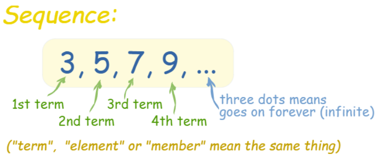
2. Sequences and series

2.1. Sequences

Content:

- Notion of a sequence
- Representation of sequences
- Bounded and unbounded sequences
- Monotone sequences
- Accumulation points

A **sequence** is an enumerated collection of objects in which repetitions are allowed and order matters.



(from mathisfun.com)

Definition

A **sequence** is a function which assigns to each natural number a unique element of a non-empty set $S \subset \mathbb{R}^n$.

Examples:

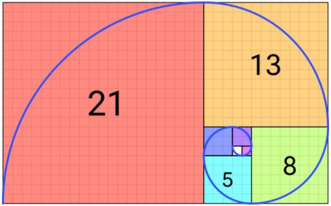
$\{1, 2, 3, \ldots,\}$, $\{1, -1, 2, -2, 3, -3, \ldots\}$, $\{0, 1, 0, 1, 0, 1, \ldots\}$.

Representation of a sequence

Notations: $\{a_1, a_2, \ldots,\}$, $\{a_n\}_{n\in\mathbb{N}}$, $\{a_n\}_{n=1}^{\infty}$, $\{a_n\}$, $\{a_n\}_{n\in\mathbb{N}}$, $\{a_n\}_{n=1}^{\infty}$.
a_1 is the 1st element, a_n is the nth element.

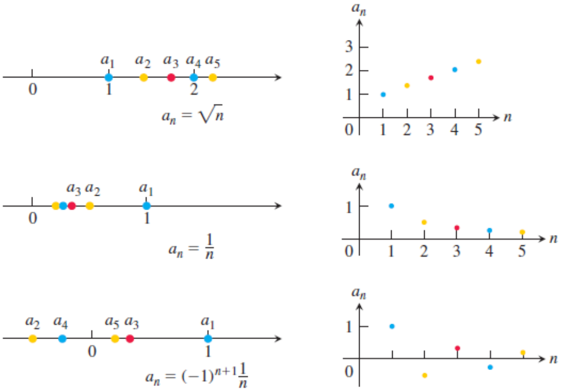
A sequence can be described by

- **listing in order**, e.g. $\{-1, 1, -1, 1, \ldots\}$;
- **a formula for the n-th term**, e.g. $\{(-1)^n\}_{n\in\mathbb{N}}$;
- **a recursion formula**, i.e. by a rule which expresses the n-th term via previous terms, i.e. $\{a_n\}_{n\in\mathbb{N}}$, where $a_1 = 0$, $a_2 = 1$, $a_n = a_{n-2} + a_{n-1}$ for $n \geq 3$ (Fibonacci sequence).



(from wikipedia.org)

- **Graphical representation**

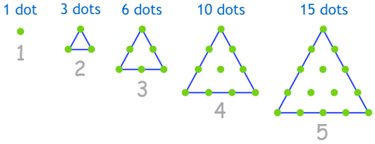


Sequences can be represented as points on the real line or as points in the plane where the horizontal axis n is the index number of the term and the vertical axis a_n is its value.

(from Thomas' Calculus)

Some special sequences

- arithmetic sequences: $\{b + d(n-1)\}_{n\in\mathbb{N}}$, where $b \in \mathbb{R}$ is the first term, d is the common difference;
- geometric sequences: $\{br^{n-1}\}_{n\in\mathbb{N}}$, where $b \in \mathbb{R}$ is the first term, r is the common ration, $r \neq 0$;
- triangular numbers: $\left\{\dfrac{n(n+1)}{2}\right\}_{n\in\mathbb{N}}$;



(from mathisfun.com)

- square numbers, cubic numbers: $\{n^2\}_{n\in\mathbb{N}}$, $\{n^3\}_{n\in\mathbb{N}}$;
- Fibonacci numbers: $\{a_n\}_{n\in\mathbb{N}}$, where $a_1 = 1$, $a_2 = 1$, $a_n = a_{n-2} + a_{n-1}$ for $n \geq 3$.

