MIT 2.853/2.854

Introduction to Manufacturing Systems

Probability

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Trick Question

I flip a coin 100 times, and it shows heads every time.

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Question: What is the probability that it will show heads on the next flip?

Another Trick Question

I flip a coin 100 times, and it shows heads every time.

Another Trick Question

I flip a coin 100 times, and it shows heads every time.

Question: *How much would you bet* that it will show heads on the next flip?

Still Another Trick Question

I flip a coin 100 times, and it shows heads every time.

Still Another Trick Question

I flip a coin 100 times, and it shows heads every time.

Question: *What odds would you demand before you bet* that it will show heads on the next flip?

Probability and Statistics

Probability $\neq$ *Statistics*

Probability and Statistics

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Probability: mathematical theory that describes uncertainty.

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Probability: mathematical theory that describes uncertainty.

Statistics: set of techniques for extracting useful information from data.

Interpretations of probability

Frequency

The probability that the outcome of an experiment is A is P...

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if the experiment is performed a large number of times and the fraction of times that the observed outcome is A is P.

Interpretations of probability
State of belief

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State of belief

The probability that the outcome of an experiment is A is P...

if that is the opinion (ie, belief or state of mind) of an observer *before* the experiment is performed.

Interpretations of probability

Example of State of Belief: Betting odds

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Example of State of Belief: Betting odds

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if *before the experiment is performed* a risk-neutral observer would be willing to bet \$1 against more than $\$\frac{1-P}{P}$.

Interpretations of probability

Example of State of Belief: Betting odds

The probability that the outcome of an experiment is A is P ...

if *before the experiment is performed* a risk-neutral observer would be willing to bet \$1 against more than $\$\frac{1-P}{P}$.

The expected value (slide 35) of the bet is greater than

$$(1 - P) \times (-1) + (P) \times \left(\frac{1-P}{P}\right) = 0$$

Interpretations of probability

Abstract measure

The probability that the outcome of an experiment is A is $P(A)$...

Interpretations of probability

Abstract measure

The probability that the outcome of an experiment is A is $P(A)$...

if $P()$ satisfies a certain set of conditions: *the axioms of probability.*

Let U be a set of *samples* . Let $\mathcal{E}_1$, $\mathcal{E}_2$, ... be subsets of U.

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Let $\emptyset$ be the *null* (or *empty*) *set* , the set that has no elements.

Interpretations of probability

Axioms of probability

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- $0 \leq P(\mathcal{E}_i) \leq 1$

Interpretations of probability

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Axioms of probability

Let U be a set of *samples* . Let $\mathcal{E}_1$, $\mathcal{E}_2$, ... be subsets of U.

Let $\emptyset$ be the *null* (or *empty*) *set* , the set that has no elements.

- $0 \leq P(\mathcal{E}_i) \leq 1$
- $P(U) = 1$
- $P(\emptyset) = 0$
- If $\mathcal{E}_i \cap \mathcal{E}_j = \emptyset$, then $P(\mathcal{E}_i \cup \mathcal{E}_j) = P(\mathcal{E}_i) + P(\mathcal{E}_j)$

Probability Basics

Discrete Sample Space

Notation, terminology:

- ω is often used as the symbol for a generic sample.

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- ω is often used as the symbol for a generic sample.
- Subsets of U are called *events.*
- $P(\mathcal{E})$ is the *probability* of $\mathcal{E}$.

Probability Basics

Discrete Sample Space

- *Example:* Throw a single die. The possible outcomes are $\{1, 2, 3, 4, 5, 6\}$. ω can be any one of those values.

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- *Example:* Consider $n(t)$, the number of parts in inventory at time t. Then

$$\omega = \{n(1), n(2), ..., n(t), ...\}$$

is a *sample path.*

Probability Basics

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Discrete Sample Space

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$$\mathcal{E} = \{\text{There are 6 parts in the buffer at time } t = 12\}$$

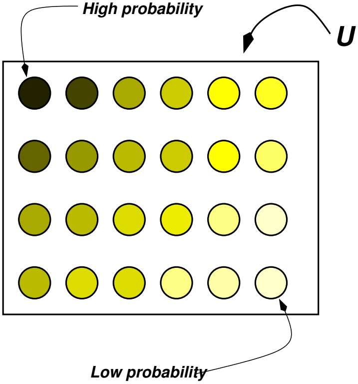
Formally, this can be written

$$\mathcal{E} = \text{the set of all } \omega \text{ such that } n(12) = 6$$

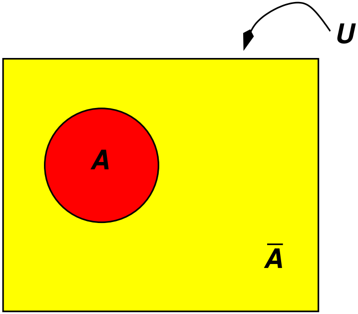
or,

$$\mathcal{E} = \{\omega | n(12) = 6\}$$

Probability Basics
Discrete Sample Space

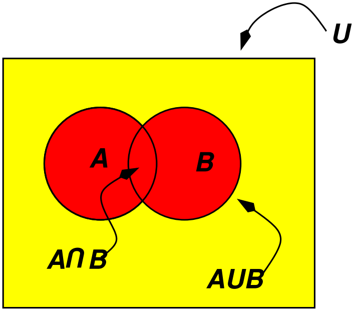


Venn diagrams



$$P(\bar{A}) = 1 - P(A)$$

Venn diagrams

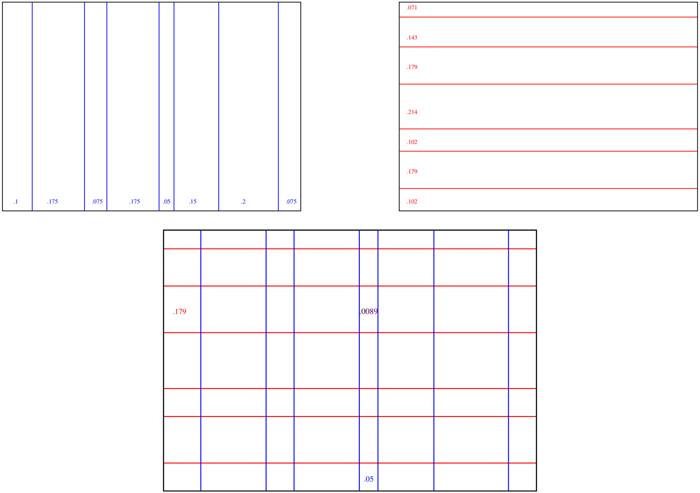


$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

A and B are *independent* if

$$P(A \cap B) = P(A)P(B).$$

grid figure to illustrate independence



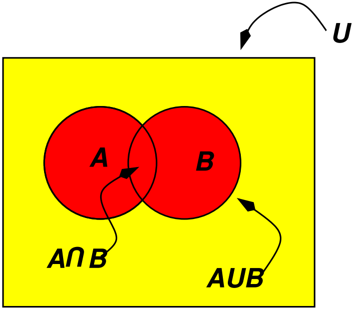
.1
.175
.075
.175
.05
.15
.2
.075
.071
.143
.179
.214
.102
.179
.102
.179
.0089
.05

Probability Basics

Conditional Probability

If $P(B) \neq 0$,

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$



We can also write $P(A \cap B) = P(A|B)P(B)$.

