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ORIGINAL PAPER

Nonlinear harmonic vibration analysis of a plate-cavity system

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Abstract Nonlinear harmonic oscillation of a plate-cavity system is analytically studied in this paper. Von-Karman theory is used to model a rectangular plate backed by an air cavity. Coupled nonlinear differential equations of system are analytically derived using Galerkin's approach. The Multiple Scales Method (MSM) is then employed to solve the corresponding nonlinear equations. Primary, secondary, and combinational resonance conditions are taken into account and the corresponding closed-form frequency-amplitude relationships are derived. A parametric study is carried out and effects of different parameters on the frequency responses are investigated.

Keywords Plate-cavity · Resonance · Nonlinear vibration · Multiple scales method

1 Introduction

Vibro-acoustic analysis of plate-cavity systems has got many attentions in recent years due to varieties of engineering applications. Numerous cases of interaction between a plate and an air cavity can be addressed in automotive, marine, aircraft, and railway engineering. Vehicular and railway coach cabins, aircraft fuselages/skin panels, acoustical instruments, and different aerospace cavities are a few applications. Many researchers have already studied the vibro-acoustic behavior of such a coupled system. Different numerical techniques, different models and cavity types have been investigated so far. Frendi and Robinson [1] studied influence of acoustic coupling on the plate vibrations in case of random and harmonic excitations. Ding and Chen [2] established a symmetrical finite-element model for structural-acoustic analysis of a thin-walled cavity under the excitation of internal acoustic source and simultaneous external structural loading. A coupled vibro-acoustic model was developed by Li and Cheng [3] to examine the structural\acoustic coupling of a flexible panel backed by a cavity having a tilted wall. Lee [4] studied nonlinear natural frequencies of a rectangular enclosure with one flexible wall. He employed FEM to examine the effect of the air-cavity depth on the nonlinear natural frequencies of the system. Nonlinear vibration of a composite plate backed by a three-dimensional cavity was investigated by Lee et al. [5]. A mixed finite-element and classical formulation was employed and the frequency ratios of a coupled vibro-acoustic system were examined. The influence of the jump phenomenon on the sound absorption of a nonlinear panel was studied by Lee et al. [6]. Harmonic Balance and Homotopy Perturbation methods were employed to examine the nonlinear frequencies of a rectangular

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tube with a flexible end [7]. Lee et al. [8] mathematically modeled a curved panel absorber backed by a cavity. Eigenvalue problem of a coupled rectangular cavity was numerically obtained and experimentally validated by Tanaka et al. [9]. An analytical method applicable for vibro-acoustic modal and forced response characteristics of a three-dimensional stiffened structure was developed by Park et al. [10]. Different methods have been so far employed to investigate the problem of a plate-cavity system. Green's function method [11], mixed finite-element and classical formulation [12], impedance-mobility approach [13], probabilistic approach [14] and variational iteration method (VIM) [15] have been employed so far by different researchers. VIM has also been used in other similar nonlinear multi degree of freedom systems such as a beam on an elastic foundation [16]. Passive and active control of an enclosure composed of flexible and rigid walls has also been recently addressed in the literature [17–20].

Nonlinear harmonic vibration of a plate-cavity system is analytically modeled and solved for the first time in present study. Large amplitude oscillation of the plate is formulated by the Von-Karman theory. Frequency responses are analytically derived for such a coupled system by use of the Multiple Scales method. Main harmonics as well as the superharmonics and subharmonic are derived in primary and secondary resonance conditions and also in the case of combinational resonance circumstance. A parametric study is then carried out and effects of different parameters on the frequency responses are evaluated. Methodology, formulation, and closed-form results can be employed by prospect researchers in extending the idea for large amplitude vibration of an air-cavity system.

2 Mathematical modeling

It is assumed that the enclosure of depth c, has a flexible wall at its bottom and five rigid walls at the other boundaries as schematically illustrated in Fig. 1. The oscillating plate is assumed to have thickness of h, length of a and width of b and to be simply supported at its four edges. An external acoustic excitation $P_E(x,y,t)$ is applied to the flexible plate. According to Von-Karman theory, plate deflection is governed by

$$D\nabla^4 W(x,y,t) + \rho\frac{\partial^2 W(x,y,t)}{\partial t^2}$$

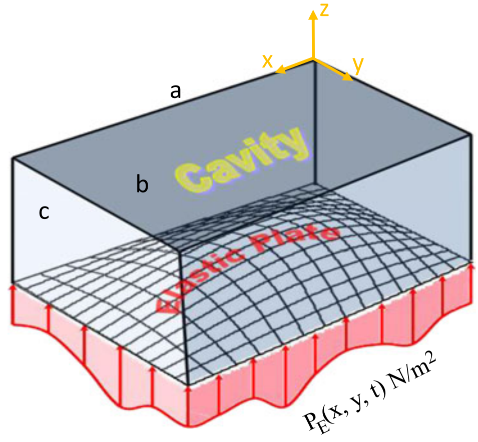

Fig. 1 Schematic configuration of a plate and an air cavity

$$= P_E(x,y,t) - P_i(x,y,t) + h\left(\frac{\partial^2\varphi}{\partial y^2}\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2\varphi}{\partial x^2}\frac{\partial^2 W}{\partial y^2} - 2\frac{\partial^2\varphi}{\partial x\partial y}\frac{\partial^2 W}{\partial x\partial y}\right) \tag{1}$$

$$\nabla^4\varphi = E\times\left[\left(\frac{\partial^2 W}{\partial x\partial y}\right)^2 - \frac{\partial^2 W}{\partial x^2}\frac{\partial^2 W}{\partial y^2}\right] \tag{2}$$

where D, ρ and E are bending stiffness, density and Young's modulus of elasticity, respectively. φ and $W(x,y,t)$ are the Airy stress function and deflection of the flexible plate. $P_i(x,y,t)$ denotes the acoustic pressure of the air cavity at $z=-c$. The acoustic pressure inside the enclosure, $P(x,y,z,t)$, is governed by the following homogeneous wave equation:

$$\nabla^2 P(x,y,z,t) - \frac{1}{c_a^2}\frac{\partial^2 P(x,y,z,t)}{\partial t^2} = 0 \tag{3}$$

where c_a is the sound speed inside the air cavity. The enclosure has a flexible wall at $z=-c$ and five rigid walls at the other boundaries. Therefore the boundary conditions for the enclosure can be presented by

$$\begin{aligned} &\left.\frac{\partial P}{\partial x}\right|_{x=0,a} = 0, \qquad \left.\frac{\partial P}{\partial y}\right|_{y=0,b} = 0 \\ &\left.\frac{\partial P}{\partial z}\right|_{z=0} = 0, \\ &\left.\frac{\partial P}{\partial z}\right|_{z=-c} = -\rho_{\mathrm{air}}\frac{\partial^2 W(x,y,t)}{\partial t^2} \end{aligned} \tag{4}$$

where ρ_{air} is the air density. Using eigenfunction expansion (Galerkin method), one can assume the plate displacement and stress function are:

$$W(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}W_{mn}(t)\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b} \tag{5}$$

$$\varphi(x,y,t)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\varphi_{mn}(t)\cos\frac{n\pi x}{a}\cos\frac{m\pi y}{b} \tag{6}$$

and accordingly by use of the separation of variables in Eq. (3) one could arrive at

$$P(x,y,z,t)=\sum_{r=0}\sum_{s=0}\left(A_{rs}\cos\frac{r\pi x}{a}\times\cos\frac{s\pi y}{b}\times\cosh\lambda_{rs}z\times T_{rs}(t)\right) \tag{7}$$

Implementing the boundary condition at $z=-c$ results in

$$\begin{aligned}\left.\frac{\partial P}{\partial z}\right|_{z=-c}&=-\rho_{\text{air}}\frac{\partial^2 W(x,y,t)}{\partial t^2}\\ &\Rightarrow\sum_{r=0}\sum_{s=0}\left[\cos\frac{r\pi x}{a}\cos\frac{s\pi y}{b}\times\lambda_{rs}A_{rs}\sinh\lambda_{rs}c\times T_{rs}(t)\right]\\ &=\rho_{\text{air}}\sum_{m=1}\sum_{n=1}\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}\ddot{W}_{mn}(t)\end{aligned} \tag{8}$$

After applying a number of mathematical simplifications one may reach to

$$\begin{aligned}&P(x,y,z,t)\\ &=\sum_{r=0}\sum_{s=0}\left[\frac{\rho_{\text{air}}[\alpha_{rs}^{11}\ddot{W}_{11}(t)+\alpha_{rs}^{12}\ddot{W}_{12}(t)+\alpha_{rs}^{21}\ddot{W}_{21}(t)]}{\lambda_{rs}\sinh\lambda_{rs}c}\times\cos\frac{r\pi x}{a}\times\cos\frac{s\pi y}{b}\times\cosh\lambda_{rs}z\right]\end{aligned} \tag{9}$$

