4. Differential calculus of functions of one real variable

4.2. Applications of derivatives: L'Hôpital's Rule

Content:

- L'Hôpital's Rule for indeterminate forms $\dfrac{0}{0}$
- L'Hôpital's Rule for indeterminate forms $\dfrac{\infty}{\infty}$
- Applications of L'Hôpital's Rule to other indeterminate forms

Indeterminate forms: $\frac{0}{0}$, $\frac{\infty}{\infty}$, $\infty \cdot 0$, $\infty - \infty$, 0^0, 1^∞.

Let (a, b) be an interval with $-\infty \leq a < b \leq +\infty$.

Theorem (L'Hôpital's Rule)

Let functions $f, g : (a, b) \to \mathbb{R}$ be differentiable on (a, b) except possibly at a point c. Suppose that $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$ and $g'(x) \neq 0$ for all $x \in (a, c) \cup (c, b)$. Then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)},$$

assuming that the limit on the right side of this equation exists.

Using L'Hôpital's Rule

To find

$$\lim_{x\to a} \frac{f(x)}{g(x)}$$

by l'Hôpital's Rule, we continue to differentiate f and g, so long as we still get the form $0/0$ at $x = a$. But as soon as one or the other of these derivatives is different from zero at $x = a$ we stop differentiating. L'Hôpital's Rule does not apply when either the numerator or denominator has a finite nonzero limit.

L'Hôpital's Rule

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Proof of l'Hôpital's Rule We first establish the limit equation for the case $x \longrightarrow a^{+}$. The method needs almost no change to apply to $x \longrightarrow a^{-}$, and the combination of these tw cases establishes the result.

Suppose that x lies to the right of a. Then $g'(x) \neq 0$, and we can apply Cauchy Mean Value Theorem to the closed interval from a to x. This step produces a number between a and x such that

$$\frac{f'(c)}{g'(c)} = \frac{f(x) - f(a)}{g(x) - g(a)}.$$

But $f(a) = g(a) = 0$, so $\dfrac{f'(c)}{g'(c)} = \dfrac{f(x)}{g(x)}$.

As x approaches a, c approaches a because it always lies between a and x. Therefore,

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = \lim_{c \to a^+} \frac{f'(c)}{g'(c)} = \lim_{x \to a^+} \frac{f'(x)}{g'(x)},$$

which establishes l'Hôpital's Rule for the case where x approaches a from above. The case where x approaches a from below is proved by applying Cauchy's Mean Value Theorem to the closed interval $[x, a]$, $x < a$. ■

(from Thomas'Calculus)

L'Hôpital's Rule

Remarks

- The assumption that $\lim_{x\to c}\frac{f'(x)}{g'(x)}$ exists is crucial and cannot be omitted.
- Condition $x \to a$ may be replaced by the one-sided limits $x \to a^+$ or $x \to a^-$.
- L'Hôpital's Rule applies to the indeterminate form $\pm\infty/\pm\infty$ in the same way as to $0/0$.

Theorem (L'Hôpital's Rule for indeterminate form ∞/∞)

Let functions $f, g : (a, b) \to \mathbb{R}$ be differentiable on (a, b) except possibly at a point c. Suppose that $\lim_{x\to c} f(x) = \pm\infty$ and $\lim_{x\to c} g(x) = \pm\infty$ and $g'(x) \neq$ for all $x \in (a, c) \cup (c, b)$. Then

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f'(x)}{g'(x)},$$

assuming that the limit on the right side of this equation exists.

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EXAMPLE The following limits involve $0/0$ indeterminate forms, so we apply l'Hôpital's Rule. In some cases, it must be applied repeatedly.

(a) $$\lim_{x\to 0}\frac{3x-\sin x}{x}=\lim_{x\to 0}\frac{3-\cos x}{1}=\left.\frac{3-\cos x}{1}\right|_{x=0}=2$$

(b) $$\lim_{x\to 0}\frac{\sqrt{1+x}-1}{x}=\lim_{x\to 0}\frac{\dfrac{1}{2\sqrt{1+x}}}{1}=\frac{1}{2}$$

(c) $$\lim_{x\to 0}\frac{\sqrt{1+x}-1-x/2}{x^2}$$ $\frac{0}{0}$; apply l'Hôpital's Rule.

$$=\lim_{x\to 0}\frac{(1/2)(1+x)^{-1/2}-1/2}{2x}$$ Still $\frac{0}{0}$; apply l'Hôpital's Rule again.

$$=\lim_{x\to 0}\frac{-(1/4)(1+x)^{-3/2}}{2}=-\frac{1}{8}$$ Not $\frac{0}{0}$; limit is found.

