28-05-2020 AM5650 - END-SEM EXAM S. TARUN PRASAD

ME17B114

"The work being submitted is my own work. I have not sought the help of any person in doing this work."

5. (a) $\dot{x} = y + y^2 x$; $\dot{y} = -x + \mu y + x^2 y$

at $(x, y) = (0, 0)$: $\dot{x} = 0$ & $\dot{y} = 0$

$\Rightarrow (0,0) \rightarrow$ fixed point

Linearising at $(0, 0)$:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & \mu \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Characteristic equation $(-\lambda)(\mu - \lambda) + 1 = 0$

$\Rightarrow \lambda^2 - \mu\lambda + 1 = 0$

$\Rightarrow \lambda = \dfrac{\mu \pm \sqrt{\mu^2 - 4}}{2}$ $\Rightarrow \lambda$ complex $\forall \mu \in (-2, 2)$

when $\mu = 0$ λ's (eigenvalues are purely imaginary)

The origin is also asymptotically stable at $\mu = 0$.

$$\frac{\partial}{\partial \mu}(\lambda)_{\mu = 0} = \frac{1}{2} > 0$$

Thus there is a Hopf bifurcation at $\mu = 0$

For finding whether it is a subcritical or supercritical Hopf bifurcation we need to find a linear transform of the form: $z = bx + i(ay)$

This is because the convential $z = x + iy$ transform yields $\bar{z}$ as a first order term in $\dot{z}$ which doesn't yield a significant outcome while performing near identity transform. Thus the transform $z = bx + iy$ shouldn't yield any $\dot{z}$ first order term of the form $\bar{z}$.

$$z = bx + aiy \Rightarrow \dot{z} = b\dot{x} + ai\dot{y}$$

Now we have, $\dfrac{z+\bar{z}}{2b} = x$ & $\dfrac{z-\bar{z}}{2ai} = y$.

$$\Rightarrow \dot{z} = b(y + y^2x) + ia(-x + \mu y + x^2y)$$

$$= b\left(\frac{z-\bar{z}}{2ia}\right) + b\left(\frac{z-\bar{z}}{2ia}\right)^2\left(\frac{z+\bar{z}}{2b}\right)$$

$$+ i\left(\frac{-a(z+\bar{z})}{2b} + \frac{a\mu(z-\bar{z})}{2ai} + a\left(\frac{z+\bar{z}}{2b}\right)^2\left(\frac{z-\bar{z}}{2ai}\right)\right)$$

Equating $\bar{z}$ coefficients $= 0$.

$$\frac{-b}{2ia} - \frac{ai}{2b} - \frac{\mu}{2} = 0$$

$$\Rightarrow a^2 - i\mu ab - b^2 = 0, \quad \forall a, b \neq 0$$

$$a = \frac{i\mu b \pm \sqrt{-\mu^2 b^2 + 4b^2}}{2}$$

$$\Rightarrow a = b\left(\frac{\mu i \pm \sqrt{4-\mu^2}}{2}\right) \rightarrow (1).$$

Now we have already proved that for a system: $\dot{z} = \lambda z + a_{20} z^2 + a_{11} z\bar{z} + a_{02}\bar{z}^2 + b_{21} z^2\bar{z} + b_{12} z\bar{z}^2 + b_{30} z^3 + b_{03}\bar{z}^3$ by performing near identity transform we can eliminate all terms except that of $z^2\bar{z}$ for purely imaginary λ. This has been shown in Assignment 6 Problem 2.

Trying to find values (a, b) for which 'z' cofficient is purely imaginary:

z coefficient: We have to get

$$\frac{b}{2ia} - \frac{ai}{2b} + \frac{\mu}{2} = 0 \rightarrow \text{purely imaginary}$$

$$(1) \Rightarrow \frac{1}{2i}\left(\frac{2}{\mu i \pm \sqrt{4-\mu^2}}\right) - \frac{i}{2}\left(\frac{\mu i \pm \sqrt{4-\mu^2}}{2}\right) + \frac{\mu}{2}.$$

$$= \left(\frac{1}{-\mu \pm i\sqrt{4-\mu^2}}\right) - \left(\frac{-\mu \pm i\sqrt{4-\mu^2}}{4}\right) + \frac{\mu}{2} = 0$$

$$= \left(\frac{-\mu \mp \left(\sqrt{4-\mu^2}\right) i}{\mu^2 + (4-\mu^2)}\right) + \frac{3\mu}{4} \mp \frac{i\sqrt{4-\mu^2}}{4}$$

$$= \frac{\mu}{2} \mp \frac{\sqrt{4-\mu^2}\, i}{2} \rightarrow \text{purely imaginary for } \mu \to 0$$

