3. Limits and continuity of functions

3.1. Limit of a function

Content:

- Notion of a limit
- Arithmetic properties of a limit
- Comparison properties of a limit
- Limits involving infinities
- Infinitesimal and infinitely large functions
- Cauchy criterium for existence of a limit
- One-sided limits
- Limits of monotonic functions
- Special limits

Definition (Cauchy or ($\varepsilon - \delta$)-definition of a limit)

Let $f : D \to \mathbb{R}$ be a function, $D \subseteq \mathbb{R}$, $x_0 \in \mathbb{R}$, $a \in \mathbb{R}$. We say that a is the **limit of $f(x)$ at x_0**, $a = \lim\limits_{x \to x_0} f(x)$ (or that **$f(x)$ tends to a as x tends to x_0**, $f(x) \to a$ as $x \to x_0$), if for any $\varepsilon > 0$ there is a $\delta > 0$ such that, for all $x \in D$

$$0 < |x - x_0| < \delta \Rightarrow |f(x) - a| < \varepsilon.$$

Symbolically: $a = \lim\limits_{x \to x_0} f(x) \iff \forall \varepsilon > 0 \exists \delta > 0 : \forall x \in D$

$$0 < |x - x_0| < \delta \Rightarrow |f(x) - a| < \varepsilon.$$

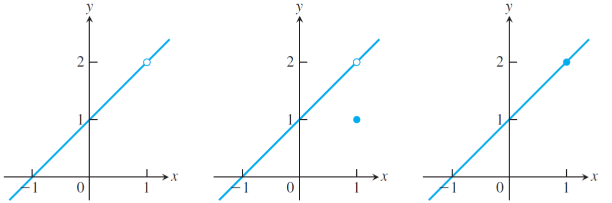
Example: how does the function $f(x) = \dfrac{x^2 - 1}{x - 1}$ behave near $x = 1$?

For any $x \neq 1$, $f(x) = x + 1$. Even though $f(1)$ is not defined, $\lim\limits_{x \to 1} f(x) = 2$: $\forall \varepsilon > 0$, take $\delta = \varepsilon$. Then $\forall x \in \mathbb{R} : 0 < |x - 1| < \delta = \varepsilon$, $|f(x) - 2| = |x + 1 - 2| = |x - 1| < \varepsilon$.

Notion of a limit

Remarks

- Informal definition: suppose $f(x)$ is defined on an open interval about x_0, except possibly at x_0 itself. If $f(x)$ can be made as close to a as we like by making x close enough, but not equal, to x_0.
- Denote a **deleted** (or **punctured**) ***r*-neighborhood of** $x_0 \in \mathbb{R}$, $r > 0$, as $\dot{U}(x_0, r) = \{x \in \mathbb{R} : 0 < |x - x_0| < r\}$. Then the above definition can be formulated as follows: $a = \lim\limits_{x \to x_0} f(x) \iff \forall \varepsilon > 0 \exists \delta > 0 : \forall x \in D \cap \dot{U}(x_0, \delta), f(x) \in U(a, \varepsilon)$.
- The limit value of a function does not depend on how the function is defined at the point being approached.



(a) $f(x) = \dfrac{x^2 - 1}{x - 1}$

(b) $g(x) = \begin{cases} \dfrac{x^2 - 1}{x - 1}, & x \neq 1 \\ 1, & x = 1 \end{cases}$

(c) $h(x) = x + 1$

The limits of $f(x)$, $g(x)$, and $h(x)$ all equal 2 as x approaches 1. However, only $h(x)$ has the same function value as its limit at $x = 1$

(from Thomas' Calculus)

Definition (Heine definition of a limit in terms od sequences)

Let $f : D \to \mathbb{R}$ be a function, $D \subseteq \mathbb{R}$, $x_0 \in \mathbb{R}$, $\Delta \in (0, +\infty]$, $a \in \mathbb{R}$. We say that a is the **limit of $f(x)$ at x_0**, if for any sequence $\{x_n\}_{n\in\mathbb{N}}$ such that $x_n \in D\ \forall n \in \mathbb{N}$ and

$$\lim_{n\to\infty} x_n = x_0 \Rightarrow \lim_{n\to\infty} f(x_n) = a.$$

Symbolically: $a = \lim\limits_{x\to x_0} f(x) \iff \forall \{x_n\}_{n\in\mathbb{N}} : x_n \in \mathbb{R} \setminus \{a\} \forall n \in ,$

$$\lim_{n\to\infty} x_n = x_0 \Rightarrow \lim_{n\to\infty} f(x_n) = a.$$

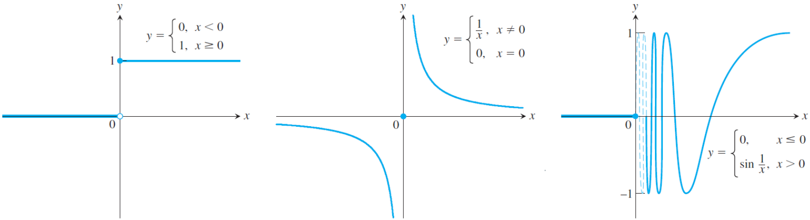
Example: $\lim\limits_{x\to 0} x^2 = 0$

For any $\{x_n\}_{n\in\mathbb{N}} \to 0$, $\lim\limits_{n\to\infty} f(x_n) = \lim\limits_{n\to\infty} x_n^2 = (\lim\limits_{n\to\infty} x_n)^2 = 0.$

