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Introduction to Manufacturing Systems

Inventory

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Storage

Storage is fundamental!

- Storage is *fundamental* in nature, management, and engineering.
  - ⋆ In nature, *energy* is stored. Life can only exist if the acquisition of energy can occur at a different time from the the expenditure of energy.
  - ⋆ In management, *products* are stored.
  - ⋆ In engineering, *energy* is stored (springs, batteries, reservoirs); etc.

Storage

Purpose of storage

- *The purpose of storage is to allow systems to survive even when important events are unsynchronized.* For example,
  - ⋆ the separation in time from the acquisition or production of something and its consumption; and
  - ⋆ the occurrence of an event at one location (such as a machine failure or a power surge) which can prevent desired performance or do damage at another.
- Storage improves system performance by *decoupling* parts of the system from one another.

Storage
Purpose of storage

- It allows production systems (of energy or goods) to be built with capacity less than the peak demand.
    - ⋆ Helps manage seasonality.
- It reduces the propagation of disturbances, and thus reduces instability and the fragility of complex, expensive systems.

Manufacturing Inventory
Motives/benefits

Storage

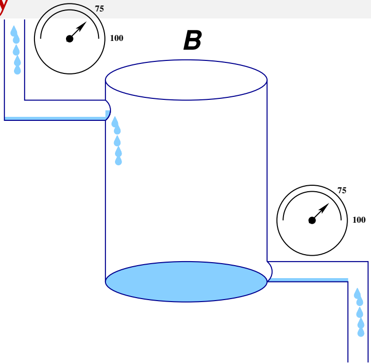
- Allows economies of scale:
  - ⋆ volume purchasing
  - ⋆ reduced set-ups
- Helps manage uncertainty:
  - ⋆ Random arrivals of customers or orders.
  - ⋆ Unreliable deliveries of raw materials.

Manufacturing Inventory
Costs

- Financial: raw materials are paid for, but no revenue comes in until the item is sold.
- Demand risk: item loses value or is unsold due to obsolescence, for example
  - ⋆ newspapers
  - ⋆ technology
  - ⋆ fashion
- Holding cost (warehouse space)
- "Shrinkage" = damage/theft/spoilage/loss

Variability and Storage

No variability



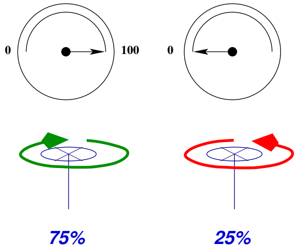
75 gal/sec in, 75 gal/sec out constantly

If we start with an empty tank, the tank is always empty (except for splashing at the bottom).

Variability and Storage

Variability from random valves

Consider a random valve:



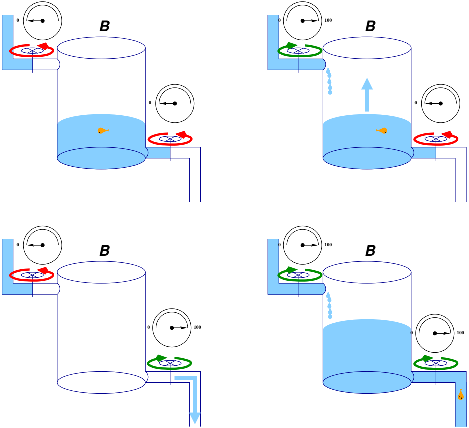
- The average period when the valve is open is 15 minutes.
- The average period when the valve is closed is 5 minutes.
- Consequently, the *average* flow rate through it is 75 gal/sec.

Variability and Storage
Variability from random valves

Four possibilities for two *unsynchronized* valves:

0 in, 0 out

100 gal/sec in, 0 out



0 in, 100 gal/sec out

100 gal/sec in, 100 gal/sec out

Variability and Inventory

Observation:

- There is *never* any water in the tank when the flow is constant.
- There is *sometimes* water in the tank when the flow is variable.

Conclusions:

1. You can't always replace random variables with their averages.
2. *Variability causes inventory!!*

Variability and Inventory

- To be more precise, *non-synchronization causes inventory.*
  - ⋆ Living things do not acquire energy at the same time they expend it. Therefore, they must store energy in the form of fat or sugar.
  - ⋆ Rivers are dammed and reservoirs are created to control the flow of water — to reduce the variability of the water supply.
  - ⋆ For solar and wind power to be successful, energy storage is required for when the sun doesn't shine and the wind doesn't blow.

Inventory Topics

- Newsvendor Problem — pure demand risk (no inventory dynamics)
- Economic Order Quantity (EOQ) — pure economies of scale (deterministic demand)
- Q, R policy — supplier lead time and random demand
- Base Stock Policy — manage a simple factory to avoid stockouts
- Other topics

Newsvendor Problem
Demand risk

- Newsvendor buys x newspapers at c dollars per paper (*cost*).

- Newspapers are sold at *customer price* $r > c$.
- Unsold newspapers are redeemed at *salvage price* $s < c$.

$$r > c > s$$

Newsvendor Problem
Demand risk

Assume demand W is a continuous random variable with distribution function

$$F(w) = \text{prob}(W \leq w);$$

Assume $f(w) = dF(w)/dw$ exists for all $w > 0$.

Problem: Choose x to maximize expected profit.

Note that $w \geq 0$ so $F(w) = 0$ and $f(w) = 0$ for $w \leq 0$.

