4. Differential calculus of functions of one real variable

4.2. Applications of derivatives: Analysis of the behavior of a function

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Content:

- Extreme values of functions
- Theorem on monotonicity of a function
- First- and second derivative tests for local extrema
- Concavity and convexity
- Theorem on concavity of a function
- First- and second derivative criteria for points of inflection
- Procedure for graphing $y = f(x)$

Definitions

A function $f : D \to \mathbb{R}$ has at $x^* \in D$

- a **global** (or **absolute**) **maximum** if $f(x^*) \geq f(x)$ for all $x \in D$; x^* is called a **global maximum point** of f;
- a **local** (or **relative**) **maximum** if there exists an $\varepsilon > 0$ such that $f(x^*) \geq f(x)$ for all $x \in U(x^*, \varepsilon) \cap D$; x^* is a **local maximum point** of f.
- a **global** (or **absolute**) **minimum** if $f(x^*) \leq f(x)$ for all $x \in D$; x^* is a **global minimum point** of f;
- a **local** (or **relative**) **minimum** if there exists an $\varepsilon > 0$ such that $f(x^*) \geq f(x)$ for all $x \in U(x^*, \varepsilon) \cap D$; x^* is a **local minimum point** of f.

Maximum and minimum values are called **absolute extreme values** of the function f. The point at which f attains its extreme values are called **points of extrema** of f.

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Definitions

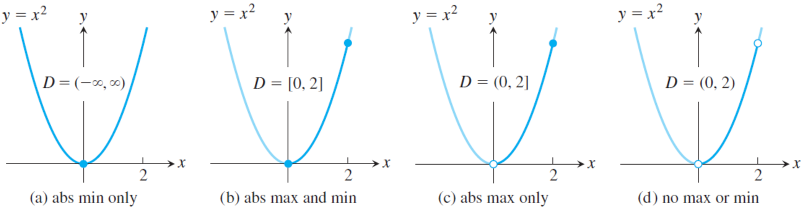
A function $f : D \to \mathbb{R}$ has at $x^* \in D$

- a **global** (or **absolute**) **strict maximum** if $f(x^*) > f(x)$ for all $x \in D$; x^* is called a **global strict maximum point** of f;
- a **local** (or **relative**) **strict maximum** if there exists an $\varepsilon > 0$ such that $f(x^*) \geq f(x)$ for all $x \in U(x^*, \varepsilon) \cap D$; x^* is a **local strict maximum point** of f.
- a **global** (or **absolute**) **strict minimum** if $f(x^*) \leq f(x)$ for all $x \in D$; x^* is a **global strict minimum point** of f;
- a **local** (or **relative**) **strict minimum** if there exists an $\varepsilon > 0$ such that $f(x^*) \geq f(x)$ for all $x \in U(x^*, \varepsilon) \cap D$; x^* is a **local strict minimum point** of f.

Strict maximum and minimum values are called **strict extreme values** of the function f. The point at which f attains its strict extreme values are called **points of strict extrema** of f.

Example:

Function rule	Domain D	Absolute extrema on D
(a) $y = x^2$	$(-\infty, \infty)$	No absolute maximum Absolute minimum of 0 at $x = 0$
(b) $y = x^2$	$[0, 2]$	Absolute maximum of 4 at $x = 2$ Absolute minimum of 0 at $x = 0$
(c) $y = x^2$	$(0, 2]$	Absolute maximum of 4 at $x = 2$ No absolute minimum
(d) $y = x^2$	$(0, 2)$	No absolute extrema



(from Thomas'Calculus)

Example:

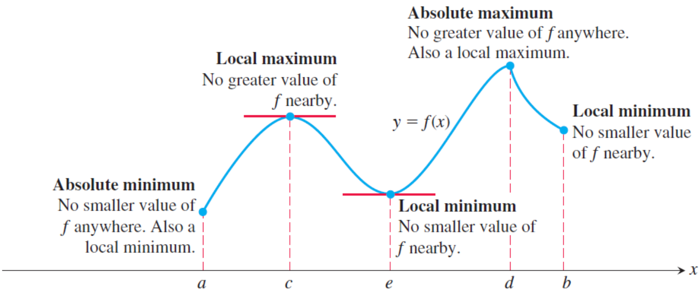


FIGURE How to identify types of maxima and minima for a function with domain $a \leq x \leq b$.

