(1.) $R(\vec{a} \cdot \vec{b}) = R(\vec{a}) \cdot R(\vec{b}) \rightarrow (1)$

$R(\vec{a} \times \vec{b}) = R(\vec{a}) \times R(\vec{b}) \rightarrow (2)$

We need to show that the above two conditions satisfy for $\vec{a}, \vec{b} \in R^3$ and R being a rotation in R^3.

$$\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \quad \vec{a} \cdot \vec{b} = [a_1 b_1 + a_2 b_2 + a_3 b_3]$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix}$$

Any rotation R in R^3 can be represented by $R = R_x(\alpha) R_y(\beta) R_z(\theta)$. If we prove that (1) & (2) are satisfied for $R = R_z(\theta)$ then (1) & (2) will be satisfied for $R = R_x(\alpha) R_y(\beta) R_z(\theta)$ as it will be equivalent to proving the same thing again.

$$\Rightarrow \text{R.H.S of (1)} = \left(\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \right)^T \cdot \left(\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \right)$$

$$= \begin{bmatrix} c_\theta a_1 - s_\theta a_2 & s_\theta a_1 + c_\theta a_2 & a_3 \end{bmatrix} \begin{bmatrix} c_\theta b_1 - s_\theta b_2 \\ s_\theta b_1 + c_\theta b_2 \\ b_3 \end{bmatrix}$$

$$= \begin{bmatrix} c_\theta^2(a_1b_1 + a_2b_2) \\ + c_\theta s_\theta(-a_1b_2 - a_2b_1 + a_1b_2 + a_2b_1) \\ + s_\theta^2(a_2b_2 + a_1b_1) + a_3b_3 \end{bmatrix}$$

$$= [a_1b_1 + a_2b_2 + a_3b_3] = \vec{a}\cdot\vec{b} = \text{L.H.S of } (1)$$

$$\text{R.H.S of } (2) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ c_\theta a_1 - s_\theta a_2 & s_\theta a_1 + c_\theta a_2 & a_3 \\ c_\theta b_1 - s_\theta b_2 & s_\theta b_1 + c_\theta b_2 & b_3 \end{vmatrix}$$

$$= \begin{bmatrix} -a_3b_2c_\theta + a_2b_3c_\theta - a_3b_1s_\theta + a_1b_3s_\theta \\ a_3b_1c_\theta - a_1b_3c_\theta - a_3b_2s_\theta + a_2b_3s_\theta \\ -a_2b_1c_\theta^2 + a_1b_2c_\theta^2 - a_2b_1s_\theta^2 + a_1b_2s_\theta^2 \end{bmatrix}$$

$$= \begin{bmatrix} (-a_3b_2 + a_2b_3)c_\theta + (-a_3b_1 + a_1b_3)s_\theta \\ (a_3b_1 - a_1b_3)c_\theta + (-a_3b_2 + a_2b_3)s_\theta \\ a_1b_2 - a_2b_1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix} = R(\vec{a}\times\vec{b}) = \text{R.H.S of } (2)$$

Thus (1) & (2) are proved