Boundedness

Remainder:

A function $f(x)$ with a domain $D \subseteq \mathbb{R}$ is

- **bounded from below** if there exists an $m \in \mathbb{R}$ such that $f(x) \geq m$ for all $x \in D$; m is called a **lower bound of** f.
- **bounded from above** if there exists an $M \in \mathbb{R}$ such that $f(x) \leq M$ for all $x \in D$; M is called an **upper bound of** f.
- **bounded** if it is bounded both from above and below; equivalently, f is bounded if there exists a $c > 0$ such that $|f(x)| \leq c$ for all $x \in D$;
- **unbounded** if it is not bounded.

Definition

A sequence of real numbers $\{a_n\}_{n\in\mathbb{N}}$ is

- **bounded from below** if there exists an $m \in \mathbb{R}$ such that $a_n \geq m$ for all $n \in \mathbb{N}$; m is called a **lower bound of** $\{a_n\}_{n\in\mathbb{N}}$.
- **bounded from above** if there exists an $M \in \mathbb{R}$ such that $a_n \leq M$ for all $n \in \mathbb{N}$; M is called an **upper bound of** $\{a_n\}_{n\in\mathbb{N}}$.
- **bounded** if it is bounded both from above and below; equivalently, it is bounded if there exists a $c \geq 0$ such that $|a_n| \leq c \ \forall n \in \mathbb{N}$;
- **unbounded** if it is not bounded, i.e. $\forall c \geq 0 \exists n_c \in \mathbb{N} : |a_{n_c}| > c$.

Infimum and supremum of a sequence

Remainder

Let S be a set in $\mathbb{R}$. The **infimum of S** (resp., **supremum of S**), is the greatest element in $\mathbb{R}$ that is less than or equal to each element of S, if such an element exists (resp., the least element in $\mathbb{R}$ that is greater than or equal to each element of S, if such an element exists).

Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers.

Definition

The **infimum** (resp., the **supremum**) **of the sequence** $\{a_n\}_{n\in\mathbb{N}}$ is the infimum (resp., supremum) of the set of its values (if exists):
$\inf\limits_{n\in\mathbb{N}}\{a_n\} = \inf\{a_1,\ldots,a_n\}$ (resp., $\sup\limits_{n\in\mathbb{N}}\{a_n\} = \sup\{a_1,\ldots,a_n\}$).

Equivalent definition

A number $a \in \mathbb{R}$ is

- the **infimum of the sequence** $\{a_n\}_{n\in\mathbb{N}}$, if $a_n \geq a$ for all $n \in \mathbb{N}$ and for any $\varepsilon > 0$ $\exists n_\varepsilon \in \mathbb{N}$ such that $a_{n_\varepsilon} < a + \varepsilon$.
- the **supremum of the sequence** $\{a_n\}_{n\in\mathbb{N}}$, if $a_n \leq a$ for all $n \in \mathbb{N}$ and for any $\varepsilon > 0$ $\exists n_\varepsilon \in \mathbb{N}$ such that $a_{n_\varepsilon} > a - \varepsilon$.

Infimum and supremum of a sequence can be infinite.

Monotonicity

Remainder:

A function $f : D \to \mathbb{R}$ $(D \subseteq \mathbb{R})$ is monotonically increasing (decreasing) if $f(x_1) \leq f(x_2)$ $(f(x_1) \geq f(x_2))$ whenever $x_1 < x_2$, for all $x_1, x_2 \in D$.

