Mathematical Notes

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Part I

Basic Mathematics

Chapter 1

Axiomatic Set Theory

Axiomatic set theory is a branch of mathematical logic that studies sets, which informally are collections of objects. Although any type of object can be collected into a set, set theory is applied most often to objects that are relevant to mathematics. The language of set theory can be used to define nearly all mathematical objects.

The modern study of set theory was initiated by Georg Cantor and Richard Dedekind in the 1870s. After the discovery of paradoxes in naive set theory, such as Russell's paradox, numerous axiom systems were proposed in the early twentieth century, of which the Zermelo–Fraenkel axioms, with or without the axiom of choice, are the best-known.

Set theory is commonly employed as a foundational system for mathematics, particularly in the form of Zermelo–Fraenkel set theory with the axiom of choice. Beyond its foundational role, set theory is a branch of mathematics in its own right, with an active research community. Contemporary research into set theory includes a diverse collection of topics, ranging from the structure of the real number line to the study of the consistency of large cardinals.

1.1 Propositional Logic

Definition 1.1 (Proposition)**.** *A **proposition** p is a variable[1] that can take the values* true *(T) or* false *(F), and no others.*

This is what a proposition is from the point of view of propositional logic. In particular, it is not the task of propositional logic to decide whether a complex statement of the form "there is extraterrestrial life" is true or not. Propositional logic already deals with the complete proposition, and it just assumes that is either true or false. It is also not the task of propositional logic to decide whether a statement of the type "in winter is colder than outside" is a proposition or not (i.e. if it has the property of being either true or false). In this particular case, the statement looks rather meaningless.

Definition 1.2 (Tautology)**.** *A proposition which is always true is called a **tautology**.*

Definition 1.3 (Contradiction)**.** *A proposition which is always false is called a **contradiction**.*

It is possible to build new propositions from given ones using *logical operators*. The simplest kind of logical operators are *unary* operators, which take in one proposition and return another proposition. There are four unary operators in total, and they differ by the truth value of the resulting proposition which, in general, depends on the truth value of p. We can represent them in a table as follows:

p	$\neg p$	$\mathrm{id}(p)$	$\top p$	$\bot p$
F	T	F	T	F
T	F	T	T	F

where $\neg$ is the *negation* operator, id is the *identity* operator, $\top$ is the *tautology* operator and $\bot$ is the *contradiction* operator. These clearly exhaust all possibilities for unary operators.

[1] By this we mean a formal expression, with no extra structure assumed.

The next step is to consider *binary* operators, i.e. operators that take in two propositions and return a new proposition. There are four combinations of the truth values of two propositions and, since a binary operator assigns one of the two possible truth values to each of those, we have 16 binary operators in total. The operators $\wedge$, $\vee$ and $\veebar$, called *and*, *or* and *exclusive or* respectively, should already be familiar to you.

p	q	$p \wedge q$	$p \vee q$	$p \veebar q$
F	F	F	F	F
F	T	F	T	T
T	F	F	T	T
T	T	T	T	F

There is one binary operator, the *implication* operator $\Rightarrow$, which is sometimes a little ill understood, unless you are already very knowledgeable about these things. Its usefulness comes in conjunction with the *equivalence* operator $\Leftrightarrow$. We have:

p	q	$p \Rightarrow q$	$p \Leftrightarrow q$
F	F	T	T
F	T	T	F
T	F	F	F
T	T	T	T

While the fact that the proposition $p \Rightarrow q$ is true whenever p is false may be surprising at first, it is just the definition of the implication operator and it is an expression of the principle "Ex falso quod libet", that is, from a false assumption anything follows. Of course, you may be wondering why on earth we would want to define the implication operator in this way. The answer to this is hidden in the following result.

Theorem 1.1. *Let p, q be propositions. Then $(p \Rightarrow q) \Leftrightarrow ((\neg q) \Rightarrow (\neg p))$.*

Proof. We simply construct the truth tables for $p \Rightarrow q$ and $(\neg q) \Rightarrow (\neg p)$.

p	q	$\neg p$	$\neg q$	$p \Rightarrow q$	$(\neg q) \Rightarrow (\neg p)$
F	F	T	T	T	T
F	T	T	F	T	T
T	F	F	T	F	F
T	T	F	F	T	T

The columns for $p \Rightarrow q$ and $(\neg q) \Rightarrow (\neg p)$ are identical and hence we are done. $\square$

Remark 1.1. We agree on decreasing binding strength in the sequence:

$$\neg\,,\ \wedge\,,\ \vee\,,\ \Rightarrow\,,\ \Leftrightarrow\,.$$

For example, $(\neg q) \Rightarrow (\neg p)$ may be written unambiguously as $\neg q \Rightarrow \neg p$.

Remark 1.2. All higher order operators $\heartsuit(p_1, \ldots, p_N)$ can be constructed from a single binary operator defined by:

p	q	$p \uparrow q$
F	F	T
F	T	T
T	F	T
T	T	F

This is called the *nand* operator and, in fact, we have $(p \uparrow q) \Leftrightarrow \neg(p \wedge q)$.

1.2 Predicate Logic

Definition 1.4 (Predicate)**.** *A* ***predicate*** *is a proposition-valued function of some variable or variables.*

Definition 1.5 (Relation)**.** *A predicate of two variables is called a* ***relation****.*

For example, $P(x)$ is a proposition for each choice of the variable x, and its truth value depends on x. Similarly, the predicate $Q(x, y)$ is, for any choice of x and y, a proposition and its truth value depends on x and y.

Just like for propositional logic, it is not the task of predicate logic to examine how predicates are built from the variables on which they depend. In order to do that, one would need some further language establishing the rules to combine the variables x and y into a predicate. Also, you may want to specify from which "set" x and y come from. Instead, we leave it completely open, and simply consider x and y formal variables, with no extra conditions imposed.

This may seem a bit weird since from elementary school one is conditioned to always ask where "x" comes from upon seeing an expression like $P(x)$. However, it is crucial that we refrain from doing this here, since we want to only later define the notion of set, using the language of propositional and predicate logic. As with propositions, we can construct new predicates from given ones by using the operators define in the previous section. For example, we might have:

$$Q(x, y, z) :\Leftrightarrow P(x) \wedge R(y, z),$$

where the symbol $:\Leftrightarrow$ means "defined as being equivalent to". More interestingly, we can construct a new proposition from a given predicate by using *quantifiers*.

Definition 1.6 (Universal Quantifier)**.** *Let $P(x)$ be a predicate. Then:*

$$\forall\, x : P(x),$$

is a proposition, which we read as "for all x, P of x (is true)", and it is defined to be true if $P(x)$ is true independently of x, false otherwise. The symbol $\forall$ is called ***universal quantifier****.*

Definition 1.7 (Existential Quantifier)**.** *Let $P(x)$ be a predicate. Then we define:*

$$\exists\, x : P(x) :\Leftrightarrow \neg(\forall\, x : \neg P(x)).$$

The proposition $\exists\, x : P(x)$ is read as "there exists (at least one) x such that P of x (is true)" and the symbol $\exists$ is called ***existential quantifier****.*

The following result is an immediate consequence of these definitions.

Corollary 1.1. *Let $P(x)$ be a predicate. Then:*

$$\forall\, x : P(x) \Leftrightarrow \neg(\exists\, x : \neg P(x)).$$

Remark 1.3. It is possible to define quantification of predicates of more than one variable. In order to do so, one proceeds in steps quantifying a predicate of one variable at each step.

Example 1.1. Let $P(x, y)$ be a predicate. Then, for fixed y, $P(x, y)$ is a predicate of one variable and we define:

$$Q(y) :\Leftrightarrow \forall\, x : P(x, y).$$

Hence we may have the following:

$$\exists\, y : \forall\, x : P(x, y) :\Leftrightarrow \exists\, y : Q(y).$$

Other combinations of quantifiers are defined analogously.

Remark 1.4. The order of quantification matters (if the quantifiers are not all the same). For a given predicate $P(x, y)$, the propositions:

$$\exists\, y : \forall\, x : P(x, y) \quad \text{and} \quad \forall\, x : \exists\, y : P(x, y)$$

are not necessarily equivalent.

Example 1.2. Consider the proposition expressing the existence of additive inverses in the real numbers. We have:

$$\forall\, x : \exists\, y : \; x + y = 0,$$

i.e. for each x there exists an inverse y such that $x + y = 0$. For 1 this is -1, for 2 it is -2 etc. Consider now the proposition obtained by swapping the quantifiers in the previous proposition:

$$\exists\, y : \forall\, x : \; x + y = 0.$$

What this proposition is saying is that there exists a real number y such that, no matter what x is, we have $x + y = 0$. This is clearly false, since if $x + y = 0$ for some x then $(x + 1) + y \neq 0$, so the same y cannot work for both x and $x + 1$, let alone every x.

Notice that the proposition $\exists\, x : P(x)$ means "there exists *at least one* x such that $P(x)$ is true". Often in mathematics we prove that "there exists *a unique* x such that $P(x)$ is true". We therefore have the following definition.

Definition 1.8 (Unique Existential Quantifier). *Let $P(x)$ be a predicate. We define the* ***unique existential quantifier*** $\exists!$ *by:*

$$\exists!\, x : P(x) :\Leftrightarrow (\exists\, x : P(x)) \wedge \forall\, y : \forall\, z : (P(y) \wedge P(z) \Rightarrow y = z).$$

This definition clearly separates the existence condition from the uniqueness condition. An equivalent definition with the advantage of brevity is:

$$\exists!\, x : P(x) :\Leftrightarrow (\exists\, x : \forall\, y : P(y) \Leftrightarrow x = y)$$

1.3 Axiomatic Systems And Theory Of Proofs

Definition 1.9 (Axiomatic System). *An* ***axiomatic system*** *is a finite sequence of propositions $a_1, a_2, \ldots, a_N$, which are called the* axioms *of the system.*

Definition 1.10 (Proof). *A* ***proof*** *of a proposition p within an axiomatic system $a_1, a_2, \ldots, a_N$ is a finite sequence of propositions $q_1, q_2, \ldots, q_M$ such that $q_M = p$ and for any $1 \leq j \leq M$ one of the following is satisfied:*

- *(A) q_j is a proposition from the list of axioms;*
- *(T) q_j is a tautology;*
- *(M) $\exists\, 1 \leq m, n < j : (q_m \wedge q_n \Rightarrow q_j)$ is true.*

Remark 1.5. If p can be proven within an axiomatic system $a_1, a_2, \ldots, a_N$, we write:

$$a_1, a_2, \ldots, a_N \vdash p$$

and we read "$a_1, a_2, \ldots, a_N$ proves p".

Remark 1.6. This definition of proof allows to easily recognise a proof. A computer could easily check that whether or not the conditions (A), (T) and (M) are satisfied by a sequence of propositions. To actually find a proof of a proposition is a whole different story.

Remark 1.7. Obviously, any tautology that appears in the list of axioms of an axiomatic system can be removed from the list without impairing the power of the axiomatic system.

An extreme case of an axiomatic system is propositional logic. The axiomatic system for propositional logic is the empty sequence. This means that all we can prove in propositional logic are tautologies.

Definition 1.11 (Consistent). *An axiomatic system $a_1, a_2, \ldots, a_N$ is said to be* ***consistent*** *if there exists a proposition q which cannot be proven from the axioms. In symbols:*

$$\exists\, q : \neg(a_1, a_2, \ldots, a_N \vdash q).$$

The idea behind this definition is the following. Consider an axiomatic system which contains contradicting propositions:

$$a_1, \ldots, s, \ldots, \neg s, \ldots, a_N.$$

Then, given *any* proposition q, the following is a proof of q within this system:

$$s, \neg s, q.$$

Indeed, s and $\neg s$ are legitimate steps in the proof since they are axioms. Moreover, $s \wedge \neg s$ is a contradiction and thus $(s \wedge \neg s) \Rightarrow q$ is a tautology. Therefore, q follows from condition (M). This shows that any proposition can be proven within a system with contradictory axioms. In other words, the inability to prove every proposition is a property possessed by no contradictory system, and hence we define a consistent system as one with this property.

Having come this far, we can now state (and prove) an impressively sounding theorem.

Theorem 1.2. *Propositional logic is consistent.*

Proof. Suffices to show that there exists a proposition that cannot be proven within propositional logic. Propositional logic has the empty sequence as axioms. Therefore, only conditions (T) and (M) are relevant here. The latter allows the insertion of a proposition q_j such that $(q_m \wedge q_n) \Rightarrow q_j$ is true, where q_m and q_n are propositions that precede q_j in the proof sequence. However, since (T) only allows the insertion of a tautology anywhere in the proof sequence, the propositions q_m and q_n must be tautologies. Consequently, for $(q_m \wedge q_n) \Rightarrow q_j$ to be true, q_j must also be a tautology. Hence, the proof sequence consists entirely of tautologies and thus only tautologies can be proven.

Now let q be any proposition. Then $q \wedge \neg q$ is a contradiction, hence not a tautology and thus cannot be proven. Therefore, propositional logic is consistent. □

Remark 1.8. While it is perfectly fine and clear how to define consistency, it is perfectly difficult to prove consistency for a given axiomatic system, propositional logic being a big exception.

Theorem 1.3. *Any axiomatic system powerful enough to encode elementary arithmetic is either inconsistent or contains an* undecidable *proposition, i.e. a proposition that can be neither proven nor disproven within the system.*

An example of an undecidable proposition is the Continuum hypothesis within the Zermelo-Fraenkel axiomatic system.

1.4 The $\in$-relation

Set theory is built on the postulate that there is a fundamental relation (i.e. a predicate of two variables) denoted $\in$ and read as "epsilon". There will be no definition of what $\in$ is, or of what a set is. Instead, we will have nine axioms concerning $\in$ and sets, and it is only in terms of these nine axioms that $\in$ and sets are defined at all. Here is an overview of the axioms. We will have:

- 2 basic existence axioms, one about the $\in$ relation and the other about the existence of the empty set;
- 4 construction axioms, which establish rules for building new sets from given ones. They are the pair set axiom, the union set axiom, the replacement axiom and the power set axiom;
- 2 further existence/construction axioms, these are slightly more advanced and newer compared to the others;
- 1 axiom of foundation, excluding some constructions as not being sets.

Using the $\in$-relation we can immediately define the following relations:

- $x \notin y :\Leftrightarrow \neg(x \in y)$
- $x \subseteq y :\Leftrightarrow \forall\, a : (a \in x \Rightarrow a \in y)$

- $x = y :\Leftrightarrow (x \subseteq y) \wedge (y \subseteq x)$
- $x \subset y :\Leftrightarrow (x \subseteq y) \wedge \neg(x = y)$

Remark 1.9. A comment about notation. Since $\in$ is a predicate of two variables, for consistency of notation we should write $\in(x, y)$. However, the notation $x \in y$ is much more common (as well as intuitive) and hence we simply define:

$$x \in y :\Leftrightarrow \in(x, y)$$

and we read "x is in (or belongs to) y" or "x is an element (or a member) of y". Similar remarks apply to the other relations $\notin$, $\subseteq$ and $=$.

1.5 Zermelo-Fraenkel Axioms Of Set Theory

Axiom on the $\in$-relation. *The expression $x \in y$ is a proposition if, and only if, both x and y are sets. In symbols:*

$$\forall\, x : \forall\, y : (x \in y) \veebar \neg(x \in y).$$

We remarked, previously, that it is not the task of predicate logic to inquire about the nature of the variables on which predicates depend. This first axiom clarifies that the variables on which the relation $\in$ depend are sets. In other words, if $x \in y$ is not a proposition (i.e. it does not have the property of being either true or false) then x and y are not both sets.

This seems so trivial that, for a long time, people thought that this not much of a condition. But, in fact, it is. It tells us when something is not a set.

Example 1.3 (Russell's paradox). Suppose that there is some u which has the following property:

$$\forall\, x : (x \notin x \Leftrightarrow x \in u),$$

i.e. u contains all the sets that are not elements of themselves, and no others. We wish to determine whether u is a set or not. In order to do so, consider the expression $u \in u$. If u is a set then, by the first axiom, $u \in u$ is a proposition.

However, we will show that his is not the case. Suppose first that $u \in u$ is true. Then $\neg(u \notin u)$ is true and thus u does not satisfy the condition for being an element of u, and hence is not an element of u. Thus:

$$u \in u \Rightarrow \neg(u \in u)$$

and this is a contradiction. Therefore, $u \in u$ cannot be true. Then, if it is a proposition, it must be false. However, if $u \notin u$, then u satisfies the condition for being a member of u and thus:

$$u \notin u \Rightarrow \neg(u \notin u)$$

which is, again, a contradiction. Therefore, $u \in u$ does not have the property of being either true or false (it can be neither) and hence it is not a proposition. Thus, our first axiom implies that u is not a set, for if it were, then $u \in u$ would be a proposition.

Remark 1.10. The fact that u as defined above is not a set means that expressions like:

$$u \in u, \quad x \in u, \quad u \in x, \quad x \notin u, \quad \text{etc.}$$

are not propositions and thus, they are not part of axiomatic set theory.

Axiom on the existence of an empty set. *There exists a set that contains no elements. In symbols:*

$$\exists\, y : \forall\, x : x \notin y.$$

Notice the use of "an" above. In fact, we have all the tools to prove that there is only one empty set. We do not need this to be an axiom.

Theorem 1.4. *There is only one empty set, and we denote it by $\varnothing$.*

Proof. Suppose that x and x' are both empty sets. Then $y \in x$ is false as x is the empty set. But then:

$$(y \in x) \Rightarrow (y \in x')$$

is true, and in particular it is true independently of y. Therefore:

$$\forall\, y : (y \in x) \Rightarrow (y \in x')$$

and hence $x \subseteq x'$. Conversely, by the same argument, we have:

$$\forall\, y : (y \in x') \Rightarrow (y \in x)$$

and thus $x' \subseteq x$. Hence $(x \subseteq x') \wedge (x' \subseteq x)$ and therefore $x = x'$. □

Axiom on pair sets. *Let x and y be sets. Then there exists a set that contains as its elements precisely x and y. In symbols:*

$$\forall\, x : \forall\, y : \exists\, m : \forall\, u : (u \in m \Leftrightarrow (u = x \vee u = y)).$$

The set m is called the *pair set* of x and y and it is denoted by $\{x, y\}$.

Remark 1.11. We have chosen $\{x, y\}$ as the notation for the pair set of x and y, but what about $\{y, x\}$? The fact that the definition of the pair set remains unchanged if we swap x and y suggests that $\{x, y\}$ and $\{y, x\}$ are the same set. Indeed, by definition, we have:

$$(a \in \{x, y\} \Rightarrow a \in \{y, x\}) \wedge (a \in \{y, x\} \Rightarrow a \in \{x, y\})$$

independently of a, hence $(\{x, y\} \subseteq \{y, x\}) \wedge (\{y, x\} \subseteq \{x, y\})$ and thus $\{x, y\} = \{y, x\}$.

The pair set $\{x, y\}$ is thus an unordered pair. However, using the axiom on pair sets, it is also possible to define an *ordered pair* (x, y) such that $(x, y) \neq (y, x)$. The defining property of an ordered pair is the following:

$$(x, y) = (a, b) \Leftrightarrow x = a \wedge y = b.$$

One candidate which satisfies this property is $(x, y) := \{x, \{x, y\}\}$, which is a set by the axiom on pair sets.

Remark 1.12. The pair set axiom also guarantees the existence of one-element sets, called *singletons*. If x is a set, then we define $\{x\} := \{x, x\}$. Informally, we can say that $\{x\}$ and $\{x, x\}$ express the same amount of information, namely that they contain x.

Axiom on union sets. *Let x be a set. Then there exists a set whose elements are precisely the elements of the elements of x. In symbols:*

$$\forall\, x : \exists\, u : \forall\, y : (y \in u \Leftrightarrow \exists\, s : (y \in s \wedge s \in x))$$

The set u is denoted by $\bigcup x$.

Example 1.4. Let a, b be sets. Then $\{a\}$ and $\{b\}$ are sets by the pair set axiom, and hence $x := \{\{a\}, \{b\}\}$ is a set, again by the pair set axiom. Then the expression:

$$\bigcup x = \{a, b\}$$

is a set by the union axiom.

Notice that, since a and b are sets, we could have immediately concluded that $\{a, b\}$ is a set by the pair set axiom. The union set axiom is really needed to construct sets with more than 2 elements.

Example 1.5. Let a, b, c be sets. Then $\{a\}$ and $\{b, c\}$ are sets by the pair set axiom, and hence $x := \{\{a\}, \{b, c\}\}$ is a set, again by the pair set axiom. Then the expression:

$$\bigcup x =: \{a, b, c\}$$

is a set by the union set axiom. This time the union set axiom was really necessary to establish that $\{a, b, c\}$ is a set, i.e. in order to be able to use it meaningfully in conjunction with the $\in$-relation.

The previous example easily generalises to a definition.

Definition 1.12 (Union Of Sets)**.** *Let $a_1, a_2, \ldots, a_N$ be sets. We define recursively for all $N \geq 2$:*

$$\{a_1, a_2, \ldots, a_{N+1}\} := \bigcup \{\{a_1, a_2, \ldots, a_N\}, \{a_{N+1}\}\}.$$

Remark 1.13. The fact that the x that appears in $\bigcup x$ has to be a set is a crucial restriction. Informally, we can say that it is only possible to take unions of as many sets as would fit into a set. The "collection" of all the sets that do not contain themselves is not a set or, we could say, does not fit into a set. Therefore it is not possible to take the union of all the sets that do not contain themselves. This is very subtle, but also very precise.

Axiom of replacement. *Let R be a functional relation and let m be a set. Then the image of m under R, denoted by $\mathrm{im}_R(m)$, is again a set.*

Of course, we now need to define the new terms that appear in this axiom. Recall that a relation is simply a predicate of two variables.

Definition 1.13 (Functional Relation)**.** *A relation R is said to be* ***functional*** *if:*

$$\forall\, x : \exists!\, y : R(x, y).$$

Definition 1.14 (Image Of A Set Under A Relational Functional Relation)**.** *Let m be a set and let R be a functional relation. The* ***image of*** *m* ***under*** *R consists of all those y for which there is an $x \in m$ such that $R(x, y)$.*

None of the previous axioms imply that the image of a set under a functional relation is again a set. The assumption that it always is, is made explicit by the axiom of replacement.

It is very likely that the reader has come across a weaker form of the axiom of replacement, called the *principle of restricted comprehension*, which says the following.

Proposition 1.1. *Let $P(x)$ be a predicate and let m be a set. Then the elements $y \in m$ such that $P(y)$ is true constitute a set, which we denote by:*

$$\{y \in m \mid P(y)\}.$$

Remark 1.14. The principle of restricted comprehension is not to be confused with the "principle" of universal comprehension which states that $\{y \mid P(y)\}$ is a set for any predicate and was shown to be inconsistent by Russell. Observe that the $y \in m$ condition makes it so that $\{y \in m \mid P(y)\}$ cannot have more elements than m itself.

Remark 1.15. If y is a set, we define the following notation:

$$\forall\, x \in y : P(x) :\Leftrightarrow \forall\, x : (x \in y \Rightarrow P(x))$$

and:

$$\exists\, x \in y : P(x) :\Leftrightarrow \neg(\forall\, x \in y : \neg P(x)).$$

Pulling the $\neg$ through, we can also write:

$$\begin{aligned}
\exists\, x \in y : P(x) &\Leftrightarrow \neg(\forall\, x \in y : \neg P(x)) \\
&\Leftrightarrow \neg(\forall\, x : (x \in y \Rightarrow \neg P(x))) \\
&\Leftrightarrow \exists\, x : \neg(x \in y \Rightarrow \neg P(x))) \\
&\Leftrightarrow \exists\, x : (x \in y \wedge P(x)),
\end{aligned}$$

where we have used the equivalence $(p \Rightarrow q) \Leftrightarrow \neg(p \wedge \neg q)$.

The principle of restricted comprehension is a consequence of the axiom of replacement.

Proof. We have two cases.

1. If $\neg(\exists\, y \in m : P(y))$, then we define: $\{y \in m \mid P(y)\} := \varnothing$.

2. If $\exists\, \hat{y} \in m : P(\hat{y})$, then let R be the functional relation:

$$R(x,y) := (P(x) \wedge x = y) \vee (\neg P(x) \wedge \hat{y} = y)$$

and hence define $\{y \in m \mid P(y)\} := \mathrm{im}_R(m)$. $\square$

Don't worry if you don't see this immediately. You need to stare at the definitions for a while and then it will become clear.

Remark 1.16. We will rarely invoke the axiom of replacement in full. We will only invoke the weaker principle of restricted comprehension, with which we are all familiar with.

We can now define the intersection and the relative complement of sets.

Definition 1.15 (Intersection)**.** *Let x be a set. Then we define the* ***intersection*** *of x by:*

$$\bigcap x := \{a \in \bigcup x \mid \forall\, b \in x : a \in b\}.$$

If $a, b \in x$ and $\bigcap x = \varnothing$, then a and b are said to be disjoint.

Definition 1.16 (Complement)**.** *Let u and m be sets such that $u \subseteq m$. Then the* ***complement*** *of u relative to m is defined as:*

$$m \setminus u := \{x \in m \mid x \notin u\}.$$

These are both sets by the principle of restricted comprehension, which is ultimately due to axiom of replacement.

Axiom on the existence of power sets. *Let m be a set. Then there exists a set, denoted by $\mathcal{P}(m)$, whose elements are precisely the subsets of m. In symbols:*

$$\forall\, x : \exists\, y : \forall\, a : (a \in y \Leftrightarrow a \subseteq x).$$

Historically, in naïve set theory, the principle of universal comprehension was thought to be needed in order to define the power set of a set. Traditionally, this would have been (inconsistently) defined as:

$$\mathcal{P}(m) := \{y \mid y \subseteq m\}.$$

To define power sets in this fashion, we would need to know, a priori, from which "bigger" set the elements of the power set come from. However, this in not possible based only on the previous axioms and, in fact, there is no other choice but to dedicate an additional axiom for the existence of power sets.

Example 1.6. Let $m = \{a, b\}$. Then $\mathcal{P}(m) = \{\varnothing, \{a\}, \{b\}, \{a, b\}\}$.

Remark 1.17. If one defines $(a, b) := \{a, \{a, b\}\}$, then the *cartesian product* $x \times y$ of two sets x and y, which informally is the set of all ordered pairs of elements of x and y, satisfies:

$$x \times y \subseteq \mathcal{P}(\mathcal{P}(\bigcup \{x, y\})).$$

Hence, the existence of $x \times y$ as a set follows from the axioms on unions, pair sets, power sets and the principle of restricted comprehension.