In the other side, substituting Eqs. (5) and (6) into (1) and (2) and multiplying the consequent equations by $\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}$ and $\cos\frac{n\pi x}{a}\cos\frac{m\pi y}{b}$, respectively, and then integrating them over the plate area yield

$$\begin{aligned}&M_{mn}\ddot{W}_{mn}(t)+K_{mn}W_{mn}(t)\\ &=\int_0^b\int_0^a P_E(x,y,t)\times\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}\,dx\,dy\\ &-\sum_{r=0}\sum_{s=0}\left[\frac{\rho_{\text{air}}\coth\lambda_{rs}c}{\lambda_{rs}}\times\big(\alpha_{rs}^{11}\ddot{W}_{11}(t)+\alpha_{rs}^{12}\ddot{W}_{12}(t)+\alpha_{rs}^{21}\ddot{W}_{21}(t)\big)\times\beta_{rs}^{mn}\right]\\ &+(\textit{Nonlinear Terms})\end{aligned} \tag{10}$$

in which

$$\beta_{rs}^{mn}=\int_0^b\int_0^a\cos\frac{r\pi x}{a}\cos\frac{s\pi y}{b}\times\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}\,dx\,dy \tag{11}$$

If one takes three first mode shapes into consideration, consequent coupled nonlinear equations of the plate-cavity system are found to be:

$$\begin{cases}\mu_1^{11}\ddot{W}_{11}(t)+\mu_2^{11}W_{11}(t)+\varrho_{11}\dot{W}_{11}(t)\\ \quad+\mu_3^{11}W_{11}(t)^3+\mu_4^{11}W_{11}(t)W_{12}(t)^2\\ \quad+\mu_5^{11}W_{11}(t)W_{21}(t)^2=P_{11}(t)\\ \mu_1^{12}\ddot{W}_{12}(t)+\mu_2^{12}W_{12}(t)+\varrho_{12}\dot{W}_{12}(t)\\ \quad+\mu_3^{12}W_{12}(t)^3+\mu_4^{12}W_{12}(t)W_{11}(t)^2\\ \quad+\mu_5^{12}W_{12}(t)W_{21}(t)^2=P_{12}(t)\\ \mu_1^{21}\ddot{W}_{21}(t)+\mu_2^{21}W_{21}(t)+\varrho_{21}\dot{W}_{21}(t)\\ \quad+\mu_3^{21}W_{21}(t)^3+\mu_4^{21}W_{21}(t)W_{11}(t)^2\\ \quad+\mu_5^{21}W_{21}(t)W_{12}(t)^2=P_{21}(t)\end{cases} \tag{12}$$

in which, $\mu_i^{mn}=(\mu_i^{mn})_{Structural}+(\mu_i^{mn})_{Acoustic}$ and μ_i^{mn} are defined in Table 1. The present study examines the nonlinear forced vibration of a general plate-cavity system subjected to an arbitrary transverse excitation. Coupled symmetric and anti-symmetric modes are taken into account, according to this fact that the coupling between the modes cannot be ignored when the system is nonlinear. This means that the first unsymmetrical modes can act as two parametrically excitation terms in the symmetrical mode equation. Another objective here in this paper is to investigate how symmetric and unsymmetrical modes can enhance or cancel each other in a coupled nonlinear system. Effects of higher order modes on this type of interaction between the symmetrical and unsymmetrical modes are anticipated to be similar.

Table 1 Definition of μ_i^{mn} in Eq. (12)

Parameter	Definition
$(\mu_1^{11})_{\text{Acoustic}}$	$\frac{16\rho_{\text{air}}}{\pi^2}\int_0^b\int_0^a[\frac{\coth\lambda_{0,0}c}{\lambda_{0,0}}-\frac{\cos\frac{2\pi y}{b}\coth\lambda_{0,2}c}{3\lambda_{0,2}}-\frac{\cos\frac{2\pi x}{a}\coth\lambda_{2,0}c}{3\lambda_{2,0}}-\frac{\cos\frac{2\pi x}{a}\cos\frac{2\pi y}{b}\coth\lambda_{2,2}c}{9\lambda_{2,2}}]\times\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}\,dx\,dy$
$(\mu_1^{12})_{\text{Acoustic}}$	$\frac{32\rho_{\text{air}}}{3\pi^2}\int_0^b\int_0^a[\frac{\cos\frac{\pi x}{a}\coth\lambda_{1,0}c}{\lambda_{1,0}}-\frac{\cos\frac{\pi x}{a}\cos\frac{2\pi y}{b}\coth\lambda_{1,2}c}{3\lambda_{1,2}}]\times\sin\frac{2\pi x}{a}\sin\frac{\pi y}{b}\,dx\,dy$
$(\mu_1^{21})_{\text{Acoustic}}$	$\frac{32\rho_{\text{air}}}{3\pi^2}\int_0^b\int_0^a[\frac{\cos\frac{\pi y}{b}\coth\lambda_{0,1}c}{\lambda_{0,1}}-\frac{\cos\frac{2\pi x}{a}\cos\frac{\pi y}{b}\coth\lambda_{2,1}c}{3\lambda_{2,1}}]\times\sin\frac{\pi x}{a}\sin\frac{2\pi y}{b}\,dx\,dy$
$\lambda_{r,s}$	$\sqrt{(\frac{r\pi}{a})^2+(\frac{s\pi}{b})^2-(\frac{\omega_{r,s}}{c_a})^2}$
$(\mu_1^{mn})_{\text{Plate}}, (\mu_2^{mn})_{\text{Plate}}$	$\frac{1}{4}\rho ab,\ \frac{\pi^4}{4}\times\frac{D((m\times a)^2+(n\times b)^2)^2}{a^3b^3}$
$(\mu_3^{11})_{\text{Plate}}, (\mu_3^{12})_{\text{Plate}}$	$\frac{\pi^4Eh}{64a^3b^3}\times(a^4+b^4),\ \frac{\pi^4Eh}{4a^3}\times b$
$(\mu_3^{21})_{\text{Plate}}, (\mu_4^{11})_{\text{Plate}}$	$\frac{\pi^4Eh}{4b^3}\times a,\ \frac{\pi^4Eh}{64a^3b^3}\times[4b^4+a^4(2+\frac{81b^4}{(4a^2+b^2)^2})]$
$(\mu_4^{21})_{\text{Plate}}, (\mu_5^{11})_{\text{Plate}}$	$\frac{\pi^4Eh}{64a^3b^3}\times[2b^4+a^4(4+\frac{81b^4}{(a^2+4b^2)^2})],\ \frac{\pi^4Eh}{64a^3b^3}\times[2b^4+a^4(4+\frac{81b^4}{(a^2+4b^2)^2})]$
$(\mu_5^{12})_{\text{Plate}}=(\mu_5^{21})_{\text{Plate}}$	$\frac{\pi^4Eh}{64}\times(\frac{81ab}{(a^2+b^2)^2})$

3 Solution procedure

According to the analytical method presented in the previous section, μ_i^{mn} in the equations denotes the coupling between the flexible plate and air cavity. In case of harmonic uniform pressure, external excitation, $P_{mn}(t)$, is found to be

$$P_{mn}(t)=\int_0^b\int_0^a P_0\cos\Omega t\times\sin\frac{n\pi x}{a}\times\sin\frac{m\pi y}{b}\,dx\,dy \tag{13}$$

in which P_0 and Ω are acoustic pressure force amplitude and the excitation frequency, respectively. Equation (12) can be rearranged into the more convenient form

$$\begin{cases}\ddot{W}_{11}(t)+\omega_{11}^2W_{11}(t)+\hat{\mu}_{11}\dot{W}_{11}(t)+\Theta_3^{11}W_{11}(t)^3+\Theta_4^{11}W_{11}(t)W_{12}(t)^2+\Theta_5^{11}W_{11}(t)W_{21}(t)^2\\ \quad=F_{11}\cos\Omega t\\ \ddot{W}_{12}(t)+\omega_{12}^2W_{12}(t)+\hat{\mu}_{12}\dot{W}_{12}(t)+\Theta_3^{12}W_{12}(t)^3+\Theta_4^{12}W_{12}(t)W_{11}(t)^2+\Theta_5^{12}W_{12}(t)W_{21}(t)^2\\ \quad=F_{12}\cos\Omega t\\ \ddot{W}_{21}(t)+\omega_{21}^2W_{21}(t)+\hat{\mu}_{21}\dot{W}_{21}(t)+\Theta_3^{21}W_{21}(t)^3+\Theta_4^{21}W_{21}(t)W_{11}(t)^2+\Theta_5^{21}W_{21}(t)W_{12}(t)^2\\ \quad=F_{21}\cos\Omega t\end{cases} \tag{14}$$

in which

$$F_{mn}=\frac{1}{\mu_1^{mn}}\int_0^b\int_0^a\lambda\rho g\times\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}\,dx\,dy \tag{15}$$