Newsvendor Problem

Model

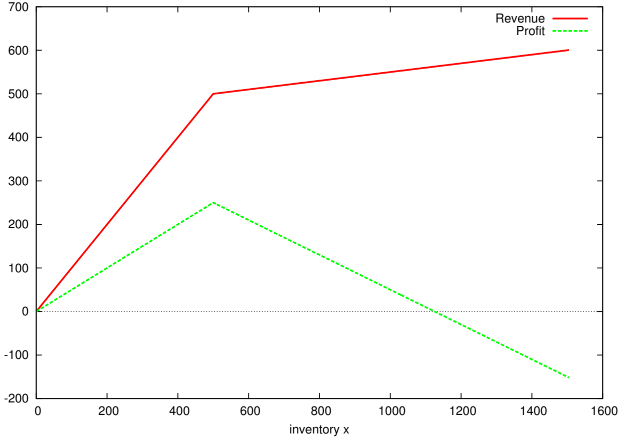
Revenue =

$$R(x,w) = \begin{cases} rx & \text{if } x \le w \\ \\ rw + s(x-w) & \text{if } x > w \end{cases}$$

Profit =

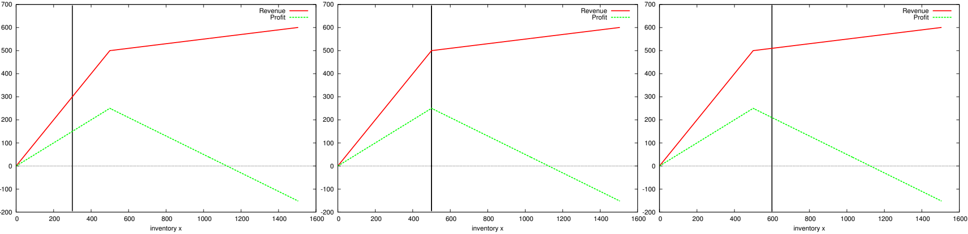
$$P(x,w) = \begin{cases} (r-c)x & \text{if } x \le w \\ \\ rw + s(x-w) - cx & \text{if } x > w \\ \\ \quad = (r-s)w - (c-s)x & \end{cases}$$

Newsvendor Problem
Model



$$r = 1.; c = .5; s = .1; w = 500.$$

Newsvendor Problem
Model

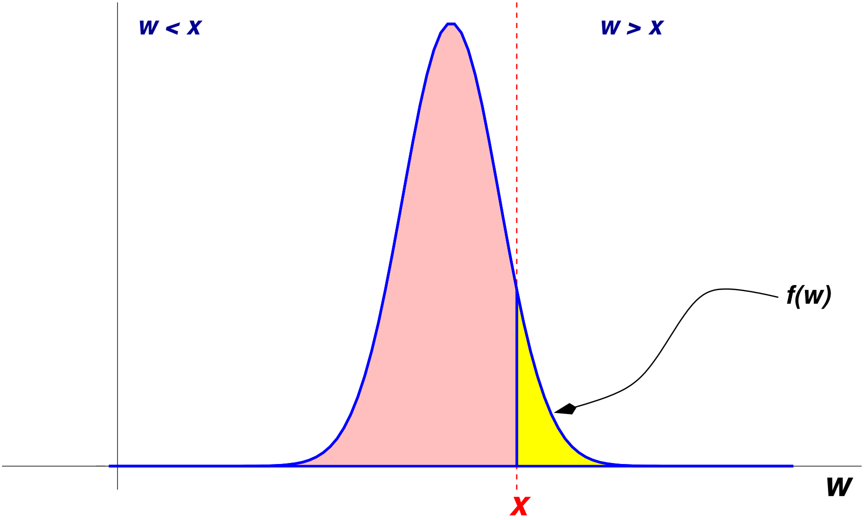


x too little $\qquad$ x just right $\qquad$ x too much

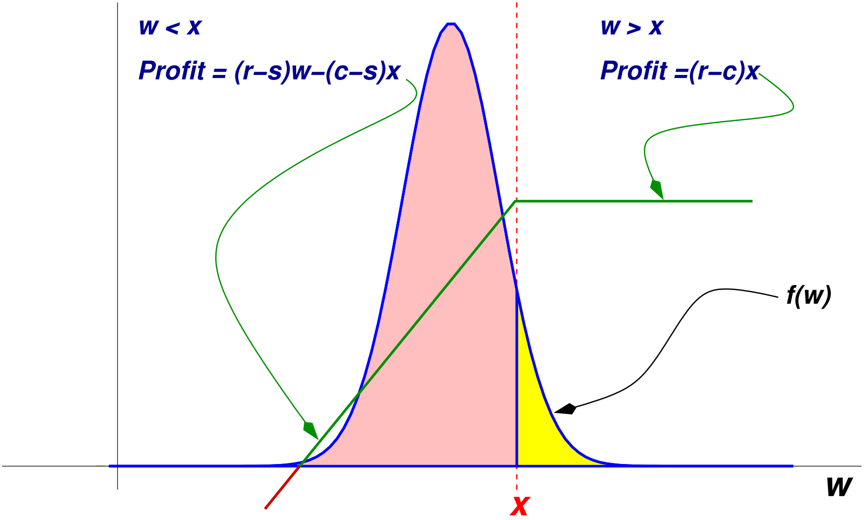
$$r = 1.; c = .5; s = .1; w = 500.$$

Note that if x is too large, the profit could be negative.