ALPEN-ADRIA UNIVERSITÄT KLAGENFURT

Remarks

- If f is the identity function $f(x) = x$, then for any $x_0 \in \mathbb{R}$, $\lim\limits_{x \to x_0} f(x) = \lim\limits_{x \to x_0} x = x_0$.
- If f is the constant function $f(x) = k$, then for any $x_0 \in \mathbb{R}$, $\lim_{x \to x_0} f(x) = \lim\limits_{x \to x_0} k = k$.
- A function may not have a limit at a particular point



(a) Unit step function $U(x)$ (b) $g(x)$ (c) $f(x)$

None of these functions has a limit as x approaches 0

(from Thomas' Calculus)

Non-existence of a limit

From the Cauchy definition, the limit $\lim\limits_{x\to x_0} f(x) \neq a$, if there exists an $\varepsilon > 0$ such that, for any $\delta > 0$ there is an $x \in D$: $0 < |x - x_0| < \delta$ and $|f(x) - a| > \varepsilon$.

From the Heine definition, the limit $\lim\limits_{x\to x_0} f(x)$ does not exist, if we can find two sequences $\{x_n\}_{n\in\mathbb{N}}$, $\{x_n'\}_{n\in\mathbb{N}}$, such that $x_n, x_n' \in D(f) \setminus \{x_0\} \forall n \in \mathbb{N}$, $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} x_n'$, while $\lim\limits_{n\to\infty} f(x_n) \neq \lim\limits_{n\to\infty} f(x_n')$.

Example: $f(x) = \sin \frac{1}{x}$, $x_0 = 0$.

Let us take $x_n = \frac{1}{\pi n}$, $x_n' = \frac{1}{\pi/2 + 2\pi n}$, $\forall n \in \mathbb{N}$. $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} x_n' = 0$, $x_n \neq 0, x_n'/ne0$, but $\lim\limits_{n\to\infty} f(x_n) = 0 \neq \lim\limits_{n\to\infty} f(x_n') = 1$.

THEOREM 1—Limit Laws If L, M, c, and k are real numbers and

$$\lim_{x\to c} f(x) = L \quad \text{and} \quad \lim_{x\to c} g(x) = M, \quad \text{then}$$

1. *Sum Rule:* $\lim_{x\to c}(f(x) + g(x)) = L + M$
2. *Difference Rule:* $\lim_{x\to c}(f(x) - g(x)) = L - M$
3. *Constant Multiple Rule:* $\lim_{x\to c}(k \cdot f(x)) = k \cdot L$
4. *Product Rule:* $\lim_{x\to c}(f(x) \cdot g(x)) = L \cdot M$
5. *Quotient Rule:* $\lim_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}, \quad M \neq 0$
6. *Power Rule:* $\lim_{x\to c} [f(x)]^n = L^n$, n a positive integer
7. *Root Rule:* $\lim_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{L} = L^{1/n}$, n a positive integer

(If n is even, we assume that $\lim_{x\to c} f(x) = L > 0$.)

(from Thomas' Calculus)

Arithmetic properties of limits

THEOREM 2—Limits of Polynomials
If $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$, then

$$\lim_{x \to c} P(x) = P(c) = a_n c^n + a_{n-1} c^{n-1} + \cdots + a_0.$$

(from Thomas' Calculus)

THEOREM 3—Limits of Rational Functions
If $P(x)$ and $Q(x)$ are polynomials and $Q(c) \neq 0$, then

$$\lim_{x \to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}.$$

(from Thomas' Calculus)

ALPEN-ADRIA UNIVERSITÄT KLAGENFURT

Examples:

Use the observations $\lim_{x\to c} k = k$ and $\lim_{x\to c} x = c$ and the fundamental rules of limits to find the following limits.

(a) $\displaystyle\lim_{x\to c}(x^3 + 4x^2 - 3)$

(b) $\displaystyle\lim_{x\to c}\frac{x^4 + x^2 - 1}{x^2 + 5}$

(c) $\displaystyle\lim_{x\to -2}\sqrt{4x^2 - 3}$

(from Thomas' Calculus)

ALPEN-ADRIA UNIVERSITÄT KLAGENFURT

Examples:

(a)
$$\lim_{x\to c}(x^3+4x^2-3) = \lim_{x\to c} x^3 + \lim_{x\to c} 4x^2 - \lim_{x\to c} 3 \qquad \text{Sum and Difference Rules}$$
$$= c^3 + 4c^2 - 3 \qquad \text{Power and Multiple Rules}$$

(b)
$$\lim_{x\to c}\frac{x^4+x^2-1}{x^2+5} = \frac{\lim_{x\to c}(x^4+x^2-1)}{\lim_{x\to c}(x^2+5)} \qquad \text{Quotient Rule}$$
$$= \frac{\lim_{x\to c} x^4 + \lim_{x\to c} x^2 - \lim_{x\to c} 1}{\lim_{x\to c} x^2 + \lim_{x\to c} 5} \qquad \text{Sum and Difference Rules}$$
$$= \frac{c^4+c^2-1}{c^2+5} \qquad \text{Power or Product Rule}$$