Axiom of infinity. *There exists a set that contains the empty set and, together with every other element y, it also contains the set $\{y\}$ as an element. In symbols:*

$$\exists\, x : \varnothing \in x \wedge \forall\, y : (y \in x \Rightarrow \{y\} \in x).$$

Let us consider one such set x. Then $\varnothing \in x$ and hence $\{\varnothing\} \in x$. Thus, we also have $\{\{\varnothing\}\} \in x$ and so on. Therefore:

$$x = \{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}, \{\{\{\varnothing\}\}\}, \ldots\}.$$

We can introduce the following notation for the elements of x:

$$0 := \varnothing, \quad 1 := \{\varnothing\}, \quad 2 := \{\{\varnothing\}\}, \quad 3 := \{\{\{\varnothing\}\}\}, \quad \ldots$$

Corollary 1.2. *The "set" $\mathbb{N} := x$ is a set according to axiomatic set theory.*

This would not be then case without the axiom of infinity since it is not possible to prove that $\mathbb{N}$ constitutes a set from the previous axioms.

Remark 1.18. At this point, one might suspect that we would need an extra axiom for the existence of the real numbers. But, in fact, we can define $\mathbb{R} := \mathcal{P}(\mathbb{N})$, which is a set by the axiom on power sets.

Remark 1.19. The version of the axiom of infinity tat we stated is the one that was first put forward by Zermelo. A more modern formulation is the following. *There exists a set that contains the empty set and, together with every other element y, it also contains the set $y \cup \{y\}$ as an element.* Here we used the notation:

$$x \cup y := \bigcup \{x, y\}.$$

With this formulation, the natural numbers look like:

$$\mathbb{N} := \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}, \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}, \ldots\}$$

This may appear more complicated than what we had before, but it is much nicer for two reasons. First, the natural number n is represented by an n-element set rather than a one-element set. Second, it generalizes much more naturally to the system of transfinite ordinal numbers where the successor operation $s(x) = x \cup \{x\}$ applies to transfinite ordinals as well as natural numbers. Moreover, the natural numbers have the same defining property as the ordinals: they are transitive sets strictly well-ordered by the $\in$-relation.

Axiom of choice. *Let x be a set whose elements are non-empty and mutually disjoint. Then there exists a set y which contains exactly one element of each element of x. In symbols:*

$$\forall\, x : P(x) \Rightarrow \exists\, y : \forall\, a \in x : \exists!\, b \in a : a \in y,$$

where $P(x) \Leftrightarrow (\exists\, a : a \in x) \wedge (\forall\, a : \forall\, b : (a \in x \wedge b \in x) \Rightarrow \bigcap \{a, b\} = \varnothing)$.

Remark 1.20. The axiom of choice is independent of the other 8 axioms, which means that one could have set theory with or without the axiom of choice. However, standard mathematics uses the axiom of choice and hence so will we. There is a number of theorems that can only be proved by using the axiom of choice. Amongst these we have:

- every vector space has a basis;
- there exists a complete system of representatives of an equivalence relation.

Axiom of foundation. *Every non-empty set x contains an element y that has none of its elements in common with x. In symbols:*

$$\forall\, x : (\exists\, a : a \in x) \Rightarrow \exists\, y \in x : \bigcap \{x, y\} = \varnothing.$$

An immediate consequence of this axiom is that there is no set that contains itself as an element.

The totality of all these nine axioms are called *ZFC set theory*, which is a shorthand for Zermelo-Fraenkel set theory with the axiom of Choice.

1.6 Maps Between Sets

A recurrent theme in mathematics is the classification of *spaces* by means of structure-preserving *maps* between them.

A space is usually meant to be some set equipped with some structure, which is usually some other set. We will define each instance of space precisely when we will need them. In the case of sets considered themselves as spaces, there is no extra structure beyond the set and hence, the structure may be taken to be the empty set.

Definition 1.17 (Map)**.** *Let A, B be sets. A **map** $\phi \colon A \to B$ is a relation such that for each $a \in A$ there exists exactly one $b \in B$ such that $\phi(a, b)$.*

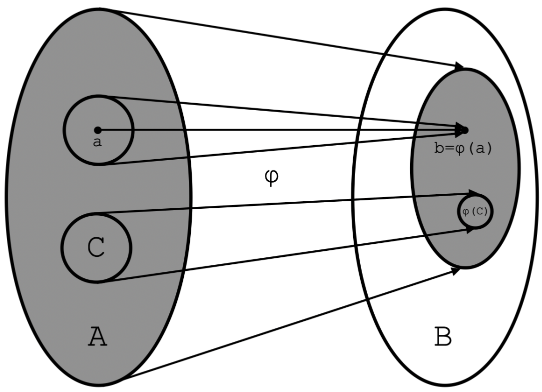
The standard notation for a map is:

$$\begin{aligned} \phi\colon A &\to B \\ a &\mapsto \phi(a) \end{aligned}$$

which is technically an abuse of notation since ϕ, being a relation of two variables, should have two arguments and produce a truth value. However, once we agree that for each $a \in A$ there exists exactly one $b \in B$ such that $\phi(a, b)$ is true, then for each a we can define $\phi(a)$ to be precisely that unique b. It is sometimes useful to keep in mind that ϕ is actually a relation.

Example 1.7. Let M be a set. The simplest example of a map is the *identity map* on M:

$$\begin{aligned} \mathrm{id}_M\colon M &\to M \\ m &\mapsto m. \end{aligned}$$



The following is standard terminology for a map $\phi\colon A \to B$:

- the set A is called the **domain** of ϕ;
- the set B is called the **codomain** or the **target** of ϕ;
- if a is an element of A, then $\phi(a) =$ b (the value of ϕ when applied to a) is called the **image of element** or the **output** of a under ϕ;
- if C is a subset of A, then $\phi(C)$ (the set of values of ϕ when applied to C) is called the **image of subset** of C under ϕ;
- the set of all elements that the map ϕ can hit in the target B (grey area in B) is called the **image** or the **range** of A under ϕ (in other words the image of a map is simply the image of its entire domain). Notice that since a map ϕ hits every point of the domain A, the whole domain A is covered by ϕ (grey area in A). However it is not necessary that the mapping will also cover the whole target B. This is why the image of a map is not necessarily equal to the whole target;
- the set of all elements of the domain A that are mapped into a given single element b of the target B is called the **fiber** of the element b under ϕ;
- the subset C of all elements of the domain A that are mapped into a subset $\phi(C)$ of the target B is called the **preimage** or the **inverse image** of $\phi(C)$ under ϕ;
- a map ϕ is called **injective** or an **injection** or **one-to-one** if distinct elements of the domain A map to distinct elements in the target B, or equivalently if each element of the target B is mapped to by at most one element of the domain A: $\forall\, a_1, a_2 \in A : \phi(a_1) = \phi(a_2) \Rightarrow a_1 = a_2$;
- a map ϕ is called **surjective** or a **surjection** or **onto** if its image is equal to the entire domain A, or equivalently if each element of the target B is mapped to by at least one element of the domain A: $\mathrm{im}_\phi(A) = B$

- a map ϕ is called **bijective** or a **bijection** or **one-to-one and onto** if it is both injective and surjective.

Definition 1.18 (Isomorphic Sets)**.** *Two sets A and B are called* ***isomorphic*** *if there exists a bijection $\phi\colon A \to B$. In this case, we write $A \cong_{\text{set}} B$.*

Remark 1.21. If there is any bijection $A \to B$ then generally there are many.

Bijections are the "structure-preserving" maps for sets. Intuitively, they pair up the elements of A and B and a bijection between A and B exists only if A and B have the same "size". This is clear for finite sets, but it can also be extended to infinite sets.

Definition 1.19 (Infinite/Finite Sets)**.** *A set A is called:*

- infinite *if there exists a proper subset $B \subset A$ such that $B \cong_{\text{set}} A$. In particular, if A is infinite, we further define A to be:*
  - ∗ countably *infinite if $A \cong_{\text{set}} \mathbb{N}$;*
  - ∗ uncountably *infinite otherwise.*
- finite *if it is not infinite. In this case, we have $A \cong_{\text{set}} \{1, 2, \dots, N\}$ for some $N \in \mathbb{N}$ and we say that the* cardinality *of A, denoted by $|A|$, is N.*

Given two maps $\phi\colon A \to B$ and $\psi\colon B \to C$, we can construct a third map, called the *composition* of ϕ and ψ, denoted by $\psi \circ \phi$ (read "psi after phi"), defined by:

$$\begin{aligned} \psi \circ \phi\colon A &\to C \\ a &\mapsto \psi(\phi(a)). \end{aligned}$$

This is often represented by drawing the following diagram

$$\begin{tikzcd} & B \arrow[dr, "\psi"] & \\ A \arrow[ur, "\phi"] \arrow[rr, "\psi\circ\phi"'] & & C \end{tikzcd}$$

and by saying that "the diagram commutes" (although sometimes this is assumed even if it is not explicitly stated). What this means is that every path in the diagram gives the same result. This might seem notational overkill at this point, but later we will encounter situations where we will have many maps, going from many places to many other places and these diagrams greatly simplify the exposition.

Proposition 1.2. *Composition of maps is associative.*

Proof. Indeed, let $\phi\colon A \to B$, $\psi\colon B \to C$ and $\xi\colon C \to D$ be maps. Then we have:

$$\begin{aligned} \xi \circ (\psi \circ \phi)\colon A &\to D \\ a &\mapsto \xi(\psi(\phi(a))) \end{aligned}$$

and:

$$\begin{aligned} (\xi \circ \psi) \circ \phi\colon A &\to D \\ a &\mapsto \xi(\psi(\phi(a))). \end{aligned}$$

Thus $\xi \circ (\psi \circ \phi) = (\xi \circ \psi) \circ \phi$. □

The operation of composition is necessary in order to defined inverses of maps.

Definition 1.20 (Inverse)**.** *Let $\phi\colon A \to B$ be a bijection. Then the* ***inverse*** *of ϕ, denoted ϕ^{-1}, is defined (uniquely) by:*

$$\phi^{-1} \circ \phi = \mathrm{id}_A$$

$$\phi \circ \phi^{-1} = \mathrm{id}_B\,.$$

Equivalently, we require the following diagram to commute:

$$
\begin{tikzcd}
A \arrow[loop left, "\mathrm{id}_A"] \arrow[rr, bend left, "\phi"] & & B \arrow[ll, bend left, "\phi^{-1}"] \arrow[loop right, "\mathrm{id}_B"]
\end{tikzcd}
$$

The inverse map is only defined for bijections. However, the notion of the pre-image, which we will often meet in topology, is defined for any map. Given the inverse map we can define the pre-image in a more systematic way as:

Definition 1.21 (Pre-image)**.** *Let $\phi\colon A \to B$ be a map and let $V \subseteq B$. Then we define the set:*

$$\mathrm{preim}_\phi(V) := \{a \in A \mid \phi(a) \in V\}$$

*called the **pre-image** of V under ϕ.*

Proposition 1.3. *Let $\phi\colon A \to B$ be a map, let $U, V \subseteq B$ and $C = \{C_j \mid j \in J\} \subseteq \mathcal{P}(B)$. Then:*

i) $\mathrm{preim}_\phi(\varnothing) = \varnothing$ *and* $\mathrm{preim}_\phi(B) = A$*;*

ii) $\mathrm{preim}_\phi(U \setminus V) = \mathrm{preim}_\phi(U) \setminus \mathrm{preim}_\phi(V)$*;*

iii) $\mathrm{preim}_\phi(\bigcup C) = \bigcup_{j\in J} \mathrm{preim}_\phi(C_j)$ *and* $\mathrm{preim}_\phi(\bigcap C) = \bigcap_{j\in J} \mathrm{preim}_\phi(C_j)$*.*

Proof. i) By definition, we have:

$$\mathrm{preim}_\phi(B) = \{a \in A : \phi(a) \in B\} = A$$

and:

$$\mathrm{preim}_\phi(\varnothing) = \{a \in A : \phi(a) \in \varnothing\} = \varnothing.$$

ii) We have:

$$\begin{aligned}
a \in \mathrm{preim}_\phi(U \setminus V) &\Leftrightarrow \phi(a) \in U \setminus V \\
&\Leftrightarrow \phi(a) \in U \wedge \phi(a) \notin V \\
&\Leftrightarrow a \in \mathrm{preim}_\phi(U) \wedge a \notin \mathrm{preim}_\phi(V) \\
&\Leftrightarrow a \in \mathrm{preim}_\phi(U) \setminus \mathrm{preim}_\phi(V)
\end{aligned}$$

iii) We have:

$$\begin{aligned}
a \in \mathrm{preim}_\phi(\textstyle\bigcup C) &\Leftrightarrow \phi(a) \in \textstyle\bigcup C \\
&\Leftrightarrow \textstyle\bigvee_{j\in J}(\phi(a) \in C_j) \\
&\Leftrightarrow \textstyle\bigvee_{j\in J}(a \in \mathrm{preim}_\phi(C_j)) \\
&\Leftrightarrow a \in \textstyle\bigcup_{j\in J} \mathrm{preim}_\phi(C_j)
\end{aligned}$$

Similarly, we get $\mathrm{preim}_\phi(\bigcap C) = \bigcap_{j\in J} \mathrm{preim}_\phi(C_j)$. □

1.7 Equivalence Relations

Definition 1.22 (Equivalence Relation)**.** *Let M be a set and let $\sim$ be a relation such that the following conditions are satisfied:*

i) reflexivity: $\forall\, m \in M : m \sim m$;

ii) symmetry: $\forall\, m, n \in M : m \sim n \Leftrightarrow n \sim n$;

iii) transitivity: $\forall\, m, n, p \in M : (m \sim n \wedge n \sim p) \Rightarrow m \sim p$.

*Then $\sim$ is called an **equivalence relation** on M.*

Example 1.8. Consider the following wordy examples.

a) $p \sim q :\Leftrightarrow p$ is of the same opinion as q. This relation is reflexive, symmetric and transitive. Hence, it is an equivalence relation.

b) $p \sim q :\Leftrightarrow p$ is a sibling of q. This relation is symmetric and transitive but not reflexive and hence, it is not an equivalence relation.

c) $p \sim q :\Leftrightarrow p$ is taller q. This relation is transitive, but neither reflexive nor symmetric and hence, it is not an equivalence relation.

d) $p \sim q :\Leftrightarrow p$ is in love with q. This relation is generally not reflexive. People don't like themselves very much. It is certainly not normally symmetric, which is the basis of much drama in literature. It is also not transitive, except in some French films.

Definition 1.23 (Equivalence Class)**.** *Let $\sim$ be an equivalence relation on the set M. Then, for any $m \in M$, we define the set:*

$$[m] := \{n \in M \mid m \sim n\}$$

called the ***equivalence class*** *of m. Note that the condition $m \sim n$ is equivalent to $n \sim m$ since $\sim$ is symmetric.*

The following are two key properties of equivalence classes.

Proposition 1.4. *Let $\sim$ be an equivalence relation on M. Then:*

i) $a \in [m] \Rightarrow [a] = [m]$*;*

ii) either $[m] = [n]$ *or* $[m] \cap [n] = \varnothing$*.*

Proof. i) Since $a \in [m]$, we have $a \sim m$. Let $x \in [a]$. Then $x \sim a$ and hence $x \sim m$ by transitivity. Therefore $x \in [m]$ and hence $[a] \subseteq [m]$. Similarly, we have $[m] \subseteq [a]$ and hence $[a] = [m]$.

ii) Suppose that $[m] \cap [n] \neq \varnothing$. That is:

$$\exists\, z : z \in [m] \wedge z \in [n].$$

Thus $z \sim m$ and $z \sim n$ and hence, by symmetry and transitivity, $m \sim n$. This implies that $m \in [n]$ and hence that $[m] = [n]$. □

Definition 1.24 (Quotient Set)**.** *Let $\sim$ be an equivalence relation on M. Then we define the* ***quotient set*** *of M by $\sim$ as:*

$$M/\!\sim\; := \{[m] \mid m \in M\}.$$

This is indeed a set since $[m] \subseteq \mathcal{P}(M)$ and hence we can write more precisely:

$$M/\!\sim\; := \{[m] \in \mathcal{P}(M) \mid m \in M\}.$$

Then clearly $M/\!\sim$ is a set by the power set axiom and the principle of restricted comprehension.

Remark 1.22. Due to the axiom of choice, there exists a complete system of representatives for $\sim$, i.e. a set R such that $R \cong_{\text{set}} M/\!\sim$.

Remark 1.23. Care must be taken when defining maps whose domain is a quotient set if one uses representatives to define the map. In order for the map to be *well-defined* one needs to show that the map is independent of the choice of representatives.

Example 1.9. Let $M = \mathbb{Z}$ and define $\sim$ by:

$$m \sim n :\Leftrightarrow n - m \in 2\mathbb{Z}.$$

It is easy to check that $\sim$ is indeed an equivalence relation. Moreover, we have:

$$[0] = [2] = [4] = \cdots = [-2] = [-4] = \cdots$$

and:

$$[1] = [3] = [5] = \cdots = [-1] = [-3] = \cdots$$

Thus we have: $\mathbb{Z}/\sim\, = \{[0],[1]\}$. We wish to define an addition $\oplus$ on $\mathbb{Z}/\sim$ by inheriting the usual addition on $\mathbb{Z}$. As a tentative definition we could have:

$$\oplus\colon \mathbb{Z}/{\sim} \times \mathbb{Z}/{\sim} \to \mathbb{Z}/{\sim}$$

being given by:

$$[a] \oplus [b] := [a+b].$$

However, we need to check that our definition does not depend on the choice of class representatives, i.e. if $[a] = [a']$ and $[b] = [b']$, then we should have:

$$[a] \oplus [b] = [a'] \oplus [b'].$$

Indeed, $[a] = [a']$ and $[b] = [b']$ means $a - a' \in 2\mathbb{Z}$ and $b - b' \in 2\mathbb{Z}$, i.e. $a - a' = 2m$ and $b - b' = 2n$ for some $m, n \in \mathbb{Z}$. We thus have:

$$\begin{aligned}[a' + b'] &= [a - 2m + b - 2n] \\ &= [(a+b) - 2(m+n)] \\ &= [a+b],\end{aligned}$$

where the last equality follows since:

$$(a+b) - 2(m+n) - (a+b) = -2(m+n) \in 2\mathbb{Z}.$$

Therefore $[a'] \oplus [b'] = [a] \oplus [b]$ and hence the operation $\oplus$ is well-defined.

Example 1.10. As a counterexample, with the same set-up as in the previous example, let us define an operation $\star$ by:

$$[a] \star [b] := \frac{a}{b}.$$

This is easily seen to be *ill-defined* since $[1] = [3]$ and $[2] = [4]$ but:

$$[1] \star [2] = \frac{1}{2} \neq \frac{3}{4} = [3] \star [4].$$

1.8 Construction of $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$

Recall that, invoking the axiom of infinity, we defined the natural numbers:

$$\mathbb{N} := \{0, 1, 2, 3, \ldots\},$$

where:

$$0 := \varnothing, \quad 1 := \{\varnothing\}, \quad 2 := \{\{\varnothing\}\}, \quad 3 := \{\{\{\varnothing\}\}\}, \quad \ldots$$

We would now like to define an addition operation on $\mathbb{N}$ by using the axioms of set theory. We will need some preliminary definitions.

Definition 1.25 (Successor Map)**.** *The* ***successor map*** *S on $\mathbb{N}$ is defined by:*

$$\begin{aligned} S\colon \mathbb{N} &\to \mathbb{N} \\ n &\mapsto \{n\}. \end{aligned}$$

Example 1.11. Consider $S(2)$. Since $2 := \{\{\varnothing\}\}$, we have $S(2) = \{\{\{\varnothing\}\}\} =: 3$. Therefore, we have $S(2) = 3$ as we would have expected.

To make progress, we also need to define the predecessor map, which is only defined on the set $\mathbb{N}^* := \mathbb{N} \setminus \{\varnothing\}$.

Definition 1.26 (Predecessor Map)**.** *The* ***predecessor map*** *P on $\mathbb{N}^*$ is defined by:*

$$\begin{aligned} P\colon \mathbb{N}^* &\to \mathbb{N} \\ n &\mapsto m \quad \textit{such that} \quad m \in n. \end{aligned}$$

Example 1.12. We have $P(2) = P(\{\{\varnothing\}\}) = \{\varnothing\} = 1$.

Definition 1.27 (n-th Power)**.** *Let $n \in \mathbb{N}$. The n**-th power** of S, denoted S^n, is defined recursively by:*

$$\begin{aligned} S^n &:= S \circ S^{P(n)} \qquad \text{if } n \in \mathbb{N}^* \\ S^0 &:= \mathrm{id}_{\mathbb{N}} \,. \end{aligned}$$

We are now ready to define addition.

Definition 1.28 (Addition Of Natural Numbers)**.** *The **addition** operation on $\mathbb{N}$ is defined as a map:*

$$\begin{aligned} +\colon \mathbb{N} \times \mathbb{N} &\to \mathbb{N} \\ (m, n) &\mapsto m + n := S^n(m). \end{aligned}$$

Example 1.13. We have:

$$2 + 1 = S^1(2) = S(2) = 3$$

and:

$$1 + 2 = S^2(1) = S(S^1(1)) = S(S(1)) = S(2) = 3.$$

Using this definition, it is possible to show that $+$ is commutative and associative. The *neutral element* of $+$ is 0 since:

$$m + 0 = S^0(m) = \mathrm{id}_{\mathbb{N}}(m) = m$$

and:

$$0 + m = S^m(0) = S^{P(m)}(1) = S^{P(P(m))}(2) = \cdots = S^0(m) = m.$$

Clearly, there exist no inverses for $+$ in $\mathbb{N}$, i.e. given $m \in \mathbb{N}$ (non-zero), there exist no $n \in \mathbb{N}$ such that $m + n = 0$. This motivates the extension of the natural numbers to the integers. In order to rigorously define $\mathbb{Z}$, we need to define the following relation on $\mathbb{N} \times \mathbb{N}$.

Let $\sim$ be the relation on $\mathbb{N} \times \mathbb{N}$ defined by:

$$(m, n) \sim (p, q) :\Leftrightarrow m + q = p + n.$$

It is easy to check that this is an equivalence relation as:

i) $(m, n) \sim (m, n)$ since $m + n = m + n$;

ii) $(m, n) \sim (p, q) \Rightarrow (p, q) \sim (m, n)$ since $m + q = p + n \Leftrightarrow p + n = m + q$;

iii) $((m, n) \sim (p, q) \wedge (p, q) \sim (r, s)) \Rightarrow (m, n) \sim (r, s)$ since we have:

$$m + q = p + n \wedge p + s = r + q,$$

hence $m + q + p + s = p + n + r + q$, and thus $m + s = r + n$.

By equipping this relation we can define the set of integers in the following way:

Definition 1.29 (Integers)**.** *We define the set of* integers *by:*

$$\mathbb{Z} := (\mathbb{N} \times \mathbb{N})/\sim \,.$$

The intuition behind this definition is that the pair (m, n) stands for "$m - n$". In other words, we represent each integer by a pair of natural numbers whose (yet to be defined) difference is precisely that integer. There are, of course, many ways to represent the same integer with a pair of natural numbers in this way. For instance, the integer -1 could be represented by $(1, 2)$, $(2, 3)$, $(112, 113)$, ...

Notice however that $(1, 2) \sim (2, 3)$, $(1, 2) \sim (112, 113)$, etc. and indeed, taking the quotient by $\sim$ takes care of this "redundancy". Notice also that this definition relies entirely on previously defined entities.

Remark 1.24. In a first introduction to set theory it is not unlikely to find the claim that the natural numbers are part of the integers, i.e. $\mathbb{N} \subseteq \mathbb{Z}$. However, according to our definition, this is obviously nonsense since $\mathbb{N}$ and $\mathbb{Z} := (\mathbb{N} \times \mathbb{N})/\sim$ contain entirely different elements. What is true is that $\mathbb{N}$ can be *embedded* into $\mathbb{Z}$, i.e. there exists an *inclusion map* ι, given by:

$$\begin{aligned} \iota \colon \mathbb{N} &\hookrightarrow \mathbb{Z} \\ n &\mapsto [(n,0)] \end{aligned}$$

and it is in this sense that $\mathbb{N}$ is included in $\mathbb{Z}$.

Definition 1.30 (Inverse Of Integer)**.** *Let* $n := [(n,0)] \in \mathbb{Z}$*. Then we define the* inverse *of* n *to be* $-n := [(0,n)]$.

We would now like to inherit the $+$ operation from $\mathbb{N}$.

Definition 1.31 (Addition Of Integers)**.** *We define the* addition of integers $+_{\mathbb{Z}} \colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ *by:*

$$[(m,n)] +_{\mathbb{Z}} [(p,q)] := [(m+p, n+q)].$$

Since we used representatives to define $+_{\mathbb{Z}}$, we would need to check that $+_{\mathbb{Z}}$ is well-defined. It is an easy exercise.

Example 1.14. $2 +_{\mathbb{Z}} (-3) := [(2,0)] +_{\mathbb{Z}} [(0,3)] = [(2,3)] = [(0,1)] =: -1$. Hallelujah!

In a similar fashion, we define the set of *rational numbers* by:

$$\mathbb{Q} := (\mathbb{Z} \times \mathbb{Z}^*)/\sim,$$

where $\mathbb{Z}^* := \mathbb{Z} \setminus \{0\}$ and $\sim$ is a relation on $\mathbb{Z} \times \mathbb{Z}^*$ given by:

$$(p,q) \sim (r,s) :\Leftrightarrow ps = qr,$$

assuming that a *multiplication* operation on the integers has already been defined.

Example 1.15. We have $(2,3) \sim (4,6)$ since $2 \times 6 = 12 = 3 \times 4$.

Similarly to what we did for the integers, here we are representing each rational number by the collection of pairs of integers (the second one in each pair being non-zero) such that their (yet to be defined) ratio is precisely that rational number. Thus, for example, we have:

$$\frac{2}{3} := [(2,3)] = [(4,6)] = \dots$$

There are many ways to construct the reals from the rationals. One is to define a set $\mathscr{A}$ of *almost homomorphisms* on $\mathbb{Z}$ and hence define:

$$\mathbb{R} := \mathscr{A}/\sim,$$

where $\sim$ is a "suitable" equivalence relation on $\mathscr{A}$.

Chapter 2

Algebraic Structures

Definition 2.1 (Algebraic Structures)**.** *A set A (called the underlying set, carrier set or domain), together with a collection of maps (called operations) on A of finite arity (typically binary operations), and a finite set of identities, known as axioms, that these operations must satisfy, is called an* ***algebraic structure****. Some algebraic structures also involve another set (called the scalar set).*

Examples of algebraic structures with a single underlying set include groups,fields and rings. Examples of algebraic structures with two underlying sets include vector spaces, modules, and algebras. In this section we will review the most important algebraic structures for our purposes.