(d) $$\lim_{x\to 0}\frac{x-\sin x}{x^3}$$ $\frac{0}{0}$; apply l'Hôpital's Rule.

$$=\lim_{x\to 0}\frac{1-\cos x}{3x^2}$$ Still $\frac{0}{0}$; apply l'Hôpital's Rule again.

$$=\lim_{x\to 0}\frac{\sin x}{6x}$$ Still $\frac{0}{0}$; apply l'Hôpital's Rule again.

$$=\lim_{x\to 0}\frac{\cos x}{6}=\frac{1}{6}$$ Not $\frac{0}{0}$; limit is found. ■

EXAMPLE Be careful to apply l'Hôpital's Rule correctly:

$$\lim_{x\to 0} \frac{1 - \cos x}{x + x^2} \qquad \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{\sin x}{1 + 2x} \qquad \text{Not } \frac{0}{0}$$

It is tempting to try to apply l'Hôpital's Rule again, which would result in

$$\lim_{x\to 0} \frac{\cos x}{2} = \frac{1}{2},$$

but this is not the correct limit. L'Hôpital's Rule can be applied only to limits that give indeterminate forms, and $\lim_{x\to 0}(\sin x)/(1 + 2x)$ does not give an indeterminate form. Instead, this limit is $0/1 = 0$, and the correct answer for the original limit is 0. ■

(from Thomas'Calculus)

EXAMPLE In this example the one-sided limits are different.

(a) $\displaystyle\lim_{x\to 0^+} \frac{\sin x}{x^2}$ $\qquad \frac{0}{0}$

$\displaystyle = \lim_{x\to 0^+} \frac{\cos x}{2x} = \infty$ $\qquad$ Positive for $x > 0$

(b) $\displaystyle\lim_{x\to 0^-} \frac{\sin x}{x^2}$ $\qquad \frac{0}{0}$

$\displaystyle = \lim_{x\to 0^-} \frac{\cos x}{2x} = -\infty$ $\qquad$ Negative for $x < 0$ ■

(from Thomas'Calculus)

Example: $f(x) = \dfrac{x - \sin x}{x}$.

$\lim_{x\to+\infty} \dfrac{x - \sin x}{x} \overset{L'H.??}{=} \lim_{x\to+\infty} \dfrac{1 - \cos x}{1}$ does not exists!

So we cannot apply L'Hôpital's Rule. On the other hand, we can apply the Sandwich theorem: $\dfrac{x-1}{x} \leq \dfrac{x - \sin x}{x} \leq \dfrac{x+1}{x}$,

$\lim_{x\to+\infty} \dfrac{x - \sin x}{x} = 1.$

EXAMPLE Find the limits of these ∞/∞ forms:

(a) $\displaystyle\lim_{x\to\pi/2} \frac{\sec x}{1 + \tan x}$ **(b)** $\displaystyle\lim_{x\to\infty} \frac{\ln x}{2\sqrt{x}}$ **(c)** $\displaystyle\lim_{x\to\infty} \frac{e^x}{x^2}$.

Solution

(a) The numerator and denominator are discontinuous at $x = \pi/2$, so we investigate the one-sided limits there. To apply l'Hôpital's Rule, we can choose I to be any open interval with $x = \pi/2$ as an endpoint.

$$\lim_{x\to(\pi/2)^-} \frac{\sec x}{1 + \tan x} \qquad \frac{\infty}{\infty} \text{ from the left so we apply l'Hôpital's Rule.}$$

$$= \lim_{x\to(\pi/2)^-} \frac{\sec x \tan x}{\sec^2 x} = \lim_{x\to(\pi/2)^-} \sin x = 1$$

The right-hand limit is 1 also, with $(-\infty)/(-\infty)$ as the indeterminate form. Therefore, the two-sided limit is equal to 1.

