \Rightarrow A)$

The main notions:

- **Variables** $x, y, z, \ldots$ represent objects, not propositions.
- **Domain** D is the set of all possible values a variable may take. Each variable take on values from a given domain.
- **Predicates**: $P, Q, \ldots,$ express properties of objects.
- Predicates are *generalizations* of propositions. They are *functions* that return T or F depending on their variables.
- Predicates **become propositions** (have truth values) when variables are replaced by values or when used with *quantifiers*.
- Logical operations from propositional logic $(\wedge, \vee, \neg, \Rightarrow, \Leftrightarrow)$ can also be applied to predicates.

Example: let x (*variable*) be an integer number (*domain D is all integer numbers*), and P denote the property "is a perfect square" (*predicate*).
P(x): "x is a perfect square" – is a predicate, not a proposition.
$P(9)$ is a true proposition, $P(8)$ is a false proposition.

Quantified statements

Universal quantifier, "For all", symbol $\forall$:
$\forall a, P(a)$ – "The property $P(a)$ holds for any element a".
$P(a)$ is a **universally quantified statement**.

Existential quantifier, "There exists", symbol $\exists$:
$\exists a, P(a)$ – "There exists an element a such that the property $P(a)$ holds".
$P(a)$ is a **existentially quantified statement**.

- $\forall a\, P(a) \Leftrightarrow P(a_1) \wedge P(a_2) \wedge \cdots \wedge P(a_n)$
- $\exists a\, P(a) \Leftrightarrow P(a_1) \vee P(a_2) \vee \cdots \vee P(a_n)$

Logical operations from propositional logic $(\wedge, \vee, \neg, \Rightarrow, \Leftrightarrow)$ can also be applied to quantified statements.

- $\neg(A \wedge B) \iff (\neg A) \vee (\neg B)$ (de Morgan's laws)
- $\neg(A \vee B) \iff (\neg A) \wedge (\neg B)$ (de Morgan's laws)
- $\neg(A \Rightarrow B) \iff A \wedge \neg B$
- $\neg(A \Leftrightarrow B) \iff (A \wedge \neg B) \vee (B \wedge \neg A)$

Examples:

A: The car is red. B: The car is fast.

$A \wedge B$: The car is red and fast.

$\neg(A \wedge B)$: The car is not red or it is not fast. (The car is not red or slow).

$A \vee B$: The car is red or fast.

$\neg(A \vee B)$: The car is not red and not fast. (The car is neither red nor fast).

$A \Rightarrow B$: If the car is red then it is fast.

$\neg(A \Rightarrow B)$: The car is red and slow.

$A \Leftrightarrow B$: The car is red iff it is fast.

$\neg(A \Leftrightarrow B)$: The car is red and slow or the car is fast and not red.

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- $\neg(\forall a, P) \iff \exists a, \neg P$
- $\neg(\exists a, P) \iff \forall a, \neg P$

Examples:

P: All students like math.

$\neg P$: There exist a student that does not like math. (Some students do not like math).

Q: There exist numbers greater than 1000.

$\neg Q$: All numbers are less than or equal to 1000.

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1.1. Elements of logic and theory of sets

1.1.2. Proof techniques

A **mathematical proof** is an inferential argument for a mathematical statement, showing that the stated assumptions logically guarantee the conclusion.

The argument may use other previously established statements, such as theorems; but every proof can, in principle, be constructed using only certain basic or original assumptions known as axioms, along with the accepted rules of inference.

Inferences are steps in reasoning, from premises to logical consequences.

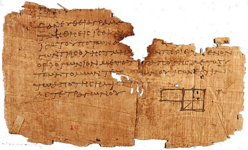
Examples (syllogism):

- All birds have wings. T
- Penguins are birds. T
- Therefore, penguins have wings. T

- All birds are black and white. F
- Penguins are birds. T
- Therefore, penguins are black and white. T

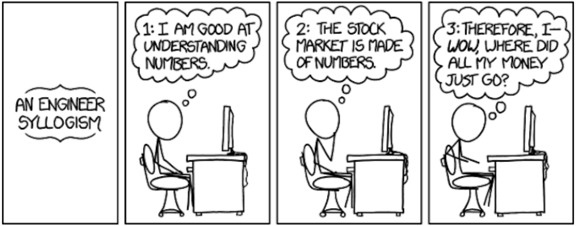
- Penguins are black and white. T
- Zebras are black and white. T
- Therefore, penguins are zebras. F

A **mathematical proof** is an inferential argument for a mathematical statement, showing that the stated assumptions logically guarantee the conclusion.



A fragment of Euclid's Elements (c. 300 BC) used for millennia to teach proof-writing techniques. (wikipedia.org)

Inferences are steps in reasoning, from premises to logical consequences.



Universally quantified statements:

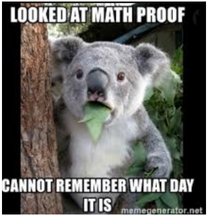
- show that it works in all cases to prove;
- find one counterexample to disprove.

Existentially statements:

- find one example to prove;
- show that none of the cases holds to disprove.

