1. Preliminaries

1.2. Set of real numbers

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- Basic operations and properties
- Completeness
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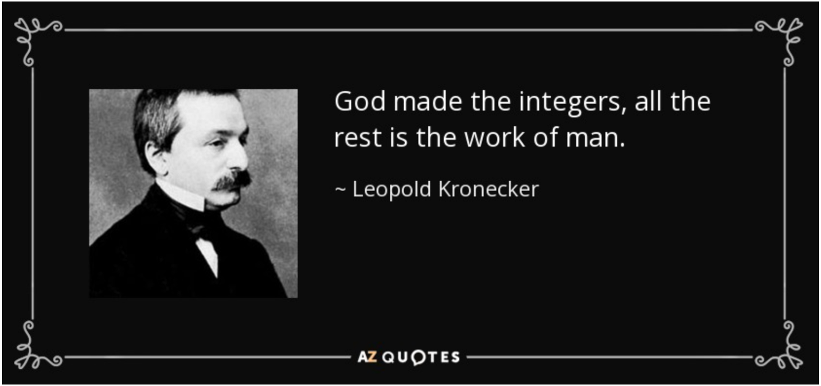
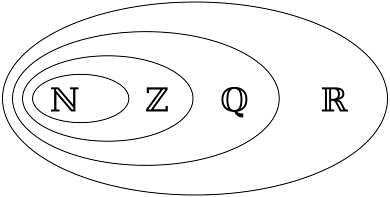
Real numbers



Real numbers

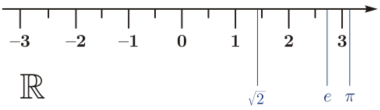
- $\mathbb{N} = \{1, 2, 3, \dots\}$ – the set of all natural numbers $(+, \cdot)$
- $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ – the set of all integer numbers $(+, \cdot, -)$
- $\mathbb{Q} = \left\{ \frac{a}{b} \middle| \, a, b \in \mathbb{Z} \text{ and } b \neq 0 \right\}$ – the set of all rational numbers $(+, \cdot, -, \div)$
- **$\mathbb{R}$ – the set of all real numbers**



Any real number can be determined by a possibly infinite decimal representation. Every conceivable decimal expansion represents a real number, although some numbers have two representations.

Examples: $-\frac{3}{4} = -0.75$, $\frac{1}{3} = 0.3333\ldots$, $1\frac{1}{7} = 0.142857142857\ldots$, $\sqrt{2} = 1.4142\ldots$, $\pi = 3.141592\ldots$, $1 = 1.00000\ldots$ (and $0.999999\ldots$).

A **number (real) line** is a picture of a graduated straight line that serves as visual representation of the real numbers. Every point of a number line is assumed to correspond to a real number, and every real number to a point.



Basic operations and properties

- Addition $+$
- Multiplication $\cdot$
- Ordering $>$

Addition

I. For any two numbers a and b, there is a uniquely defined number $a + b$ called their **sum**, which satisfies the following properties:

I.1) *commutativity:* for any two numbers a, b,

$$a + b = b + a;$$

I.2) *associativity:* for any three numbers a, b, c,

$$a + (b + c) = (a + b) + c;$$

I.3) *additive identity:* there is a number denoted 0 and called *zero* such that, for every number a,

$$a + 0 = a;$$

I.4) *additive inverses:* for any number a, there is a number denoted $-a$ and called the *opposite number or additive inverse* , such that

$$a + (-a) = 0.$$

II. For any two numbers a and b, there is a uniquely defined number ab (or $a \cdot b$) called their **product**, which satisfies the following properties:

II.1) *commutativity:* for any two numbers a and b,

$$a \cdot b = b \cdot a;$$

II.2) *associativity:* for any three numbers a, b, and c,

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c;$$

II.3) *multiplicative identity:* there is a number denoted 1 and called *one* such that, for every number a,

$$a \cdot 1 = a;$$

II.4) *multiplicative inverse:* for any number $a \neq 0$, there is a number denoted $\frac{1}{a}$ and called the *multiplicative inverse (reciprocal)*, s. t.

$$a \cdot \left(\frac{1}{a}\right) = 1.$$

III. Relation between addition and multiplication:

- *distributive property:* for any three numbers a, b, and c,

$$(a+b) \cdot c = a \cdot c + b \cdot c.$$

Total ordering

IV. For any number a, we can define one of the relations: $a > 0$ (a greater than 0), $a = 0$ (a equals 0), or $a < 0$ (a less than 0), in such a way, that the condition $a > 0$ is equivalent to the condition $-a < 0$. Meanwhile, if $a > 0$, $b > 0$, then

IV.1) $a + b > 0$;

IV.2) $a \cdot b > 0$.

