Multivariable Calculus Notes

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1 Multiple Integrals

These notes were taken from chapter 2 in [1]. I don't go over the definition or construction of integrals here, but if you're interested, please read the book.

1.1 Double Integrals

We assume that a domain of integration D is always *closed*. This means that D is bounded by a closed curve and the points lying on the boundary belongs to D.

A *double integral* of a function $f(x, y)$ over the region or domain D is notated

$$\iint_D f(x, y)\, dx\, dy$$

Sometimes instead of writing $dy\, dy$, it is written ds which represents the area of a infinitesimally small subregion of D.

Some properties

1. If $f(x, y) \geq 0$, then $\iint_D f(x, y)\, ds$ is the volume of the solid bounded by the surface $z = f(x, y)$, the plane $z = 0$, and a cylindrical surface whose generators are parallel to the z-axis, while the directrix is the boundary of the domain D.

2. For two functions $f(x, y)$ and $g(x, y)$, and any constant c,

$$\iint_D [cf(x, y) + g(x, y)]\, ds = c \iint_D f(x, y)\, ds + \iint_D g(x, y)\, ds.$$

3. If a region D is divided into disjoint regions D_1 and D_2, then

$$\iint_D f(x, y)\, ds = \iint_{D_1} f(x, y)\, ds + \iint_{D_2} f(x, y)\, ds.$$

1.2 Calculating Double Integrals

In order to explicitly compute double integrals, we consider two kinds of closed regions in the xy-plane:

- *Type I region*: bounded by the continuous functions $y = \phi_1(x)$ and $y = \phi_2(x)$, and the parallel lines $x = a$ and $x = b$, where $\phi_1(x) \leq \phi_2(x)$ and $a \leq b$. The book calls this kind of region *regular in the y-direction*.

- *Type II region*: bounded by the continuous functions $x = \psi_1(y)$ and $x = \psi_2(y)$, and the parallel lines $y = c$ and $y = d$, where $\psi_2(y)$ is always to the right of $\psi_1(y)$ for all y in the domain of interest, and $c \leq d$. The book calls this kind of region *regular in the x-direction*.

If D is a type I region, then it can be computed as a *two-fold iterated integral*:

$$\iint_D f(x,y)\,dy\,dx = \int_a^b \left(\int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy \right) dx.$$

Similarly, if D is a type II region, then it can be computed by the iterated integral:

$$\iint_D f(x,y)\,dy\,dx = \int_c^d \left(\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,dx \right) dy.$$

References

[1] Piskunov, Nikolai. *Differential and integral calculus: volume 2*. Moscow: Mir Publishers, 1974.