Newsvendor Problem
Normally distributed demand



Newsvendor Problem
Normally distributed demand



Newsvendor Problem

Expected profit

Expected Profit $= EP(x) = E_w P(x,w) =$

$$\int_{-\infty}^{\infty} P(x,w) f(w) dw =$$

$$\int_{-\infty}^{x} [(r-s)w - (c-s)x] f(w) dw +$$

$$\int_{x}^{\infty} (r-c) x f(w) dw$$

Newsvendor Problem
Expected profit

or $EP(x)$

$$= (r-s)\int_{-\infty}^{x} wf(w)dw - (c-s)x\int_{-\infty}^{x} f(w)dw$$

$$+(r-c)x\int_{x}^{\infty} f(w)dw$$

$$= (r-s)\int_{-\infty}^{x} wf(w)dw$$

$$-(c-s)xF(x)+(r-c)x(1-F(x))$$

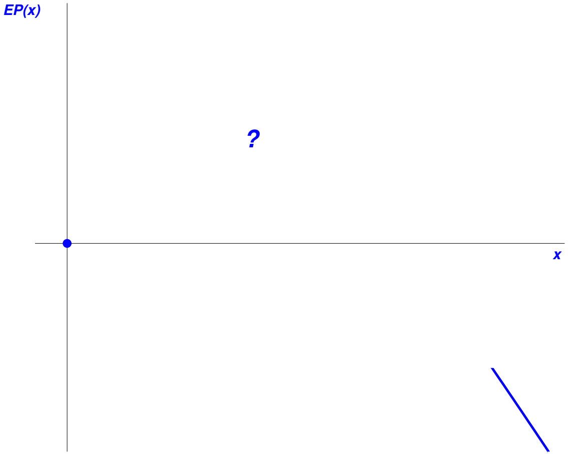
Newsvendor Problem
Expected profit

- When $x = 0$, EP is 0.
- When $x \rightarrow \infty$,
  - ⋆ the first term goes to a finite constant, $(r - s)Ew$;
  - ⋆ the middle term, $-(c - s)xF(x), \rightarrow -\infty$.
  - ⋆ the last term goes to 0;

Therefore when $x \rightarrow \infty$, $EP \rightarrow -\infty$.

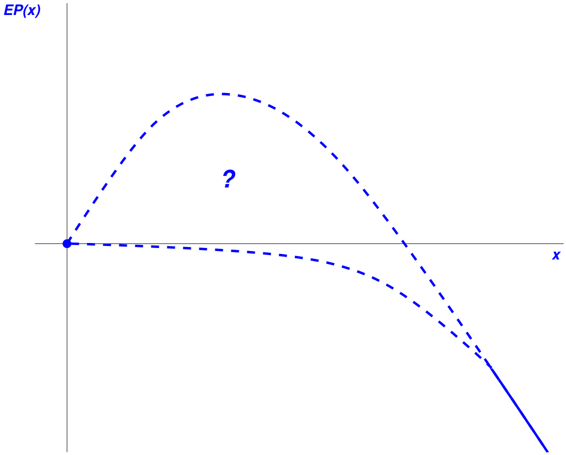
Newsvendor Problem

Expected profit



Newsvendor Problem

Expected profit



Newsvendor Problem
Expected profit

$$EP(x) = (r-s)\int_{-\infty}^{x} wf(w)dw - (c-s)xF(x) + (r-c)x(1-F(x))$$

$$\frac{dEP}{dx} = (r-s)xf(x) - (c-s)F(x) - (c-s)xf(x)$$
$$+(r-c)(1-F(x)) - (r-c)xf(x)$$

$$= xf(x)(r-s+s-c-r+c)$$
$$+r-c+(s-c-r+c)F(x)$$

$$= r-c-(r-s)F(x)$$

Note also that $\dfrac{d^2EP}{dx^2} = -(r-s)f(x)$.

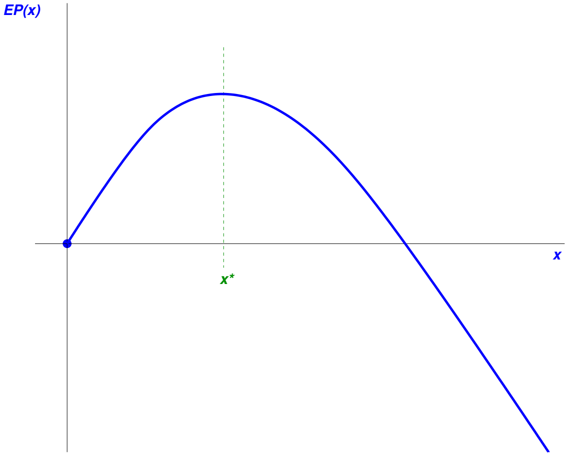
Newsvendor Problem

Expected profit

Note that

- $d^2EP/dx^2 = -(r-s)f(x) < 0$ for all $x > 0$. Therefore EP is concave and has a maximum.
- $dEP/dx = r - c > 0$ when $x = 0$. Therefore EP is increasing at $x = 0$, so $EP(x)$ is positive for some x.
- x^* satisfies $\dfrac{dEP}{dx}(x^*) = 0$.

Newsvendor Problem
Expected profit



Newsvendor Problem

Expected profit

Therefore $\boxed{F(x^*) = \dfrac{r-c}{r-s}.}$

Note that $0 \leq \dfrac{r-c}{r-s} \leq 1$. Therefore there is an x^*.

Recall the definition of $F()$: $F(x^*)$ is the probability that $W \leq x^*$.

