Theoretical part

Problem 1 (6 Points)

Decide whether the following statements are true or false.

	true	false	
1) Any differentiable at a point function has a finite limit at this point.	○	○	(1 Point)
2) Let A, B, C be non-empty sets such that $C \subset A$ and $C \subset B$. Then $(A \setminus B) \cap C = \emptyset$.	○	○	(1 Point)
3) The function $f : [0, 2] \to \mathbb{R}$, $f(x) = (x-1)^4$ is injective, but not surjective.	○	○	(1 Point)
4) Any function which is differentiable on a closed interval is continuous on this interval.	○	○	(1 Point)
5) If the series $\sum_{n=1}^{\infty} a_n$ converges, then the series $\sum_{n=1}^{\infty} \|a_n\|$ is also converging.	○	○	(1 Point)
6) If the function is bounded on a closed interval then it is integrable over this interval.	○	○	(1 Point)

Problem 2 (8 Points)

a) Formulate the definition of the limit of a function. (2 Points)

b) Formulate the Theorem on existence of a limit of a function. (2 Points)

c) Give an example of a divergent sequence which has at least one convergent subsequence. (2 Points)

Problem 3 (6 Points)

a) Formulate the definition of a derivative of a function. (2 Points)

b) Formulate Lagrange's Mean value theorem. (3 Points)

Problem 4 (5 Points)

Formulate the Comparison convergence test for infinite series. (3 Points)

Bonus question: Prove it. (5 Bonus Points)

Problem 5 (4 Points)

a) Formulate Taylor's Theorem for an infinitely differentiable function $f : [a, b] \to \mathbb{R}$. (3 Points)

b) Write down *one* of the common Taylor series (e.g., for $f(x) = e^x$, $\sin x$, $\cos x$, $\frac{1}{1-x}$, etc.) (2 Points)

Problem 6 (6 Points)

Formulate the Substitution Rule for definite integrals. (3 Points)

Practical part

Problem 1 (6 Points)

Consider the sequence $\{a_n\}_{n\in\mathbb{N}}$ with $a_n = \frac{1}{2}\left(a_{n-1} + \frac{c}{a_{n-1}}\right)$ for all $n \in \mathbb{N}$, where $c > 0$ and $a_0 = M > 0$.

a) Prove that $a_n \geq \sqrt{c}\ \forall n \in \mathbb{N}$. (2 Points)
   Hint: use the fact that $t + \frac{1}{t} \geq 2\ \forall t \geq 0$.

b) Using a), prove that $\{a_n\}_{n\in\mathbb{N}}$ monotonically decreases. (2 Points)

c) Determine whether the sequence $\{a_n\}_{n\in\mathbb{N}}$ is bounded or unbounded. (1 Point)

d) Prove that $\{a_n\}_{n\in\mathbb{N}}$ has a finite limit. (1 Point)

e) Prove that $\lim\limits_{n\to\infty} a_n = \sqrt{c}$. (2 Points)

Problem 2 (8 Points)

Consider the function $f : [-1, 3] \to \mathbb{R}$, $f(x) = x^3 - 3x^2 + 1$.

a) Find intervals of monotonicity of f. (2 Points)

b) Find all local and global extrema of f. (2 Points)

c) Find the equation of the tangential line at the points $x = 0$ and $x = 1$. (2 Points)

Problem 3 (6 Points)

Find $\frac{dy}{dx}$ if

a) $y = \frac{e^{2x}}{e^{x^2}}$. (2 Points)

b) $y = f(\sin^2 x) + f(\cos^2 x)$, where f is a differentiable function. (3 Points)

c) **Bonus question**: Find $\frac{dy}{dx}$ if $y^5 + y^3 + y - x = 1$ and calculate the value of $\frac{dy}{dx}$ when $x = 0$. (4 Bonus Points)

Problem 4 (5 Points)

Find the following limits:

a) $\lim_{x \to 4} \frac{\sqrt{1+2x}-3}{\sqrt{x}-2}$. (2 Points)

b) $\lim_{x \to 0} \frac{\sin x - x\cos x}{\sin^3 x}$. (2 Points)

Problem 5 (4 Points)

Find the following integrals:

a) $\int\limits_0^1 \left(\frac{1}{\sqrt{x}} - \pi \sin(\pi x) \right) dx.$ (3 Points)

b) $\int x \ln^2 x \, dx.$ (3 Points)