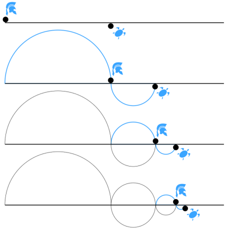
2. Sequences and series

2.3. Series



Content:

- Basic definitions
- Some common series
- Properties of convergent series
- Necessary convergence condition
- Cauchy convergence criterium
- Series with non-negative terms
- Alternating series
- Absolutely converging series and their properties
- Properties of conditionally convergent series
- Convergence tests

Basic definitions

Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers. Consider a new sequence $\{S_n\}_{n\in\mathbb{N}}$:

$$S_1 = a_1,\ S_2 = a_1 + a_2,\ S_3 = a_1 + a_2 + a_3,\ \ldots,\ S_n = \sum_{j=1}^{n} a_j.$$

Definition

We define an **infinite series** as

$$\sum_{j=1}^{\infty} a_j = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{j=1}^{n} a_j.$$

- a_n is the **n-th term** of the series, S_n is the **n-th partial sum**; $\sum_{j=n+1}^{\infty}$ is the **n-th remainder**.
- If the sequence $\{S_n\}_{n\in\mathbb{N}}$ converges, the series $\sum_{j=1}^{\infty} a_j$ is called to be **convergent** (or summable), and $\sum_{j=1}^{\infty} a_j = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{j=1}^{n} a_j$. is called the **sum of series**.
- If $\{S_n\}_{n\in\mathbb{N}}$ diverges (i.e. $\lim_{n\to\infty} S_n$ is infinite or does not exists), the series $\sum_{j=1}^{\infty} a_j$ is said to be **divergent**. (e.g., $\sum\limits_{j=1}^{\infty}(-1)^j$)

Examples of series

The series $\sum_{j=1}^{+\infty} \frac{1}{j^2}$ converges.

Indeed, consider the sequence $\{S_n\}_{n\in\mathbb{N}}$ with $S_n = \sum_{j=1}^{n} \frac{1}{j^2}$.

Remainder: Cauchy convergence criterium

A sequence of real numbers $\{a_n\}_{n\in\mathbb{N}}$ converges if and only if it is a Cauchy sequence, i.e. $\forall\varepsilon > 0\ \exists N_\varepsilon \in \mathbb{N}$: $\forall m, n > N_\varepsilon$, $|a_m - a_n| < \varepsilon$.

Observe that, $\forall j \in \mathbb{N} \setminus \{1\}$, $\frac{1}{j^2} < \frac{1}{j^2 - j} = \frac{1}{j(j-1)} = \frac{1}{j-1} - \frac{1}{j}$.

Consider $|S_m - S_n|$ for arbitrary $m \geq n > 1$:

$$\begin{aligned} |S_m - S_n| &= \sum_{j=n+1}^{m} \frac{1}{j^2} < \sum_{j=n+1}^{m} \left(\frac{1}{j-1} - \frac{1}{j}\right) \\ &= \frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+1} - \frac{1}{n+2} + \cdots + \frac{1}{m-1} - \frac{1}{m} = \frac{1}{n} - \frac{1}{m}. \end{aligned}$$

$\forall\varepsilon > 0$, let us take $N_\varepsilon > 1/\varepsilon$. Then $\forall m \geq n > N_\varepsilon$,

$|S_m - S_n| < \frac{1}{n} - \frac{1}{m} < \frac{1}{n} < \frac{1}{N_\varepsilon} < \varepsilon$.

Examples of series

Geometric series

The **geometric series** $\sum_{j=0}^{+\infty} q^j$ converges for any $q \in (-1, 1)$.

Moreover, in this case $\sum_{j=0}^{+\infty} q^j = \dfrac{1}{1-q}$.

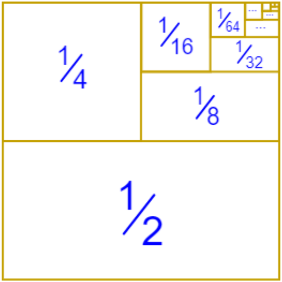
Consider the sequence $\{S_n\}_{n\in\mathbb{N}}$ with $S_n = \sum_{j=0}^{n} q^j$, $|q| < 1$. Then

$$(1-q)S_n = (1-q)\sum_{j=0}^{n} q^j = \sum_{j=0}^{n}\left(q^j - q^{j+1}\right)$$
$$= 1 - q + q - q^2 + q^2 - q^3 + \cdots + q^n - q^{n+1} = 1 - q^{n+1}.$$