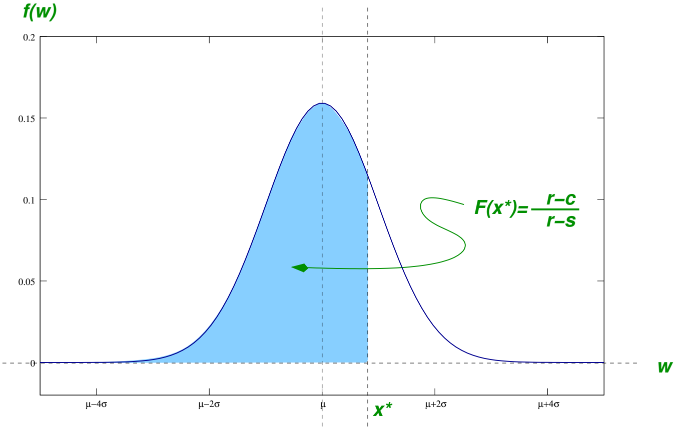
So the equation in the box means

Buy enough stock to satisfy demand $100K\%$ of the time, where

$$K = \frac{r-c}{r-s}.$$

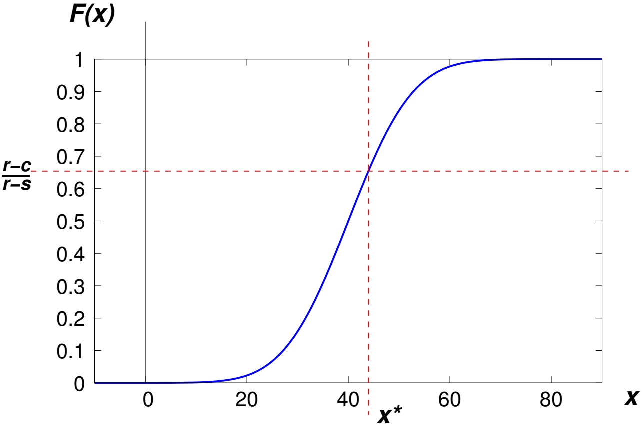
Newsvendor Problem

Expected profit



Newsvendor Problem

Expected profit



Newsvendor Problem
Expected profit

Note that $F(x^*) = \dfrac{r-c}{r-s} = \dfrac{r-c}{(r-c)+(c-s)}$.

$r - c > 0$ is the marginal profit when $x < w$.

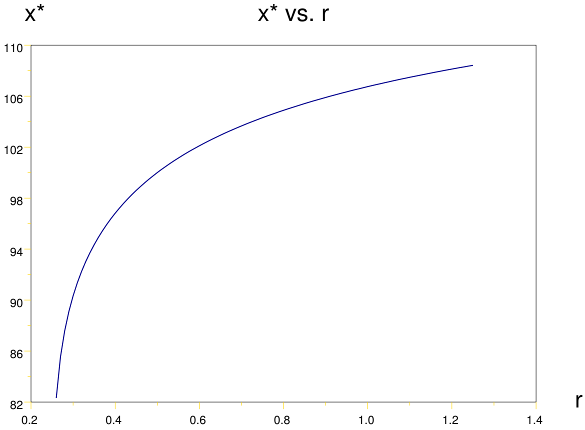
$c - s > 0$ is the marginal loss when $x > w$.

