(a) See Figure 1.

(b) $\lambda_{12} = \lambda_{23} = \lambda_{34} = .1; \lambda_{41} = .5.$

To determine λ_{14}, note that the rate the machine leaves state 1 (given that it is in state 1) is $\lambda_{14} + \lambda_{12}$ (since it leaves when the first of the two exit processes occurs). The probability that it goes to state 4 during $[t, t + \delta t]$ given that it leaves state 1 during $[t, t + \delta t]$ is

$$\frac{\lambda_{14}\delta t}{(\lambda_{14} + \lambda_{12})\delta t} = \frac{\lambda_{14}}{\lambda_{14} + \lambda_{12}} = .001$$

Therefore λ_{14} represents .1% of the probability of leaving state 1 and λ_{12} represents 99.9% of the probability of leaving state 1. Or, $\lambda_{14} = (1/999)\lambda_{12} = 0.0001001$. Similarly $\lambda_{24} = (1/9)\lambda_{23} = 0.01111$.

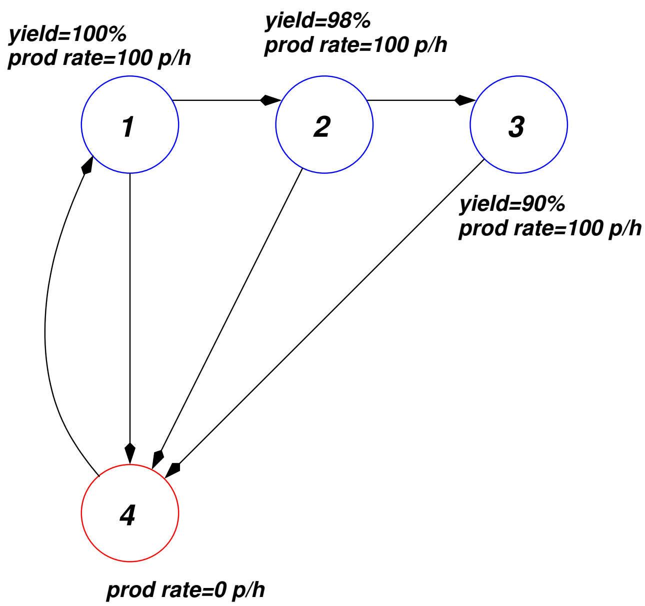

Figure 1: Problem 1a

(c) Balance equations:

$$\begin{aligned}
\lambda_{41}\mathbf{p_4} &= (\lambda_{12} + \lambda_{14})\mathbf{p_1} \\
\lambda_{12}\mathbf{p_1} &= (\lambda_{23} + \lambda_{24})\mathbf{p_2} \\
\lambda_{23}\mathbf{p_2} &= \lambda_{34}\mathbf{p_3} \\
\lambda_{14}\mathbf{p_1} + \lambda_{\mathbf{24}}\mathbf{p_2} + \lambda_{\mathbf{34}}\mathbf{p_3} &= \lambda_{41}\mathbf{p_4}
\end{aligned}$$

Normalization equation:

$$\mathbf{p_1} + \mathbf{p_2} + \mathbf{p_3} + \mathbf{p_4} = \mathbf{1}$$

Solution:

$$\begin{aligned}
\mathbf{p_2} &= \tfrac{\lambda_{34}}{\lambda_{23}}\mathbf{p_3} \\
\mathbf{p_1} &= \tfrac{(\lambda_{23}+\lambda_{24})}{\lambda_{12}}\mathbf{p_2} = \tfrac{(\boldsymbol{\lambda_{23}}+\boldsymbol{\lambda_{24}})}{\boldsymbol{\lambda_{12}}}\tfrac{\boldsymbol{\lambda_{34}}}{\boldsymbol{\lambda_{23}}}\mathbf{p_3} \\
\mathbf{p_4} &= \tfrac{(\lambda_{12}+\lambda_{14})}{\lambda_{41}}\mathbf{p_1} = \tfrac{(\boldsymbol{\lambda_{12}}+\boldsymbol{\lambda_{14}})}{\boldsymbol{\lambda_{41}}}\tfrac{(\boldsymbol{\lambda_{23}}+\boldsymbol{\lambda_{24}})}{\boldsymbol{\lambda_{12}}}\tfrac{\boldsymbol{\lambda_{34}}}{\boldsymbol{\lambda_{23}}}\mathbf{p_3} \\
1 &= \tfrac{(\lambda_{23}+\lambda_{24})}{\lambda_{12}}\tfrac{\lambda_{34}}{\lambda_{23}}\mathbf{p_3} + \tfrac{\boldsymbol{\lambda_{34}}}{\boldsymbol{\lambda_{23}}}\mathbf{p_3} + \mathbf{p_3} + \tfrac{(\boldsymbol{\lambda_{12}}+\boldsymbol{\lambda_{14}})}{\boldsymbol{\lambda_{41}}}\tfrac{(\boldsymbol{\lambda_{23}}+\boldsymbol{\lambda_{24}})}{\boldsymbol{\lambda_{12}}}\tfrac{\boldsymbol{\lambda_{34}}}{\boldsymbol{\lambda_{23}}}\mathbf{p_3}
\end{aligned}$$

