28-05-2020 AM5650 - END-SEM EXAM S. TARUN PRASAD

ME17B114

"The work being submitted is my own work. I have not sought the help of any person in doing this work."

(2) $(1+4p^2x^2)\ddot{x} + \Lambda x + 4p^2\dot{x}^2 x = 0$

For $\dot{x} = v$, transformation,

we get $\dot{x} = v$ & $\dot{v} = -\dfrac{\Lambda x + 4p^2 x v^2}{1+4p^2x^2} \rightarrow (2)$

$\hookrightarrow (1)$

$$\frac{(2)}{(1)} \Rightarrow \frac{dv}{dx} = \frac{-(\Lambda x + 4p^2 x v^2)}{v(1+4p^2x^2)}.$$

$$\Rightarrow \frac{1}{2}(1+4p^2x^2)\,d(v^2) + 4p^2 x v^2\,dx + \Lambda x\,dx = 0$$

$$\Rightarrow (1+4p^2x^2)v^2 + \Lambda x^2 = h.$$

(Where h is the constant of integration)

$$\Rightarrow v^2 = \frac{h - \Lambda x^2}{1+4p^2x^2}$$

For v to exist for $\Lambda > 0$,

$$h - \Lambda x^2 > 0 \Rightarrow -\sqrt{\frac{h}{\Lambda}} \leq x \leq \sqrt{\frac{h}{\Lambda}}$$

Thus x is bounded for $\Lambda > 0$

Substituting $x = \varepsilon^{1/2} u$ in main equation,

$$(1+4\varepsilon p^2u^2)\cancel{\varepsilon^{1/2}}\ddot{u} + \Lambda \cancel{\varepsilon^{1/2}} u + 4\varepsilon p^2 \dot{u}^2 u \cancel{\varepsilon^{1/2}} = 0$$

$$\Rightarrow \ddot{u} + \Lambda u = -4p^2\varepsilon(u^2\ddot{u} + u\dot{u}^2)$$

for $\Lambda = \omega_0^2$

$$\Rightarrow \ddot{u} + \omega_0^2 u = -4p^2\varepsilon(u^2\ddot{u} + u\dot{u}^2).$$

(a) $\ddot{u} + \omega_0^2 u = -4p^2\varepsilon(u^2\ddot{u} + u\dot{u}^2)$

Let $u = u_0 + \varepsilon u_1$ (two-term straightforward expansion)

$$\Rightarrow \ddot{u}_0 + \omega_0^2 u_0 + \varepsilon\ddot{u}_1 + \omega_0^2\varepsilon u_1$$
$$= -4p^2\varepsilon\big((u_0^2 + \varepsilon^2 u_1^2 + 2\varepsilon u_0 u_1)(\ddot{u}_0 + \varepsilon\ddot{u}_1)$$
$$+ (u_0 + \varepsilon u_1)(\dot{u}_0^2 + \varepsilon^2\dot{u}_1^2 + 2\varepsilon\dot{u}_0\dot{u}_1)\big).$$

$\varepsilon = 0$.

$$\Rightarrow \ddot{u}_0 + \omega_0^2 u_0 = 0.$$
$$\Rightarrow u_0 = U\cos(\omega_0 t + \beta).$$

$\varepsilon = 1$

$$\ddot{u}_1 + \omega_0^2 u_1 = -4p^2(u_0^2\ddot{u}_0 + u_0\dot{u}_0^2)$$

$$\ddot{u}_1 + \omega_0^2 u_1 = -4p^2\big(U^2\cos^2(\omega_0 t + \beta)(-\omega_0^2 U\cos\omega_0 t)$$
$$+ U\cos(\omega_0 t + \beta)(-U\omega_0\sin(\omega_0 t + \beta))^2\big)$$
$$= -4p^2\omega_0^2 U^3\big(-\cos^3(\omega_0 t + \beta) + \sin^2(\omega_0 t + \beta)\cos(\omega_0 t + \beta)\big)$$
$$= 4p^2\omega_0^2 U^3\big(\cos(2\omega_0 t + 2\beta)\cos(\omega_0 t + \beta)\big)$$
$$= -4p^2\omega_0^2 U^3\cos(\omega_0 t + \beta)\big(1 - 2\cos^2(\omega_0 t + \beta)\big)$$
$$= -4p^2\omega_0^2 U^3\big(\cos(\omega_0 t + \beta) - 2\cos^3(\omega_0 t + \beta)\big)$$