An absolute maximum is also a local maximum. Being the largest value overall, it is also the largest value in its immediate neighborhood. Hence, *a list of all local maxima will automatically include the absolute maximum if there is one*. Similarly, *a list of all local minima will include the absolute minimum if there is one*.

(from Thomas'Calculus)

Weierstrass Extreme values theorem

THEOREM 1—The Extreme Value Theorem If f is continuous on a closed interval $[a, b]$, then f attains both an absolute maximum value M and an absolute minimum value m in $[a, b]$. That is, there are numbers x_1 and x_2 in $[a, b]$ with $f(x_1) = m$, $f(x_2) = M$, and $m \leq f(x) \leq M$ for every other x in $[a, b]$.

(from Thomas'Calculus)

Extreme values theorem

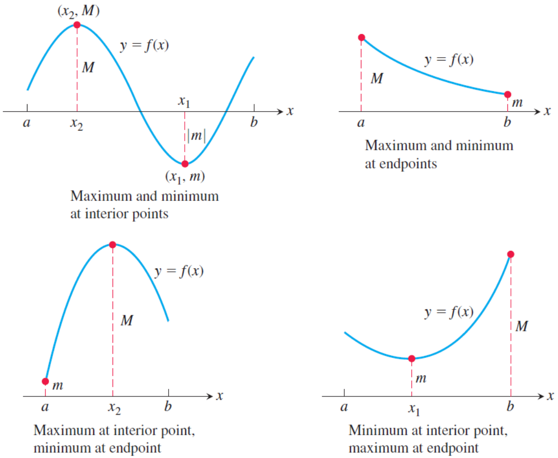


FIGURE Some possibilities for a continuous function's maximum and minimum on a closed interval $[a, b]$.

(from Thomas'Calculus)

Local extremum theorem

Theorem (Fermat's local extremum theorem)

If a function $f : D \to \mathbb{R}$ has a local maximum or minimum value at an interior point x^* of D and if f' is defined at x^*, then

$$f'(x^*) = 0.$$

Proof To prove that $f'(c)$ is zero at a local extremum, we show first that $f'(c)$ cannot be positive and second that $f'(c)$ cannot be negative. The only number that is neither positive nor negative is zero, so that is what $f'(c)$ must be.

To begin, suppose that f has a local maximum value at $x = c$ (Figure 4.6) so that $f(x) - f(c) \leq 0$ for all values of x near enough to c. Since c is an interior point of f's domain, $f'(c)$ is defined by the two-sided limit

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c}.$$

This means that the right-hand and left-hand limits both exist at $x = c$ and equal $f'(c)$. When we examine these limits separately, we find that

$$f'(c) = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} \leq 0. \qquad \text{Because } (x - c) > 0 \text{ and } f(x) \leq f(c) \tag{1}$$

Similarly,

$$f'(c) = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \geq 0. \qquad \text{Because } (x - c) < 0 \text{ and } f(x) \leq f(c) \tag{2}$$

Together, Equations (1) and (2) imply $f'(c) = 0$.

(from Thomas'Calculus)

Local extremum theorem

Remark

Fermat's local extremum theorem says that a function's first derivative is always zero at an interior point where the function has a local extreme value and the derivative is defined. Thus, the only points where a function f can possibly have an extreme value (local or global) are:

- interior points where $f' = 0$;
- interior points where f' is undefined;
- endpoints of the domain of f.

An interior point of the domain of a function f where f' is zero or undefined is a **critical point** of f.

A function may have a critical point without having a local extreme value there!