Definition

A sequence of real numbers $\{a_n\}_{n\in\mathbb{N}}$ is

- **monotonically increasing** if $a_{n+1} \geq a_n$ for all $n \in \mathbb{N}$;
- **monotonically decreasing** if $a_{n+1} \leq a_n$ for all $n \in \mathbb{N}$;
- **monotonic** if it is either decreasing or increasing;
- **strictly monotonically increasing** if $a_{n+1} > a_n$ for all $n \in \mathbb{N}$;
- **strictly monotonically decreasing** if $a_{n+1} < a_n$ for all $n \in \mathbb{N}$.

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Examples:

1) $\left\{\dfrac{n}{n+1}\right\}_{n\in\mathbb{N}}$: bounded, strictly monotonically increasing.

Indeed, let $a_n = \dfrac{n}{n+1}$, $n \in \mathbb{N}$. Boundedness: $a_n = 1 - \dfrac{1}{n+1}$, $\dfrac{1}{2} \le a_n \le 1\ \forall n \in \mathbb{N}$.

Monotonicity: $a_{n+1} - a_n = \dfrac{n+1}{n+2} - \dfrac{n}{n+1} = \dfrac{1}{(n+1)(n+2)} > 0 \Rightarrow a_{n+1} > a_n,\ \forall n \in \mathbb{N}$.

$\inf\limits_{n\in\mathbb{N}} \dfrac{n}{n+1} = \dfrac{1}{2}$, $\sup\limits_{n\in\mathbb{N}} \dfrac{n}{n+1} = 1$.

2) $\{(-1)^n\}_{n\in\mathbb{N}}$: bounded, not monotonic,
$\inf\limits_{n\in\mathbb{N}} (-1)^n = -1$, $\sup\limits_{n\in\mathbb{N}} (-1)^n = 1$.

3) $\{n(-1)^n\}_{n\in\mathbb{N}}$: unbounded, not monotonic, $\inf\limits_{n\in\mathbb{N}} n(-1)^n = -\infty$, $\sup\limits_{n\in\mathbb{N}} n(-1)^n = +\infty$.

4) $\left\{\dfrac{(-1)^n}{n}\right\}_{n\in\mathbb{N}}$: bounded, not monotonic, $\inf\limits_{n\in\mathbb{N}} \dfrac{(-1)^n}{n} = -1$, $\sup\limits_{n\in\mathbb{N}} \dfrac{(-1)^n}{n} = \dfrac{1}{2}$.

5) $\left\{\dfrac{1}{n}\right\}_{n\in\mathbb{N}}$: bounded from above, str.mon.decreasing, $\inf\limits_{n\in\mathbb{N}} \dfrac{1}{n} = 0$, $\sup\limits_{n\in\mathbb{N}} \dfrac{1}{n} = 1$.

6) $\{n\}_{n\in\mathbb{N}}$: bounded from above, str.mon.decreasing, $\inf\limits_{n\in\mathbb{N}} n = 1$, $\sup\limits_{n\in\mathbb{N}} n = +\infty$.

7) $\left\{n \sin\dfrac{\pi n}{2}\right\}_{n\in\mathbb{N}}$: unbounded, not monotonic, $\inf\limits_{n\in\mathbb{N}} n = -\infty$, $\sup\limits_{n\in\mathbb{N}} n = +\infty$.

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Examples:

8) $\{a_n\}_{n\in\mathbb{N}}$, **with** $a_1 = 1$, $a_{n+1} = \dfrac{a_n^2+1}{2}$: bounded, not strictly monotonic (constant), $\inf\limits_{n\in\mathbb{N}} a_n = \sup\limits_{n\in\mathbb{N}} a_n = 1$.

9) $\{a_n\}_{n\in\mathbb{N}}$, **with** $a_1 = 2$, $a_{n+1} = \dfrac{a_n^2+1}{2}$: bounded from below, monotonically increasing, $\inf\limits_{n\in\mathbb{N}} a_n = 2$, $\sup\limits_{n\in\mathbb{N}} a_n = +\infty$.

