(2.) The trace and determinant are 2 parameters which remain constant during similarity transformation. We can similarity transform R to a diagonal matrix whose diagonal values are the eigen values.

Thus $tr(R) = \lambda_1 + \lambda_2 + \lambda_3$ & $det(R) = \lambda_1 \lambda_2 \lambda_3$.

The characteristic polynomial will also remain the same during a similarity transform.

$$\begin{vmatrix} \lambda_1 - \lambda & 0 & 0 \\ 0 & \lambda_2 - \lambda & 0 \\ 0 & 0 & \lambda_3 - \lambda \end{vmatrix} = 0 \Rightarrow -\lambda^3 + \lambda_1\lambda_2\lambda_3 + (\lambda_1 + \lambda_2 + \lambda_3)\lambda^2 - (\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3) = 0.$$

$$\Rightarrow \lambda^3 - (\lambda_1 + \lambda_2 + \lambda_3) + (\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1] - \lambda_1\lambda_2\lambda_3 = 0.$$

Given initially, characterisic polynomial:

$$\lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0.$$

$$\Rightarrow \quad I_1 = \lambda_1 + \lambda_2 + \lambda_3 \rightarrow tr(R).$$

$$I_2 = \lambda_1\lambda_3 + \lambda_1\lambda_2 + \lambda_2\lambda_3 = \lambda_1\lambda_2\lambda_3\left(\frac{\lambda_1\lambda_3 + \lambda_1\lambda_2 + \lambda_2\lambda_3}{\lambda_1\lambda_2\lambda_3}\right)$$

$$= det(R)\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3}\right)$$

The eigen-values of R^{-1} will be the reciprocals of the eigen-values of $R \Rightarrow I_2 = det(R)\, tr(R^{-1})$

As R is orthogonal, $R^{-1} = R^T$ & $det(R) = 1$.

$\Rightarrow \boxed{I_2 = 1(tr(R^T) = tr(R) = I_1}$, Also $\boxed{I_3 = \lambda_1\lambda_2\lambda_3 = det(R) = 1}$

(b). Let λ be an eigenvalue and $\vec{v}$ be the corresponding eigenvector for a matrix R.

$\Rightarrow \quad R\vec{v} = \lambda\vec{v} \Rightarrow \|R\vec{v}\| = \|\lambda\vec{v}\|$

$\Rightarrow \quad \vec{v}^T R^T R \vec{v} = \vec{v}^T R^{-1} R \vec{v} = \|\vec{v}\| = \lambda \|v\|$

$\Rightarrow \lambda = 1$ as $R^{-1} = R^T$ due to R being orthogonal

The trace of a rotation matrix can be shown to be $2\cos\theta + 1$ as the matrix can be expressed as $\begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$ about ~~its~~ an axis.

Thus, $\quad \lambda_1 + \lambda_2 + \lambda_3 = 1 + \lambda_2 + \lambda_3 = 2\cos\phi + 1$.

$\Rightarrow \lambda_2 + \lambda_3 = 2\cos\phi$.

Also $\lambda_1\lambda_2\lambda_3 = 1 \Rightarrow \lambda_2 = \frac{1}{\lambda_3}$.

$\Rightarrow \lambda_2^2 + \frac{1}{\lambda_2} = 2\cos\phi \Rightarrow \lambda_2^2 - 2\cos\phi\,\lambda_2 + 1 = 0$.

$\Rightarrow \lambda_2, \lambda_3 = \frac{2\cos\phi \pm \sqrt{4\cos^2\phi - 4}}{2} = \cos\phi \pm \sqrt{-\sin^2\phi}$.

$\Rightarrow \lambda_2, \lambda_3 = \cos\phi \pm i\sin\phi = e^{\pm i\phi}$

Eigenvalues of $R = 1, e^{i\phi}, e^{-i\phi}$.

(c) $\vec{R}\vec{v} = \lambda\vec{v}$

$R^{-1}R\vec{v} = R^{-1}\lambda\vec{v}$

As $\lambda = 1$ is the only ~~~~ real eigenvalue & $R^{-1} = R^T$

$\Rightarrow \vec{v} = R^T\vec{v}$ & $R\vec{v} = \vec{v}$.

$\Rightarrow (R - R^T)\vec{v} = 0$.

$\Rightarrow \vec{v} = K(R - R^T)^\vee$ as $R - R^T$ is a skew symmetric matrix

Rotation matrices have a magnitude of $2\sin\phi$ where ϕ is the angle of rotation about the eigenvector for real eigen value. $\Rightarrow K = \dfrac{1}{2\sin\phi}$.

$$\Rightarrow \vec{v} = \frac{(R - R^T)^\vee}{2\sin\phi}.$$

The remaining two eigen vectors are the solutions of $R\vec{v} = e^{\pm\phi}\vec{v}$. for variable $\vec{v}$.

Geometrically, the eigen-vector for real eigen value is the axis of rotation in the real space whereas the eigenvectors for the complex eigenvalues are the axes of rotation in the complex space