Probability Basics

Conditional Probability

$$P(A|B) = P(A \cap B)/P(B)$$

Example: Throw a die.

Probability Basics

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- A is the event of getting an odd number (1, 3, 5).

Probability Basics

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- A is the event of getting an odd number (1, 3, 5).
- B is the event of getting a number less than or equal to 3 (1, 2, 3).

Probability Basics

Conditional Probability

$$P(A|B) = P(A \cap B)/P(B)$$

Example: Throw a die.Let

- A is the event of getting an odd number (1, 3, 5).
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Then $P(A) = P(B) = 1/2$,

Probability Basics

Conditional Probability

$$P(A|B) = P(A \cap B)/P(B)$$

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- A is the event of getting an odd number (1, 3, 5).
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Then $P(A) = P(B) = 1/2, P(A \cap B) = P(1, 3) = 1/3$.

Probability Basics

Conditional Probability

$$P(A|B) = P(A \cap B)/P(B)$$

Example: Throw a die.Let

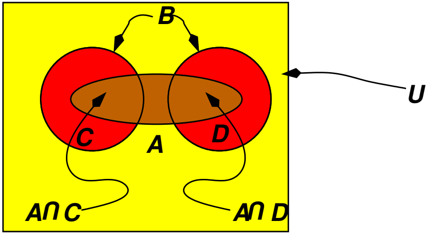
- A is the event of getting an odd number (1, 3, 5).
- B is the event of getting a number less than or equal to 3 (1, 2, 3).

Then $P(A) = P(B) = 1/2, P(A \cap B) = P(1, 3) = 1/3.$

Also, $P(A|B) = P(A \cap B)/P(B) = 2/3.$

Probability Basics

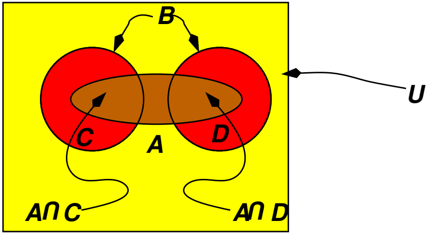
Law of Total Probability



- Let $B = C \cup D$ and assume $C \cap D = \emptyset$. Then

Probability Basics

Law of Total Probability

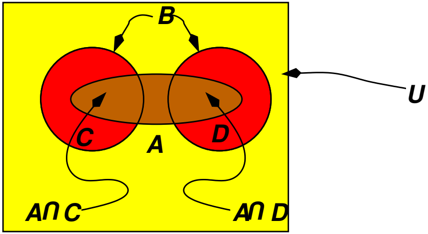


- Let $B = C \cup D$ and assume $C \cap D = \emptyset$. Then

  $$P(A|C) = \frac{P(A \cap C)}{P(C)}$$

Probability Basics

Law of Total Probability

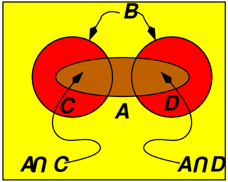


- Let $B = C \cup D$ and assume $C \cap D = \emptyset$. Then
  $P(A|C) = \dfrac{P(A \cap C)}{P(C)}$ and $P(A|D) = \dfrac{P(A \cap D)}{P(D)}$.

Probability Basics

Also,

- $P(C|B) = \dfrac{P(C \cap B)}{P(B)} = \dfrac{P(C)}{P(B)}$ because $C \cap B = C$.



Probability Basics

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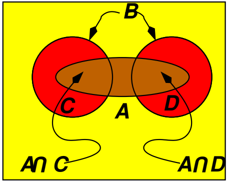
- $P(C|B) = \dfrac{P(C \cap B)}{P(B)} = \dfrac{P(C)}{P(B)}$ because $C \cap B = C$.

  Similarly, $P(D|B) = \dfrac{P(D)}{P(B)}$



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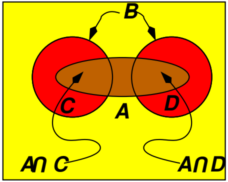

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- $A \cap B = A \cap (C \cup D) = (A \cap C) \cup (A \cap D)$

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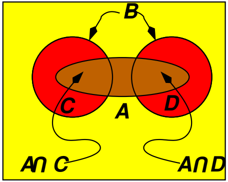

- $A \cap B = A \cap (C \cup D) = (A \cap C) \cup (A \cap D)$

- Therefore

  $P(A \cap B) = P(A \cap (C \cup D))$

Probability Basics

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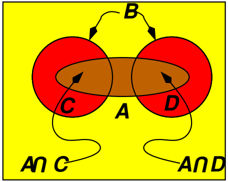

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- $A \cap B = A \cap (C \cup D) = (A \cap C) \cup (A \cap D)$

- Therefore

$$\begin{aligned} P(A \cap B) &= P(A \cap (C \cup D)) \\ &= P(A \cap C) + P(A \cap D) \end{aligned}$$

Probability Basics

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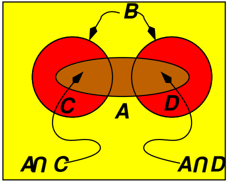

- $A \cap B = A \cap (C \cup D) = (A \cap C) \cup (A \cap D)$

- Therefore

  $P(A \cap B) = P(A \cap (C \cup D))$

  $\qquad\qquad = P(A \cap C) + P(A \cap D)$ because $(A \cap C)$ and $(A \cap D)$ are disjoint.