Choose x^* so that the fraction of time you do not buy too much is

$$\frac{\text{marginal profit}}{\text{marginal profit} + \text{marginal loss}}$$

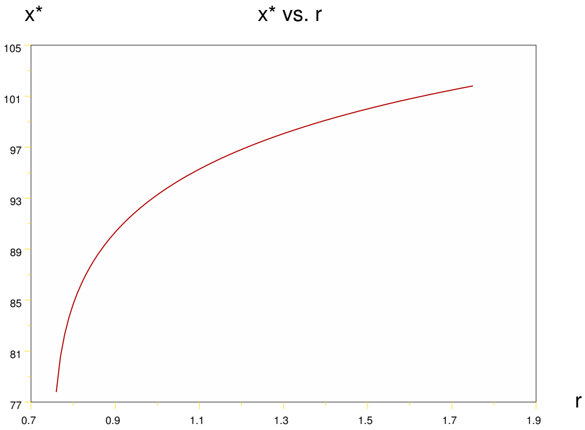
Newsvendor Problem

Example 1: r varying, $c = .25, s = 0, \mu_w = 100, \sigma_w = 10$



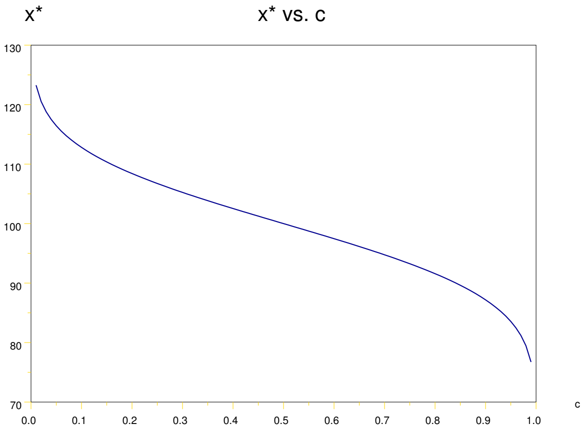
Newsvendor Problem

Example 2: r varying, $c = .75, s = 0, \mu_w = 100, \sigma_w = 10$



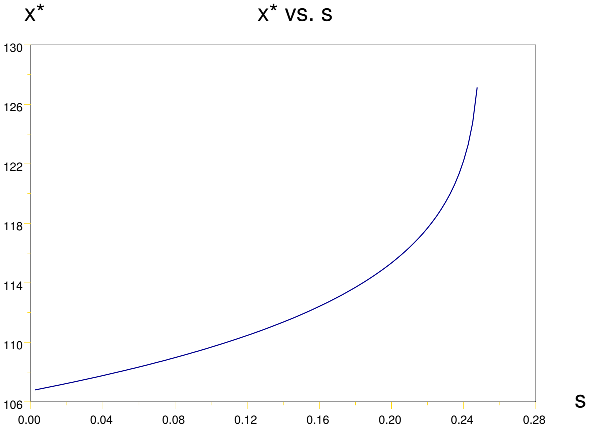
Newsvendor Problem

$r = 1$, c **varying**, $s = 0, \mu_w = 100, \sigma_w = 10$



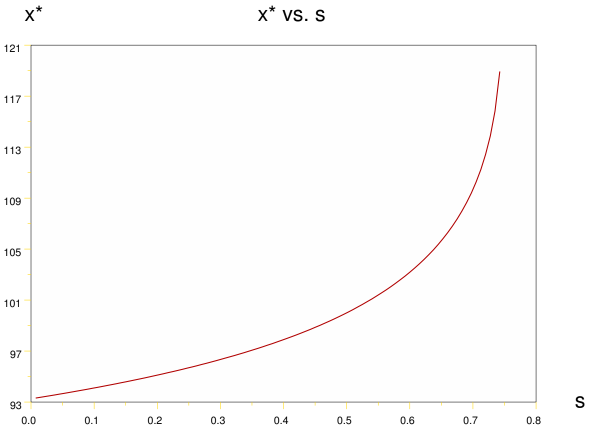
Newsvendor Problem

Example 1: $r = 1, c = .25$, s varying, $\mu_w = 100, \sigma_w = 10$



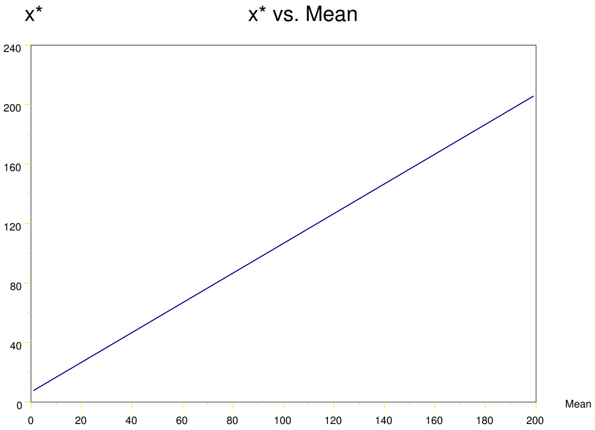
Newsvendor Problem

Example 2: $r = 1, c = .75$, s varying, $\mu_w = 100, \sigma_w = 10$



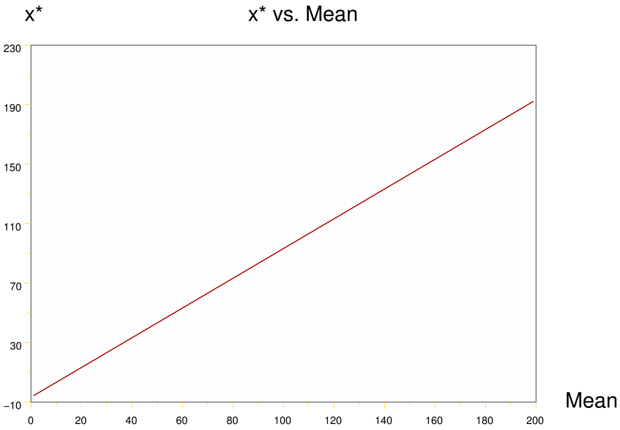
Newsvendor Problem

Example 1: $r = 1, c = .25, s = 0$, μ_w varying, $\sigma_w = 10$



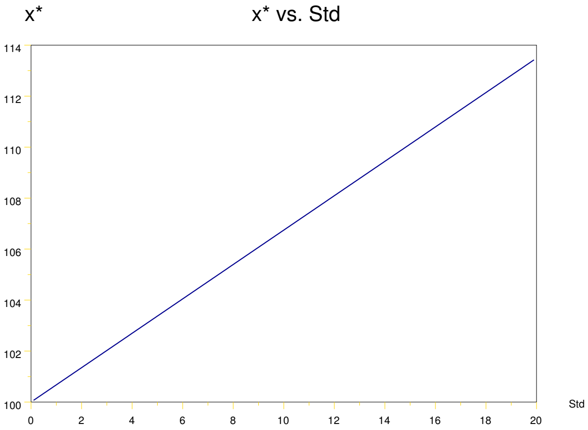
Newsvendor Problem

Example 2: $r = 1, c = .75, s = 0,$ μ_w varying, $\sigma_w = 10$



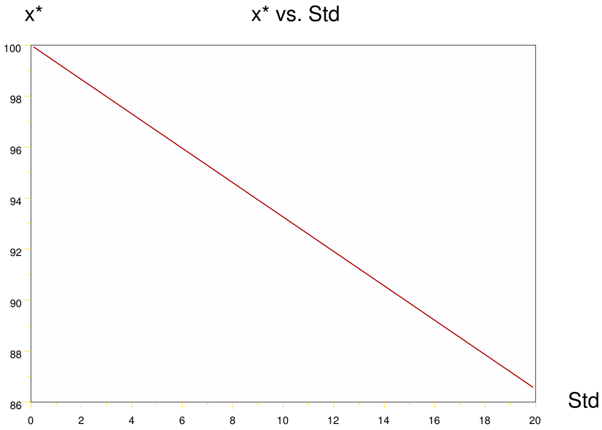
Newsvendor Problem

Example 1: $r = 1, c = .25, s = 0, \mu_w = 100$, σ_w varying



Newsvendor Problem

Example 2: $r = 1, c = .75, s = 0, \mu_w = 100,\ \sigma_w$ varying



Newsvendor Problem

Why does x^* look linear in μ_w and σ_w?

- x^* is the solution to $F(x^*) = \dfrac{r-c}{r-s}$.
- Assume demand w is $N(\mu_w, \sigma_w)$. Then, for any demand w,

$$F(w) = \Phi\left(\frac{w-\mu_w}{\sigma_w}\right)$$

where Φ is the standard normal cumulative distribution function.

Newsvendor Problem

Why does x^* look linear in μ_w and σ_w?

- Therefore

$$\Phi\left(\frac{x^* - \mu_w}{\sigma_w}\right) = \frac{r - c}{r - s}$$

- or

$$\frac{x^* - \mu_w}{\sigma_w} = \Phi^{-1}\left(\frac{r - c}{r - s}\right)$$

- or

$$\boxed{x^* = \mu_w + \sigma_w \Phi^{-1}\left(\frac{r - c}{r - s}\right)}$$

Newsvendor Problem

Why does x^* look linear in μ_w and σ_w? And should it?

Is x^* actually linear in μ_w and σ_w?

