Parameter estimation Assignment

(1) Let $(X_1, X_2, --)$ be a random sample of size n taken from a Normal population with parameters: mean $= \theta_1$ and variance $= \theta_2$. Find maximum likelihood estimates of these two parameters.

pdf of normal distribution -

$$f(x) = \frac{1}{\sqrt{2\pi}\sqrt{\theta_2}} e^{-1/2\left(\frac{x-\theta_1}{\sqrt{\theta_2}}\right)^2}$$

where $\theta_2 = \sigma^2$
$\theta_1 = \mu$

according to question $x_1, x_2, -- x_n$ are random values from the distribution which makes likelihood function as follows -

$$L = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\theta_2}} e^{-1/2 \frac{(x_i-\theta_1)^2}{\theta_2}}$$

Taking log on both sides -

$$\log(L) = \log\left((\sqrt{2\pi\theta_2})^{-n} \prod_{i=1}^{n} e^{-1/2 \frac{(x_i-\theta_1)^2}{\theta_2}}\right)$$

$$\log(L) = \frac{-n}{2}\log(2\pi\theta_2) + \left(\frac{-1}{2\theta_2}\right)\sum_{i=1}^{n}(x_i-\theta_1)^2$$

differentiate w.r.t 'θ_1'.

$$\frac{1}{L}\frac{\partial L}{\partial\theta_1} = \frac{-1}{2\theta_2}\sum_{i=1}^{n} 2(x_i-\theta_1)(-1)$$

$$\frac{1}{L}\frac{\partial L}{\partial\theta_1} = \frac{1}{2\theta_2}\sum_{i=1}^{n} 2(x_i-\theta_1)$$

equating $\frac{\partial L}{\partial\theta_1} = 0$

$$\frac{\partial L}{\partial\theta_1} = \frac{L}{2\theta_2}\sum_{i=1}^{n} 2(x_i-\theta_1) = 0$$

either $L=0$ or $\frac{1}{2\theta_2} \sum_{i=1}^{n} \alpha(x_i-\theta_1)=0$.

$L=0$ (not possible)

$$\frac{1}{2\theta_2} \sum_{i=1}^{n} 2(x_i-\theta_1)=0$$

$$\cancel{2} \sum_{i=1}^{n} x_i = \cancel{2} \sum_{i=1}^{n} \theta_1$$

$$n\theta_1 = \sum_{i=1}^{n} x_i$$

$$\theta_1 = \frac{\sum_{i=1}^{n} x_i}{n}$$

$$\boxed{\theta_1 = \frac{1}{n} \sum_{i=1}^{n} x_i}$$

θ_1 = sample mean

now differentiate w.r.t. 'θ_2':

$$\frac{\partial L}{\partial \theta_2}\left(\frac{1}{L}\right) = \frac{-n}{2} \frac{2\pi}{2\pi\theta_2} + \sum_{i=1}^{n} (x_i-\theta_1)^2 \left(\frac{1}{2\theta_2^2}\right)$$

putting $\frac{\partial L}{\partial \theta_2}=0$

$$\frac{-n}{2\theta_2} + \sum_{i=1}^{n} (x_i-\theta_1)^2 \frac{1}{2\theta_2^2}=0$$

$$\sum_{i=1}^{n} (x_i-\theta_1)^2 = n\theta_2$$

$$\boxed{\theta_2 = \frac{1}{n} \sum_{i=1}^{n} (x_i-\theta_1)^2}$$

θ_2 = sample variance

classmate

Date

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(2) Let $x_1, x_2, \ldots x_n$ be a random sample from $B(m, \theta)$ distribution where $\theta \in (0, 1)$ is unknown and m is a known positive integer. compute value of θ using MLE.

PMF of binomial distribution -

$$P(X = k) = {}^mC_k \, \theta^k (1-\theta)^{m-k}$$

Let $x_1, x_2, \ldots x_n$ be random sample from $B(m, \theta)$ distribution where for a x_i, it represents a number of successes in ith trial.

$$L(\theta) = \prod_{i=1}^{n} {}^mC_{x_i} \, \theta^{x_i} (1-\theta)^{m-x_i}$$

Taking log on both sides

$$\log L = \log\left(\prod_{i=1}^{n} {}^mC_{x_i} \, \theta^{x_i} (1-\theta)^{m-x_i}\right)$$

$$\log L = \sum_{i=1}^{n} \left(\log {}^mC_{x_i} + x_i \log\theta + (m-x_i)\log(1-\theta)\right)$$

differentiate w.r.t. θ and equate to 0.

$$\frac{1}{L}\frac{dL}{d\theta} = \frac{1}{\theta}\sum_{i=1}^{n} x_i - \frac{1}{(1-\theta)}\sum_{i=1}^{n} (m-x_i)$$

$$\frac{\partial L}{\partial \theta} = 0$$

$$L\left(\frac{1}{\theta}\sum_{i=1}^{n} x_i - \frac{1}{1-\theta}\sum_{i=1}^{n} m-x_i\right) = 0$$

L can't be 0

$$\frac{1}{\theta}\sum_{i=1}^{n} x_i = \frac{1}{1-\theta}\sum_{i=1}^{n} m-x_i$$

$$(1-\theta)\sum_{i=1}^{n} x_i = \theta nm - \theta\sum_{i=1}^{n} x_i$$

$\theta = \frac{\sum x_i}{nm}$ where $i = 1, 2, \ldots n$

$$\boxed{\theta = \frac{1}{m}\left(\frac{1}{n}\sum x_i\right)}$$

$\theta = \frac{\text{sample mean}}{m}$

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