(4)(a) $\vec{v}_{rot} = \vec{v}\cos\phi + (\vec{k}\cdot\vec{v})\vec{k}(1-\cos\phi) + (\vec{k}\times\vec{v})\sin\phi$ — ①

$$K = \hat{k} = \begin{bmatrix} 0 & -k_z & k_y \\ k_z & 0 & -k_x \\ -k_y & k_x & 0 \end{bmatrix} \quad \text{where} \quad \vec{k} = \begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix}.$$

As $K^2\vec{v} = \vec{k}\times(\vec{k}\times\vec{v}) = (\vec{k}\cdot\vec{v})\vec{k} - (\vec{k}\cdot\vec{k})\vec{v}$

$$\Rightarrow (\vec{k}\cdot\vec{v})\vec{k} = K^2\vec{v} + \vec{v}.$$

① $\Rightarrow$ $\vec{v}_{rot} = \vec{v}\cos\phi + K^2\vec{v}(1-\cos\phi) + \vec{v}(1-\cos\phi) + (\vec{k}\times\vec{v})\sin\phi$

$$= \vec{v} + K^2\vec{v}(1-\cos\phi) + (\vec{k}\times\vec{v})\sin\phi.$$

Also $K\vec{v} = \vec{k}\times\vec{v}$.

① $\Rightarrow$ $\vec{v}_{rot} = \vec{v} + K^2(1-\cos\phi)\vec{v} + K\sin\phi\,\vec{v}.$

$$= (I_3 + K^2(1-\cos\phi) + K\sin\phi)\,\vec{v},$$

As $\vec{v}_{rot} = R_K(\phi)\,\vec{v}$

$$\Rightarrow R_K(\phi) = (I_3 + K^2(1-\cos\phi) + K\sin\phi).$$

H.P

(a)(i). The physical significance of the equation is that $\vec{v}_{rot}$ is a vector in R_3 obtained by rotating a vector $\vec{v}$ in R_3 about an axis $\vec{k}$ in R_3 by an angle ϕ.

(c.) $\cancel{K}$ $K = \begin{bmatrix} 0 & -k_z & k_y \\ k_z & 0 & -k_x \\ -k_y & k_x & 0 \end{bmatrix}$.

Characteristic polynomial $\rightarrow$ $\begin{vmatrix} -\lambda & -k_z & k_y \\ k_z & -\lambda & -k_x \\ -k_y & k_x & -\lambda \end{vmatrix} = 0.$

$$\Rightarrow -\lambda(\lambda^2 + k_x^2) + k_z(-\lambda k_z - k_x k_y) + k_y(k_z k_x - \lambda k_y) = 0$$

$$\Rightarrow -\lambda^3 - \lambda(k_x^2 + k_y^2 + k_z^2) - \cancel{k_x k_y k_z} + \cancel{k_x k_y k_z} = 0$$

As k is a unit vector $\Rightarrow \lambda^3 = -\lambda$.

$$\Rightarrow \lambda = 0 \text{ or } \lambda^2 = -1 \Rightarrow \lambda = 0, \pm i.$$

For $\lambda = 0$, $K\vec{v} = 0$

$$\Rightarrow \begin{bmatrix} 0 & -k_z & k_y \\ k_z & 0 & -k_x \\ -k_y & k_x & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0.$$

$$\Rightarrow \begin{aligned} k_z b &= k_y c \\ k_z a &= k_x c. \\ k_x b &= k_y a. \end{aligned} \Rightarrow \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} k_x / k_z \\ k_y / k_z \\ 1 \end{bmatrix}$$

$\Rightarrow$ Eigenvector for $\lambda = 0$ $\rightarrow$ $\begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix}$

For $\lambda = \pm i$ the eigenvectors are the solutions for $\vec{v}$ in $K\vec{v} = \pm i \vec{v}$.