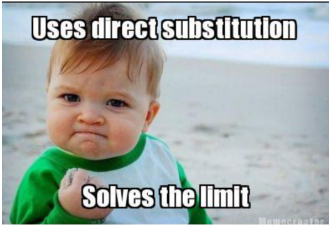
(c)
$$\lim_{x\to -2}\sqrt{4x^2-3} = \sqrt{\lim_{x\to -2}(4x^2-3)} \qquad \text{Root Rule with } n=2$$
$$= \sqrt{\lim_{x\to -2} 4x^2 - \lim_{x\to -2} 3} \qquad \text{Difference Rule}$$
$$= \sqrt{4(-2)^2-3} \qquad \text{Product and Multiple Rules}$$
$$= \sqrt{16-3} = \sqrt{13}$$

Examples:

The following calculation illustrates Theorems 2 and 3:

$$\lim_{x\to -1} \frac{x^3 + 4x^2 - 3}{x^2 + 5} = \frac{(-1)^3 + 4(-1)^2 - 3}{(-1)^2 + 5} = \frac{0}{6} = 0$$

(from Thomas' Calculus)



(from facebook.com)

ALPEN-ADRIA UNIVERSITÄT KLAGENFURT

Examples:

Evaluate $\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - x}$.

Solution We cannot substitute $x = 1$ because it makes the denominator zero. We test the numerator to see if it, too, is zero at $x = 1$. It is, so it has a factor of $(x - 1)$ in common with the denominator. Canceling this common factor gives a simpler fraction with the same values as the original for $x \neq 1$:

$$\frac{x^2 + x - 2}{x^2 - x} = \frac{(x - 1)(x + 2)}{x(x - 1)} = \frac{x + 2}{x}, \qquad \text{if } x \neq 1.$$

Using the simpler fraction, we find the limit of these values as $x \to 1$ by Theorem 3:

$$\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - x} = \lim_{x \to 1} \frac{x + 2}{x} = \frac{1 + 2}{1} = 3.$$

(from Thomas' Calculus)

Comparison properties

Sandwich (or squeeze) theorem

Let $f, g, h : D \to \mathbb{R}$ be such that $g(x) \leq f(x) \leq h(x)$ for all x in some open interval containing x_0, except possibly at $x = x_0$ itself. Suppose also that $\lim\limits_{x \to x_0} g(x) = \lim\limits_{x \to x_0} h(x) = a$. Then $\lim\limits_{x \to x_0} f(x) = a$.

Limit passage in inequalities

Let $f, g, h : D \to \mathbb{R}$ be such that $f(x) \leq h(x)$ for all x in some open interval containing x_0, except possibly at $x = x_0$ itself. Then $\lim\limits_{x \to x_0} f(x) \leq \lim\limits_{x \to x_0} h(x)$, provided that both limits exist.

ALPEN-ADRIA UNIVERSITÄT KLAGENFURT

Examples:

The Sandwich Theorem helps us establish several important limit rules:

(a) $\lim_{\theta \to 0} \sin \theta = 0$ **(b)** $\lim_{\theta \to 0} \cos \theta = 1$

(c) For any function f, $\lim_{x \to c} |f(x)| = 0$ implies $\lim_{x \to c} f(x) = 0$.

Solution

(a) $-|\theta| \leq \sin\theta \leq |\theta|$ for all θ

Since $\lim_{\theta \to 0}(-|\theta|) = \lim_{\theta \to 0} |\theta| = 0$, we have

$$\lim_{\theta \to 0} \sin \theta = 0.$$

(b) $0 \leq 1 - \cos \theta \leq |\theta|$ for all θ , and we have $\lim_{\theta \to 0}(1 - \cos \theta) = 0$ or

$$\lim_{\theta \to 0} \cos \theta = 1.$$

(c) Since $-|f(x)| \leq f(x) \leq |f(x)|$ and $-|f(x)|$ and $|f(x)|$ have limit 0 as $x \to c$, it follows that $\lim_{x \to c} f(x) = 0$. ■

(from Thomas' Calculus)

Limits as $x \to \pm\infty$

ALPEN-ADRIA UNIVERSITÄT KLAGENFURT

Definition

Let $f : D \to \mathbb{R}$ be a function, $D = (\Delta, +\infty)$, $\Delta \in \bar{\mathbb{R}}$, $a \in \mathbb{R}$. We say that a is the **limit of $f(x)$ as x tends to plus infinity**, $a = \lim\limits_{x \to +\infty} f(x)$ if, for every $\varepsilon > 0$, there exists an $M \in \mathbb{R}$ such that $x > M \Rightarrow |f(x) - a| < \varepsilon$.

Definition

Let $f : D \to \mathbb{R}$ be a function, $D = (-\infty, \Delta)$, $\Delta \in \bar{\mathbb{R}}$, $a \in \mathbb{R}$. We say that a is the **limit of $f(x)$ as x tends to minus infinity**, $a = \lim\limits_{x \to -\infty} f(x)$ if, for every $\varepsilon > 0$, there exists an $m \in \mathbb{R}$ such that $x < m \Rightarrow |f(x) - a| < \varepsilon$.