Probability Basics

Law of Total Probability

- Or, from the definition of conditional probability,
  $P(A|B)P(B) = P(A|C)P(C) + P(A|D)P(D)$

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  or,

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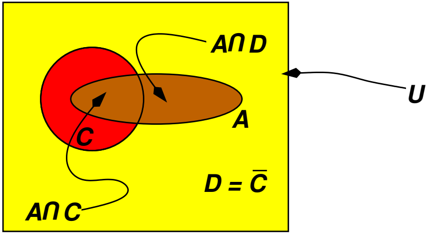
or,

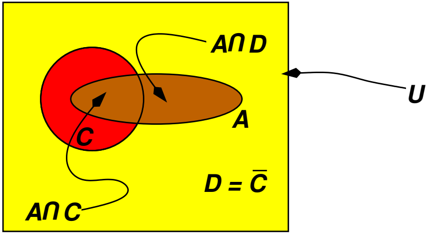
$$P(A|B) = P(A|C)P(C|B) + P(A|D)P(D|B)$$

Probability Basics

Law of Total Probability



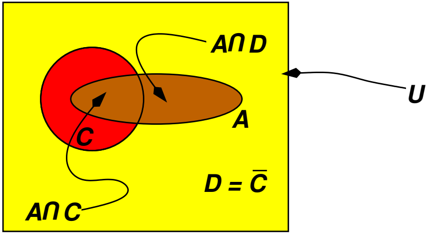
An important case is when $C \cup D = B = U$, so that $A \cap B = A$.



An important case is when $C \cup D = B = U$, so that $A \cap B = A$. Then $P(A) = P(A \cap C) + P(A \cap D)$ or

Probability Basics

Law of Total Probability

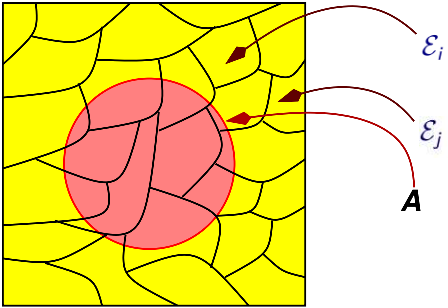


An important case is when $C \cup D = B = U$, so that $A \cap B = A$. Then $P(A) = P(A \cap C) + P(A \cap D)$ or

$$\boxed{P(A) = P(A|C)P(C) + P(A|D)P(D)}$$

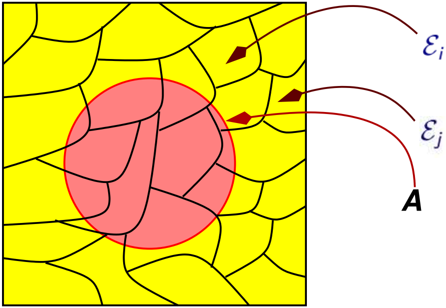
Probability Basics

Law of Total Probability



Probability Basics

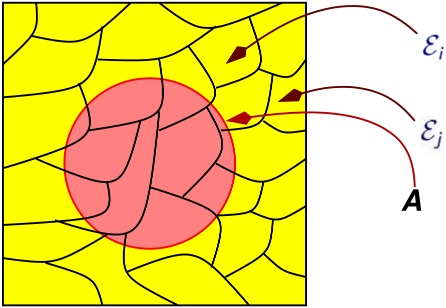
Law of Total Probability



More generally, if A and $\mathcal{E}_1, \ldots \mathcal{E}_k$ are events and

Probability Basics

Law of Total Probability



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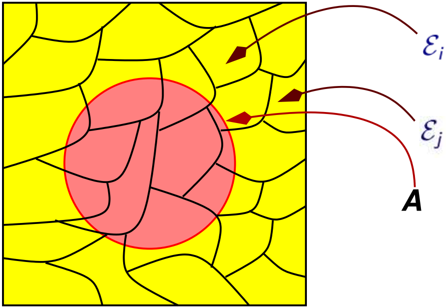
Probability Basics

Law of Total Probability



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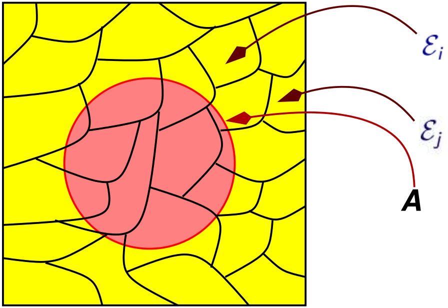
$$\mathcal{E}_i \text{ and } \mathcal{E}_j = \emptyset, \text{ for all } i \neq j$$

and

$$\bigcup_j \mathcal{E}_j = \text{ the universal set}$$

Probability Basics
Law of Total Probability



More generally, if A and $\mathcal{E}_1, \ldots \mathcal{E}_k$ are events and

$\mathcal{E}_i$ and $\mathcal{E}_j = \emptyset$, for all $i \neq j$

and

$$\bigcup_j \mathcal{E}_j = \text{ the universal set}$$

(ie, the set of $\mathcal{E}_j$ sets is *mutually exclusive* and *collectively exhaustive*) then ...

Probability Basics

Law of Total Probability

$$\sum_j P(\mathcal{E}_j) = 1$$

Probability Basics
Law of Total Probability

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and

$$\boxed{P(A) = \textstyle\sum_j P(A|\mathcal{E}_j)P(\mathcal{E}_j).}$$

Probability Basics

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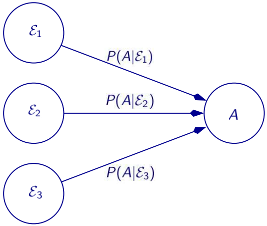

Probability Basics

Law of Total Probability

Example

$A = \{$I will have a cold tomorrow.$\}$
$\mathcal{E}_1 = \{$It is raining today.$\}$
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(Assume $\mathcal{E}_1 \cup \mathcal{E}_2 \cup \mathcal{E}_3 = U$ and $\mathcal{E}_1 \cap \mathcal{E}_2 = \mathcal{E}_1 \cap \mathcal{E}_3 = \mathcal{E}_2 \cap \mathcal{E}_3 = \emptyset$.)

Then $A \cap \mathcal{E}_1 = \{$I will have a cold tomorrow *and* it is raining today$\}$.

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And $P(A|\mathcal{E}_1)$ is the probability I will have a cold tomorrow *given* that it is raining today.

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etc.

