3. Limits and continuity of functions

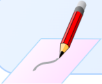
3.2. Continuity of a function

Content:

- Definitions of continuity
- Continuous function and their properties
- Main types of discontinuities
- Asymptotes
- Extremum points. Weierstrass Extreme value theorem
- Intermediate value theorem and its applications

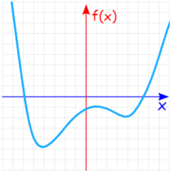
Notion of a continuous function

A function is continuous when its graph is a single unbroken curve …

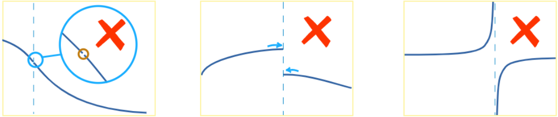


… that you could draw without lifting your pen from the paper.

That is not a formal definition, but it helps you understand the idea.



(from mathisfun.com)



Not Continuous (hole) | **Not** Continuous (jump) | **Not** Continuous (vertical asymptote)

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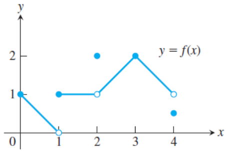


The function is not continuous at $x = 1, x = 2$, and $x = 4$

EXAMPLE At which numbers does the function f in Figure 2.35 appear to be not continuous? Explain why. What occurs at other numbers in the domain?

Solution First we observe that the domain of the function is the closed interval $[0, 4]$, so we will be considering the numbers x within that interval. From the figure, we notice right away that there are breaks in the graph at the numbers $x = 1$, $x = 2$, and $x = 4$. The breaks appear as jumps, which we identify later as "jump discontinuities." These are numbers for which the function is not continuous, and we discuss each in turn.

(from Thomas'Calculus)

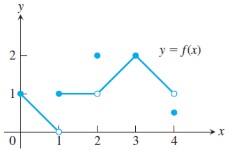
Numbers at which the graph of f has breaks:

At $x = 1$, the function fails to have a limit. It does have both a left-hand limit, $\lim_{x\to 1^-} f(x) = 0$, as well as a right-hand limit, $\lim_{x\to 1^+} f(x) = 1$, but the limit values are different, resulting in a jump in the graph. The function is not continuous at $x = 1$.

At $x = 2$, the function does have a limit, $\lim_{x\to 2} f(x) = 1$, but the value of the function is $f(2) = 2$. The limit and function values are not the same, so there is a break in the graph and f is not continuous at $x = 2$.

At $x = 4$, the function does have a left-hand limit at this right endpoint, $\lim_{x\to 4^-} f(x) = 1$, but again the value of the function $f(4) = \frac{1}{2}$ differs from the value of the limit. We see again a break in the graph of the function at this endpoint and the function is not continuous from the left.

(from Thomas'Calculus)



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(from Thomas'Calculus)

Numbers at which the graph of f has no breaks:

At $x = 0$, the function has a right-hand limit at this left endpoint, $\lim_{x\to 0^+} f(x) = 1$, and the value of the function is the same, $f(0) = 1$. So no break occurs in the graph of the function at this endpoint, and the function is continuous from the right at $x = 0$.

At $x = 3$, the function has a limit, $\lim_{x\to 3} f(x) = 2$. Moreover, the limit is the same value as the function there, $f(3) = 2$. No break occurs in the graph and the function is continuous at $x = 3$.

At all other numbers $x = c$ in the domain, which we have not considered, the function has a limit equal to the value of the function at the point, so $\lim_{x\to c} f(x) = f(c)$. For example, $\lim_{x\to 5/2} f(x) = f\left(\frac{5}{2}\right) = \frac{3}{2}$. No breaks appear in the graph of the function at any of these remaining numbers and the function is continuous at each of them. ∎

(from Thomas'Calculus)

Notion of continuity

In the definition of a limit of a function $f : D \to \mathbb{R}$ at x_0, it is assumed that $x_0 \notin D$. What if $x_0 \in D$?

Lemma

Let $f : D \to \mathbb{R}$, $x_0 \in D$. Then the function f has a limit at x_0 if and only if $\lim\limits_{x \to x_0} f(x) = f(x_0)$

Definitions

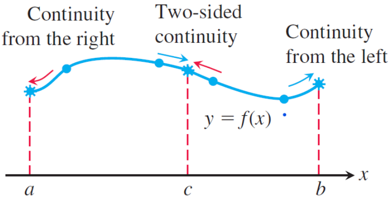
A function $f : D \to \mathbb{R}$ is said to be

- **continuous at $x_0 \in D$** if $\lim\limits_{x \to x_0} f(x) = f(x_0)$;
- **right-continuous at $x_0 \in D$** (or **continuous from the right**) if $\lim\limits_{x \to x_0^+} f(x) = f(x_0)$;
- **left-continuous at $x_0 \in D$** (or **continuous from the left**) if $\lim\limits_{x \to x_0^-} f(x) = f(x_0)$;
- **continuous over a closed interval $[a, b] \subseteq D$**, if it is right-continuous at a, left-continuous at b, and continuous at every $x_0 \in (a, b)$;
- **discontinuous at $y_0 \in \operatorname{int} D$** if is not continuous at y_0; y_0 is called a **point of discontinuity** of f.