10) $\{a_n\}_{n\in\mathbb{N}}$, **with** $a_1 = 1/2$, $a_{n+1} = \dfrac{a_n^2+1}{2}$: bounded, monotonically increasing, $\inf\limits_{n\in\mathbb{N}} a_n = \dfrac{1}{2}$, $\sup\limits_{n\in\mathbb{N}} a_n = 1$.

11) $\left\{(1+1/n)^n\right\}_{n\in\mathbb{N}}$: bounded from below, strictly monotonically increasing.
$a_1 = 2$, $a_2 = 2.25$, $a_3 \approx 2,37...$, $a_4 \approx 2.44...$, $a_4 \approx 2.48...$, $a_6 \approx 2.52...$, $a_7 \approx 2.54...$

$$a_n \to e \approx 2.718281828... \text{ as } n \to \infty.$$

Boundedness and monotonicity

Examples:

12) $\left\{ \dfrac{n-3}{n^2+1} \right\}_{n\in\mathbb{N}}$: boundedness? monotonicity? Let $a_n = \dfrac{n-3}{n^2+1}$, $n \in \mathbb{N}$.

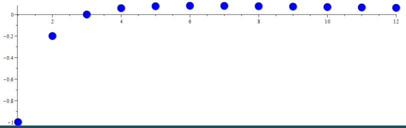
Boundedness from above:

$$a_n = \frac{n}{n^2+1} \underbrace{-\frac{3}{n^2+1}}_{<0} < \frac{n}{n^2+1} < \frac{n}{n^2} = \frac{1}{n} \leq 1.$$

Boundedness from below: $a_n > 0 \ \forall n \geq 0$, $a_1 = -1$, $a_2 = -\dfrac{1}{5}$. Therefore, $a_n \geq \min\{0, -1, -1/5\} = -1 \ \forall n \in \mathbb{N}$.

Monotonicity: $a_1 = -1 < a_2 = -\dfrac{1}{5} < a_3 = 0 < a_4 = \dfrac{1}{17} < a_5 = \dfrac{1}{13} <$

$a_6 = \dfrac{3}{37} > a_7 = \dfrac{2}{25} <>???$



Examples:

$\left\{ \frac{n-3}{n^2+1} \right\}_{n\in\mathbb{N}}$. For which values $n \in \mathbb{N}$ $a_n > a_{n+1}$?

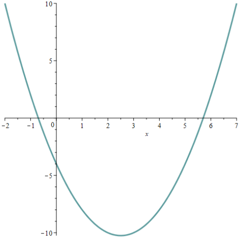
$$a_n > a_{n+1} \iff \frac{n-3}{n^2+1} > \frac{n+1-3}{(n+1)^2+1} = \frac{n-2}{(n+1)^2+1}$$
$$\iff (n-3)((n+1)^2+1) > (n-2)(n^2+1)$$
$$\iff n^2-5n-4>0.$$

Consider the function $f(x) = x^2 - 5x - 4$.

$f(x) = 0 \iff x_{1,2} = \dfrac{5 \pm \sqrt{41}}{2}$

$\Rightarrow x_1 \approx -0.7$, $x_2 \approx 5.7$.

$f(x) = (x - x_1)(x - x_2) \Rightarrow n^2 - 5n - 4 > 0$ holds for all $n \geq 6$, i.e. starting from $n = 6$ the sequence becomes strictly decreasing.