Remarks

- Intuitively, $a = \lim\limits_{x \to +\infty} f(x)$ (resp., $a = \lim\limits_{x \to -\infty} f(x)$) if, as x moves increasingly far from the origin in the positive (resp., negative) direction, $f(x)$ gets arbitrarily close to a.
- $\forall k \in \mathbb{R}$, $\lim\limits_{x \to \pm\infty} k = k$; $\lim\limits_{x \to \pm\infty} \dfrac{1}{x} = 0$.
- Limits at infinity have arithmetic and comparison properties similar to those of finite limits.

ALPEN-ADRIA UNIVERSITÄT KLAGENFURT

Examples:

(a) $\lim_{x\to\infty} \frac{1}{x} = 0$ **(b)** $\lim_{x\to-\infty} \frac{1}{x} = 0.$

Solution

(a) Let $\epsilon > 0$ be given. We must find a number M such that for all x

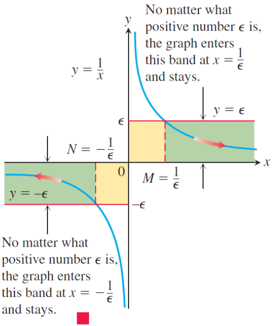
$$x > M \quad \Rightarrow \quad \left|\frac{1}{x} - 0\right| = \left|\frac{1}{x}\right| < \epsilon.$$

The implication will hold if $M = 1/\epsilon$ or any larger positive number
This proves $\lim_{x\to\infty}(1/x) = 0$.

(b) Let $\epsilon > 0$ be given. We must find a number N such that for all x

$$x < N \quad \Rightarrow \quad \left|\frac{1}{x} - 0\right| = \left|\frac{1}{x}\right| < \epsilon.$$

The implication will hold if $N = -1/\epsilon$ or any number less than $-1/\epsilon$
This proves $\lim_{x\to-\infty}(1/x) = 0$.



(from Thomas' Calculus)

Examples:

(a) $$\lim_{x\to\infty}\left(5+\frac{1}{x}\right) = \lim_{x\to\infty} 5 + \lim_{x\to\infty}\frac{1}{x} \qquad \text{Sum Rule}$$
$$= 5 + 0 = 5 \qquad \text{Known limits}$$

(b) $$\lim_{x\to-\infty}\frac{\pi\sqrt{3}}{x^2} = \lim_{x\to-\infty} \pi\sqrt{3}\cdot\frac{1}{x}\cdot\frac{1}{x}$$
$$= \lim_{x\to-\infty}\pi\sqrt{3}\cdot\lim_{x\to-\infty}\frac{1}{x}\cdot\lim_{x\to-\infty}\frac{1}{x} \qquad \text{Product Rule}$$
$$= \pi\sqrt{3}\cdot 0\cdot 0 = 0 \qquad \text{Known limits}$$

(from Thomas' Calculus)

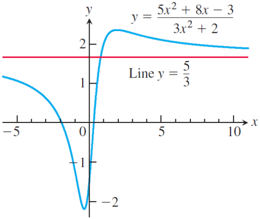
Examples:

(a) $$\lim_{x\to\infty} \frac{5x^2 + 8x - 3}{3x^2 + 2} = \lim_{x\to\infty} \frac{5 + (8/x) - (3/x^2)}{3 + (2/x^2)}$$ Divide numerator and denominator by x^2.

$$= \frac{5 + 0 - 0}{3 + 0} = \frac{5}{3}$$

(b) $$\lim_{x\to-\infty} \frac{11x + 2}{2x^3 - 1} = \lim_{x\to-\infty} \frac{(11/x^2) + (2/x^3)}{2 - (1/x^3)}$$ Divide numerator and denominator by x^3.

$$= \frac{0 + 0}{2 - 0} = 0$$



(from Thomas' Calculus)

Remark

To determine the limit of a rational function as $x \to \pm\infty$, we first divide the numerator and denominator by the highest power of x in the denominator. The result then depends on the degrees of the polynomials involved.

ALPEN-ADRIA UNIVERSITÄT KLAGENFURT

Examples:

$\lim\limits_{x\to+\infty} \dfrac{x^2+x+1}{x^2-1} =?$. Calculate using the Heine definition of a limit.

Let us take any sequence $\{x_n\}_{n\to N}$ such that $x_n \in \mathbb{R} \setminus \{\pm 2\}$ and $x_n \to +\infty$. Then

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} \frac{x_n^2+x_n+1}{x_n^2-1} = \frac{1+\lim\limits_{n\to\infty}\dfrac{1}{x_n}+\lim\limits_{n\to\infty}\dfrac{1}{x_n^2}}{1-\lim\limits_{n\to\infty}\dfrac{1}{x_n^2}} = 1.$$

ALPEN-ADRIA UNIVERSITÄT KLAGENFURT

Examples:

Find $\lim_{x\to\infty}\left(x - \sqrt{x^2+16}\right)$.

