1. Preliminaries

1.3. Set of complex numbers

Content:

- Definition and properties
- Complex plane
- Operations, relations, properties
- Polar complex system
- Euler's formula
- Fields and rings
- Fundamental theorem of algebra

Complex numbers

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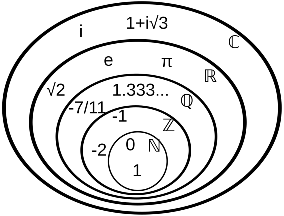
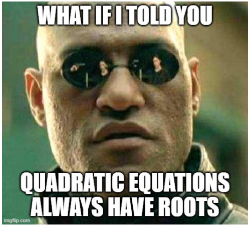
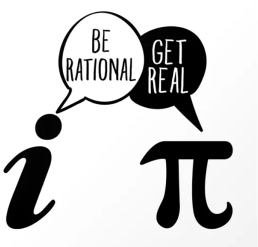
- $\mathbb{N} = \{1, 2, 3, \dots\}$ – the set of all natural numbers $(+, \cdot)$
- $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ – the set of all integer numbers $(+, \cdot, -)$
- $\mathbb{Q} = \left\{ \frac{a}{b} \middle| \, a, b \in \mathbb{Z} \text{ and } b \neq 0 \right\}$ – the set of all rational numbers $(+, \cdot, -, \div)$
- $\mathbb{R}$ – the set of all real numbers
- $\mathbb{C} = \{a + ib | a, b \in \mathbb{R}\}$ **– the set of all complex numbers** $(i = \sqrt{-1})$**.**





Real-life applications of complex numbers

- Signal Processing
- Electronics (AC Circuit Analysis)
- Electromagnetism
- Computer science engineering
- Mechanical and Civil Engineering
- Control systems
- Quantum Mechanics
- Quadratic equation



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A **complex number** is a number of the form $a + bi$, where a and b are real numbers, and i is an indeterminate satisfying $i^2 = -1$. a is called the **real part** and b is called the **imaginary part**.

Notations: if $z = a + ib$,

$$a := \Re(z) = Re(z),\ b := \Im(z) = Im(z)$$

Example: $z = 2 + 3i$ is a complex number, $Re(z) = 2$, $Im(z) = 3$.

$\mathbb{C} = \{a + ib | a, b \in \mathbb{R}\}$ is the **set of all complex numbers** ($i = \sqrt{-1}$).

A real number a can be regarded as a complex number $a + 0i$, whose imaginary part is 0.

Two complex numbers $a_1 + b_1 i$ and $a_2 + b_2 i$ are **equal** iff both their real and imaginary parts are equal, i.e.
$a_1 + ib_1 = a_2 + ib_2 \iff a_1 = a_2$ and $b_1 = b_2$.

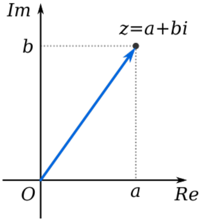
Complex plane

Argand Diagrams

There are two geometric representations of the complex number $z = x + iy$:

- as the point in the xy-plane;
- as the vector from the origin to the above point.

In each representation, the x-axis is called the **real axis** and the y-axis is the **imaginary axis**. Both representations are **Argand diagrams** for $x + iy$.



A complex number $z = a + ib$ can be identified with an ordered pair $(Re(z), Im(z))$ of real numbers.
The set of complex numbers can be treated as the set of all ordered pairs of real numbers (a, b) with the following operations:

- $(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2)$
- $(a_1, b_1) \cdot (a_2, b_2) = (a_1 a_2 - b_1 b_2, a_1 b_2 + a_2 b_1)$

Addition and multiplication

For any two complex numbers $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$, their **sum** is defined as

- $z_1 + z_2 = a_1 + ib_1 + a_2 + ib_2 = (a_1 + a_2) + i(b_1 + b_2)$,

and their **product** is defined as

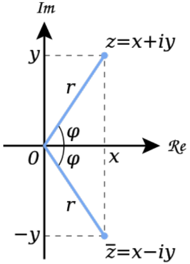
- $z_1 \cdot z_2 = (a_1 + ib_1) \cdot (a_2 + ib_2) = a_1 a_2 - b_1 b_2 + i(a_1 b_2 + a_2 b_1)$.