One has to be careful with the terminology since it changes depending on the area of mathematics. For example, in the context of universal algebra, the set A with this structure is called an algebra, while, in other contexts, it is (somewhat ambiguously) called an algebraic structure, the term algebra being reserved for specific algebraic structures that are vector spaces over a field or modules over a commutative ring.

The properties of specific algebraic structures are studied in abstract algebra. The general theory of algebraic structures has been formalized in universal algebra. The language of category theory is used to express and study relationships between different classes of algebraic and non-algebraic objects. This is because it is sometimes possible to find strong connections between some classes of objects, sometimes of different kinds. For example, Galois theory establishes a connection between certain fields and groups: two algebraic structures of different kinds.

In this chapter we will introduce the basic algebraic structures by giving their definitions and some of their key properties. In later chapter we get into depth in various topics of algebraic structures.

2.1 Groups

Definition 2.2 (Group)**.** *A* ***group*** *is a tuple $(G, \cdot)$, where G is a set (called the underlying set of the group) and $\cdot$ is a map (called operation) $G \times G \to G$ satisfying the following four group axioms:*

- *Closure:* $\forall\, a, b \in G : a \cdot b \in G$;
- *Associativity:* $\forall\, a, b, c \in G : (a \cdot b) \cdot c = a \cdot (b \cdot c)$;
- *Neutral Element:* $\exists\, e \in G : \forall\, a \in G : a \cdot e = e \cdot a = a$;
- *Inverse Element:* $\forall\, a \in G : \exists\, a^{-1} \in G : a \cdot a^{-1} = a^{-1} \cdot a = e$;

The identity element e of a group G is often written as 1 a notation inherited from the multiplicative identity. If a group is abelian, then one may choose to denote the group operation by $+$ and the identity element by 0.

The result of the group operation may depend on the order of the operands. In other words, the result of combining element a with element b need not yield the same result as combining element b with element a, so the equation $a \cdot b = b \cdot a$ may not be true for every two elements a and b.

Definition 2.3 (Abelian Group)**.** *A group G is called **Abelian** if on top of the four group axioms it also satisfies the axiom of commutativity:*

- *Commutativity:* $\forall\, a, b \in G : a \cdot b = b \cdot a;$

Commutativity always holds in the group of integers under addition, because $a + b = b + a$ for any two integers (commutativity of addition). The symmetry group is an example of a group that is not abelian.

2.2 Fields

Definition 2.4 (Field)**.** *An **(algebraic) field** is a triple $(K, +, \cdot)$, where K is a set and $+, \cdot$ are maps $K \times K \to K$ satisfying the following axioms:*

- $(K, +)$ *is an abelian group, i.e.*
  - *i) Closure:* $\forall\, a, b \in K : a + b \in K;$
  - *ii) Associativity:* $\forall\, a, b, c \in K : (a + b) + c = a + (b + c);$
  - *iii) Neutral Element:* $\exists\, 0 \in K : \forall\, a \in K : a + 0 = 0 + a = a;$
  - *iv) Inverse Element:* $\forall\, a \in K : \exists\, {-a} \in K : a + (-a) = (-a) + a = 0;$
  - *v) Commutativity:* $\forall\, a, b \in K : a + b = b + a;$
- $(K^*, \cdot)$, *where* $K^* := K \setminus \{0\}$, *is an abelian group, i.e.*
  - *vi) Closure:* $\forall\, a, b \in K^* : a \cdot b \in K^*;$
  - *vii) Associativity:* $\forall\, a, b, c \in K^* : (a \cdot b) \cdot c = a \cdot (b \cdot c);$
  - *viii) Neutral Element:* $\exists\, 1 \in K^* : \forall\, a \in K^* : a \cdot 1 = 1 \cdot a = a;$
  - *ix) Inverse Element:* $\forall\, a \in K^* : \exists\, a^{-1} \in K^* : a \cdot a^{-1} = a^{-1} \cdot a = 1;$
  - *x) Commutativity:* $\forall\, a, b \in K^* : a \cdot b = b \cdot a;$
- *the maps $+$ and $\cdot$ satisfy the distributive property:*
  - *xi)* $\forall\, a, b, c \in K : (a + b) \cdot c = a \cdot c + b \cdot c;$

Remark 2.1. In the above definition, we included axiom v for the sake of clarity, but in fact it can be proven starting from the other axioms.

2.3 Vector Spaces

Definition 2.5 (K-Vector Space)**.** *Let $(K, +, \cdot)$ be a field. A K**-vector space**, or **vector space over** K or **linear space over** K is a triple $(V, \oplus, \odot)$, where V is a set and*

$$\begin{aligned} \oplus &: V \times V \to V \\ \odot &: K \times V \to V \end{aligned}$$

are maps satisfying the following axioms:

- $(V, \oplus)$ *is an abelian group i.e.*
  - *i) Closure:* $\forall\, v, w \in V : v \oplus w \in K;$
  - *ii) Associativity:* $\forall\, v, w, z \in V : (v \oplus w) \oplus z = v \oplus (w \oplus z);$
  - *iii) Neutral Element:* $\exists\, 0 \in V : \forall\, v \in V : v \oplus 0 = 0 \oplus v = v;$
  - *iv) Inverse Element:* $\forall\, v \in V : \exists - v \in V : v \oplus (-v) = (-v) \oplus v = 0;$
  - *v) Commutativity:* $\forall\, v, w \in V : v \oplus w = w \oplus v;$
- *the map $\odot$ is an* action *of K on $(V, \oplus)$:*
  - *vi) Distributivity Of Scalar Multiplication - Vector Addition:* $\forall\, \lambda \in K : \forall\, v, w \in V : \lambda \odot (v \oplus w) = (\lambda \odot v) \oplus (\lambda \odot w);$

vii) Distributivity Of Scalar Multiplication - Field Addition: $\forall\,\lambda,\mu \in K : \forall\, v \in V : (\lambda+\mu)\odot v = (\lambda\odot v)\oplus(\mu\odot v)$;

viii) Compatibility Of Scalar Multiplication - Field Multiplication $\forall\,\lambda,\mu \in K : \forall\, v \in V : (\lambda\cdot\mu)\odot v = \lambda\odot(\mu\odot v)$;

ix) Neutral Element Of Scalar Multiplication $\forall\, v \in V : 1\odot v = v$.

The elements of a vector space are called *vectors*, while the elements of K are often called *scalars*, and the map $\odot$ is called *scalar multiplication*.

2.3.1 Linear Maps

As usual by now, we will look at the structure-preserving maps between vector spaces.

Definition 2.6 (Linear Maps)**.** *Let* $(V,\oplus,\odot)$, $(W,\boxplus,\boxdot)$ *be vector spaces over the same field* K *and let* $f\colon V \to W$ *be a map. We say that* f *is a* ***linear map****, and we denote it as* $f\colon V \xrightarrow{\sim} W$, *if for all* $v_1, v_2 \in V$ *and all* $\lambda \in K$

$$f((\lambda\odot v_1)\oplus v_2) = (\lambda\boxdot f(v_1))\boxplus f(v_2).$$

From now on, we will drop the special notation for the vector space operations and suppress the dot for scalar multiplication. For instance, we will write the equation above as $f(\lambda v_1 + v_2) = \lambda f(v_1) + f(v_2)$, hoping that this will not cause any confusion.

Definition 2.7 ($\mathrm{Hom}(V,W)$)**.** *Let* V *and* W *be vector spaces over the same field* K. *We define the set* $\mathrm{Hom}(V,W)$ *as the set of all linear maps between* V *and* W*:*

$$\mathrm{Hom}(V,W) := \{f \mid f\colon V \xrightarrow{\sim} W\}$$

$\mathrm{Hom}(V,W)$ can itself be made into a vector space over K by defining:

$$\begin{aligned}\diamondplus\colon \mathrm{Hom}(V,W)\times\mathrm{Hom}(V,W) &\to \mathrm{Hom}(V,W)\\ (f,g) &\mapsto f\diamondplus g\end{aligned}$$

where

$$\begin{aligned}f\diamondplus g\colon V &\xrightarrow{\sim} W\\ v &\mapsto (f\diamondplus g)(v) := f(v)+g(v),\end{aligned}$$

and

$$\begin{aligned}\diamonddot\colon K\times\mathrm{Hom}(V,W) &\to \mathrm{Hom}(V,W)\\ (\lambda,f) &\mapsto \lambda\diamonddot f\end{aligned}$$

where

$$\begin{aligned}\lambda\diamonddot f\colon V &\xrightarrow{\sim} W\\ v &\mapsto (\lambda\diamonddot f)(v) := \lambda f(v).\end{aligned}$$

It is easy to check that both $f\diamondplus g$ and $\lambda\diamonddot f$ are indeed linear maps from V to W. For instance, we have:

$$\begin{aligned}(\lambda\diamonddot f)(\mu v_1+v_2) &= \lambda f(\mu v_1+v_2) && \text{(by definition)}\\ &= \lambda(\mu f(v_1)+f(v_2)) && \text{(since } f \text{ is linear)}\\ &= \lambda\mu f(v_1)+\lambda f(v_2) && \text{(by axioms i and iii)}\\ &= \mu\lambda f(v_1)+\lambda f(v_2) && \text{(since } K \text{ is a field)}\\ &= \mu(\lambda\diamonddot f)(v_1)+(\lambda\diamonddot f)(v_2)\end{aligned}$$

so that $\lambda\diamonddot f \in \mathrm{Hom}(V,W)$. One should also check that $\diamondplus$ and $\diamonddot$ satisfy the vector space axioms.

Definition 2.8 (Endomorphisms)**.** *Let* V *be a vector space. An* ***endomorphism*** *of* V *is a linear map* $V \to V$.

Definition 2.9 (End(V)). *Let V be a vector space. We define the set* End(V) *as the set of all endomorphisms of V:*

$$\mathrm{End}(V) := \mathrm{Hom}(V, V)$$

It is easy to show that End(V) can again itself be made into a vector space over K.

Definition 2.10 (Linear Isomorphism). *A bijective linear map is called a* ***linear isomorphism*** *of vector spaces.*

Definition 2.11 (Isomorphic Vector Spaces). *Two vector spaces are said to be* ***isomorphic*** *if there exists a linear isomorphism between them. We write $V \cong_{\mathrm{vec}} W$.*

Definition 2.12 (Automorphism). *Let V be a vector space. An* ***automorphism*** *of V is a linear isomorphism $V \to V$.*

Definition 2.13 (Aut(V)). *Let V be a vector space. We define the set* Aut(V) *as the set of all automorphisms of V:*

$$\mathrm{Aut}(V) := \{f \in \mathrm{End}(V) \mid f \textit{ is an isomorphism}\}$$

Remark 2.2. Note that Aut(V) **cannot** be made into a vector space . It is however a group under the operation of composition of linear maps.

Definition 2.14 (Dual Vector Space). *Let V be a vector space over K. The* ***dual*** *vector space to V is*

$$V^* := \mathrm{Hom}(V, K),$$

where K is considered as a vector space over itself.

The dual vector space to V is the vector space of linear maps from V to the underlying field K, which are variously called *linear functionals*, *covectors*, or *one-forms* on V. The dual plays a very important role, in that from a vector space and its dual, we will construct the tensor space.

Definition 2.15 (Bilinear Maps). *Let V, W, Z be vector spaces over K. A map $f: V \times W \to Z$ is said to be* ***bilinear*** *if*

- $\forall\, w \in W : \forall\, v_1, v_2 \in V : \forall\, \lambda \in K : f(\lambda v_1 + v_2, w) = \lambda f(v_1, w) + f(v_2, w);$
- $\forall\, v \in V : \forall\, w_1, w_2 \in W : \forall\, \lambda \in K : f(v, \lambda w_1 + w_2) = \lambda f(v, w_1) + f(v, w_2);$

i.e. if the maps $v \mapsto f(v, w)$, for any fixed w, and $w \mapsto f(v, w)$, for any fixed v, are both linear as maps $V \to Z$ and $W \to Z$, respectively.

Remark 2.3. Compare this with the definition of a linear map $f: V \times W \xrightarrow{\sim} Z$:

$$\forall\, x, y \in V \times W : \forall\, \lambda \in K : f(\lambda x + y) = \lambda f(x) + f(y).$$

More explicitly, if $x = (v_1, w_1)$ and $y = (v_2, w_2)$, then:

$$f(\lambda(v_1, w_1) + (v_2, w_2)) = \lambda f((v_1, w_1)) + f((v_2, w_2)).$$

A bilinear map out of $V \times W$ is *not* the same as a linear map out of $V \times W$. In fact, bilinearity is just a special kind of non-linearity.

Example 2.1. The map $f: \mathbb{R}^2 \to \mathbb{R}$ given by $(x, y) \mapsto x + y$ is linear but not bilinear, while the map $(x, y) \mapsto xy$ is bilinear but not linear.

We can immediately generalise the above to define *multilinear* maps out of a Cartesian product of vector spaces.

Definition 2.16 (Tensors). *Let V be a vector space over K. A (p, q)-**tensor** T on V is a multilinear map*

$$T\colon \underbrace{V^* \times \cdots \times V^*}_{p \text{ copies}} \times \underbrace{V \times \cdots \times V}_{q \text{ copies}} \to K.$$

Remark 2.4. By convention, a $(0, 0)$ on V is just an element of K, and hence $T^0_0 V = K$.

Definition 2.17 (Covariant / Contravariant Tensor). *A type $(p,0)$ tensor is called a* ***covariant*** p*-***tensor***, while a tensor of type $(0,q)$ is called a* ***contravariant*** q*-***tensor***.*

Definition 2.18 ($T^p_q V$). *We define the set of all (p,q)-tensors T as:*

$$T^p_q V := \underbrace{V \otimes \cdots \otimes V}_{p \text{ copies}} \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{q \text{ copies}} := \{T \mid T \text{ is a } (p,q)\text{-tensor on } V\}.$$

Remark 2.5. Note that to define $T^p_q V$ as a set, we should be careful and invoke the principle of restricted comprehension, i.e. we should say where the Ts are coming from. In general, say we want to build a set of maps $f\colon A \to B$ satisfying some property p. Recall that the notation $f\colon A \to B$ is hiding the fact that is a relation (indeed, a functional relation), and a relation between A and B is a subset of $A \times B$. Therefore, we ought to write:

$$\{f \in \mathcal{P}(A \times B) \mid f\colon A \to B \text{ and } p(f)\}.$$

In the case of $T^p_q V$ we have:

$$T^p_q V := \big\{T \in \mathcal{P}\big(\underbrace{V^* \times \cdots \times V^*}_{p \text{ copies}} \times \underbrace{V \times \cdots \times V}_{q \text{ copies}} \times K\big) \mid T \text{ is a } (p,q)\text{-tensor on } V\big\},$$

although we will not write this down every time.
The set $T^p_q V$ can be equipped with a K-vector space structure by defining

$$\begin{aligned} \oplus\colon T^p_q V \times T^p_q V &\to T^p_q V \\ (T,S) &\mapsto T \oplus S \end{aligned}$$

and

$$\begin{aligned} \odot\colon K \times T^p_q V &\to T^p_q V \\ (\lambda,T) &\mapsto \lambda \odot T, \end{aligned}$$

where $T \oplus S$ and $\lambda \odot T$ are defined pointwise, as we did with $\mathrm{Hom}(V,W)$.

We now define an important way of obtaining a new tensor from two given ones.

Definition 2.19 (Tensor Product). *Let $T \in T^p_q V$ and $S \in T^r_s V$. The* ***tensor product*** *of T and S is the tensor $T \otimes S \in T^{p+r}_{q+s} V$ defined by:*

$$\begin{aligned} &(T \otimes S)(\omega_1, \dots, \omega_p, \omega_{p+1}, \dots, \omega_{p+r}, v_1, \dots, v_q, v_{q+1}, \dots, v_{q+s}) \\ &\qquad := T(\omega_1, \dots, \omega_p, v_1, \dots, v_q)\, S(\omega_{p+1}, \dots, \omega_{p+r}, v_{q+1}, \dots, v_{q+s}), \end{aligned}$$

with $\omega_i \in V^*$ and $v_i \in V$.

Some examples are in order.

Example 2.2. a) $T^0_1 V := \{T \mid T\colon V \xrightarrow{\sim} K\} = \mathrm{Hom}(V,K) =: V^*$. Note that here multilinear is the same as linear since the maps only have one argument.

b) $T^1_1 V \equiv V \otimes V^* := \{T \mid T \text{ is a bilinear map } V^* \times V \to K\}$. We claim that this is the same as $\mathrm{End}(V^*)$. Indeed, given $T \in V \otimes V^*$, we can construct $\widehat{T} \in \mathrm{End}(V^*)$ as follows:

$$\begin{aligned} \widehat{T}\colon V^* &\xrightarrow{\sim} V^* \\ \omega &\mapsto T(-,\omega) \end{aligned}$$

where, for any fixed ω, we have

$$\begin{aligned} T(-,\omega)\colon V &\xrightarrow{\sim} K \\ v &\mapsto T(v,\omega). \end{aligned}$$

The linearity of both $\widehat{T}$ and $T(-,\omega)$ follows immediately from the bilinearity of T. Hence $T(-,\omega) \in V^*$ for all ω, and $\widehat{T} \in \mathrm{End}(V^*)$. This correspondence is invertible, since can reconstruct T from $\widehat{T}$

by defining

$$\begin{aligned} T\colon V \times V^* &\to K \\ (v,\omega) &\mapsto T(v,\omega) := (\widehat{T}(\omega))(v). \end{aligned}$$

The correspondence is in fact linear, hence an isomorphism, and thus

$$T^1_1 V \cong_{\mathrm{vec}} \operatorname{End}(V^*).$$

Other examples we would like to consider are

c) $T^0_1 V \overset{?}{\cong}_{\mathrm{vec}} V$: while you will find this stated as true in some physics textbooks, it is in fact *not true* in general;

d) $T^1_1 V \overset{?}{\cong}_{\mathrm{vec}} \operatorname{End}(V)$: This is also not true in general;

e) $(V^*)^* \overset{?}{\cong}_{\mathrm{vec}} V$: This only holds if V is finite-dimensional (we will define the dimensions of a vector space in the next section).

2.3.2 Basis Of Vector Spaces

Given a vector space without any additional structure, the only notion of basis that we can define is a so-called Hamel basis.

Definition 2.20 (Hamel Basis)**.** *Let $(V, +, \cdot)$ be a vector space over K. A subset $\mathcal{B} \subseteq V$ is called a **Hamel basis** for V if*

- *every finite subset $\{b_1, \ldots, b_N\}$ of $\mathcal{B}$ is linearly independent, i.e.*

$$\sum_{i=1}^{N} \lambda^i b_i = 0 \;\Rightarrow\; \lambda^1 = \cdots = \lambda^N = 0;$$

- *$\mathcal{B}$ is a* generating *or* spanning set *of V, i.e.*

$$\forall\, v \in V : \exists\, v^1, \ldots, v^M \in K : \exists\, b_1, \ldots, b_M \in \mathcal{B} : v = \sum_{i=1}^{M} v^i b_i.$$

Remark 2.6. We can write the second condition more succinctly by defining

$$\operatorname{span}_K(\mathcal{B}) := \left\{ \sum_{i=1}^{n} \lambda^i b_i \;\middle|\; \lambda^i \in K \wedge b_i \in \mathcal{B} \wedge n \geq 1 \right\}$$

and thus writing $V = \operatorname{span}_K(\mathcal{B})$.

Remark 2.7. Once we have a basis $\mathcal{B}$, the expansion of $v \in V$ in terms of elements of $\mathcal{B}$ is, in fact, unique. Hence we can meaningfully speak of the *components* of v in the basis $\mathcal{B}$.

Remark 2.8. Note that we have been using superscripts for the elements of K, and these should not be confused with exponents.

The following characterisation of a Hamel basis is often useful.

Proposition 2.1. *Let V be a vector space and $\mathcal{B}$ a Hamel basis of V. Then $\mathcal{B}$ is a minimal spanning and maximal independent subset of V, i.e., if $S \subseteq V$, then*

- $\operatorname{span}(S) = V \;\Rightarrow\; |S| \geq |\mathcal{B}|$*;*
- *S is linearly independent $\;\Rightarrow\; |S| \leq |\mathcal{B}|$.*

Definition 2.21 (Dimension Of Vector Space)**.** *Let V be a vector space. The **dimension** of V is $\dim V := |\mathcal{B}|$, where $\mathcal{B}$ is a Hamel basis for V.*

Even though we will not prove it, it is the case that every Hamel basis for a given vector space has the same cardinality, and hence the notion of dimension is well-defined.

Proposition 2.2. *If $\dim V < \infty$ and $S \subseteq V$, then we have the following:*

- *if $\operatorname{span}_K(S) = V$ and $|S| = \dim V$, then S is a Hamel basis of V;*
- *if S is linearly independent and $|S| = \dim V$, then S is a Hamel basis of V.*

Theorem 2.1. *If $\dim V < \infty$, then $(V^*)^* \cong_{\text{vec}} V$.*

Remark 2.9. Note that while we need the concept of basis to state this result (since we require $\dim V < \infty$), the isomorphism that we have constructed is independent of any choice of basis.

Remark 2.10. While a choice of basis often simplifies things, when defining new objects it is important to do so without making reference to a basis. If we do define something in terms of a basis (e.g. the dimension of a vector space), then we have to check that the thing is well-defined, i.e. it does not depend on which basis we choose.

If V is finite-dimensional, then V^* is also finite-dimensional and $V \cong_{\text{vec}} V^*$. Moreover, given a basis $\mathcal{B}$ of V, there is a spacial basis of V^* associated to $\mathcal{B}$.

Definition 2.22 (Dual Basis)**.** *Let V be a finite-dimensional vector space with basis $\mathcal{B} = \{e_1, \dots, e_{\dim V}\}$. The **dual basis** to $\mathcal{B}$ is the unique basis $\mathcal{B}' = \{\epsilon^1, \dots, \epsilon^{\dim V}\}$ of V^* such that*

$$\forall\, 1 \leq i, j \leq \dim V : \quad \epsilon^i(e_j) = \delta^i_j := \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Remark 2.11. If V is finite-dimensional, then V is isomorphic to both V^* and $(V^*)^*$. In the case of V^*, an isomorphism is given by sending each element of a basis $\mathcal{B}$ of V to a different element of the dual basis $\mathcal{B}'$, and then extending linearly to V. You will (and probably already have) read that a vector space is *canonically* isomorphic to its double dual, but *not* canonically isomorphic to its dual, because an arbitrary choice of basis on V is necessary in order to provide an isomorphism.

Finally by using a basis (and its dual) we can define the components of a tensor as follows.

Definition 2.23 (Components Of A Tensor)**.** *Let V be a finite-dimensional vector space over K with basis $\mathcal{B} = \{e_1, \dots, e_{\dim V}\}$ and dual basis $\{\epsilon^1, \dots, \epsilon^{\dim V}\}$ and let $T \in T^p_q V$. We define the **components** of T in the basis $\mathcal{B}$ to be the numbers*

$$T^{a_1 \dots a_p}{}_{b_1 \dots b_q} := T(\epsilon^{a_1}, \dots, \epsilon^{a_p}, e_{b_1}, \dots, e_{b_q}) \in K,$$

where $1 \leq a_i, b_j \leq \dim V$.

Just as with vectors, the components completely determine the tensor. Indeed, we can reconstruct the tensor from its components by using the basis:

$$T = \underbrace{\sum_{a_1=1}^{\dim V} \cdots \sum_{b_q=1}^{\dim V}}_{p+q \text{ sums}} T^{a_1 \dots a_p}{}_{b_1 \dots b_q} e_{a_1} \otimes \cdots \otimes e_{a_p} \otimes \epsilon^{b_1} \otimes \cdots \otimes \epsilon^{b_q},$$

where the e_{a_i}s are understood as elements of $T^1_0 V \cong_{\text{vec}} V$ and the $\epsilon^{b_i}s$ as elements of $T^0_1 V \cong_{\text{vec}} V^*$. Note that each summand is a (p, q)-tensor and the (implicit) multiplication between the components and the tensor product is the scalar multiplication in $T^p_q V$.

Notational Conventions

From now on, we will employ the Einstein's summation convention, which consists in suppressing the summation sign when the indices to be summed over each appear once as a subscript and once as a superscript in the same term. For example, we write

$$v = v^a e_a, \qquad \omega = \omega_a \epsilon^a \qquad \text{and} \qquad T = T^{ab}{}_c e_a \otimes e_b \otimes \epsilon^c$$

instead of

$$v = \sum_{a=1}^{d} v^a e_a, \qquad \omega = \sum_{a=1}^{d} \omega_a \epsilon^a \qquad \text{and} \qquad T = \sum_{a=1}^{d}\sum_{b=1}^{d}\sum_{c=1}^{d} T^{ab}{}_c e_a \otimes e_b \otimes \epsilon^c.$$

Indices that are summed over are called *dummy indices*; they always appear in pairs and clearly it doesn't matter which particular letter we choose to denote them, provided it doesn't already appear in the expression. Indices that are not summed over are called *free indices*; expressions containing free indices represent multiple expressions, one for each value of the free indices; free indices must match on both sides of an equation. The ranges over which the indices run are usually understood and not written out.

The convention on which indices go upstairs and which downstairs (which we have already been using) is that:

- the basis vectors of V carry downstairs indices;
- the basis vectors of V^* carry upstairs indices;
- all other placements are enforced by the Einstein's summation convention.

For example, since the components of a vector must multiply the basis vectors and be summed over, the Einstein's summation convention requires that they carry upstair indices.

Example 2.3. Using the summation convention, we have:

a) $\epsilon^a(v) = \epsilon^a(v^b e_b) = v^b \epsilon^a(e_b) = v^b \delta^a_b = v^a$;

b) $\omega(e_b) = (\omega_a \epsilon^a)(e_b) = \omega_a \epsilon^a(e_b) = \omega_b$;

c) $\omega(v) = \omega_a \epsilon^a(v^b e_b) = \omega_a v^a$;

where $v \in V$, $\omega \in V^*$, $\{e_i\}$ is a basis of V and $\{\epsilon^j\}$ is the dual basis to $\{e_i\}$.

Remark 2.12. The Einstein's summation convention should only be used when dealing with linear spaces and multilinear maps. The reason for this is the following. Consider a map $\phi\colon V \times W \to Z$, and let $v = v^i e_i \in V$ and $w = w^i \widetilde{e}_i \in W$. Then we have:

$$\phi(v, w) = \phi\left(\sum_{i=1}^{d} v^i e_i, \sum_{j=1}^{\widetilde{d}} w^j \widetilde{e}_j\right) = \sum_{i=1}^{d}\sum_{j=1}^{\widetilde{d}} \phi(v^i e_i, w^j \widetilde{e}_j) = \sum_{i=1}^{d}\sum_{j=1}^{\widetilde{d}} v^i w^j \phi(e_i, \widetilde{e}_j).$$

Note that by suppressing the greyed out summation signs, the second and third term above are indistinguishable. But this is only true if ϕ is bilinear! Hence the summation convention should not be used (at least, not without extra care) in other areas of mathematics.