Notion of continuity

Remarks

- From the properties of the limit, it follows immediately that a function f is continuous at an interior point x_0 of its domain if and only if it is both right-continuous and left-continuous at x_0.
- This definition of continuity over an interval applies to the infinite closed intervals $[a, +\infty)$ and $(-\infty, b]$ as well, but only one endpoint is involved.
- A function f can be continuous, right-continuous, or left-continuous only at a point x_0 for which $f(x_0)$ is defined.

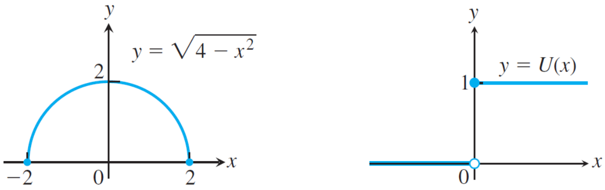


Continuity at points a, b, and c.

Examples:

1) The function $f(x) = \sqrt{4 - x^2}$ is continuous over its domain $[-2, 2]$. It is right-continuous at $x = -2$, and left-continuous at $x = 2$.

2) The unit step function $U(x)$ is right-continuous at $x = 0$, but is neither left-continuous nor continuous there. It has a *jump discontinuity* at $x = 0$.



(from Thomas'Calculus)

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Continuity Test

A function $f(x)$ is continuous at a point $x = c$ if and only if it meets the following three conditions.

1. $f(c)$ exists (c lies in the domain of f).
2. $\lim_{x \to c} f(x)$ exists (f has a limit as $x \to c$).
3. $\lim_{x \to c} f(x) = f(c)$ (the limit equals the function value).

(from Thomas'Calculus)

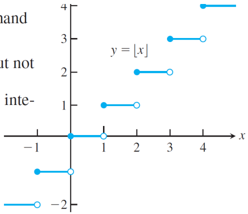
Examples:

The function $y = \lfloor x \rfloor$ is discontinuous at every integer because the left- and right-hand limits are not equal as $x \to n$: $\lim_{x \to n^-} \lfloor x \rfloor = n - 1$ and $\lim_{x \to n^+} \lfloor x \rfloor = n$.

Since $\lfloor n \rfloor = n$, the greatest integer function is right-continuous at every integer n (but not left-continuous).

The greatest integer function is continuous at every real number other than the integers. For example, $\lim_{x \to 1.5} \lfloor x \rfloor = 1 = \lfloor 1.5 \rfloor$.

In general, if $n - 1 < c < n$, n an integer, then $\lim_{x \to c} \lfloor x \rfloor = n - 1 = \lfloor c \rfloor$.



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Examples:

Find parameter $a \in \mathbb{R}$ that makes f continuous in $x_0 = 2$. Is it then continuous on $\mathbb{R}$?

$$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = \begin{cases} a \cdot 2^x, & \text{if } x \leq 2, \\ x^2 - 2ax + 8, & \text{if } x > 2. \end{cases}$$

Solution:

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} a \cdot 2^x = 4a.$$

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} x^2 - 2ax + 8 \cdot 2^x = 12 - 4a.$$

Use definition:

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) \Rightarrow 4a = 12 - 4a \Rightarrow a = \frac{3}{2}.$$

Equivalent definitions of continuity

Definition (($\varepsilon - \delta$)-definition of continuity)

A function $f : D \to \mathbb{R}$ is **continuous at** $x_0 \in D$ if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon.$$

Definition (Definition of continuity in terms of neighborhoods)

A function $f : D \to \mathbb{R}$ is **continuous at** $x_0 \in D$ if for any ε-neighborhood of $f(x_0)$, $U(f(x_0), \varepsilon)$, there exists a δ-neighborhood of x_0, $U(x_0, \delta)$, such that
$x \in U(x_0, \delta) \cap D \Rightarrow f(x) \in U(f(x_0), \varepsilon)$.

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Definition

If a function $f : D \to \mathbb{R}$ is continuous at every point in its domain, it is called a **continuous function**.
If a function is discontinuous at one or more points of its domain, we say it is a **discontinuous function**.

Remark

A function always has a specified domain, so if we change the domain, we change the function, and this may change its continuity property as well.

Examples: 1) The function $f(x) = \dfrac{1}{x}$ is a continuous function because it is continuous at every point of its domain. It has a point of discontinuity at $x = 0$, however, because it is not defined there; that is, it is discontinuous on any interval containing $x = 0$.
2) The identity function $f(x) = x$ and constant functions are continuous everywhere.