Solution Both of the terms x and $\sqrt{x^2+16}$ approach infinity as $x \to \infty$, so what happens to the difference in the limit is unclear (we cannot subtract ∞ from ∞ because the symbol does not represent a real number). In this situation we can multiply the numerator and the denominator by the conjugate radical expression to obtain an equivalent algebraic result:

$$\lim_{x\to\infty}\left(x - \sqrt{x^2+16}\right) = \lim_{x\to\infty}\left(x - \sqrt{x^2+16}\right)\frac{x+\sqrt{x^2+16}}{x+\sqrt{x^2+16}}$$

$$= \lim_{x\to\infty}\frac{x^2-(x^2+16)}{x+\sqrt{x^2+16}} = \lim_{x\to\infty}\frac{-16}{x+\sqrt{x^2+16}}.$$

As $x \to \infty$, the denominator in this last expression becomes arbitrarily large, so we see that the limit is 0. We can also obtain this result by a direct calculation using the Limit Laws:

$$\lim_{x\to\infty}\frac{-16}{x+\sqrt{x^2+16}} = \lim_{x\to\infty}\frac{-\dfrac{16}{x}}{1+\sqrt{\dfrac{x^2}{x^2}+\dfrac{16}{x^2}}} = \frac{0}{1+\sqrt{1+0}} = 0.$$

(from Thomas' Calculus)

Examples:

Find $\lim\limits_{x \to -\infty} \dfrac{2x^5 - 6x^4 + 1}{3x^2 + x - 7}$.

Solution We are asked to find the limit of a rational function as $x \to -\infty$, so we divide the numerator and denominator by x^2, the highest power of x in the denominator:

$$\begin{aligned}
\lim_{x \to -\infty} \frac{2x^5 - 6x^4 + 1}{3x^2 + x - 7} &= \lim_{x \to -\infty} \frac{2x^3 - 6x^2 + x^{-2}}{3 + x^{-1} - 7x^{-2}} \\
&= \lim_{x \to -\infty} \frac{2x^2(x - 3) + x^{-2}}{3 + x^{-1} - 7x^{-2}} \\
&= -\infty, \qquad x^{-n} \to 0,\ x - 3 \to -\infty
\end{aligned}$$

because the numerator tends to $-\infty$ while the denominator approaches 3 as $x \to -\infty$. ■

(from Thomas' Calculus)

ALPEN-ADRIA UNIVERSITÄT KLAGENFURT

Statement (limits of rational functions)

Let $f(x) = \dfrac{P(x)}{Q(x)}$, where $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$,
$Q(x) = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0$,
$a_i, b_j \in \mathbb{R} \forall i \in \{1, 2, \ldots, n\}$, $j \in \{1, 2, \ldots, m\}$, $a_n \neq 0, b_m \neq 0$. Then

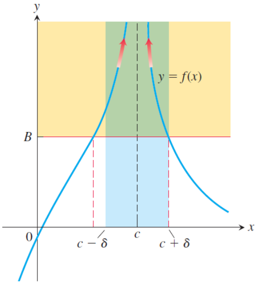
$$\lim_{x\to+\infty} f(x) = \begin{cases} 0 \text{ if } n < m, \\ +\infty \text{ if } n < m \text{ and } a_n b_m > 0, \\ -\infty \text{ if } n < m \text{ and } a_n b_m < 0, \\ \dfrac{a_n}{b_m} \text{ if } n = m; \end{cases}$$

$$\lim_{x\to-\infty} f(x) = \begin{cases} 0 \text{ if } n < m, \\ +\infty \text{ if } n < m \text{ and } (-1)^{n+m} a_n b_m > 0, \\ -\infty \text{ if } n < m \text{ and } (-1)^{n+m} a_n b_m < 0, \\ \dfrac{a_n}{b_m} \text{ if } n = m. \end{cases}$$

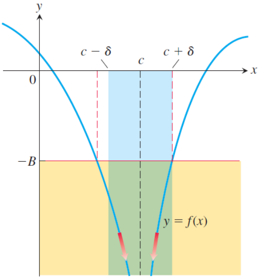
$$\lim_{x\to+\infty} f(x) = \lim_{t\to 0^+} f\left(\frac{1}{t}\right), \quad \lim_{x\to-\infty} f(x) = \lim_{t\to 0^-} f\left(\frac{1}{t}\right).$$

Definition

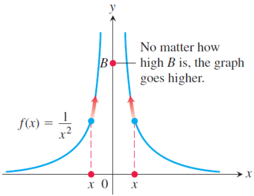
Let $f : D \to \mathbb{R}$ be a function, $x_0 \in \mathbb{R}$. We say that f **tends to** (or **approaches**) **plus infinity** (resp., **minus infinity**) **as x tends to x_0**, $\lim\limits_{x \to x_0} f(x) = +\infty$, if for any $K > 0$ there exists a $\delta > 0$ such that, for all x, $0 < |x - x_0| < \delta \Rightarrow f(x) > K$ (resp., $f(x) < -K$).



For $c - \delta < x < c + \delta$, the graph of $f(x)$ lies above the line $y = B$.



For $c - \delta < x < c + \delta$, the graph of $f(x)$ lies below the line $y = -B$.