3^{rd} order terms:

$$-\frac{b}{8a^2 b}\left(z^2 - 2z\bar{z} + \bar{z}^2\right)(z+\bar{z}) + \frac{ai}{8b^2 ai}\left(z^2 + 2z\bar{z} + \bar{z}^2\right)(z-\bar{z})$$

$$= \frac{1}{8}\Big(-\left(z^3 - 2z^2\bar{z} + z\bar{z}^2 + z^2\bar{z} - 2z\bar{z}^2 + \bar{z}^3\right)/a^2$$
$$+ \left(z^3 + 2z^2\bar{z} + z\bar{z}^2 - z^2\bar{z} - 2z\bar{z}^2 - \bar{z}^3\right)/b^2\Big).$$

Coefficient of $z^2\bar{z} = b_{21}$

$$= \frac{1}{8}\left(\frac{1}{a^2} + \frac{1}{b^2}\right).$$

For $a = 1$, we have $b^{-1} = \frac{\mu i \pm \sqrt{4-\mu^2}}{2}$

$$\Rightarrow z^2\bar{z} \text{ coefficient} = \frac{1}{8}\left(1 + \frac{-\mu^2 + 4 - \mu^2 \pm i2\mu\sqrt{4-\mu^2}}{4}\right)$$

$$= \frac{1}{32}\left(8 - 2\mu^2 \pm i2\mu\sqrt{4-\mu^2}\right)$$

At $\mu \to 0$ coefficient of $z^2\bar{z} = \frac{1}{4}$

$\Rightarrow$ For the near identity tranform:

$z = \tilde{z} + \alpha \tilde{z}^p \bar{\tilde{z}}^q$

We get, from assignment 6 problem 2

$$\frac{d\tilde{z}}{dt} = \lambda \tilde{z} + b_{21} \tilde{z}^2 \bar{\tilde{z}}$$

$$= \mp i \tilde{z} + \frac{1}{4} \tilde{z}^2 \bar{\tilde{z}} \Rightarrow \text{Subcritical Hopf Bifurcation}$$

Thus we have shown that it shows subcritical Hopf bifurcation in the limit $\mu \to 0$.

(b) $\dot{x} = -y - x^3$; $\dot{y} = x + \mu y - y^3$.

at $(x, y) = (0, 0)$: $\dot{x} = 0$ & $\dot{y} = 0$

$\Rightarrow (0,0) \to$ fixed point

Linearising at $(0,0)$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & \mu \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Characteristic equation: $(-\lambda)(\mu - \lambda) + 1 = 0$

$\Rightarrow \lambda^2 - \mu\lambda + 1 = 0 \Rightarrow \lambda = \dfrac{\mu \pm \sqrt{\mu^2 - 4}}{2}$

When ~~$\lambda =$~~ $\mu=0$, λ's (eigenvalues) are purely imaginary. Also, the origin is asymptotically stable at the origin. ~~for $\mu=0$~~

$$\frac{\partial}{\partial \mu}(\lambda)\Big|_{\mu=0} = \frac{1}{2} > 0$$

Thus there is a Hopf Bifurcation at the origin. For classifying whether it is a supercritical or subcritical Hopf Bifurcation we follow the same approach of finding a linear transform of the form: $z = bx + iay$ such that $\dot{z}$ does not contain any $\bar{z}$ terms and the coefficient of z term is purely imaginary.

$$z = bx + aiy \Rightarrow \dot{z} = b\dot{x} + ai\dot{y}$$

We have, $\dfrac{z+\bar{z}}{2b} = x$ & $\dfrac{z-\bar{z}}{2ai} = y$

$$\dot{z} = b(-y - x^3) + ia(x + \mu y - y^3)$$

$$= b\left(\frac{\bar{z}-z}{2ai}\right) - b\left(\frac{z+\bar{z}}{2b}\right)^3 \quad \text{(contd.)}$$