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Theorem (Properties of continuous functions)

Let $f, g : D \to \mathbb{R}$ be continuous at $x_0 \in D$. Then the following algebraic combinations are continuous at $x = x_0$:

- *sums and differences*: $f \pm g$;
- *constant multiples*: $k \cdot f$, $\forall k \in \mathbb{R}$;
- *products*: $f \cdot g$;
- *quotients*: f/g, provided that $g(x_0) \neq 0$;
- *powers*: f^n, $\forall n \in \mathbb{N}$;
- *roots*: $\sqrt[n]{f}$, provided it is defined on an open interval containing x_0, $n \in \mathbb{N}$.

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Reminder:

If f and g are functions, the **composite function** $f \circ g$ ("f composed with g") is defined by

$$(f \circ g)(x) = f(g(x)).$$

Theorem (Composite of continuous functions)

Let $f : D_f \to \mathbb{R}$ be continuous at $x_0 \in D_f$, and $g : D_g \to \mathbb{R}$ be continuous at $f(x_0)$. Then the composite $g \circ f$ is continuous at x_0.

Continuity of inverse functions

Reminder:

Suppose that f is an injective function on a domain $D(f)$ with range $R(f)$. The **inverse function** f^{-1} is defined by

$$f^{-1}(b) = a \text{ if } f(a) = b.$$

Equivalent definition

Suppose that f is an injective function on a domain $D(f)$ with range $R(f)$. A function $g : R(f) \to D(f)$ is the **inverse function** for f, $g = f^{-1}$, if the following property holds:

$$\forall x \in D(f)\ g(f(x)) = x \text{ and } \forall y \in R(f)\ f(g(y)) = y.$$

The function is invertible iff it is bijective.

Continuity of inverse functions

Lemma

A continuous $f : [a, b] \to \mathbb{R}$ is injective iff it is strictly monotonic. In this case, the range of $R(f)f$ is $[f(a), f(b)]$ if f is increasing, $R(f) = [f(b), f(a)]$ if f is decreasing, and $f : [a, b] \to R(f)$ is bijective.

Theorem (Continuity of inverse functions)

Let $f : D \to \mathbb{R}$ be continuous and monotonic on an interval $(a, b) \in D \cap R(f)$. Then it is invertible on (a, b), and its inverse function f^{-1} is continuous.

Limits of continuous functions

Theorem (Limits of continuous functions)

Let $g : D_g \to \mathbb{R}$ be continuous at $x_0 \in D_g$, and $\lim\limits_{x\to x_0} f(x) = a$.

Then $\lim\limits_{x\to x_0} g(f(x)) = g(a) = g\left(\lim\limits_{x\to x_0} f(x)\right)$.

Properties of continuous functions

Examples:

1) Every polynomial function $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ is continuous in $\mathbb{R}$ because $\lim\limits_{x\to x_0} P(x) = P(x_0)$ for all $x_0 \in \mathbb{R}$.

2) If $P(x)$ and $Q(x)$ are polynomials, then the rational function $\dfrac{P(x)}{Q(x)}$ is continuous whenever defined $(Q(x) \neq 0)$.

3) The function $f(x) = |x|$ is continuous in $\mathbb{R}$. If $x > 0$, we have $f(x) = x$, a polynomial. If $x < 0$, we have $f(x) = -x$, another polynomial. Finally, at the origin, $\lim\limits_{x\to 0} |x| = 0 = |0|$.

4) All six trigonometric functions are continuous wherever they are defined. For example, $y = \tan x$ is continuous on $\cdots \cup (\pi/2, \pi/2) \cup (\pi/2, 3\pi/2) \cup \ldots$.

5) The exponential function $f(x) = a^x$ $(a > 0)$ is continuous in $\mathbb{R}$.

6) The logarithmic function $f(x) = \log_a x$ $(a > 0, a \neq 1)$ is continuous on $(0, +\infty)$.

Theorem

Every elementary function is continuous on its domain.

Examples:

7)

Show that the following functions are continuous on their natural domains.

(a) $y = \sqrt{x^2 - 2x - 5}$ **(b)** $y = \dfrac{x^{2/3}}{1 + x^4}$

(c) $y = \left|\dfrac{x - 2}{x^2 - 2}\right|$ **(d)** $y = \left|\dfrac{x \sin x}{x^2 + 2}\right|$

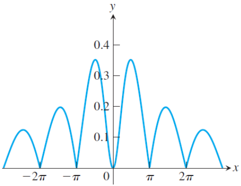
Solution

(a) The square root function is continuous on $[0, \infty)$ because it is a root of the continuous identity function $f(x) = x$. The given function is then the composite of the polynomial $f(x) = x^2 - 2x - 5$ with the square root function $g(t) = \sqrt{t}$, and is continuous on its natural domain.

(b) The numerator is the cube root of the identity function squared; the denominator is an everywhere-positive polynomial. Therefore, the quotient is continuous.