Properties:

- commutativity for $+, \cdot$;
- associativity for $+, \cdot$;
- identity element for $+, \cdot$;
- opposite element for $+, \cdot$;
- distributivity;
- multiplication by a real number: $\forall c \in \mathbb{R}$, $cz = c(a + ib) = ca + icb$;
- subtraction can be performed as $z_1 - z_2 = z_1 + (-z_2) = (a_1 + a_2) - i(b_1 + b_2)$.

Complex conjugate and absolute value

For any complex number $z = x + yi$ its **complex conjugate**, $\bar{z}$ (or z^*) is defined as $\bar{z} = x - yi$, and its **absolute value** (or modulus), $|z|$, is defined as $|z| = \sqrt{\mathrm{Re}(z)^2 + \mathrm{Im}(z)^2} = \sqrt{x^2 + y^2}$.



Geometrically, z is the "reflection" of z about the real axis. The absolute value of a complex number z is the distance r of z from the origin.

$\forall z, z_1, z_2 \in \mathbb{C}$,

- $z \in \mathbb{R} \iff \bar{z} = z$;
- $\overline{\overline{z}} = z$;
- $\mathrm{Re}(\overline{z}) = \mathrm{Re}(z)$, $|\overline{z}| = |z|$;
- $\mathrm{Im}(\overline{z}) = -\,\mathrm{Im}(z)$;
- $z \cdot \overline{z} = x^2 + y^2 = |z|^2 = |\overline{z}|^2$;
- $\mathrm{Re}(z) = \dfrac{z + \overline{z}}{2}$, $\mathrm{Im}(z) = \dfrac{z - \overline{z}}{2i}$;
- $\overline{z_1 \pm z_2} = \overline{z_1} \pm \overline{z_2}$;
- $\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$;
- for $z_2 \neq 0$, $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\overline{z_1}}{\overline{z_2}}$, that is for $z \neq 0$ $\overline{\dfrac{1}{z}} = \dfrac{1}{\overline{z}}$;
- *Corollary from the triangle inequality:* $||z_1| - |z_2|| \leq |z_1 + z_2|$.

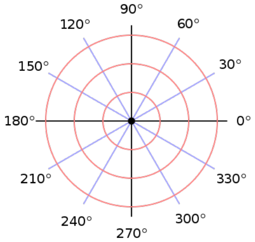
- **opposite element** (reciprocal): for $z \in \mathbb{C} \setminus \{0\}$,

$$\frac{1}{z} = \frac{x}{x^2 + y^2} - \frac{y}{x^2 + y^2} i,$$

- **division**: for $z_1 \in \mathbb{C}$, $z_2 \in \mathbb{C} \setminus \{0\}$,

$$\frac{z_1}{z_2} = \frac{(x_1 x_2 + y_1 y_2) + (y_1 x_2 - x_1 y_2) i}{x_2^2 + y_2^2}.$$

The **polar coordinate system** is a two-dimensional coordinate system in which each point on a plane is determined by a distance from a reference point and an angle from a reference direction.



(from wikipedia.org)

Equations relating polar and Cartesian coordinates:

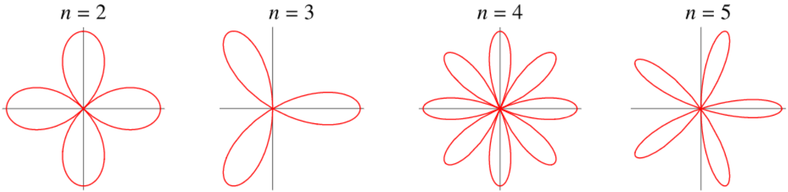
$$x = r\cos\theta,\ y = r\sin\theta,\ r^2 = x^2 + y^2,\ \tan\theta = \frac{y}{x}.$$

