1. Preliminaries

1.1. Elements of logic and theory of sets

1.1.3. Sets

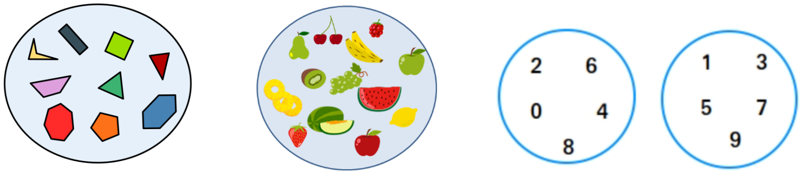
Content:

- Sets: basic notions
- Types of sets
- Basic operations with sets
- Algebra of sets
- Further operations with sets

Sets

A **set** is a well-defined collection of objects. The items present in a set are called **elements** (or members) of a set.

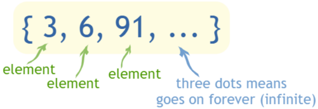
B. Bolzano (1781 –1848)



(from wikipedia.org, mathsisfun.com, basic-mathematics.com)

Basic notations and definitions

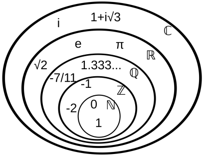
- $A, B \ldots, X, Y$ – sets
- $a, b \ldots, x, y$ – elements of sets
- $a \in A$ – a is an element of set A
- $a \notin A$ – a is not an element of set A
- $A = \{a, b, c, \ldots\}$ – set A consists of elements a, b, c, and other elements defined by some rule
- $\emptyset$ – null or empty set (the set with no elements)
- U (or X) – universal set



(from mathsisfun.com)

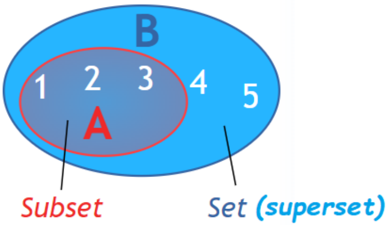
Special sets of numbers in mathematics

- $\mathbb{N} = \{1, 2, 3, \dots\}$ – the set of all natural numbers (sometimes with 0);
- $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ – the set of all integer numbers;
- $\mathbb{Q} = \left\{ \frac{a}{b} \,\middle|\, a, b \in \mathbb{Z} \text{ and } b \neq 0 \right\}$ – the set of all rational numbers;
- $\mathbb{R}$ the set of all real numbers, including all rational numbers and all irrational numbers (which include algebraic numbers such as $\sqrt{2}$ that cannot be rewritten as fractions, as well as transcendental numbers such as π and e);
- $\mathbb{C} = \{a + ib | a, b \in \mathbb{R}\}$ – the set of all complex numbers ($i = \sqrt{-1}$).



(from wikipedia.org)

Set A is a **subset** of B, $A \subseteq B$, if any element of A is contained in B. In this case, set B is **superset** of A, $B \supseteq A$.
If $A \subset B$, but $A \neq B$, then A is called a **proper subset** of B, $A \subset B$ or $A \subsetneq B$, and B is a **proper superset** of A, $B \supset A$ or $B \supsetneq A$.



(from mathsisfun.com)

Example: $A = \{1, 2, 3\}$, $B = \{1, 2, 3, 4, 5\}$. $A \subset B$, $B \supset A$.

Sets A and B are **equal**, $A = B$, if the consist of the same elements. If two sets have at least one element that is different, then they are **unequal** sets, $A \neq B$.

Example: $A = \{1, 2, 3\}$, $B = \{3, 1, 2\}$, $C = \{1, 2\}$. $A = B$, $A \neq C$, $B \neq C$.

Two sets A and B are said to be **equivalent**, $A \sim B$, when they have the same number of elements, though the elements are different.
The number of elements is called the **cardinality** (or the cardinal number) of the set, $n(A)$.

Two sets are said to be **overlapping** if at least one element from set A is present in set B. Otherwise, they called to be **disjoint**.

Types of sets

A set that does not contain any element is called an **empty set** or a **null** set. A

A set with a single element is a **singleton**.