(from Thomas' Calculus)

ALPEN-ADRIA UNIVERSITÄT KLAGENFURT

Examples:

$$\text{Prove that } \lim_{x \to 0} \frac{1}{x^2} = \infty.$$

Solution Given $B > 0$, we want to find $\delta > 0$ such that

$$0 < |x - 0| < \delta \quad \text{implies} \quad \frac{1}{x^2} > B.$$

Now,

$$\frac{1}{x^2} > B \qquad \text{if and only if} \qquad x^2 < \frac{1}{B}$$

or, equivalently,

$$|x| < \frac{1}{\sqrt{B}}.$$

Thus, choosing $\delta = 1/\sqrt{B}$ (or any smaller positive number), we see that

$$|x| < \delta \quad \text{implies} \quad \frac{1}{x^2} > \frac{1}{\delta^2} \geq B.$$

Therefore, by definition,

$$\lim_{x \to 0} \frac{1}{x^2} = \infty.$$

■

(from Thomas' Calculus)

Infinitesimal and infinitely large functions

ALPEN-ADRIA UNIVERSITÄT KLAGENFURT

Definitions

- A function $\alpha : D \to \mathbb{R}$ is called to be **infinitesimal** as $x \to x_0 \in D$, if $\lim\limits_{x \to x_0} \alpha(x) = 0$.
- A function $f : D \to \mathbb{R}$ is called to be **infinitely large** as $x \to x_0 \in D$, if $\lim\limits_{x \to x_0} f(x) = \infty$.

Lemma

A function $f : D \to \mathbb{R}$ has a limit $a \in \mathbb{R}$ if and only if $f(x) = a + \alpha(x)$ for all $x \in D$, where $\alpha(x)$ is infinitesimal as $x \to x_0$.

Theorem

A sum and a product of a finite number of infinitesimal as $x \to x_0 \in D$ functions is an infinitesimal as $x \to x_0 \in D$ function. A product of an infinitesimal as $x \to x_0 \in D$ function with a bounded in D function is an infinitesimal as $x \to x_0 \in D$ function.

Lemma

Let $f : D \to \mathbb{R}$ be infinitely large as $x \to x_0 \in D$. Then $\dfrac{1}{f}$ is infinitesimal as $x \to x_0$.

Cauchy criterium

ALPEN-ADRIA UNIVERSITÄT KLAGENFURT

Theorem (Cauchy criterium for existence of a finite limit)

A function $f : D \to \mathbb{R}$ has a finite limit at x_0 if and only if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that, for any $x', x'' \in U(x_0, \delta) \cap D$, $|f(x'') - f(x')| < \varepsilon$.

Remark

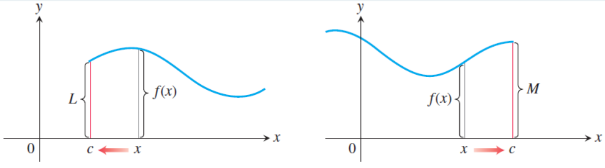
- If $x_0 \in \mathbb{R}$, the Cauchy condition can be formulated as follows: for any $\varepsilon > 0$ there exists a $\delta > 0$ such that, for any $x', x'' \in D$, if $|x' - x_0| < \delta$ and $|x'' - x_0| < \delta$ then $|f(x'') - f(x')| < \varepsilon$.
- If $x_0 = \infty$, the Cauchy condition can be formulated as follows: for any $\varepsilon > 0$ there exists a $\delta > 0$ such that, for any $x', x'' \in D$, if $|x'| > \delta$ and $|x''| > \delta$ then $|f(x'') - f(x')| < \varepsilon$.

ALPEN-ADRIA UNIVERSITÄT KLAGENFURT

Definition

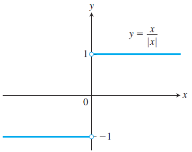
Suppose $x_0 \in \mathbb{R}$, $\Delta \in (x_0, +\infty]$, $a \in \mathbb{R}$.

- The function $f : (x_0, \Delta) \to \mathbb{R}$ has a **right-hand** (or **right-sided**) **limit** a **at** x_0, $\lim\limits_{x \to x_0^+} f(x) = a$ (or $\lim\limits_{x \searrow x_0} f(x) = a$), if $\forall \varepsilon > 0\ \exists \delta > 0 : \forall x \in (x_0, x_0 + \delta)$, $|f(x) - a| < \varepsilon$.
- The function $f(-\Delta, x_0) \to \mathbb{R}$ has a **left-hand** (or **left-sided**) **limit** a **at** x_0, $\lim\limits_{x \to x_0^-} f(x) = a$ (or $\lim\limits_{x \nearrow x_0} f(x) = a$), if $\forall \varepsilon > 0$ $\exists \delta > 0 : \forall x \in (x_0 - \delta, x_0)$, $|f(x) - a| < \varepsilon$.



(a) $\lim\limits_{x \to c^+} f(x) = L$ (b) $\lim\limits_{x \to c^-} f(x) = M$

(a) Right-hand limit as x approaches c. (b) Left-hand limit as x approaches c.



Different right-hand and left-hand limits at the origin.

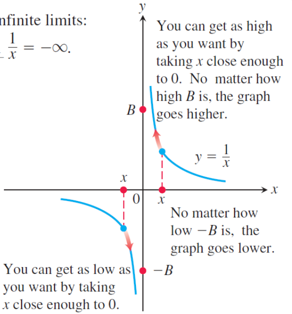
(from Thomas' Calculus)

One-sided limits

Remarks

- Notions of one-sided limits extend the notion of an ordinary limits to functions that may be undefined on one side of x_0. If f fails to have a two-sided limit at x_0, it may still have a one-sided limit.
- One-sided limits have all the properties of ordinary limits, e.g. arithmetic and comparison properties.
- Similarly to ordinary limits, one can introduce notions of one-sided infinite limits.