How to Find the Absolute Extrema of a Continuous Function f on a Finite Closed Interval

1. Evaluate f at all critical points and endpoints.
2. Take the largest and smallest of these values.

(from Thomas'Calculus)

EXAMPLE 2 Find the absolute maximum and minimum values of $f(x) = x^2$ on $[-2, 1]$.

Solution The function is differentiable over its entire domain, so the only critical point is where $f'(x) = 2x = 0$, namely $x = 0$. We need to check the function's values at $x = 0$ and at the endpoints $x = -2$ and $x = 1$:

Critical point value: $f(0) = 0$

Endpoint values: $f(-2) = 4$

$f(1) = 1.$

The function has an absolute maximum value of 4 at $x = -2$ and an absolute minimum value of 0 at $x = 0$. ■

(from Thomas'Calculus)

Monotonic functions

Theorem on monotonicity of a function

Suppose that f is continuous on $[a, b]$ and differentiable on (a, b). Then the following properties hold:

- if $f'(x) > 0$ at each $x \in (a, b)$, then f is strictly increasing on $[a, b]$;
- if $f'(x) < 0$ at each $x \in (a, b)$, then f is strictly decreasing on $[a, b]$;
- if $f'(x) \geq 0$ at each $x \in (a, b)$, then f is non-decreasing on $[a, b]$;
- if $f'(x) \leq 0$ at each $x \in (a, b)$, then f is non-increasing on $[a, b]$.

This result follows from Lagrange's Mean value theorem.

EXAMPLE Find the critical points of $f(x) = x^3 - 12x - 5$ and identify the open intervals on which f is increasing and on which f is decreasing.

Solution The function f is everywhere continuous and differentiable. The first derivative

$$\begin{aligned} f'(x) &= 3x^2 - 12 = 3(x^2 - 4) \\ &= 3(x + 2)(x - 2) \end{aligned}$$

is zero at $x = -2$ and $x = 2$. These critical points subdivide the domain of f to create nonoverlapping open intervals $(-\infty, -2)$, $(-2, 2)$, and $(2, \infty)$ on which f' is either positive or negative. We determine the sign of f' by evaluating f' at a convenient point in each subinterval.

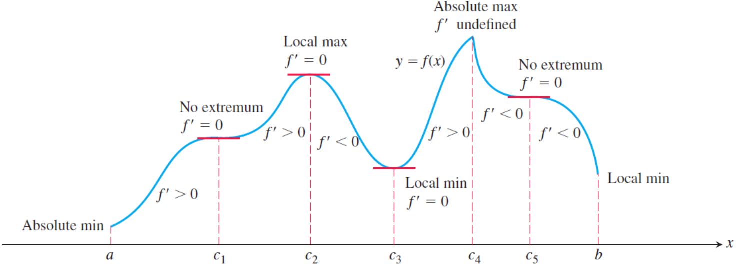
Interval	$-\infty < x < -2$	$-2 < x < 2$	$2 < x < \infty$
f' evaluated	$f'(-3) = 15$	$f'(0) = -12$	$f'(3) = 15$
Sign of f'	$+$	$-$	$+$
Behavior of f	increasing	decreasing	increasing

(from Thomas'Calculus)

First Derivative Test for Local Extrema

Suppose that c is a critical point of a continuous function f, and that f is differentiable at every point in some interval containing c except possibly at c itself. Moving across this interval from left to right,

1. if f' changes from negative to positive at c, then f has a local minimum at c;
2. if f' changes from positive to negative at c, then f has a local maximum at c;
3. if f' does not change sign at c (that is, f' is positive on both sides of c or negative on both sides), then f has no local extremum at c.



(from Thomas'Calculus)

EXAMPLE Find the critical points of $f(x) = x^{1/3}(x - 4) = x^{4/3} - 4x^{1/3}$. Identify the open intervals on which f is increasing and decreasing. Find the function's local and absolute extreme values.