(contd. from prev page):

$$+ ai\left(\frac{z+\bar{z}}{2b} + \frac{\mu(z-\bar{z})}{2ai} - \left(\frac{z-\bar{z}}{2ia}\right)^3\right).$$

$\Rightarrow$ Equating $\bar{z}$ coefficients $= 0$.

$$\frac{b}{2ai} + \frac{ai}{2b} - \frac{\mu}{2} = 0.$$

$$b^2 - a^2 - iab\mu = 0. \quad \forall\ a, b \neq 0.$$

$$\Rightarrow a^2 + (ib\mu)a - b^2 = 0.$$

$$\Rightarrow a = \frac{-ib\mu \pm \sqrt{-b^2\mu^2 + 4b^2}}{2}$$

$$\Rightarrow a = b\left(\frac{-\mu i \pm \sqrt{4-\mu^2}}{2}\right) \longrightarrow (1)$$

Trying to find values (a,b) if they produce a 'z' coefficient which is purely imaginary:

$$\frac{-b}{2ai} + \frac{ai}{2b} + \frac{\mu}{2} \rightarrow \text{should be purely imaginary}$$

$$(1) \Rightarrow \frac{-1}{2i}\left(\frac{2}{-\mu i \pm \sqrt{4-\mu^2}}\right) + \frac{i}{2}\left(\frac{-\mu i \pm \sqrt{4-\mu^2}}{2}\right) + \frac{\mu}{2}.$$

$$= -\left(\frac{1}{\mu \pm i\sqrt{4-\mu^2}}\right) + \left(\frac{\mu \pm i\sqrt{4-\mu^2}}{4}\right) + \frac{\mu}{2}.$$

$$-\left(\frac{\mu \mp i\sqrt{4-\mu^2}}{\mu^2+4-\mu^2}\right) + \frac{3\mu}{4} \pm \frac{i\sqrt{4-\mu^2}}{4}$$

$$= \frac{2\mu}{4} \pm \frac{2i\sqrt{4-\mu^2}}{4} = \frac{\mu}{2} \pm \frac{i\sqrt{4-\mu^2}}{2}$$

In the limit $\mu \to 0$ we get z coefficient

$= \pm i$

3^{rd} order terms:

$$\frac{-b}{8b^3}(z+\bar{z})^3 + \frac{a}{8a^3}(z-\bar{z})^3$$

$$= \frac{1}{8}\left(-\left(\frac{z^3+\bar{z}^3+3z^2\bar{z}+3z\bar{z}^2}{b^2}\right) + \frac{z^3-\bar{z}^3-3z^2\bar{z}+3z\bar{z}^2}{a^2}\right)$$

Coefficient of $z^2\bar{z} = b_{21}$

$$b_{21} = \frac{1}{8}\left(-\frac{3}{b^2} - \frac{3}{a^2}\right)$$

For $a=1$, we have $\frac{1}{b} = \frac{-\mu i \pm \sqrt{4-\mu^2}}{2}$

$$\Rightarrow b_{21} = -\frac{3}{8}\left(\frac{-\mu^2+4-\mu^2 \mp i2\mu\sqrt{4-\mu^2}}{4} + 1\right)$$

$$= -\frac{3}{32}\left(8-2\mu^2 \mp i2\mu\sqrt{4-\mu^2}\right)$$

At $\mu \to 0$, coefficient of $z^2\bar{z} = b_{21} = \frac{-3}{4}$

$\Rightarrow$ For the near-identity transform:

$z = \tilde{z} + \alpha \tilde{z}^p \bar{\tilde{z}}^q$

We get from assignment 6 problem 2,

$$\frac{d\tilde{z}}{dt} = \lambda \tilde{z} + b_{21} \tilde{z}^2 \bar{\tilde{z}} = \pm i \tilde{z} - \frac{3}{4} \tilde{z}^2 \bar{\tilde{z}}$$