(c) The quotient $(x - 2)/(x^2 - 2)$ is continuous for all $x \neq \pm\sqrt{2}$, and the function is the composition of this quotient with the continuous absolute value function

(d) Because the sine function is everywhere-continuous, the numerator term $x \sin x$ is the product of continuous functions, and the denominator term $x^2 + 2$ is an everywhere-positive polynomial. The given function is the composite of a quotient of continuous functions with the continuous absolute value function ■



(from Thomas'Calculus)

Example:
As an application of Theorem on limits of continuous functions, we have the following calculations.

(a) $$\lim_{x\to\pi/2}\cos\left(2x+\sin\left(\frac{3\pi}{2}+x\right)\right)=\cos\left(\lim_{x\to\pi/2}2x+\lim_{x\to\pi/2}\sin\left(\frac{3\pi}{2}+x\right)\right)$$
$$=\cos(\pi+\sin 2\pi)=\cos\pi=-1.$$

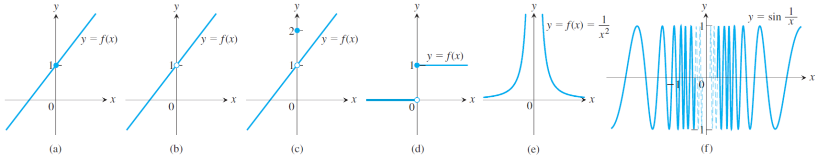
(b) $$\lim_{x\to 1}\sin^{-1}\left(\frac{1-x}{1-x^2}\right)=\sin^{-1}\left(\lim_{x\to 1}\frac{1-x}{1-x^2}\right)$$ Arcsine is continuous.
$$=\sin^{-1}\left(\lim_{x\to 1}\frac{1}{1+x}\right)$$ Cancel common factor $(1-x)$.
$$=\sin^{-1}\frac{1}{2}=\frac{\pi}{6}$$

(c) $$\lim_{x\to 0}\sqrt{x+1}\,e^{\tan x}=\lim_{x\to 0}\sqrt{x+1}\cdot\exp\left(\lim_{x\to 0}\tan x\right)$$ Exponential is continuous.
$$=1\cdot e^0=1$$ ∎

(from Thomas'Calculus)

A function $f : D \to \mathbb{R}$, $D = \dot{U}(x_0, \Delta)$, $x_0 \in \mathbb{R}$, $\Delta \in (0, +\infty]$, has

- a **removable discontinuity at x_0** if the one-sided limits of f at x_0 exist and are equal and finite, and $\lim\limits_{x \to x_0} f(x) \neq f(x_0)$;
- a **jump discontinuity at x_0** (or step discontinuity, or discontinuity of the first kind), if the one-sided limits of f at x_0 exist and are finite, but not equal, so that $\nexists \lim\limits_{x \to x_0} f(x)$;
- an **infinite discontinuity at x_0** (or essential discontinuity, or discontinuity of the second kind), if at least one of the one-sided limits of f at x_0 does not exist or infinite.



The function in (a) is continuous at $x = 0$; the functions in (b) through (f) are not.

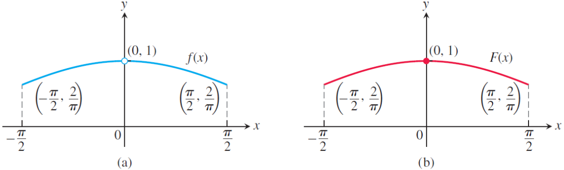
(from Thomas'Calculus)

Continuous extension to a point

If a function f has a removable discontinuity at x_0, we can extend the function's domain to include the point $x = x_0$ in such a way that the extended function is continuous at $x = x_0$. We define the new function as

$$F(x) = \begin{cases} f(x), & \text{if } x \neq x_0, \\ \lim\limits_{x \to x_0} f(x), & \text{if } x = x_0. \end{cases}$$

The function F is called the **continuous extension** of f to $x \neq x_0$.



The graph (a) of $f(x) = (\sin x)/x$ for $-\pi/2 \leq x \leq \pi/2$ does not include the point $(0, 1)$ because the function is not defined at $x = 0$. (b) We can remove the discontinuity from the graph by defining the new function $F(x)$ with $F(0) = 1$ and $F(x) = f(x)$ everywhere else. Note that $F(0) = \lim_{x \to 0} f(x)$.

Definition

A line $y = b$ is a **horizontal asymptote** of the graph of a function $y = f(x)$ if either $\lim\limits_{x \to +\infty} = b$ or $\lim\limits_{x \to -\infty} = b$.

Definition

A line $x = a$ is a **vertical asymptote** of the graph of a function $y = f(x)$ if either $\lim\limits_{x \to a^+} = \pm\infty$ or $\lim\limits_{x \to a^-} = \pm\infty$.

Definition

If the degree of the numerator of a rational function is 1 greater than the degree of the denominator, the graph has an **oblique** or **slant line asymptote**.