Probability Basics

Law of Total Probability

Then

$\{$I will have a cold tomorrow.$\}=$
$\{$I will have a cold tomorrow *and* it is raining today$\}$ $\cup$
$\{$I will have a cold tomorrow *and* it is snowing today$\}$ $\cup$
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Probability Basics

Law of Total Probability

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{I will have a cold tomorrow.}=
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{I will have a cold tomorrow *and* it is sunny today}

so

P({I will have a cold tomorrow.})=
P({I will have a cold tomorrow *and* it is raining today}) $+$
P({I will have a cold tomorrow *and* it is snowing today}) $+$
P({I will have a cold tomorrow *and* it is sunny today})

Probability Basics

Law of Total Probability

$P(\{\text{I will have a cold tomorrow.}\}) =$

Probability Basics
Law of Total Probability

$P(\{\text{I will have a cold tomorrow.}\}) =$

$P(\{\text{I will have a cold tomorrow} \mid \text{it is raining today}\})P(\{\text{it is raining today}\}) +$

Probability Basics

Law of Total Probability

$P(\{\text{I will have a cold tomorrow.}\})=$

$P(\{\text{I will have a cold tomorrow} \mid \text{it is raining today}\})P(\{\text{it is raining today}\}) +$

$P(\{\text{I will have a cold tomorrow} \mid \text{it is snowing today}\})P(\{\text{it is snowing today}\}) +$

Probability Basics

Law of Total Probability

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$P(\{\text{I will have a cold tomorrow} \mid \text{it is sunny today}\})\ P(\{\text{it is sunny today}\})$

or

Probability Basics
Law of Total Probability

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or

$$P(A) = P(A|\mathcal{E}_1)P(\mathcal{E}_1) + P(A|\mathcal{E}_2)P(\mathcal{E}_2) + P(A|\mathcal{E}_3)P(\mathcal{E}_3)$$

Probability Basics

Random Variables

Let V be a vector space. Then a *random variable* X is a mapping (a function) from U to V.

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Example: V could be the real number line.

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Typical notation :

- Upper case letters (X) are usually used for random variables and corresponding lower case letters (x) are usually used for possible values of random variables.

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Typical notation :

- Upper case letters (X) are usually used for random variables and corresponding lower case letters (x) are usually used for possible values of random variables.
- Random variables ($X(\omega)$) are usually not written as functions; the argument (ω) of the random variable is usually not written. *This sometimes causes confusion.*

Flip of a Coin

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Let U=$\{$H,T$\}$. Let $\omega =$ H if we flip a coin and get heads; $\omega =$ T if we flip a coin and get tails.

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$X(H) = 1$

Assume the coin is fair. *(No tricks this time!)* Then
$P(\omega = \text{ T }) = P(X = 0) = 1/2$
$P(\omega = \text{ H }) = P(X = 1) = 1/2$

Flip of Three Coins

Let $U = \{$HHH, HHT, HTH, HTT, THH, THT, TTH, TTT$\}$.

Flip of Three Coins

Let $U = \{$HHH, HHT, HTH, HTT, THH, THT, TTH, TTT$\}$.

Let $\omega =$ HHH if we flip 3 coins and get 3 heads;

Probability Basics

Random Variables

Flip of Three Coins

Let $U = \{$HHH, HHT, HTH, HTT, THH, THT, TTH, TTT$\}$.

Let $\omega =$ HHH if we flip 3 coins and get 3 heads; $\omega =$HHT if we flip 3 coins and get 2 heads and *then* one tail, etc.

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Probability Basics

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Probability Basics

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- $P(\omega) = 1/8$ for all ω.

Let X be the *number* of heads. Then $X = 0$, 1, 2, or 3.

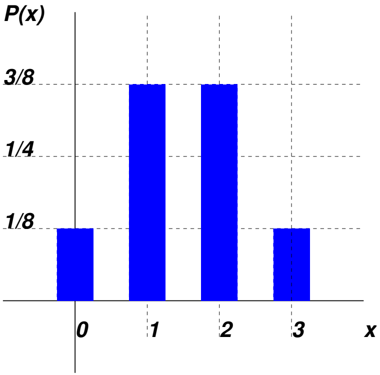
- $P(X = 0)$=1/8; $P(X = 1)$=3/8; $P(X = 2)$=3/8; $P(X = 3)$=1/8.

There are 4 distinct values of X.

Probability Basics

Probability Distributions

Let $X(\omega)$ be a random variable. Then $P(X(\omega) = x)$ is the *probability distribution* of X (usually written $P(x)$). For three coin flips:



Probability Basics

Probability Distributions

Shorthand:

- Instead of writing $P(X(\omega) = x)$, people often write $P(x)$ if the meaning is unambiguous.

Mean and Variance:

- *Mean (average):* $\bar{x} = \mu_x = E(X) = \sum_x xP(x)$

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Probability Basics

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- *Variance:* $V_x = \sigma_x^2 = E(X - \mu_x)^2 = \sum_x (x - \mu_x)^2 P(x)$
- *Standard deviation (sd):* $\sigma_x = \sqrt{V_x}$
- *Coefficient of variation (cv):* σ_x / μ_x

For three coin flips:

$$\bar{x} = 1.5$$

$$V_x = 0.75$$

$$\sigma_x = 0.866$$

$$\text{cv} = 0.577$$

- A function of a random variable is a random variable.

Probability Basics

Functions of a Random Variable

- A function of a random variable is a random variable.

- *Special case: linear function*

  For every ω, let $Y(\omega) = aX(\omega) + b$. Then

  - $\bar{Y} = a\bar{X} + b$.

- A function of a random variable is a random variable.

- *Special case: linear function*

  For every ω, let $Y(\omega) = aX(\omega) + b$. Then

  - $\bar{Y} = a\bar{X} + b$.
  - $V_Y = a^2 V_X$; $\sigma_Y = |a|\sigma_X$.

Discrete Random Variables

1. Bernoulli

Flip a biased coin.

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X^B is a *Bernoulli random variable.*

Discrete Random Variables

2. Binomial

The sum of N independent Bernoulli random variables X_i^B with the same parameter p is a *binomial* random variable X^b.

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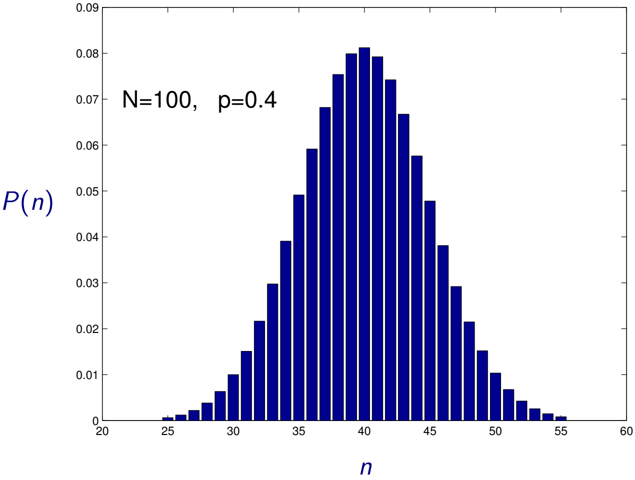
$$X^b = \sum_{i=0}^{N} X_i^B$$

$$P(X^b = x) = \frac{N!}{x!(N-x)!} p^x (1-p)^{(N-x)}$$

Discrete Random Variables

2. Binomial probability distribution



Discrete Random Variables

3. Geometric

The number of independent Bernoulli random variables X_i^B with the same parameter p tested *until the first 1 appears* is a *geometrically distributed* random variable X^g.