This gives $F_{12}=F_{21}=0$. That means that the external excitation parts are vanished in both unsymmetrical modes due to symmetrical excitation. However, a different condition of combinational resonances may exist due to coupling between the symmetrical and unsymmetrical mode shapes. Periodic solutions are to be determined in case of combinational resonance in the next sections. In order to solve the nonlinear equations, Multiple Scales method [21–23] is employed in this section. The solution of Eq. (14) is then assumed to be

$$\begin{cases} W_{11}(t) = \varepsilon W_{11,1}(T_0, T_2, \ldots) \\ \quad + \varepsilon^3 W_{11,3}(T_0, T_2, \ldots) + \cdots \\ W_{12}(t) = \varepsilon W_{12,1}(T_0, T_2, \ldots) \\ \quad + \varepsilon^3 W_{12,3}(T_0, T_2, \ldots) + \cdots \\ W_{21}(t) = \varepsilon W_{21,1}(T_0, T_2, \ldots) \\ \quad + \varepsilon^3 W_{21,3}(T_0, T_2, \ldots) + \cdots \end{cases} \tag{16}$$

where ε is a small, dimensionless parameter. Time scales are defined as $T_n = \varepsilon^n t$. Therefore

$$\begin{aligned} \frac{d}{dt} &= D_0 + \varepsilon D_1 + \varepsilon^2 D_2 + \cdots \\ \frac{d^2}{dt^2} &= D_0^2 + 2D_0 D_1 \varepsilon + (D_1^2 + 2D_0 D_2)\varepsilon^2 + \cdots \end{aligned} \tag{17}$$

in which $D_n = \frac{d}{dT_n}$. Substituting Eqs. (16) and (17) into (14) results in

$$\begin{aligned} &\left(D_0^2 + 2D_0 D_1 \varepsilon + \left(D_1^2 + 2D_0 D_2\right)\varepsilon^2 + \cdots\right) \\ &\quad \times \left(\varepsilon W_{11,1} + \varepsilon^3 W_{11,3} + \cdots\right) \\ &\quad + \omega_{11}^2 \left(\varepsilon W_{11,1} + \varepsilon^3 W_{11,3} + \cdots\right) \\ &= -\varepsilon^2 \mu_{11} \left(D_0 + \varepsilon D_1 + \varepsilon^2 D_2 + \cdots\right) \\ &\quad \times \left(\varepsilon W_{11,1} + \varepsilon^3 W_{11,3} + \cdots\right) \\ &\quad - \Theta_3^{11} \left(\varepsilon W_{11,1} + \varepsilon^3 W_{11,3} + \cdots\right)^3 \\ &\quad - \Theta_4^{11} \left(\varepsilon W_{11,1} + \varepsilon^3 W_{11,3} + \cdots\right) \\ &\quad \times \left(\varepsilon W_{12,1} + \varepsilon^3 W_{12,3} + \cdots\right)^2 \\ &\quad - \Theta_5^{11} \left(\varepsilon W_{11,1} + \varepsilon^3 W_{11,3} + \cdots\right) \\ &\quad \times \left(\varepsilon W_{21,1} + \varepsilon^3 W_{21,3} + \ldots\right)^2 \\ &\quad + F_{11} \cos \Omega t \end{aligned} \tag{18}$$

$$\begin{aligned} &\left(D_0^2 + 2D_0 D_1 \varepsilon + \left(D_1^2 + 2D_0 D_2\right)\varepsilon^2 + \cdots\right) \\ &\quad \times \left(\varepsilon W_{12,1} + \varepsilon^3 W_{12,3} + \cdots\right) \\ &\quad + \omega_{12}^2 \left(\varepsilon W_{12,1} + \varepsilon^3 W_{12,3} + \cdots\right) \\ &= -\varepsilon^2 \mu_{12} \left(D_0 + \varepsilon D_1 + \varepsilon^2 D_2 + \cdots\right) \\ &\quad \times \left(\varepsilon W_{12,1} + \varepsilon^3 W_{12,3} + \cdots\right) \\ &\quad - \Theta_3^{12} \left(\varepsilon W_{12,1} + \varepsilon^3 W_{12,3} + \cdots\right)^3 \\ &\quad - \Theta_4^{12} \left(\varepsilon W_{12,1} + \varepsilon^3 W_{12,3} + \cdots\right) \\ &\quad \times \left(\varepsilon W_{11,1} + \varepsilon^3 W_{11,3} + \cdots\right)^2 \\ &\quad - \Theta_5^{12} \left(\varepsilon W_{12,1} + \varepsilon^3 W_{12,3} + \cdots\right) \\ &\quad \times \left(\varepsilon W_{21,1} + \varepsilon^3 W_{21,3} + \cdots\right)^2 \end{aligned} \tag{19}$$

$$\begin{aligned} &\left(D_0^2 + 2D_0 D_1 \varepsilon + \left(D_1^2 + 2D_0 D_2\right)\varepsilon^2 + \cdots\right) \\ &\quad \times \left(\varepsilon W_{21,1} + \varepsilon^3 W_{21,3} + \cdots\right) \\ &\quad + \omega_{21}^2 \left(\varepsilon W_{21,1} + \varepsilon^3 W_{21,3} + \cdots\right) \\ &= -\varepsilon^2 \mu_{21} \left(D_0 + \varepsilon D_1 + \varepsilon^2 D_2 + \cdots\right) \\ &\quad \times \left(\varepsilon W_{21,1} + \varepsilon^3 W_{21,3} + \cdots\right) \\ &\quad - \Theta_3^{21} \left(\varepsilon W_{21,1} + \varepsilon^3 W_{21,3} + \cdots\right)^3 \\ &\quad - \Theta_4^{21} \left(\varepsilon W_{21,1} + \varepsilon^3 W_{21,3} + \cdots\right) \\ &\quad \times \left(\varepsilon W_{11,1} + \varepsilon^3 W_{11,3} + \cdots\right)^2 \\ &\quad - \Theta_5^{21} \left(\varepsilon W_{21,1} + \varepsilon^3 W_{21,3} + \cdots\right) \\ &\quad \times \left(\varepsilon W_{12,1} + \varepsilon^3 W_{12,3} + \ldots\right)^2 \end{aligned} \tag{20}$$

3.1 Primary resonances

We study the primary resonance case in this section when $\Omega \to \omega_{11}$. We can consequently assume that

$$\Omega = \omega_{11} + \varepsilon^2 \sigma \tag{21}$$

where σ is the deviation parameter. By substituting Eq. (21) into (18) and vanishing the coefficients of ε^1, one can arrive at

$$\begin{aligned} &D_0^2 W_{mn,1} + \omega_{mn}^2 W_{mn,1} = 0 \\ &\quad \Rightarrow \quad W_{mn,1} = A_{mn}(T_2) e^{i\omega_{mn} T_0} + c.c \end{aligned} \tag{22}$$

where $c.c$ denotes the complex conjugate of the preceding terms. Vanishing of the coefficients of ε^3 leads to

$$
\begin{aligned}
&D_0^2 W_{11,3}+\omega_{11}^2 W_{11,3}\\
&\quad=-\left(D_1^2+2D_0D_2\right)W_{11,1}-\mu_{11}D_0W_{11,1}\\
&\qquad-\Theta_3^{11}W_{11,1}^3-\Theta_4^{11}W_{11,1}W_{12,1}^2\\
&\qquad-\Theta_5^{11}W_{11,1}W_{21,1}^2+f_{11}\cos\Omega T_0\\
&D_0^2 W_{12,3}+\omega_{12}^2 W_{12,3}\\
&\quad=-\left(D_1^2+2D_0D_2\right)W_{12,1}-\mu_{12}D_0W_{12,1}\\
&\qquad-\Theta_3^{12}W_{12,1}^3-\Theta_4^{12}W_{12,1}W_{11,1}^2\\
&\qquad-\Theta_5^{12}W_{12,1}W_{21,1}^2\\
&D_0^2 W_{21,3}+\omega_{21}^2 W_{21,3}\\
&\quad=-\left(D_1^2+2D_0D_2\right)W_{21,1}-\mu_{21}D_0W_{21,1}\\
&\qquad-\Theta_3^{21}W_{21,1}^3-\Theta_4^{21}W_{21,1}W_{11,1}^2\\
&\quad-\Theta_5^{21}W_{21,1}W_{12,1}^2
\end{aligned}
\tag{23}
$$