The number b is called to be **greater than a**, $b > a$, or, equally, the number a is called to be **less than b**, $a < b$, if $b - a > 0$. The existence of the properties "greater" or "less" means for any pair of real numbers a and b means that the set or real numbers is **ordered**.

Completeness

Two sets $A \subset \mathbb{R}$ and $B \subset \mathbb{R}$ are called to be a **cut** $A|B$ of the set of real numbers $\mathbb{R}$ if $A \cup B = \mathbb{R}$; $A \neq \emptyset$ and $B \neq \emptyset$; $\forall a \in A, b \in B$, $a < b$.

Dedekind cut : for any $\alpha \in \mathbb{R}$, let $A := \{x : x \leq \alpha\}$, $B\{x : x > \alpha\}$. In this case, we say that the cut is generated by α, $\alpha = A|B$. It has the following properties:

- The set A has the greatest number (α), while there is no smallest in the set B.
- The number which generates the cut is unique.

V. For every cut $A|B$ of the set of real numbers there exists a number α which generates this cut: $\alpha = A|B$.

Each real number can be defined by the corresponding Dedekind cut!

Completeness: *no "gaps" or "missing points" in the real number line.*

Axiomatic definition

The set of real numbers is a non-trivial set of elements satisfying properties I–V, i.e.:

- The set $\mathbb{R}$ is a set with well-defined operations of addition and multiplication.
- The set $\mathbb{R}$ is ordered, meaning that there is a total order $>$.
- The order is complete, meaning that there are no "gaps" or "missing points" in the real numbers line.

For any numbers $a \in \mathbb{R}$ and $b \in \mathbb{R}$, we call the number $a + (-b)$ to be a **difference** between a and b and denote it $a - b$,

$$a - b := a + (-b).$$

For any numbers $a \in \mathbb{R}$ and $b \in \mathbb{R}$, $b \neq 0$, we call the number $a \cdot \frac{1}{b}$ to be a **quotient** of a and b and denote it $a : b$ or $\frac{a}{b}$, or a/b,

$$\frac{a}{b} := a \cdot \frac{1}{b},\ b \neq 0.$$

Subtraction and division properties

- The zero element is unique.
- $\forall a \in \mathbb{R}$, its opposite $-a$ is unique.
- $\forall a \in \mathbb{R}$, $-(-a) = a$.
- $\forall a \in \mathbb{R}$, $a - a = 0$.
- $\forall a, b \in \mathbb{R}$, $-a - b = -(b + a)$
- $\forall a, b \in \mathbb{R}$, the solution of the equation $a + x = b$ exists and is unique, $x = b - a$.
- The identity element is unique.
- $\forall a \in \mathbb{R}$, $a \neq 0$, its inverse $\frac{1}{a}$ is unique.
- $\forall a \in \mathbb{R}$, $a \neq 0$, $\frac{1}{1/a} = a$.
- $\forall a \in \mathbb{R}$, $a \neq 0$, $\frac{a}{a} = 1$.
- $\forall a, b \in \mathbb{R}$, $a \neq 0$, $b \neq 0$, $\frac{1}{a} \cdot \frac{1}{b} = \frac{1}{a \cdot b}$.
- $\forall a, b \in \mathbb{R}$, $a \neq 0$, the solution of the equation $a \cdot x = b$ exists and is unique, $x = \frac{b}{a}$.