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Examples:

Fibonacci sequence: $\{a_n\}_{n\in\mathbb{N}}$ **with** $a_1 = 0, a_2 = 1,$ $a_n = a_{n-2} + a_{n-1}$ **for** $n \geq 3$. It is:

- bounded from below: $a_n \geq 0\ \forall n \in \mathbb{N}$ (prove by mathematical induction);
- not bounded from above: $\nexists M : a_n \leq M$ for all $n \in \mathbb{N}$.
  Assume contrary: $\exists M > 0 : a_n \leq M \forall n \in \mathbb{N}$. Consider the element a_k with $k \geq M + 2$.
  Then $a_k = \underbrace{a_k - a_{k-1}}_{=a_{k-2}\geq 1} + \underbrace{a_{k-1} - a_{k-2}}_{=a_{k-3}\geq 1} + \cdots + \underbrace{a_4 - a_3}_{=a_2=1} + \underbrace{a_3 - a_2}_{=a_1=0} + \underbrace{a_2}_{=1} \geq (k-1) \geq M + 1$. Thus, we get the contradiction: $M + 1 \leq a_k \leq M$.
- monotonically increasing: $a_{n+1} - a_n = a_{n-1} \geq 0$ for all $n \in \mathbb{N}$;
- not strictly monotonic: $a_2 = a_3 = 1$.

Operations with sequences

Definition

Given two sequences of real numbers $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$, the sequences $\{a_n + b_n\}_{n\in\mathbb{N}}$, $\{a_n - b_n\}_{n\in\mathbb{N}}$, $\{a_n \cdot b_n\}_{n\in\mathbb{N}}$, and (in case $b_n \neq 0 \forall n \in \mathbb{N}$) $\left\{\frac{a_n}{b_n}\right\}_{n\in\mathbb{N}}$ are called the **sum**, **difference**, **product** and **quotient** of these sequence, respectively.

Neighbourhood in $\mathbb{R}$

Definition

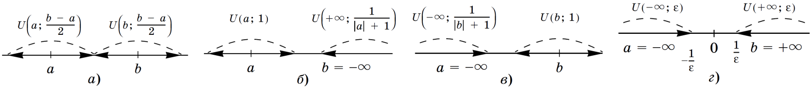
Given a real number $a \in \mathbb{R}$ and a number $\varepsilon > 0$, an **ε-neighbourhood of the number a** is the interval $(a-\varepsilon, a+\varepsilon)$:

$$U(a,\varepsilon) := (a-\varepsilon, a+\varepsilon).$$

For $a = \pm\infty$, $U(+\infty,\varepsilon) := \left(\frac{1}{\varepsilon}, +\infty\right]$, $U(-\infty,\varepsilon) := \left[-\infty, -\frac{1}{\varepsilon}\right)$.

Lemma

For any $a, b \in \bar{\mathbb{R}}$ there exist $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that $U(a,\varepsilon_1) \cap U(b,\varepsilon_2) = \emptyset$.



Accumulation point

Definition

- A number $\alpha \in \mathbb{R}$ is **accumulation point** of a sequence $\{a_n\}_{n\in\mathbb{N}}$, if any ε-neighborhood of α contains infinitely many elements of $\{a_n\}_{n\in\mathbb{N}}$, i.e. $|\alpha - a_k| < \varepsilon$ for infinitely many $k \in \mathbb{N}$.
- We say that $+\infty$ (resp., $-\infty$) is an accumulation point of a sequence $\{a_n\}_{n\in\mathbb{N}}$, if for any $M > 0$ there are infinitely numbers $k \in \mathbb{N}$ such that $a_k > M$ (resp., $a_k < -M$).

Sufficient condition: $\alpha \in \mathbb{R}$ is accumulation point of a sequence $\{a_n\}_{n\in\mathbb{N}}$, if for any $\varepsilon > 0$ and $n \in \mathbb{N}$ there is an integer $k > n$ such that $|\alpha - a_k| < \varepsilon$. A sequence is unbounded from above (resp., from below) iff $+\infty$ (resp., $-\infty$) is an accumulation point.

Examples: accumulation points of $\{(-1)^n\}_{n\in\mathbb{N}}$ are 1 and -1;

$\left\{\frac{1}{n}\right\}_{n\in\mathbb{N}}$ is 0;

$\{(1+(-1)^n)^n\}_{n\in\mathbb{N}}$ are 0 and $+\infty$.