Matrix Representations

Having chosen a basis for V and the dual basis for V^*, it is very tempting to think of $v = v^i e_i \in V$ and $\omega = \omega_i \epsilon^i \in V^*$ as d-tuples of numbers. In order to distinguish them, one may choose to write vectors as *columns* of numbers and covectors as *rows* of numbers:

$$v = v^i e_i \quad \leftrightsquigarrow \quad v \mathrel{\hat{=}} \begin{pmatrix} v^1 \\ \vdots \\ v^d \end{pmatrix}$$

and

$$\omega = \omega_i \epsilon^i \quad \leftrightsquigarrow \quad \omega \mathrel{\hat{=}} (\omega_1, \ldots, \omega_d).$$

Given $\phi \in \operatorname{End}(V) \cong_{\mathrm{vec}} T^1_1 V$, recall that we can write $\phi = \phi^i{}_j \, e_i \otimes \epsilon^j$, where $\phi^i{}_j := \phi(\epsilon^i, e_j)$ are the components of ϕ with respect to the chosen basis. It is then also very tempting to think of ϕ as a square array of numbers:

$$\phi = \phi^i{}_j\, e_i \otimes \epsilon^j \quad \rightsquigarrow \quad \phi \mathrel{\hat{=}} \begin{pmatrix} \phi^1{}_1 & \phi^1{}_2 & \cdots & \phi^1{}_d \\ \phi^2{}_1 & \phi^2{}_2 & \cdots & \phi^2{}_d \\ \vdots & \vdots & \ddots & \vdots \\ \phi^d{}_1 & \phi^d{}_2 & \cdots & \phi^d{}_d \end{pmatrix}$$

The convention here is to think of the i index on $\phi^i{}_j$ as a *row index*, and of j as a *column index* (we cannot stress enough that this is pure convention). Hence, once we start using the "matrix representation" (although technically it shouldn't be called representation since as we will see the word "representation" means something else), we can then express all the linear maps ϕ of V (a.k.a all the elements of $\mathrm{End}(V)$) as $n \times n$ matrices. This coincides with the usual picture we have in physics, where all the vectors are represented by a column vector of size n and all the linear transformations are represented by $n \times n$ matrices that act on v and produce another vector w (hence the $\mathrm{End}(V)$).

Going one step further, notice that not all matrices have an inverse. This coincides with the fact that not all linear maps have an inverse. Since $\mathrm{End}(V)$ contains all linear maps, it also contains maps that are not linear isomorphisms (a.k.a maps that are not bijections, a.k.a maps that do not have an inverse). However, if we restrict ourselves more, from linear maps to linear isomorphisms then we move from $\mathrm{End}(V)$ to $\mathrm{Aut}(V) := \{f \in \mathrm{End}(V) \mid f \text{ is an isomorphism}\}$. And if we switch again to the "matrix representation", now we are dealing with matrices that do have an inverse. We call the set of all these matrices "General Linear Group" and we denote by $GL(V)$ (we can indeed equip this set with matrix multiplication and show that it is closed under the operation, hence the "group" in the name).

Its usefulness stems from the following example.

Example 2.4. If $\dim V < \infty$, then we have $\mathrm{End}(V) \cong_{\mathrm{vec}} T^1_1 V$. Explicitly, if $\phi \in \mathrm{End}(V)$, we can think of $\phi \in T^1_1 V$, using the same symbol, as

$$\phi(\omega, v) := \omega(\phi(v)).$$

Hence the components of $\phi \in \mathrm{End}(V)$ are $\phi^a{}_b := \epsilon^a(\phi(e_b))$.

Similarly, $\omega(v) = \omega_m v^m$ can be thought of as the *dot product* $\omega \cdot v \equiv \omega^T v$, and

$$\phi(v, w) = w_a\, \phi^a{}_b\, v^b \quad \leftrightsquigarrow \quad \omega^T \phi v.$$

The last expression is could mislead you into thinking that the transpose is a "good" notion, but in fact it is not. It is very bad notation. It almost pretends to be basis independent, but it is not at all.

Now consider $\phi, \psi \in \mathrm{End}(V)$. Let us determine the components of $\phi \circ \psi$. We have:

$$\begin{aligned} (\phi \circ \psi)^a{}_b &:= (\phi \circ \psi)(\epsilon^a, e_b) \\ &:= \epsilon^a((\phi \circ \psi)(e_b)) \\ &= \epsilon^a((\phi(\psi(e_b))) \\ &= \epsilon^a(\phi(\psi^m{}_b\, e_m)) \\ &= \psi^m{}_b \epsilon^a(\phi(e_m)) \\ &:= \psi^m{}_b\, \phi^a{}_m. \end{aligned}$$

The multiplication in the last line is the multiplication in the field K, and since that's commutative, we have $\psi^m{}_b\, \phi^a{}_m = \phi^a{}_m\, \psi^m{}_b$. However, in light of the convention introduced in the previous remark, the latter is preferable. Indeed, if we think of the superscripts as row indices and of the subscripts as column indices, then $\phi^a{}_m\, \psi^m{}_b$ is the entry in row a, column b, of the matrix product $\phi\psi$.

The moral of the story is that you should try your best *not* to think of vectors, covectors and tensors as arrays of numbers. Instead, always try to understand them from the abstract, intrinsic, component-free point of view.

2.3.3 Change Of Basis

Let V be a vector space over K with $d = \dim V < \infty$ and let $\{e_1, \dots, e_d\}$ be a basis of V. Consider a new basis $\{\widetilde{e}_1, \dots, \widetilde{e}_d\}$. Since the elements of the new basis are also elements of V, we can expand them in terms of the old basis. We have:

$$\widetilde{e}_a = \sum_{b=1}^{d} A^b{}_a e_b = A^b{}_a e_b$$

for some $A^b{}_a \in K$. Similarly, we have

$$e_a = \sum_{m=1}^{d} B^m{}_a \widetilde{e}_m = B^m{}_a \widetilde{e}_m$$

for some $B^m{}_a \in K$. It is a standard linear algebra result that the matrices A and B, with entries $A^b{}_a$ and $B^b{}_a$ respectively, are invertible and, in fact, $A^{-1} = B$. Note that in index notation, the equation $AB = I$ reads $A^a{}_m B^m{}_b = \delta^a_b$.

We now investigate how the components of vectors and covectors change under a change of basis.

a) Let $v = v^a e_a = \widetilde{v}^a \widetilde{e}_a \in V$. Then:

$$v^a = \epsilon^a(v) = \epsilon^a(\widetilde{v}^b \widetilde{e}_b) = \widetilde{v}^b \epsilon^a(\widetilde{e}_b) = \widetilde{v}^b \epsilon^a(A^m{}_b e_m) = A^m{}_b \widetilde{v}^b \epsilon^a(e_m) = A^a{}_b \widetilde{v}^b.$$

b) Let $\omega = \omega_a \epsilon^a = \widetilde{\omega}_a \widetilde{\epsilon}^a \in V^*$. Then:

$$\omega_a := \omega(e_a) = \omega(B^m{}_a \widetilde{e}_m) = B^m{}_a \omega(\widetilde{e}_m) = B^m{}_a \widetilde{\omega}_m.$$

Summarising, for $v \in V$, $\omega \in V^*$ and $\widetilde{e}_a = A^b{}_a e_b$, we have:

$$\begin{aligned} v^a &= A^a{}_b \widetilde{v}^b & \qquad \omega_a &= B^b{}_a \widetilde{\omega}_b \\ \widetilde{v}^a &= B^a{}_b v^b & \qquad \widetilde{\omega}_a &= A^b{}_a \omega_b \end{aligned}$$

The result for tensors is a combination of the above, depending on the type of tensor.

c) Let $T \in T^p_q V$. Then:

$$T^{a_1 \dots a_p}{}_{b_1 \dots b_q} = A^{a_1}{}_{m_1} \cdots A^{a_p}{}_{m_p} B^{n_1}{}_{b_1} \cdots B^{n_q}{}_{b_q} \widetilde{T}^{m_1 \dots m_p}{}_{n_1 \dots n_q},$$

i.e. the upstair indices transform like vector indices, and the downstair indices transform like covector indices.

Coming back (once again) to the "matrix representation", let's see now one of the biggest misunderstandings that might come up when we want to perform a change of basis for tensors.

Recall that, if $\phi \in T^1_1 V$, then we can arrange the components $\phi^a{}_b$ in matrix form:

$$\phi = \phi^a{}_b \, e_a \otimes \epsilon^b \qquad \rightsquigarrow \qquad \phi \mathrel{\hat{=}} \begin{pmatrix} \phi^1{}_1 & \phi^1{}_2 & \cdots & \phi^1{}_d \\ \phi^2{}_1 & \phi^2{}_2 & \cdots & \phi^2{}_d \\ \vdots & \vdots & \ddots & \vdots \\ \phi^d{}_1 & \phi^d{}_2 & \cdots & \phi^d{}_d \end{pmatrix}$$

Similarly, if we have $g \in T^0_2 V$, its components are $g_{ab} := g(e_a, e_b)$ and we can write

$$g = g_{ab} \, epsilon^a \otimes epsilon^b \qquad \rightsquigarrow \qquad g \mathrel{\hat{=}} \begin{pmatrix} g_{11} & g_{12} & \cdots & g_{1d} \\ g_{21} & g_{22} & \cdots & g_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ g_{d1} & g_{d2} & \cdots & g_{dd} \end{pmatrix}$$

Needless to say that these two objects could not be more different if they tried. Indeed

- ϕ is an endomorphism of V; the first index in $\phi^a{}_b$ transforms like a vector index, while the second index transforms like a covector index;
- g is a *bilinear form* on V; both indices in g_{ab} transform like covector indices.

In linear algebra, you may have seen the two different transformation laws for these objects:

$$\phi \to A^{-1}\phi A \qquad \text{and} \qquad g \to A^T g A,$$

where A is the change of basis matrix. However, once we fix a basis, the matrix representations of these two objects are indistinguishable. It is then very tempting to think that what we can do with a matrix, we can just as easily do with another matrix.

For instance, if we have a rule to calculate the determinant of a square matrix, we should be able to apply it to both of the above matrices. However, the notion of determinant is *only* defined for endomorphisms. The only way to see this is to give a basis-independent definition, i.e. a definition that does not involve the "components of a matrix". Let's do that!

2.3.4 Determinants

In your previous course on linear algebra, you may have met the determinant of a square matrix as a number calculated by applying a mysterious rule. Using the mysterious rule, you may have shown, with a lot of work, that for example, if we exchange two rows or two columns, the determinant changes sign. But, as we have seen, matrices are the result of pure convention. Hence, one more polemic remark is in order.

We will need some preliminary definitions. (We will define n-forms in a more proper way in next chapters, so for now do not spend a lot of time on them)

Definition 2.24 (*n-form*)**.** *Let V be a d-dimensional vector space. An n-**form** on V is a $(0, n)$-tensor ω that is* totally antisymmetric*, i.e.*

$$\forall\, \pi \in S_n : \ \omega(v_1, v_2, \ldots, v_n) = \operatorname{sgn}(\pi)\, \omega(v_{\pi(1)}, v_{\pi(2)}, \ldots, v_{\pi(n)}).$$

Note that a 0-form is a scalar, and a 1-form is a covector. A d-form is also called a *top form*, and one can show that for two top forms ω and ω' the following holds

$$\forall\, \omega, \omega' \in \Lambda^d V : \exists\, c \in K : \ \omega = c\,\omega',$$

i.e. there is essentially only one top form on V, up to a scalar factor.

Definition 2.25 (Choice Of Volume)**.** *A choice of top form on V is called a choice of **volume form** on V. A vector space with a chosen volume form is then called a* vector space with volume*.*

This terminology is due to the next definition.

Definition 2.26 (Volume)**.** *Let $\dim V = d$ be the dimension of vector space V and let $v_1, \ldots, v_d \in V$, be d vectors in V. Then the **volume** spanned by $v_1, \ldots, v_d$ is*

$$\operatorname{vol}(v_1, \ldots, v_d) := \omega(v_1, \ldots, v_d).$$

where ω is the (chosen) top form.

Intuitively, the antisymmetry condition on ω makes sure that $\operatorname{vol}(v_1, \ldots, v_d)$ is zero whenever the set $\{v_1, \ldots, v_d\}$ is not linearly independent. Indeed, in that case $v_1, \ldots, v_d$ could only span a $(d-1)$-dimensional hypersurface in V at most, which should have 0 volume.

Remark 2.13. You may have rightfully thought that the notion of volume would require some extra structure on V, such as a notion of length or angles, and hence an inner product. But instead, we only need a top form.

We are finally ready to define the determinant.

Definition 2.27 (Determinant). *Let V be a d-dimensional vector space and let $\phi \in \operatorname{End}(V) \cong_{\mathrm{vec}} T^1_1 V$. The* determinant *of ϕ is*

$$\det \phi := \frac{\omega(\phi(e_1), \ldots, \phi(e_d))}{\omega(e_1, \ldots, e_d)}$$

for some top form ω and some basis $\{e_1, \ldots, e_d\}$ of V.

The first thing we need to do is to check that this is well-defined. That $\det \phi$ is independent of the choice of ω is clear, since if ω, ω' are top forms, then there is a $c \in K$ such that $\omega = c\,\omega'$, and hence

$$\frac{\omega(\phi(e_1), \ldots, \phi(e_d))}{\omega(e_1, \ldots, e_d)} = \frac{\cancel{c}\,\omega'(\phi(e_1), \ldots, \phi(e_d))}{\cancel{c}\,\omega'(e_1, \ldots, e_d)}.$$

The independence from the choice of basis is more cumbersome to show, but it does hold, and thus $\det \phi$ is well-defined.

It is very important to notice that ϕ needs to be an endomorphism because we need to apply ω to $\phi(e_1), \ldots, \phi(e_d)$, and thus ϕ needs to output a vector. Which means that the determinant can only be defined for endomorphisms.

Of course, under the identification of ϕ as a matrix, this definition coincides with the usual definition of determinant, and all your favourite results about determinants can be derived from it. However once we switch to "matrix representation" as we said one is not able to distinguish between an endomorphism $\phi \in T^1_1 V$ and the so called "bilinear form" $g \in T^0_2 V$, hence one might think that they can calculate the determinant of the second guy. Let's see why such a determinant is not well defined.

In your linear algebra course, you may have shown the the determinant is basis-independent as follows: if A denotes the change of basis matrix, then

$$\det(A^{-1}\phi A) = \det(A^{-1}) \det(\phi) \det(A) = \det(A^{-1}A) \det(\phi) = \det(\phi)$$

since scalars commute, and $\det(A^{-1}A) = \det(I) = 1$.

Recall that the transformation rule for a bilinear form g under a change of basis is $g \to A^T g A$. The determinant of g then transforms as

$$\det(A^T g A) = \det(A^T) \det(g) \det(A) = (\det A)^2 \det(g)$$

i.e. it not invariant under a change of basis. It is not a well-defined object, and thus we should not use it.

We will later meet quantities X that transform as

$$X \to \frac{1}{(\det A)^2} X$$

under a change of basis, and hence they are also not well-defined. However, we obviously have

$$\det(g) X \to \frac{\cancel{(\det A)^2}}{\cancel{(\det A)^2}} \det(g) X = \det(g) X$$

so that the product $\det(g)X$ is a well-defined object. It seems that two wrongs make a right!

In order to make this mathematically precise, we will have to introduce *principal fibre bundles.* Using them, we will be able to give a bundle definition of tensor and of *tensor densities* which are, loosely speaking, quantities that transform with powers of $\det A$ under a change of basis. We will see all of that in later chapters.

2.4 Rings

Definition 2.28 (Ring)**.** *A **ring** is a triple $(R, +, \cdot)$, where R is a set and $+, \cdot \colon R \times R \to R$ are maps satisfying the following axioms*

- *$(R, +)$ is an abelian group:*
  - *i) Closure: $\forall\, a, b \in R : a + b \in R$;*
  - *ii) Associativity: $\forall\, a, b, c \in R : (a + b) + c = a + (b + c)$;*
  - *iii) Neutral Element: $\exists\, 0 \in R : \forall\, a \in R : a + 0 = 0 + a = a$;*
  - *iv) Inverse Element: $\forall\, a \in R : \exists\, {-a} \in R : a + (-a) = (-a) + a = 0$;*
  - *v) Commutativity: $\forall\, a, b \in R : a + b = b + a$;*
- *the operation $\cdot$ is closed and associative in $R^* := R \setminus \{0\}$:*
  - *vi) Closure: $\forall\, a, b \in R^* : a \cdot b \in R^*$;*
  - *vii) Associativity: $\forall\, a, b, c \in R^* : (a \cdot b) \cdot c = a \cdot (b \cdot c)$;*
- *the maps $+$ and $\cdot$ satisfy the distributive properties:*
  - *viii) $\forall\, a, b, c \in R : (a + b) \cdot c = a \cdot c + b \cdot c$;*
  - *ix) $\forall\, a, b, c \in R : a \cdot (b + c) = a \cdot b + a \cdot c$.*

Note that since $\cdot$ is not required to be commutative, axioms viii and ix are both necessary. In the case of fields where $\cdot$ was commutative, ix followed from viii and commutativity of $\cdot$.

Definition 2.29 (Commutative / Unital / Division Rings)**.** *A ring $(R, +, \cdot)$ is said to be*

- ***commutative** if $\forall\, a, b \in R : a \cdot b = b \cdot a$;*
- ***unital** if $\exists\, 1 \in R : \forall\, a \in R : 1 \cdot a = a \cdot 1 = a$;*
- *a **division** (or **skew**)* ring *if it is unital and*

$$\forall\, a \in R \setminus \{0\} : \exists\, a^{-1} \in R \setminus \{0\} : \; a \cdot a^{-1} = a^{-1} \cdot a = 1.$$

In a unital ring, an element for which there exists a multiplicative inverse is said to be a *unit*. The set of units of a ring R is denoted by R^* (not to be confused with the vector space dual) and forms a group under multiplication. Then, R is a division ring iff $R^* = R \setminus \{0\}$.

Example 2.5. The sets $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ are all rings under the usual operations. They are also all fields, except $\mathbb{Z}$.

In general, if $(A, +, \cdot, \bullet)$ is an algebra, then $(A, +, \bullet)$ is a ring.

2.5 Modules

Definition 2.30 (R-Module)**.** *Let $(R, +, \cdot)$ be a unital ring. An R-**module** is a triple $(M, \oplus, \odot)$ where M is a set and*

$$\begin{aligned} \oplus &\colon M \times M \to M \\ \odot &\colon \; R \times M \to M \end{aligned}$$

are maps satisfying the following axioms:

- *$(M, \oplus)$ is an abelian group i.e*
  - *i) Closure: $\forall\, m, n \in M : m \oplus n \in M$;*
  - *ii) Associativity: $\forall\, m, n, s \in M : (m \oplus n) \oplus s = m \oplus (n \oplus s)$;*
  - *iii) Neutral Element: $\exists\, 0 \in M : \forall\, m \in M : m \oplus 0 = 0 \oplus m = m$;*
  - *iv) Inverse Element: $\forall\, m \in M : \exists\, -m \in M : m \oplus (-m) = (-m) \oplus m = 0$;*

v) Commutativity: $\forall\, m, n \in M : m \oplus n = n \oplus m$*;*

- *the map* $\odot$ *is an* action *of* R *on* $(M, \oplus)$*:*
  vi) Distributivity Of Scalar Multiplication - Vector Addition: $\forall\, r \in R : \forall\, m, n \in M : r \odot (m \oplus n) = (r \odot m) \oplus (r \odot n)$*;*
  vii) Distributivity Of Scalar Multiplication - Field Addition: $\forall\, r, s \in K : \forall\, m \in V : (r + s) \odot m = (r \odot m) \oplus (s \odot m)$*;*
  viii) Compatibility Of Scalar Multiplication - Field Multiplication $\forall\, r, s \in R : \forall\, m \in M : (r \cdot s) \odot m = r \odot (s \odot m)$*;*
  ix) Neutral Element Of Scalar Multiplication $\forall\, m \in M : 1 \odot m = m$*.*

So, modules are simply vector spaces over rings instead of fields. For this reason, most definitions we had for vector spaces carry over unaltered to modules.

Example 2.6. Any ring R is trivially a module over itself, in the sense that every field K is a vector space over itself.

In the following, we will usually denote $\oplus$ by $+$ and suppress the $\odot$, as we did with vector spaces.

Definition 2.31 (Direct Sum Of Modules)**.** *The* ***direct sum*** *of two* R*-modules* M *and* N *is the* R*-module* $M \oplus N$*, which has* $M \times N$ *as its underlying set and operations (inherited from* M *and* N*) defined component-wise.*

Note that while we have been using $\oplus$ to temporarily distinguish two "plus-like" operations in different spaces, the symbol $\oplus$ is the standard notation for the direct sum.

Definition 2.32 (Finitely Generated / Free / Projective Modules)**.** *An* R*-module* M *is said to be*

- ***finitely generated*** *if it has a finite generating set;*
- ***free*** *is it has a basis;*
- ***projective*** *if it is a direct summand of a free* R*-module* F*, i.e.*

$$M \oplus Q = F$$

  for some R*-module* Q*.*

Example 2.7. Clearly, every free module is also projective.

Definition 2.33 (R-Linear Maps)**.** *Let* M *and* N *be two* R*-modules. A map* $f : M \to N$ *is said to be an* R***-linear map****, or an* R***-module homomorphism****, if*

$$\forall\, r \in R : \forall\, m_1, m_2 \in M : \; f(rm_1 + m_2) = rf(m_1) + f(m_2),$$

where it should be clear which operations are in M *and which in* N*.*

Definition 2.34 (Module Isomorphisms)**.** *A bijective module homomorphism is said to be a* ***module isomorphism****.*

Definition 2.35 (Isomorphic Modules)**.** *Two modulesare said to be* ***isomorphic*** *if there exists a module isomorphism between them. We write* $M \cong_{\text{mod}} N$*.*

Proposition 2.3. *If a finitely generated module* R*-module* F *is free, and* $d \in \mathbb{N}$ *is the cardinality of a finite basis, then*

$$F \cong_{\text{mod}} = \underbrace{R \oplus \cdots \oplus R}_{d \text{ copies}} =: R^d.$$

One can show that if $R^d \cong_{\text{mod}} R^{d'}$, then $d = d'$ and hence, the concept of dimension is well-defined for finitely generated, free modules.

Theorem 2.2. *Let P, Q be finitely generated (projective) modules over a commutative ring R. Then*

$$\mathrm{Hom}_R(P, Q) := \{\phi\colon P \xrightarrow{\sim} Q \mid \phi \text{ is } R\text{-linear}\}$$

is again a finitely generated (projective) R-module, with operations defined pointwise.

The proof is exactly the same as with vector spaces. As an example, we can use this to define the dual of a module.

2.5.1 Basis Of Modules

The key fact that sets modules apart from vector spaces is that, unlike a vector space, an R-module need not have a basis, unless R is a division ring. This is actually a well-known theorem that we will state but not prove.

Theorem 2.3. *If D is a division ring, then any D-module V admits a basis.*

Corollary 2.1. *Every vector space has a basis, since any field is also a division ring.*

2.6 Algebras

Definition 2.36 (Algebra)**.** *Let K be a field, and let A be a vector space over K equipped with an additional bilinear map (called binary operation or product) $\bullet\colon A \times A \to A$. The quadruple $(A, +, \cdot, \bullet)$ is called an **algebra** over a field K.*

Definition 2.37 (Associative / Unital / Commutative Algebra)**.** *An algebra $(A, +, \cdot, \bullet)$ is said to be*

i) ***Associative*** *if* $\forall\, v, w, z \in A : v \bullet (w \bullet z) = (v \bullet w) \bullet z$*;*

ii) ***Unital*** *if* $\exists\, \mathbf{1} \in A : \forall\, v \in V : \mathbf{1} \bullet v = v \bullet \mathbf{1} = v$*;*

iii) ***Commutative*** *or* abelian *if* $\forall\, v, w \in A : v \bullet w = w \bullet v$.

Definition 2.38 (Derivation)**.** *Let A and B be algebras. A **derivation** on A is a linear map $D\colon A \xrightarrow{\sim} B$ satisfying the Leibniz rule*

$$D(v \bullet_A w) = D(v) \bullet_B w +_B v \bullet_B D(w).$$

for all $v, w \in A$.

Chapter 3

Lie Algebras

We already defined in the previous chapter that an algebra is a vector space A with an additional bilinear map (called binary operation or product) $\bullet\colon A \times A \to A$. A very important class of algebras, that we will also see later, are the so-called Lie algebras, in which the product $v \bullet w$ is called "Lie bracket" and denoted as $[v, w]$. In general Lie algebras are just a very specific class of algebras, hence we might have them introduced in the previous chapter under "algebras". However, since they are so important, and lengthy, we will introduce them separately in their own chapter.

Lie algebras are closely related to Lie groups, which are groups that are also smooth manifolds: any Lie group gives rise to a Lie algebra, which is its tangent space at the identity. Conversely, to any finite-dimensional Lie algebra over real or complex numbers, there is a corresponding connected Lie group unique up to finite coverings. This correspondence allows one to study the structure and classification of Lie groups in terms of Lie algebras (we will see all of that as we proceed in the notes).

In physics, Lie groups appear as symmetry groups of physical systems, and their Lie algebras (tangent vectors near the identity) may be thought of as infinitesimal symmetry motions. Thus Lie algebras and their representations are used extensively in physics, notably in quantum mechanics and particle physics.

3.1 Lie Algebras

Definition 3.1 (Lie Algebra)**.** *A **Lie algebra** A is an algebra whose product $[-,-]$, called* Lie bracket*, satisfies*

i) bilinearity: $A \times A \to A$: $[av + w, z] = a[v, w] + [v, z]$

ii) antisymmetry: $\forall\, v \in A : [v, v] = 0$;

iii) the Jacobi identity: $\forall\, v, w, z \in A : [v, [w, z]] + [w, [z, v]] + [z, [v, w]] = 0$.