Subtraction and division properties (cont'd)

- $\forall a, b, c, d \in \mathbb{R},\ b, d \neq 0,\ \frac{a}{b} = \frac{c}{d} \iff ad = bc$;
  corollary (basic fraction property): $\forall a, b, c \in \mathbb{R},\ b, c \neq 0$, $\frac{a}{b} = \frac{ac}{bc}$.
- $\forall a, b, c \in \mathbb{R},\ a(b - c) = ab - ac$
- $\forall a \in \mathbb{R},\ a \cdot 0 = 0$
- if $ab = 0$, then $a = 0$, or $b = 0$, or both of them
- $\forall a, b \in \mathbb{R},\ (-a)b = -ab,\ (-a)(-b) = ab$
- $\forall a, b, c, d \in \mathbb{R},\ b, d \neq 0,\ \frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$
- $\forall a, b, c, d \in \mathbb{R},\ b, d \neq 0,\ \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$
- the inverse for $\frac{a}{b}$ is $\frac{b}{a}$, i.e. $\frac{a}{b} \cdot \frac{b}{a} = 1$
- $\forall a, b, c, d \in \mathbb{R},\ b, c, d \neq 0,\ \frac{a}{b} : \frac{c}{d} = \frac{ad}{bc}$
- $\forall a \in ,n \in \mathbb{N},\ a^n := \underbrace{aa \dots a}_{n \text{ times}};\ a^0 := 1$
- $\forall a \in ,m, n \in \mathbb{Z},\ a^m a^n = a^{m+n},\ (a^m)^n = a^{mn}$ (for $m \leq 0$ or $n \neq 0$ important $a \neq 0$)

Ordering properties

$\forall a, b, c, d \in \mathbb{R}$,

- if $a > b$ and $b > c$, then $a > c$
- if $a > b$ then $a + c > b + c$
- only one of the following relations possible: $a > b$, $a = b$, $a < b$
- if $a < b$ then $-a > -b$
- if $a < b$ and $c \leq d$, then $a + c < b + d$
- if $a < b$ and $c \geq d$, then $a - c < b - d$
- if $a < b$ and $c < 0$, then $ac > bc$
- if $a > 0$ then $\frac{1}{a} > 0$
- if $0 < a < b$ then $0 < \frac{1}{b} < \frac{1}{a}$
- if $a > 0$ and $b > 0$ then $a < b \iff a^2 < b^2$

Absolute value

For any $a \in \mathbb{R}$, the **absolute value of** a, $|a|$, is defined as

$$|a| = \begin{cases} a, & \text{if } a \geq 0, \\ -a, & \text{if } a < 0. \end{cases}$$

Properties: $\forall a, b \in \mathbb{R}, c \in \mathbb{R} \setminus \{0\}$

- $|a| \geq 0$, $|a| = 0 \iff a = 0$,
- $|a| = |-a|$, $a \leq |a|$, $-a \leq |a|$, $|a| = \max\{a, -a\}$
- $|a + b| \leq |a| + |b|$ (triangle inequality)
- $||a| - |b|| \leq |a - b|$ (corollary from the triangle inequality)
- $|a \cdot b| = |a| \cdot |b|$, $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$

An **n-th root of a number a** is a number r which, when raised to the power n, yields x: $a^n = x$. It is denoted as $\sqrt[n]{a}$ or $a^{1/n}$, i.e.

$$(\sqrt[n]{a})^n := a.$$

Here n is a positive integer, sometimes called the degree of the root.

Properties: let $n, m \in \mathbb{N}$, $a, b \geq 0$. Then

- $\sqrt[n]{\sqrt[m]{a}} = \sqrt[mn]{a}$
- $\sqrt[n]{a} = \sqrt[mn]{a^m}$
- $\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$
- for $b \neq 0$, $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$
- $(\sqrt[n]{a})^m = a^{\frac{m}{n}} = \sqrt[n]{a^m}$

Let $a > 0$, $r \in \mathbb{Q}$, i.e. $r = \frac{m}{n}$ with some $m, n \in \mathbb{N}$. Then

$$a^r := \sqrt[n]{a^m}.$$

Properties: let $r_1, r_2 \in \mathbb{Q}$, $a, b > 0$. Then

- $a^{-r} = \frac{1}{a^r}$
- $a^{r_1} a^{r_2} = a^{r_1 + r_2}$
- $(a^{r_1})^{r_2} = a^{r_1 r_2}$
- $(ab)^r = a^r b^r$