Solution The function f is continuous at all x since it is the product of two continuous functions, $x^{1/3}$ and $(x - 4)$. The first derivative

$$f'(x) = \frac{d}{dx}\left(x^{4/3} - 4x^{1/3}\right) = \frac{4}{3}x^{1/3} - \frac{4}{3}x^{-2/3} = \frac{4}{3}x^{-2/3}(x - 1) = \frac{4(x - 1)}{3x^{2/3}}$$

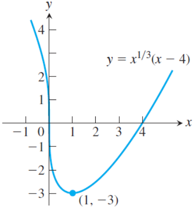
is zero at $x = 1$ and undefined at $x = 0$. There are no endpoints in the domain, so the critical points $x = 0$ and $x = 1$ are the only places where f might have an extreme value.

Interval	$x < 0$	$0 < x < 1$	$x > 1$
Sign of f'	$-$	$-$	$+$
Behavior of f	decreasing	decreasing	increasing

Corollary to the Mean Value Theorem implies that f decreases on $(-\infty, 0)$, decreases on $(0, 1)$, and increases on $(1, \infty)$. The First Derivative Test for Local Extrema tells us that f does not have an extreme value at $x = 0$ (f' does not change sign) and that f has a local minimum at $x = 1$ (f' changes from negative to positive).

The value of the local minimum is $f(1) = 1^{1/3}(1 - 4) = -3$. This is also an absolute minimum since f is decreasing on $(-\infty, 1)$ and increasing on $(1, \infty)$. Figure 4.22 shows this value in relation to the function's graph.

Note that $\lim_{x\to 0} f'(x) = -\infty$, so the graph of f has a vertical tangent at the origin.



(from Thomas'Calculus)

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EXAMPLE Find the critical points of $f(x) = (x^2 - 3)e^x$.
Identify the open intervals on which f is increasing and decreasing. Find the function's local and absolute extreme values.

Solution The function f is continuous and differentiable for all real numbers, so the critical points occur only at the zeros of f'.

Using the Derivative Product Rule, we find the derivative

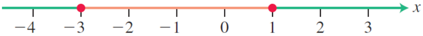
$$f'(x) = (x^2 - 3)\cdot\frac{d}{dx}e^x + \frac{d}{dx}(x^2 - 3)\cdot e^x = (x^2 - 3)\cdot e^x + (2x)\cdot e^x = (x^2 + 2x - 3)e^x.$$

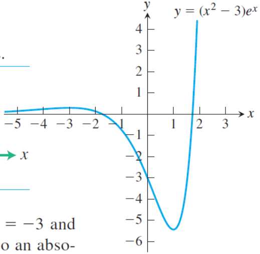
Since e^x is never zero, the first derivative is zero if and only if

$$x^2 + 2x - 3 = 0 \qquad (x + 3)(x - 1) = 0.$$

The zeros $x = -3$ and $x = 1$ partition the x-axis into open intervals as follows.

Interval	$x < -3$	$-3 < x < 1$	$1 < x$
Sign of f'	$+$	$-$	$+$
Behavior of f	increasing	decreasing	increasing





We can see from the table that there is a local maximum (about 0.299) at $x = -3$ and a local minimum (about -5.437) at $x = 1$. The local minimum value is also an absolute minimum because $f(x) > 0$ for $|x| > \sqrt{3}$. There is no absolute maximum. The function increases on $(-\infty, -3)$ and $(1, \infty)$ and decreases on $(-3, 1)$.

(from Thomas'Calculus)

Second derivative test for local extrema

THEOREM —Second Derivative Test for Local Extrema Suppose f'' is continuous on an open interval that contains $x = c$.

1. If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x = c$.
2. If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at $x = c$.
3. If $f'(c) = 0$ and $f''(c) = 0$, then the test fails. The function f may have a local maximum, a local minimum, or neither.