Note that the zeros above represent the additive identity element in A, not the zero scalar

Some remarks are in order

Remark 3.1. The antisymmetry condition immediately implies $[v, w] = -[w, v]$ for all $v, w \in A$ since

$$[v + w, v + w] = [v, v] + [v, w] + [w, v] + [w, w] = [v, w] + [w, v] = 0 \implies [v, w] = -[w, v]$$

Remark 3.2. Notice that the Lie bracket is not defined as the usual commutator $[v, w] = vw - wv$, but is defined very abstractly by the 3 conditions. In other words, anything that satisfies these 3 conditions can be defined as a Lie bracket. Of course one example is the commutator (you can check it yourself)

Remark 3.3. Notice that we specifically defined the Lie algebra on top of a vector space A. One can construct an algebra over a ring, by imposing all the axioms on a module instead of a vector space. However, in this notes we will stick with Lie algebras on top of a vector space, and more specifically on top of a complex vector space (i.e where the K field is the complex and to the real numbers), since they are more related to our purposes. In general, same definitions apply for an algebra over a ring with the appropriate changes when needed.

Now let's give some examples of Lie algebras.

Example 3.1. The usual cross product between vectors $u \times w$ in $\mathbb{R}^3$ can be shown that satisfies all the requirements for a Lie bracket, hence the vector space $\mathbb{R}^3$ equipped with the cross product is actually a Lie algebra.

Example 3.2. Let V be a vector space. Recall that we defined the set $\text{End}(V)$ as the set of all endomorphisms of V, i.e the set of all linear transformations that send V back to itself. Now we define the following Lie bracket:

$$\begin{aligned}[-,-]\colon \text{End}(V) \times \text{End}(V) &\to \text{End}(V) \\ (\phi, \psi) &\mapsto [\phi, \psi] := \phi \circ \psi - \psi \circ \phi.\end{aligned}$$

It is instructive to check that this is actually a Lie bracket. Hecne, $(\text{End}(V), +, \cdot, [-,-])$ is a Lie algebra. In this case, the Lie bracket is typically called the *commutator*.

An example of this algebra is the set of all $n \times n$ matrices over a field K, with their commutator $[v, w] = vw - wv$, (where here the composition is the usual matrix multiplication). Of course the set of $n \times n$ matrices can be seen as linear transformations of the vectors of K-vector space V hence they belong to the $\text{End}(V)$.

One of the most important topics on Lie algebras is the classification of them. While it is possible to classify Lie algebras more generally, we will only consider the classification of finite-dimensional complex Lie algebras, i.e. Lie algebras $(L, [-,-])$ where L is a finite-dimensional $\mathbb{C}$-vector space.
If A, B are Lie subalgebras of a Lie algebra $(L, [-,-])$ over K, then

$$[A, B] := \text{span}_K\big(\{[x, y] \in L \mid x \in A \text{ and } y \in B\}\big)$$

is again a Lie subalgebra of L.

Definition 3.2. *A Lie algebra L is said to be* abelian *if*

$$\forall\, x, y \in L : \ [x, y] = 0.$$

Equivalently, $[L, L] = 0$, where 0 denotes the trivial Lie algebra $\{0\}$.

Abelian Lie algebras are highly non-interesting as Lie algebras: since the bracket is identically zero, it may as well not be there. Even from the classification point of view, the vanishing of the bracket implies that, given any two abelian Lie algebras, every linear isomorphism between their underlying vector spaces is automatically a Lie algebra isomorphism. Therefore, for each $n \in \mathbb{N}$, there is (up to isomorphism) only one abelian n-dimensional Lie algebra.

Definition 3.3. *An* ideal *I of a Lie algebra L is a Lie subalgebra such that $[I, L] \subseteq I$, i.e.*

$$\forall\, x \in I : \forall\, y \in L : \ [x, y] \in I.$$

The ideals 0 and L are called the trivial ideals *of L.*

Definition 3.4. *A Lie algebra L is said to be*

- simple *if it is non-abelian and it contains no non-trivial ideals;*
- semi-simple *if it contains no non-trivial abelian ideals.*

Remark 3.4. Note that any simple Lie algebra is also semi-simple. The requirement that a simple Lie algebra be non-abelian is due to the 1-dimensional abelian Lie algebra, which would otherwise be the only simple Lie algebra which is not semi-simple.

Definition 3.5. *Let L be a Lie algebra. The Lie subalgebra*

$$L' := [L, L]$$

is called the derived subalgebra *of L.*

We can form a sequence of Lie subalgebras

$$L \supseteq L' \supseteq L'' \supseteq \cdots \supseteq L^{(n)} \supseteq \cdots$$

called the *derived series* of L.

Definition 3.6. *A Lie algebra L is* solvable *if there exists $k \in \mathbb{N}$ such that $L^{(k)} = 0$.*

Recall that the direct sum of vector spaces $V \oplus W$ has $V \times W$ as its underlying set and operations defined componentwise.

Definition 3.7. *Let L_1 and L_2 be Lie algebras. The* direct sum *$L_1 \oplus_{\mathrm{Lie}} L_2$ has $L_1 \oplus L_2$ as its underlying vector space and Lie bracket defined as*

$$[x_1 + x_2, y_1 + y_2]_{L_1 \oplus_{\mathrm{Lie}} L_2} := [x_1, y_1]_{L_1} + [x_2, y_2]_{L_2}$$

for all $x_1, y_1 \in L_1$ and $x_2, y_2 \in L_2$. Alternatively, by identifying L_1 and L_2 with the subspaces $L_1 \oplus 0$ and $0 \oplus L_2$ of $L_1 \oplus L_2$ respectively, we require

$$[L_1, L_2]_{L_1 \oplus_{\mathrm{Lie}} L_2} = 0.$$

In the following, we will drop the "Lie" subscript and understand $\oplus$ to mean $\oplus_{\mathrm{Lie}}$ whenever the summands are Lie algebras.

There is a weaker notion than the direct sum, defined only for Lie algebras.

Definition 3.8. *Let R and L be Lie algebras. The* semi-direct sum *$R \oplus_s L$ has $R \oplus L$ as its underlying vector space and Lie bracket satisfying*

$$[R, L]_{R \oplus_s L} \subseteq R,$$

i.e. R is an ideal of $R \oplus_s L$.

We are now ready to state Levi's decomposition theorem.

Theorem 3.1 (Levi)**.** *Any finite-dimensional complex Lie algebra L can be decomposed as*

$$L = R \oplus_s (L_1 \oplus \cdots \oplus L_n)$$

where R is a solvable Lie algebra and $L_1, \ldots, L_n$ are simple Lie algebras.

As of today, no general classification of solvable Lie algebras is known, except for some special cases (e.g. in low dimensions). In contrast, the finite dimensional, simple, complex Lie algebras have been classified completely.

Proposition 3.1. *A Lie algebra is semi-simple if, and only if, it can be expressed as a direct sum of simple Lie algebras.*

Hence, the simple Lie algebras are the basic building blocks from which one can build any semi-simple Lie algebra. Then, by Levi's theorem, the classification of simple Lie algebras easily extends to a classification of all semi-simple Lie algebras.

3.1.1 The adjoint map and the Killing form

Definition 3.9. *Let L be a Lie algebra over k and let $x \in L$. The* adjoint map *with respect to x is the K-linear map*

$$\begin{aligned} \mathrm{ad}_x \colon L &\xrightarrow{\sim} L \\ y &\mapsto \mathrm{ad}_x(y) := [x, y]. \end{aligned}$$

The linearity of ad_x follows from the linearity of the bracket in the second argument, while the linearity in the first argument of the bracket implies that the map

$$\begin{aligned} \mathrm{ad} \colon L &\xrightarrow{\sim} \mathrm{End}(L) \\ x &\mapsto \mathrm{ad}(x) := \mathrm{ad}_x\,. \end{aligned}$$

itself is also linear. In fact, more is true. Recall that $\mathrm{End}(L)$ is a Lie algebra with bracket

$$[\phi, \psi] := \phi \circ \psi - \psi \circ \phi.$$

Then, we have the following.

Proposition 3.2. *The map* $\mathrm{ad}\colon L \xrightarrow{\sim} \mathrm{End}(L)$ *is a Lie algebra homomorphism.*

Proof. It remains to check that ad preserves the brackets. Let $x, y, z \in L$. Then

$$\begin{aligned}
\mathrm{ad}_{[x,y]}(z) &:= [[x, y], z] && \text{(definition of ad)}\\
&= -[[y, z], x] - [[z, x], y] && \text{(Jacobi's identity)}\\
&= [x, [y, z]] - [y, [x, z]] && \text{(anti-symmetry)}\\
&= \mathrm{ad}_x(\mathrm{ad}_y(z)) - \mathrm{ad}_y(\mathrm{ad}_x(z))\\
&= (\mathrm{ad}_x \circ \mathrm{ad}_y - \mathrm{ad}_y \circ \mathrm{ad}_x)(z)\\
&= [\mathrm{ad}_x, \mathrm{ad}_y](z).
\end{aligned}$$

Hence, we have $\mathrm{ad}([x, y]) = [\mathrm{ad}(x), \mathrm{ad}(y)]$. □

Definition 3.10. *Let L be a Lie algebra over K. The* Killing form *on L is the K-bilinear map*

$$\begin{aligned}
\kappa\colon L \times L &\to K\\
(x, y) &\mapsto \kappa(x, y) := \mathrm{tr}(\mathrm{ad}_x \circ \mathrm{ad}_y),
\end{aligned}$$

where tr *is the usual trace on the vector space* $\mathrm{End}(L)$*.*

Note that the Killing form is not a "form" in the sense that we defined previously. In fact, since L is finite-dimensional, the trace is cyclic and thus κ is symmetric, i.e.

$$\forall\, x, y \in L : \ \kappa(x, y) = \kappa(y, x).$$

An important property of κ is its associativity with respect to the bracket.

Proposition 3.3. *Let L be a Lie algebra. For any $x, y, z \in L$, we have*

$$\kappa([x, y], z) = \kappa(x, [y, z]).$$

Proof. This follows easily from the fact that ad is a homomorphism.

$$\begin{aligned}
\kappa([x, y], z) &:= \mathrm{tr}(\mathrm{ad}_{[x,y]} \circ \mathrm{ad}_z)\\
&= \mathrm{tr}([\mathrm{ad}_x, \mathrm{ad}_y] \circ \mathrm{ad}_z)\\
&= \mathrm{tr}((\mathrm{ad}_x \circ \mathrm{ad}_y - \mathrm{ad}_y \circ \mathrm{ad}_x) \circ \mathrm{ad}_z)\\
&= \mathrm{tr}(\mathrm{ad}_x \circ \mathrm{ad}_y \circ \mathrm{ad}_z) - \mathrm{tr}(\mathrm{ad}_y \circ \mathrm{ad}_x \circ \mathrm{ad}_z)\\
&= \mathrm{tr}(\mathrm{ad}_x \circ \mathrm{ad}_y \circ \mathrm{ad}_z) - \mathrm{tr}(\mathrm{ad}_x \circ \mathrm{ad}_z \circ \mathrm{ad}_y)\\
&= \mathrm{tr}(\mathrm{ad}_x \circ (\mathrm{ad}_y \circ \mathrm{ad}_z - \mathrm{ad}_z \circ \mathrm{ad}_y))\\
&= \mathrm{tr}(\mathrm{ad}_x \circ [\mathrm{ad}_y, \mathrm{ad}_z])\\
&= \mathrm{tr}(\mathrm{ad}_x \circ \mathrm{ad}_{[y,z]})\\
&=: \kappa(x, [y, z]),
\end{aligned}$$

where we used the cyclicity of the trace. □

We can use κ to give a further equivalent characterisation of semi-simplicity.

Proposition 3.4 (Cartan's criterion)**.** *A Lie algebra L is semi-simple if, and only if, the Killing form κ is non-degenerate, i.e.*

$$(\forall\, y \in L : \kappa(x, y) = 0) \Rightarrow x = 0.$$

Hence, if L is semi-simple, then κ is a pseudo inner product on L. Recall the following definition from linear algebra.

Definition 3.11. *A linear map* $\phi\colon V \xrightarrow{\sim} V$ *is said to be* symmetric *with respect to the pseudo inner product* $B(-,-)$ *on* V *if*

$$\forall\, v, w \in V:\; B(\phi(v), w) = B(v, \phi(w)).$$

If, instead, we have

$$\forall\, v, w \in V:\; B(\phi(v), w) = -B(v, \phi(w)),$$

then ϕ *is said to be* anti-symmetric *with respect to* B*.*

The associativity property of κ with respect to the bracket can be restated by saying that, for any $z \in L$, the linear map ad_z is anti-symmetric with respect to κ, i.e.

$$\forall\, x, y \in L:\; \kappa(\mathrm{ad}_z(x), y) = -\kappa(x, \mathrm{ad}_z(y)).$$

In order to do computations, it is useful to introduce a basis $\{E_i\}$ on L.

Definition 3.12. *Let* L *be a Lie algebra over* K *and let* $\{E_i\}$ *be a basis. Then, we have*

$$[E_i, E_j] = C^k{}_{ij} E_k$$

for some $C^k{}_{ij} \in K$*. The numbers* $C^k{}_{ij}$ *are called the* structure constants *of* L *with respect to the basis* $\{E_i\}$*.*

In terms of the structure constants, the anti-symmetry of the Lie bracket reads

$$C^k{}_{ij} = -C^k{}_{ji}$$

while the Jacobi identity becomes

$$C^n{}_{im} C^m{}_{jk} + C^n{}_{jm} C^m{}_{ki} + C^n{}_{km} C^m{}_{ij} = 0.$$

We can now express both the adjoint maps and the Killing form in terms of components with respect to a basis.

Proposition 3.5. *Let* L *be a Lie algebra and let* $\{E_i\}$ *be a basis. Then*

i) $(\mathrm{ad}_{E_i})^k{}_j = C^k{}_{ij}$

ii) $\kappa_{ij} = C^m{}_{ik} C^k{}_{jm}$

where $C^k{}_{ij}$ *are the structure constants of* L *with respect to* $\{E_i\}$*.*

Proof. i) Denote by $\{\varepsilon^i\}$ the dual basis to $\{E_i\}$. Then, we have

$$\begin{aligned}
(\mathrm{ad}_{E_i})^k{}_j &:= \varepsilon^k(\mathrm{ad}_{E_i}(E_j)) \\
&= \varepsilon^k([E_i, E_j]) \\
&= \varepsilon^k(C^m{}_{ij} E_m) \\
&= C^m{}_{ij} \varepsilon^k(E_m) \\
&= C^k{}_{ij},
\end{aligned}$$

since $\varepsilon^k(E_m) = \delta^k_m$.

ii) Recall from linear algebra that if V is finite-dimensional, for any $\phi \in \mathrm{End}(V)$ we have $\mathrm{tr}(\phi) = \Phi^k{}_k$, where Φ is the matrix representing the linear map in any basis. Also, recall that the matrix representing $\phi \circ \psi$ is the product $\Phi\Psi$. Using these, we have

$$\begin{aligned}
\kappa_{ij} &:= \kappa(E_i, E_j) \\
&= \mathrm{tr}(\mathrm{ad}_{E_i} \circ \mathrm{ad}_{E_j}) \\
&= (\mathrm{ad}_{E_i} \circ \mathrm{ad}_{E_j})^k{}_k \\
&= (\mathrm{ad}_{E_i})^m{}_k (\mathrm{ad}_{E_j})^k{}_m \\
&= C^m{}_{ik} C^k{}_{jm},
\end{aligned}$$

where we used the same notation for the linear maps and their matrices. □

3.1.2 The fundamental roots and the Weyl group

We will now focus on finite-dimensional semi-simple complex Lie algebras, whose classification hinges on the existence of a special type of subalgebra.

Definition 3.13. *Let L be a d-dimensional Lie algebra. A* Cartan subalgebra *H of L is a maximal Lie subalgebra of L with the following property: there exists a basis $\{h_1, \dots, h_r\}$ of H which can be extended to a basis $\{h_1, \dots, h_r, e_1, \dots, e_{d-r}\}$ of L such that $e_1, \dots, e_{d-r}$ are eigenvectors of* $\mathrm{ad}(h)$ *for any $h \in H$, i.e.*

$$\forall\, h \in H : \exists\, \lambda_\alpha(h) \in \mathbb{C} : \; \mathrm{ad}(h)e_\alpha = \lambda_\alpha(h)e_\alpha,$$

for each $1 \leq \alpha \leq d - r$.

The basis $\{h_1, \dots, h_r, e_1, \dots, e_{d-r}\}$ is known as a *Cartan-Weyl basis* of L.

Theorem 3.2. *Let L be a finite-dimensional semi-simple complex Lie algebra. Then*

i) L possesses a Cartan subalgebra;

ii) all Cartan subalgebras of L have the same dimension, called the rank *of L;*

iii) any of Cartan subalgebra of L is abelian.

Note that we can think of the λ_α appearing above as a map $\lambda_\alpha \colon H \to \mathbb{C}$. Moreover, for any $z \in \mathbb{C}$ and $h, h' \in H$, we have

$$\begin{aligned}
\lambda_\alpha(zh + h')e_\alpha &= \mathrm{ad}(zh + h')e_\alpha \\
&= [zh + h', e_\alpha] \\
&= z[h, e_\alpha] + [h', e_\alpha] \\
&= z\lambda_\alpha(h)e_\alpha + \lambda_\alpha(h')e_\alpha \\
&= (z\lambda_\alpha(h) + \lambda_\alpha(h'))e_\alpha,
\end{aligned}$$

Hence λ_α is a $\mathbb{C}$-linear map $\lambda_\alpha \colon H \xrightarrow{\sim} \mathbb{C}$, and thus $\lambda_\alpha \in H^*$.

Definition 3.14. *The maps $\lambda_1, \dots, \lambda_{d-r} \in H^*$ are called the* roots *of L. The collection*

$$\Phi := \{\lambda_\alpha \mid 1 \leq \alpha \leq d - r\} \subseteq H^*$$

is called the root set *of L.*

One can show that if λ_α were the zero map, then we would have $e_\alpha \in H$. Thus, we must have $0 \notin \Phi$. Note that a consequence of the anti-symmetry of each $\mathrm{ad}(h)$ with respect to the Killing form κ is that

$$\lambda \in \Phi \;\Rightarrow\; -\lambda \in \Phi.$$

Hence Φ is not a linearly independent subset of H^*.

Definition 3.15. *A set of* fundamental roots *$\Pi := \{\pi_1, \dots, \pi_f\}$ is a subset $\Pi \subseteq \Phi$ such that*

a) Π is a linearly independent subset of H^*;

b) for each $\lambda \in \Phi$, there exist $n_1, \dots, n_f \in \mathbb{N}$ and $\varepsilon \in \{+1, -1\}$ such that

$$\lambda = \varepsilon \sum_{i=1}^{f} n_i \pi_i.$$

We can write the last equation more concisely as $\lambda \in \mathrm{span}_{\varepsilon,\mathbb{N}}(\Pi)$. Observe that, for any $\lambda \in \Phi$, the coefficients of $\pi_1, \dots, \pi_f$ in the expansion above always have the same sign. Indeed, we have $\mathrm{span}_{\varepsilon,\mathbb{N}}(\Pi) \neq \mathrm{span}_{\mathbb{Z}}(\Pi)$.

Theorem 3.3. *Let L be a finite-dimensional semi-simple complex Lie algebra. Then*

i) a set $\Pi \subseteq \Phi$ of fundamental roots always exists;

ii) we have $\mathrm{span}_{\mathbb{C}}(\Pi) = H^*$*, that is,* Π *is a basis of* H^**.*

Corollary 3.1. *We have* $|\Pi| = r$*, where* r *is the rank of* L*.*

Proof. Since Π is a basis, $|\Pi| = \dim H^* = \dim H = r$. □

We would now like to use κ to define a pseudo inner product on H^*. We know from linear algebra that a pseudo inner product $B(-,-)$ on a finite-dimensional vector space V over K induces a linear isomorphism

$$\begin{aligned} i \colon V &\xrightarrow{\sim} V^* \\ v &\mapsto i(v) := B(v,-) \end{aligned}$$

which can be used to define a pseudo inner product $B^*(-,-)$ on V^* as

$$\begin{aligned} B^* \colon V^* \times V^* &\to K \\ (\phi, \psi) &\mapsto B^*(\phi,\psi) := B(i^{-1}(\phi), i^{-1}(\psi)). \end{aligned}$$

We would like to apply this to the restriction of κ to the Cartan subalgebra. However, a pseudo inner product on a vector space is not necessarily a pseudo inner product on a subspace, since the non-degeneracy condition may fail when considered on a subspace.

Proposition 3.6. *The restriction of* κ *to* H *is a pseudo inner product on* H*.*

Proof. Bilinearity and symmetry are automatically satisfied. It remains to show that κ is non-degenerate on H.

i) Let $\{h_1, \ldots, h_r, e_{r+1}, \ldots, e_d\}$ be a Cartan-Weyl basis of L and let $\lambda_\alpha \in \Phi$. Then

$$\begin{aligned} \lambda_\alpha(h_j)\kappa(h_i, e_\alpha) &= \kappa(h_i, \lambda_\alpha(h_j)e_\alpha) \\ &= \kappa(h_i, [h_j, e_\alpha]) \\ &= \kappa([h_i, h_j], e_\alpha) \\ &= \kappa(0, e_\alpha) \\ &= 0. \end{aligned}$$

Since $\lambda_\alpha \neq 0$, there is some h_j such that $\lambda_\alpha(h_j) \neq 0$ and hence

$$\kappa(h_i, e_\alpha) = 0.$$

By linearity, we have $\kappa(h, e_\alpha) = 0$ for any $h \in H$ and any e_α.

ii) Let $h \in H \subseteq L$. Since κ is non-degenerate on L, we have

$$\big(\forall\, x \in L : \kappa(h, x) = 0\big) \Rightarrow h = 0.$$

Expand $x \in L$ in the Cartan-Weyl basis as

$$x = h' + e$$

where $h' := x^i h_i$ and $e := x^\alpha e_\alpha$. Then, we have

$$\kappa(h, x) = \kappa(h, h') + x^\alpha \kappa(h, e_\alpha) = \kappa(h, h').$$

Thus, the non-degeneracy condition reads

$$\big(\forall\, h' \in H : \kappa(h, h') = 0\big) \Rightarrow h = 0,$$

which is what we wanted. □

We can now define

$$\begin{aligned} \kappa^* \colon H^* \times H^* &\to \mathbb{C} \\ (\mu, \nu) &\mapsto \kappa^*(\mu,\nu) := \kappa(i^{-1}(\mu), i^{-1}(\nu)), \end{aligned}$$

where $i\colon H \xrightarrow{\sim} H^*$ is the linear isomorphism induced by κ.

Remark 3.5. If $\{h_i\}$ is a basis of H, the components of κ^* with respect to the dual basis satisfy

$$(\kappa^*)^{ij}\kappa_{jk} = \delta^i_k.$$

Hence, we can write

$$\kappa^*(\mu,\nu) = (\kappa^*)^{ij}\mu_i\nu_j,$$

where $\mu_i := \mu(h_i)$.

We now turn our attention to the real subalgebra $H^*_{\mathbb{R}} := \operatorname{span}_{\mathbb{R}}(\Pi)$. Note that we have the following chain of inclusions

$$\Pi \subseteq \Phi \subseteq \operatorname{span}_{\varepsilon,\mathbb{N}}(\Pi) \subseteq \underbrace{\operatorname{span}_{\mathbb{R}}(\Pi)}_{H^*_{\mathbb{R}}} \subseteq \underbrace{\operatorname{span}_{\mathbb{C}}(\Pi)}_{H^*}.$$

The restriction of κ^* to $H^*_{\mathbb{R}}$ leads to a surprising result.

Theorem 3.4. *i) For any $\alpha,\beta \in H^*_{\mathbb{R}}$, we have $\kappa^*(\alpha,\beta) \in \mathbb{R}$.*

ii) $\kappa^*\colon H^*_{\mathbb{R}} \times H^*_{\mathbb{R}} \to \mathbb{R}$ is an inner product on $H^*_{\mathbb{R}}$.

This is indeed a surprise! Upon restriction to $H^*_{\mathbb{R}}$, instead of being weakened, the non-degeneracy of κ^* gets strengthened to positive definiteness. Now that we have a proper real inner product, we can define some familiar notions from basic linear algebra, such as lengths and angles.

Definition 3.16. *Let $\alpha,\beta \in H^*_{\mathbb{R}}$. Then, we define*

i) the length *of α as $|\alpha| := \sqrt{\kappa^*(\alpha,\alpha)}$;*

ii) the angle *between α and β as $\varphi := \cos^{-1}\left(\dfrac{\kappa^*(\alpha,\beta)}{|\alpha||\beta|}\right)$.*

We need one final ingredient for our classification result.

Definition 3.17. *For any $\lambda \in \Phi \subseteq H^*_{\mathbb{R}}$, define the linear map*

$$\begin{aligned} s_\lambda\colon H^*_{\mathbb{R}} &\xrightarrow{\sim} H^*_{\mathbb{R}} \\ \mu &\mapsto s_\lambda(\mu), \end{aligned}$$

where

$$s_\lambda(\mu) := \mu - 2\frac{\kappa^*(\lambda,\mu)}{\kappa^*(\lambda,\lambda)}\,\lambda.$$

The map s_λ is called a Weyl transformation *and the set*

$$W := \{s_\lambda \mid \lambda \in \Phi\}$$

is a group under composition of maps, called the Weyl group.

Theorem 3.5. *i) The Weyl group W is generated by the fundamental roots in Π, in the sense that for some $1 \leq n \leq r$, with $r = |\Pi|$,*

$$\forall\, w \in W : \exists\, \pi_1,\ldots,\pi_n \in \Pi : \; w = s_{\pi_1} \circ s_{\pi_2} \circ \cdots \circ s_{\pi_n};$$

ii) Every root can be produced from a fundamental root by the action of W, i.e.

$$\forall\, \lambda \in \Phi : \exists\, \pi \in \Pi : \exists\, w \in W : \; \lambda = w(\pi);$$

iii) The Weyl group permutes the roots, that is,

$$\forall\, \lambda \in \Phi : \forall\, w \in W : \; w(\lambda) \in \Phi.$$

3.1.3 Dynkin diagrams and the Cartan classification

Consider, for any $\pi_i, \pi_j \in \Pi$, the action of the Weyl transformation

$$s_{\pi_i}(\pi_j) := \pi_j - 2\frac{\kappa^*(\pi_i, \pi_j)}{\kappa^*(\pi_i, \pi_i)}\,\pi_i.$$

Since $s_{\pi_i}(\pi_j) \in \Phi$ and $\Phi \subseteq \operatorname{span}_{\varepsilon,\mathbb{N}}(\Pi)$, for all $1 \leq i \neq j \leq r$ we must have

$$-2\frac{\kappa^*(\pi_i, \pi_j)}{\kappa^*(\pi_i, \pi_i)} \in \mathbb{N}.$$

Definition 3.18. *The* Cartan matrix *of a Lie algebra is the $r \times r$ matrix C with entries*

$$C_{ij} := 2\frac{\kappa^*(\pi_i, \pi_j)}{\kappa^*(\pi_i, \pi_i)},$$

where the C_{ij} should not be confused with the structure constants ${C^k}_{ij}$.