$\boxed{\text{Using the substitution } \cos^3\theta = \frac{\cos 3\theta}{4} + \frac{3}{4}\cos\theta}$

$$= -\cancel{4p^2\omega_0^2 U^3}\big(\cancel{\cos(\omega_0 t + \beta)} - .5\cos(3\omega_0 t + 3\beta) - \cancel{1.5}\ 0.5\cos(\omega_0 t + 3\beta)\big)$$

$$\ddot{u}_1 + \omega_0^2 u_1 = C\big(\cos(\omega_0 t + \beta) + \cos(3\omega_0 t + 3\beta)\big) \rightarrow (1)$$

where $C = 2p^2\omega_0^2 U^3$).

Equating ① is an inhomogenous equation whose general solution is the sum of a homogenous solution and a particular solution.
The homogenous solution is easily obtainable and for the particular solution we develop two particular solutions for the two terms in RHS for convenience in solving and we will superimpose them later for the general solution.
The two equations thus to solve for the particular solution are thus:

$$\ddot{u}_1 + \omega_0^2 u_1 = c\cos(\omega_0 t \mp \beta) \quad — ②$$

$$\& \quad \ddot{u}_1 + \omega_0^2 u_1 = c\cos(3\omega_0 t + 3\beta) - ③$$

The ~~gen~~ solution for ② & ③ should be of the forms: $Kt\sin(\omega_0 t \pm \beta)$

& $K\Xi\cos(3\omega_0 t + 3\beta)$ for ② & ③ respectively

Solving for k in (2),

$$\frac{\partial^2}{\partial t^2}(kt \sin(\omega_0 t + \beta))$$
$$= -kt\omega_0^2 \sin(\omega_0 t + \beta) + C\cos(\omega_0 t + \beta)$$

$$= \frac{\partial}{\partial t}\left(k\sin(\omega_0 t + \beta) + kt\omega_0 \cos(\omega_0 t + \beta)\right)$$

$$= +k\omega_0 \cos(\omega_0 t + \beta) + k\omega_0 \cos(\omega_0 t + \beta) - kt\omega_0^2 \sin(\omega_0 t + \beta)$$

$$\Rightarrow 2k\omega_0 = C$$

$$\Rightarrow k = \frac{C}{2\omega_0}$$

$$\Rightarrow u_1 = \frac{Ct}{2\omega_0}\sin(\omega_0 t + \beta) = p^2\omega_0 a^3 t \sin(\omega_0 t$$

Solving for k in (3),

$$\ddot{u}_1 + \omega_0^2 u_1 = C\cos(3\omega_0 t + 3\beta)$$

Let $u_1 = k\cos(3\omega_0 t + 3\beta)$

$$\Rightarrow \frac{\partial}{\partial t^2}(k\cos(3\omega_0 t + 3\beta)) = (C - k\omega_0^2)\cos(3\omega_0 t + 3\beta)$$

$$\Rightarrow -9k\omega_0^2 = C - k\omega_0^2$$

$$\Rightarrow -8k\omega_0^2 = C$$

$$\Rightarrow k = \frac{-C}{8\omega_0^2}$$

$$\Rightarrow u_1 = \frac{-c}{8\omega_0^2} \cos(3\omega_0 t + 3\beta)$$

$$= \frac{-p^2 u^3}{4} \cos(3\omega_0 t + 3\beta)$$

$$\Rightarrow u_1 = U_1 \cos(\omega_0 t + \beta_1)$$
$$+ p^2 \omega_0 U^3 t \sin(\omega_0 t + \beta) - \frac{c}{8\omega_0^2} \cos(3\omega_0 t + 3\beta)$$

$$u_0 = U \cos(\omega_0 t + \beta)$$

$$u = u_0 + \varepsilon u_1$$
$$= U\cos(\omega_0 t + \beta)$$
$$+ \varepsilon \left[p^2 \omega_0 U^3 t \sin(\omega_0 t + \beta) - \frac{p^2 u^3}{4} \cos(3\omega_0 t + 3\beta) \right.$$

The two term straightforward expansion has a secular Term $\varepsilon p^2 \omega_0 u^3 t \sin(\omega_0 t + \beta)$. Thus the expansion is not periodic and x_1 doesn't provide a small correction to x_0. The more the increase in number of terms the solution is less uniformly valid as terms like $t^m \cos(\omega_0 t + \beta)$ and $t^m (\sin(\omega_0 t + \beta))$ will get added to the solution.