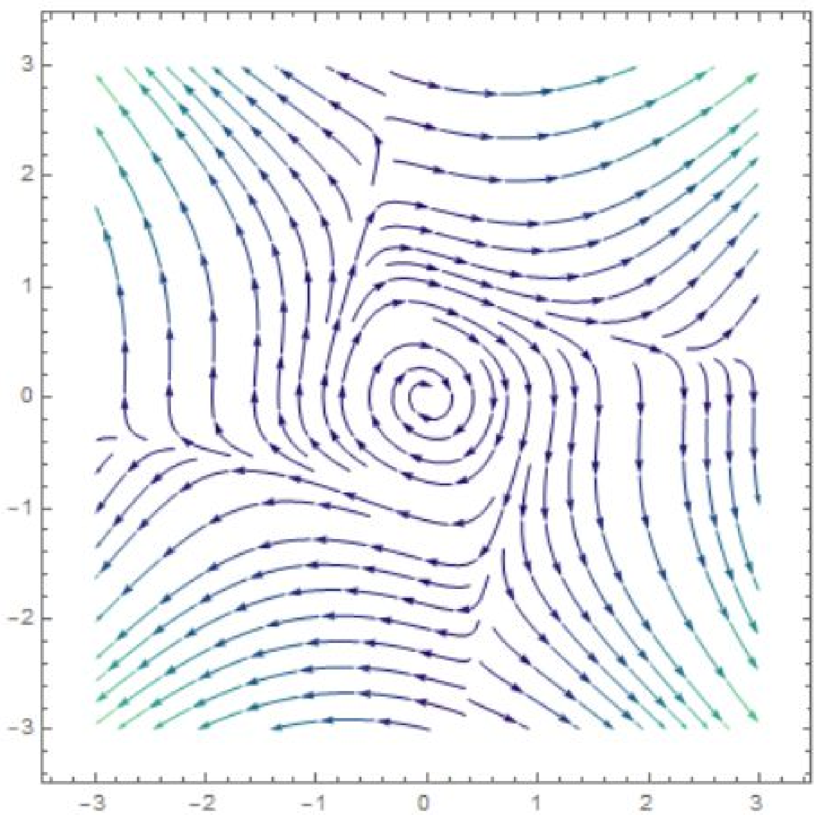
This is the normal form for supercritical hopf bifurcation. Thus we have shown that the system shows supercritical Hopf bifurcation when $\mu \to 0$.

The external source used is:
http://minitorn.tlu.ee/~jaagup/uk/dynsys/ds2/limit/Hopf/Hopf.html

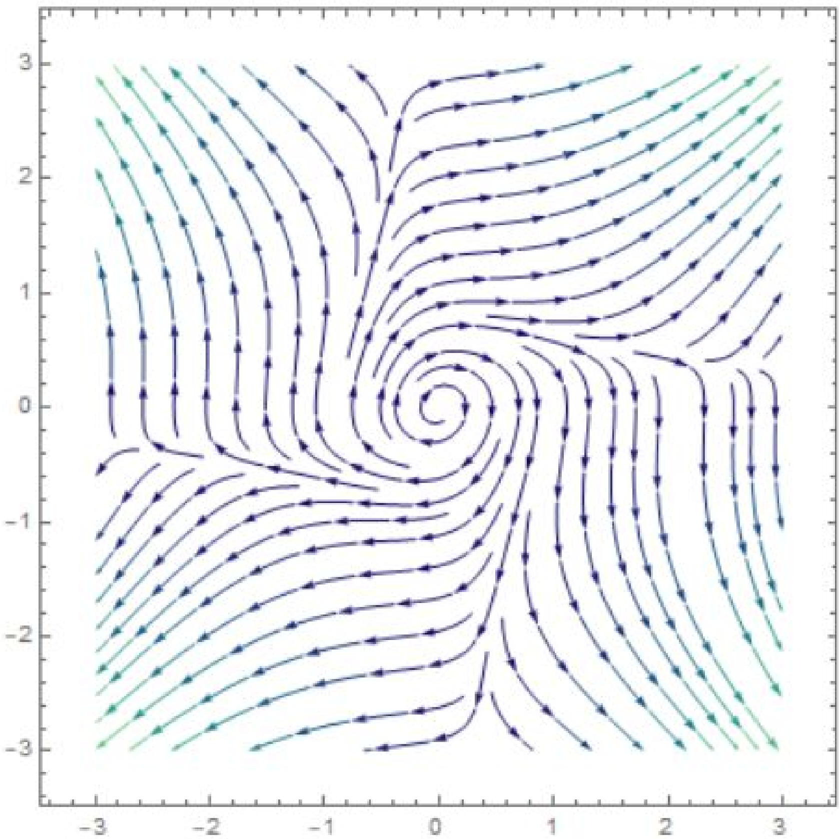
Phase Portraits:

