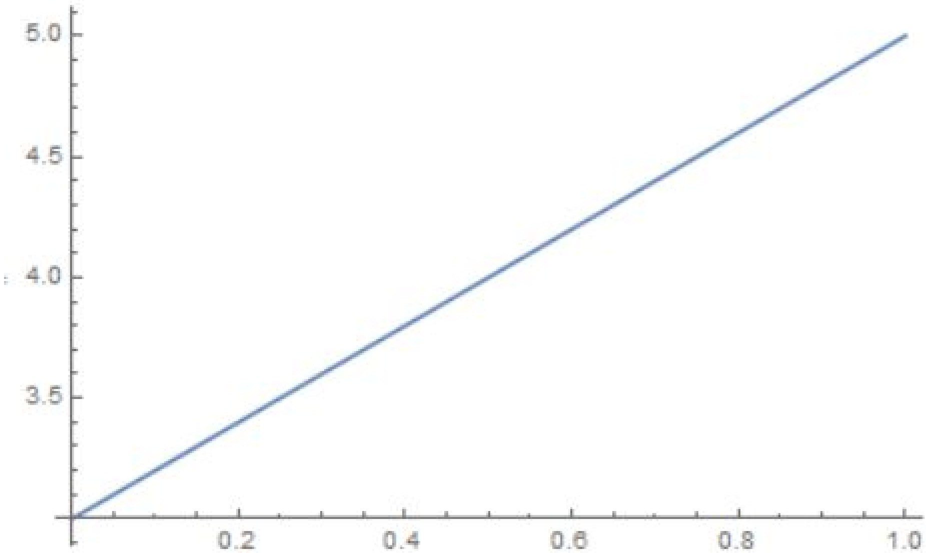
28-05-2020 AM5650-END-SEM EXAM S.TARUN PRASAD

ME17B114

"The work being submitted is my own work. I have not sought the help of any person in doing this work."

Plot of a(t) = 3b+ct in 4(a)(i) :



Plot of a(t) in 4(a)(ii) :

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Plot[Piecewise[{{0, 0 < x < 0.5},
  {2, 0.5 < x < 1}}], {x, 0, 1}]
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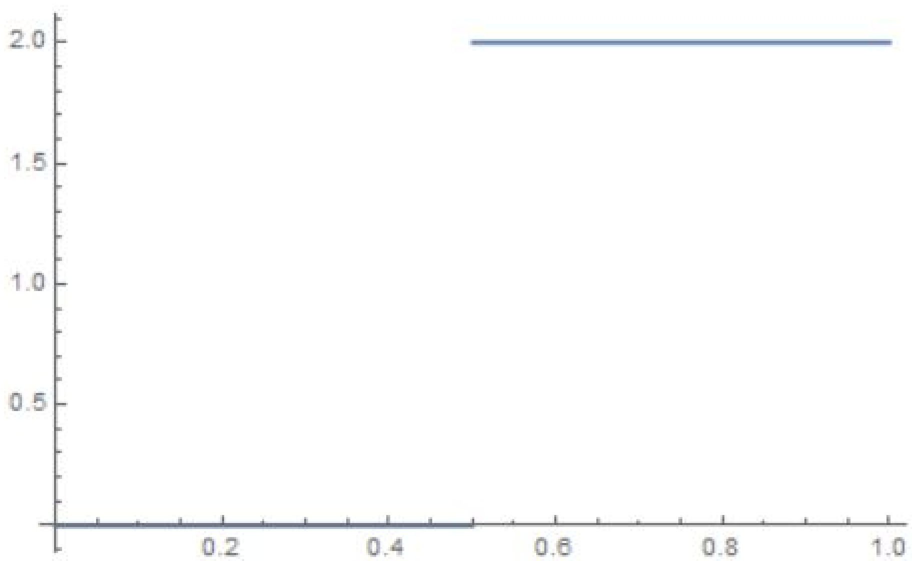
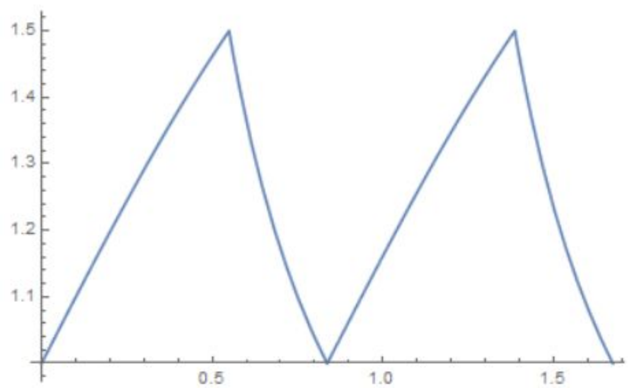