(b) $\ddot{u} + \omega_0^2 u = -4p^2\varepsilon(u^2\ddot{u} + u\dot{u}^2)$.

We introduce $\tau = \omega t$.

$$\Rightarrow \frac{d}{dt} = \omega\frac{d}{d\tau} \quad \& \quad \frac{d^2}{dt^2} = \omega^2\frac{d^2}{d\tau^2}.$$

$$\Rightarrow \omega^2\ddot{u} + \omega_0^2 u = -4p^2\varepsilon\omega^2(u^2\ddot{u} + u\dot{u}^2).$$

Let $u = u_0(\tau) + \varepsilon u_1(\tau) + \ldots$

& $\omega = \omega_0 + \varepsilon\omega_1 + \ldots$

$$\Rightarrow (\omega_0 + \varepsilon\omega_1 + \ldots)^2(u_0'' + \varepsilon u_1'' + \ldots) + \omega_0^2(u_0 + \varepsilon u_1 + \ldots)$$

$$= -4p^2\varepsilon(\omega_0 + \varepsilon\omega_1 + \ldots)^2$$

$$\times\left((u_0 + \varepsilon u_1 + \ldots)^2(\ddot{u}_0 + \varepsilon\ddot{u}_1 + \ldots) + (u_0 + \varepsilon u_1 + \ldots)(\dot{u}_0 + \varepsilon\dot{u}_1 + \ldots)^2\right).$$

ε^0 terms:

$$\omega_0^2 u_0'' + \omega_0^2 u_0 = 0.$$

$$\Rightarrow u_0 = U_0\cos(\tau + \beta_0).$$

ε^1 terms:

$$\omega_0^2 u_1'' + 2\omega_0\omega_1 u_0'' = -4p^2\omega_0^2(u_0^2\ddot{u}_0 + u_0\dot{u}_0^2)$$
$$+ \omega_0^2 u_1$$

$$\Rightarrow \omega_0^2(u_1'' + u_1) =$$

$$2\omega_0\omega_1 U_0\cos(\tau + \beta_0)$$
$$-4p^2\omega_0^2 U_0^3\left(-\cos^3(\tau + \beta_0) + \cos(\tau + \beta_0)\sin^2(\tau + \beta_0)\right)$$

$$\Rightarrow \omega_0^2(U_1'' + U_1) = (2\omega_0\omega_1 U_0 - 4p^2\omega_0^2 U_0^3)\cos(\tau + \beta_0)$$

$$+ 8\cancel{4}p^2\omega_0^2 U_0^3 (\cos^3(\tau + \beta_0))$$

Using the substitution,

$$\cos^3\theta = \frac{\cos 3\theta}{4} + \frac{3}{4}\cos\theta$$

$$\Rightarrow \omega_0^2(u_1''+u_1) = (2\omega_0\omega_1 u_0 - 4p^2\omega_0^2 u_0^3)\cos(\tau+\beta_0)$$
$$+ 2p^2\omega_0^2 u_0^3(\cos(3\tau+3\beta_0)) + 6p^2\omega_0^2 u_0^3\cos(\tau+\beta_0)$$

$$\Rightarrow (u_1''+u_1) = \left(\frac{\omega_1}{\omega_0} + 2p^2u_0^3\right)\cos(\tau+\beta_0))$$
$$+2p^2u_0^3\cos(3\tau+\beta_0\times 3)$$

Following the similar substitution for obtaining particular solution as in part (a), we get the general solution to be,

$$u_1 = U_0\cos(\tau+\beta_1) + \left(\frac{\omega_1}{\omega_0}U_0 + p^2u_0^3\right)\tau\sin(\tau+\beta_0)$$
$$-\frac{p^2U_0^3}{4}(\cos(3\tau+3\beta_0)).$$