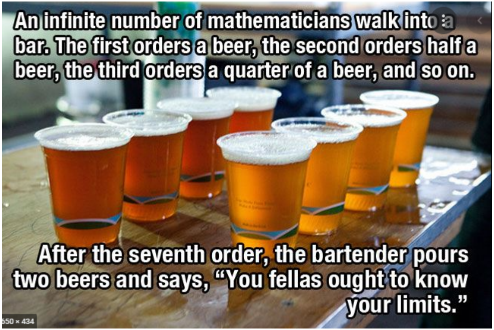
Thus, $\sum_{j=0}^{+\infty} q^j = \lim_{n\to\infty} \dfrac{1-q^{n+1}}{1-q} = \dfrac{1}{1-q}$ as $|q| < 1$.

Corollary

$$\sum_{j=0}^{+\infty} \frac{1}{2^j} = 2.$$



(from wikipedia.org)



(from pinterest.com)

Harmonic series

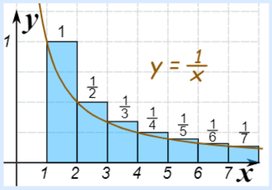
The **harmonic series** $\sum_{j=1}^{+\infty} \frac{1}{j}$ is divergent.

Consider the 2^n-th partial sum:

$$S_{2^n} = \sum_{j=1}^{2^n} \frac{1}{j} = 1 + \frac{1}{2} + \underbrace{\left(\frac{1}{3} + \frac{1}{4}\right)}_{> \frac{1}{2}} + \underbrace{\left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)}_{> \frac{1}{2}}$$

$$+ \cdots + \underbrace{\left(\frac{1}{2^{n-1}+1} + \frac{1}{2^{n-1}+2} + \cdots + \frac{1}{2^n}\right)}_{> \frac{1}{2}} = 1 + \frac{n}{2}.$$

Thus, $\lim_{n\to\infty} S_n \to +\infty$.



1/x vs harmonic series area

(from mathisfun.com)

Properties of convergent series

Let $\sum_{j=1}^{+\infty} a_j$ and $\sum_{j=1}^{+\infty} b_j$ be convergent series. Then

- The sum of these series converges and
  $\sum_{j=1}^{+\infty} (a_j + b_j) = \sum_{j=1}^{+\infty} a_j + \sum_{j=1}^{+\infty} b_j$.
- For any $c \in \mathbb{R}$, the product of the series $\sum_{j=1}^{+\infty} a_j$ with the number c converges and $\sum_{j=1}^{+\infty} ca_j = c \sum_{j=1}^{+\infty} a_j$.
- The series $\sum_{j=1}^{+\infty} a_j$ converges iff any its remainder $r_m = \sum_{j=m+1}^{+\infty} a_j$ converges, moreover, $S = S_m + r_m$, where $S = \sum_{j=1}^{+\infty} a_j$,
  $S_m = \sum_{j=1}^{m} a_j$.

Necessary convergence condition

ALPEN-ADRIA UNIVERSITÄT KLAGENFURT

Theorem (Necessary condition for convergence of a series)

If the series $\sum_{j=1}^{+\infty} a_j$ converges, then $\lim_{n\to\infty} a_n = 0$.

Proof: Denote $S_n = \sum_{j=1}^{n} a_j$ and consider sequences S_n, $n \in \mathbb{N}$, and S_n, $n \in \mathbb{N} \setminus \{1\}$. The series $\sum_{j=1}^{+\infty} a_j$ converges, therefore, $\exists S \in \mathbb{R}$: $\lim_{n\to+\infty} S_n = S$. Because of the uniqueness of a limit, $\lim_{n\to+\infty} S_{n-1} = S$. Then $\lim_{n\to+\infty} a_n = \lim_{n\to+\infty} (S_n - S_{n-1}) = 0$.

Remarks

- The above condition is only necessary, but not sufficient For example, the harmonic series $\sum_{j=1}^{+\infty} \frac{1}{j}$ is divergent.
- The above condition implies that if $\lim_{n\to+\infty} a_n \neq 0$, then $\sum_{j=1}^{+\infty} a_j$ diverges. For example, the geometric series $\sum_{j=1}^{+\infty} q^j$ diverges if $|q| \geq 1$.