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The number of independent Bernoulli random variables X_i^B with the same parameter p tested *until the first 1 appears* is a *geometrically distributed* random variable X^g.

$$
\begin{array}{cccccccccc}
1 & 2 & 3 & 4 & \ldots & k-4 & k-3 & k-2 & k-1 & k \\
0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & \mathbf{1} \\
\multicolumn{10}{c}{\longleftarrow \quad k \quad \longrightarrow}
\end{array}
$$

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\longleftarrow & & & & k & & & & & \longrightarrow
\end{array}
$$

$$
\boxed{X^g = k \text{ if } X_1^B = 0,\ X_2^B = 0,\ \ldots,\ X_{k-1}^B = 0,\ X_k^B = 1}
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To calculate $P(X^g = k)$, observe that $P(X^g = 1) = p$, so $P(X^g > 1) = 1 - p$.

Discrete Random Variables

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To calculate $P(X^g = k)$, observe that $P(X^g = 1) = p$, so $P(X^g > 1) = 1 - p$. Also, observe that $\{X^g > k\}$ is a subset of $\{X^g > k - 1\}$.

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because

$$P(X^g > k | X^g > k - 1) = P(X_1^B = 0, \ldots, X_k^B = 0 \;| X_1^B = 0, \ldots, X_{k-1}^B = 0)$$

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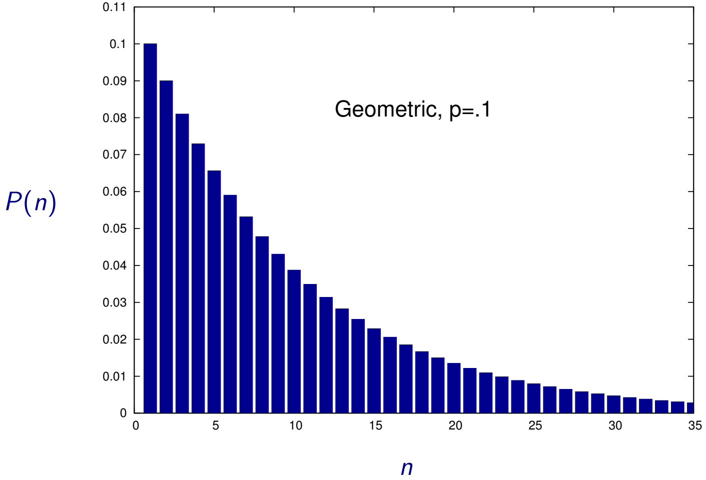
and $P(X^g = k) = P(\{X^g > k-1\} \text{ and } \{X_k^B = 1\}) = (1-p)^{k-1} p$.

Discrete Random Variables

3. Geometric probability distribution



Discrete Random Variables

4. Poisson Distribution

$$P(X^P = n) = e^{-\lambda} \frac{\lambda^n}{n!}$$

Discussion later.

Continuous Random Variables

Philosophical Issues

1. *Mathematically* , continuous and discrete random variables are very different.

Continuous Random Variables
Philosophical Issues

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2. *Quantitatively* , however, some continuous models are very close to some discrete models.

Continuous Random Variables

Philosophical Issues

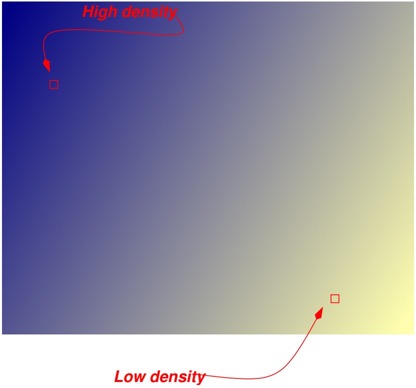
1. *Mathematically* , continuous and discrete random variables are very different.
2. *Quantitatively* , however, some continuous models are very close to some discrete models.
3. Therefore, which kind of model to use for a given system is a matter of *convenience* .

Continuous Random Variables
Philosophical Issues

Example: The production process for small metal parts (nuts, bolts, washers, etc.) might better be modeled as a continuous flow than as a large number of discrete parts.

Continuous Random Variables

Philosophical Issues

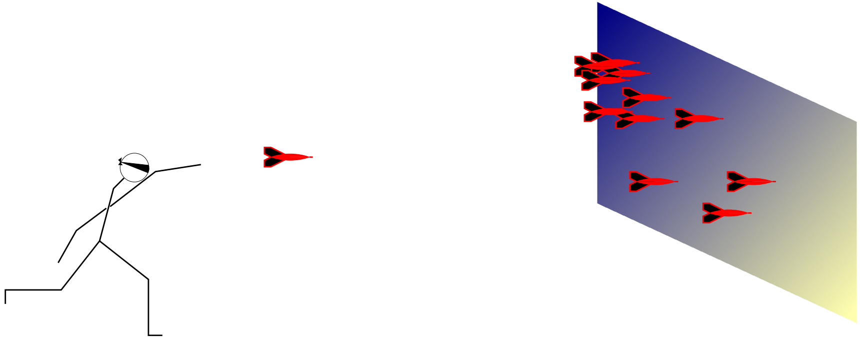


The probability of a two-dimensional random variable being in a small square is the *probability density* times the area of the square. (The definition is similar in higher-dimensional spaces.)

Compare with slide 14.