or

$$\begin{aligned}
\lambda_{12}\lambda_{23}\lambda_{41} &= \lambda_{41}(\lambda_{23}+\lambda_{24})\lambda_{34}\mathbf{p_3} + \boldsymbol{\lambda_{12}}\boldsymbol{\lambda_{41}}\boldsymbol{\lambda_{34}}\mathbf{p_3} + \boldsymbol{\lambda_{12}}\boldsymbol{\lambda_{23}}\boldsymbol{\lambda_{41}}\mathbf{p_3} \\
&\quad +(\lambda_{12}+\lambda_{14})(\lambda_{23}+\lambda_{24})\lambda_{23}\mathbf{p_3} \\
&= [\lambda_{41}(\lambda_{23}+\lambda_{24})\lambda_{34} + \lambda_{12}\lambda_{41}\lambda_{34} + \lambda_{12}\lambda_{23}\lambda_{41} \\
&\quad + (\lambda_{12}+\lambda_{14})(\lambda_{23}+\lambda_{24})\lambda_{23}]\,\mathbf{p_3}
\end{aligned}$$

You could then plug the numbers in for λ_{ij} to evaluate $\mathbf{p_3}$, and then evaluate all the other probabilities from it. However, this is messy if you are not a computer.

The easy way suggested by the hint starts by guessing $\mathbf{p_3'} = \mathbf{1}$. Then we have

$$\begin{aligned}
\mathbf{p_2'} &= \tfrac{\lambda_{34}}{\lambda_{23}}\mathbf{p_3'} \\
\mathbf{p_1'} &= \tfrac{(\lambda_{23}+\lambda_{24})}{\lambda_{12}}\mathbf{p_2'} \\
\mathbf{p_4'} &= \tfrac{(\lambda_{12}+\lambda_{14})}{\lambda_{41}}\mathbf{p_1'}
\end{aligned}$$

We calculate $\nu = \mathbf{p_1'} + \mathbf{p_2'} + \mathbf{p_3'} + \mathbf{p_4'}$ and

$$\begin{aligned}
\mathbf{p_1} &= \mathbf{p_1'}/\nu \\
\mathbf{p_2} &= \mathbf{p_2'}/\nu \\
\mathbf{p_3} &= \mathbf{p_3'}/\nu \\
\mathbf{p_4} &= \mathbf{p_4'}/\nu
\end{aligned}$$

or

$$\begin{aligned}
\mathbf{p_1} &= 0.3333 \\
\mathbf{p_2} &= 0.3000 \\
\mathbf{p_3} &= 0.3000 \\
\mathbf{p_4} &= 0.0667
\end{aligned}$$

(d) The total production rate is given by $100(\mathbf{p_1} + \mathbf{p_2} + \mathbf{p_3})$. The yield is

$$\frac{1.0\mathbf{p_1} + \mathbf{.98p_2} + .9\mathbf{p_3}}{\mathbf{p_1} + \mathbf{p_2} + \mathbf{p_3}}$$

The good production rate is $100(1.0\mathbf{p_1} + \mathbf{.98p_2} + .9\mathbf{p_3})$

$$\begin{aligned} \text{total production rate} &= 93.33 \text{ parts per hour} \\ \text{yield} &= 0.9614 \\ \text{good production rate} &= 89.73 \text{ parts per hour} \end{aligned}$$

(e)

$$\begin{aligned} \mathbf{p_1} &= 0.4761 \\ \mathbf{p_2} &= 0.4285 \\ \mathbf{p_3} &= 0.0 \\ \mathbf{p_4} &= 0.0953 \end{aligned}$$

$$\begin{aligned} \text{total production rate} &= 90.47 \text{ parts per hour} \\ \text{yield} &= 0.9905 \\ \text{good production rate} &= 89.61 \text{ parts per hour} \end{aligned}$$

By avoiding a low-yield state, we increased the yield. However, we are now spending less time in up states, so the total production rate goes down, and the good production rate actually goes down slightly. Even so, this may be a good idea because we get about the same good production rate and we waste less raw material.