Substituting Eq. (22) into (23) results in

$$
\begin{aligned}
&D_0^2 W_{11,3}+\omega_{11}^2 W_{11,3}\\
&\quad=-2i\omega_{11}A_{11}'e^{i\omega_{11}T_0}-\mu_{11}i\omega_{11}A_{11}e^{i\omega_{11}T_0}\\
&\qquad-\Theta_3^{11}\left(A_{11}^3e^{3i\omega_{11}T_0}+3A_{11}^2\overline{A}_{11}e^{i\omega_{11}T_0}\right)\\
&\qquad-\Theta_4^{11}\left(A_{11}A_{12}^2e^{i(\omega_{11}+2\omega_{12})T_0}\right.\\
&\qquad+2A_{11}A_{12}\overline{A}_{12}e^{i\omega_{11}T_0}\\
&\qquad\left.+A_{11}\overline{A}_{12}^2e^{i(\omega_{11}-2\omega_{12})T_0}\right)\\
&\qquad-\Theta_5^{11}\left(A_{11}A_{21}^2e^{i(\omega_{11}+2\omega_{21})T_0}\right.\\
&\qquad+2A_{11}A_{21}\overline{A}_{21}e^{i\omega_{11}T_0}\\
&\qquad\left.+A_{11}\overline{A}_{21}^2e^{i(\omega_{11}-2\omega_{21})T_0}\right)+\frac{1}{2}f_{11}e^{i\Omega T_0}+c.c\\
&D_0^2 W_{12,3}+\omega_{12}^2 W_{12,3}\\
&\quad=-2i\omega_{12}A_{12}'e^{i\omega_{12}T_0}-\mu_{12}i\omega_{12}A_{12}e^{i\omega_{12}T_0}\\
&\qquad-\Theta_3^{12}\left(A_{12}^3e^{3i\omega_{12}T_0}+3A_{12}^2\overline{A}_{12}e^{i\omega_{12}T_0}\right)\\
&\qquad-\Theta_4^{12}\left(A_{12}A_{11}^2e^{i(\omega_{12}+2\omega_{11})T_0}\right.\\
&\qquad+2A_{12}A_{11}\overline{A}_{11}e^{i\omega_{12}T_0}\\
&\qquad\left.+A_{12}\overline{A}_{11}^2e^{i(\omega_{12}-2\omega_{11})T_0}\right)\\
&\qquad-\Theta_5^{12}\left(A_{12}A_{21}^2e^{i(\omega_{12}+2\omega_{21})T_0}\right.\\
&\qquad+2A_{12}A_{21}\overline{A}_{21}e^{i\omega_{12}T_0}\\
&\qquad\left.+A_{12}\overline{A}_{21}^2e^{i(\omega_{12}-2\omega_{21})T_0}\right)+c.c\\
&D_0^2 W_{21,3}+\omega_{21}^2 W_{21,3}\\
&\quad=-2i\omega_{21}A_{21}'e^{i\omega_{21}T_0}-\mu_{21}i\omega_{21}A_{21}e^{i\omega_{21}T_0}\\
&\qquad-\Theta_3^{21}\left(A_{21}^3e^{3i\omega_{21}T_0}+3A_{21}^2\overline{A}_{21}e^{i\omega_{21}T_0}\right)\\
&\qquad-\Theta_4^{21}\left(A_{21}A_{11}^2e^{i(\omega_{21}+2\omega_{11})T_0}\right.\\
&\qquad+2A_{21}A_{11}\overline{A}_{11}e^{i\omega_{21}T_0}\\
&\qquad\left.+A_{21}\overline{A}_{11}^2e^{i(\omega_{21}-2\omega_{11})T_0}\right)\\
&\qquad-\Theta_5^{21}\left(A_{21}A_{12}^2e^{i(\omega_{21}+2\omega_{12})T_0}\right.\\
&\qquad+2A_{21}A_{12}\overline{A}_{12}e^{i\omega_{21}T_0}\\
&\qquad\left.+A_{21}\overline{A}_{12}^2e^{i(\omega_{21}-2\omega_{12})T_0}\right)+c.c
\end{aligned}
\tag{24}
$$

Eliminating secular terms in Eqs. (24) yields

$$
\begin{aligned}
&-2i\omega_{11}A_{11}'-\mu_{11}i\omega_{11}A_{11}-3\Theta_3^{11}A_{11}^2\overline{A}_{11}\\
&\quad-2\Theta_4^{11}A_{11}A_{12}\overline{A}_{12}-2\Theta_5^{11}A_{11}A_{21}\overline{A}_{21}\\
&\quad+\frac{1}{2}f_{11}e^{i\sigma T_2}=0\\
&\quad-2i\omega_{12}A_{12}'-\mu_{12}i\omega_{12}A_{12}-3\Theta_3^{12}A_{12}^2\overline{A}_{12}\\
&\quad-2\Theta_4^{12}A_{12}A_{11}\overline{A}_{11}-2\Theta_5^{12}A_{12}A_{21}\overline{A}_{21}=0\\
&\quad-2i\omega_{21}A_{21}'-\mu_{21}i\omega_{21}A_{21}-3\Theta_3^{21}A_{21}^2\overline{A}_{21}\\
&\quad-2\Theta_4^{21}A_{21}A_{11}\overline{A}_{11}-2\Theta_5^{21}A_{21}A_{12}\overline{A}_{12}=0
\end{aligned}
\tag{25}
$$

By defining $A_{ij}=\frac{1}{2}a_{ij}e^{i\theta_{ij}}$ in the latter equation and separating real and imaginary parts, one can arrive at

$$
\begin{aligned}
&a_{11}\omega_{11}\theta_{11}'-\frac{3}{8}\Theta_3^{11}a_{11}^3\\
&\quad-\frac{1}{4}\Theta_4^{11}a_{11}a_{12}^2-\frac{1}{4}\Theta_5^{11}a_{11}a_{21}^2\\
&\quad+\frac{1}{2}f_{11}\cos[\sigma T_2-\theta_{11}]=0\\
&-\omega_{11}a_{11}'-\frac{1}{2}\omega_{11}\mu_{11}a_{11}+\frac{1}{2}f_{11}\sin[\sigma T_2-\theta_{11}]=0\\
&a_{12}\omega_{12}\theta_{12}'-\frac{3}{8}\Theta_3^{12}a_{12}^3\\
&\quad-\frac{1}{4}\Theta_4^{12}a_{12}a_{11}^2-\frac{1}{4}\Theta_5^{12}a_{12}a_{21}^2=0\\
&-\omega_{12}a_{12}'-\frac{1}{2}\omega_{12}\mu_{12}a_{12}=0
\end{aligned}
\tag{26}
$$

$$a_{21}\omega_{21}\theta'_{21}-\frac{3}{8}\Theta_3^{21}a_{21}^3$$
$$-\frac{1}{4}\Theta_4^{21}a_{21}a_{11}^2-\frac{1}{4}\Theta_5^{21}a_{21}a_{12}^2=0$$
$$-\omega_{21}a'_{21}-\frac{1}{2}\omega_{21}\mu_{21}a_{21}=0$$

In case of the steady-state condition, $a'_{11}=a'_{12}=a'_{21}=\gamma'_{11}=\theta'_{12}=\theta'_{21}=0$, and it then reads from Eqs. (26):

$$a_{12}=a_{21}=0,$$
$$\begin{cases} a_{11}\omega_{11}\sigma-\frac{3}{8}\Theta_3^{11}a_{11}^3+\frac{1}{2}f_{11}\cos[\gamma_{11}]=0 \\ -\frac{1}{2}\omega_{11}\mu_{11}a_{11}+\frac{1}{2}f_{11}\sin[\gamma_{11}]=0 \end{cases} \tag{27}$$

where $\sigma T_2-\theta_{11}=\gamma_{11}$. We can now eliminate γ_{11} in Eq. (27) and find the closed-form frequency response of the system for the first primary resonance case.

In a similar way, two other primary resonance cases, $\Omega\to\omega_{12}$ and $\Omega\to\omega_{21}$, can be examined. In the second and third cases, the following equations are, respectively, obtained:

$$\Omega=\omega_{12}+\varepsilon^2\sigma\colon a_{11}=a_{21}=0,$$
$$\begin{cases} a_{12}\omega_{12}\sigma-\frac{3}{8}\Theta_3^{12}a_{12}^3-\Theta_4^{12}\Lambda_{11}^2a_{12} \\ \quad -\frac{1}{2}\Theta_4^{12}\Lambda_{11}^2a_{12}\cos[2\gamma_{12}]=0 \\ -\frac{1}{2}\omega_{12}\mu_{12}a_{12}-\frac{1}{2}\Theta_4^{12}\Lambda_{11}^2a_{12}\sin[2\gamma_{12}]=0 \end{cases} \tag{28}$$

$$\Omega=\omega_{21}+\varepsilon^2\sigma\colon a_{11}=a_{12}=0,$$
$$\begin{cases} a_{21}\omega_{21}\sigma-\frac{3}{8}\Theta_3^{21}a_{21}^3-\Theta_4^{21}\Lambda_{11}^2a_{21} \\ \quad -\frac{1}{2}\Theta_4^{21}\Lambda_{11}^2a_{21}\cos[2\gamma_{21}]=0 \\ -\frac{1}{2}\omega_{21}\mu_{21}a_{21}-\frac{1}{2}\Theta_4^{21}\Lambda_{11}^2a_{21}\sin[2\gamma_{21}]=0 \end{cases} \tag{29}$$

where $\Lambda_{11}=\frac{f_{11}}{2(\omega_{11}^2-\Omega^2)}$.

