(5). As A is a skew symmetric matrix,

$$A^T = -A.$$

We need to show that $R_1, R_2, R_3, R_4 \in SO_3$..

Hence they should satisfy $R^T = R^{-1}$ & $\det(R) = 1$.

(i) (a) $R_1 = (I-A)(I+A)^{-1}$

For $R_1^T = R_1^{-1} \Rightarrow R_1 R_1^T = I$

$$\Rightarrow R_1 R_1^T = (I-A)(I+A)^{-1} \left((I-A)(I+A)^{-1} \right)^T.$$

$$= (I-A)(I+A)^{-1} \left((I+A)^{-1}\right)^T (I-A)^T$$

$$= (I-A)(I+A)^{-1} \left((I+A)^T\right)^{-1} (I-A)^T$$

$$= (I-A)(I+A)^{-1} (I-A)^{-1} (I+A).$$

$$= (I-A)\left((I-A)(I+A)\right)^{-1}(I+A).$$

$$= (I-A)(I + A - A - A^2)^{-1}(I+A)$$

$$= (I-A)(I - A + A - A^2)^{-1}(I+A)$$

$$= (I-A)\left((I+A)(I-A)\right)^{-1}(I+A).$$

$$= (I-A)(I-A)^{-1}(I+A)^{-1}(I+A) = I.$$

$$\det(R_1) = \det\left((I-A)(I+A)^{-1}\right) = \det(I-A)\det\left((I+A)^{-1}\right)$$

$$= \det(I-A) \times \frac{1}{\det(I+A)} = \frac{\det(I^T + A^T)}{\det(I+A)}.$$

$$= \frac{\det\left((I+A)^T\right)}{\det(I+A)} = 1.$$

Hence $R_1 \in SO_3$

(b.) $R_2 = (I+A)(I-A)^{-1}$.

$$R_1^{-1} = \left((I-A)(I+A)^{-1} \right)^{-1} = (I+A)(I-A)^{-1} = R_2.$$

$\Rightarrow R_2 \in SO_3$.

(c.) $R_3 = (I-A)^{-1}(I+A) = \left((I+A)^{-1}\right)^T (I-A)^T$

$$= \left((I-A)(I+A)^{-1} \right)^T = R_1^T = R_1^{-1} = R_2$$

$\Rightarrow R_3 \in SO_3$.

(d.) $R_4 = (I+A)^{-1}(I-A) = \left((I-A)^{-1}\right)^T (I+A)^T$

$$= \left((I+A)(I-A)^{-1} \right)^T = R_2^T = R_2^{-1} = R_1.$$

(ii) From part (i) we know that,

$$R_1 = R_4 = R_2^{-1} = R_3^{-1}.$$

(iii). $R_2 = (I+A)(I-A)^{-1} = (I+I-I+A)(I-A)^{-1}$.

$$= (2I - (I-A))(I-A)^{-1}.$$

$$= 2(I-A)^{-1} - I.$$

$$A = \begin{bmatrix} 0 & -k_3 \tan\left(\frac{\phi}{2}\right) & k_2 \tan\left(\frac{\phi}{2}\right) \\ k_3 \tan\left(\frac{\phi}{2}\right) & 0 & -k_1 \tan\left(\frac{\phi}{2}\right) \\ -k_2 \tan\left(\frac{\phi}{2}\right) & k_1 \tan\left(\frac{\phi}{2}\right) & 0 \end{bmatrix}.$$

$$I - A = \begin{bmatrix} 1 & k_3 \tan\left(\frac{\phi}{2}\right) & -k_2 \tan\left(\frac{\phi}{2}\right) \\ -k_3 \tan\left(\frac{\phi}{2}\right) & 1 & k_1 \tan\left(\frac{\phi}{2}\right) \\ k_2 \tan\left(\frac{\phi}{2}\right) & -k_1 \tan\left(\frac{\phi}{2}\right) & 1 \end{bmatrix}$$

$$(I-A)^{-1} = \frac{\operatorname{adj}(I-A)}{\det(I-A)}$$

$$\det(I-A) = k_1^2\tan^2\left(\frac{\phi}{2}\right) + 1 + k_3^2\tan^2\left(\frac{\phi}{2}\right) + k_1 k_2 k_3 \tan^3\frac{\phi}{2}$$

$$- k_1 k_2 k_3 \tan^3\frac{\phi}{2} + k_2^2 \tan^2\frac{\phi}{2}$$

$$= 1 + \tan^2\frac{\phi}{2}\left(k_1^2 + k_2^2 + k_3^2\right)$$

$$\Rightarrow (I-A)^{-1} = \frac{\begin{bmatrix} \left(1 + k_1^2\tan^2\left(\frac{\phi}{2}\right)\right) & \tan\frac{\phi}{2}\left(k_3 + k_1 k_2 \tan\left(\frac{\phi}{2}\right)\right) & \tan\frac{\phi}{2}\left(k_3 k_1 \tan \ldots - k_2\right) \\ \tan\frac{\phi}{2}\left(-k_3 + k_1 k_2 \tan\frac{\phi}{2}\right) & 1 + k_2^2\tan^2\frac{\phi}{2} & \tan\frac{\phi}{2}\left(k_1 + k_2 k_3 \tan\frac{\phi}{2} \ldots\right) \\ \tan\frac{\phi}{2}\left(k_2 + k_1 k_3 \tan\frac{\phi}{2}\right) & \tan\frac{\phi}{2}\left(-k_1 + k_2 k_3 \tan\frac{\phi}{2}\right) & 1 + k_3^2\tan^2\frac{\phi}{2} \end{bmatrix}}{1 + \tan^2\frac{\phi}{2}\left(k_1^2 + k_2^2 + k_3^2\right)}$$

This value when substituted in $2(I-A)^{-1} - I$ gives R_2.

This is not valid for $\tan\frac{\phi}{2}$ tending to ∞.

This happens at $\phi = 2 \times \frac{\pi}{2} = \pi$ or $2 \times \frac{\pi}{2} = -\pi$