Continuous Random Variables

Philosophical Issues



Continuous Random Variables
Spaces

Dimensionality

- Continuous random variables can be defined
  - ⋆ in one, two, three, ..., infinite dimensional spaces;
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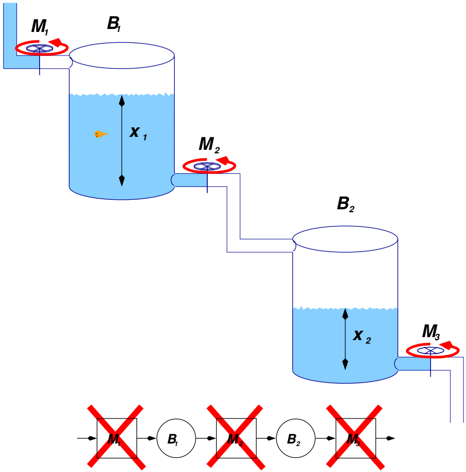
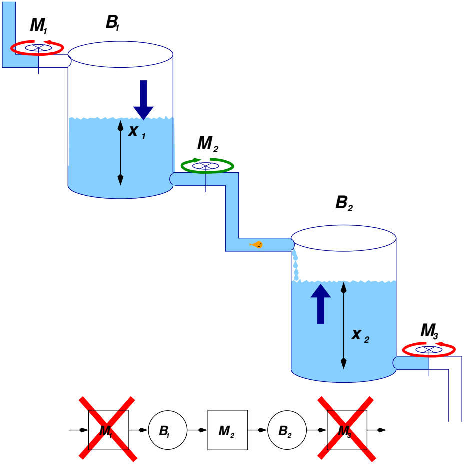
Continuous Random Variables

No change in water levels



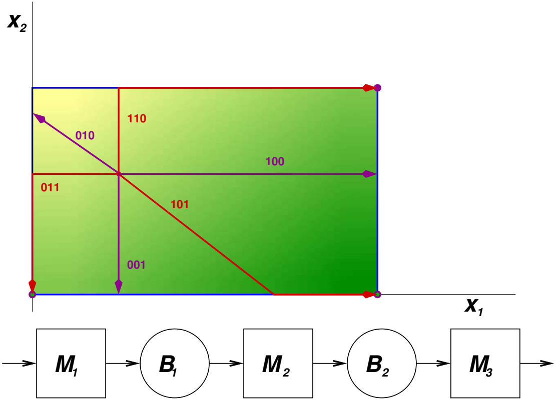
Continuous Random Variables

One kind of change in water levels



Continuous Random Variables

Trajectories

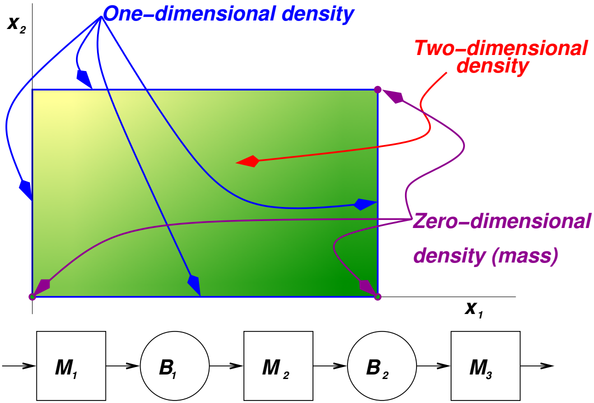


Trajectories of buffer levels in the three-machine line if the machine states stay constant for a long enough time period.

Notation: 110 means M_1 and M_2 are operational and M_3 is down, 100 means M_1 is operational, M_2 and M_3 are down, etc.

Continuous Random Variables

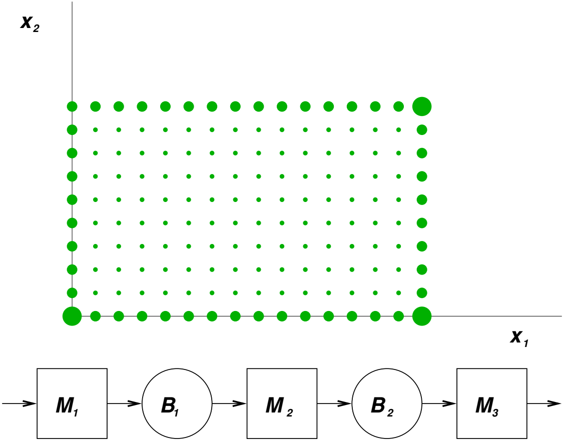
Two-dimensional probability distribution



Probability distribution of the amount of material in each of the two buffers.

Continuous Random Variables

Discrete approximation of the probability distribution



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Continuous Random Variables

Densities and Distributions

In one dimension, $F()$ is the *cumulative probability distribution of* X if

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Continuous Random Variables

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Continuous Random Variables

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$f()$ is the *density function of* X if

$$F(x) = \int_{-\infty}^{x} f(t)dt$$

Therefore,

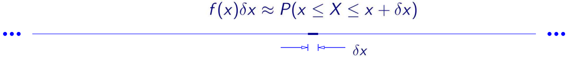
$$f(x) = \frac{dF}{dx}$$

wherever F is differentiable.

Continuous Random Variables

Densities and Distributions

Fact: $f(x)\delta x \approx P(x \leq X \leq x + \delta x)$ for sufficiently small δx.



Continuous Random Variables

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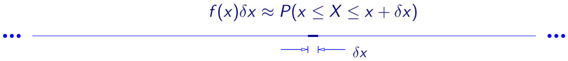

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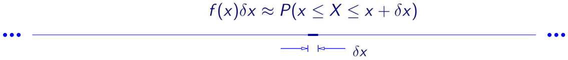

Fact: $F(b) - F(a) = \int_a^b f(t)dt$

Definition: Expected value of $x = \bar{x} = \int_{-\infty}^{\infty} tf(t)dt$

Continuous Random Variables

Standard Normal Distribution

The density function of the *normal* (or *gaussian*) distribution with mean 0 and variance 1 (the *standard normal*) is given by

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

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(There is no closed form expression for $F(x)$.)

Continuous Random Variables

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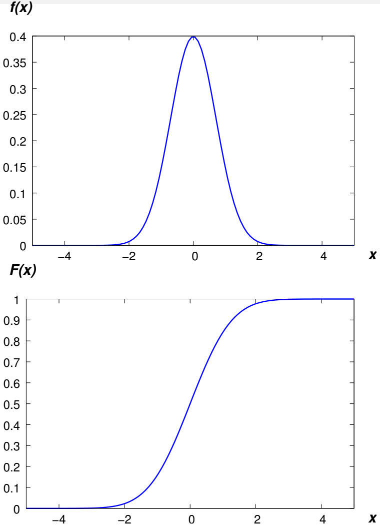

Continuous Random Variables
Normal Distribution

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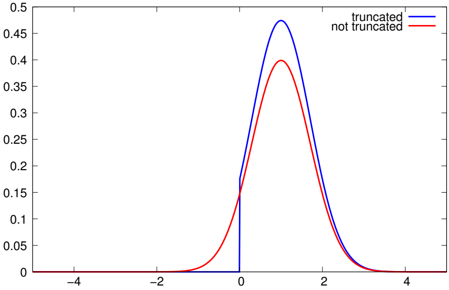
Consequently, $N(\mu, \sigma)$ easy to compute from $N(0, 1)$. This is why $N(0, 1)$ is tabulated in books.