3.2 Secondary and combination resonance cases

In this case, F_{11} is assumed to be εf_{11} in Eq. (14). Equating coefficients of the same powers of ε results in ε^1:

$$\begin{aligned} &D_0^2W_{11,1}+\omega_{11}^2W_{11,1}=f_{11}\cos[\Omega T_0] \\ &\quad\Rightarrow W_{11,1}=A_{11}(T_2)e^{i\omega_{11}T_0} \\ &\qquad +\Lambda_{11}e^{i\Omega T_0}+c.c \\ &D_0^2W_{12,1}+\omega_{12}^2W_{12,1}=0 \\ &\quad\Rightarrow W_{12,1}=A_{12}(T_2)e^{i\omega_{12}T_0}+c.c \\ &D_0^2W_{21,1}+\omega_{21}^2W_{21,1}=0 \\ &\quad\Rightarrow W_{21,1}=A_{21}(T_2)e^{i\omega_{21}T_0}+c.c \end{aligned} \tag{30}$$

ε^3:

$$\begin{aligned} &D_0^2W_{11,3}+\omega_{11}^2W_{11,3} \\ &\quad=-\left(D_1^2+2D_0D_2\right)W_{11,1}-\mu_{11}D_0W_{11,1} \\ &\qquad-\Theta_3^{11}W_{11,1}^3-\Theta_4^{11}W_{11,1}W_{12,1}^2 \\ &\qquad-\Theta_5^{11}W_{11,1}W_{21,1}^2 \\ &D_0^2W_{12,3}+\omega_{12}^2W_{12,3} \\ &\quad=-\left(D_1^2+2D_0D_2\right)W_{12,1}-\mu_{12}D_0W_{12,1} \\ &\qquad-\Theta_3^{12}W_{12,1}^3-\Theta_4^{12}W_{12,1}W_{11,1}^2 \\ &\qquad-\Theta_5^{12}W_{12,1}W_{21,1}^2 \\ &D_0^2W_{21,3}+\omega_{21}^2W_{21,3} \\ &\quad=-\left(D_1^2+2D_0D_2\right)W_{21,1}-\mu_{21}D_0W_{21,1} \\ &\qquad-\Theta_3^{21}W_{21,1}^3-\Theta_4^{21}W_{21,1}W_{11,1}^2 \\ &\qquad-\Theta_5^{21}W_{21,1}W_{12,1}^2 \end{aligned} \tag{31}$$

Substituting Eqs. (30) into (31) gives

$$\begin{aligned} &D_0^2W_{11,3}+\omega_{11}^2W_{11,3} \\ &\quad=-2i\omega_{11}A'_{11}e^{i\omega_{11}T_0}-\mu_{11}\big(i\omega_{11}A_{11}e^{i\omega_{11}T_0} \\ &\qquad+i\Omega\Lambda_{11}e^{i\Omega T_0}\big) \\ &\qquad-\Theta_3^{11}\big[A_{11}^3e^{3i\omega_{11}T_0}+\Lambda_{11}^3e^{3i\Omega T_0} \\ &\qquad+3A_{11}^2\Lambda_{11}e^{i(2\omega_{11}+\Omega)T_0} \\ &\qquad+3A_{11}\Lambda_{11}^2e^{i(\omega_{11}+2\Omega)T_0}+3\overline{A}_{11}A_{11}^2e^{i\omega_{11}T_0} \\ &\qquad+3\overline{A}_{11}\Lambda_{11}^2e^{i(-\omega_{11}+2\Omega)T_0} \\ &\qquad+6\overline{A}_{11}A_{11}\Lambda_{11}e^{i\Omega T_0}+3A_{11}^2\Lambda_{11}e^{i(2\omega_{11}-\Omega)T_0} \\ &\qquad+3\Lambda_{11}^3e^{i\Omega T_0} \end{aligned}$$

$$
\begin{aligned}
&+6A_{11}\Lambda_{11}^2 e^{i\omega_{11}T_0}\big] \\
&-\Theta_4^{11}\big[A_{12}^2 A_{11} e^{i(2\omega_{12}+\omega_{11})T_0} \\
&+2A_{12}\overline{A}_{12}A_{11}e^{i\omega_{11}T_0}+A_{12}^2\Lambda_{11}e^{i(2\omega_{12}+\Omega)T_0} \\
&+2A_{12}\overline{A}_{12}\Lambda_{11}e^{i\Omega T_0}+\overline{A}_{12}^2 A_{11}e^{i(-2\omega_{12}+\omega_{11})T_0} \\
&+\overline{A}_{12}^2\Lambda_{11}e^{i(-2\omega_{12}+\Omega)T_0}\big] \\
&-\Theta_5^{11}\big[A_{21}^2 A_{11}e^{i(2\omega_{21}+\omega_{11})T_0} \\
&+A_{21}^2\Lambda_{11}e^{i(2\omega_{21}+\Omega)T_0}+2A_{21}\overline{A}_{21}A_{11}e^{i\omega_{11}T_0} \\
&+2A_{21}\overline{A}_{21}\Lambda_{11}e^{i\Omega T_0}+\overline{A}_{21}^2 A_{11}e^{i(-2\omega_{21}+\omega_{11})T_0} \\
&+\overline{A}_{21}^2\Lambda_{11}e^{i(-2\omega_{21}+\Omega)T_0}\big]+c.c
\end{aligned}
\tag{32}
$$

$$
\begin{aligned}
&D_0^2 W_{12,3}+\omega_{12}^2 W_{12,3} \\
&\quad=-2i\omega_{12}A'_{12}e^{i\omega_{12}T_0} \\
&\quad-\mu_{12}i\omega_{12}A_{12}e^{i\omega_{12}T_0} \\
&\quad-\Theta_3^{12}\big[A_{12}^3 e^{3i\omega_{12}T_0}+3\overline{A}_{12}A_{12}^2 e^{i\omega_{12}T_0}\big] \\
&\quad-\Theta_4^{12}\big[A_{11}^2 A_{12}e^{i(2\omega_{11}+\omega_{12})T_0} \\
&\quad+A_{12}\Lambda_{11}^2 e^{i(\omega_{12}+2\Omega)T_0} \\
&\quad+2A_{12}A_{11}\Lambda_{11}e^{i(\omega_{11}+\omega_{12}+\Omega)T_0} \\
&\quad+2A_{12}A_{11}\overline{A}_{11}e^{i\omega_{12}T_0} \\
&\quad+2A_{12}A_{11}\Lambda_{11}e^{i(\omega_{11}+\omega_{12}-\Omega)T_0} \\
&\quad+2A_{12}\overline{A}_{11}\Lambda_{11}e^{i(-\omega_{11}+\omega_{12}+\Omega)T_0} \\
&\quad+2A_{12}\Lambda_{11}^2 e^{i\omega_{12}T_0} \\
&\quad+A_{12}\overline{A}_{11}^2 e^{i(-2\omega_{11}+\omega_{12})T_0} \\
&\quad+A_{12}\Lambda_{11}^2 e^{i(-2\Omega+\omega_{12})T_0} \\
&\quad+2A_{12}\overline{A}_{11}\Lambda_{11}e^{i(-\omega_{11}+\omega_{12}-\Omega)T_0}\big] \\
&\quad-\Theta_5^{12}\big[A_{12}A_{21}^2 e^{i(2\omega_{21}+\omega_{12})T_0} \\
&\quad+2A_{12}A_{21}\overline{A}_{21}e^{i\omega_{12}T_0} \\
&\quad+A_{12}\overline{A}_{21}^2 e^{i(-2\omega_{21}+\omega_{12})T_0}\big]+c.c
\end{aligned}
\tag{33}
$$

$$
\begin{aligned}
&D_0^2 W_{21,3}+\omega_{21}^2 W_{21,3} \\
&\quad=-2i\omega_{21}A'_{21}e^{i\omega_{21}T_0}-\mu_{21}i\omega_{21}A_{21}e^{i\omega_{21}T_0} \\
&\quad-\Theta_3^{21}\big[A_{21}^3 e^{3i\omega_{21}T_0}+3\overline{A}_{21}A_{21}^2 e^{i\omega_{21}T_0}\big] \\
&\quad-\Theta_4^{21}\big[A_{11}^2 A_{21}e^{i(2\omega_{11}+\omega_{21})T_0} \\
&\quad+A_{21}\Lambda_{11}^2 e^{i(\omega_{21}+2\Omega)T_0} \\
&\quad+2A_{21}A_{11}\Lambda_{11}e^{i(\omega_{11}+\omega_{21}+\Omega)T_0} \\
&\quad+2A_{21}A_{11}\overline{A}_{11}e^{i\omega_{21}T_0} \\
&\quad+2A_{21}A_{11}\Lambda_{11}e^{i(\omega_{11}+\omega_{21}-\Omega)T_0} \\
&\quad+2A_{21}\overline{A}_{11}\Lambda_{11}e^{i(-\omega_{11}+\omega_{21}+\Omega)T_0} \\
&\quad+2A_{21}\Lambda_{11}^2 e^{i\omega_{21}T_0}+A_{21}\overline{A}_{11}^2 e^{i(-2\omega_{11}+\omega_{21})T_0} \\
&\quad+A_{21}\Lambda_{11}^2 e^{i(-2\Omega+\omega_{21})T_0} \\
&\quad+2A_{21}\overline{A}_{11}\Lambda_{11}e^{i(-\omega_{11}+\omega_{21}-\Omega)T_0}\big] \\
&\quad-\Theta_5^{21}\big[A_{21}A_{12}^2 e^{i(2\omega_{12}+\omega_{21})T_0} \\
&\quad+2A_{21}A_{12}\overline{A}_{12}e^{i\omega_{21}T_0} \\
&\quad+A_{21}\overline{A}_{12}^2 e^{i(-2\omega_{12}+\omega_{21})T_0}\big]+c.c
\end{aligned}
\tag{34}
$$