Theorem 3.6. *To every simple finite-dimensional complex Lie algebra there corresponds a unique Cartan matrix and vice versa (up to relabelling of the basis elements).*

Of course, not every matrix can be a Cartan matrix. For instance, since $C_{ii} = 2$ (no summation implied), the diagonal entries of C are all equal to 2, while the off-diagonal entries are either zero or negative. In general, $C_{ij} \neq C_{ji}$, so the Cartan matrix is not symmetric, but if $C_{ij} = 0$, then necessarily $C_{ji} = 0$.
We have thus reduced the problem of classifying the simple finite-dimensional complex Lie algebras to that of finding all the Cartan matrices. This can, in turn, be reduced to the problem of determining all the inequivalent Dynkin diagrams.

Definition 3.19. *Given a Cartan matrix C, the ij-th* bond number *is*

$$n_{ij} := C_{ij}C_{ji} \qquad \text{(no summation implied).}$$

Note that we have

$$\begin{aligned} n_{ij} &= 4\,\frac{\kappa^*(\pi_i, \pi_j)}{\kappa^*(\pi_i, \pi_i)}\,\frac{\kappa^*(\pi_j, \pi_i)}{\kappa^*(\pi_j, \pi_j)} \\ &= 4\left(\frac{\kappa^*(\pi_i, \pi_j)}{|\pi_i||\pi_j|}\right)^2 \\ &= 4\cos^2\varphi, \end{aligned}$$

where φ is the angle between π_i and π_j. For $i \neq j$, the angle φ is neither zero nor 180°, hence $0 \leq \cos^2\varphi < 1$, and therefore

$$n_{ij} \in \{0, 1, 2, 3\}.$$

Since $C_{ij} \leq 0$ for $i \neq j$, the only possibilities are

C_{ij}	C_{ji}	n_{ij}
0	0	0
-1	-1	1
-1	-2	2
-1	-3	3

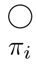
Note that while the Cartan matrices are not symmetric, swapping any pair of C_{ij} and C_{ji} gives a Cartan matrix which represents the same Lie algebra as the original matrix, with two elements from the Cartan-Weyl basis swapped. This is why we have not included $(-2, -1)$ and $(-3, -1)$ in the table above.
If $n_{ij} = 2$ or 3, then the corresponding fundamental roots have different lengths, i.e. either $|\pi_i| < |\pi_j|$ or $|\pi_i| > |\pi_j|$. We also have the following result.

Proposition 3.7. *The roots of a simple Lie algebra have, at most, two distinct lengths.*

The redundancy of the Cartan matrices highlighted above is nicely taken care of by considering Dynkin diagrams.

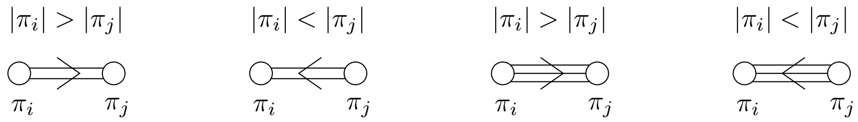
Definition 3.20. *A* Dynkin diagram *associated to a Cartan matrix is constructed as follows.*

1. *Draw a circle for every fundamental root in $\pi_i \in \Pi$;*



2. *Draw n_{ij} lines between the circles representing the roots π_i and π_j;*

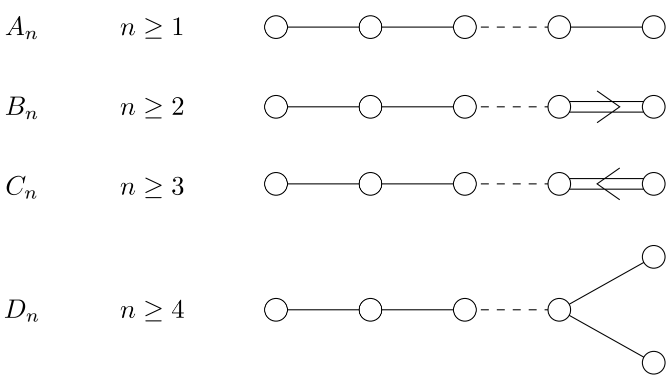
3. *If $n_{ij} = 2$ or 3, draw an arrow on the lines from the longer root to the shorter root.*



Dynkin diagrams completely characterise any set of fundamental roots, from which we can reconstruct the entire root set by using the Weyl transformations. The root set can then be used to produce a Cartan-Weyl basis.
We are now finally ready to state the much awaited classification theorem.

Theorem 3.7 (Killing, Cartan)**.** *Any simple finite-dimensional complex Lie algebra can be reconstructed from its set of fundamental roots Π, which only come in the following forms.*

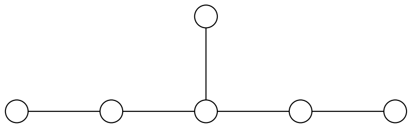
i) There are 4 infinite families



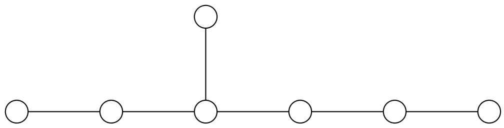
where the restrictions on n ensure that we don't get repeated diagrams (the diagram D_2 is excluded since it is disconnected and does not correspond to a simple Lie algebra)

ii) five exceptional cases

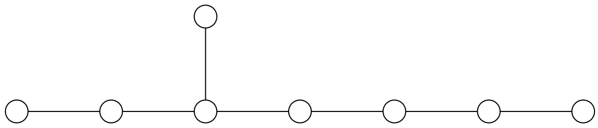
E_6



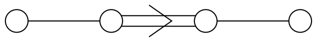
E_7



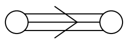
E_8



F_4



G_2



and no other. These are all the possible (connected) Dynkin diagrams.

At last, we have achieved a classification of all simple finite-dimensional complex Lie algebras. The finite-dimensional semi-simple complex Lie algebras are direct sums of simple Lie algebras, and correspond to disconnected Dynkin diagrams whose connected components are the ones listed above.

Chapter 4

Topology

4.1 Topological Spaces

We will now discuss topological spaces based on our previous development of set theory. As we will see, a topology on a set provides the weakest structure in order to define the two very important notions of convergence of sequences to points in a set, and of continuity of maps between two sets. The definition of topology on a set is, at first sight, rather abstract. But on the upside it is also extremely simple. This definition is the result of a historical development, it is the simplest definition of topology that mathematician found to be useful.

Definition 4.1 (Topology)**.** *Let M be a set. A* ***topology*** *on M is a set $\mathcal{O} \subseteq \mathcal{P}(M)$ such that:*

i) $\varnothing \in \mathcal{O}$ *and* $M \in \mathcal{O}$*;*

ii) $\{U, V\} \subseteq \mathcal{O} \Rightarrow \bigcap\{U, V\} \in \mathcal{O}$*;*

iii) $C \subseteq \mathcal{O} \Rightarrow \bigcup C \in \mathcal{O}$*.*

Definition 4.2 (Topological Space)**.** *Let M be a set and $\mathcal{O}$ a topology on the set M. The pair $(M, \mathcal{O})$ is called a* ***topological space****. If we write "let M be a topological space" then some topology $\mathcal{O}$ on M is assumed.*

Remark 4.1. Unless $|M| = 1$, there are (usually many) different topologies $\mathcal{O}$ that one can choose on the set M.

$\lvert M \rvert$	Number of topologies
1	1
2	4
3	29
4	355
5	6,942
6	209,527
7	9,535,241

Example 4.1. Let $M = \{a, b, c\}$. Then $\mathcal{O} = \{\varnothing, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$ is a topology on M since:

i) $\varnothing \in \mathcal{O}$ and $M \in \mathcal{O}$;

ii) Clearly, for any $S \in \mathcal{O}$, $\bigcap\{\varnothing, S\} = \varnothing \in \mathcal{O}$ and $\bigcap\{S, M\} = S \in \mathcal{O}$. Moreover, $\{a\} \cap \{b\} = \varnothing \in \mathcal{O}$, $\{a\} \cap \{a, b\} = \{a\} \in \mathcal{O}$, and $\{b\} \cap \{a, b\} = \{b\} \in \mathcal{O}$;

iii) Let $C \subseteq \mathcal{O}$. If $M \in C$, then $\bigcup C = M \in \mathcal{O}$. If $\{a, b\} \in C$ (or $\{a\}, \{b\} \in C$) but $M \notin C$, then $\bigcup C = \{a, b\} \in \mathcal{O}$. If either $\{a\} \in C$ or $\{b\} \in C$, but $\{a, b\} \notin C$ and $M \notin C$, then $\bigcup C = \{a\} \in \mathcal{O}$ or $\bigcup C = \{b\} \in \mathcal{O}$, respectively. Finally, if none of the above hold, then $\bigcup C = \varnothing \in \mathcal{O}$.

Example 4.2. Let M be a set. Then $\mathcal{O} = \{\varnothing, M\}$ is a topology on M. Indeed, we have:

i) $\varnothing \in \mathcal{O}$ and $M \in \mathcal{O}$;

ii) $\bigcap\{\varnothing, \varnothing\} = \varnothing \in \mathcal{O}$, $\bigcap\{\varnothing, M\} = \varnothing \in \mathcal{O}$, and $\bigcap\{M, M\} = M \in \mathcal{O}$;

iii) If $M \in C$, then $\bigcup C = M \in \mathcal{O}$, otherwise $\bigcup C = \varnothing \in \mathcal{O}$.

This is called the *chaotic topology* and can be defined on any set.

Example 4.3. Let M be a set. Then $\mathcal{O} = \mathcal{P}(M)$ is a topology on M. Indeed, we have:

i) $\varnothing \in \mathcal{P}(M)$ and $M \in \mathcal{P}(M)$;

ii) If $U, V \in \mathcal{P}(M)$, then $\bigcap\{U, V\} \subseteq M$ and hence $\bigcap\{U, V\} \in \mathcal{P}(M)$;

iii) If $C \subseteq \mathcal{P}(M)$, then $\bigcup C \subseteq M$, and hence $\bigcup C \in \mathcal{P}(M)$.

This is called the *discrete topology* and can be defined on any set.

We now give some common terminology regarding topologies.

Definition 4.3 (Coarser / Finer Topology). *Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be two topologies on a set M. If $\mathcal{O}_1 \subset \mathcal{O}_2$, then we say that $\mathcal{O}_1$ is a* ***coarser*** *(or* weaker*) topology than $\mathcal{O}_2$. Equivalently, we say that $\mathcal{O}_2$ is a* ***finer*** *(or* stronger*) topology than $\mathcal{O}_1$.*

Clearly, the chaotic topology is the coarsest topology on any given set, while the discrete topology is the finest.

Definition 4.4 (Open / Closed Subsets). *Let $(M, \mathcal{O})$ be a topological space. A subset S of M is said to be* ***open*** *(with respect to $\mathcal{O}$) if $S \in \mathcal{O}$ and* ***closed*** *(with respect to $\mathcal{O}$) if $M \setminus S \in \mathcal{O}$.*

Notice that the notions of open and closed sets, as defined, are not mutually exclusive. A set could be both or neither, or one and not the other.

Example 4.4. Let $(M, \mathcal{O})$ be a topological space. Then $\varnothing$ is open since $\varnothing \in \mathcal{O}$. However, $\varnothing$ is also closed since $M \setminus \varnothing = M \in \mathcal{O}$. Similarly for M.

Example 4.5. Let $M = \{a, b, c\}$ and let $\mathcal{O} = \{\varnothing, \{a\}, \{a, b\}, \{a, b, c\}\}$. Then $\{a\}$ is open but not closed, $\{b, c\}$ is closed but not open, and $\{b\}$ is neither open nor closed.

We will now define what is called the standard topology on $\mathbb{R}^d$, where:

$$\mathbb{R}^d := \underbrace{\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}}_{d \text{ times}}.$$

We will need the following auxiliary definition.

Definition 4.5 (Open Balls). *For any $x \in \mathbb{R}^d$ and any $r \in \mathbb{R}^+ := \{s \in \mathbb{R} \mid s > 0\}$, we define the* ***open ball*** *of radius r around the point x:*

$$B_r(x) := \left\{y \in \mathbb{R}^d \mid \sqrt{\textstyle\sum_{i=1}^{d}(y_i - x_i)^2} < r\right\},$$

where $x := (x_1, x_2, \ldots, x_d)$ and $y := (y_1, y_2, \ldots, y_d)$, with $x_i, y_i \in \mathbb{R}$.

Remark 4.2. The quantity $\sqrt{\sum_{i=1}^{d}(y_i - x_i)^2}$ is usually denoted by $\|y - x\|_2$, where $\|\cdot\|_2$ is the 2-norm on $\mathbb{R}^d$. However, the definition of a norm on a set requires the set to be equipped with a vector space structure (which we haven't defined yet), while our construction does not. Moreover, our construction can be proven to be independent of the particular norm used to define it, i.e. any other norm will induce the same topological structure.

Definition 4.6 (Standard Topology). *The* ***standard topology*** *on $\mathbb{R}^d$, denoted $\mathcal{O}_{\text{std}}$, is defined by:*

$$U \in \mathcal{O}_{\text{std}} :\Leftrightarrow \forall\, p \in U : \exists\, r \in \mathbb{R}^+ : B_r(p) \subseteq U.$$

Of course, simply calling something a topology, does not automatically make it into a topology. We have to prove that $\mathcal{O}_{\text{std}}$ as we defined it, does constitute a topology.

Proposition 4.1. *The pair $(\mathbb{R}^d, \mathcal{O}_{\text{std}})$ is a topological space.*

Proof. i) First, we need to check whether $\varnothing \in \mathcal{O}_{\text{std}}$, i.e. whether is true:

$$\forall\, p \in \varnothing : \exists\, r \in \mathbb{R}^+ : B_r(p) \subseteq \varnothing$$

This proposition is of the form $\forall\, p \in \varnothing : Q(p)$, which was defined as being equivalent to:

$$\forall\, p : p \in \varnothing \Rightarrow Q(p).$$

However, since $p \in \varnothing$ is false, the implication is true independent of p. Hence the initial proposition is true and thus $\varnothing \in \mathcal{O}_{\text{std}}$.
Second, by definition, we have $B_r(x) \subseteq \mathbb{R}^d$ independent of x and r, hence:

$$\forall\, p \in \mathbb{R}^d : \exists\, r \in \mathbb{R}^+ : B_r(p) \subseteq \mathbb{R}^d$$

is true and thus $\mathbb{R}^d \in \mathcal{O}_{\text{std}}$.

ii) Let $U, V \in \mathcal{O}_{\text{std}}$ and let $p \in U \cap V$. Then:

$$p \in U \cap V :\Leftrightarrow p \in U \wedge p \in V$$

and hence, since $U, V \in \mathcal{O}_{\text{std}}$, we have:

$$\exists\, r_1 \in \mathbb{R}^+ : B_{r_1}(p) \subseteq U \quad \wedge \quad \exists\, r_2 \in \mathbb{R}^+ : B_{r_2}(p) \subseteq V.$$

Let $r = \min\{r_1, r_2\}$. Then:

$$B_r(p) \subseteq B_{r_1}(p) \subseteq U \quad \wedge \quad B_r(p) \subseteq B_{r_2}(p) \subseteq V$$

and hence $B_r(p) \subseteq U \cap V$. Therefore $U \cap V \in \mathcal{O}_{\text{std}}$.

iii) Let $C \subseteq \mathcal{O}_{\text{std}}$ and let $p \in \bigcup C$. Then, $p \in U$ for some $U \in C$ and, since $U \in \mathcal{O}_{\text{std}}$, we have:

$$\exists\, r \in \mathbb{R}^+ : B_r(p) \subseteq U \subseteq \bigcup C.$$

Therefore, $\mathcal{O}_{\text{std}}$ is indeed a topology on $\mathbb{R}^d$. □

4.2 Construction Of New Topologies From Given Ones

Definition 4.7 (Induced Topology)**.** *Let $(M, \mathcal{O})$ be a topological space and let $N \subset M$. Then we call the* ***induced topology*** *on N the topology:*

$$\mathcal{O}|_N := \{U \cap N \mid U \in \mathcal{O}\} \subseteq \mathcal{P}(N)$$

Of course we need to prove that this is indeed a topology.

Proof. i) Since $\varnothing \in \mathcal{O}$ and $\varnothing = \varnothing \cap N$, we have $\varnothing \in \mathcal{O}|_N$. Similarly, we have $M \in \mathcal{O}$ and $N = M \cap N$, and thus $N \in \mathcal{O}|_N$.

ii) Let $U, V \in \mathcal{O}|_N$. Then, by definition:

$$\exists\, S \in \mathcal{O} : U = S \cap N \quad \wedge \quad \exists\, T \in \mathcal{O} : V = T \cap N.$$

We thus have:

$$U \cap V = (S \cap N) \cap (T \cap N) = (S \cap T) \cap N.$$

Since $S, T \in \mathcal{O}$ and $\mathcal{O}$ is a topology, we have $S \cap T \in \mathcal{O}$ and hence $U \cap V \in \mathcal{O}|_N$.

iii) Let $C := \{S_\alpha \mid \alpha \in \mathcal{A}\} \subseteq \mathcal{O}|_N$. By definition, we have:

$$\forall\, \alpha \in \mathcal{A} : \exists\, U_\alpha \in \mathcal{O} : S_\alpha = U_\alpha \cap N.$$

Then, using the notation:

$$\bigcup_{\alpha\in\mathcal{A}} S_\alpha := \bigcup C = \bigcup\{S_\alpha \mid \alpha\in\mathcal{A}\}$$

and De Morgan's law, we have:

$$\bigcup_{\alpha\in\mathcal{A}} S_\alpha = \bigcup_{\alpha\in\mathcal{A}} (U_\alpha \cap N) = \Big(\bigcup_{\alpha\in\mathcal{A}} U_\alpha\Big) \cap N.$$

Since $\mathcal{O}$ is a topology, we have $\bigcup_{\alpha\in\mathcal{A}} U_\alpha \in \mathcal{O}$ and hence $\bigcup C \in \mathcal{O}|_N$.
Thus $\mathcal{O}|_N$ is a topology on N.

□

Example 4.6. Consider $(\mathbb{R}, \mathcal{O}_{\text{std}})$ and let:

$$N = [-1,1] := \{x\in\mathbb{R} \mid -1 \leq x \leq 1\}.$$

Then $(N, \mathcal{O}_{\text{std}}|_N)$ is a topological space. The set $(0,1]$ is clearly not open in $(\mathbb{R}, \mathcal{O}_{\text{std}})$ since $(0,1] \notin \mathcal{O}_{\text{std}}$. However, we have:

$$(0,1] = (0,2) \cap [-1,1]$$

where $(0,2) \in \mathcal{O}_{\text{std}}$ and hence $(0,1] \in \mathcal{O}_{\text{std}}|_N$, i.e. the set $(0,1]$ is open in $(N, \mathcal{O}_{\text{std}}|_N)$.

Definition 4.8 (Quotient Topology). *Let $(M,\mathcal{O})$ be a topological space and let $\sim$ be an equivalence relation on M. Then, the quotient set:*

$$M/\!\sim\; = \{[m] \in \mathcal{P}(M) \mid m \in M\}$$

can be equipped with the ***quotient topology*** *$\mathcal{O}_{M/\sim}$ defined by:*

$$\mathcal{O}_{M/\sim} := \{U \in M/\!\sim\; \mid \bigcup U = \bigcup_{[a]\in U} [a] \in \mathcal{O}\}.$$

An equivalent definition of the quotient topology is as follows. Let $q\colon M \to M/\!\sim$ be the map:

$$\begin{aligned} q\colon M &\to M/\!\sim \\ m &\mapsto [m] \end{aligned}$$

Then we have:

$$\mathcal{O}_{M/\sim} := \{U \in M/\!\sim\; \mid \text{preim}_q(U) \in \mathcal{O}\}.$$

Example 4.7. The *circle* (or 1-sphere) is defined as the set $S^1 := \{(x,y)\in\mathbb{R}^2 \mid x^2+y^2=1\}$ equipped with the subset topology inherited from $\mathbb{R}^2$. The open sets of the circle are (unions of) open arcs, i.e. arcs without the endpoints. Individual points on the circle are clearly not open since there is no open set of $\mathbb{R}^2$ whose intersection with the circle is a single point. However, an individual point on the circle is a closed set since its complement is an open arc.
An alternative definition of the circle is the following. Let $\sim$ be the equivalence relation on $\mathbb{R}$ defined by:

$$x \sim y :\Leftrightarrow \exists\, n \in \mathbb{Z} : x = y + 2\pi n.$$

Then the circle can be defined as the set $S^1 := \mathbb{R}/\!\sim$ equipped with the quotient topology.

Definition 4.9 (Product Topology). *Let $(A,\mathcal{O}_A)$ and $(B,\mathcal{O}_B)$ be topological spaces. Then a topology on $A\times B$ is defined by the set $\mathcal{O}_{A\times B}$ called the* ***product topology*** *as:*

$$U \in \mathcal{O}_{A\times B} :\Leftrightarrow \forall\, p\in U : \exists\,(S,T)\in\mathcal{O}_A\times\mathcal{O}_B : S\times T\subseteq U$$

Remark 4.3. This definition can easily be extended to n-fold cartesian products:

$$U \in \mathcal{O}_{A_1\times\cdots\times A_n} :\Leftrightarrow \forall\, p\in U : \exists\,(S_1,\ldots,S_n)\in\mathcal{O}_{A_1}\times\cdots\times\mathcal{O}_{A_n} : S_1\times\cdots\times S_n\subseteq U.$$

Remark 4.4. Using the previous definition, one can check that the standard topology on $\mathbb{R}^d$ satisfies:

$$\mathcal{O}_{\text{std}} = \mathcal{O}_{\underbrace{\mathbb{R}\times\mathbb{R}\times\cdots\times\mathbb{R}}_{d \text{ times}}}.$$

Therefore, a more minimalistic definition of the standard topology on $\mathbb{R}^d$ would consist in defining $\mathcal{O}_{\text{std}}$ only for $\mathbb{R}$ (i.e. $d = 1$) and then extending it to $\mathbb{R}^d$ by the product topology.

4.3 Convergence & Continuity

Definition 4.10 (Sequence)**.** *Let M be a set. A* ***sequence*** *(of points) in M is a function $q\colon \mathbb{N} \to M$.*

Definition 4.11 (Convergence)**.** *Let $(M, \mathcal{O})$ be a topological space. A sequence q in M is said to* ***converge*** *against a* limit point *$a \in M$ if:*

$$\forall\, U \in \mathcal{O} : a \in U \Rightarrow \exists\, N \in \mathbb{N} : \forall\, n > N : q(n) \in U.$$

Remark 4.5. An open set U of M such that $a \in U$ is called an *open neighbourhood* of a. If we denote this by $U(a)$, then the previous definition of convergence can be rewritten as:

$$\forall\, U(a) : \exists\, N \in \mathbb{N} : \forall\, n > N : q(n) \in U.$$

Example 4.8. Consider the topological space $(M, \{\varnothing, M\})$. Then every sequence in M converges to every point in M. Indeed, let q be any sequence and let $a \in M$. Then, q converges against a if:

$$\forall\, U \in \{\varnothing, M\} : a \in U \Rightarrow \exists\, N \in \mathbb{N} : \forall\, n > N : q(n) \in U.$$

This proposition is vacuously true for $U = \varnothing$, while for $U = M$ we have $q(n) \in M$ independent of n. Therefore, the (arbitrary) sequence q converges to the (arbitrary) point $a \in M$.

Example 4.9. Consider the topological space $(M, \mathcal{P}(M))$. Then only definitely constant sequences converge, where a sequence is *definitely constant* with value $c \in M$ if:

$$\exists\, N \in \mathbb{N} : \forall\, n > N : q(n) = c.$$

This is immediate from the definition of convergence since in the discrete topology all singleton sets (i.e. one-element sets) are open.

Example 4.10. Consider the topological space $(\mathbb{R}^d, \mathcal{O}_{\text{std}})$. Then, a sequence $q\colon \mathbb{N} \to \mathbb{R}^d$ converges against $a \in \mathbb{R}^d$ if:

$$\forall\, \varepsilon > 0 : \exists\, N \in \mathbb{N} : \forall\, n > N : \|q(n) - a\|_2 < \varepsilon.$$

Example 4.11. Let $M = \mathbb{R}$ and let $q = 1 - \frac{1}{n+1}$. Then, since q is not definitely constant, it is not convergent in $(\mathbb{R}, \mathcal{P}(\mathbb{R}))$, but it is convergent in $(\mathbb{R}, \mathcal{O}_{\text{std}})$.

Definition 4.12 (Continuity)**.** *Let $(M, \mathcal{O}_M)$ and $(N, \mathcal{O}_N)$ be topological spaces and let $\phi\colon M \to N$ be a map. Then, ϕ is said to be* ***continuous*** *(with respect to the topologies $\mathcal{O}_M$ and $\mathcal{O}_N$) if:*

$$\forall\, S \in \mathcal{O}_N\,,\ \operatorname{preim}_\phi(S) \in \mathcal{O}_M,$$

where $\operatorname{preim}_\phi(S) := \{m \in M : \phi(m) \in S\}$ *is the pre-image of S under the map ϕ.*

Informally, one says that ϕ is continuous if the pre-images of open sets are open.

Example 4.12. If M is equipped with the discrete topology, or N with the chaotic topology, then any map $\phi\colon M \to N$ is continuous. Indeed, let $S \in \mathcal{O}_N$. If $\mathcal{O}_M = \mathcal{P}(M)$ (and $\mathcal{O}_N$ is any topology), then we have:

$$\operatorname{preim}_\phi(S) = \{m \in M : \phi(m) \in S\} \subseteq M \in \mathcal{P}(M) = \mathcal{O}_M.$$

If instead $\mathcal{O}_N = \{\varnothing, N\}$ (and $\mathcal{O}_M$ is any topology), then either $S = \varnothing$ or $S = N$ and thus, we have:

$$\operatorname{preim}_\phi(\varnothing) = \varnothing \in \mathcal{O}_M \quad \text{and} \quad \operatorname{preim}_\phi(N) = M \in \mathcal{O}_M.$$

Example 4.13. Let $M = \{a, b, c\}$ and $N = \{1, 2, 3\}$, with respective topologies:

$$\mathcal{O}_M = \{\varnothing, \{b\}, \{a, c\}, \{a, b, c\}\} \quad \text{and} \quad \mathcal{O}_N = \{\varnothing, \{2\}, \{3\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\},$$

and let $\phi\colon M \to N$ by defined by:

$$\phi(a) = 2, \quad \phi(b) = 1, \quad \phi(c) = 2.$$

Then ϕ is continuous. Indeed, we have:

$$\begin{array}{ccc} \mathrm{preim}_\phi(\varnothing) = \varnothing, & \mathrm{preim}_\phi(\{2\}) = \{a, c\}, & \mathrm{preim}_\phi(\{3\}) = \varnothing, \\ \mathrm{preim}_\phi(\{1, 3\}) = \{b\}, & \mathrm{preim}_\phi(\{2, 3\}) = \{a, c\}, & \mathrm{preim}_\phi(\{1, 2, 3\}) = \{a, b, c\}, \end{array}$$

and hence $\mathrm{preim}_\phi(S) \in \mathcal{O}_M$ for all $S \in \mathcal{O}_N$.