Continuous Random Variables

Truncated Normal Density (1)



$f_T(x)\delta x = P(x \leq X \leq x + \delta x) = \dfrac{f(x)}{1 - F(0)}\delta x$ where $F()$ and $f()$ are the normal distribution and density functions with parameters μ and σ.

Continuous Random Variables
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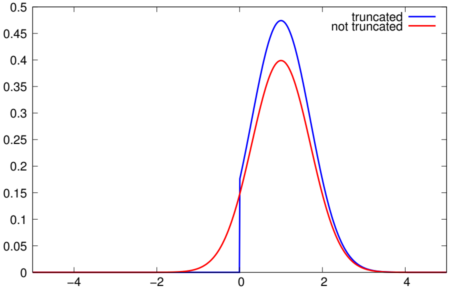

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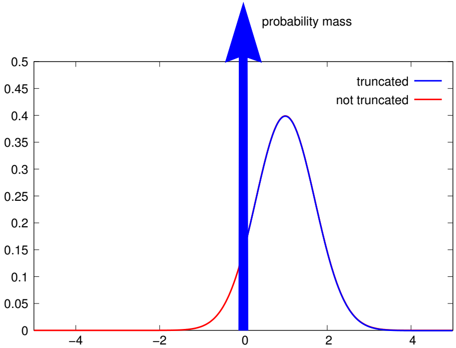
Note: μ and σ are the parameters of $f(x)$, *not* $f_T(x)$.

Continuous Random Variables

Truncated Normal Density (2)



$f_{T'}(x)\delta x = P(x \leq X \leq x + \delta x) = f(x)\delta x$ for $x > 0$ and $P(X = 0) = F(0)$ where $F()$ and $f()$ are the normal distribution and density functions with parameters μ and σ.

Continuous Random Variables

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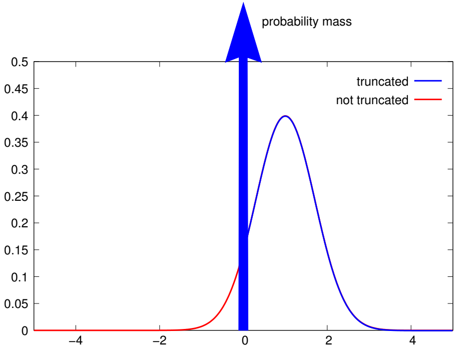

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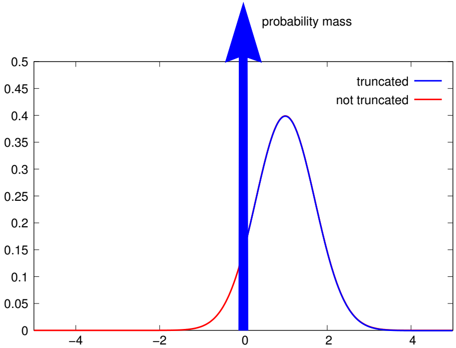

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Here again, μ and σ are the parameters of $f(x)$, *not* $f_{T'}(x)$.

For *both* kinds of truncation, $f_T(x)$ and $f_{T'}(x)$ are close to $f(x)$ when $\mu \gg \sigma$, and not otherwise.

Continuous Random Variables

Law of Large Numbers

Let $\{X_k\}$ be a sequence of independent identically distributed *(i.i.d.)* random variables that have finite mean μ. Let S_n be the sum of the first n X_ks, so

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That is, *the average approaches the mean.*

Continuous Random Variables

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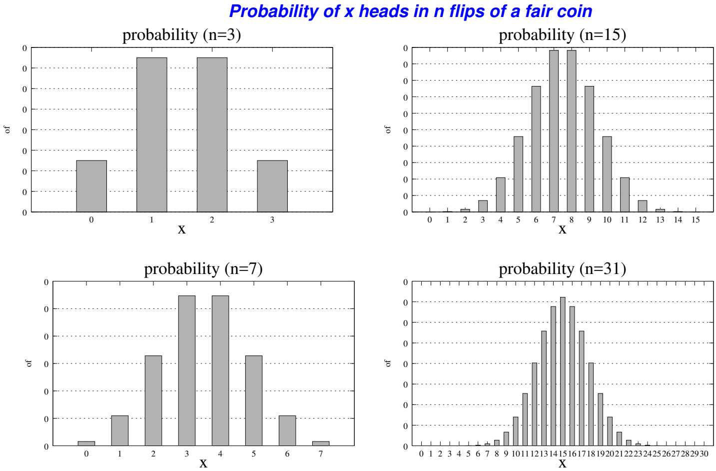
If we define A_n as S_n/n, the average of the first n X_ks, then this is equivalent to:

As $n \to \infty$, $P(A_n) \to N(\mu, \sigma/\sqrt{n})$.

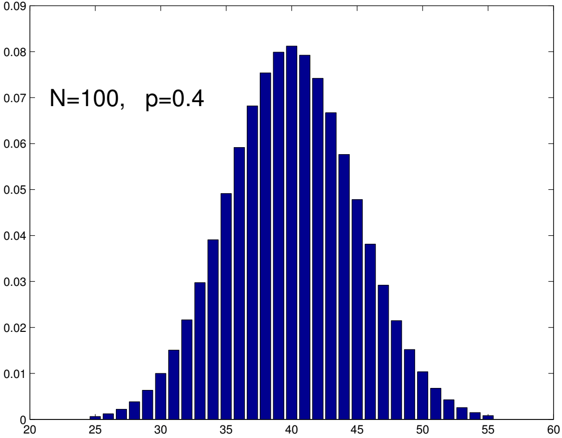
Continuous Random Variables

Coin flip examples



Continuous Random Variables

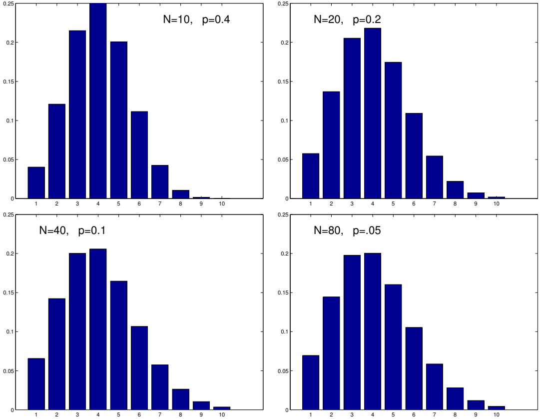
Binomial probability distribution approaches normal for large N.