(b) $$\lim_{x\to\infty} \frac{\ln x}{2\sqrt{x}} = \lim_{x\to\infty} \frac{1/x}{1/\sqrt{x}} = \lim_{x\to\infty} \frac{1}{\sqrt{x}} = 0 \qquad \frac{1/x}{1/\sqrt{x}} = \frac{\sqrt{x}}{x} = \frac{1}{\sqrt{x}}$$

(c) $$\lim_{x\to\infty} \frac{e^x}{x^2} = \lim_{x\to\infty} \frac{e^x}{2x} = \lim_{x\to\infty} \frac{e^x}{2} = \infty$$

(from Thomas'Calculus)

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Further examples:

1) $\forall k \in \mathbb{N}$,

$$\lim_{x\to+\infty} \frac{x^k}{e^x} \overset{L'H.\infty/\infty}{=} \lim_{x\to+\infty} \frac{kx^{k-1}}{e^x} \overset{L'H.\infty/\infty}{=} \dots \overset{L'H.\infty/\infty}{=} \lim_{x\to+\infty} \frac{k!}{e^x} = 0.$$

2) $\forall k \in \mathbb{N}$, $\displaystyle\lim_{x\to+\infty} \frac{\ln x^k}{x} \overset{L'H.\infty/\infty}{=} \lim_{x\to+\infty} \frac{k(lnx)^{k-1}\frac{1}{x}}{1} =$

$$\lim_{x\to+\infty} \frac{k(lnx)^{k-1}}{x} \overset{L'H.\infty/\infty}{=} \dots \overset{L'H.\infty/\infty}{=} \lim_{x\to+\infty} \frac{k!}{x} = 0.$$

3) $\displaystyle\lim_{x\to+\infty} \frac{e^x+e^{-x}}{e^x-e^{-x}} \overset{L'H.\infty/\infty}{=} \lim_{x\to+\infty} \frac{e^x-e^{-x}}{e^x+e^{-x}} \overset{L'H.\infty/\infty}{=} \lim_{x\to+\infty} \frac{e^x+e^{-x}}{e^x-e^{-x}} =$

...

$$\lim_{x\to+\infty} \frac{e^x+e^{-x}}{e^x-e^{-x}} = \{y=e^x\} = \lim_{y\to+\infty} \frac{y+1/y}{y-1/y} = \lim_{y\to+\infty} \frac{1-y^{-2}}{1+y^{-2}} = 1.$$

Alternatively,

$$\lim_{x\to+\infty} \frac{e^x+e^{-x}}{e^x-e^{-x}} = \lim_{x\to+\infty} \frac{e^{2x}+1}{e^{2x}-1} \overset{L'H.\infty/\infty}{=} \lim_{x\to+\infty} \frac{2e^{2x}}{2e^{2x}} = 1.$$

Further examples:

4)

- An arbitrarily large number of applications may never lead to an answer even without repeating:

$$\lim_{x\to\infty}\frac{x^{\frac{1}{2}}+x^{-\frac{1}{2}}}{x^{\frac{1}{2}}-x^{-\frac{1}{2}}}=\lim_{x\to\infty}\frac{\frac{1}{2}x^{-\frac{1}{2}}-\frac{1}{2}x^{-\frac{3}{2}}}{\frac{1}{2}x^{-\frac{1}{2}}+\frac{1}{2}x^{-\frac{3}{2}}}=\lim_{x\to\infty}\frac{-\frac{1}{4}x^{-\frac{3}{2}}+\frac{3}{4}x^{-\frac{5}{2}}}{-\frac{1}{4}x^{-\frac{3}{2}}-\frac{3}{4}x^{-\frac{5}{2}}}=\cdots.$$

This situation too can be dealt with by a transformation of variables, in this case $y=\sqrt{x}$:

$$\lim_{x\to\infty}\frac{x^{\frac{1}{2}}+x^{-\frac{1}{2}}}{x^{\frac{1}{2}}-x^{-\frac{1}{2}}}=\lim_{y\to\infty}\frac{y+y^{-1}}{y-y^{-1}}=\lim_{y\to\infty}\frac{1-y^{-2}}{1+y^{-2}}=\frac{1}{1}=1.$$