Four fundamental proof techniques:

- Direct proof
- Proof by contradiction
- Proof by mathematical induction
- Proof by contrapositive



Direct proof of statement $P \Rightarrow Q$

1. Assume that P is true.
2. Use P to show that Q must be true:

$$P \Rightarrow Q.$$

Example:

Theorem. If a and b are consecutive integers, then $a + b$ is odd.

Proof:

P : a and b are consecutive integers, Q : $a + b$ is odd.

$$P \Rightarrow b = a + 1 \Rightarrow: \; a + b = 2a + 1 \Rightarrow: \; \exists \text{ integer } k\!: \; a + b = 2k + 1 \Rightarrow Q.$$

Proof by contradiction of statement $P \Rightarrow Q$

1. Assume that P is true.
2. Assume that $\neg Q$ is true.
3. Use $\neg Q$ to show that P must be false:

$$\neg Q \wedge P = F.$$

Example:

Theorem. If a and b are consecutive integers, then $a+b$ is odd.

Proof:

P : a and b are consecutive integers, Q : $a+b$ is odd.

Assume $\neg Q$: $a+b$ is not odd $\Rightarrow \nexists$ integer k: $a+b = 2k+1$.

However, $P \Rightarrow a+b = 2a+1$ and a is integer.

$\Rightarrow \neg Q \wedge P = F \Rightarrow \neg Q \equiv F \Rightarrow Q \equiv T$.

Mathematical induction

To prove that a proposition $F(n)$ holds for every natural number $n = 1, 2, 3, \ldots$:

1. Verify that $F(1)$ is true.
2. For every $k \geq 1$, prove that if $F(k)$ is true then $F(k+1)$ is also true.
3. Then, by the **principle of mathematical induction**, $F(n)$ is true for all $n \geq 1$.

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Mathematical induction

To prove that a proposition $F(n)$ holds for every natural number $n = 1, 2, 3, \dots$:

1. Verify that $F(1)$ is true.
2. For every $k \geq 1$, prove that if $F(k)$ is true then $F(k+1)$ is also true.
3. Then, by the **principle of mathematical induction**, $F(n)$ is true for all $n \geq 1$.

Example:

Theorem. If a and b are consecutive integers, then $a + b$ is odd.

Proof: rewrite as $F(n) : \forall n \in \mathbb{N}\, P(n) \Rightarrow Q(n)$ *is odd*, where

$P(n)$: $a = n$ and $b = n + 1$ with integer n, $Q(n)$: $a + b$.

Step 1. $P(1) \Rightarrow Q(1)$ *is odd*: $1 + 2 = 3$ is odd, i.e. $F(1) \equiv T$.

Step 2. Assume $F(k) \equiv T$ for $k \geq 1$, i.e. $Q(k) = k + (k + 1)$ is odd. Then $Q(k+1) = (k+1) + (k+2) = Q(k) + 2$. Thus, $Q(k)$ is odd implies $Q(k+1)$ is odd, i.e. $(F(k) \equiv T) \Rightarrow (F(k+1) \equiv T)$.

Step 3. By the principle of mathematical induction, $F(n) \equiv T \forall n \in \mathbb{N}$.

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Mathematical induction

To prove that a proposition $F(n)$ holds for every natural number $n = 1, 2, 3, \ldots$:

1. Verify that $F(1)$ is true.
2. For every $k \geq 1$, prove that if $F(k)$ is true then $F(k+1)$ is also true.
3. Then, by the **principle of mathematical induction**, $F(n)$ is true for all $n \geq 1$.

Another example:

Theorem. For any integer n, $1 + 3 + 5 + \cdots + (2n-1) = n^2$.

Proof: $F(n) : 1 + 3 + 5 + \cdots + (2n-1) = n^2$.

Step 1. $F(1) : 1 = 1^2$ (T)

Step 2. Let $F(k) \equiv T$ for $k \geq 1$, i.e. $1 + 3 + \cdots + (2k-1) = k^2$. Then for $n = k+1$, $\underbrace{1 + 3 + \cdots + (2k-1)}_{=k^2 \text{ because of } F(k)} + (2(k+1)-1) = k^2 + 2k + 1 = (k+1)^2$.

Thus, $(F(k) \equiv T) \Rightarrow (F(k+1) \equiv T)$.

Step 3. By the principle of mathematical induction, $F(n) \equiv T \forall n \in \mathbb{N}$.

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Proof by contrapositive of statement $P \Rightarrow Q$

1. Assume that $\neg Q$ is true.
2. Show that $\neg P$ must be true.
3. Observe that $P \Rightarrow Q$ by contraposition:

$$(P \Rightarrow Q) \Leftrightarrow (\neg Q \Rightarrow \neg P).$$

Example:

Theorem. If a and b are consecutive integers, then $a + b$ is odd.

Proof:

P : a and b are consecutive integers, Q : $a + b$ is odd.

$\neg Q \Rightarrow \neg P$: *if $a + b$ is not odd, then a and b are not consecutive integers.*

Assume $\neg Q \equiv T \Rightarrow \nexists k \in \mathbb{N}\colon\ a + b = 2k + 1$.

Thus, $a + b \neq k + (k + 1)$ for all integers k.

This implies that a and b cannot be consecutive integers.

$(\neg Q \Rightarrow \neg P) \Leftrightarrow (P \Rightarrow Q)$.