(from Thomas'Calculus)

Generalization:

Let $f : (a, b \to \mathbb{R})$ be n-times continuously differentiable, $n \in \mathbb{N}$, $x_0 \in (a, b)$, $f'(x_0) = \cdots = f^{(n-1)}(x_0) = 0$ and $f^{(n)}(x_0) \neq 0$. Then

- if n is even and $n \geq 2$, then f has a local extremum at x_0:
  - if $f^{(n)}(x_0) > 0$, then x_0 is a point of local minimum;
  - if $f^{(n)}(x_0) < 0$, then x_0 is a point of local maximum;
- if n is odd, then f has no extremum at x_0.

Concavity and convexity

Definition

A function $f : (a, b) \to \mathbb{R}$ is said to be **concave** (or **concave down**) if, for any $x_1, x_2 \in (a, b)$ and for any $\alpha \in [0, 1]$,
$f((1-\alpha)x_1 + \alpha x_2) \geq (1-\alpha)f(x_1) + \alpha f(x_2)$.
If the inequality is strict, then the function is called **strictly concave**.

Geometrically, this means that the line segment between any two points on the graph of the function lies below the graph between the two points.

Another definition

A differentiable function $f : (a, b) \to \mathbb{R}$ is said to be **concave** if any its tangent lies above the graph of the function, i.e. if
$f(x_2) \leq f(x_1) + f'(x_1)(x_2 - x_1)$ for all $x_1, x_2 \in (a, b)$.
If the inequality is strict, then the function is called **strictly concave**.

Concavity and convexity

Definition

A function $f : (a, b) \to \mathbb{R}$ is said to be **convex** (or **concave up**) if, for any $x_1, x_2 \in (a, b)$ and for any $\alpha \in [0, 1]$,
$f((1-\alpha)x_1 + \alpha x_2) \leq (1-\alpha)f(x_1) + \alpha f(x_2)$.
If the inequality is strict, then the function is called **strictly convex**.

Geometrically, this means that the line segment between any two points on the graph of the function lies above the graph between the two points.

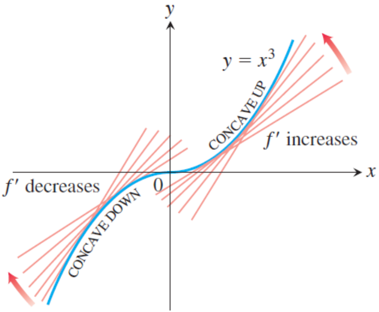
Another definition

A differentiable function $f : (a, b) \to \mathbb{R}$ is said to be **convex** if any its tangent lies below the graph of the function, i.e. if
$f(x_2) \leq f(x_1) + f'(x_1)(x_2 - x_1)$ for all $x_1, x_2 \in (a, b)$.
If the inequality is strict, then the function is called **strictly convex**.

Theorem (Theorem on concavity of a function)

Let $f : (a, b) \to \mathbb{R}$ be differentiable on (a, b). Then

- f is (strictly) concave on (a, b) if and only if its derivative function f' is (strictly) monotonically decreasing on (a, b);
- f is (strictly) convex on (a, b) if and only if its derivative function f' is (strictly) monotonically increasing on (a, b).



Concavity and convexity

Theorem (Second derivative test for concavity)

Let $f : (a, b) \to \mathbb{R}$ be twice-differentiable. Then

- if $f''(x) \leq 0$ for all $x \in (a, b)$ then f is concave on (a, b);
- if $f''(x) \geq 0$ for all $x \in (a, b)$ then f is convex on (a, b).

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Definition

Let $f : (a, b) \to \mathbb{R}$ be differentiable. A point $c \in (a, b)$ is called a **point of inflection** of f, if there exists an $\varepsilon > 0$ such that one of the following properties holds:

- f is concave on $(c - \varepsilon, c)$ and convex on $(c, c + \varepsilon)$;
- f is convex on $(c - \varepsilon, c)$ and concave on $(c, c + \varepsilon)$.

DEFINITION A point $(c, f(c))$ where the graph of a function has a tangent line and where the concavity changes is a **point of inflection**.

(from Thomas'Calculus)

First derivative criterium for points of inflection

Let $f : (a, b) \to \mathbb{R}$ be differentiable. Then $c \in (a, b)$ is a point of inflection of f iff f' has a extremum at c.