Extended real line

Often, it is convenient to extend the set of real numbers with elements $+\infty$ (**plus infinity**) and $-\infty$ (**minus infinity**), assuming that, by definition,

- $-\infty < +\infty$;
- $(+\infty) + (+\infty) = +\infty$, $(-\infty) + (-\infty) = -\infty$, $(+\infty) - (-\infty) = +\infty$, $(-\infty) - (+\infty) = -\infty$;
- $(+\infty) \cdot (+\infty) = (-\infty) \cdot (-\infty) = +\infty$, $(+\infty) \cdot (-\infty) = (-\infty) \cdot (+\infty) = -\infty$;
- $\forall a \in \mathbb{R}$, $a + (+\infty) = +\infty + a = +\infty$, $a + (-\infty) = -\infty + a = -\infty$;
- $\forall a > 0$, $a \cdot (+\infty) = (+\infty) \cdot a = +\infty$, $a \cdot (-\infty) = (-\infty) \cdot a = -\infty$;
- $\forall a < 0$, $a \cdot (+\infty) = (+\infty) \cdot a = -\infty$, $a \cdot (-\infty) = (-\infty) \cdot a = +\infty$.

The set of real numbers with elements $+\infty$ and $-\infty$ is called **affinely extended real number system**
$\bar{\mathbb{R}} = [-\infty, +\infty] = \mathbb{R} \cup \{-\infty, +\infty\}$.

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A subset of the real line is called an **interval** if it contains at least two numbers and all the real numbers lying between any two of its elements.

TABLE Types of intervals

Notation	Set description	Type	Picture
(a, b)	$\{x \mid a < x < b\}$	Open	
$[a, b]$	$\{x \mid a \le x \le b\}$	Closed	
$[a, b)$	$\{x \mid a \le x < b\}$	Half-open	
$(a, b]$	$\{x \mid a < x \le b\}$	Half-open	
(a, ∞)	$\{x \mid x > a\}$	Open	
$[a, \infty)$	$\{x \mid x \ge a\}$	Closed	
$(-\infty, b)$	$\{x \mid x < b\}$	Open	
$(-\infty, b]$	$\{x \mid x \le b\}$	Closed	
$(-\infty, \infty)$	$\mathbb{R}$ (set of all real numbers)	Both open and closed	

From Thomas' Calculus.

The process of finding the interval or intervals of numbers that satisfy an inequality in x is called **solving the inequality**.

Neighbourhood in $\mathbb{R}$

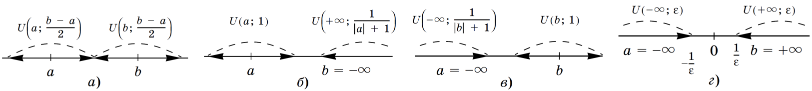
Given a real number $a \in \mathbb{R}$ and a number $\varepsilon > 0$, an **ε-neighbourhood of the number a** is the interval $(a-\varepsilon, a+\varepsilon)$:

$$U(a,\varepsilon) := (a-\varepsilon, a+\varepsilon).$$

For $a = \pm\infty$, $U(+\infty,\varepsilon) := \left(\frac{1}{\varepsilon}, +\infty\right]$, $U(-\infty,\varepsilon) := \left[-\infty, -\frac{1}{\varepsilon}\right)$.

Lemma

For any $a, b \in \bar{\mathbb{R}}$ there exist $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that $U(a,\varepsilon_1) \cap U(b,\varepsilon_2) = \emptyset$.



Bounded sets in $\mathbb{R}$

A set $S \subset \mathbb{R}$ is called **bounded from above** if there exists a number $k \in \mathbb{R}$ (not necessarily in S) such that $k \geq s$ for all $s \in S$.
The number k is called an **upper bound of S**.
The terms **bounded from below** and **lower bound** are similarly defined.
A set S is **bounded** if it has both upper and lower bounds, i.e. if there exists $k_1 \in \mathbb{R}$ and $k_2 \in \mathbb{R}$ such that $k_1 \leq s \leq k_2$ for all $s \in S$.