From Eqs. (32) to (34), it is found that the secondary resonant condition may happen when $\Omega \to \frac{1}{3}\omega_{11}$ or $\Omega \to 3\omega_{11}$ and combination resonance can exist when $\Omega \to \omega_{11}+2\omega_{12}$ or $\Omega \to \omega_{11}+2\omega_{21}$.

3.2.1 *Super-harmonic resonance: $\Omega \to \frac{1}{3}\omega_{11}$*

In this case we assume

$$
3\Omega=\omega_{11}+\varepsilon^2\sigma \tag{35}
$$

Eliminating secular terms in Eqs. (32) to (34) and then separating the resultant real and imaginary parts yield

$$
\begin{aligned}
&\omega_{11}a_{11}\theta'_{11}-\Theta_3^{11}\Lambda_{11}^3\cos[\sigma T_2-\theta_{11}] \\
&\quad-\frac{3}{8}\Theta_3^{11}a_{11}^3-3\Theta_3^{11}\Lambda_{11}^2 a_{11}-\frac{1}{4}\Theta_4^{11}a_{12}^2 a_{11} \\
&\quad-\frac{1}{4}\Theta_5^{11}a_{21}^2 a_{11}=0 \\
&-\omega_{11}a'_{11}-\frac{1}{2}\mu_{11}\omega_{11}a_{11} \\
&\quad-\Theta_3^{11}\Lambda_{11}^3\sin[\sigma T_2-\theta_{11}]=0 \\
&\omega_{12}a_{12}\theta'_{12}-\frac{3}{8}\Theta_3^{12}a_{12}^3-\frac{1}{4}\Theta_4^{12}a_{11}^2 a_{12} \\
&\quad-\Theta_4^{12}\Lambda_{11}^2 a_{12}-\frac{1}{4}\Theta_5^{12}a_{21}^2 a_{12}=0 \\
&-\omega_{12}a'_{12}-\frac{1}{2}\mu_{12}\omega_{12}a_{12}=0 \\
&\omega_{21}a_{21}\theta'_{21}-\frac{3}{8}\Theta_3^{21}a_{21}^3-\frac{1}{4}\Theta_4^{21}a_{11}^2 a_{21}
\end{aligned}
\tag{36}
$$

$$- \Theta_4^{21} \Lambda_{11}^2 a_{21} - \frac{1}{4} \Theta_5^{21} a_{12}^2 a_{21} = 0$$

$$-\omega_{21} a'_{21} - \frac{1}{2} \mu_{21} \omega_{21} a_{21} = 0$$

One can now obtain the following expressions for the steady-state condition:

$$a_{12} = a_{21} = 0$$

$$\sigma = \frac{3\Theta_3^{11} a_{11}^2}{8\omega_{11}} + \frac{3\Theta_3^{11} \Lambda_{11}^2}{\omega_{11}} \pm \sqrt{\left(\frac{\Theta_3^{11} \Lambda_{11}^3}{a_{11}\omega_{11}}\right)^2 - \frac{1}{4}\mu_{11}^2} \tag{37}$$

3.2.2 Sub-harmonic resonance: $\Omega \to 3\omega_{11}$

Going through the same procedure, the following closed-form expression is obtained for the corresponding frequency response:

$$a_{12} = a_{21} = 0$$

$$\sigma = \frac{9\Theta_3^{11} a_{11}^2}{8\omega_{11}} + \frac{9\Theta_3^{11} \Lambda_{11}^2}{\omega_{11}} \pm \sqrt{\left(\frac{9\Theta_3^{11} \Lambda_{11} a_{11}}{4\omega_{11}}\right)^2 - \frac{9}{4}\mu_{11}^2} \tag{38}$$

3.2.3 Combination resonance

We start with the first resonance case when $\Omega \to \omega_{11} + 2\omega_{12}$. The corresponding deviation is then defined by

$$\Omega = \omega_{11} + 2\omega_{12} + \varepsilon^2 \sigma \tag{39}$$

Separating real and imaginary parts of the secular terms in this case results in

$$\begin{aligned}
&\omega_{11} a_{11} \theta'_{11} - \frac{1}{4} \Theta_4^{11} \Lambda_{11} a_{12}^2 \cos[\sigma T_2 - 2\theta_{12} - \theta_{11}] \\
&\quad - \frac{3}{8} \Theta_3^{11} a_{11}^3 - 3\Theta_3^{11} \Lambda_{11}^2 a_{11} \\
&\quad - \frac{1}{4} \Theta_4^{11} a_{12}^2 a_{11} - \frac{1}{4} \Theta_5^{11} a_{21}^2 a_{11} = 0 \\
&-\omega_{11} a'_{11} - \frac{1}{2} \mu_{11} \omega_{11} a_{11} \\
&\quad - \frac{1}{4} \Theta_4^{11} \Lambda_{11} a_{12}^2 \sin[\sigma T_2 - 2\theta_{12} - \theta_{11}] = 0 \\
&\omega_{12} a_{12} \theta'_{12} - \frac{1}{2} \Theta_4^{12} \Lambda_{11} a_{11} a_{12} \cos[\sigma T_2 - 2\theta_{12} - \theta_{11}] \\
&\quad - \frac{3}{8} \Theta_3^{12} a_{12}^3 - \Theta_4^{12} \Lambda_{11}^2 a_{12} - \frac{1}{4} \Theta_4^{12} a_{11}^2 a_{12} \\
&\quad - \frac{1}{4} \Theta_5^{12} a_{21}^2 a_{12} = 0 \\
&-\omega_{12} a'_{12} - \frac{1}{2} \mu_{12} \omega_{12} a_{12} \\
&\quad - \frac{1}{2} \Theta_4^{12} \Lambda_{11} a_{11} a_{12} \sin[\sigma T_2 - 2\theta_{12} - \theta_{11}] = 0
\end{aligned} \tag{40}$$

In case of steady-state condition i.e. $a'_{11} = a'_{12} = a'_{21} = \gamma' = 0$, one can obtain

$$\begin{aligned}
&\sigma - \frac{2}{\omega_{12} a_{12}} \bigg[\frac{1}{2} \Theta_4^{12} \Lambda_{11} a_{11} a_{12} \cos[\gamma] \\
&\quad + \frac{3}{8} \Theta_3^{12} a_{12}^3 + \Theta_4^{12} \Lambda_{11}^2 a_{12} + \frac{1}{4} \Theta_4^{12} a_{11}^2 a_{12} \bigg] \\
&\quad - \frac{1}{\omega_{11} a_{11}} \bigg[\frac{1}{4} \Theta_4^{11} \Lambda_{11} a_{12}^2 \cos[\gamma] \\
&\quad + \frac{3}{8} \Theta_3^{11} a_{11}^3 + 3\Theta_3^{11} \Lambda_{11}^2 a_{11} \\
&\quad + \frac{1}{4} \Theta_4^{11} a_{12}^2 a_{11} \bigg] = 0 \\
&\mu_{11} \omega_{11} a_{11} + \frac{1}{2} \Theta_4^{11} \Lambda_{11} a_{12}^2 \sin[\gamma] = 0 \\
&\mu_{12} \omega_{12} a_{12} + \Theta_4^{12} \Lambda_{11} a_{11} a_{12} \sin[\gamma] = 0
\end{aligned} \tag{41}$$

where $\gamma = \sigma T_2 - 2\theta_{12} - \theta_{11}$. One can obtain

$$\begin{aligned}
&\sin[\gamma] = \frac{-\mu_{12}\omega_{12}}{\Theta_4^{12} \Lambda_{11} a_{11}}; \\
&\cos[\gamma] = \pm \sqrt{1 - \left(\frac{\mu_{12}\omega_{12}}{\Theta_4^{12} \Lambda_{11} a_{11}}\right)^2}
\end{aligned} \tag{42}$$

$$a_{12} = \left(\sqrt{\frac{2\mu_{11}\omega_{11}\Theta_4^{12}}{\mu_{12}\omega_{12}\Theta_4^{11}}}\right) a_{11} \tag{43}$$

The corresponding frequency response is then explicitly obtained by substituting Eqs. (42) and (43) into (41).