Second derivative criterium for points of inflection

Let $f : (a, b) \to \mathbb{R}$ be twice-differentiable. Then $c \in (a, b)$ is a point of inflection of f iff f'' change its sign at c.

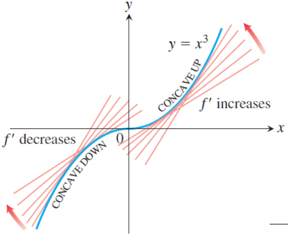
EXAMPLE

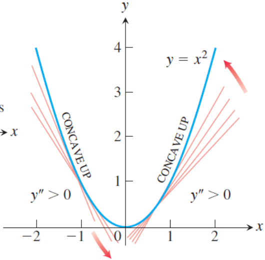
(a) The curve $y = x^3$ is concave down on $(-\infty, 0)$ where $y'' = 6x < 0$ and concave up on $(0, \infty)$ where $y'' = 6x > 0$.

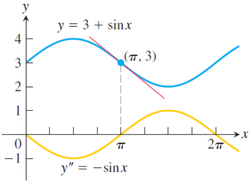
(b) The curve $y = x^2$ is concave up on $(-\infty, \infty)$ because its second derivative $y'' = 2$ is always positive. ■

EXAMPLE Determine the concavity of $y = 3 + \sin x$ on $[0, 2\pi]$.

Solution The first derivative of $y = 3 + \sin x$ is $y' = \cos x$, and the second derivative is $y'' = -\sin x$. The graph of $y = 3 + \sin x$ is concave down on $(0, \pi)$, where $y'' = -\sin x$ is negative. It is concave up on $(\pi, 2\pi)$, where $y'' = -\sin x$ is positive ■







(from Thomas'Calculus)

Concavity and convexity

EXAMPLE The graph of $f(x) = x^{5/3}$ has a horizontal tangent at the origin because $f'(x) = (5/3)x^{2/3} = 0$ when $x = 0$. However, the second derivative

$$f''(x) = \frac{d}{dx}\left(\frac{5}{3}x^{2/3}\right) = \frac{10}{9}x^{-1/3}$$

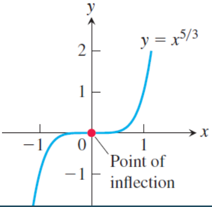
fails to exist at $x = 0$. Nevertheless, $f''(x) < 0$ for $x < 0$ and $f''(x) > 0$ for $x > 0$, so the second derivative changes sign at $x = 0$ and there is a point of inflection at the origin.

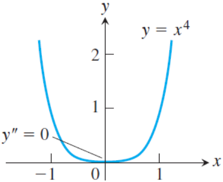
EXAMPLE The curve $y = x^4$ has no inflection point at $x = 0$ Even though the second derivative $y'' = 12x^2$ is zero there, it does not change sign.

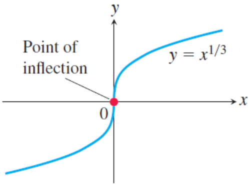
EXAMPLE The graph of $y = x^{1/3}$ has a point of inflection at the origin because the second derivative is positive for $x < 0$ and negative for $x > 0$:

$$y'' = \frac{d^2}{dx^2}\left(x^{1/3}\right) = \frac{d}{dx}\left(\frac{1}{3}x^{-2/3}\right) = -\frac{2}{9}x^{-5/3}.$$

However, both $y' = x^{-2/3}/3$ and y'' fail to exist at $x = 0$, and there is a vertical tangent







Concavity and convexity

To study the motion of an object moving along a line as a function of time, we often are interested in knowing when the object's acceleration, given by the second derivative, is positive or negative. The points of inflection on the graph of the object's position function reveal where the acceleration changes sign.

EXAMPLE A particle is moving along a horizontal coordinate line (positive to the right) with position function $s(t) = 2t^3 - 14t^2 + 22t - 5, \quad t \geq 0.$
Find the velocity and acceleration, and describe the motion of the particle.