A set in $\mathbb{R}$ is bounded if it is contained in a finite interval.

A set $S \subset \mathbb{R}$ is called **unbounded from above** if for any number $k \in \mathbb{R}$ there exists an $s \in S$ such that $s > k$.
A set $S \subset \mathbb{R}$ is called **unbounded** if it is not bounded.

Examples: Bounded sets: $(0, 1)$, $[1, 2]$, the set of values of the function $\sin x$;
Bounded from above or below sets: $(-5, +\infty)$, $\mathbb{N}$;
Unbounded sets: $\mathbb{Q}$, $\mathbb{Z}$.

The least and the greatest elements

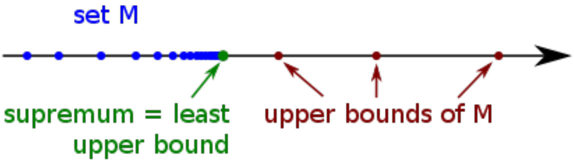
Let S be a set in $\mathbb{R}$. A number $a \in S$ is called the **least element (or minimum) of S**, $a = \min S$, if $s \geq a$ for any $s \in S$. A number $b \in S$ is called the **greatest element (or maximum) of S**, $b = \max S$, if $s \leq b$ for any $s \in S$.

Important: least/greatest elements should belong to S, while lower/upper bounds may be any element in $\mathbb{R}$:

A number $a \in \mathbb{R}$ is called a **lower bound of S**, if $s \geq a$ for any $s \in S$. A number $b \in \mathbb{R}$ is called an **upper bound of S**, if $s \leq b$ for any $s \in S$.

Infimum and supremum

Let S be a set in $\mathbb{R}$.
The **infimum of S**, $\inf S$ or $\inf_{s\in S}\{s\}$ (or the greatest lower bound, GLB), is the greatest element in $\mathbb{R}$ that is less than or equal to each element of S (if such an element exists).
The **supremum of S**, $\sup S$ or $\sup_{s\in S}\{s\}$ (or the least upper bound, LUP), is the least element in $\mathbb{R}$ that is greater than or equal to each element of S (if such an element exists).



Infimum and supremum (formal definitions)

Let S be a set in $\mathbb{R}$. A number $\alpha \in \mathbb{R}$ is the **infimum of S** if:

- $s \geq \alpha \ \forall s \in S$;
- $\forall \alpha' > \alpha \ \exists s \in S$: $s < \alpha'$.

Let S be a set in $\mathbb{R}$. A number $\beta \in \mathbb{R}$ is the **supremum of S** if:

- $s \leq \beta \ \forall s \in S$;
- $\forall \beta' < \beta \ \exists s \in S$: $s > \beta'$.

Equivalent definitions:

Let S be a set in $\mathbb{R}$. A number $\alpha \in \mathbb{R}$ is the **infimum of S** if:

- $s \geq \alpha \ \forall s \in S$;
- $\forall \varepsilon > 0 \ \exists s \in S$: $s < \alpha + \varepsilon$.

Let S be a set in $\mathbb{R}$. A number $\beta \in \mathbb{R}$ is the **supremum of S** if:

- $s \leq \beta \ \forall s \in S$;
- $\forall \varepsilon > 0 \ \exists s \in S$: $s > \beta - \varepsilon$.

Infimum and supremum (formal definitions)

Let S be a set in $\mathbb{R}$. A number $\alpha \in \mathbb{R}$ is the **infimum of S** if:

- $s \geq \alpha \ \forall s \in S$;
- $\forall \alpha' > \alpha \ \exists s \in S$: $s < \alpha'$.

Let S be a set in $\mathbb{R}$. A number $\beta \in \mathbb{R}$ is the **supremum of S** if:

- $s \leq \beta \ \forall s \in S$;
- $\forall \beta' < \beta \ \exists s \in S$: $s > \beta'$.