Example 4.14. Consider $(\mathbb{R}^d, \mathcal{O}_{\mathrm{std}})$ and $(\mathbb{R}^s, \mathcal{O}_{\mathrm{std}})$. Then $\phi\colon \mathbb{R}^d \to \mathbb{R}^s$ is continuous with respect to the standard topologies if it satisfies the usual ε-δ definition of continuity:

$$\forall\, a \in \mathbb{R}^d : \forall\, \varepsilon > 0 : \exists\, \delta > 0 : \forall\, 0 < \|x - a\|_2 < \delta : \|\phi(x) - \phi(a)\|_2 < \varepsilon.$$

Definition 4.13 (Homeomorphism)**.** *Let $(M, \mathcal{O}_M)$ and $(N, \mathcal{O}_N)$ be topological spaces. A bijection $\phi\colon M \to N$ is called a* ***homeomorphism*** *if both $\phi\colon M \to N$ and $\phi^{-1}\colon N \to M$ are continuous.*

Remark 4.6. Homeo(morphism)s are the structure-preserving maps in topology.

If there exists a homeomorphism ϕ between $(M, \mathcal{O}_M)$ and $(N, \mathcal{O}_N)$,

$$\begin{tikzcd} M \arrow[r, bend left, "\phi"] & N \arrow[l, bend left, "\phi^{-1}"] \end{tikzcd}$$

then ϕ provides a one-to-one pairing of the open sets of M with the open sets of N.

Definition 4.14 (Isomorphic Topological Spaces)**.** *If there exists a homeomorphism between two topological spaces $(M, \mathcal{O}_M)$ and $(N, \mathcal{O}_N)$, we say that the two spaces are* ***homeomorphic*** *or* ***topologically isomorphic*** *and we write $(M, \mathcal{O}_M) \cong_{\mathrm{top}} (N, \mathcal{O}_N)$.*

Clearly, if $(M, \mathcal{O}_M) \cong_{\mathrm{top}} (N, \mathcal{O}_N)$, then $M \cong_{\mathrm{set}} N$.

4.4 Invariant Topological Properties

Definition 4.15 (Invariant Topological Properties)**.** *A property of a topological space is called an* ***invariant*** *if any two homeomorphic topological spaces share the property.*

In this section we will mention some of the (almost uncountable) invariant topological properties of topological spaces. A *classification* of topological spaces would be a list of topological invariants such that any two spaces which share these invariants are homeomorphic. As of now, no such list is known!

4.4.1 Separation Properties

Definition 4.16 (T1 Topological Space)**.** *A topological space $(M, \mathcal{O})$ is said to be* ***T1*** *if for any two distinct points $p, q \in M$, $p \neq q$:*

$$\exists\, U(p) \in \mathcal{O} : q \notin U(p).$$

Definition 4.17 (T2 or Hausdorff Topological Space)**.** *A topological space $(M, \mathcal{O})$ is said to be* ***T2*** *or* ***Hausdorff*** *if, for any two distinct points, there exist non-intersecting open neighbourhoods of these two points:*

$$\forall\, p, q \in M : p \neq q \Rightarrow \exists\, U(p), V(q) \in \mathcal{O} : U(p) \cap V(q) = \varnothing.$$

Example 4.15. The topological space $(\mathbb{R}^d, \mathcal{O}_{\mathrm{std}})$ is T2 and hence also T1.

Example 4.16. The Zariski topology on an algebraic variety is T1 but not T2.

Example 4.17. The topological space $(M, \{\varnothing, M\})$ does not have the T1 property since for any $p \in M$, the only open neighbourhood of p is M and for any other $q \neq p$ we have $q \in M$. Moreover, since this space is not T1, it cannot be T2 either.

Remark 4.7. There are many other "T" properties, including a *T2½* property which differs from T2 in that the neighbourhoods are closed.

Definition 4.18 (Cover)**.** *Let $(M, \mathcal{O})$ be a topological space. A set $C \subseteq \mathcal{P}(M)$ is called a **cover** (of M) if:*

$$\bigcup C = M.$$

Additionally, it is said to an open *cover if $C \subseteq \mathcal{O}$.*

Definition 4.19 (Open Cover)**.** *Let $(M, \mathcal{O})$ be a topological space. A cover $C \subseteq \mathcal{P}(M)$ is said to be an **open cover** if $C \subseteq \mathcal{O}$.*

Definition 4.20 (Subcover)**.** *Let C be a cover. Then any subset $\widetilde{C} \subseteq C$ such that $\widetilde{C}$ is still a cover, is called a **subcover**. Additionally, it is said to be a* finite *subcover if it is finite as a set.*

Definition 4.21 (Compact Topological Space)**.** *A topological space $(M, \mathcal{O})$ is said to be **compact** if every open cover has a finite subcover.*

Definition 4.22 (Compact Subset)**.** *Let $(M, \mathcal{O})$ be a topological space. A subset $N \subseteq M$ is called **compact** if the topological space $(N, \mathcal{O}|_N)$ is compact.*

Determining whether a set is compact or not is not an easy task. Fortunately though, for $\mathbb{R}^d$ equipped with the standard topology $\mathcal{O}_{\text{std}}$, the following theorem greatly simplifies matters.

Theorem 4.1 (Heine-Borel)**.** *Let $\mathbb{R}^d$ be equipped with the standard topology $\mathcal{O}_{\text{std}}$. Then, a subset of $\mathbb{R}^d$ is compact if, and only if, it is closed and bounded.*

A subset S of $\mathbb{R}^d$ is said to be *bounded* if:

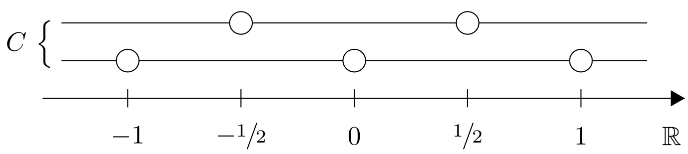
$$\exists\, r \in \mathbb{R}^+ : S \subseteq B_r(0).$$

Example 4.18. The interval $[0, 1]$ is compact in $(\mathbb{R}, \mathcal{O}_{\text{std}})$. The one-element set containing $(-1, 2)$ is a cover of $[0, 1]$, but it is also a finite subcover and hence $[0, 1]$ is compact from the definition. Alternatively, $[0, 1]$ is clearly closed and bounded, and hence it is compact by the Heine-Borel theorem.

Example 4.19. The set $\mathbb{R}$ is not compact in $(\mathbb{R}, \mathcal{O}_{\text{std}})$. To prove this, it suffices to show that there exists a cover of $\mathbb{R}$ that does not have a finite subcover. To this end, let:

$$C := \{(n, n+1) \mid n \in \mathbb{Z}\} \cup \{(n + \tfrac{1}{2}, n + \tfrac{3}{2}) \mid n \in \mathbb{Z}\}.$$

This corresponds to the following picture.



It is clear that removing even one element from C will cause C to fail to be an open cover of $\mathbb{R}$. Therefore, there is no finite subcover of C and hence, $\mathbb{R}$ is not compact.

Theorem 4.2. *Let $(M, \mathcal{O}_M)$ and $(N, \mathcal{O}_N)$ be compact topological spaces. Then $(M \times N, \mathcal{O}_{M \times N})$ is a compact topological space.*

The above theorem easily extends to finite cartesian products.

Definition 4.23 (Refinement)**.** *Let $(M, \mathcal{O})$ be a topological space and let C be a cover. A **refinement** of C is a cover R such that:*

$$\forall\, U \in R : \exists\, V \in C : U \subseteq V.$$

Any subcover of a cover is a refinement of that cover, but the converse is not true in general. A refinement R is said to be:

- *open* if $R \subseteq \mathcal{O}$;
- *locally finite* if for any $p \in M$ there exists a neighbourhood $U(p)$ such that the set:

$$\{U \in R \mid U \cap U(p) \neq \varnothing\}$$

  is finite as a set.

Compactness is a very strong property. Hence often times it does not hold, but a weaker and still useful property, called paracompactness, may still hold.

Definition 4.24 (Paracompact Topological Space)**.** *A topological space $(M, \mathcal{O})$ is said to be **paracompact** if every open cover has an open refinement that is locally finite.*

Corollary 4.1. *If a topological space is compact, then it is also paracompact.*

Remark 4.8. Paracompactness is, informally, a rather natural property since every example of a non-paracompact space looks artificial. One such example is the *long line* (or *Alexandroff line*). To construct it, we first observe that we could "build" $\mathbb{R}$ by taking the interval $[0, 1)$ and stacking countably many copies of it one after the other. Hence, in a sense, $\mathbb{R}$ is equivalent to $\mathbb{Z} \times [0, 1)$. The long line L is defined analogously as $L : \omega_1 \times [0, 1)$, where ω_1 is an uncountably infinite set. The resulting space L is not paracompact.

Theorem 4.3. *Let $(M, \mathcal{O}_M)$ be a paracompact space and let $(N, \mathcal{O}_N)$ be a compact space. Then $M \times N$ (equipped with the product topology) is paracompact.*

Corollary 4.2. *Let $(M, \mathcal{O}_M)$ be a paracompact space and let $(N_i, \mathcal{O}_{N_i})$ be compact spaces for every $1 \leq i \leq n$. Then $M \times N_1 \times \cdots \times N_n$ is paracompact.*

Definition 4.25 (Partition Of Unity)**.** *Let $(M, \mathcal{O}_M)$ be a topological space. A **partition of unity** of M is a set $\mathcal{F}$ of continuous maps from M to the interval $[0, 1]$ such that for each $p \in M$ the following conditions hold:*

i) there exists $U(p)$ such that the set $\{f \in \mathcal{F} \mid \forall\, x \in U(p) : f(x) \neq 0\}$ is finite;

ii) $\sum_{f \in \mathcal{F}} f(p) = 1$.

If C is an open cover, then $\mathcal{F}$ is said to be subordinate *to the cover C if:*

$$\forall\, f \in \mathcal{F} : \exists\, U \in C : f(x) \neq 0 \Rightarrow x \in U.$$

Theorem 4.4. *Let $(M, \mathcal{O}_M)$ be a Hausdorff topological space. Then $(M, \mathcal{O}_M)$ is paracompact if, and only if, every open cover admits a partition of unity subordinate to that cover.*

4.4.2 Connectedness And Path-Connectedness

Definition 4.26 (Connected Topological Space)**.** *A topological space $(M, \mathcal{O})$ is said to be **connected** unless there exist two non-empty, non-intersecting open sets A and B such that $M = A \cup B$.*

Example 4.20. Consider $(\mathbb{R} \setminus \{0\}, \mathcal{O}_{\text{std}}|_{\mathbb{R}\setminus\{0\}})$, i.e. $\mathbb{R} \setminus \{0\}$ equipped with the subset topology inherited from $\mathbb{R}$. This topological space is not connected since $(-\infty, 0)$ and $(0, \infty)$ are open, non-empty, non-intersecting sets such that $\mathbb{R} \setminus \{0\} = (-\infty, 0) \cup (0, \infty)$.

Theorem 4.5. *The interval $[0, 1] \subseteq \mathbb{R}$ equipped with the subset topology is connected.*

Theorem 4.6. *A topological space $(M, \mathcal{O})$ is connected if, and only if, the only subsets that are both open and closed are $\varnothing$ and M.*

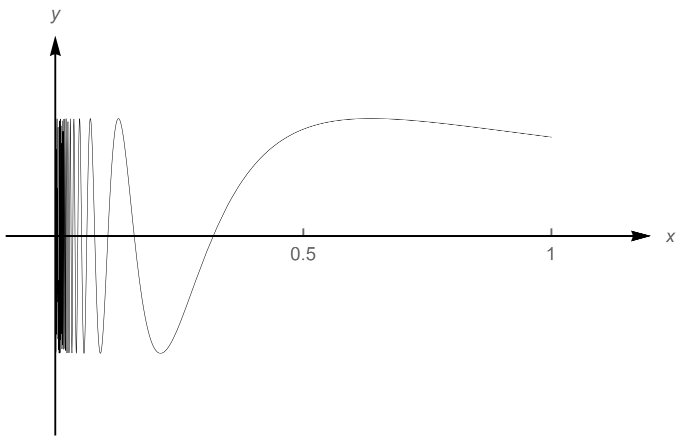
Definition 4.27 (Path - Connected Topological Space)**.** *A topological space $(M, \mathcal{O})$ is said to be **path-connected** if for every pair of points $p, q \in M$ there exists a continuous curve $\gamma\colon [0, 1] \to M$ such that $\gamma(0) = p$ and $\gamma(1) = q$.*

Example 4.21. The space $(\mathbb{R}^d, \mathcal{O}_{\text{std}})$ is path-connected. Indeed, let $p, q \in \mathbb{R}^d$ and let:

$$\gamma(\lambda) := p + \lambda(q - p).$$

Then γ is continuous and satisfies $\gamma(0) = p$ and $\gamma(1) = q$.

Example 4.22. Let $S := \{(x, \sin(\frac{1}{x})) \mid x \in (0, 1]\} \cup \{(0, 0)\}$ be equipped with the subset topology inherited from $\mathbb{R}^2$.



The space $(S, \mathcal{O}_{\text{std}}|_S)$ is connected but not path-connected.

Theorem 4.7. *If a topological space is path-connected, then it is also connected.*

4.4.3 Homotopic Curves And The Fundamental Group

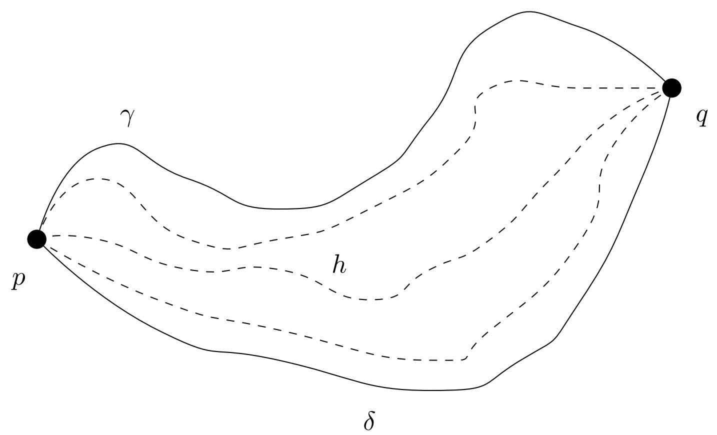
Definition 4.28 (Homotopic Curves)**.** *Let $(M, \mathcal{O})$ be a topological space. Two curves $\gamma, \delta \colon [0, 1] \to M$ such that:*

$$\gamma(0) = \delta(0) \quad \textit{and} \quad \gamma(1) = \delta(1)$$

*are said to be **homotopic** if there exists a continuous map $h \colon [0, 1] \times [0, 1] \to M$ such that for all $\lambda \in [0, 1]$:*

$$h(0, \lambda) = \gamma(\lambda) \quad \textit{and} \quad h(1, \lambda) = \delta(\lambda).$$

Pictorially, two curves are homotopic if they can be continuously deformed into one another.



Proposition 4.2. *Let $\gamma \sim \delta :\Leftrightarrow$ "γ and δ are homotopic". Then, $\sim$ is an equivalence relation.*

Definition 4.29 (Space Of Loops)**.** *Let $(M, \mathcal{O})$ be a topological space. Then, for every $p \in M$, we define the **space of loops** at p by:*

$$\mathscr{L}_p := \{\gamma \colon [0, 1] \to M \mid \gamma \textit{ is continuous and } \gamma(0) = \gamma(1)\}.$$

Definition 4.30 (Concatenation)**.** *Let $\mathscr{L}_p$ be the space of loops at $p \in M$. We define the* ***concatenation*** *operation $*\colon \mathscr{L}_p \times \mathscr{L}_p \to \mathscr{L}_p$ by:*

$$(\gamma * \delta)(\lambda) := \begin{cases} \gamma(2\lambda) & \text{if } 0 \leq \lambda \leq \frac{1}{2} \\ \delta(2\lambda - 1) & \text{if } \frac{1}{2} \leq \lambda \leq 1 \end{cases}$$

Definition 4.31 (Fundamental Group)**.** *Let $(M, \mathcal{O})$ be a topological space. The* ***fundamental group*** *$\pi_1(p)$ of $(M, \mathcal{O})$ at $p \in M$ is the set:*

$$\pi_1(p) := \mathscr{L}_p/\sim\, = \{[\gamma] \mid \gamma \in \mathscr{L}_p\},$$

where $\sim$ is the homotopy equivalence relation, together with the map

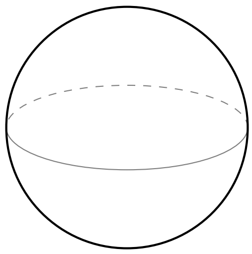
$$\begin{aligned} \bullet\colon \pi_1(p) \times \pi_1(p) &\to \pi_1(p) \\ (\gamma, \delta) &\mapsto [\gamma] \bullet [\delta] := [\gamma * \delta]. \end{aligned}$$

Observe that while all the previously discussed topological properties are "boolean-valued", i.e. a topological space is either Hausdorff or not Hausdorff, either connected or not connected, and so on, the fundamental group is a "group-valued" property, i.e. the value of the property is not "either yes or no", but a group.

Example 4.23. The 2-sphere is defined as the set:

$$S^2 := \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$$

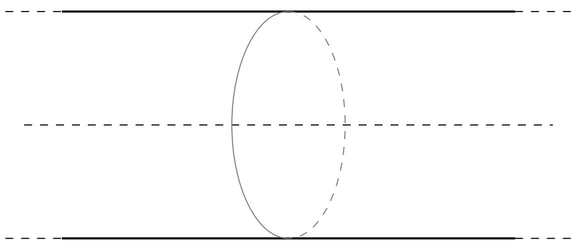
equipped with the subset topology inherited from $\mathbb{R}^3$.



The sphere has the property that all the loops at any point are homotopic, hence the fundamental group (at every point) of the sphere is the trivial group:

$$\forall\, p \in S^2 : \pi_1(p) = 1 := \{[\gamma_e]\}.$$

Example 4.24. The cylinder is defined as $C := \mathbb{R} \times S^1$ equipped with the product topology.

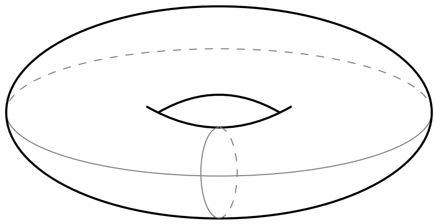


A loop in C can either go around the cylinder (i.e. around its central axis) or not. If it does not, then it can be continuously deformed to a point (the identity loop). If it does, then it cannot be deformed to the identity loop (intuitively because the cylinder is infinitely long) and hence it is a homotopically different loop. The number of times a loop winds around the cylinder is called the *winding number*. Loops with different winding numbers are not homotopic.

Moreover, loops with different *orientations* are also not homotopic and hence we have:

$$\forall\, p \in C : (\pi_1(p), \bullet) \cong_{\mathrm{grp}} (\mathbb{Z}, +).$$

Example 4.25. The 2-torus is defined as the set $T^2 := S^1 \times S^1$ equipped with the product topology.



A loop in T^2 can intuitively wind around the cylinder-like part of the torus as well as around the hole of the torus. That is, there are two independent winding numbers and hence:

$$\forall\, p \in T^2 : \pi_1(p) \cong_{\mathrm{grp}} \mathbb{Z} \times \mathbb{Z},$$

where $\mathbb{Z} \times \mathbb{Z}$ is understood as a group under pairwise addition.

Chapter 5

Topological Manifolds

5.1 Topological Manifolds

Definition 5.1 (Topological Manifold)**.** *A paracompact, Hausdorff, topological space* $(M, \mathcal{O})$ *is called a* d***-dimensional topological manifold*** *if for every point* $p \in M$ *there exist a neighbourhood* $U(p)$ *and a homeomorphism* $x \colon U(p) \to x(U(p)) \subseteq \mathbb{R}^d$*. We also write* $\dim M = d$.

Intuitively, a d-dimensional manifold is a topological space which locally (i.e. around each point) looks like $\mathbb{R}^d$. Note that, strictly speaking, what we have just defined are *real* topological manifolds. We could define *complex* topological manifolds as well, simply by requiring that the map x be a homeomorphism onto an open subset of $\mathbb{C}^d$.

Proposition 5.1. *Let* M *be a* d*-dimensional manifold and let* $U, V \subseteq M$ *be open, with* $U \cap V \neq \varnothing$*. If* x *and* y *are two homeomorphisms*

$$x \colon U \to x(U) \subseteq \mathbb{R}^d \qquad \text{and} \qquad y \colon V \to y(V) \subseteq \mathbb{R}^{d'},$$

then $d = d'$.

This ensures that the concept of dimension is indeed well-defined, i.e. it is the same at every point, at least on each connected component of the manifold.

Example 5.1. Trivially, $\mathbb{R}^d$ is a d-dimensional manifold for any $d \geq 1$. The space S^1 is a 1-dimensional manifold while the spaces S^2, C and T^2 are 2-dimensional manifolds.

Definition 5.2 (Topological Submanifold)**.** *Let* $(M, \mathcal{O})$ *be a topological manifold and let* $N \subseteq M$*. Then* $(N, \mathcal{O}|_N)$ *is called a* ***submanifold*** *of* $(M, \mathcal{O})$ *if it is a manifold in its own right.*

Example 5.2. The space S^1 is a submanifold of $\mathbb{R}^2$ while the spaces S^2, C and T^2 are submanifolds of $\mathbb{R}^3$.

Definition 5.3 (Product Manifold)**.** *Let* $(M, \mathcal{O}_M)$ *and* $(N, \mathcal{O}_N)$ *be topological manifolds of dimension* m *and* n*, respectively. Then,* $(M \times N, \mathcal{O}_{M \times N})$ *is a topological manifold of dimension* $m + n$ *called the* ***product manifold****.*

Example 5.3. We have $T^2 = S^1 \times S^1$ not just as topological spaces, but as topological manifolds as well. This is a special case of the n-torus:

$$T^n := \underbrace{S^1 \times S^1 \times \cdots \times S^1}_{n \text{ times}},$$

which is an n-dimensional manifold.

Example 5.4. The cylinder $C = S^1 \times \mathbb{R}$ is a 2-dimensional manifold.

5.2 Charts & Atlases

Definition 5.4 (Chart)**.** *Let* $(M, \mathcal{O})$ *be a* d*-dimensional manifold. Then, a pair* (U, x) *where* $U \in \mathcal{O}$ *and* $x \colon U \to x(U) \subseteq \mathbb{R}^d$ *is a homeomorphism, is said to be a* ***chart*** *of the manifold.*

Definition 5.5 (Components / Co-Ordinates Of A Chart)**.** *The **component functions (or maps)** of $x\colon U \to x(U) \subseteq \mathbb{R}^d$ are the maps:*

$$\begin{aligned} x^i\colon U &\to \mathbb{R} \\ p &\mapsto \mathrm{proj}_i(x(p)) \end{aligned}$$

*for $1 \leq i \leq d$, where $\mathrm{proj}_i(x(p))$ is the i-th component of $x(p) \in \mathbb{R}^d$. The $x^i(p)$ are called the **co-ordinates** of the point $p \in U$ with respect to the chart (U, x).*

Definition 5.6 (Atlas)**.** *An **atlas** of a manifold M is a collection $\mathscr{A} := \{(U_\alpha, x_\alpha) \mid \alpha \in \mathcal{A}\}$ of charts such that:*

$$\bigcup_{\alpha \in \mathcal{A}} U_\alpha = M.$$

Definition 5.7 ($\mathcal{C}^0$-Compatible Charts)**.** *Two charts (U, x) and (V, y) are said to be **$\mathcal{C}^0$-compatible** if either $U \cap V = \varnothing$ or the map:*

$$y \circ x^{-1}\colon x(U \cap V) \to y(U \cap V)$$

is continuous.

Note that $y \circ x^{-1}$ is a map from a subset of $\mathbb{R}^d$ to a subset of $\mathbb{R}^d$.

$$\begin{tikzcd} & U \cap V \subseteq M \arrow[dl, "x"'] \arrow[dr, "y"] & \\ x(U \cap V) \subseteq \mathbb{R}^d \arrow[rr, "y \circ x^{-1}"'] & & y(U \cap V) \subseteq \mathbb{R}^d \end{tikzcd}$$

Since the maps x and y are homeomorphisms, the composition map $y \circ x^{-1}$ is also a homeomorphism and hence continuous. Therefore, any two charts on a topological manifold are $\mathcal{C}^0$-compatible. This definition my thus seem redundant since it applies to every pair of charts. However, it is just a "warm up" since we will later refine this definition and define the *differentiability* of maps on a manifold in terms of $\mathcal{C}^k$-compatibility of charts.

Definition 5.8 (Chart Transition Map)**.** *The map $y \circ x^{-1}$ (and its inverse $x \circ y^{-1}$) is called the **chart transition map**.*

Definition 5.9 ($\mathcal{C}^0$-Atlas)**.** *A **$\mathcal{C}^0$-atlas** of a manifold is an atlas of pairwise $\mathcal{C}^0$-compatible charts.*

Note that any atlas is also a $\mathcal{C}^0$*-atlas.*

Definition 5.10 (Maximal Atlas)**.** *A $\mathcal{C}^0$-atlas $\mathscr{A}$ is said to be a **maximal atlas** if for every $(U, x) \in \mathscr{A}$, we have $(V, y) \in \mathscr{A}$ for all (V, y) charts that are $\mathcal{C}^0$-compatible with (U, x).*

Example 5.5. Not every $\mathcal{C}^0$-atlas is a maximal atlas. Indeed, consider $(\mathbb{R}, \mathcal{O}_{\text{std}})$ and the atlas $\mathscr{A} := (\mathbb{R}, \mathrm{id}_\mathbb{R})$. Then $\mathscr{A}$ is not maximal since $((0,1), \mathrm{id}_\mathbb{R})$ is a chart which is $\mathcal{C}^0$-compatible with $(\mathbb{R}, \mathrm{id}_\mathbb{R})$ but $((0,1), \mathrm{id}_\mathbb{R}) \notin \mathscr{A}$.

