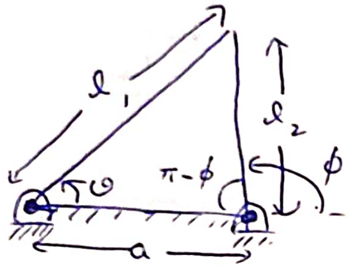
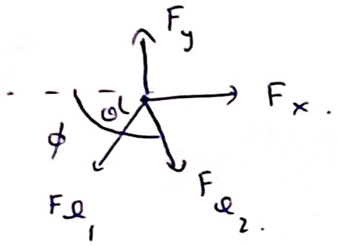
(1)



$$\cos\theta = \frac{l_1^2 + a^2 - l_2^2}{2 l_1 a}$$

$$\cos \pi - \phi = \frac{l_2^2 + a^2 - l_1^2}{2 l_2 a}.$$



$$\Rightarrow \cos\phi = \frac{l_1^2 - a^2 - l_2^2}{2 l_2 a}$$

$$F_x = F_{l_1}\left(\frac{l_1^2 + a^2 - l_2^2}{2 l_1 a}\right) + \left(\frac{l_1^2 \cancel{+} a^2 - l_2^2}{2 l_2 a.}\right) \times F_{l_2}$$

$$F_y = F_{l_1}\left(1 - \left(\frac{l_1^2 + a^2 - l_2^2}{2 l_1 a}\right)^2\right)^{1/2} + F_{l_2}\left(1 - \left(\frac{l_1^2 - a^2 - l_2^2}{2 l_2 a}\right)^2\right)^{1/2}$$

$$\Rightarrow \underbrace{\begin{bmatrix} \dfrac{l_1^2 + a^2 - l_2^2}{2 l_1 a} & \dfrac{l_1^2 - a^2 - l_2^2}{2 l_2 a.} \\ \left(1 - \left(\dfrac{l_1^2 + a^2 - l_2^2}{2 l_1 a}\right)^2\right)^{1/2} & \left(1 - \left(\dfrac{l_1^2 - a^2 - l_2^2}{2 l_2 a}\right)^2\right)^{1/2} \end{bmatrix}}_{T} \underbrace{\begin{bmatrix} F_{l_1} \\ F_{l_2} \end{bmatrix}}_{F_l} = \underbrace{\begin{bmatrix} F_x \\ F_y \end{bmatrix}}_{F_x}$$

When the transformation matrix becomes non-invertible then a given $F = [F_x \ F_y]^T$ cannot be supported. This happens when T is not defined or its determinant $= 0$.

T is not defined for $l_1 = 0$ or $l_2 = 0$ or $a = 0$.

T's determinant $= 0$

$$(l_1^2 + a^2 - l_2^2)\left((2l_2 a)^2 - (l_1^2 - a^2 - l_2^2)^2\right)^{1/2}$$

$$= (l_1^2 - a^2 - l_2^2)\left((2l_1 a)^2 - (l_1^2 + a^2 - l_2^2)^2\right)^{1/2}$$

$$\Rightarrow (l_1^2 + a^2 - l_2^2)(2l_2 a + a^2 + l_2^2 - l_1^2)^{1/2} \cancel{(\;)}$$

$$(-a^2 - l_2^2 + 2l_2 a + l_1^2)^{1/2}$$

$$= (l_1^2 - a^2 - l_2^2)(2l_1 a + l_1^2 + a^2 - l_2^2)^{1/2}$$

$$(2l_1 a - l_1^2 - a^2 + l_2^2)^{1/2}$$

$$\Rightarrow (l_1^2 + a^2 - l_2^2)\left((l_2 + a)^2 - l_1^2\right)^{1/2}\left(-(l_2 - a)^2 + l_1^2\right)^{1/2}$$

$$= (l_1^2 - a^2 - l_2^2)\left((l_1 + a)^2 - l_2^2\right)$$

$$\left(l_2^2 - (l_1 - a)^2\right)^{1/2}$$

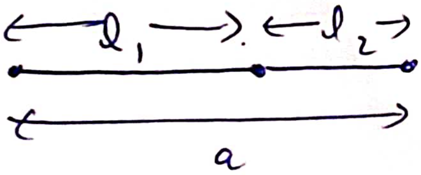
This happens when

~~$l_2 + a = l_1$~~, $l_1 + l_2 = a$

or $l_1 + a = l_2$

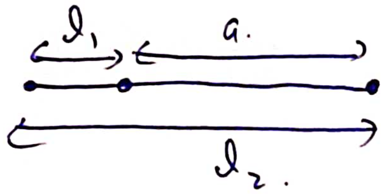
or $l_2 + a = l_1$

Case I:



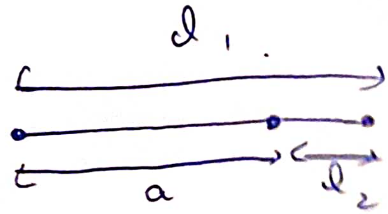
$l_1 + l_2 = a$

Case II



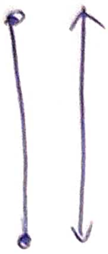
$l_1 + a = l_2$

Case III



$l_2 + a = l_1$

This is also true when any of $a, \cancel{d_1, d_2} = 0$ which is a subset of the first three conditions



$d_1 = d_2$ → case 4.

