Probability

Problem 2

Problem Solution

They can only show 8 if they land on (2,6), (6,2), (3,5), (5,3) and (4,4). Out of the 5 possibilities, 4 of them do not have 4s, so the probability is 4/5.

Problem 3

Problem Solution

This is just taking the discrete expectation.
(.8)(.02)+(.19)(.1)+(.01)(.2)=0.37

Problem 4

Problem Solution

a) What is the probability that there are 6 or fewer heads?
$\binom{12}{1}(.45)(.55)^{11}+\binom{12}{2}(.45)^2(.55)^{10}+\binom{12}{3}(.45)^3(.55)^9+\binom{12}{4}(.45)^4(.55)^8+\binom{12}{5}(.45)^5(.55)^7+\binom{12}{6}(.45)^6(.55)^6+\binom{12}{0}(0.55)^{12}=0.739$
b) What is the expected number of heads?
(0.45)(12)=5.4
c) What is the standard deviation of the number of heads?

$$\sqrt{(12 * 0.45 * .55)} = 1.723$$

Stochastic Process

Problem 5

Problem Solution

Figure 1 shows the transition graph of a three-state discrete time Markov process.

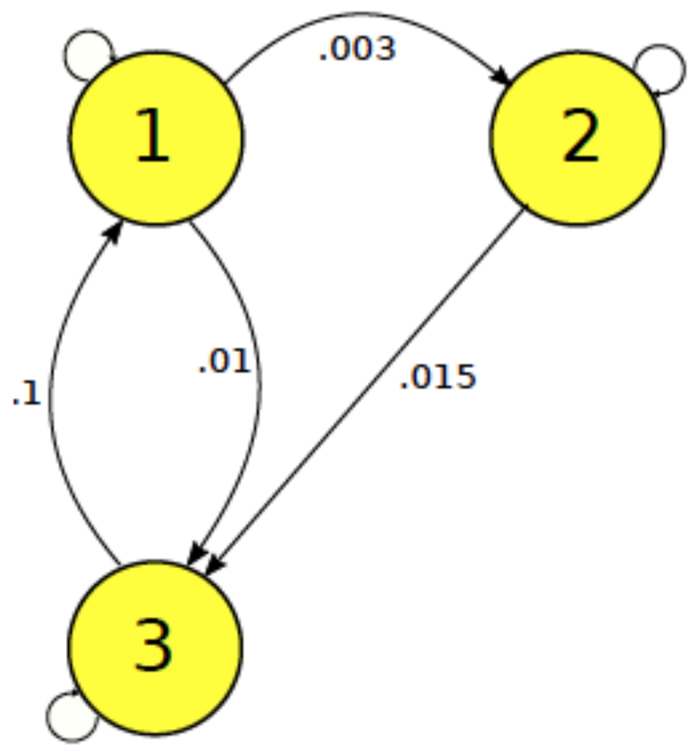

Figure 1: Discrete time Markov process transition graph

a) Write the values of all the non-zero transition probabilities $P_{ij} = \mathrm{P}(\mathrm{X}(\mathrm{t}+1) = \mathrm{i}|\mathrm{X}(\mathrm{t}) = \mathrm{j})$ for i, j = 1, 2, 3:

P_{11}=1-0.01-0.003=0.987
P_{12} =0.003
P_{13}=0.01
P_{22}=1-0.015=0.985
P_{23}=0.15
P_{31}=0.1
P_{33}=1-0.1=0.9

b) Suppose the system is in state 1 at time t=0. What are the probabilities that the system will be in states 1, 2, 3 at time t=1?

Let $\pi_i(t)$ be the probability that the system is in state i at time t. Then $\pi_1(0)=1$ and $\pi_2(0)=\pi_3(0)=0$

$$\pi_1 = P_{11}\pi_1(0) + P_{21}\pi_2(0) + P_{31}\pi_3(0) = 0.987$$
$$\pi_2 = P_{12}\pi_1(0) + P_{22}\pi_2(0) + P_{32}\pi_3(0) = 0.003$$
$$\pi_3 = P_{13}\pi_1(0) + P_{23}\pi_2(0) + P_{33}\pi_3(0) = 0.01$$

c) Let π_i be the steady-state probabilities of state i. Write the steady-state transition equations. (We will not ask for this on the exam, but you will need to know how to do this to solve part d)

$$\pi_1 = P_{11}\pi_1 + P_{21}\pi_2 + P_{31}\pi_3$$
$$\pi_2 = P_{12}\pi_1 + P_{22}\pi_2 + P_{32}\pi_3$$
$$\pi_3 = P_{13}\pi_1 + P_{23}\pi_2 + P_{33}\pi_3$$

d) Find the steady-state probabilities

By using the normalization equation
$\pi_1+\pi_2+\pi_3=1$
$\pi_1=0.7519$
$\pi_2=0.1504$
$\pi_3=0.0977$

Queuing Theory

Problem 6

Problem Solution

a) Write a formula for F(N), the probability that there are N customers of fewer in the line.

$$F(N) = \sum_{n=0}^{N} (1-\rho)\rho^n \text{ where } \rho = \lambda/\mu.$$

b) Suppose customers arrive once every 30 seconds on the average and the sandwich maker can make a sandwich in 29 seconds on average. What is the average number of customers in the line?

In this case, $\rho = 29/30 = .9667$.

$$\bar{n} = \frac{\rho}{1-\rho} = \frac{(29/30)}{(1/30)} = 29$$

c) What is the probability that there are 5 or fewer customers in the line?

$$F(5) = \sum_{n=0}^{5} (1-\rho)\rho^n = .1841$$

That sum is easy to calculate using a spreadsheet program.

d) The performance described in the previous part is unsatisfactory, and that is why the sandwich maker was fired. How fast must the new sandwich maker work (that is, what is the required μ) so that the probability that there are 5 customers or fewer in the line is greater than 0.9?

Using the spreadsheet program, we can search for μ. We find that μ muse be at least 0.04893 so the average sandwich-making time is not greater than 20.44 seconds.

e) What will be the average number of customers in the system with this new sandwich maker?

Using the formula from b), there will be on average 2.14 customers in the system.

Inventory

Problem 7

Problem Solution

The resort's oil supplier will charge them \$3.00 per gallon in February. They estimate that each gallon that is consumed creates \$6.00 in revenue. Since February is the last month of the ski season, any oil that is left at the end of the month must be discarded in an environmentally responsible manner. Consequently, any oil that remains on March 1 costs \$2.00 to remove. In other words, it has a negative salvage value: -\$2.00 per gallon.

They would like to maximize their net profit. How much oil should they order for February?

This is a newsvendor problem. The solution is the value of x that satisfies

$$F(x) = \frac{r-c}{r-s}$$

where $F()$ is the triangular cumulative distribution function, r is the revenue for each gallon, c is the price that the resort has to pay for oil, and s is the salvage value of the oil.

In this problem, $r = 6, c = 3$, and $s = -2$.

According to Wikipedia, the cumulative distribution function of a triangular random variable is

$$F(x) = \begin{cases} 0 & \text{for } x < a \\ \frac{(x-a)^2}{(b-a)(c-a)} & \text{for } a \le x \le c \\ 1 - \frac{(b-x)^2}{(b-a)(b-c)} & \text{for } c \le x \le b \\ 1 & \text{for } b \le x \end{cases}$$

Therefore we have to solve

$$F(x) = \frac{6-3}{6+2} = \frac{3}{8} = .375$$

The solution is 2063.5 gallons.