Cauchy convergence criterium

ALPEN-ADRIA UNIVERSITÄT KLAGENFURT

Theorem (Cauchy convergence criterium)

The series $\sum_{j=1}^{+\infty} a_j$ converges iff for any $\varepsilon > 0$ there exists an $n_\varepsilon \in \mathbb{N}$ such that, for any $n \geq n_\varepsilon$ and any $p \in \mathbb{N} \cup \{0\}$, $|a_n + a_{n+1} + \cdots + a_{n+p}| < \varepsilon$.

Remark

Necessary convergence condition follows from Cauchy convergence criterium with $p = 0$.

Example: For the series $\sum_{j=1}^{+\infty} \frac{1}{j}$:, $a_n + a_{n+1} + \cdots + a_{2n-1} =$

$$\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n-1} > \frac{1}{2n} + \frac{1}{2n} + \cdots + \frac{1}{2n} = \frac{n}{2n} = \frac{1}{2}.$$

Thus, for $\varepsilon = 1/2$, any $n_\varepsilon = n$ and $p = n-1$, $|a_n + a_{n+1} + \cdots + a_{n+p}| > \varepsilon$.

Lemma

A series $\sum_{j=1}^{+\infty} a_j$ with non-negative terms $a_j \geq 0 \ \forall j \in \mathbb{N}$ converges iff there exists a convergent subsequence of the sequence $\{S_n\}_{n\in\mathbb{N}}$.

Proof: $a_j \geq 0 \ \forall j \in \mathbb{N} \Rightarrow \{S_n\}_{n\in\mathbb{N}}$ is strictly monotonic, therefore, it converges iff it has a convergent subsequence.

Lemma

For the convergence of a series $\sum_{j=1}^{+\infty} a_j$ with non-negative terms $a_j \geq 0 \ \forall j \in \mathbb{N}$,

- it is necessary that $\{S_n\}_{n\in\mathbb{N}}$ is bounded from above;
- it is sufficient that at least one subsequence of $\{S_n\}_{n\in\mathbb{N}}$ is bounded from above; in this case, $\sum_{j=1}^{+\infty} a_j = \sup_{k\in\mathbb{N}}\{S_{n_k}\}$.

ALPEN-ADRIA UNIVERSITÄT KLAGENFURT

Theorem (Comparison convergence test)

Let $0 \le a_n \le b_n$, for (almost) all $n \in \mathbb{N}$. Then the convergence of the series $\sum_{j=1}^{+\infty} b_j$ implies the convergence of $\sum_{j=1}^{+\infty} a_j$, and the divergence of $\sum_{j=1}^{+\infty} a_j$ implies the convergence of $\sum_{j=1}^{+\infty} b_j$.

Example: $\sum_{j=1}^{+\infty} \frac{\sin^2(j\alpha)}{2^j}$ converges for any $\alpha \in \mathbb{R}$ because

$0 \le \frac{\sin^2(j\alpha)}{2^j} \le \frac{1}{2^j}$, and the series $\sum_{j=1}^{+\infty} \frac{1}{2^j}$ converges.

More convergence tests for series with non-negative terms: when studying absolute convergent series.

Alternating series

Definition

A series $\sum_{j=1}^{+\infty} a_j$ is called **alternating** if $a_j = (-1)^j b_j$ with some $b_j \in \mathbb{R}$, for all $j \in \mathbb{N}$.

Theorem (Leibniz convergence test)

Let $\sum_{j=1}^{+\infty} a_j$ be an alternating series, and let $\{|a_n|\}_{n\in\mathbb{N}}$ be a monotonically decreasing infinitesimal sequence. Then $\sum_{j=1}^{+\infty} a_j$ converges and $|r_n| = |\sum_{j=1}^{+\infty} a_j - S_n| \leq a_{n+1}\ \forall n \in \mathbb{N}$.

Example: $\sum_{j=1}^{+\infty} \frac{(-1)^j}{j}$ converges.

Absolutely converging series

ALPEN-ADRIA UNIVERSITÄT KLAGENFURT

Definition

A series $\sum_{j=1}^{+\infty} a_j$ is said to be **absolutely converging** if the series $\sum_{j=1}^{+\infty} |a_j|$.

Theorem

Any absolutely convergent series converges.

But not every convergent series converges absolutely, e.g. $\sum_{j=1}^{+\infty} \frac{(-1)^j}{j}$ converges.

Definition

A series $\sum_{j=1}^{+\infty} a_j$ is said to be **conditionally converging** if it converges but does not converge absolutely.