FIGURE : One-sided infinite limits:
$\lim_{x\to 0^+} \frac{1}{x} = \infty$ and $\lim_{x\to 0^-} \frac{1}{x} = -\infty$.



(from Thomas' Calculus)

Theorem (Existence of a limit)

A function $f(x)$ has a limit as x approaches x_0 if and only if it has left-hand and right-hand limits there and these one-sided limits are equal:

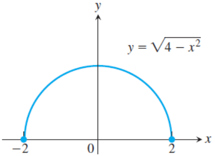
$$\lim_{x\to x_0} f(x) = a \Leftrightarrow \lim_{x\to x_0^+} f(x) = \lim_{x\to x_0^-} f(x) = a.$$

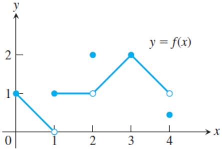
Examples:

The domain of $f(x) = \sqrt{4 - x^2}$ is $[-2, 2]$; its graph is the semicircle . We have

$$\lim_{x\to -2^+} \sqrt{4 - x^2} = 0 \quad \text{and} \quad \lim_{x\to 2^-} \sqrt{4 - x^2} = 0.$$

The function does not have a left-hand limit at $x = -2$ or a right-hand limit at $x = 2$. It does not have a two-sided limit at either -2 or 2 because each point does not belong to an open interval over which f is defined. ■





For the function graphed in Figure

At $x = 0$: $\lim_{x\to 0^+} f(x) = 1$,
$\lim_{x\to 0^-} f(x)$ and $\lim_{x\to 0} f(x)$ do not exist. The function is not defined to the left of $x = 0$.

At $x = 1$: $\lim_{x\to 1^-} f(x) = 0$ even though $f(1) = 1$,
$\lim_{x\to 1^+} f(x) = 1$,
$\lim_{x\to 1} f(x)$ does not exist. The right- and left-hand limits are not equal.

At $x = 2$: $\lim_{x\to 2^-} f(x) = 1$,
$\lim_{x\to 2^+} f(x) = 1$,
$\lim_{x\to 2} f(x) = 1$ even though $f(2) = 2$.

At $x = 3$: $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = \lim_{x\to 3} f(x) = f(3) = 2$.

At $x = 4$: $\lim_{x\to 4^-} f(x) = 1$ even though $f(4) \neq 1$,
$\lim_{x\to 4^+} f(x)$ and $\lim_{x\to 4} f(x)$ do not exist. The function is not defined to the right of $x = 4$.

At every other point c in $[0, 4]$, $f(x)$ has limit $f(c)$. ■

(from Thomas' Calculus)

Examples:

Prove that $\lim_{x\to 0^+} \sqrt{x} = 0$.

Solution Let $\epsilon > 0$ be given. Here $c = 0$ and $L = 0$, so we want to find a $\delta > 0$ such that for all x

$$0 < x < \delta \quad \Rightarrow \quad |\sqrt{x} - 0| < \epsilon,$$

or

$$0 < x < \delta \quad \Rightarrow \quad \sqrt{x} < \epsilon.$$

Squaring both sides of this last inequality gives

$$x < \epsilon^2 \quad \text{if} \quad 0 < x < \delta.$$

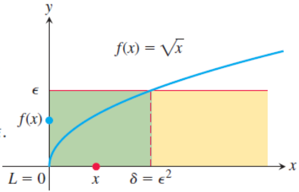
If we choose $\delta = \epsilon^2$ we have

$$0 < x < \delta = \epsilon^2 \quad \Rightarrow \quad \sqrt{x} < \epsilon,$$

or

$$0 < x < \epsilon^2 \quad \Rightarrow \quad |\sqrt{x} - 0| < \epsilon.$$

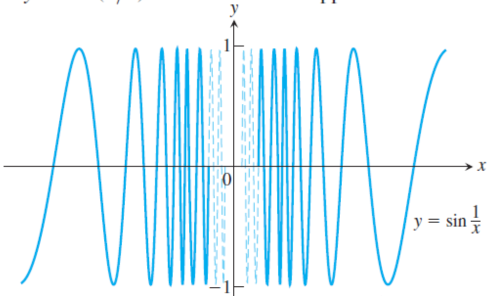
According to the definition, this shows that $\lim_{x\to 0^+} \sqrt{x} = 0$



(from Thomas' Calculus)

Examples:

Show that $y = \sin(1/x)$ has no limit as x approaches zero from either side



Solution As x approaches zero, its reciprocal, $1/x$, grows without bound and the values of $\sin(1/x)$ cycle repeatedly from -1 to 1. There is no single number L that the function's values stay increasingly close to as x approaches zero. This is true even if we restrict x to positive values or to negative values. The function has neither a right-hand limit nor a left-hand limit at $x = 0$. ■

(from Thomas' Calculus)

ALPEN-ADRIA UNIVERSITÄT KLAGENFURT

Examples:

These examples illustrate that rational functions can behave in various ways near zeros of the denominator.