4 Numerical results

Numerical results based on the closed-form frequency responses are presented in this section. For this purpose, a case study is constructed for a steel plate with

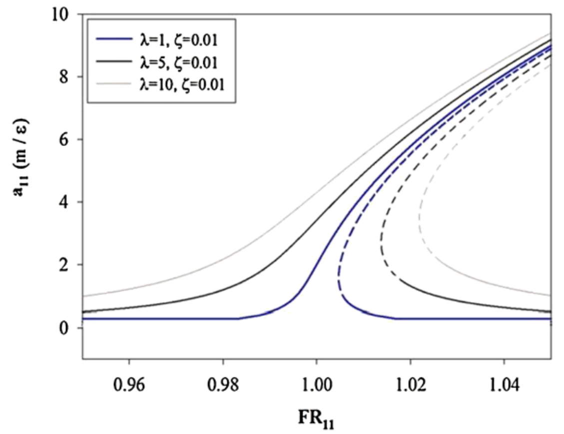

Fig. 2 Primary resonance, $\Omega \to \omega_{11}$, frequency-response curves for different levels of excitation

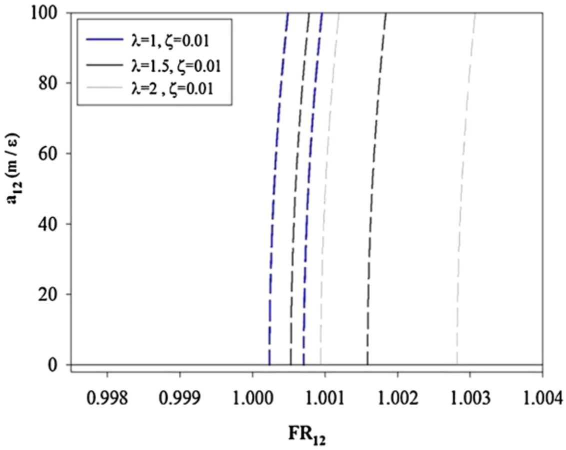

Fig. 3 Primary resonance, $\Omega \to \omega_{12}$, frequency-response curves for different levels of excitation

a length of $a = 26$ m, a width of $b = 2.8$ m, and a thickness of $h = 10$ mm. The material properties are assumed to be $E = 210 \times 10^9$ Pa, $\nu = 0.3$ and $\rho = 7800$ kg/m^3. The air cavity has a depth of $c = 3.3$ m, and the sound speed is assumed to be 340 m/s. In case of the primary resonance ($\Omega \to \omega_{11}$), frequency-response curves of the plate-cavity system are shown in Fig. 2, for different levels of acoustic excitation. For the two other primary resonance cases, $\Omega \to \omega_{12}$ and $\Omega \to \omega_{21}$, amplitude-frequency ratio curves are illustrated in Figs. 3 and 4, respectively. As it is seen, for the steady-state condition, zero is the only stable solution in these cases. In other words, in the second and third primary resonance cases, the solutions of the equations of linear and nonlinear systems exactly meet each other. Additionally, it is seen that the excitation amplitude has no effect on the stability of the solution. Frequency response of the plate-cavity system for the super-harmonic resonance, $\Omega \to \frac{1}{3}\omega_{11}$, is shown in Fig. 5. Stable zone of the curves increases with increasing the value of excitation amplitude. For the case of subharmonic resonance, $\Omega \to 3\omega_{11}$, amplitude is plotted against the frequency ratio in Fig. 6. Unlike the super-harmonic resonance, the stable zone becomes smaller in higher values of excitation amplitude. Moreover, zero is another stable steady-state solution in this case. In secondary resonance cases, amplitude-frequency responses of the second and third modes of the plate-cavity system, $(1, 2)$ and $(2, 1)$, are identical to the linear system. In other words, for

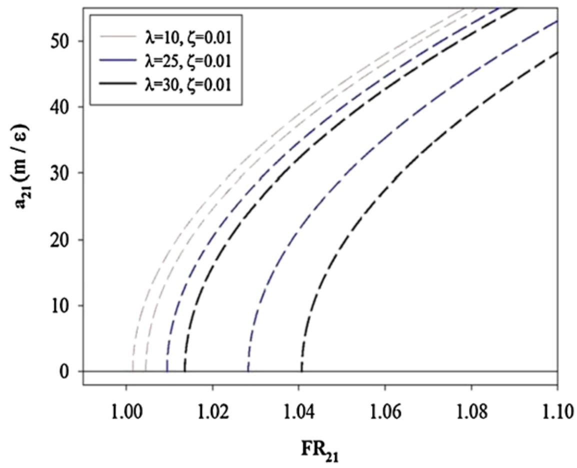

Fig. 4 Primary resonance, $\Omega \to \omega_{21}$, frequency-response curves for different levels of excitation

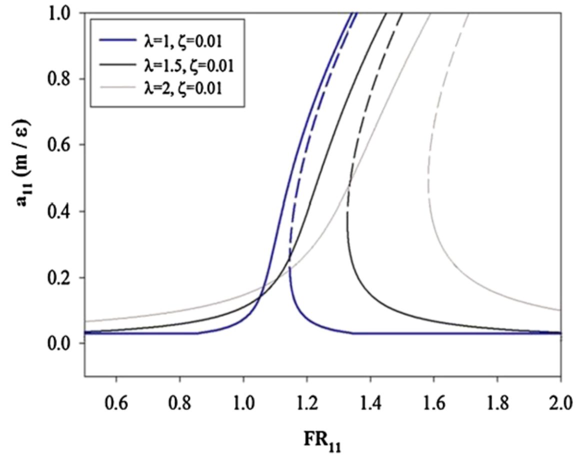

Fig. 5 Super-harmonic resonance, $\Omega \to \frac{1}{3}\omega_{11}$, frequency-response curves for different levels of excitation

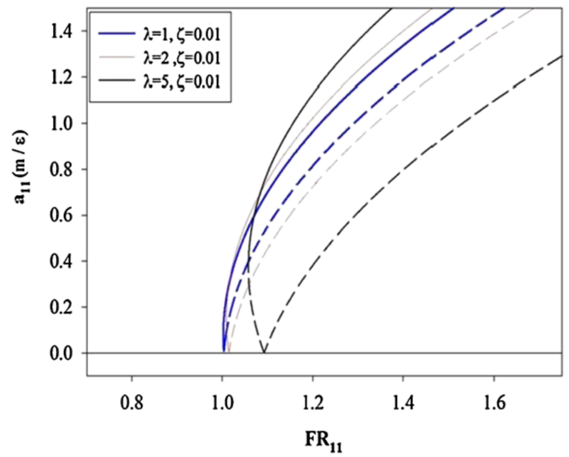

Fig. 6 Subharmonic resonance, $\Omega \to 3\omega_{11}$, frequency-response curves for different levels of excitation

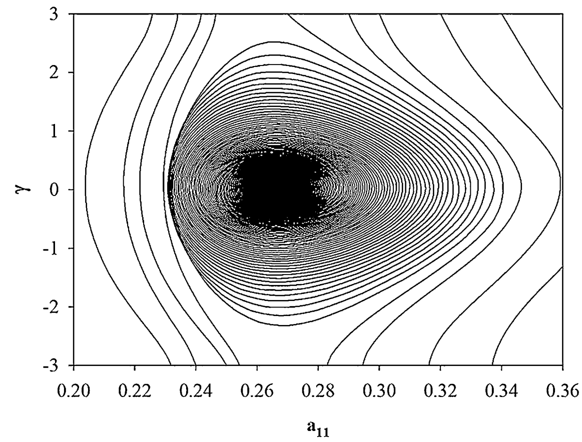

Fig. 8 State-plane for combination resonance (neighborhood of point A)

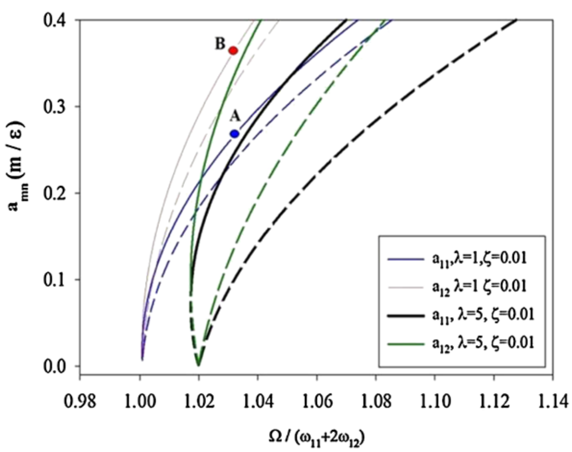

Fig. 7 Combination resonance, $\Omega \to \omega_{11} + 2\omega_{12}$, frequency response curves for different levels of excitation