Properties of absolutely convergent series

Let $\sum_{j=1}^{+\infty} a_j$ and $\sum_{j=1}^{+\infty} b_j$ be absolutely convergent series. Then

- The sum of the series $\sum_{j=1}^{+\infty} (a_j + b_j)$ converges absolutely.
- For any $c \in \mathbb{R}$, the product $\sum_{j=1}^{+\infty} c a_j$ converges absolutely.
- The product of the series $\left(\sum_{j=1}^{+\infty} a_j\right)\left(\sum_{k=1}^{+\infty} b_k\right) = \sum_{j,k=1}^{+\infty} a_j b_k$ converges absolutely, and the sum of their products equals the product of sums.

Unconditionally convergent series

Definition

A series $\sum_{j=1}^{+\infty} a_j$ is **unconditionally convergent** if any permutation creates a series with the same convergence as the original series.

Proposition

Absolutely convergent series are unconditionally convergent.

Properties of conditionally convergent series

Given a series $\sum_{j=1}^{+\infty} a_j$, denote $a_n^+ := a_n$ for all $n \in \mathbb{N} : a_n \geq 0$, and $a_n^- := a_n$ for all $n \in \mathbb{N} : a_n < 0$.

Lemma

If the series $\sum_{j=1}^{+\infty} a_j$ converges conditionally, then $\sum_{j=1}^{+\infty} a_j^+$ and $\sum_{j=1}^{+\infty} a_j^-$ are divergent series.

Theorem (Riemann series theorem)

If the series $\sum_{j=1}^{+\infty} a_j$ converges conditionally, then for any $A \in \overline{\mathbb{R}}$ there exists a permutation of this series creating a series convergent to A.

Properties of conditionally convergent series

Example: $\ln(2) = \sum_{j=1}^{+\infty} \frac{(-1)^{j+1}}{j} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$. But

$$
\begin{aligned}
&\left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} + \frac{1}{6}\right) - \frac{1}{8} + \left(\frac{1}{5} - \frac{1}{10}\right) - \frac{1}{12} + \dots \\
&\qquad = \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \dots \\
&\qquad = \frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots\right) = \frac{\ln(2)}{2}.
\end{aligned}
$$

ALPEN-ADRIA UNIVERSITÄT KLAGENFURT

Theorem (Comparison convergence test)

Let $0 \le c_j \le |a_j| \le b_j$, for (almost) all $j \in \mathbb{N}$. Then

- the convergence of the series $\sum\limits_{j=1}^{+\infty} b_j$ implies the convergence and absolute convergence of $\sum\limits_{j=1}^{+\infty} a_j$;
- the divergence of $\sum\limits_{j=1}^{+\infty} c_j$ implies the divergence of $\sum\limits_{j=1}^{+\infty} a_j$.

Theorem (D'Alembert's ratio test)

Let $\sum_{j=1}^{+\infty} a_j$ be a series and there exits an $N \in \mathbb{N}$ such that $a_n \neq 0$ for all $n \geq N$.

- If there exists an $L \in [0, 1)$ and an $n_0 \geq N$ such that, for all $n > n_0$, $\left|\frac{a_{n+1}}{a_n}\right| \leq L$, then $\sum_{j=1}^{+\infty} a_j$ is absolutely convergent.
- If there exists an $n_0 \geq N$ such that, for all $n > n_0$, $\left|\frac{a_{n+1}}{a_n}\right| \geq 1$, then $\sum_{j=1}^{+\infty} a_j$ is divergent.

Proof:

- For any $j \geq n_0 + 1$, $|a_j| \leq L|a_{j-1}| \leq L^2|a_{j-2}| \leq \cdots \leq L^{j-(n_0+1)}|a_{n_0+1}|$. Therefore, $\sum_{j=n_0+1}^{+\infty} |a_j| \leq |a_{n_0+1}| \sum_{j=n_0+1}^{\infty} L^{j-(n_0+1)} = |a_{n_0+1}| \sum_{j=1}^{\infty} L^k$. The latter series converges to $\frac{|a_{n_0+1}|}{1-L}$ because $L < 1$. Hence, the Comparison convergence test implies the absolute convergence of $\sum_{j=1}^{+\infty} a_j$.
- For any $j \geq n_0 + 1$, $|a_j| \geq |a_{j-1}| \geq \cdots \geq |a_{n_0+1}|$, therefore $\{a_j\}_{j \in \mathbb{N}}$ cannot be an infinitesimal sequence, so that the necessary condition for convergence of a series is not satisfied.