A **finite** set is a set that has a finite number of elements. Otherwise, it is **infinite**.

A set is **countable** if either it is finite or it can be made in one to one correspondence with the set of natural numbers $\mathbb{N} = 0, 1, 2, 3, ...$ Otherwise, the set is **uncountable**.

Power set is the set of all subsets that a set could contain.

Algebra of sets

Basic operations and relations:

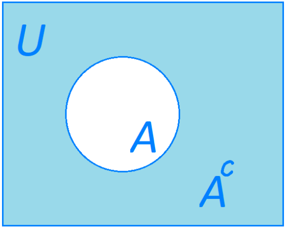
- complement A^c;
- intersection $\cap$;
- union $\cup$;
- inclusion $\subset$;
- equality $=$.

Negation

Let A be a set in U. The **complement of A in U**, $\bar{A}$, A^c, A' is the set of elements that are in U and not in A:

$$A^c = \{x \in U : x \notin A\}.$$

Venn diagram:



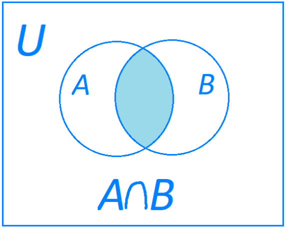
Example:

$$U = \mathbb{Z},$$
$$A = \{1, 2, \ldots, \}.$$
$$A^c = \{\ldots, -3, -2, -1, 0\}.$$

Let A, B be sets in U. The **intersection of sets A and B**, $A \cap B$, is the set of elements that exist in both set A and set B:

$$A \cap B = \{x \in U : x \in A \text{ and } x \in B\}.$$

Venn diagram:



Example:

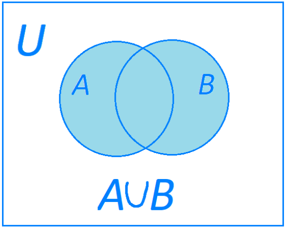
$A = \{1, 2, 3\}$.
$B = \{1, 2, 5\}$.
$A \cap B = \{1, 2\}$.

Union

Let A, B be sets in U. The **union of sets A and B**, $A \cup B$, $A \cup B$, is the set of elements that exist in set A, set B, or both A and B:

$$A \cup B = \{x \in U : x \in A \text{ or } x \in B\}.$$

Venn diagram:



Example:

$A = \{1, 2, 3\}.$
$B = \{1, 2, 5\}.$
$A \cup B = \{1, 2, 3, 5\}.$

Let A, B, C be sets in U.

Idempotent laws	$A \cup A = A,\ A \cap A = A$
Associative laws	$A \cap (B \cap C) = (A \cap B) \cap C$ $A \cup (B \cup C) = (A \cup B) \cup C$
Commutative laws	$A \cap B = B \cap A,\ A \cup B = B \cup A$
Distributive laws	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Identity laws	$A \cup \emptyset = A,\ A \cap U = A$
Domination laws	$A \cap \emptyset = \emptyset,\ A \cup U = U$
Double complement	$(A^c)^c = A$

Algebra of sets (cont'd)

Complement laws	$A \cap A^c = \emptyset$, $A \cup A^c = U$, $U^c = \emptyset$, $\emptyset =^c U$
De Morgan's laws:	$(A \cap B)^c = A^c \cup B^c$ $(A \cup B)^c = A^c \cap B^c$
Absorption laws	$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$
Conditional identities	$A \subseteq B \iff A^c \cup B = U$ $A = B \iff (A \subseteq B)$ and $(B \subseteq A)$

Further operations with sets

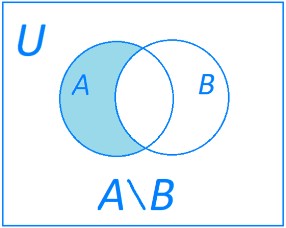
Further operations:

- complement A^c;
- intersection $\cap$;
- union $\cup$;
- set difference $\setminus$;
- symmetric difference $\triangle$;
- Cartesian product $A \times B$.