We find an equation for the asymptote by dividing numerator by denominator to express f as a linear function plus a remainder that goes to zero as $x \to \pm\infty$. **Hint for computing**: $y = mx + n$ is an oblique asymptote of f if $\lim\limits_{x \to \pm\infty} \dfrac{f(x)}{x} = m$ and $\lim\limits_{x \to \pm\infty} (f(x) - mx) = n$.

Asymptotes

Examples:

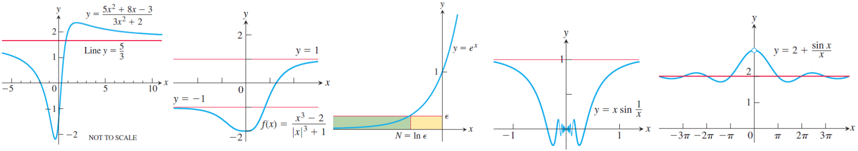
1) $f(x) = \dfrac{5x^2+8x-3}{3x^2+2}$ has $y = \dfrac{5}{3}$ as a horizontal asymptote on both the right and the left because $\lim\limits_{x\to+\infty} f(x) = \lim\limits_{x\to-\infty} f(x) = \dfrac{5}{3}$.

2) $f(x) = \dfrac{x^3-2}{|x|^3+1}$ has two horizontal asymptotes $y=-1$ and $y=1$.

3) $f(x) = e^x$ has the horizontal asymptote $y=0$.

4) $f(x) = x\sin\dfrac{1}{x}$ has the horizontal asymptote $y=1$.

5) $f(x) = 2 + \dfrac{\sin x}{x}$ has the horizontal asymptote $y=2$ (prove using the Sandwich theorem).



(from Thomas'Calculus)

Examples: 6) $f(x) = \dfrac{x^2-3}{2x-4}$ has the vertical asymptote $x = 2$ ($\lim\limits_{x\to 2^\pm} = \pm\infty$) and oblique asymptote $y = \dfrac{x}{2} + 1$.

Solution We are interested in the behavior as $x \to \pm\infty$. We divide $(2x-4)$ into (x^2-3):

$$\begin{array}{r} \frac{x}{2} + 1 \\ 2x-4\,\overline{)\,x^2 - 3} \\ \underline{x^2 - 2x} \\ 2x - 3 \\ \underline{2x - 4} \\ 1 \end{array}$$

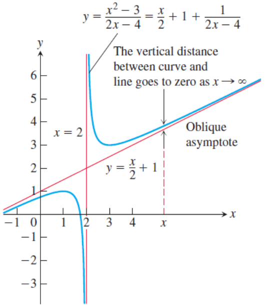
This tells us that

$$f(x) = \frac{x^2-3}{2x-4} = \underbrace{\left(\frac{x}{2}+1\right)}_{\text{linear } g(x)} + \underbrace{\left(\frac{1}{2x-4}\right)}_{\text{remainder}}.$$

As $x \to \pm\infty$, the remainder, whose magnitude gives the vertical distance between the graphs of f and g, goes to zero, making the slanted line

$$g(x) = \frac{x}{2} + 1$$

an asymptote of the graph of f ■



(from Thomas'Calculus)

Asymptotes

Examples: 7) $f(x) = \dfrac{x+3}{x+2}$ has the vertical asymptote $x = -2$ and horizontal asymptote $y = 1$.

Solution We are interested in the behavior as $x \to \pm\infty$ and the behavior as $x \to -2$, where the denominator is zero.

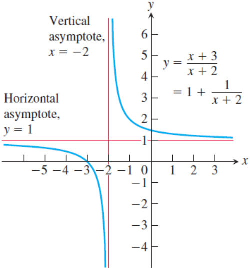
The asymptotes are quickly revealed if we recast the rational function as a polynomial with a remainder, by dividing $(x + 2)$ into $(x + 3)$:

$$\begin{array}{r|l} & \phantom{x+{}}1 \\ x+2 & x+3 \\ & \underline{x+2} \\ & \phantom{x+{}}1 \end{array}$$

This result enables us to rewrite y as:

$$y = 1 + \frac{1}{x+2}.$$

As $x \to \pm\infty$, the curve approaches the horizontal asymptote $y = 1$; as $x \to -2$, the curve approaches the vertical asymptote $x = -2$. We see that the curve in question is the graph of $f(x) = 1/x$ shifted 1 unit up and 2 units left ■



(from Thomas'Calculus)

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Examples: 8) $f(x) = -\dfrac{8}{x^2-4}$ has the vertical asymptotes $x = \pm 2$ and horizontal asymptote $y = 0$.

Solution We are interested in the behavior as $x \to \pm\infty$ and as $x \to \pm 2$, where the denominator is zero. Notice that f is an even function of x, so its graph is symmetric with respect to the y-axis.