Again, an alternative approach is to multiply numerator and denominator by $x^{1/2}$ before applying L'Hôpital's rule:

$$\lim_{x\to\infty}\frac{x^{\frac{1}{2}}+x^{-\frac{1}{2}}}{x^{\frac{1}{2}}-x^{-\frac{1}{2}}}=\lim_{x\to\infty}\frac{x+1}{x-1}=\lim_{x\to\infty}\frac{1}{1}=1.$$

(from wikipedia.org)

Applications of L'Hôpital's Rule

Indeterminate form	Conditions	Transformation to $0/0$
$\frac{0}{0}$	$\lim\limits_{x\to c} f(x) = 0, \ \lim\limits_{x\to c} g(x) = 0$	—
$\frac{\infty}{\infty}$	$\lim\limits_{x\to c} f(x) = \infty, \ \lim\limits_{x\to c} g(x) = \infty$	$\lim\limits_{x\to c} \frac{f(x)}{g(x)} = \lim\limits_{x\to c} \frac{1/g(x)}{1/f(x)}$
$0 \cdot \infty$	$\lim\limits_{x\to c} f(x) = 0, \ \lim\limits_{x\to c} g(x) = \infty$	$\lim\limits_{x\to c} f(x)g(x) = \lim\limits_{x\to c} \frac{f(x)}{1/g(x)}$
$\infty - \infty$	$\lim\limits_{x\to c} f(x) = \infty, \ \lim\limits_{x\to c} g(x) = \infty$	$\lim\limits_{x\to c}(f(x) - g(x)) = \lim\limits_{x\to c} \frac{1/g(x) - 1/f(x)}{1/(f(x)g(x))}$
0^0	$\lim\limits_{x\to c} f(x) = 0^+, \lim\limits_{x\to c} g(x) = 0$	$\lim\limits_{x\to c} f(x)^{g(x)} = \exp \lim\limits_{x\to c} \frac{g(x)}{1/\ln f(x)}$
1^∞	$\lim\limits_{x\to c} f(x) = 1, \ \lim\limits_{x\to c} g(x) = \infty$	$\lim\limits_{x\to c} f(x)^{g(x)} = \exp \lim\limits_{x\to c} \frac{\ln f(x)}{1/g(x)}$
∞^0	$\lim\limits_{x\to c} f(x) = \infty, \ \lim\limits_{x\to c} g(x) = 0$	$\lim\limits_{x\to c} f(x)^{g(x)} = \exp \lim\limits_{x\to c} \frac{g(x)}{1/\ln f(x)}$

(from wikipedia.org)

EXAMPLE Find the limits of these $\infty \cdot 0$ forms:

(a) $\lim\limits_{x\to\infty}\left(x \sin \dfrac{1}{x}\right)$ **(b)** $\lim\limits_{x\to 0^+} \sqrt{x} \ln x$

Solution

a. $\lim\limits_{x\to\infty}\left(x \sin \dfrac{1}{x}\right) = \lim\limits_{h\to 0^+}\left(\dfrac{1}{h}\sin h\right) = \lim\limits_{h\to 0^+} \dfrac{\sin h}{h} = 1$ $\infty \cdot 0$; let $h = 1/x$.

b.
$$\lim_{x\to 0^+} \sqrt{x}\ln x = \lim_{x\to 0^+} \frac{\ln x}{1/\sqrt{x}} \qquad \infty\cdot 0 \text{ converted to } \infty/\infty$$

$$= \lim_{x\to 0^+} \frac{1/x}{-1/2x^{3/2}} \qquad \text{l'Hôpital's Rule applied}$$

$$= \lim_{x\to 0^+}\left(-2\sqrt{x}\right) = 0$$

(from Thomas'Calculus)

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EXAMPLE Find the limit of this $\infty - \infty$ form:

$$\lim_{x \to 0} \left(\frac{1}{\sin x} - \frac{1}{x} \right).$$

Solution If $x \to 0^+$, then $\sin x \to 0^+$ and $\dfrac{1}{\sin x} - \dfrac{1}{x} \to \infty - \infty.$

Similarly, if $x \to 0^-$, then $\sin x \to 0^-$ and $\dfrac{1}{\sin x} - \dfrac{1}{x} \to -\infty - (-\infty) = -\infty + \infty.$

Neither form reveals what happens in the limit. To find out, we first combine the fractions:

$$\frac{1}{\sin x} - \frac{1}{x} = \frac{x - \sin x}{x \sin x}. \qquad \text{Common denominator is } x \sin x.$$