Let A, B be sets in U. The **difference (or relative complement) of sets A and B**, $A \setminus B$, is the set of elements that exist in set A but not in set B:

$$A \setminus B = \{x \in U : x \in A \text{ and } x \notin B\}.$$

Venn diagram:



Example:

$A = \{1, 2, 3\}$.
$B = \{1, 2, 5\}$.
$A \setminus B = \{3\}$,
$B \setminus A = \{5\}$.

Properties:

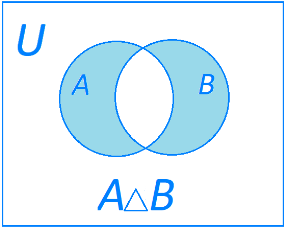
- $A \setminus B = A \cap B^c$, $(A \setminus B)^c = A^c \cup B = A^c \cup (B \cap A)$, $A^c \setminus B^c = B \setminus A$
- $A \setminus A = \emptyset$, $A \setminus \emptyset = A$, $\emptyset \setminus A = \emptyset$;
- $U \setminus A = A^c$, $A \setminus U = \emptyset$
- $C \setminus (A \cap B) = (C \setminus A) \cup (C \setminus B)$,
- $C \setminus (A \cup B) = (C \setminus A) \cap (C \setminus B)$,
- $C \setminus (B \setminus A) = (C \cap A) \cup (C \setminus B)$,
- $(B \setminus A) \cap C = (B \cap C) \setminus A = B \cap (C \setminus A)$,
- $(B \setminus A) \cup C = (B \cup C) \setminus (A \setminus C)$.

Symmetric difference

Let A, B be sets in U. The **symmetric difference (or disjunctive union) of sets A and B**, $A \triangle B$, is the set of elements which are either in A or in B, but not in their intersection:

$$A \triangle B = \{x : x \in A \text{ or } x \in B \text{ and } x \notin A \cap B\}.$$

Venn diagram:



Example:

$A = \{1, 2, 3\}$.
$B = \{1, 2, 5\}$.
$A \triangle B = \{3, 5\}$.

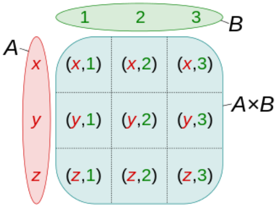
Properties :

- $A \triangle B = (A \setminus B) \cup (B \setminus A)$;
- $A \triangle B = (A \cup B) \setminus (A \cap B)$;
- $A \triangle B = B \triangle A$, $(A \triangle B) \triangle C = A \triangle (B \triangle C)$;
- $A \cap (B \triangle C) = (A \cap B) \triangle (A \cap C)$;
- $A \triangle \emptyset = A$, $A \triangle A = \emptyset$, $A \triangle U = A^c$;
- $A \triangle B = A^c \triangle B^c$;
- $A \triangle B = \emptyset$ iff $A = B$.

Cartesian product

Let A, B be sets in U. The **Cartesian product of of sets A and B**, $A \times B$, the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$:

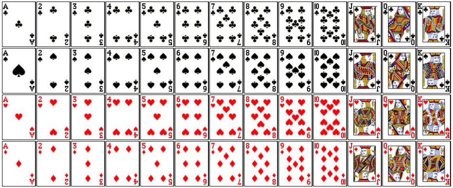
$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}.$$



from wikipedia.org

Example:

$A = \{1, 2, 3\}$.
$B = \{1, 2, 5\}$.
$A \times B = \{(1,1), (1,2), (1,5), (2,1), (2,2), (2,5), (3,1), (3,2), (3,5)\}$.



Cartesian product

Properties:

- $A \times B \neq B \times A$
  (unless $A = B$ or $A = \emptyset$ or $B = \emptyset$);
- $(A \times B) \times C \neq A \times (B \times C)$ *(unless one of the involved sets is empty)*;
- $A \times (B \cap C) = (A \times B) \cap (A \times C)$,
  $A \times (B \cup C) = (A \times B) \cup (A \times C)$,
  $A \times (B \setminus C) = (A \times B) \setminus (A \times C)$.
- $(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$ *In most cases, the above statement is not true if we replace intersection with union:*

$$(A \cup B) \times (C \cup D) \neq (A \times C) \cup (B \times D)$$

- $|A \times B| = |A| \cdot |B|$.