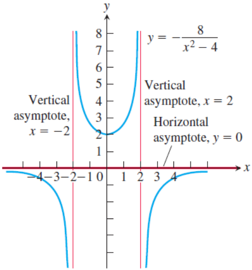
(a) *The behavior as* $x \to \pm\infty$. Since $\lim_{x\to\infty} f(x) = 0$, the line $y = 0$ is a horizontal asymptote of the graph to the right. By symmetry it is an asymptote to the left as well (Figure 2.66). Notice that the curve approaches the x-axis from only the negative side (or from below). Also, $f(0) = 2$.

(b) *The behavior as* $x \to \pm 2$. Since

$$\lim_{x\to 2^+} f(x) = -\infty \qquad \text{and} \qquad \lim_{x\to 2^-} f(x) = \infty,$$

the line $x = 2$ is a vertical asymptote both from the right and from the left. By symmetry, the line $x = -2$ is also a vertical asymptote.

There are no other asymptotes because f has a finite limit at all other points. ■

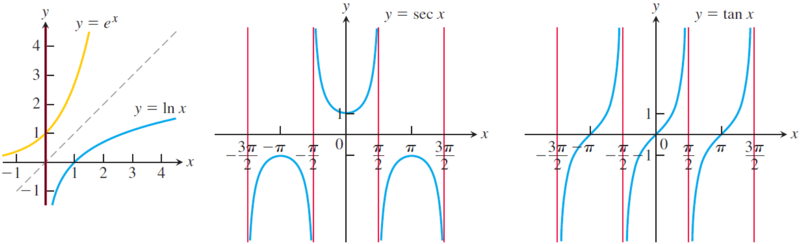


(from Thomas'Calculus)

Asymptotes

Examples: 9) $f(x) = \ln x$ has the vertical asymptotes $x = 0$.

10) $f(x) = \sec x = \dfrac{1}{\cos x}$ and $f(x) = \tan x = \dfrac{\sin}{\cos x}$ have the vertical asymptotes $x = \dfrac{\pi k}{2}$, $k \in \mathbb{Z}$.



(from Thomas'Calculus)

Dominant terms

In Example 6 we saw that by long division we could rewrite the function $f(x) = \dfrac{x^2-3}{2x-4}$ as a linear function plus a remainder term:

$$f(x) = \left(\frac{x}{2}+1\right) + \left(\frac{1}{2x-4}\right).$$

This tells us immediately that

$$f(x) \approx \frac{x}{2} + 1 \qquad \text{For } |x| \text{ large, } \frac{1}{2x-4} \text{ is near 0.}$$

$$f(x) \approx \frac{1}{2x-4} \qquad \text{For x near 2, this term is very large in absolute value.}$$

If we want to know how f behaves, this is the way to find out. It behaves like $y = (x/2) + 1$ when $|x|$ is large and the contribution of $1/(2x-4)$ to the total value of f is insignificant. It behaves like $1/(2x| - 4)$ when x is so close to 2 that $1/(2x-4)$ makes the dominant contribution.

We say that $(x/2) + 1$ **dominates** when x is numerically large, and we say that $1/(2x-4)$ dominates when x is near 2. **Dominant terms** like these help us predict a function's behavior.

(from Thomas'Calculus)

Let $f(x) = 3x^4 - 2x^3 + 3x^2 - 5x + 6$ and $g(x) = 3x^4$. Show that although f and g are quite different for numerically small values of x, they are virtually identical for $|x|$ very large, in the sense that their ratios approach 1 as $x \to \infty$ or $x \to -\infty$.

Solution The graphs of f and g behave quite differently near the origin but appear as virtually identical on a larger scale

We can test that the term $3x^4$ in f, represented graphically by g, dominates the polynomial f for numerically large values of x by examining the ratio of the two functions as $x \to \pm\infty$. We find that

$$\begin{aligned}\lim_{x\to\pm\infty} \frac{f(x)}{g(x)} &= \lim_{x\to\pm\infty} \frac{3x^4 - 2x^3 + 3x^2 - 5x + 6}{3x^4} \\ &= \lim_{x\to\pm\infty} \left(1 - \frac{2}{3x} + \frac{1}{x^2} - \frac{5}{3x^3} + \frac{2}{x^4}\right) \\ &= 1,\end{aligned}$$

which means that f and g appear nearly identical when $|x|$ is large. ■

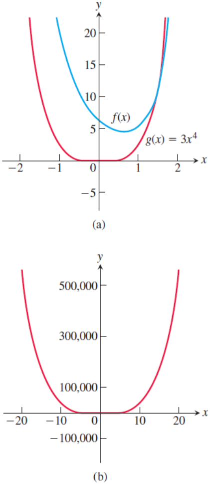


FIGURE The graphs of f and g are (a) distinct for $|x|$ small, and (b) nearly identical for $|x|$ large

(from Thomas'Calculus)

Further properties of continuous functions

Lemma (Boundedness of continuous functions)

If a function $f : D \to \mathbb{R}$ is continuous at $x_0 \in D$, then there exists a neighborhood of x_0, $U(x_0)$, such that the function f is bounded on $U(x_0) \cap D$.