Some plane curves expressed in terms of both polar coordinate and Cartesian coordinate equations:

Polar equation	Cartesian equivalent
$r\cos\theta = 2$	$x = 2$
$r^2\cos\theta\sin\theta = 4$	$xy = 4$
$r^2\cos^2\theta - r^2\sin^2\theta = 1$	$x^2 - y^2 = 1$
$r = 1 + 2r\cos\theta$	$y^2 - 3x^2 - 4x - 1 = 0$
$r = 1 - \cos\theta$	$x^4 + y^4 + 2x^2y^2 + 2x^3 + 2xy^2 - y^2 = 0$

(from Thomas's Calculus.)

Some curves are more simply expressed with polar coordinates; others are not.

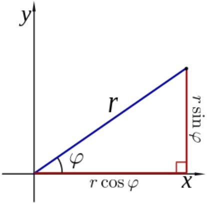


Rose curve $r = a\cos(n\theta)$ (from wolfram.com)

Polar form of complex numbers

Expressing $x = r\cos\varphi$, $y = r\sin\varphi$, we get the **polar form** of any complex number $z = x + iy$:

$$z = r\cos\varphi + ir\sin\varphi = r(\cos\varphi + i\sin\varphi).$$



(from wikipedia.org)

- the **absolute value** of z is $|z| = r = \sqrt{x^2 + y^2}$,
- the **argument** of z is $\varphi = \arg(x + yi) = \begin{cases} \arctan\frac{y}{x} & \text{if } x > 0 \\ \arctan\frac{y}{x} + \pi & \text{if } x < 0, y > 0 \\ \arctan\frac{y}{x} - \pi & \text{if } x < 0, y < 0. \end{cases}$
- $\mathrm{Re} z = r\cos\varphi$, $\mathrm{Im} z = r\sin\varphi$, $\bar{z} = r(\cos\varphi - i\sin\varphi)$.

Euler's formula

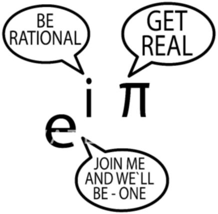
For any real number φ,

$$e^{i\varphi} = \cos\varphi + i\sin\varphi,$$

When $\varphi = \pi$, Euler's formula may be rewritten as

$$e^{i\pi} + 1 = 0,$$

which is known as **Euler's identity**.



Multiplication and division in polar form

For any $z_1 = r_1(\cos\varphi_1 + i\sin\varphi_1)$, $z_2 = r_2(\cos\varphi_1 + i\sin\varphi_2)$,

- $z_1 z_2 = r_1 r_2(\cos(\varphi_1+\varphi_2) + i\sin(\varphi_1+\varphi_2))$;
- $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}\left(\cos(\varphi_1-\varphi_2) + i\sin(\varphi_1-\varphi_2)\right)$ (provided that $z_2 \neq 0$);
- $\dfrac{1}{z_2} = \dfrac{1}{|z_2|}\left(\cos\varphi_2 - i\sin\varphi_2\right)$.

From Euler's formula,

- $z_1 z_2 = r_1 r_2 e^{i(\varphi_1+\varphi_2)}$,
- $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2} e^{i(\varphi_1-\varphi_2)}$.

Integer exponents

$\forall n \in \mathbb{N}, z \in \mathbb{C},$

- **Binomial expansion:**

$$z^n = (x+iy)^n = \sum_{k=0}^{n} C_n^k x^{n-k} (iy)^k,$$

where $C_n^k = \begin{pmatrix} n \\ k \end{pmatrix} = \dfrac{n!}{k!(n-k)!}.$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

(from wikipedia.org)

- **De Moivre's formula:**

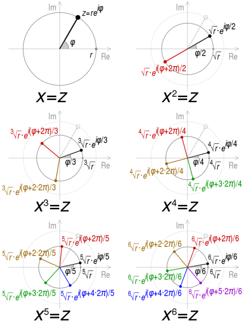
$$z^n = (r(\cos\varphi + i\sin\varphi))^n = r^n(\cos(n\varphi) + i\sin(n\varphi)).$$

$\forall n \in \mathbb{N}, z \in \mathbb{C},$

- The **n-th roots of a complex number:**

$$z^{1/n} = \sqrt[n]{r}\left(\cos\left(\frac{\varphi + 2k\pi}{n}\right) + i \sin\left(\frac{\varphi + 2k\pi}{n}\right)\right),$$

for $0 \leq k \leq n-1$. (Here $\sqrt[n]{r}$ is the usual (positive) n-th root of the positive real number r.)



