Indian Institute of Technology Madras

Department of Applied Mechanics

Nonlinear Dynamics (AM5650)

Take-home end semester Exam

Assigned: 10:00AM, 26th May 2020. Due: 10:00AM, 2nd June 2020

Due: in 7 days | Jan-May 2020 | Max mark: 60

Instructions:

(i) You may consult any notes, homework solutions or text books available online or in printed form. Identify the resource material that you used for each problem at the end of the work for each problem.

(ii) Please write the answers to the following questions, scan or take a photograph of your work and upload on to moodle. Each problem solution should be uploaded as a separate pdf file.

(iii) All questions carry equal marks.

(iv) It is expected that you will do the work yourself without consulting any other person, either online or offline. Please hand-write and sign the following honour code on top of your work. "The work being submitted is my own work. I have not sought the help of any person in doing this work." Work submitted without the signed honour statement will not be graded.

(v) An oral examination may be conducted to judge your understanding of the solutions and ascertain completeness of this evaluation.

(vi) Any transgression of the above instructions could result in serious ramifications including a fail grade in this and other courses.

1. For small ϵ, determine two terms in the expansion of each root of the following equations:

   (a) $x^3 - (2+\epsilon)\,x^2 - (1-\epsilon)\,x + 2 + 3\epsilon = 0$

   (b) $x^3 - (3+\epsilon)\,x - 2 + \epsilon = 0$

   (c) $\epsilon\,(x^3 + x^2) + 4x^2 - 3x - 1 = 0$

2. Consider the motion of a particle on a rotating parabola. It is governed by

$$\left(1 + 4p^2x^2\right)\ddot{x} + \Lambda x + 4p^2\dot{x}^2x = 0$$

   Here p and Λ are constants. Show that for $\Lambda > 0$, the motion is bounded. (Hint: see section 2.4.3 in Nayfeh and Mook). Let $\Lambda = \omega_0^2$ and introduce the scaling $x = \epsilon^{1/2}u$, and show that $u\,(t)$ is governed by

$$\ddot{u} + \omega_0^2 u = -4p^2\epsilon\left(u^2\ddot{u} + u\dot{u}^2\right), \qquad \epsilon \ll 1$$

(a) Determine a two-term straight forward expansion and discuss its uniformity.

(b) Determine a first-order uniform expansion using the Poincaré-Lindstedt technique.

(c) Use the method of multiple scales to determine a first-order uniform expansion.

3. Consider the cubic differential equation

$$\dot{x} = c + dx - x^3 = F(c, d, x)$$

which depends on two real parameters c and d. Discuss the equilibrium points of the system, and find the bifurcation values of the parameters c and d where a change in number of solutions takes place. Show the bifurcation set in (c, d) plane and show the qualitative/representative phase portraits for the system. Make sure to identify key changes in phase portraits, e.g., saddle-node, pitchfork bifurcations.

4. (a) Consider the first-order time periodic systems $\dot{x} = a(t)x$. [where $a(t) = a(t+T)$, $T > 0$]. Consider two different cases for $a(t)$ given by

   (i) $a(t) = 3b + ct, \quad (0 < t < T)$

   (ii) $a(t) = \begin{cases} 0 & (0 < t < T/2) \\ b & (T/2 < t < T) \end{cases}$

   Plot the function $a(t)$ in each case. Then, find the period propagator K for each of the problems and determine the stability of the equilibrium point for all real values of constants b and c.

   (b) Consider the period-1 differential equation

$$\dot{x} = -x + c(t), \qquad c(t) = c(t+1)$$

   where $c(t)$ is an arbitrary continuous function of period-1. Show that this system has a unique periodic solution. Use Fourier series to develop an expression for the periodic solution.

   (c) A model for a population that becomes susceptible to epidemics can be constructed as follows. The population size $p(t)$ is initially governed by $\dot{p} = ap - bp^2$ [Model 1] and it grows to a certain value Q, $Q < a/b$. At this population level, the epidemic strikes and the population dynamics is governed by $\dot{p} = Ap - Bp^2$ [Model 2], where $Q > A/B$. What is the significance of requirements, $Q < a/b$ and $Q > A/B$? The population now falls to the value q, where $A/B < q < Q$. At this point in time, the epidemic ceases and the population growth is governed by Model 1. This cycle is repeated indefinitely.

(i) Sketch curves in (t, p) plane to illustrate the fluctuations in population with time. Show that the time T_1 for the population to increase from q to Q is given by

$$T_1 = \frac{1}{a} \ln \frac{Q\,(a - bq)}{q\,(a - bQ)}$$

(ii) Find the time T_2 taken for the population to fall from Q to q under the influence of Model 2, and deduce the period of the the population cycle.

5. Consider the following two systems. (i) Show that the systems undergo Hopf bifurcation at $\mu = 0$. (ii) Then, identify whether the system exhibits sub-critical or super-critical Hopf bifurcation. (iii) Finally, for each of the two cases, draw two phase diagrams, one each for a value of μ less than and greater than the critical value of μ at which the bifurcation occurs.

   (a) $\dot{x} = y + y^2 x$; $\dot{y} = -x + \mu y + x^2 y$

   (b) $\dot{x} = -y - x^3$; $\dot{y} = x + \mu y - y^3$

6. Consider the iterated map $x_{n+1} = r \sin^2 \pi x_n$. (i) Compute the bifurcation diagram, the fixed point solution (x^*) plotted as a function of parameter r. (ii) Indicate the superstable equilibrium points on the bifurcation diagram. (iii) Then, numerically calculate the critical value of r where period-2 fixed point is observed. (iv) At this value of r, graph the function $f(x; r)$ and infer why the bifurcation to a period-2 solution occurs at this value of r. (v) Calculate the Lyapunov exponent as a function of r for $0 < r < 1$ using the code provided. Comment on the value of r when chaos will be observed. You may start with the Matlab programs provided to you for the logistic map problem.

Best wishes. Stay safe and stay healthy.