Answer: yes and no.

Yes if we accept the assumption that the demand is normal. But ...

No in reality because the demand cannot be normal.

For what values of μ_w and σ_w is x^* closest to linear in μ_w and σ_w?

Answer: When $\mu_w >> \sigma_w$. If so, the normal distribution predicts a very small probability for w being negative. Therefore the normal is a good approximation for the truncated normal.

Newsvendor Problem

x^* Increasing or Decreasing in σ_w

What determines whether x^* increases or decreases in σ_w?

- Note that

$$\Phi^{-1}(k) > 0 \text{ if } k > .5$$

  and

$$\Phi^{-1}(k) < 0 \text{ if } k < .5$$

- Therefore, since $x^* = \mu_w + \sigma_w \Phi^{-1}\left(\frac{r-c}{r-s}\right)$,

  - ⋆ x^* increases with σ_w if $\frac{r-c}{r-s} > .5$, and
  - ⋆ x^* decreases with σ_w if $\frac{r-c}{r-s} < .5$.

Newsvendor Problem

A General Strategy

Question:

- Can we extend this strategy to manage inventory in other settings?
- In particular, suppose the stock remaining at the end of the day today can be used in the future. Does the result of the newsvendor problem suggest a possible *heuristic* extension?

Answer: Yes.

"Build enough inventory to satisfy demand $100K\%$ of the time, for some K."

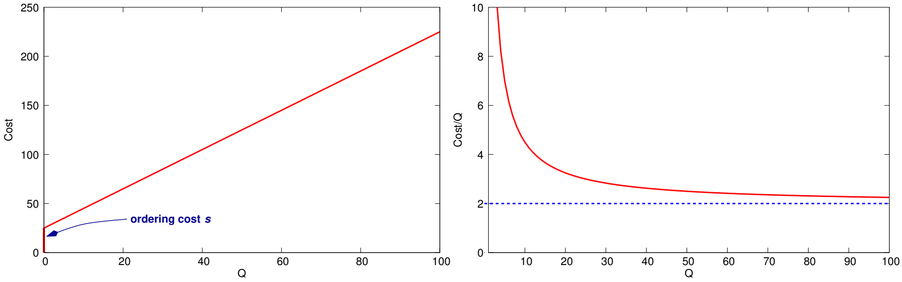
Economic Order Quantity
How much to order and when

- *Economic Order Quantity*
- Motivation: economy of scale in ordering.
- Tradeoff:
  - ⋆ Each time an order is placed, the company incurs a fixed cost in addition to the cost of the goods. *This motivates infrequent ordering.*
  - ⋆ The order must be large enough to satisfy demand until the next order is placed. This determines how much inventory must be stored.

Economic Order Quantity
Assumptions

- No randomness.
- No delay between ordering and arrival of goods.
- No backlogs.
- Goods are required at an annual rate of λ units per year. Inventory is therefore depleted at the rate of λ units per year.
- It costs h to store one unit for one year. h is the *holding cost* (\$/unit/year).
- If the company orders Q units, it must pay $s + cQ$ for the order. s is the *ordering cost* (\$). c is the unit cost (\$/unit).

Economic Order Quantity
Assumptions



Total cost and cost per unit of an order

$$s = 25, c = 2$$

Economic Order Quantity
Problem

- Find a strategy for ordering materials that will minimize the total cost *per year.*

- There are two costs to consider: the *annual* ordering cost and the *annual* holding cost.

Economic Order Quantity
Scenario

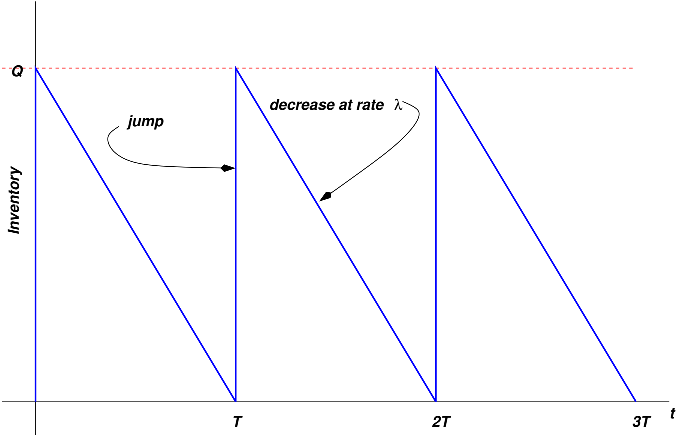
- At time 0, inventory level is 0.
- Q units are ordered and the inventory level jumps instantaneously to Q.
- Material is depleted at rate λ.
- Since the problem is totally deterministic, we can wait until the inventory goes to zero before we order next. There is no danger that the inventory will go to zero earlier than we expect it to.

Economic Order Quantity
Scenario

- Because of the very simple assumptions, we can assume that the optimal strategy does not change over time.
- Therefore the policy is to order Q each time the inventory goes to zero.
- We must determine the optimal Q.

Economic Order Quantity

Scenario



$$T = Q/\lambda \text{ (years)}$$

Economic Order Quantity
Formulation

- The number of orders in a year is $1/T = \lambda/Q$. Therefore, the ordering cost in a year is $s\lambda/Q$.
- The average inventory is $Q/2$. Therefore the average inventory holding cost is $hQ/2$.
- Therefore, we must minimize the annual cost

$$C = \frac{hQ}{2} + \frac{s\lambda}{Q}$$

over Q.