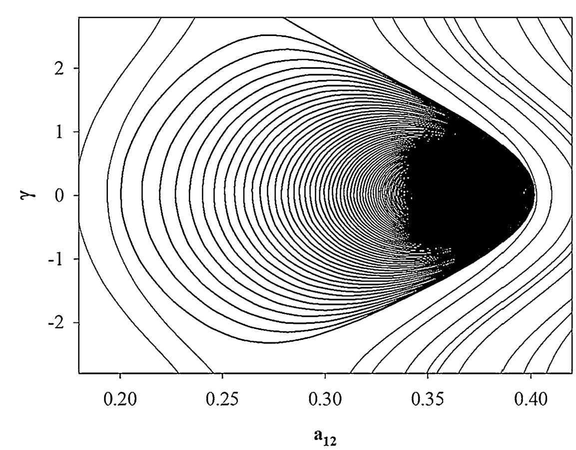

Fig. 9 State-plane for combination resonance (neighborhood of point B)

the steady-state condition, when the excitation frequency, Ω, tends to $\frac{1}{3}\omega_{11}$ and $3\omega_{11}$, a_{12} and a_{21} both will vanish. Figure 7 presents the amplitude-frequency response curves for the first combination resonance when $\Omega \to \omega_{11} + 2\omega_{12}$. It is seen that both W_{11} and W_{12} are excited and a_{12} becomes larger than the first mode amplitude, a_{11}. This fact is again valid for both excitation amplitudes examined in this resonance case. It can be concluded that the stable zone becomes smaller as excitation amplitude increases. In order to determine stable points on the frequency response curves, stability analysis is numerically performed. As an example, points A and B are captured in Fig. 7. Stability analysis is done for points A and B and state-plane graphs are illustrated in Figs. 8 and 9, respectively. The attraction zone is recognized in the state-plane graphs and it is found that both points are stable ones. Actually, if initial conditions fall into the corresponding attraction zone, the motion ends up to these steady-state conditions. Finally, for the second combination resonance, $\Omega \to \omega_{11} + 2\omega_{21}$, results are shown in Fig. 10. Unlike the first case, a_{11} is larger than the third mode steady-state amplitude, a_{21}. In order to capture the acoustic resonance in the frequency response of the system, wider range of frequency is swept and the first structural and acoustic resonant peaks are illustrated in Fig. 11. The effects of the excitation level on the frequency curves are shown in Fig. 12. As seen, magnitude of the excitation has

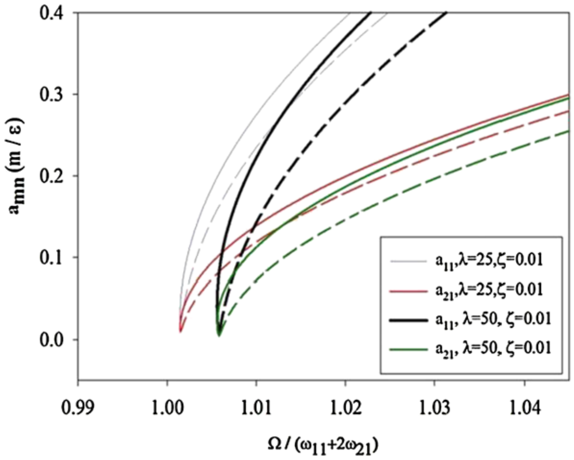

Fig. 10 Combination resonance, $\Omega \to \omega_{11} + 2\omega_{21}$, frequency-response curves for different levels of excitation

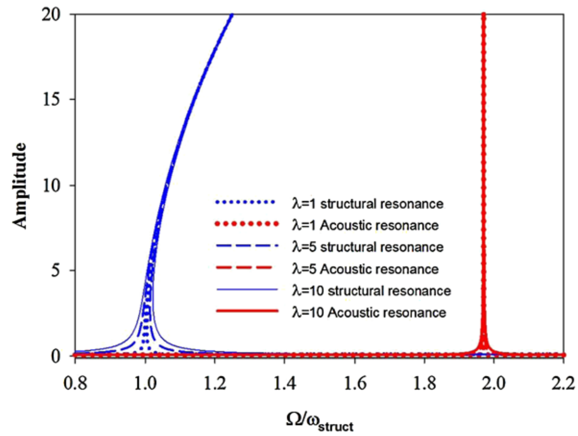

Fig. 12 Effect of the excitation amplitude on the frequency curves; primary resonance condition; units are m/ε and Pa/ε, respectively, for deflection and acoustic pressure, $\xi = 0.01$

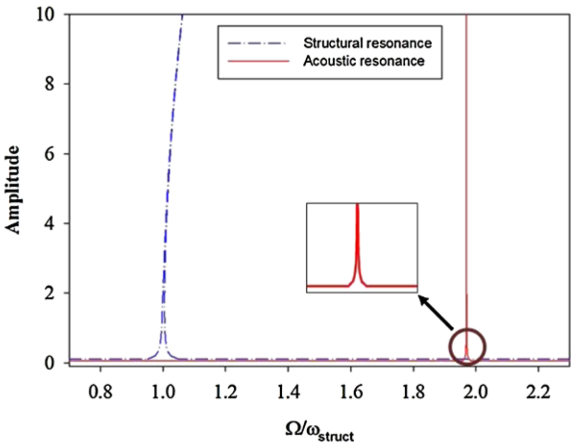

Fig. 11 Structural and acoustic resonant peaks in frequency response of the system; primary resonance condition units are m/ε and Pa/ε, respectively, for deflection and acoustic pressure ($\lambda = 1$, $\xi = 0.01$)

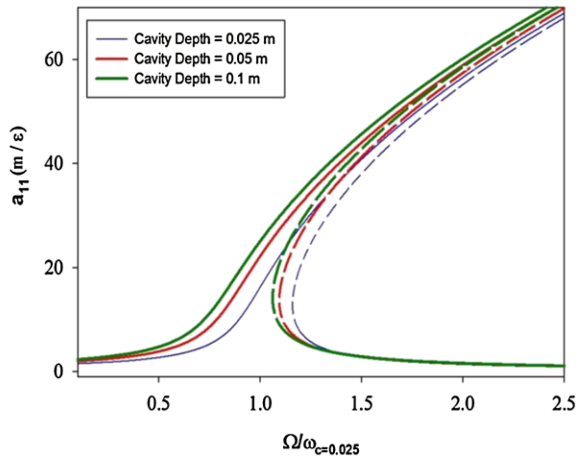

Fig. 13 Effect of cavity depth on frequency response of the system; primary resonance ($\lambda = 100$, $\xi = 0.01$)

a negligible influence on the acoustic backbone frequency curves.

To examine the effect of cavity depth on the frequency responses of the plate-cavity system, a parametric study is carried out and the corresponding results are illustrated in Fig. 13. The flexible plate length and width are assumed to be $a = b = 1$ m. It can be concluded that the smaller cavity depth results in higher natural frequency. Consequently, resonance condition occurs at the higher value of excitation frequency in the case of shallower cavities.

5 Conclusions

Nonlinear forced vibration of a plate-cavity system was analytically studied in this paper. Galerkin method was used to derive the coupled nonlinear equations of the system. In order to solve the nonlinear equations of plate-cavity system, Multiple Scales method was employed. Closed form expressions were obtained for the frequency-amplitude relationship in different resonance conditions. Four secondary resonance conditions were captured to be $\frac{1}{3}\omega_{11}$, $3\omega_{11}$, $\omega_{11} + 2\omega_{12}$ and $\omega_{11} + 2\omega_{21}$. According to the obtained results, following remarks can be made.

1. For the second and third primary resonance cases, $\Omega \to \omega_{12}$ and $\Omega \to \omega_{21}$, The only stable steady-state solution is zero, just like linear systems. A different story is committed when $\Omega \to \omega_{11}$. A nonzero stable steady-state solution exists under such a condition.
2. For the first primary resonance case, $\Omega \to \omega_{11}$, excitation amplitude has a negligible effect on the general trend of stable zone.
3. Stable zone enhances with increasing the value of excitation amplitude, in the case of super-harmonic resonance. However, the reverse trend is observed in case of sub-harmonic resonance condition.
4. In case of combinational resonance, $\Omega \to \omega_{11} + 2\omega_{12}$, increasing the excitation amplitude tends to decrease the stable zone.
5. For the combination resonance case, $\Omega \to \omega_{11} + 2\omega_{21}$, both W_{11} and W_{12} are excited and unlike the first combinational case, a_{11} is larger than the third mode steady-state amplitude, a_{21}.

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