Then we apply l'Hôpital's Rule to the result:

$$\begin{aligned}
\lim_{x \to 0} \left(\frac{1}{\sin x} - \frac{1}{x} \right) &= \lim_{x \to 0} \frac{x - \sin x}{x \sin x} && \frac{0}{0} \\
&= \lim_{x \to 0} \frac{1 - \cos x}{\sin x + x \cos x} && \text{Still } \frac{0}{0} \\
&= \lim_{x \to 0} \frac{\sin x}{2 \cos x - x \sin x} = \frac{0}{2} = 0. && \blacksquare
\end{aligned}$$

(from Thomas'Calculus)

EXAMPLE Apply l'Hôpital's Rule to show that $\lim_{x\to 0^+}(1+x)^{1/x} = e$.

Solution The limit leads to the indeterminate form 1^∞. We let $f(x) = (1+x)^{1/x}$ and find $\lim_{x\to 0^+} \ln f(x)$. Since

$$\ln f(x) = \ln(1+x)^{1/x} = \frac{1}{x}\ln(1+x),$$

l'Hôpital's Rule now applies to give

$$\begin{aligned} \lim_{x\to 0^+} \ln f(x) &= \lim_{x\to 0^+} \frac{\ln(1+x)}{x} && \frac{0}{0} \\ &= \lim_{x\to 0^+} \frac{\frac{1}{1+x}}{1} && \text{l'Hôpital's Rule applied} \\ &= \frac{1}{1} = 1. \end{aligned}$$

Therefore, $\lim_{x\to 0^+}(1+x)^{1/x} = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} e^{\ln f(x)} = e^1 = e.$ ■

(from Thomas'Calculus)

Further examples:

1) $\lim\limits_{x\to 0^+} x\ln x = \lim\limits_{x\to 0^+} \dfrac{\ln x}{\frac{1}{x}} \overset{L'H.\infty/\infty}{=} \lim\limits_{x\to 0^+} \dfrac{x^{-1}}{-x^{-2}} = \lim\limits_{x\to 0^+}(-x) = 0;$

2) $\lim\limits_{x\to 0^+} x^x = \lim\limits_{x\to 0^+} \left(e^{\ln x}\right)^x = \lim\limits_{x\to 0^+} e^{x\ln x} = e^0 = 1;$

3) Let $a \in \mathbb{R}$, $c \in \overline{\mathbb{R}}$, $\lim\limits_{x\to c} f(x) = +\infty$. Then $\lim\limits_{x\to c}\left(1 + \dfrac{a}{f(x)}\right)^{f(x)} = e^a$.

Proof:

$$\lim_{x\to c}\left(1 + \frac{a}{f(x)}\right)^{f(x)} = \lim_{x\to c} e^{f(x)\ln\left(1+\frac{a}{f(x)}\right)} = e^{\lim\limits_{x\to c}\left(f(x)\ln\left(1+\frac{a}{f(x)}\right)\right)} =$$

$$e^{\lim\limits_{x\to c}\left(\frac{\ln\left(1+\frac{a}{f(x)}\right)}{\frac{1}{f(x)}}\right)} \overset{L'H.0/0}{=} e^{\lim\limits_{x\to c}\left(\frac{\left(1+\frac{a}{f(x)}\right)^{-1}\cdot a\cdot\left(\frac{1}{f(x)}\right)'}{\left(\frac{1}{f(x)}\right)'}\right)} = a.$$

Further examples:

4) $$\lim_{x\to+\infty}\left(\frac{x+5}{x+1}\right)^{2x+6} = \lim_{x\to+\infty}\left(\frac{x+1+4}{x+1}\right)^{2x+6} =$$

$$\lim_{x\to+\infty}\left(1+\frac{4}{x+1}\right)^{2x+6} = \lim_{x\to+\infty}\left(\left(1+\frac{4}{x+1}\right)^{x+1}\right)^{\dfrac{2x+6}{x+1}} =$$

$$\lim_{x\to+\infty}(e^4)^{\dfrac{2x+6}{x+1}} = e^{4\lim\limits_{x\to+\infty}\dfrac{2x+6}{x+1}} = e^8.$$