Corollary from D'Alembert's ratio test

Let $\sum_{j=1}^{+\infty} a_j$ be a series.

- If $\overline{\lim_{n\to\infty}} \left|\frac{a_{n+1}}{a_n}\right| < 1$ then the series is absolutely convergent.
- If $\underline{\lim_{n\to\infty}} \left|\frac{a_{n+1}}{a_n}\right| > 1$ then the series is divergent.
- If there exists an $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$, then the series is absolutely convergent for $L < 1$ and divergent for $L > 1$. if $L = 1$ then the test is inconclusive, because there exist both convergent and divergent series that satisfy this case.

ALPEN-ADRIA UNIVERSITÄT KLAGENFURT

Theorem (Cauchy's root test or Cauchy's radical test)

Let $\sum_{j=1}^{+\infty} a_j$ be a series.

- If there exists an $L \in [0, 1)$ and an $n_0 \in \mathbb{N}$ such that, for all $n > n_0$, $\sqrt[n]{|a_n|} \leq L$, then $\sum_{j=1}^{+\infty} a_j$ is absolutely convergent.
- If there exists an $n_0 \geq N$ such that, for all $n > n_0$, $\sqrt[n]{|a_n|} \geq 1$, then $\sum_{j=1}^{+\infty} a_j$ is divergent.

Proof:

- For any $j \geq n_0 + 1$, $\sqrt[j]{|a_j|} \leq L \Rightarrow |a_j| \leq L^j$. Therefore, $\sum_{j=n_0+1}^{+\infty} |a_j| \leq \sum_{j=n_0+1}^{\infty} L^j \leq \sum_{j=0}^{\infty} L^j$. The latter series converges to $\frac{1}{1-L}$ because $L < 1$. Hence, the Comparison convergence test implies the absolute convergence of $\sum_{j=1}^{+\infty} a_j$.
- For any $j \geq n_0 + 1$, $\sqrt[j]{|a_j|} > 1 \Rightarrow |a_j| > 1$, therefore $\{a_j\}_{j \in \mathbb{N}}$ cannot be an infinitesimal sequence, so that the necessary condition for convergence of a series is not satisfied.

Corollary from Cauchy's root test

Let $\sum_{j=1}^{+\infty} a_j$ be a series.

- If $\overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|} < 1$ then the series is absolutely convergent.
- If $\underline{\lim_{n\to\infty}} \sqrt[n]{|a_n|} > 1$ then the series is divergent.
- If there exists an $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$, then the series is absolutely convergent for $L < 1$ and divergent for $L > 1$. if $L = 1$ then the test is inconclusive, because there exist both convergent and divergent series that satisfy this case.

ALPEN-ADRIA UNIVERSITÄT KLAGENFURT

Examples:

1) $\sum\limits_{j=1}^{+\infty} \frac{1}{j!}$: $a_n = \frac{1}{n!}$, $\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{n+1} < 1 \forall n \in \mathbb{N} \Rightarrow$ the series is absolutely convergent by D'Alembert's ratio test.

2) $\sum\limits_{j=1}^{+\infty} \frac{x}{j!}$, $x \in \mathbb{R} \setminus \{0\}$: $a_n = \frac{x}{n!}$, $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0 < 1 \Rightarrow$ the series is absolutely convergent by D'Alembert's ratio test.

3) $\sum\limits_{j=1}^{+\infty} \frac{1}{j^j}$: $a_n = \frac{1}{n^n}$, $\sqrt[n]{|a_n|} = \frac{1}{n} > 1 \forall n \in \mathbb{N} \setminus \{1\} \Rightarrow$ the series is absolutely convergent by Cauchy's root test.

4) $\sum\limits_{j=1}^{+\infty} \frac{x^j}{j}$: $a_n = \frac{x^n}{n}$, $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = |x| \lim\limits_{n\to\infty} \frac{1}{\sqrt[n]{|n|}} = |x|$ the series is absolutely convergent for $|x| < 1$ and divergent for $|x| > 1$ by Cauchy's root test.

5) $\sum\limits_{j=1}^{+\infty} \frac{1}{j}$, $\sum\limits_{j=1}^{+\infty} \frac{1}{j^2}$: both series satisfy both the conditions $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$ and $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = 1$, but the first series is divergent and the second one is absolutely convergent.