Lemma (Sign-preserving property)

If a function $f : D \to \mathbb{R}$ is continuous at $x_0 \in D$ and $f(x_0) > 0$ (resp., $f(x_0) < 0$), then there exists a neighborhood of x_0, $U(x_0)$, such that $f(x) > 0$ (resp., $f(x) < 0$) for all $x \in U(x_0) \cap D$.

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Definition

A function $f : D \to \mathbb{R}$ has a

- **global** (or **absolute**) **maximum** at $x^* \in D$ if $f(x^*) \geq f(x)$ for all $x \in D$. The point x^* is a **global maximum point** of f;
- **local** (or **relative**) **maximum** at $x^* \in D$ if there exists an $\varepsilon > 0$ such that $f(x^*) \geq f(x)$ for all $x \in U(x^*, \varepsilon) \cap D$. The point x^* is a **local maximum point** of f.
- **global** (or **absolute**) **minimum** at $x^* \in D$ if $f(x^*) \leq f(x)$ for all $x \in D$. The point x^* is a **global minimum point** of f;
- has a **local** (or **relative**) **minimum** at $x^* \in D$ if there exists an $\varepsilon > 0$ such that $f(x^*) \geq f(x)$ for all $x \in U(x^*, \varepsilon) \cap D$. The point x^* is a **local minimum point** of f.

Extremum points of a function

Remarks

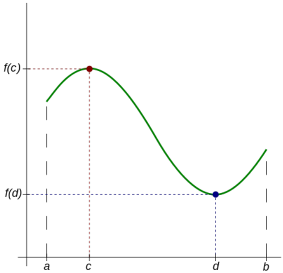
- The points of minima and maxima are called the **points of extrema** of f.
- If the inequalities are strict for all $x \neq x^*$, then x^* is called a **strict** extremum point.
- If a function f has a maximal (resp., minimal) value then $\max_D f = \sup_D f$ (resp., $\min_D f = \inf_D f$).

Extremum points of a function

Theorem (Weierstrass Extreme value theorem)

Any continuous on a closed interval $[a, b]$ function is bounded on $[a, b]$ and attains a maximum and a minimum, each at least once, i.e. there exist $x_1, x_2 \in [a, b]$: $f(x_1) \leq f(x) \leq f(x_2)$ for all $x \in [a, b]$.

As a consequence, $R(f) = [f(x_1), f(x_2)]$.



(from wikipedia.org)

Extremum points of a function

Theorem (Weierstrass Extreme value theorem)

Any continuous on a closed interval $[a, b]$ function is bounded on $[a, b]$ and attains a maximum and a minimum, each at least once, i.e. there exist $x_1, x_2 \in [a, b]$: $f(x_1) \leq f(x) \leq f(x_2)$ for all $x \in [a, b]$.

As a consequence, $R(f) = [f(x_1), f(x_2)]$. **Examples**:

- Is there exists a continuous bijection $f : (0, 1] \to [1, +\infty)$? Yes, e.g. $f(x) = x^{-2}$.
- Is there exists a continuous bijection $f : [0, 1] \to [1, +\infty)$? No, because of Extreme value theorem: $[1, +\infty)$ is unbounded, so $f : [0, 1] \to [1, +\infty)$ cannot be surjective (and therefore, cannot be bijective).

Further properties of continuous functions

Let $f : D \to \mathbb{R}$ be a function.

Zeros of a function

Is there an $x_0 \in D$ such that $f(x_0) = 0$?

Solvability problem

Given an $y_0 \in \mathbb{R}$, is there an $x_0 \in D$ such that $f(x_0) = y_0$?
$\Leftrightarrow$ there is a zero of the function $g(x) := f(x) - y_0$.

Equality problem

Given $f, g : D \to \mathbb{R}$, is there an $x_0 \in D$ such that $f(x_0) = g(x_0)$?
$\Leftrightarrow$ there is a zero of the function $g(x) := f(x) - g(x)$.
Important special case: Fixed point problem.
Given $f : D \to \mathbb{R}$, is there an $x_0 \in D$ such that $f(x_0) = x_0$?

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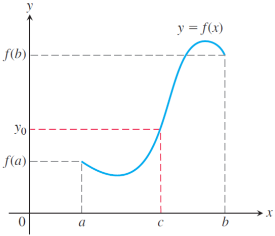
Theorem (Bolzano-Cauchy Intermediate value theorem)

Let a function f be continuous on a closed interval $[a, b]$. Then for any $y_0 \in (f(a), f(b))$ there exists an $x_0 \in [a, b]$ such that $y_0 = f(x_0)$.



(from Thomas'Calculus)

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Theorem (Bolzano-Cauchy Intermediate value theorem)

Let a function f be continuous on a closed interval $[a, b]$. Then for any $y_0 \in (f(a), f(b))$ there exists an $x_0 \in [a, b]$ such that $y_0 = f(x_0)$.