$a = 0$

(2.) (a) $\begin{vmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)^2 = 0.$

$\Rightarrow \lambda = 1, 1.$

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{v} = 1 \times \begin{bmatrix} a \\ b \end{bmatrix}.$$

$\Rightarrow a = a$ & $b = b$.

Hence there are no constraints on the eigenvector. Thus every vector $[a, b]^T$ for all $a, b \in \mathbb{R}$ is an eigenvector and $\lambda = 1$ is the eigenvalue.

(b.) $\begin{vmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)^2 = 0.$

$\Rightarrow \lambda = 1, 1$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 1 \times \begin{bmatrix} a \\ b \end{bmatrix}$$

$\Rightarrow \quad a + b = a \quad \& \quad b = b.$

$\Rightarrow \quad b = 0.$

$\Rightarrow$ $[a, 0]$ is an eigenvector for all $a \in \mathbb{R}$ and $\lambda = 1$ is the corresponding eigenvalue.

(3.) (a.) For a skew symmetric matrix,

$$Ax = \lambda x \Rightarrow \bar{x}^T A x = \lambda \bar{x}^T x = \lambda \|x\|^2.$$

$$(\bar{x}^T A)(x) = x^T (\bar{x}^T A)^T$$

as $\bar{x}^T A$ is a row vector & x is a column vector.

$$\Rightarrow \bar{x}^T A x = x^T A^T \bar{x} = -x^T A \bar{x}$$

$$\Rightarrow \bar{x}^T A x = \boxed{-x^T A \bar{x} = \lambda \bar{x}^T x}.$$

$\cancel{\Rightarrow x^T A \bar{x} = \lambda \|x\|^2}$.

Taking conjugate on both sides $\Rightarrow \bar{x}^T A x = -\bar{\lambda} \bar{x}^T x$ as A is real.

$$= -\bar{\lambda} \|x\|^2$$

$$\Rightarrow \lambda = -\bar{\lambda} \Rightarrow \lambda = 0 \text{ or complex}$$

Since λ & $\bar{\lambda}$ are ~~eigen~~ both eigenvalues $\Rightarrow$ for a 3×3 matrix the eigenvalues have to be $ci, -ci, 0$ or $0, 0, 0$. Thus the rank of a 3×3 real matrix is 2 or when it is a zero 3×3 matrix, it is 0.

(b) $$\begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$\Rightarrow \quad -cy + bz = 0.$$
$$cx - az = 0.$$
$$-bx + ay = 0.$$

$$\Rightarrow \quad x = \frac{a}{b} y. \qquad z = \frac{c}{b} y. \quad \Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{a}{b} \\ 1 \\ \frac{c}{b} \end{bmatrix} y.$$

Nullspace of the matrix is given by $\begin{bmatrix} \frac{a}{b} \\ 1 \\ \frac{c}{b} \end{bmatrix}$ $\forall\, a, b, c \in \mathbb{R}$.

4.) (a) A similarity transform is an operation on a matrix performed to change the basis of the matrix without changing properties such as the determinant

$$A = P^{-1} B P.$$

A is said to be a similar matrix to B.

P first changes the basis an then B performs the transformation P^{-1} reverts the basis geometrically

(b). Let λ be an eigenvalue ~~of~~ A and v be the corresponding eigenvector of a rank 2 tensor A,

$$\Rightarrow \quad Av = \lambda v.$$

~~Post~~ Pre Multiplying v by P on both sides $\Rightarrow$ $APv = \lambda Pv.$

Pre - multiplying LHS & RHS by P^{-1}

$$\Rightarrow \quad P^{-1}APv = \lambda P^{-1}Pv = \lambda v$$

Thus the eigenvalues remain the same

$$\cancel{Av = \lambda v}.$$

(5.) (a) $\begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & -2 \\ 0 & 1 & -3 \end{bmatrix} \rightarrow A.$

$|A-\lambda I| = 0 \Rightarrow \begin{vmatrix} -\lambda & 0 & -1 \\ 1 & -\lambda & -2 \\ 0 & 1 & -3-\lambda \end{vmatrix} = 0.$

$\Rightarrow \quad \cancel{\lambda^2 + 2\lambda} \; 1 + 2\lambda = -(3+\lambda)\lambda^2.$

$\Rightarrow \quad \lambda^3 + 3\lambda^2 + 2\lambda + 1 = 0.$

This resembly

$\lambda^3 - tr(A)\lambda^2 + 2\lambda + (-1)^3 \det A = 0$

$\Rightarrow$ For $n \times n \Rightarrow \lambda^n - tr(A)\lambda^{n-1} + \dots + (-1)^n \det A =$

(b.) The eigenvalues for a matrix and its Transpose are the same.

$AA^T x = A\lambda x = \lambda A x = \lambda^2 x.$

Similarly $A^T A x = A^T \lambda x = \lambda A^T x = \lambda^2 x.$

Thus the eigen value of AA^T & $A^T A$ is the square of the eigenvalue of A.