EXAM STRUCTURE

(preliminary version, the number of questions/problems/points may vary)

Exam consists of a theoretical part and a practical part.

THEORETICAL PART

Any auxiliaries (incl. notes and calculators) are prohibited!
Duration: 40 min.
Content:

I) 5 true/false questions; 1 point for each correct answer, 5 points in total.
Example question: *Decide whether the following statements are true. Argument if yes, give a counterexample if not.*

1. *The product of two strictly decreasing functions is a strictly decreasing function.*
Answer: False. Counterexample: $f_1 : (-\infty; 0) \to (-\infty; 0)$, $f_1(x) = \dfrac{1}{x}$, $f_2 : (-\infty; 0) \to (0, +\infty)$, $f_2(x) = x^2$. f_1, f_2 are strictly decreasing on $(-\infty; 0)$, $(f_1 \cdot f_2)(x) = x$ is strictly increasing on $(-\infty; 0)$.

2. *Any quadratic polynomial has two complex roots.*
Answer: True. Justification: fundamental theorem of algebra.

II) 4 questions on formulating definitions and statements, giving examples; 3–5 points for each correct answer, 12–20 points in total.
Example questions:

1. *Formulate the definition of a strictly increasing function.*
Answer: Let $f : D \to \mathbb{R}$ be a function, $D \subseteq \mathbb{R}$, x_1 and x_2 be any two points in D. The function f is strictly increasing in D if $f(x_2) > f(x_1)$ whenever $x_1 < x_2$.
2. *Formulate the Fundamental theorem of algebra.*
Answer: Fundamental theorem of algebra: every single-variable polynomial with complex coefficients
$$p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, \quad a_0, a_1, \ldots, a_{n-1} \in \mathbb{C}, n \geq 1,$$
has at least one complex root.

PRACTICAL PART

Notes are allowed, handwritten, max. 1 A4 sheet (two-sided). Calculators are prohibited!
Duration: 70 min.
Content: 5 problems; 5–10 points for each correct answer (with justified solution). 25–40 points in total.
Example problems:

1. *Consider the sequence $\{a_n\}_{n\in\mathbb{N}}$ with $a_n = \dfrac{1}{2}\left(a_{n-1} + \dfrac{c}{a_{n-1}}\right)$ for all $n \in \mathbb{N}$, where $c > 0$ and $a_0 = M > 0$.*

1.1) Prove that $a_n \geq \sqrt{c}\ \forall n \in \mathbb{N}$. (2 point)
***Hint:** use the fact that $t + \dfrac{1}{t} \geq 2\ \forall t \geq 0$.*

1.2) Using 1.2), prove that $\{a_n\}_{n\in\mathbb{N}}$ monotonically decreases. (2 points)

1.3) Determine whether the sequence $\{a_n\}_{n\in\mathbb{N}}$ is bounded or unbounded. (1 point)

1.4) Prove that $\{a_n\}_{n\in\mathbb{N}}$ has a finite limit. (1 point)

1.5) Prove that $\lim\limits_{n\to\infty} a_n = \sqrt{c}$. (2 points).

Solution:
1.1) By mathematical induction, $a_n > 0\ \forall n \in \mathbb{N}$. Indeed, $a_0 = M > 0$. For an arbitrary $k \in \mathbb{N}$, assume that $a_k > 0$. Then $a_{k+1} = \frac{1}{2}\left(a_k + \frac{c}{a_k}\right) > 0$ because $a_k > 0$ and $c > 0$. By the principle of mathematical induction, $a_n > 0\ \forall n \in \mathbb{N}$.
Applying the fact $t + \frac{1}{t} \geq 2\ \forall t \geq 0$, we get

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{k}{a_n}\right) = \frac{\sqrt{k}}{2}\left(\frac{a_n}{\sqrt{k}} + \frac{\sqrt{k}}{a_n}\right) \geq \frac{\sqrt{k}}{2}2 = \sqrt{k}, \forall n \in \mathbb{N}.$$

1.2) Since $a_n \geq \sqrt{k}\ \forall n \in \mathbb{N}$, $a_{n+1} = \frac{1}{2}\left(a_n + \frac{k}{a_n}\right) = \frac{a_n}{2}\left(1 + \frac{k}{a_n^2}\right) \leq \frac{a_n}{2}\left(1 + \frac{k}{k}\right) = a_n$, $\forall n \in \mathbb{N}$. Thus, $a_{n+1} \geq a_n\ \forall n \in \mathbb{N}$, so $\{a_n\}_{n\in\mathbb{N}}$ monotonically decreases.
1.3) Since $\{a_n\}_{n\in\mathbb{N}}$ monotonically decreases, we have $a_n \leq a_0 = M\ \forall n \in \mathbb{N}$.
1.4) The sequence $\{a_n\}_{n\in\mathbb{N}}$ is bounded from below and monotonically decreasing, therefore, it has a finite limit (by the Weierstrass theorem).
1.5) Denote $\lim\limits_{n\to\infty} a_n = x$. Then $\lim\limits_{n\to\infty} a_{n-1} = x$, and

$$x = \lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{1}{2}\left(a_{n-1} + \frac{k}{a_{n-1}}\right)\right) = \frac{1}{2}\left(x + \frac{k}{x}\right).$$

Solving the above equation for $x \geq 0$ (because of $a_n \geq 0 \forall n \in \mathbb{N}$), we get $x = \sqrt{k}$.

2. *Compute $\int\limits_0^{1/2} e^x \sin(\pi x)\mathrm{d}x$ (4 points).*

   Solution:

$$\int\limits_0^{1/2} e^x \sin(\pi x)\mathrm{d}x = \left\{\begin{array}{cc} u = e^x & \mathrm{d}u = e^x\mathrm{d}x \\ \mathrm{d}v = \sin(\pi x)\mathrm{d}x & v = -\frac{1}{\pi}\cos(\pi x) \end{array}\right\} = -\frac{e^x}{\pi}\cos(\pi x)\Big|_0^{1/2} + \frac{1}{\pi}\int\limits_0^{1/2} e^x\cos(\pi x)\mathrm{d}x$$

$$= \underbrace{-\frac{e^{1/2}}{\pi}\cos(\pi/2)}_{=0} + \underbrace{\frac{e^0}{\pi}\cos(0)}_{=1/\pi} + \frac{1}{\pi}\int\limits_0^{1/2} e^x\cos(\pi x)\mathrm{d}x = \left\{\begin{array}{cc} u = e^x & \mathrm{d}u = e^x\mathrm{d}x \\ \mathrm{d}v = \cos(\pi x)\mathrm{d}x & v = \frac{1}{\pi}\sin(\pi x) \end{array}\right\}$$

$$= \frac{1}{\pi} + \frac{e^x}{\pi^2}\sin(\pi x)\Big|_0^{1/2} - \frac{1}{\pi^2}\int\limits_0^{1/2} e^x\sin(\pi x)\mathrm{d}x = \frac{1}{\pi} + \frac{\sqrt{e}}{\pi^2} - \frac{1}{\pi^2}\int\limits_0^{1/2} e^x\sin(\pi x)\mathrm{d}x.$$

Denote $A := \int\limits_0^{1/2} e^x\sin(\pi x)\mathrm{d}x$. Then

$$A = \frac{1}{\pi} + \frac{\sqrt{e}}{\pi^2} - \frac{A}{\pi^2},$$

therefore,

$$A = \frac{\pi + \sqrt{e}}{\pi^2 + 1}.$$

Answer: $A = \frac{\pi + \sqrt{e}}{\pi^2 + 1}$.