Examples:
For $\mathbb{R}^+ = (0, +\infty)$, $\inf \mathbb{R}^+ = 0$, $\nexists \sup \mathbb{R}^+$ ($\sup \mathbb{R}^+ = +\infty$).
For $S = [a, b]$, $\inf S = a$, $\sup S = b$, $\min S = a$, $\max S = b$.
For $S = (a, b)$, $\inf S = a$, $\sup S = b$, $\nexists \min S, \nexists \max S$.
For $S = \{a, b\}$, $a < b$, $\inf S = \min S = a$, $\sup S = \max S = b$.
For $S = \{x \in \mathbb{R} : x^2 < \pi\}$, $\inf S = -\sqrt{\pi}$, $\sup S = \sqrt{\pi}$, $\nexists \min S, \nexists \max S$.
For $S = \{x \in \mathbb{Z} : x^2 < \pi\}$, $\inf S = -1$, $\sup S = 1$, $\min S = -1$, $\max S = 1$.
If $\xi = A|B$ (a cut), then $\xi = \sup A = \inf B$.

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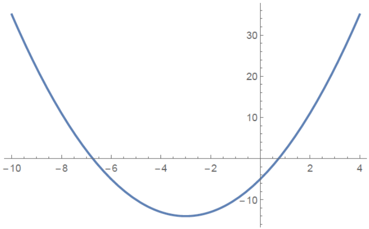
Examples:

Let $S = \{y \in \mathbb{R} : y = x^2 + 6x - 5,\ x \in \mathbb{R}\}$.
Define $f(x) = x^2 + 6x - 5$.
Then $f(x) = x^2 + 6x - 5 = (x^2 + 6x + 9) - 9 - 5 = (x + 3)^2 - 14 \geq -14$
and $f(-3) = -14$.
$\inf S = \min S = -14$,
the function $f(x)$ can be arbitrary large $\Rightarrow \nexists \max S, \sup S$ $(\sup S = +\infty)$.



Existence of infimum and supremum

Theorem (least-upper-bound property)

Any non-empty set of real numbers that has an upper bound must have a least upper bound in real numbers.

Theorem

Any bounded from above non-empty set in $\mathbb{R}$ has a supremum, and any bounded from below non-empty set in $\mathbb{R}$ has an infimum.

If the set S is unbounded from below, we say that $\inf S = -\infty$.
If the set S is unbounded from above, we say that $\sup S = +\infty$.

Any non-empty set in $\mathbb{R}$ has infimum and supremum from $\bar{\mathbb{R}}$.

Properties of infima and suprema

Let us define the following operations with sets $S_1, S_2 \subseteq \mathbb{R}$:

- $S_1 + S_2 = \{s : s = s_1 + s_2,\ s_1 \in S_1, s_2 \in S_2\}$;
  $S_1 - S_2 = \{s : s = s_1 - s_2,\ s_1 \in S_1, s_2 \in S_2\}$;
- for $\lambda \in \mathbb{R}$, $\lambda S_1 = \{s : s = \lambda s_1,\ s_1 \in S_1\}$;
- $S_1 \cdot S_2 = \{s : s = s_1 \cdot s_2,\ s_1 \in S_1, s_2 \in S_2\}$.

Then the following properties hold:

- $\sup(S_1 + S_2) = \sup S_1 + \sup S_2$, $\inf(S_1 + S_2) = \inf S_1 + \inf S_2$;
- $\sup(S_1 - S_2) = \sup S_1 - \inf S_2$;
- if $\lambda \geq 0$, then $\sup \lambda S_1 = \lambda \sup S_1$, $\inf \lambda S_1 = \lambda \inf S_1$; if $\lambda < 0$, then $\sup \lambda S_1 = \lambda \inf S_1$, $\inf \lambda S_1 = \lambda \sup S_1$;
- if S_1 and S_2 consist of non-negative numbers, them $\sup(S_1 \cdot S_2) = \sup S_1 \cdot \sup S_2$, $\inf(S_1 \cdot S_2) = \inf S_1 \cdot \inf S_2$.

Theorem (Archimedean property)

Given any real number $a \in \mathbb{R}$, there exists a natural number $n \in \mathbb{N}$, such that $n > a$.

Corollary

Given any two real numbers $a, b \in \mathbb{R}$, $0 < a < b$ there exists a natural number $n \in \mathbb{N}$, such that $na > b$.