(from wikipedia.org)

Rings and fields

A set S with two operations $+$ and $\cdot$ is called to be **a ring**, if these operations are well-defined (associates with each ordered pair of elements of S a uniquely determined element of S) and satisfy the following properties (**ring axioms**):

- Associativity of addition: $a + (b + c) = (a + b) + c$.
- Associativity of multiplication: $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.
- Commutativity of addition: $a + b = b + a$.
- Additive identity: there exists $0 \in S$ such that $a + 0 = 0 + a = a$.
- Multiplicative identity: there exists $1 \in S$ such that $a \cdot 1 = 1 \cdot a = a$.
- Additive inverses: for every $a \in S$, there exists an element in S, denoted $-a$, called the additive inverse of a, such that $a + (-a) = 0$.
- Distributivity of multiplication over addition: $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$.

If, additionally, the commutativity of multiplication holds ($a \cdot b = b \cdot a$) then S is called to be a (**commutative ring**).

Rings and fields

The set $(S, +, \cdot)$ is called to be a **field**, if it is a commutative ring with multiplicative inverses, i.e.,

- for every $a \neq 0$ in S, there exists an element in S, denoted by a^{-1} or $1/a$, called the multiplicative inverse of a, such that $a \cdot a^{-1} = a^{-1} \cdot a = 1$.

Examples:

- $\mathbb{Z}, +, \cdot$ is a ring;
- $\mathbb{Q}, +, \cdot$ is a field;
- $\mathbb{R}, +, \cdot$ is a field;
- $\mathbb{C}, +, \cdot$ is a field;
- The sets $\mathrm{Pol}\mathbb{R}$, $\mathrm{Pol}\mathbb{C}$, $\mathrm{Pol}\mathbb{Z}$ (polynomials with coefficients in $\mathbb{R}, \mathbb{C}, \mathbb{Z}$) are rings.

Fundamental theorem of algebra

Fundamental theorem of algebra (d'Alembert–Gauss theorem)

Every single-variable polynomial with complex coefficients

$$p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, \quad a_0, a_1, \ldots, a_{n_1} \in \mathbb{C}, n \geq 1,$$

has at least one complex root.

Let F be a field and $p(x)$ be a polynomial in one variable with coefficients in F. An element $x_0 \in F$ is a **root of multiplicity** k of $p(x)$ if there is a polynomial $s(x)$ such that $s(x_0) \neq 0$ and $p(x) = (x - x_0)^k s(x)$. If $k = 1$, then x_0 is called a **simple root**. If $k \geqslant 2$, then x_0 is called a **multiple root**.

Fundamental theorem of algebra (equivalent formulation)

Every non-zero, single-variable, degree n polynomial with complex coefficients has, counted with multiplicity, exactly n complex roots.

Properties of polynomials

- If x_0 is a root of the polynomial

  $$p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, \quad a_0, a_1, \ldots, a_{n_1} \in \mathbb{C}, n \geq 1, ,$$

  then there exists an $n-1$ degree polynomial $q(x)$ such that $p(x) = (x - x_0)q(x)$.
- Every polynomial

  $$p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, \quad a_0, a_1, \ldots, a_{n_1} \in \mathbb{C}, n \geq 1,$$

  can be represented as

  $$p(x) = (x - x_1)(x - x_2) \cdot \cdots \cdot (x - x_n).$$

  The factors $(x - x_j)$ are called linear factors. The numbers of factors $x - x_j$ equals to the multiplicity of the root x_j.