(a) $\lim_{x\to 2} \frac{(x-2)^2}{x^2-4} = \lim_{x\to 2} \frac{(x-2)^2}{(x-2)(x+2)} = \lim_{x\to 2} \frac{x-2}{x+2} = 0$

(b) $\lim_{x\to 2} \frac{x-2}{x^2-4} = \lim_{x\to 2} \frac{x-2}{(x-2)(x+2)} = \lim_{x\to 2} \frac{1}{x+2} = \frac{1}{4}$

(c) $\lim_{x\to 2^+} \frac{x-3}{x^2-4} = \lim_{x\to 2^+} \frac{x-3}{(x-2)(x+2)} = -\infty$ The values are negative for $x > 2$, x near 2.

(d) $\lim_{x\to 2^-} \frac{x-3}{x^2-4} = \lim_{x\to 2^-} \frac{x-3}{(x-2)(x+2)} = \infty$ The values are positive for $x < 2$, x near 2.

(e) $\lim_{x\to 2} \frac{x-3}{x^2-4} = \lim_{x\to 2} \frac{x-3}{(x-2)(x+2)}$ does not exist. See parts (c) and (d).

(f) $\lim_{x\to 2} \frac{2-x}{(x-2)^3} = \lim_{x\to 2} \frac{-(x-2)}{(x-2)^3} = \lim_{x\to 2} \frac{-1}{(x-2)^2} = -\infty$

(from Thomas' Calculus)

Monotonic functions

Definition

Let $f : D \to \mathbb{R}$ be a function, $D \subseteq \mathbb{R}$, x_1 and x_2 be any two points in D. The function f is

- **increasing on** D if $f(x_2) \geq f(x_1)$ whenever $x_1 < x_2$;
- **decreasing on** D if $f(x_2) \leq f(x_1)$ whenever $x_1 < x_2$.

If the inequality is strict, then f is **strictly increasing on** D or **strictly decreasing on** D

Theorem

Let function $f : D \to \mathbb{R}$ increases on D, $\alpha = \inf D$, $\beta = \sup D$, $\alpha \notin D, \beta \notin D$. Then the function f has a right-hand limit at α and a left-hand limit at β, and

$$\lim_{x \to \alpha^+} f(x) = \inf_{x \in D} f(x), \quad \lim_{x \to \beta^-} f(x) = \sup_{x \in D} f(x).$$

Special limits

- $\lim\limits_{x\to\infty}\left(1+\dfrac{1}{x}\right)^x = e;$
- $\lim\limits_{x\to 0}\dfrac{\sin x}{x} = 1.$

Some other useful limits

- $\lim\limits_{x\to+\infty}\dfrac{1}{x} = 0^+ = 0,\ \lim\limits_{x\to+\infty}\dfrac{1}{x} = 0^- = 0,\ \lim\limits_{x\to 0^+}\dfrac{1}{x} = +\infty,$
  $\lim\limits_{x\to 0^-}\dfrac{1}{x} = -\infty;$
- $\lim\limits_{x\to+\infty} a^{-x} = \begin{cases} 0, & a > 1 \\ 1, & a = 1 \\ \infty, & 0 < a < 1 \end{cases};$
- $\lim\limits_{x\to+\infty}\sqrt[x]{x} = \lim\limits_{x\to\infty} x^{1/x} = 1.$

ALPEN-ADRIA UNIVERSITÄT KLAGENFURT

Examples:

1) $\lim\limits_{x\to 0} \dfrac{\cos x - 1}{x} = 0.$

Solution: $\lim\limits_{x\to 0} \dfrac{\cos x - 1}{x} = \lim\limits_{x\to 0} \dfrac{1 - 2\sin^2 \frac{x}{2} - 1}{x} = -\lim\limits_{x\to 0} \dfrac{2\sin^2 \frac{x}{2}}{x} =$
$\left\{\theta := \dfrac{x}{2}\right\} - \lim\limits_{\theta\to 0} \dfrac{\sin^2\theta}{\theta} = -\lim\limits_{\theta\to 0} \dfrac{\sin\theta}{\theta}\sin\theta = -1\cdot 0 = 0.$

2) $\lim\limits_{x\to 0} \dfrac{\sin 2x}{5x} = \dfrac{2}{5}.$

Solution: $\lim\limits_{x\to 0} \dfrac{\sin 2x}{5x} = \lim\limits_{x\to 0} \dfrac{(2/5)\sin 2x}{(2/5)5x} = \dfrac{2}{5}\lim\limits_{x\to 0} \dfrac{\sin 2x}{2x} = \{\theta := 2x\} =$
$\dfrac{2}{5}\lim\limits_{\theta\to 0} \dfrac{\sin\theta}{\theta} = \dfrac{2}{5}\cdot 1 = \dfrac{2}{5}.$

3) $\lim\limits_{x\to 0} \dfrac{\tan x \sec 2x}{3x} = \dfrac{1}{3}.$

Solution: $\lim\limits_{x\to 0} \dfrac{\tan x \sec 2x}{3x} = \lim_{x\to 0} \dfrac{1}{3}\cdot\dfrac{1}{x}\cdot\dfrac{\sin x}{\cos x}\cdot\dfrac{1}{\cos 2x} =$
$\dfrac{1}{3}\lim_{x\to 0}\cdot\dfrac{\sin x}{x}\cdot\dfrac{1}{\cos x}\cdot\dfrac{1}{\cos 2x} = \dfrac{1}{3}\cdot 1\cdot 1\cdot 1 = \dfrac{1}{3}.$