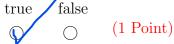
#### Theoretical part

#### Problem 1 (6 Points)

Decide whether the following statements are true or false.

1) Any differentiable at a point function has a finite limit at this point.



2) Let  $A,\,B,\,C$  be non-empty sets such that  $C\subset A$  and  $C\subset B$ . Then  $(A\setminus B)\cap C=\emptyset$ .



3) The function  $f:[0,2]\to\mathbb{R},\, f(x)=(x-1)^4$  is injective, but not surjective.



4) Any function which is differentiable on a closed interval is continuous on this interval.



5) If the series  $\sum_{n=1}^{\infty} a_n$  converges, then the series  $\sum_{n=1}^{\infty} |a_n|$  is also converging.



6) If the function is bounded on a closed interval then it is integrable over this interval.



## Problem 2 (8 Points)

- a) Formulate the definition of the limit of a function. (2 Points)
- b) Formulate the Theorem on existence of a limit of a function. (2 Points)
- c) Give an example of a divergent sequence which has at least one convergent subsequence. (2 Points)

## Problem 3 (6 Points)

a) Formulate the definition of a derivative of a function. (2 Points)

b) Formulate Lagrange's Mean value theorem. (3 Points)

Problem 4 (5 Points)
Formulate the Comparison convergence test for infinite series. (3 Points)

Bonus question: Prove it. (5 Bonus Points)

## Problem 5 (4 Points)

- a) Formulate Taylor's Theorem for an infinitely differentiable function  $f:[a,b]\to\mathbb{R}.$  (3 Points)
- b) Write down <u>one</u> of the common Taylor series (e.g., for  $f(x) = e^x$ ,  $\sin x$ ,  $\cos x$ ,  $\frac{1}{1-x}$ , etc.) (2 Points)

Problem 6 (6 Points)
Formulate the Substitution Rule for definite integrals.

(3 Points)

## Practical part

#### Problem 1 (6 Points)

Consider the sequence  $\{a_n\}_{n\in\mathbb{N}}$  with  $a_n = \frac{1}{2}\left(a_{n-1} + \frac{c}{a_{n-1}}\right)$  for all  $n\in\mathbb{N}$ , where c>0 and  $a_0=M>0$ .

- a) Prove that  $a_n \ge \sqrt{c} \ \forall n \in \mathbb{N}$ . (2 Points) **Hint:** use the fact that  $t + \frac{1}{t} \ge 2 \ \forall t \ge 0$ .
- b) Using a), prove that  $\{a_n\}_{n\in\mathbb{N}}$  monotonically decreases. (2 Points)
- c) Determine whether the sequence  $\{a_n\}_{n\in\mathbb{N}}$  is bounded or unbounded.

  (1 Point)
- d) Prove that  $\{a_n\}_{n\in\mathbb{N}}$  has a finite limit. (1 Point)
- e) Prove that  $\lim_{n\to\infty} a_n = \sqrt{c}$ . (2 Points)

Problem 2 (8 Points) Consider the function  $f: [-1,3] \to \mathbb{R}, f(x) = x^3 - 3x^2 + 1$ .

a) Find intervals of monotonicity of f. (2 Points)

b) Find all local and global extrema of f. (2 Points)

c) Find the equation of the tangential line at the points x = 0 and x = 1. (2 Points)

# Problem 3 (6 Points) Find $\frac{dy}{dx}$ if

a) 
$$y = \frac{e^{2x}}{e^{x^2}}$$
. (2 Points)

- b)  $y = f(\sin^2 x) + f(\cos^2 x)$ , where f is a differentiable function. (3 Points)
- c) Bonus question: Find  $\frac{dy}{dx}$  if  $y^5 + y^3 + y x = 1$  and calculate the value of  $\frac{dy}{dx}$  when x = 0. (4 Bonus Points)

# Problem 4 (5 Points) Find the following limits:

a) 
$$\lim_{x \to 4} \frac{\sqrt{1+2x}-3}{\sqrt{x}-2}$$
. (2 Points)

b) 
$$\lim_{x \to 0} \frac{\sin x - x \cos x}{\sin^3 x}.$$
 (2 Points)

## Problem 5 (4 Points) Find the following integrals:

a) 
$$\int_{0}^{1} \left( \frac{1}{\sqrt{x}} - \pi \sin(\pi x) \right) dx.$$
 (3 Points)

b) 
$$\int x \ln^2 x \, dx$$
. (3 Points)