To eliminate secular terms and to obtain a uniform solution we need,

$$\frac{\omega_1}{\omega_0}U_0 + p^2u_0^3 = 0 \Rightarrow \omega_1 = -p^2u_0^2\omega_0$$
$$\Rightarrow U_1 = U_0\cos(\tau+\beta_1) - \frac{p^2u_0^3}{4}\cos(3\tau+3\beta_0)$$

$$\Rightarrow \omega = \omega_0(1-p^2\varepsilon U_0^2) + \cdots$$
$$u = U_0\cos(\tau+\beta_0) + \varepsilon\left(U_0\cos(\tau+\beta_1) - \frac{p^2U_0^3}{4}\cos(3\tau+3\beta_0)\right) + \cdots$$

Substituting for $\tau = \omega t$ with ω value,

$$u = U_0 \cos\left(\left((1 - \varepsilon p^2 U_0^2)\omega_0 + \dots\right) t + \beta_0\right)$$

$$+ \varepsilon\left(U_0 \cos\left(\left((1 - \varepsilon p^2 U_0^2)\omega_0 + \dots\right) t + \beta_1\right)\right.$$

$$\left. - \frac{p^2 U_0^3}{4} \cos\left(3\left((1 - \varepsilon p^2 U_0^2)\omega_0 + \dots\right) t + 3\beta_0\right)\right)$$

$$+ \dots\dots$$

This is the first order uniform expansion obtained by using Poincaré-Lindstedt technique.

(c) $\ddot{u} + \omega_0^2 u = -4p^2\varepsilon\left(u^2\ddot{u} + u\dot{u}^2\right)$ — (1).

Let $u(t;\varepsilon) = u(t, \varepsilon t, \varepsilon^2 t, \ldots; \varepsilon)$

$= u(T_0, T_1, T_2, \ldots; \varepsilon)$

where $T_1 = \varepsilon t$, $T_0 = t$, $T_2 = \varepsilon^2 t$, ...

$$\Rightarrow \frac{d}{dt} = \frac{\partial}{\partial T_0} + \varepsilon\frac{\partial}{\partial T_1} + \varepsilon^2\frac{\partial}{\partial T_2} + \ldots$$

$$\frac{d^2}{dt^2} = \frac{\partial^2}{\partial T_0^2} + 2\varepsilon\frac{\partial^2}{\partial T_0 \partial T_1} + \varepsilon^2\left(\frac{2\partial^2}{\partial T_0 \partial T_2} + \frac{\partial^2}{\partial T_1^2}\right) + \ldots$$

$$\Rightarrow (1) \rightarrow \frac{\partial^2 u}{\partial T_0^2} + 2\varepsilon\frac{\partial^2 u}{\partial T_0 \partial T_1} + \varepsilon^2\left(\frac{2\partial^2 u}{\partial T_0 T_2} + \frac{\partial^2 u}{\partial T_1^2}\right) + \omega_0^2 u + \ldots$$

$$= -4p^2\varepsilon\left(u^2\left(\frac{\partial^2 u}{\partial T_0^2} + 2\varepsilon\frac{\partial^2 u}{\partial T_0\partial T_1} + \varepsilon^2\left(\frac{2\partial^2 u}{\partial T_0 T_2} + \frac{\partial^2 u}{\partial T_1^2}\right) + \ldots\right)\right.$$

$$\left. + u\left(\frac{\partial u}{\partial T_0} + \varepsilon\frac{\partial u}{\partial T_1} + \varepsilon^2\frac{\partial u}{\partial T_2} + \ldots\right)^2\right)$$

Substituting $u = u_0(T_0, T_1, \ldots) + \varepsilon u_1(T_0, T_1, \ldots) + \ldots$

$$\Rightarrow \frac{\partial^2 u_0}{\partial T_0^2} + \varepsilon\frac{\partial^2 u_1}{\partial T_0^2} + 2\varepsilon\frac{\partial^2 u_0}{\partial T_0\partial T_1} + \omega_0^2 u_0 + \varepsilon\omega_0^2 u_1 + \ldots$$

$$= -4p^2\varepsilon\left(u_0^2\frac{\partial^2 u_0}{\partial T_0^2} + u_0\left(\frac{\partial u_0}{\partial T_0}\right)^2\right) + \ldots$$