ALPEN-ADRIA UNIVERSITÄT KLAGENFURT

Examples:

6) $\sum_{j=1}^{+\infty} \frac{2^j}{j^2}$: $a_n = \frac{2^n}{n^2}$, $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 2 \lim_{n\to\infty} \frac{1}{\sqrt[n]{|n|^2}} = 2 > 1$ the series is divergent by Cauchy's root test.

7) $\sum_{j=1}^{+\infty} \frac{j}{e^j}$: $a_n = \frac{n}{e^n}$,

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{n+1}{e^{n+1}}}{\frac{n}{e^n}}\right| = \frac{1}{e} \lim_{n\to\infty} \left|\frac{n+1}{n}\right| = \frac{1}{e} < 1 \Rightarrow$$ the series is absolutely convergent by D'Alembert's ratio test.

8) Let $0 < q_1 < q_2 < 1$, consider $\sum_{j=1}^{+\infty} a_j$ with $a_n = \begin{cases} q_1^n \text{ if } n \text{ is } \textit{even}, \\ q_2^n \text{ if } n \text{ is } \textit{odd}. \end{cases}$.

$\sqrt[n]{|a_n|} = \begin{cases} q_1 \text{ if } n \text{ is } \textit{even}, \\ q_2 \text{ if } n \text{ is } \textit{odd}. \end{cases} \Rightarrow \overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|} < 1 \Rightarrow$ the series is absolutely convergent by Cauchy's root test.

Remark 1: $\lim_{n\to\infty} \sqrt[n]{|a_n|}$ does not exist!

Remark 2: D'Alembert's ratio test is not informative.

Convergence tests

Theorem (Dirichlet's test)

Let $\{a_n\}_{n\in\mathbb{N}}$ be a monotonically decreasing sequence of real numbers, $\lim\limits_{n\to\infty} a_n = 0$, and $\{b_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers such that there exists an $M > 0$: $\left|\sum\limits_{j=1}^{N} b_j\right| \leq M$ for any $N \in \mathbb{N}$.

Then the series $\sum\limits_{j=1}^{+\infty} a_j b_j$ is convergent.

Theorem (Abel's test)

Let $\sum\limits_{j=1}^{+\infty} a_j$ be a convergent series, and $\{b_n\}_{n\in\mathbb{N}}$ be a bounded monotone sequence of real numbers. Then the series $\sum\limits_{j=1}^{+\infty} a_j b_j$ is convergent.

Examples:

1) $\displaystyle\sum_{j=1}^{+\infty} \frac{\sin j\alpha}{j}$: if $\alpha \neq 2\pi m$, $m \in \mathbb{Z}$, then $\displaystyle\sum_{j=1}^{n} \sin j\alpha = \sum_{j=1}^{n} \frac{2\sin\frac{\alpha}{2}\sin j\alpha}{2\sin\frac{\alpha}{2}} =$

$$\frac{\sum_{j=1}^{n}\left(\cos\left(j-\frac{1}{2}\right)\alpha - \cos\left(j+\frac{1}{2}\right)\alpha\right)}{2\sin\frac{\alpha}{2}} = \frac{\cos\frac{1}{2}\alpha - \cos\left(n+\frac{1}{2}\right)\alpha}{2\sin\frac{\alpha}{2}} =$$

$\dfrac{\sin\frac{n+1}{2}\alpha\sin\frac{n}{2}\alpha}{\sin\frac{\alpha}{2}}$. Therefore, $\displaystyle\left|\sum_{j=1}^{n}\sin j\alpha\right| \leq \frac{1}{\left|\sin\frac{\alpha}{2}\right|}$.

if $\alpha \neq 2\pi m$, $m \in \mathbb{Z}$, then $\displaystyle\sum_{j=1}^{n}\sin j\alpha = 0$. Therefore, $\displaystyle\sum_{j=1}^{n}\sin j\alpha$ are bounded for any $\alpha \in \mathbb{R}$. Since the sequence $\left\{\frac{1}{n}\right\}_{n\in\mathbb{N}}$ monotonically decreases and converges to 0, Dirichlet's test implies the convergence of the series for any $\alpha \in \mathbb{R}$.

Examples:

2) $\displaystyle\sum_{j=1}^{+\infty} \frac{\sin j\alpha \cos \frac{\pi}{j}}{\ln \ln j}$:

the series $\displaystyle\sum_{j=1}^{+\infty} \frac{\sin j\alpha}{\ln \ln j}$ converges by Dirichlet's test, the sequence $\{cos\frac{\pi}{n}\}_{n\in\mathbb{N}}$ is monotone and bounded. Therefore, the given series converges by Abel's test.