Plot for 4(c) for two time periods:

$$
\begin{aligned}
&\texttt{Plot}\Big[\texttt{Piecewise}\Big[\Big\{\Big\{\frac{2.\grave{}\ e^{2.\grave{}\ x}}{1.\grave{} + e^{2.\grave{}\ x}},\ 0 < x < \texttt{Log[3^0.5]}\Big\},\\
&\quad \Big\{\frac{0.5\grave{}\ e^{x}}{(-1.1547005383792517\grave{} + 0.\grave{}\ i) + e^{x}},\ \texttt{Log[3^0.5]} < x < \texttt{Log[4/(3^0.5)]}\Big\},\\
&\quad \Big\{-\frac{2.\grave{}\ e^{2.\grave{}\ x}}{-5.333333333333335\grave{} - 1.\grave{}\ e^{2.\grave{}\ x}},\ \texttt{Log[4/(3^0.5)]} < x < \texttt{Log[4/(3^0.5)]} + \texttt{Log[3^0.5]}\Big\},\\
&\quad \Big\{\frac{0.5\grave{}\ e^{x}}{(-2.666666666666667\grave{} + 0.\grave{}\ i) + e^{x}},\\
&\quad \texttt{Log[4/(3^0.5)]} + \texttt{Log[3^0.5]} < x < \texttt{Log[4/(3^0.5)]} + \texttt{Log[4/(3^0.5)]}\Big\}\Big\}\Big],\ \{x, 0, 2 * \texttt{Log[4/(3^0.5)]}\}\Big]
\end{aligned}
$$



(4) (a) $\dot{x} = a(t)x$

(i.) $\dot{x} = (3b + ct)x$, $(0 < t < T)$

$$\Rightarrow K = \int_0^T (3b + ct)\,dt$$

$$\Rightarrow K = 3bT + \frac{cT^2}{2}$$

$x = 0$ is the equilibrium point

$\dot{x} = (3b + ct)x \Rightarrow \dot{x} = x_0 e^{3bt + \frac{ct^2}{2}}$

for a small displacement x_0 from $x = 0$ at $t = 0$

The stability of $x = 0$ depends on sign of K

for $3bT + \frac{cT^2}{2} > 0 \Rightarrow x = 0 \rightarrow$ unstable

$\Rightarrow b > \frac{(-cT)}{3} \Rightarrow x = 0 \rightarrow$ unstable

& $b < (-cT/3) \Rightarrow x = 0 \rightarrow$ stable

(b.) $a(t) = \begin{cases} 0, & (0 < t < T/2) \\ b, & (T/2 < t < T) \end{cases}$

$$\Rightarrow K = \frac{bT}{2}$$

$x = 0 \rightarrow$ equilibrium point.