Economic Order Quantity
Formulation

Then

$$\frac{dC}{dQ} = \frac{h}{2} - \frac{s\lambda}{Q^2} = 0$$

or

$$Q^* = \sqrt{\frac{2s\lambda}{h}}$$

Economic Order Quantity
Examples

In the following graphs, the base case is

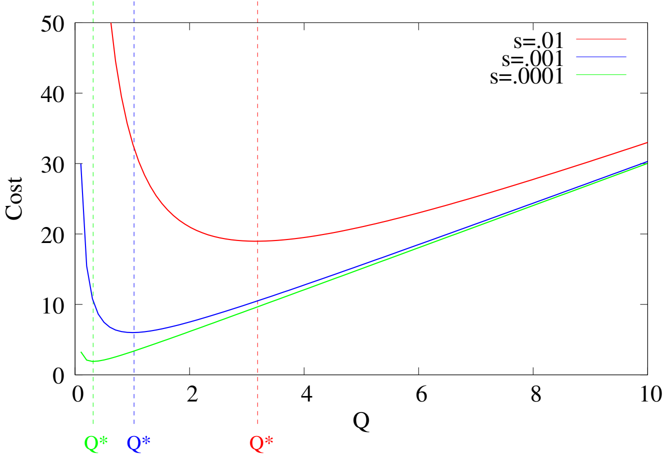
- $\lambda = 3000$
- $s = .001$
- $h = 6$

Note that

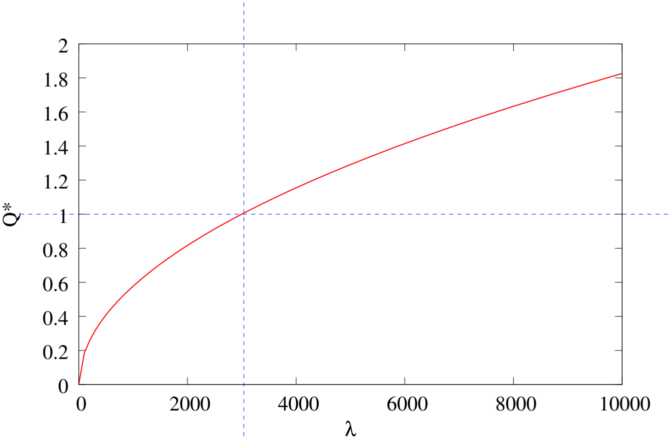
$$Q^* = 1$$

Economic Order Quantity

Examples

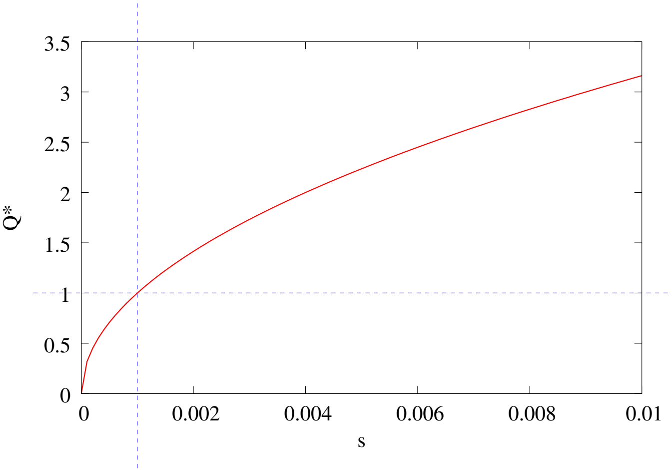


Economic Order Quantity
Examples

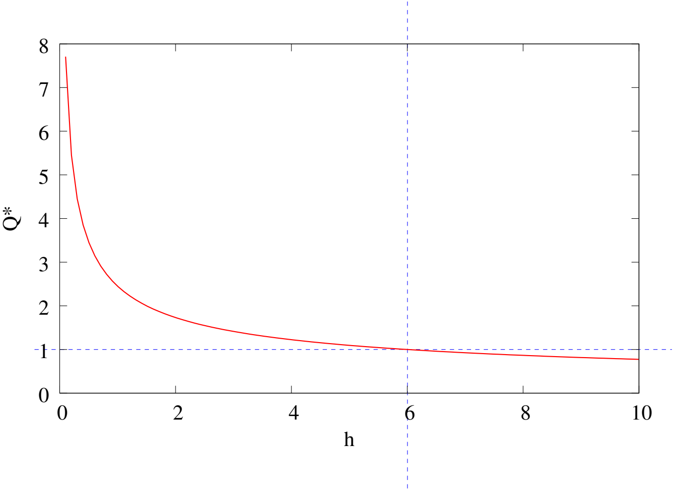


Economic Order Quantity

Examples



Economic Order Quantity
Examples



Supplier Lead Time + Random Demand
Issues

Issue: Supplies are not delivered instantaneously. The time between order and delivery, the lead time $L > 0$.

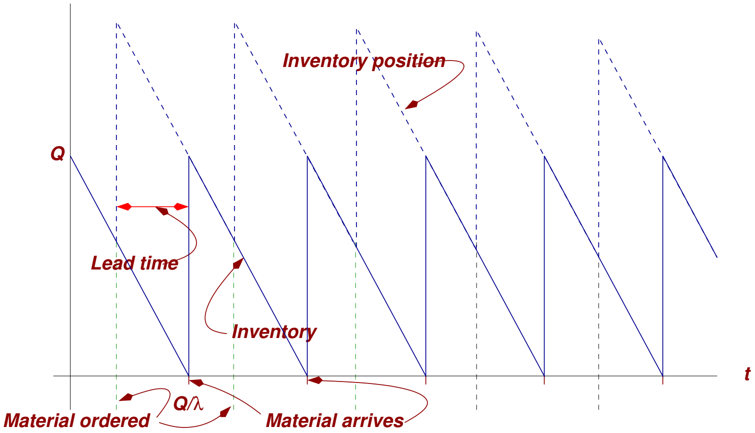
- If everything were deterministic, this would not be a problem. Just order earlier.

Issue: ... but demand is random.

- *Problem:* The demand that occurs between when the order is placed and and when the goods are delivered might be large enough to cause stockout.