Solution The velocity is $v(t) = s'(t) = 6t^2 - 28t + 22 = 2(t-1)(3t-11)$,
and the acceleration is $a(t) = v'(t) = s''(t) = 12t - 28 = 4(3t - 7)$.
When the function $s(t)$ is increasing, the particle is moving to the right; when $s(t)$ is decreasing, the particle is moving to the left.

Notice that the first derivative ($v = s'$) is zero at the critical points $t = 1$ and $t = 11/3$.

Interval	$0 < t < 1$	$1 < t < 11/3$	$11/3 < t$
Sign of $v = s'$	$+$	$-$	$+$
Behavior of s	increasing	decreasing	increasing
Particle motion	right	left	right

The particle is moving to the right in the time intervals $[0, 1)$ and $(11/3, \infty)$, and moving to the left in $(1, 11/3)$. It is momentarily stationary (at rest) at $t = 1$ and $t = 11/3$.

The acceleration $a(t) = s''(t) = 4(3t - 7)$ is zero when $t = 7/3$.

Interval	$0 < t < 7/3$	$7/3 < t$
Sign of $a = s''$	$-$	$+$
Graph of s	concave down	concave up

The particle starts out moving to the right while slowing down, and then reverses and begins moving to the left at $t = 1$ under the influence of the leftward acceleration over the time interval $[0, 7/3)$. The acceleration then changes direction at $t = 7/3$ but the particle continues moving leftward, while slowing down under the rightward acceleration. At $t = 11/3$ the particle reverses direction again: moving to the right in the same direction as the acceleration, so it is speeding up.

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Procedure for graphing $y = f(x)$

1. Find the domain of f (and, if possible, the range of f).
2. Symmetries (odd/even/neither or both; periodicity).
3. Continuity.
4. Intercepts ($x = 0$, $f(0) =?$; $y = 0$, $x =?$ (if possible)).
5. Behavior at the endpoints of domain (also at $\pm\infty$).
6. Differentiability.
7. Compute the first derivative and critical points of f. Find intervals of monotonicity, extreme points, extreme values (make a table).
8. Compute the second derivative and critical points of f'. Find intervals of concavity, points of inflection, values at these points (make a table).
9. Identify any asymptotes that may exist (vertical $x = a$ if $\lim_{x\to a^\pm} f(x) = \pm\infty$; horizontal $y = b$ if $\lim_{x\to\pm\infty} f(x) = b$; oblique $y = mx + n$ if $\lim_{x\to\pm\infty}(f(x) - (mx + n)) = 0$.
10. Plot key points, confirm the range, sketch the curve together with any asymptotes that exist.

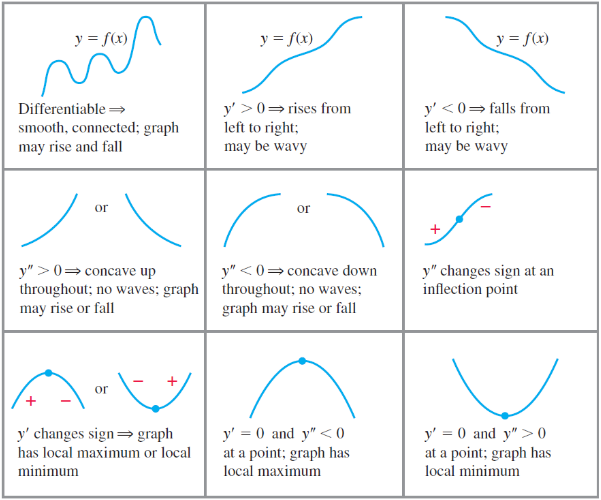
Examples (see Thomas'Calculus, Ch. 4.4)

1) $f(x) = \dfrac{(x+1)^2}{x^2+1}$;

2) $f(x) = \dfrac{x^2+4}{2x}$;

3) $f(x) = e^{1/x}$.

Graphical behavior of functions from derivatives



(from Thomas'Calculus)