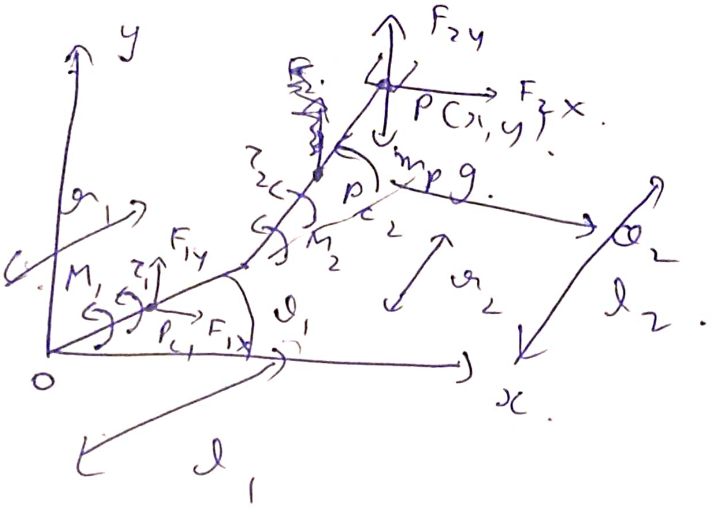
Assignment 6- Question 1

(1.)



Lagrangian equation of motion

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = Q^{nc}.$$

$$L = T - V.$$

Calculation of T:

Link 1: Only rotates about O with $\dot{\theta}_1$

$$T_1 = \frac{I_0 \dot{\theta}_1^2}{2} = \frac{(I_{c_1} + m_1 r_1^2)\dot{\theta}_1^2}{2}.$$

Link 2.

$$\vec{P}_{c_2} = \left(l_1\cos\theta_1 + \cancel{\tfrac{l_2}{2}}^{\,r_2}\cos(\theta_1+\theta_2)\right)\hat{i} + \left(l_2\sin\theta_1 + \cancel{\tfrac{l_2}{2}}^{\,r_2}\sin(\theta_1+\theta_2)\right)\hat{j}$$

$$\vec{V}_{c_2} = -\left(l_1\sin\theta_1\,\dot\theta_1 + \cancel{\tfrac{l_2}{2}}^{\,r_2}\sin(\theta_1+\theta_2)(\dot\theta_1+\dot\theta_2)\right)\hat{i}$$
$$+ \left(l_1\cos\theta_1\,\dot\theta_1 + \cancel{\tfrac{l_2}{2}}^{\,r_2}\cos(\theta_1+\theta_2)(\dot\theta_1+\dot\theta_2)\right)\hat{j}.$$

$$|\vec{V}_{c_2}|^2 = (l_1\dot\theta_1)^2 + \left(\cancel{\tfrac{l_2}{2}}^{\,r_2}(\dot\theta_1+\dot\theta_2)\right)^2$$
$$+ 2l_1 r_2(\dot\theta_1+\dot\theta_2)\left(c\theta_1 c\theta_{12} - s\theta_1 s\theta_{12}\right)\dot\theta_1$$

$$\Rightarrow |\vec{V}_{c_2}|^2 = (l_1\dot\theta_1)^2 + r_2^2(\dot\theta_1+\dot\theta_2)^2 + 2l_1 r_2(\dot\theta_1+\dot\theta_2)\cos\theta_2\,\dot\theta_1$$

Link 2 translates with the above velocity as well as rotates with $(\dot\theta_1+\dot\theta_2)$ about P_{c_2}.

$$\Rightarrow T_2 = \frac{1}{2}m_2\left[(l_1\dot\theta_1)^2 + (r_2(\dot\theta_1+\dot\theta_2))^2 + 2l_1 r_2(\dot\theta_1+\dot\theta_2)\dot\theta_1\cos\theta_2\right]$$
$$+ \frac{1}{2}I_{c_2}(\dot\theta_1+\dot\theta_2)^2$$

Payloads:-

The link has a velocity of

$$(l_1\dot\theta_1)^2 + (l_2(\dot\theta_1+\dot\theta_2))^2 + 2l_1 l_2(\dot\theta_1+\dot\theta_2)\cos\theta_2\,\dot\theta_1$$

$$T_p = \frac{1}{2}m_p\left((l_1\dot\theta_1)^2 + (l_2(\dot\theta_1+\dot\theta_2))^2 + 2l_1 l_2(\dot\theta_1+\dot\theta_2)\cos\theta_2\,\dot\theta_1\right).$$

$$T = \frac{I_{c_1}\dot{\theta}_1^2}{2} + \frac{I_{c_2}(\dot{\theta}_1+\dot{\theta}_2)^2}{2}$$

$$+ \frac{m_1 r_1^2}{2}\dot{\theta}_1^2 + \frac{m_2}{2}\Big[(l_1\dot{\theta}_1)^2 + (r_2(\dot{\theta}_1+\dot{\theta}_2))^2 + 2l_1 r_2\cos\theta_2 (\dot{\theta}_1+\dot{\theta}_2)\dot{\theta}_1\Big]$$

$$+ \frac{m_p}{2}\Big[(l_1\dot{\theta}_1)^2 + l_2^2(\dot{\theta}_1+\dot{\theta}_2)^2 + 2l_1 l_2\cos\theta_2(\dot{\theta}_1+\dot{\theta}_2)\dot{\theta}_1\Big]$$