Other possible issues: Lead time could be random, production quantity could be random, etc.

Supplier Lead Time + Random Demand
Inventory Position



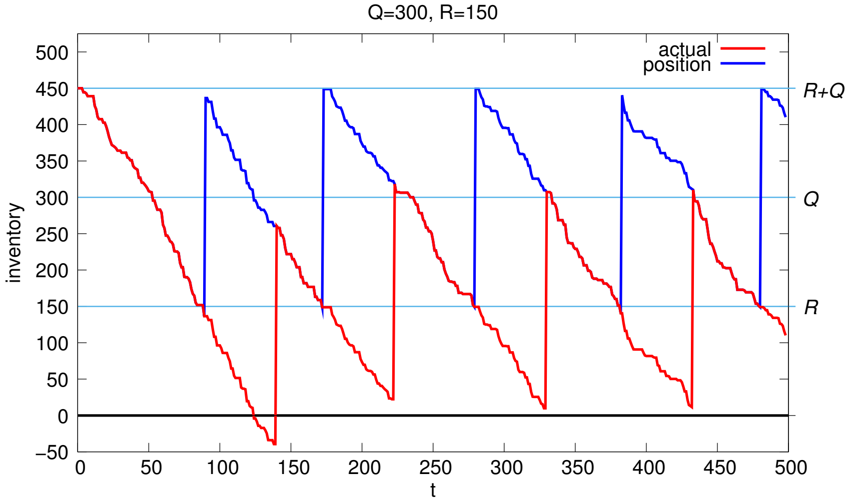
Supplier Lead Time + Random Demand

Q, R Policy

- Fixed ordering cost, like in EOQ problem.
- Random demand, like in newsvendor problem.
  - ⋆ *But the inventory retains its value. It is not scrapped.*
- Make to stock — no advance ordering by customers.
- *Policy:* when the inventory level goes below R, order a quantity Q.
- *Problem:* find Q and R to minimize ordering + holding costs while not losing too many sales due to stockout.

Random Demand

Q, R Policy



Random Demand

R

First we describe how to choose R.

- Let L be the number of days between when an order is placed and when the shipment arrives, the order lead time. L is constant.
- Let D be the total demand that arrives between when an order is placed and when the shipment arrives (L days). D is a random variable.
- The order is placed when the inventory level goes below R.
- The factory would like R to be greater than the demand D that arrives before the raw material arrives.
- But since the demand is random, the best we can do is to choose R such that

$$P(D > R) \leq \epsilon$$

where $P()$ is the probability and ϵ is a small number.

Random Demand

R

It is convenient to write the last equation as

$$P(D < R) \geq 1 - \epsilon$$

If L is large enough, the Central Limit Theorem says that D is approximately normal regardless of the distribution of the daily demand.

Assume that D is $N(\mu, \sigma)$, a normal random variable with mean μ and standard deviation σ.

That is, the probability distribution of the demand is

$$P(D < d) = \Phi\left(\frac{d - \mu}{\sigma}\right)$$

Random Demand
R

Then we will choose R to satisfy

$$\Phi\left(\frac{R-\mu}{\sigma}\right) \geq 1-\epsilon$$

The smallest value of R that satisfies this is

$$R^* = \mu + \sigma\Phi^{-1}(1-\epsilon)$$

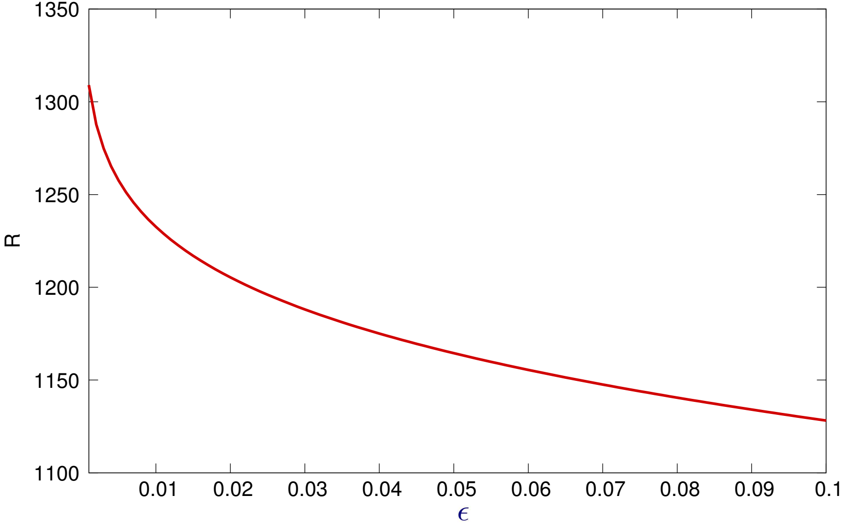
Recall that in the EOQ formulation, λ was the yearly demand. Here, μ is the mean demand for L days. Therefore,

$$\mu = \frac{L\lambda}{365}$$

A graph of R^* as a function of ϵ with $\mu = 1000$ and $\sigma = 100$ is on the next slide.

Random Demand

R



Random Demand
Q

Now we describe how to choose Q.

- The *expected* time between orders is $T = Q/\lambda$.
- The expected number of orders in a year is λ/Q. Therefore, the expected ordering cost in a year is $s\lambda/Q$.
- Assume that the holding cost for material in transit is the same as for inventory in the factory. The average inventory position is $(R+Q)/2$ and the average inventory holding cost is $h(R+Q)/2$.

Random Demand

Q

- Therefore, we choose Q to minimize the approximate expected annual cost

$$C = \frac{h}{2}(R + Q) + \frac{s\lambda}{Q}$$

But since R is constant, this is essentially the same as the EOQ problem for determining Q.

$$Q^* = \sqrt{\frac{2s\lambda}{h}}$$