(a)

```
(* Ensem 5.1 for μ=-.25 *)
StreamPlot[{y + x*y^2, -x - 0.25*y + y*x^2}, {x, -3, 3}, {y, -3, 3},
 StreamColorFunction → "BlueGreenYellow"]
```

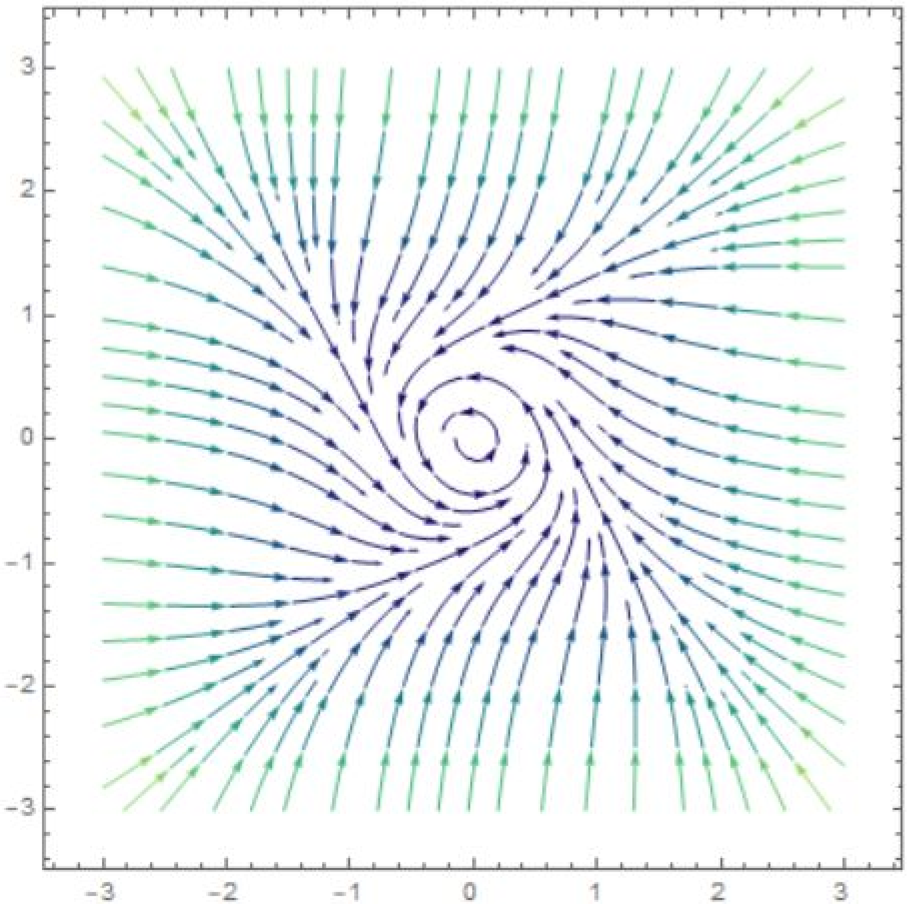


```
(* Ensem 5.1 for μ=+.25 *)
StreamPlot[{y + x*y^2, -x + 0.25*y + y*x^2}, {x, -3, 3}, {y, -3, 3},
 StreamColorFunction → "BlueGreenYellow"]
```



(b)

```
(* Ensem 5.2 for μ=+.25 *)
StreamPlot[{-y - x^3, x + 0.25 * y - y^3}, {x, -3, 3}, {y, -3, 3},
 StreamColorFunction → "BlueGreenYellow"]
```



```
(* Ensem 5.2 for μ=-.25 *)
StreamPlot[{-y - x^3, x - 0.25*y - y^3}, {x, -3, 3}, {y, -3, 3},
 StreamColorFunction → "BlueGreenYellow"]
```