Continuous Random Variables

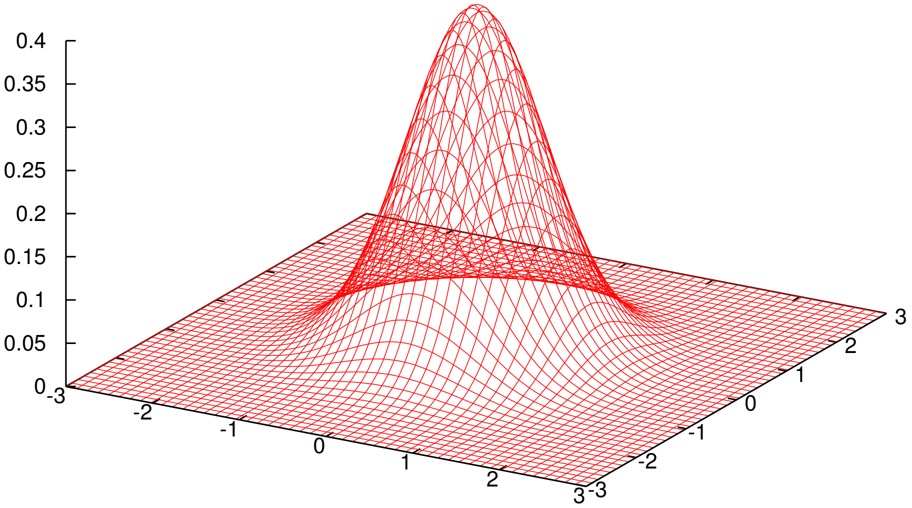
Binomial distributions

Note the resemblance to a *truncated* normal in these examples.



Normal Density Function

... in Two Dimensions



More Continuous Distributions

Uniform

$$f(x) = \frac{1}{b-a} \quad \text{for } a \leq x \leq b$$

$$f(x) = 0 \qquad\qquad \text{otherwise}$$

More Continuous Distributions

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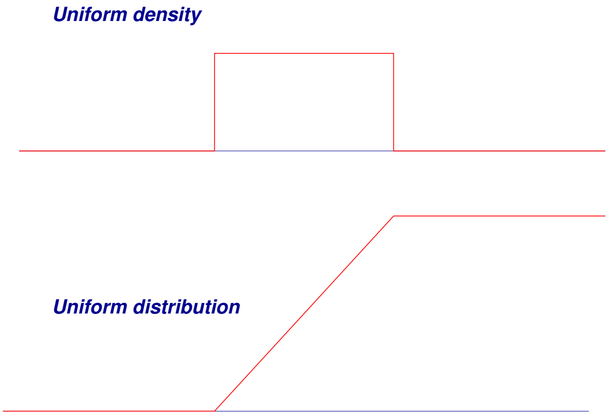
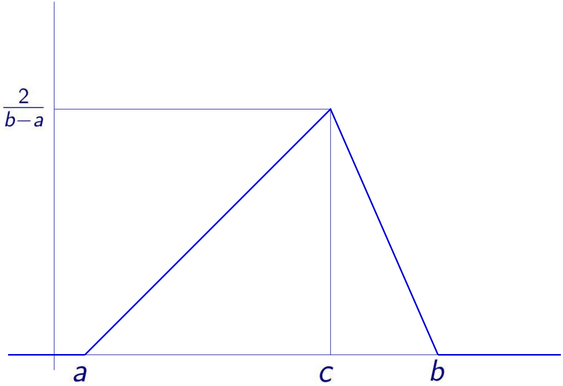


More Continuous Distributions

Triangular

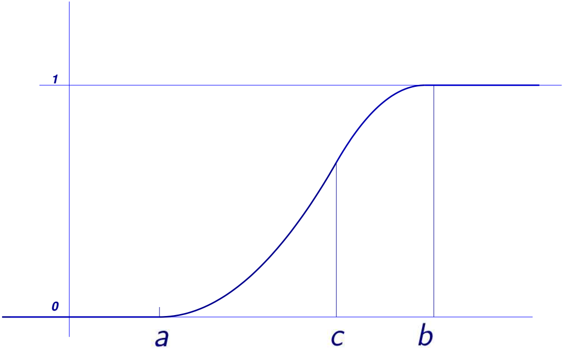
Probability density function



More Continuous Distributions

Triangular

Cumulative distribution function



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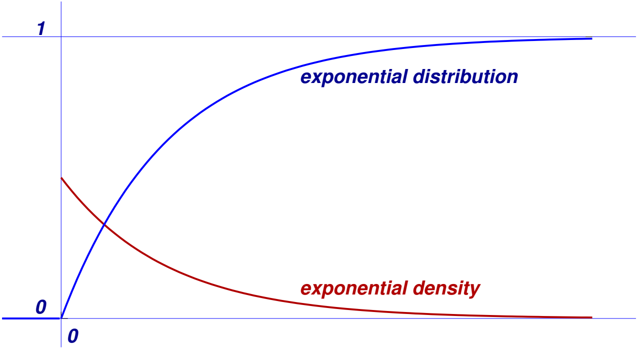

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  Suppose an exponentially distributed process is started at time 0 and the event of interest has not occurred yet at time x. Then the probability distribution of the time after x at which it occurs is the same as the original exponential distribution. The process has no "memory" of when it was actually started.

Another Discrete Random Variable

Poisson Distribution

$$P(X^P = x) = e^{-\lambda t}\frac{(\lambda t)^x}{x!}$$

is the probability that x events happen in $[0, t]$ if the events are independent and the times between them are exponentially distributed with parameter λ.

Typical examples: arrivals and services at queues. *(Next lecture!)*

NOT Random

...but almost

- A *pseudo-random number generator* is a set of numbers $X_0, X_1, \ldots$ where there is a function F such that

$$X_{n+1} = F(X_n)$$

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  - and the sequence $U_0, U_1, \ldots$ (where $U_i = X_i / X^{\max}$) *looks like* a set of uniformly distributed, independent random variables.
    - That is, statistical tests say that the probability of the sequence *not* being independent uniform random variables is very small.

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- Pseudo-random number generators are used extensively in *simulation*.