Differential Operator

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The operator representing the computation of a derivative,

$$\tilde{D} \equiv \frac{d}{dx}, \tag{1}$$

sometimes also called the Newton-Leibniz operator. The second derivative is then denoted $\tilde{D}^2$, the third $\tilde{D}^3$, etc. The integral is denoted $\tilde{D}^{-1}$.

The differential operator satisfies the identity

$$\left(2x - \frac{d}{dx}\right)^n 1 = H_n\left(x\right), \tag{2}$$

where $H_n(x)$ is a Hermite polynomial (Arfken 1985, p. 718), where the first few cases are given explicitly by

$$H_1(x) = 2x - \frac{\delta 1}{\delta x} \tag{3}$$
$$= 2x \tag{4}$$
$$H_2(x) = 2x(2x) - \frac{\delta(2x)}{\delta x} \tag{5}$$
$$= 4x^2 - 2 \tag{6}$$
$$H_3(x) = 2x(4x^2 - 2) - \frac{\delta(4x^2 - 2)}{\delta x} \tag{7}$$
$$= 8x^3 - 12x. \tag{8}$$

The symbol ϑ can be used to denote the operator

$$\vartheta \equiv x\frac{d}{dx} \tag{9}$$

(Bailey 1935, p. 8). A fundamental identity for this operator is given by

$$\left(x\tilde{D}\right)^n = \sum_{k=0}^{n} S(n,k)\, x^k \tilde{D}^k, \tag{10}$$

where *S(n,k)* is a Stirling number of the second kind (Roman 1984, p. 144), giving

$$\left(x\tilde{D}\right)^1 = x\tilde{D} \tag{11}$$
$$\left(x\tilde{D}\right)^2 = x\tilde{D} + x^2\tilde{D}^2 \tag{12}$$
$$\left(x\tilde{D}\right)^3 = x\tilde{D} + 3x^2\tilde{D}^2 + x^3\tilde{D}^3 \tag{13}$$
$$\left(x\tilde{D}\right)^4 = x\tilde{D} + 7x^2\tilde{D}^2 + 6x^3\tilde{D}^3 + x^4\tilde{D}^4 \tag{14}$$

and so on (OEIS A008277). Special cases include

$$\vartheta^n e^x = e^x \sum_{k=0}^{n} S(n,k) x^k \tag{15}$$

$$\vartheta^n \cos x = \cos x \sum_{k=0}^{n} (-1)^k \ S(n,2k) x^{2k} + \sin x \sum_{k=1}^{n} (-1)^k \ S(n,2k-1) x^{2k-1} \tag{16}$$

$$\vartheta^n \sin x = \cos x \sum_{k=1}^{n} (-1)^{k+1} \ S(n,2k-1) x^{2k-1} + \sin x \sum_{k=0}^{n} (-1)^k \ S(n,2k) x^{2k}. \tag{17}$$

A shifted version of the identity is given by

$$[\,(x-a)\tilde{D}]^{\,n} = \sum_{k=0}^{n} S(n,k)(x-a)^k \ \tilde{D}^k \tag{18}$$

(Roman 1984, p. 146).