$\Rightarrow b > 0 \Rightarrow x = 0 \rightarrow$ unstable $\forall$ c

$b < 0 \Rightarrow x = 0 \rightarrow$ stable $\forall$ c

4(b) $\dot{x} = -x + c(t)$

$\dot{x}_h = -x_h \Rightarrow x_h = x_0 e^{t_0 - t}$

$\dot{x}_p = -x_p + c(t)$

Let $x_p = A(t) e^{t_0 - t}$

$\Rightarrow \dot{A} e^{t_0 - t} - A e^{t_0 - t} = -A(t) e^{t_0 - t} + c(t)$

$\Rightarrow \dot{A} = c(t) e^{t - t_0}$

$\Rightarrow A = e^{-t_0} \int_{t_0}^{t} e^{-t} c(t)\, dt$

$\Rightarrow x = e^{t_0 - t} \left[x_0 + e^{-t_0} \int_{t_0}^{t} e^{-t} c(t)\, dt \right]$

$x(t_0) = x_0 = 1 [x_0 + 0]$

$x(t_0 + T) = e^{-T} \left[x_0 + e^{-t_0} \int_{t_0}^{t_0+T} e^{-t} c(t)\, dt \right]$

We need need to show $x(t_0 + T) = x_0$

for some T

$\Rightarrow x_0 = \frac{e^{-T}}{1 - e^{-T}} \left(e^{-t_0} \int_{t_0}^{t_0+T} e^{-t} c(t)\, dt \right)$ $\quad e^{-T} \neq 1$

Thus the system has a unique periodic solution satisfying the above equation

$\Rightarrow x(t) = e^{-t} \left[\frac{e^{-T}}{1 - e^{-T}} \int_{t_0}^{t_0+T} e^{-t} c(t)\, dt + \int_{t_0}^{t} e^{-t} c(t)\, dt \right]$

(c) $\dot{p} = ap - bp^2$

For this $\dot{p} > 0$ only for $p < a/b$.

Thus using the growth rate the population can only grow till a value Q which follows $Q < a/b$. $\left[\frac{a}{b} \rightarrow \text{equilibrium point for } \dot{p} = ap - bp^2 \right.$

$\dot{p} = Ap - Bp^2$

For this $\dot{p} < 0$ only for $p > A/B$

Thus using this rate the population can only decrease till a value q which follows

$q > A/B$ $\left[A/B \rightarrow \text{equilibrium point for } \dot{p} = Ap - Bp^2 \right]$.

(i) $\dot{p} = ap - bp^2$

$$\Rightarrow \int_q^Q \frac{dp}{ap - bp^2} = \int_0^{T_1} dt = \int_q^Q \frac{dp}{p(a - bp)}$$

$$\frac{1}{p(a-bp)} = \frac{c_1}{p} + \frac{c_2}{a-bp} = c_1 a + p(c_2 - c_1 b)$$

$$\Rightarrow c_1 = \frac{1}{a} \ \& \ c_2 = \frac{b}{a}.$$

$$\Rightarrow T_1 = \frac{1}{a}\int_q^Q \frac{dp}{p} + \frac{b}{a}\int_q^Q \frac{dp}{a-bp}$$

Let $u = a - bp \Rightarrow du = -b\,dp$

$$\Rightarrow T_1 = \frac{1}{a}\left(\ln\frac{Q}{q} - \int_{a-bq}^{a-bQ}\frac{du}{u}\right)$$

$$\Rightarrow T_1 = \frac{1}{a}\ln\left(\frac{Q(a-bq)}{q(a-bQ)}\right).$$

(ii) $\dot{p} = Ap - Bp^2$

$$\Rightarrow \int_Q^q \frac{dp}{Ap - Bp^2} = \int_0^{T_2} dt = T_2$$

$$\Rightarrow T_2 = \frac{1}{A}\int_Q^q \frac{dp}{p} + \frac{B}{A}\int_Q^q \frac{dp}{A - Bp}$$

$$= \frac{1}{A}\left(\ln\frac{q}{Q} - \int_{A-BQ}^{A-Bq}\frac{du}{u}\right) \text{ for } u = A - Bp$$

$$\Rightarrow T_2 = \frac{1}{A}\ln\left(\frac{q(A-BQ)}{Q(A-Bq)}\right).$$

For the entire cycle,

$$T = T_1 + T_2 = \frac{1}{a}\ln\left(\frac{Q(a-bq)}{q(a-bQ)}\right) + \frac{1}{A}\ln\left(\frac{q(A-BQ)}{Q(A-Bq)}\right)$$