Potential energy

$$V = g\Big(m_1 r_1\sin\theta_1 + m_2(l_1\sin\theta_1 + r_2\sin(\theta_1+\theta_2)) + m_p(l_1\sin\theta_1 + l_2\sin(\theta_1+\theta_2))\Big).$$

$$\Rightarrow L = \frac{I_{c_1}\dot{\theta}_1^2}{2} + \frac{I_{c_2}(\dot{\theta}_1+\dot{\theta}_2)^2}{2} + \frac{m_1 r_1^2\dot{\theta}_1^2}{2}$$

$$+ \frac{m_2}{2}\Big((l_1\dot{\theta}_1)^2 + (r_2(\dot{\theta}_1+\dot{\theta}_2))^2 + 2l_1 r_2\cos\theta_2(\dot{\theta}_1+\dot{\theta}_2)(\dot{\theta}_1)\Big)$$

$$+ \frac{m_p}{2}\Big((l_1\dot{\theta}_1)^2 + (l_2(\dot{\theta}_1+\dot{\theta}_2))^2 + 2l_1 l_2\cos\theta_2(\dot{\theta}_1+\dot{\theta}_2)\dot{\theta}_1\Big)$$

Thus we get

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{q}}\right) - \left(\frac{\partial L}{\partial q}\right)$$ given by $$\frac{\partial L}{\partial \dot{q}} = \left[\frac{\partial L}{\partial \dot{\theta}_1}, \frac{\partial L}{\partial \dot{\theta}_2}\right]$$

and $$\frac{\partial L}{\partial q} = \left[\frac{\partial L}{\partial \theta_1}, \frac{\partial L}{\partial \theta_2}\right]$$

as,

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{bmatrix}\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} - (m_2 l_1 r_2 \sin\theta_2 + m_p l_1 l_2 \sin\theta_2)\begin{bmatrix} \dot{\theta}_2 & \dot{\theta}_2 + \dot{\theta}_1 \\ -\dot{\theta}_1 & 0 \end{bmatrix}\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

$$+ g\left(l_1 \cos\theta_1 + r_2 \cos(\theta_2 + \theta_1)\right)(m_2 + m_p)$$

where $m_{11}, m_{12}, m_{12}, m_{22}$ are as per convention.

$$P^{NC} = \begin{bmatrix} F_{1x} \\ F_{1y} \end{bmatrix} \cdot \begin{bmatrix} -r_1 \dot{\theta}_1 \sin\theta_1 \\ r_1 \dot{\theta}_1 \cos\theta_1 \end{bmatrix} + \begin{bmatrix} F_{2x} \\ F_{2y} \end{bmatrix} \cdot \begin{bmatrix} -l_1 \dot{\theta}_1 \sin\theta_1 - l_2 \sin(\theta_1+\theta_2)(\dot{\theta}_1 + \dot{\theta}_2) \\ l_1 \dot{\theta}_1 \cos\theta_1 + l_2 \cos(\theta_1+\theta_2)(\dot{\theta}_1 + \dot{\theta}_2) \end{bmatrix}$$

$$+ \begin{bmatrix} M_1 + \tau_1 - \tau_2 \\ M_2 + \tau_2 \end{bmatrix} \cdot \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix}$$

$$= \dot{\theta}_1 \Big(- F_{1x} r_1 \sin\theta_1 + F_{1y} r_1 \cos\theta_1 - F_{2x}\left(l_1 \sin\theta_1 + l_2 \sin(\theta_1+\theta_2)\right) + F_{2y}\left(l_1 \cos\theta_1 + l_2 \cos(\theta_1+\theta_2)\right) + (M_1 + M_2 + \tau_1\Big)$$

$$+ \dot{\theta}_2 \left(- l_2 \sin(\theta_1+\theta_2) + l_2 \cos(\theta_1+\theta_2) \right) + (M_2 + \tau_2)$$

$$\Rightarrow Q^{NC} = \begin{bmatrix} -F_{1x} r_1 \sin\theta_1 + F_{1y} r_1 \cos\theta_1 - F_{2x}\left(l_2 \sin(\theta_1+\theta_2) + l_1 \sin\theta_1\right) + F_{2y}\left(l_2 \cos(\theta_1+\theta_2) + l_1 \cos\theta_1\right) + (M_1 + M_2 + \tau) \\ \left(-F_{2x}\, l_2 \sin(\theta_1+\theta_2)\right) + F_{2y}\left(l_2 \cos(\theta_1+\theta_2)\right) + \left(M_2 + \tau_2\right) \end{bmatrix}$$

Thus we have obtained all the terms in the equation of motion given by

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = Q^{NC}$$