13-10-2020 DIY Lecture 16 Assignment S. Tarun Prasad

(15) ME17B114

(1.) Properties of $g = J_{vp}^T J_{vp}$:

→ It is a square matrix and is diagonizable.

→ $\lambda_i \geq 0$ where λ_i is an eigenvalue of g.

→ g has linearly independent eigen-vectors.

→ g is a symmetric matrix.

→ g can be ~~...~~ expressed as $x^T M x \geq 0 \;\forall\; x \in R^{\dim(g)}$ and thus is positive semi definite.

(2.)(a) To prove: $o_i^* \cdot o_j^* = \delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$

Here o_i^*, o_j^* are the eigenvectors of given matrix 'g'.

As g is a symmetric matrix $\Rightarrow g^T = g$

Let λ_i & λ_j be the corresponding eigenvalues of 'g' for eigenvectors o_i^* & o_j^* respectively.

$\Rightarrow \quad g\, o_i^* = \lambda_i\, o_i^* \rightarrow (1)$

$g\, o_j^* = \lambda_j\, o_j^* \rightarrow (2)$

$g\, o_i^* \cdot o_j^* = (\lambda_i o_i^*) \cdot o_j^* = (g\, o_i^*)^T o_j^*$

$\Rightarrow (\lambda_i o_i^*) \cdot o_j^* = o_i^{*T} g^T o_j^* = o_i^{*T} g\, o_j^* = \lambda_j\, o_i^T o_j^*$

$\Rightarrow \lambda_i\, o_i^* \cdot o_j^* = \lambda_j\, o_i^* \cdot o_j^* \rightarrow (3)$

For ③ to be satisfied either,

$\lambda_i = \lambda_j$ or $o_i^* \cdot o_j^* = 0$.

When $i = j \Rightarrow \lambda_i = \lambda_j$ while $o_i^* \cdot o_i^* = 1$.

When $i \neq j$ $o_i^* \cdot o_j^* = 0$.

$$\Rightarrow o_i^* \cdot o_j^* = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases} = \delta_{ij}.$$

(b.) To prove : $Vp_i^* \cdot Vp_j^* = 0$ when $i \neq j$

Here $Vp_i^* = g_{vp} o_i^*$ & $Vp_j^* = g_{vp} o_j^*$

$$Vp_i^* \cdot Vp_j^* = (g_{vp} o_i^*) \cdot (g_{vp} o_j^*).$$

$$= (g_{vp} o_i^*)^T (g_{vp} o_j^*).$$

$$= o_i^{*T} g_{vp}^T g_{vp} o_j^* = o_i^{*T} g\, o_j^*$$

$$= \lambda_j (o_i^* \cdot o_j^*).$$

From part (a) we have proved that

$o_i^* \cdot o_j^* = 0$ when $i \neq j$.

Thus $Vp_i^* \cdot Vp_j^* = 0$ when $i \neq j$.

(3.)

To prove $\frac{\partial}{\partial \theta}\left(\theta^T g \theta + \lambda(1-\theta^T\theta)\right) = 0$

$\theta^T\theta = \sum_{i=1}^{n} \theta_i^2$ where g is $n \times n$ & θ is $n \times 1$

& $\theta^T g \theta = \sum_{i=1}^{n}\sum_{j=1}^{n} \theta_i g_{ij} \theta_j$

$\Rightarrow \frac{\partial}{\partial \theta_k}\left[\sum_{i=1}^{n}\sum_{j=1}^{n} \theta_i g_{ij}\theta_j + \lambda - \lambda \sum_{i=1}^{n}\sum_{j=1}^{n} \theta_i^2\right] = 0$

$\Rightarrow \sum_{i=1}^{n}\sum_{j=1}^{n} g_{ij}\frac{\partial}{\partial \theta_k}(\theta_i\theta_j) + \underbrace{\frac{\partial \lambda}{\partial \theta_k}}_{0} - \lambda\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial}{\partial \theta_k}(\theta_i^2) = 0$

$\Rightarrow \sum_{j=1}^{n} g_{kj}\theta_j + \sum_{i=1}^{n} g_{ik}\theta_i - 2\lambda\theta_k = 0$

as g is symmetric

$\Rightarrow 2\left(\sum_{i=1}^{n} g_{ik}\theta_i - \lambda\theta_k\right) = 0$

L.H.S is equivalent to $g\theta - \lambda\theta = 0$.

$\Rightarrow \frac{\partial}{\partial \theta}\left(\theta^T g \theta + \lambda(1-\theta^T\theta)\right) = 0$.