Collecting ε^0 terms:

$$\frac{\partial^2 u_0}{\partial T_0^2} + \omega_0^2 u_0 = 0$$

$$\Rightarrow \quad u_0 = a_0 \cos(\omega_0 T_0 + \beta_0)$$

where,
$a_0 = a_0(T_1, T_2, \dots)$
$b_0 = b_0(T_1, T_2, \dots)$

Collecting ε^1 terms:

$$\varepsilon \frac{\partial^2 u_1}{\partial T_0^2} + \frac{2\partial^2 u_0}{\partial T_0 \partial T_1} + \omega_0^2 u_1$$

$$= -4p^2 \varepsilon \left(u_0^2 \frac{\partial u_0^2}{\partial T_0^2} + u_0 \left(\frac{\partial u_0}{\partial T_0}\right)^2 \right)$$

Substituting u_0,

$$\left(\frac{\partial^2 u_1}{\partial T_0^2} + \omega_0^2 u_1\right) + 2\frac{\partial a_0}{\partial T_1}\left(-\omega_0 \sin(\omega_0 T_0 + \beta_0)\right)$$

$$+ 2a_0 \frac{\partial \beta_0}{\partial T_1} \left(-\omega_0 \cos(\omega_0 T_0 + \beta_0)\right)$$

$$= -4p^2 a_0^3 \omega_0^2 \left(-\cos^3(\omega_0 T_0 + \beta_0) + \sin^2(\omega_0 T_0 + \beta_0) \cos(\omega_0 T_0 + \beta_0) \right)$$

$$= -4p^2 a_0^3 \omega_0^2 \left(\cos(\omega_0 T_0 + \beta_0) - 2\cos^3(\omega_0 T_0 + \beta_0) \right)$$

$$= -4p^2 a_0^3 \omega_0^2 \left(\cos(\omega_0 T_0 + \beta_0) - \frac{2}{4}\cos(3\omega_0 T_0 + 3\beta_0) - \frac{6}{4}\cos(\omega_0 T_0 + \beta_0) \right)$$

$$\Rightarrow \frac{\partial^2 u_1}{\partial T_0^2} + \omega_0^2 u_1 = 2\frac{\partial a_0}{\partial T_1}\omega_0 \sin(\omega_0 T_0 + \beta_0)$$

$$+ \cos(\omega_0 T_0 + \beta_0)\left[2a_0\omega_0\frac{\partial \beta_0}{\partial T_1} + 2p^2 a_0^3 \omega_0^2\right]$$

$$+ \cos(3\omega_0 T_0 + \beta_0 \times 3) \times (2p^2 a_0^3 \omega_0^2)$$

To eliminate secular terms,

$$\frac{\partial a_0}{\partial T_1} = 0 \Rightarrow a_0 = a_0(T_2, T_3, \ldots)$$

& $$\frac{\partial \beta_0}{\partial T_1} = -p^2 a_0^2 \omega_0 \Rightarrow \beta_0 = -p^2 a_0^2 \omega_0 T_1 + \beta_0(T_2, T_3, T_4 \ldots)$$

Declaring the dependence on secular terms as zero,

$$\frac{\partial^2 u_1}{\partial T_0^2} + \omega_0^2 u_1 = 2p^2 a_0^3 \omega_0^2 \cos(3\omega_0 T_0 + 3\beta_0).$$

$$\Rightarrow u_1 = a_0 \cos(\omega_0 \tau + \beta_0) - \frac{p^2 a_0^3 \omega_0^2}{4} \cos(3\omega_0 T_0 + 3\beta_0)$$

$$\Rightarrow u = a_0(T_2, T_3, \dots)\cos\left(\omega_0 T_0 - \omega_0 p^2 a_0^2 (T_2, T_3 \dots) T_1 + \beta(T_2, T_3 \dots)\right)$$

$$+ \varepsilon \left(a_0(T_2, T_3, \dots)\cos\left(\omega_0 T_0 - \omega_0 p^2 a_0^2 (T_2, T_3, \dots) T_1 + \beta(T_2, T_3 \dots)\right) - \frac{p^2 a_0^3 (T_2, T_3 \dots)\omega_0^2}{4} \cos\left(3\omega_0 T_0 - 3\omega_0 p^2 a_0^2 (T_2, T_3 \dots) T_1 + 3\beta(T_2, T_3 \dots)\right) \right)$$

This is the first order uniform expansion obtained from multiple scales method