Corollary

Let a function f be continuous on a closed interval $[a, b]$ and $f(a) \cdot f(b) < 0$. Then there exists an $x_0 \in [a, b]$ such that $f(x_0) = 0$.

Corollary

Let a function f be continuous on a closed interval $[a, b]$ and $m = \inf f$, $M = \sup f$. Then $R(f) = [m, M]$.

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Remarks

- *Geometrically*, the Intermediate value theorem says that any horizontal line $y = y_0$ crossing the y-axis between the numbers $f(a)$ and $f(b)$ will cross the curve $y = f(x)$ at least once over the interval $[a, b]$.
- *Consequence for graphing: Connectedness*. Theorem implies that the graph of a function continuous on an interval cannot have any breaks over the interval and represents a connected (single, unbroken) curve, in particular, it does not have jumps or separate branches.

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Applications

Let $f : D \to \mathbb{R}$ be a function.

- *Zeros of a function*: is there exist an $x^* \in D$ such that $f(x^*) = 0$? (x^* is called a **zero of the function** f). The Intermediate value theorem implies that if f is continuous, then any interval on which f changes sign contains a zero of the function, i.e. there exists an x^* such that $f(x^*) = 0$.
- *Solutions of an equation*: given a $y^* \in \mathbb{R}$, is there exist an $x^* \in D$ such that $f(x^*) = y^*$? Defining $g(x) = f(x) - y^*$, we reduce this problem to finding zeros of g.
- *Equal values*: given a $g : D \to \mathbb{R}$, is there exist an $x^* \in D$ such that $f(x^*) = g(x^*)$? Defining $h(x) = f(x) - g(x)$, we reduce this problem to finding zeros of h.
  Special case: *Fixed points problem*: is there exist an $x^* \in D$ such that $f(x^*) = x^*$?

Example: the equation $x^3 - x - 1 = 0$ has a root between 1 and 2 because $f(1) < 0$ and $f(2) > 0$.

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Example: show that $\exists x_0 \in [-\pi, \pi] : \sin x_0 + \frac{x_0}{2} = \sqrt{2}$.

Solution:

$f(x) := \sin x + \frac{x}{2}$ is continuous in $\mathbb{R}$.

$f(-\pi) = -\frac{\pi}{2} < \sqrt{2}$; $f(\pi) = \frac{\pi}{2} > \sqrt{2}$.

Intermediate value theorem implies the existence of $x_0 \in [-\pi, \pi]$ such that $f(x_0) = \sqrt{2}$.

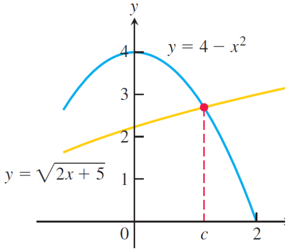
Example:

Use the Intermediate Value Theorem to prove that the equation

$$\sqrt{2x+5} = 4 - x^2$$

has a solution

Solution We rewrite the equation as $\sqrt{2x+5} + x^2 = 4$, and set $f(x) = \sqrt{2x+5} + x^2$. Now $g(x) = \sqrt{2x+5}$ is continuous on the interval $[-5/2, \infty)$ since it is the composite of the square root function with the nonnegative linear function $y = 2x + 5$. Then f is the sum of the function g and the quadratic function $y = x^2$, and the quadratic function is continuous for all values of x. It follows that $f(x) = \sqrt{2x+5} + x^2$ is continuous on the interval $[-5/2, \infty)$. By trial and error, we find the function values $f(0) = \sqrt{5} \approx 2.24$ and $f(2) = \sqrt{9} + 4 = 7$, and note that f is also continuous on the finite closed interval $[0, 2] \subset [-5/2, \infty)$. Since the value $y_0 = 4$ is between the numbers 2.24 and 7, by the Intermediate Value Theorem there is a number $c \in [0, 2]$ such that $f(c) = 4$. That is, the number c solves the original equation. ■



(from Thomas'Calculus)

Bisection method

Let $f : [a, b] \in \mathbb{R}$ be continuous and $f(a) \cdot f(b) \leq 0$. Then there exists an $x^* \in [a, b]$: $f(x_0) = 0$ (by the Intermediate value theorem). How to find x^*? Approximate solution: to construct (recursively) a sequence $(a_j)_{j \in \mathbb{N}}$ converging to x^*.

Algorithm (bisection method): define $a_1 := a$, $b_1 := b$, and let a_{j+1}, b_{j+1} be defined recursively using a_j, b_j:

- if $f(a_j) \cdot f\left(\dfrac{a_j + b_j}{2}\right) \leq 0$, then $a_{j+1} := a_j$, $b_{j+1} := \dfrac{a_j + b_j}{2}$;
- if $f(a_j) \cdot f\left(\dfrac{a_j + b_j}{2}\right) > 0$, then $a_{j+1} := \dfrac{a_j + b_j}{2}$, $b_{j+1} := b_j$.