We can now look at "objects on" topological manifolds from two points of view. For instance, consider a curve on a d-dimensional manifold M, i.e. a map $\gamma\colon \mathbb{R} \to M$. We now ask whether this curve is continuous, as it should be if models the trajectory of a particle on the "physical space" M.

A first answer is that $\gamma\colon \mathbb{R} \to M$ is continuous if it is continuous as a map between the topological spaces $\mathbb{R}$ and M.

However, the answer that may be more familiar to you from undergraduate physics is the following. We consider only a portion (open subset U) of the physical space M and, instead of studying the map $\gamma\colon \mathrm{preim}_\gamma(U) \to U$ directly, we study the map:

$$x \circ \gamma\colon \mathrm{preim}_\gamma(U) \to x(U) \subseteq \mathbb{R}^d,$$

where (U, x) is a chart of M. More likely, you would be checking the continuity of the co-ordinate maps $x^i \circ \gamma$, which would then imply the continuity of the "real" curve $\gamma\colon \mathrm{preim}_\gamma(U) \to U$ (real, as opposed to its co-ordinate representation).

$$\begin{tikzcd}
 & y(U) \subseteq \mathbb{R}^d & \\
\mathrm{preim}_\gamma(U) \subseteq \mathbb{R} \arrow[ur, "y\circ\gamma"] \arrow[r, "\gamma"] \arrow[dr, "x\circ\gamma"'] & U \subseteq M \arrow[u, "y"'] \arrow[d, "x"] & \\
 & x(U) \subseteq \mathbb{R}^d \arrow[uu, bend right=60, "y\circ x^{-1}"'] &
\end{tikzcd}$$

At some point you may wish to use a different "co-ordinate system" to answer a different question. In this case, you would chose a different chart (U, y) and then study the map $y \circ \gamma$ or its co-ordinate maps. Notice however that some results (e.g. the continuity of γ) obtained in the previous chart (U, x) can be immediately "transported" to the new chart (U, y) via the chart transition map $y \circ x^{-1}$. Moreover, the map $y \circ x^{-1}$ allows us to, intuitively speaking, forget about the inner structure (i.e. U and the maps γ, x and x) which, in a sense, is the real world, and only consider $\mathrm{preim}_\gamma(U) \subseteq \mathbb{R}$ and $x(U), y(U) \subseteq \mathbb{R}^d$ together with the maps between them, which is our representation of the real world.

As we already said, for a topological manifold $(M, \mathcal{O})$, the concept of a $\mathcal{C}^0$-atlas is fully redundant since every atlas is also a $\mathcal{C}^0$-atlas. We will now generalise the notion of a $\mathcal{C}^0$-atlas, or more precisely, the notion of $\mathcal{C}^0$-compatibility of charts, to something which is non-trivial and non-redundant.

Definition 5.11 (❀-Atlas)**.** *An atlas $\mathscr{A}$ for a topological manifold is called a* ***❀-atlas*** *if any two charts $(U, x), (V, y) \in \mathscr{A}$ are ❀-compatible, where the symbol ❀ is being used as a placeholder for any of the following:*

- ❀ $= \mathcal{C}^0$*: this just reduces to the previous definition;*
- ❀ $= \mathcal{C}^k$*: the transition maps are k-times continuously differentiable as maps between open subsets of $\mathbb{R}^{\dim M}$;*
- ❀ $= \mathcal{C}^\infty$*: the transition maps are smooth (infinitely many times differentiable); equivalently, the atlas is $\mathcal{C}^k$ for all $k \geq 0$;*
- ❀ $= \mathcal{C}^\omega$*: the transition maps are (real) analytic, which is stronger than being smooth;*
- ❀ $=$ *complex: if* $\dim M$ *is even, M is a* complex manifold *if the transition maps are continuous and satisfy the Cauchy-Riemann equations; its complex dimension is $\frac{1}{2}\dim M$.*

In other words, either $U \cap V = \varnothing$ or if $U \cap V \neq \varnothing$, then the transition map $y \circ x^{-1}$ from $x(U \cap V)$ to $y(U \cap V)$ must be ❀.

$$\begin{tikzcd}
 & U \cap V \subseteq M \arrow[dl, "x"'] \arrow[dr, "y"] & \\
x(U \cap V) \subseteq \mathbb{R}^{\dim M} \arrow[rr, "y\circ x^{-1}"'] & & y(U \cap V) \subseteq \mathbb{R}^{\dim M}
\end{tikzcd}$$

Theorem 5.1 (Whitney)**.** *Any maximal $\mathcal{C}^k$-atlas, with $k \geq 1$, contains a $\mathcal{C}^\infty$-atlas. Moreover, any two maximal $\mathcal{C}^k$-atlases that contain the same $\mathcal{C}^\infty$-atlas are identical.*

An immediate implication is that if we can find a $\mathcal{C}^1$-atlas for a manifold, then we can also assume the existence of a $\mathcal{C}^\infty$-atlas for that manifold. This is not the case for topological manifolds in general: a space with a $\mathcal{C}^0$-atlas may not admit any $\mathcal{C}^1$-atlas. But if we have at least a $\mathcal{C}^1$-atlas, then we can obtain a $\mathcal{C}^\infty$-atlas simply by removing charts, keeping only the ones which are $\mathcal{C}^\infty$-compatible.

Hence, for the purposes of this course, we will not distinguish between $\mathcal{C}^k$ $(k \geq 1)$ and $\mathcal{C}^\infty$-manifolds in the above sense.

We now give the explicit definition of a $\mathcal{C}^k$-manifold.

Definition 5.12 ($\mathcal{C}^k$-Manifold)**.** *A $\mathcal{C}^k$**-manifold** is a triple $(M, \mathcal{O}, \mathscr{A})$, where $(M, \mathcal{O})$ is a topological manifold and $\mathscr{A}$ is a maximal $\mathcal{C}^k$-atlas.*

Definition 5.13 (Smooth Manifold)**.** *A $\mathcal{C}^\infty$-manifold is called a **smooth manifold**.*

Remark 5.1. A given topological manifold can carry different incompatible atlases.

Note that while we only defined compatibility of charts, it should be clear what it means for two atlases of the same type to be compatible.

Definition 5.14 (Compatible / Incompatible Atlases)**.** *Two ❀-atlases $\mathscr{A}$, $\mathscr{B}$ are **compatible** if their union $\mathscr{A} \cup \mathscr{B}$ is again a ❀-atlas, and are **incompatible** otherwise.*

Alternatively, we can define the compatibility of two atlases in terms of the compatibility of any pair of charts, one from each atlas.

Example 5.6. Let $(M, \mathcal{O}) = (\mathbb{R}, \mathcal{O}_{\text{std}})$. Consider the two atlases $\mathscr{A} = \{(\mathbb{R}, \mathrm{id}_{\mathbb{R}})\}$ and $\mathscr{B} = \{(\mathbb{R}, x)\}$, where $x\colon a \mapsto \sqrt[3]{a}$. Since they both contain a single chart, the compatibility condition on the transition maps is easily seen to hold (in both cases, the only transition map is $\mathrm{id}_{\mathbb{R}}$). Hence they are both $\mathcal{C}^\infty$-atlases. Consider now $\mathscr{A} \cup \mathscr{B}$. The transition map $\mathrm{id}_{\mathbb{R}} \circ x^{-1}$ is the map $a \mapsto a^3$, which is smooth. However, the other transition map, $x \circ \mathrm{id}_{\mathbb{R}}^{-1}$, is the map x, which is not even differentiable once (the first derivative at 0 does not exist). Consequently, $\mathscr{A}$ and $\mathscr{B}$ are not even $\mathcal{C}^1$-compatible.

The previous example shows that we can equip the real line with (at least) two different incompatible $\mathcal{C}^\infty$-structures. This looks like a disaster as it implies that there is an arbitrary choice to be made about which differentiable structure to use. Fortunately, the situation is not as bad as it looks, as we will see in the next sections.

5.3 Differentiable Manifolds

Definition 5.15 (Differentiable Map)**.** *Let $\phi\colon M \to N$ be a map, where $(M, \mathcal{O}_M, \mathscr{A}_M)$ and $(N, \mathcal{O}_N, \mathscr{A}_N)$ are $\mathcal{C}^k$-manifolds. Then ϕ is said to be ($\mathcal{C}^k$-)**differentiable at** $p \in M$ if for some charts $(U, x) \in \mathscr{A}_M$ with $p \in U$ and $(V, y) \in \mathscr{A}_N$ with $\phi(p) \in V$, the map $y \circ \phi \circ x^{-1}$ is k-times continuously differentiable at $x(p) \in x(U) \subseteq \mathbb{R}^{\dim M}$ in the usual sense.*

$$\begin{tikzcd} U \subseteq M \arrow[r, "\phi"] \arrow[d, "x"] & V \subseteq N \arrow[d, "y"] \\ x(U) \subseteq \mathbb{R}^{\dim M} \arrow[r, "y \circ \phi \circ x^{-1}"] & y(V) \subseteq \mathbb{R}^{\dim N} \end{tikzcd}$$

The above diagram shows a typical theme with manifolds. We have a map $\phi\colon M \to N$ and we want to define some property of ϕ at $p \in M$ analogous to some property of maps between subsets of $\mathbb{R}^d$. What we typically do is consider some charts (U, x) and (V, y) as above and define the desired property of ϕ at $p \in U$ in terms of the corresponding property of the downstairs map $y \circ \phi \circ x^{-1}$ at the point $x(p) \in \mathbb{R}^d$. Notice that in the previous definition we only require that *some* charts from the two atlases satisfy the stated property. So we should worry about whether this definition depends on which charts we pick. In fact, this “lifting” of the notion of differentiability from the chart representation of ϕ to the manifold level is well-defined.

Proposition 5.2. *The definition of differentiability is well-defined.*

Proof. We want to show that if $y \circ \phi \circ x^{-1}$ is differentiable at $x(p)$ for some $(U, x) \in \mathscr{A}_M$ with $p \in U$ and $(V, y) \in \mathscr{A}_N$ with $\phi(p) \in V$, then $\widetilde{y} \circ \phi \circ \widetilde{x}^{-1}$ is differentiable at $\widetilde{x}(p)$ for all charts $(\widetilde{U}, \widetilde{x}) \in \mathscr{A}_M$ with $p \in \widetilde{U}$ and $(\widetilde{V}, \widetilde{y}) \in \mathscr{A}_N$ with $\phi(p) \in \widetilde{V}$.

$$\begin{tikzcd}
\widetilde{x}(U \cap \widetilde{U}) \subseteq \mathbb{R}^{\dim M} \arrow[r, "\widetilde{y}\circ\phi\circ\widetilde{x}^{-1}"] & \widetilde{y}(V \cap \widetilde{V}) \subseteq \mathbb{R}^{\dim N} \\
U \cap \widetilde{U} \subseteq M \arrow[u, "\widetilde{x}"] \arrow[r, "\phi"] \arrow[d, "x"] & V \cap \widetilde{V} \subseteq N \arrow[u, "\widetilde{y}"] \arrow[d, "y"] \\
x(U \cap \widetilde{U}) \subseteq \mathbb{R}^{\dim M} \arrow[r, "y\circ\phi\circ x^{-1}"] \arrow[uu, bend left=60, "\widetilde{x}\circ x^{-1}"] & y(V \cap \widetilde{V}) \subseteq \mathbb{R}^{\dim N} \arrow[uu, bend right=60, "\widetilde{y}\circ y^{-1}"']
\end{tikzcd}$$

Consider the map $\widetilde{x} \circ x^{-1}$ in the diagram above. Since the charts (U, x) and $(\widetilde{U}, \widetilde{x})$ belong to the same $\mathcal{C}^k$-atlas $\mathscr{A}_M$, by definition the transition map $\widetilde{x} \circ x^{-1}$ is $\mathcal{C}^k$-differentiable as a map between subsets of $\mathbb{R}^{\dim M}$, and similarly for $\widetilde{y} \circ y^{-1}$. We now notice that we can write:

$$\widetilde{y} \circ \phi \circ \widetilde{x}^{-1} = (\widetilde{y} \circ y^{-1}) \circ (y \circ \phi \circ x^{-1}) \circ (\widetilde{x} \circ x^{-1})^{-1}$$

and since the composition of $\mathcal{C}^k$ maps is still $\mathcal{C}^k$, we are done. □

This proof shows the significance of restricting to $\mathcal{C}^k$-atlases. Such atlases only contain charts for which the transition maps are $\mathcal{C}^k$, which is what makes our definition of differentiability of maps between manifolds well-defined.

The same definition and proof work for smooth ($\mathcal{C}^\infty$) manifolds, in which case we talk about *smooth maps*. As we said before, this is the case we will be most interested in.

Example 5.7. Consider the smooth manifolds $(\mathbb{R}^d, \mathcal{O}_{\text{std}}, \mathscr{A}_d)$ and $(\mathbb{R}^{d'}, \mathcal{O}_{\text{std}}, \mathscr{A}_{d'})$, where $\mathscr{A}_d$) and $\mathscr{A}_{d'}$) are the maximal atlases containing the charts $(\mathbb{R}^d, \mathrm{id}_{\mathbb{R}^d})$ and $(\mathbb{R}^{d'}, \mathrm{id}_{\mathbb{R}^{d'}})$ respectively, and let $f\colon \mathbb{R}^d \to \mathbb{R}^{d'}$ be a map. The diagram defining the differentiability of f with respect to these charts is

$$\begin{tikzcd}
\mathbb{R}^d \arrow[r, "f"] \arrow[d, "\mathrm{id}_{\mathbb{R}^d}"] & \mathbb{R}^{d'} \arrow[d, "\mathrm{id}_{\mathbb{R}^{d'}}"] \\
\mathbb{R}^d \arrow[r, "\mathrm{id}_{\mathbb{R}^{d'}} \circ f \circ (\mathrm{id}_{\mathbb{R}^d})^{-1}"] & \mathbb{R}^{d'}
\end{tikzcd}$$

and, by definition, the map f is smooth as a map between manifolds if, and only if, the map $\mathrm{id}_{\mathbb{R}^{d'}} \circ f \circ (\mathrm{id}_{\mathbb{R}^d})^{-1} = f$ is smooth in the usual sense.

Example 5.8. Let $(M, \mathcal{O}, \mathscr{A})$ be a d-dimensional smooth manifold and let $(U, x) \in \mathscr{A}$. Then $x\colon U \to x(U) \subseteq \mathbb{R}^d$ is smooth. Indeed, we have

$$\begin{tikzcd}
U \arrow[r, "x"] \arrow[d, "x"] & x(U) \arrow[d, "\mathrm{id}_{x(U)}"] \\
x(U) \subseteq \mathbb{R}^d \arrow[r, "\mathrm{id}_{x(U)} \circ x \circ x^{-1}"] & x(U) \subseteq \mathbb{R}^d
\end{tikzcd}$$

Hence $x\colon U \to x(U)$ is smooth if, and only if, the map $\mathrm{id}_{x(U)} \circ x \circ x^{-1} = \mathrm{id}_{x(U)}$ is smooth in the usual sense, which it certainly is.

The coordinate maps $x^i := \mathrm{proj}_i \circ x \colon U \to \mathbb{R}$ are also smooth. Indeed, consider the diagram

$$\begin{tikzcd}
U \arrow[r, "x^i"] \arrow[d, "x"] & \mathbb{R} \arrow[d, "\mathrm{id}_{\mathbb{R}}"] \\
x(U) \subseteq \mathbb{R}^d \arrow[r, "\mathrm{id}_{\mathbb{R}} \circ x^i \circ x^{-1}"] & \mathbb{R}
\end{tikzcd}$$

Then, x^i is smooth if, and only if, the map

$$\mathrm{id}_{\mathbb{R}} \circ x^i \circ x^{-1} = x^i \circ x^{-1} = \mathrm{proj}_i$$

is smooth in the usual sense, which it certainly is.

5.3.1 Classification Of Differentiable Structures

Definition 5.16 (Diffeomorphism)**.** *Let $\phi \colon M \to N$ be a bijective map between smooth manifolds. If both ϕ and ϕ^{-1} are smooth, then ϕ is said to be a* ***diffeomorphism****.*

Diffeomorphisms are the structure preserving maps between smooth manifolds.

Definition 5.17 (Diffeomorphic Manifolds)**.** *Two manifolds $(M, \mathcal{O}_M, \mathscr{A}_M)$, $(N, \mathcal{O}_N, \mathscr{A}_N)$ are said to be* ***diffeomorphic*** *if there exists a diffeomorphism $\phi \colon M \to N$ between them. We write $M \cong_{\mathit{diff}} N$.*

Note that if the differentiable structure is understood (or irrelevant), we typically write M instead of the triple $(M, \mathcal{O}_M, \mathscr{A}_M)$.

Remark 5.2. Being diffeomorphic is an equivalence relation. In fact, it is customary to consider diffeomorphic manifolds to be *the same* from the point of view of differential geometry. This is similar to the situation with topological spaces, where we consider homeomorphic spaces to be the same from the point of view of topology. This is typical of all structure preserving maps.

Armed with the notion of diffeomorphism, we can now ask the following question: how many smooth structures on a given topological space are there, up to diffeomorphism?
The answer is quite surprising: it depends on the dimension of the manifold!

Theorem 5.2 (Radon-Moise)**.** *Let M be a manifold with $\dim M = 1, 2,$ or 3. Then there is a unique smooth structure on M up to diffeomorphism.*

Recall that in a previous example, we showed that we can equip $(\mathbb{R}, \mathcal{O}_{\text{std}})$ with two incompatible atlases $\mathscr{A}$ and $\mathscr{B}$. Let $\mathscr{A}_{\max}$ and $\mathscr{B}_{\max}$ be their extensions to maximal atlases, and consider the smooth manifolds $(\mathbb{R}, \mathcal{O}_{\text{std}}, \mathscr{A}_{\max})$ and $(\mathbb{R}, \mathcal{O}_{\text{std}}, \mathscr{B}_{\max})$. Clearly, these are different manifolds, because the atlases are different, but since $\dim \mathbb{R} = 1$, they must be diffeomorphic.

The answer to the case $\dim M > 4$ (we emphasize $\dim M \neq 4$) is provided by *surgery theory*. This is a collection of tools and techniques in topology with which one obtains a new manifold from given ones by performing surgery on them, i.e. by cutting, replacing and gluing parts in such a way as to control topological invariants like the fundamental group. The idea is to understand all manifolds in dimensions higher than 4 by performing surgery systematically. In particular, using surgery theory, it has been shown that there are only finitely many smooth manifolds (up to diffeomorphism) one can make from a topological manifold.

This is not as neat as the previous case, but since there are only finitely many structures, we can still enumerate them, i.e. we can write an exhaustive list.

While finding all the differentiable structures may be difficult for any given manifold, this theorem has an immediate impact on a physical theory that models spacetime as a manifold. For instance, some physicists believe that spacetime should be modelled as a 10-dimensional manifold (we are neither proposing nor condemning this view). If that is indeed the case, we need to worry about which differentiable structure we equip our 10-dimensional manifold with, as each different choice will likely lead to different predictions. But since there are only finitely many such structures, physicists can, at least in principle, devise and perform finitely many experiments to distinguish between them and determine which is the right one, if any.

We now turn to the special case $\dim M = 4$. The result is that if M is a non-compact topological manifold, then there are uncountably many non-diffeomorphic smooth structures that we can equip M with. In particular, this applies to $(\mathbb{R}^4, \mathcal{O}_{\text{std}})$.

5.4 Tangent Spaces

In this section, whenever we say "manifold", we mean a (real) d-dimensional differentiable manifold, unless we explicitly say otherwise. We will also suppress the differentiable structure in the notation.

Definition 5.18 ($\mathcal{C}^\infty(M)$ Vector Space)**.** *Let M be a manifold. We define the infinite-dimensional vector space over $\mathbb{R}$ of all smooth functions on M with underlying set*

$$\mathcal{C}^\infty(M) := \{f\colon M \to \mathbb{R} \mid f \text{ is smooth}\}$$

and operations defined pointwise, i.e. for any $p \in M$,

$$\begin{aligned}(f+g)(p) &:= f(p) + g(p)\\ (\lambda f)(p) &:= \lambda f(p).\end{aligned}$$

A routine check shows that this is indeed a vector space.

Definition 5.19 (Smooth Curve)**.** *A **smooth curve** on M is a smooth map $\gamma\colon \mathbb{R} \to M$, where $\mathbb{R}$ is understood as a 1-dimensional manifold.*

Definition 5.20 (Directional Derivative Operator)**.** *Let $\gamma\colon \mathbb{R} \to M$ be a smooth curve through $p \in M$; w.l.o.g. let $\gamma(0) = p$. The **directional derivative operator** at p along γ is the linear map*

$$\begin{aligned}X_{\gamma,p}\colon \mathcal{C}^\infty(M) &\xrightarrow{\sim} \mathbb{R}\\ f &\mapsto (f \circ \gamma)'(0),\end{aligned}$$

where $\mathbb{R}$ is understood as a 1-dimensional vector space over the field $\mathbb{R}$.

Note that $f \circ \gamma$ is a map $\mathbb{R} \to \mathbb{R}$, hence we can calculate the usual derivative and evaluate it at 0.

Remark 5.3. In differential geometry, $X_{\gamma,p}$ is called the *tangent vector* to the curve γ at the point $p \in M$. Intuitively, $X_{\gamma,p}$ is the velocity γ at p. Consider the curve $\delta(t) := \gamma(2t)$, which is the same curve parametrised twice as fast. We have, for any $f \in \mathcal{C}^\infty(M)$:

$$X_{\delta,p}(f) = (f \circ \delta)'(0) = 2(f \circ \gamma)'(0) = 2X_{\gamma,p}(f)$$

by using the chain rule. Hence $X_{\gamma,p}$ scales like a velocity should.

Definition 5.21 (Tangent Space)**.** *Let M be a manifold and $p \in M$. The **tangent space** to M at p is the vector space over $\mathbb{R}$ with underlying set*

$$T_pM := \{X_{\gamma,p} \mid \gamma \text{ is a smooth curve through } p\},$$

addition

$$\begin{aligned}\oplus\colon T_pM \times T_pM &\to T_pM\\ (X_{\gamma,p}, X_{\delta,p}) &\mapsto X_{\gamma,p} \oplus X_{\delta,p},\end{aligned}$$

and scalar multiplication

$$\begin{aligned}\odot\colon \mathbb{R} \times T_pM &\to T_pM\\ (\lambda, X_{\gamma,p}) &\mapsto \lambda \odot X_{\gamma,p},\end{aligned}$$

both defined pointwise, i.e. for any $f \in \mathcal{C}^\infty(M)$,

$$\begin{aligned}(X_{\gamma,p} \oplus X_{\delta,p})(f) &:= X_{\gamma,p}(f) + X_{\delta,p}(f)\\ (\lambda \odot X_{\gamma,p})(f) &:= \lambda X_{\gamma,p}(f).\end{aligned}$$

Note that the outputs of these operations do not look like elements in T_pM, because they are not of the form $X_{\sigma,p}$ for some curve σ. Hence, we need to show that the above operations are, in fact, well-defined.

Proposition 5.3. *Let* $X_{\gamma,p}, X_{\delta,p} \in T_pM$ *and* $\lambda \in \mathbb{R}$. *Then, we have* $X_{\gamma,p} \oplus X_{\delta,p} \in T_pM$ *and* $\lambda \odot X_{\gamma,p} \in T_pM$.

Since the derivative is a local concept, it is only the behaviour of curves near p that matters. In particular, if two curves γ and δ agree on a neighbourhood of p, then $X_{\gamma,p}$ and $X_{\delta,p}$ are the same element of T_pM. Hence, we can work *locally* by using a chart on M.

Proof. Let (U, x) be a chart on M, with U a neighbourhood of p.

i) Define the curve

$$\sigma(t) := x^{-1}((x \circ \gamma)(t) + (x \circ \delta)(t) - x(p)).$$

Note that σ is smooth since it is constructed via addition and composition of smooth maps and, moreover:

$$\begin{aligned}\sigma(0) &= x^{-1}(x(\gamma(0)) + x(\delta(0)) - x(p))\\ &= x^{-1}(x(p)) + x(p) - x(p))\\ &= x^{-1}(x(p))\\ &= p.\end{aligned}$$

Thus σ is a smooth curve through p. Let $f \in \mathcal{C}^\infty(U)$ be arbitrary. Then we have

$$\begin{aligned}X_{\sigma,p}(f) &:= (f \circ \sigma)'(0)\\ &= [f \circ x^{-1} \circ ((x \circ \gamma) + (x \circ \delta) - x(p))]'(0)\\ &= [\partial_a(f \circ x^{-1})(x(p))]\,((x^a \circ \gamma) + (x^a \circ \delta) - x^a(p))'(0)\\ &= [\partial_a(f \circ x^{-1})(x(p))]\,((x^a \circ \gamma)'(0) + (x^a \circ \delta)'(0))\\ &= (f \circ x^{-1} \circ x \circ \gamma)'(0) + (f \circ x^{-1} \circ x \circ \delta)'(0)\\ &= (f \circ \gamma)'(0) + (f \circ \delta)'(0)\\ &=: (X_{\gamma,p} \oplus X_{\delta,p})(f).\end{aligned}$$

Therefore $X_{\gamma,p} \oplus X_{\delta,p} = X_{\sigma,p} \in T_pM$.

ii) The second part is straightforward. Define $\sigma(t) := \gamma(\lambda t)$. This is again a smooth curve through p and we have:

$$\begin{aligned}X_{\sigma,p}(f) &:= (f \circ \sigma)'(0)\\ &= f'(\sigma(0))\,\sigma'(0)\\ &= \lambda f'(\gamma(0))\,\gamma'(0)\\ &= \lambda(f \circ \gamma)'(0)\\ &:= (\lambda \odot X_{\gamma,p})(f)\end{aligned}$$

for any $f \in \mathcal{C}^\infty(U)$. Hence $\lambda \odot X_{\gamma,p} = X_{\sigma,p} \in T_pM$. □

Hence indeed T_pM is a vector space.

The question is, what exactly $X_{\gamma,p}$ is mathematically speaking? Since it's **a** map of the form:

$$X_{\gamma,p} \colon \mathcal{C}^\infty(M) \xrightarrow{\sim} \mathbb{R}$$

it's clear that it's an element of $\mathrm{Hom}(\mathcal{C}^\infty(M), R)$, i.e an element of the dual vector space of $\mathcal{C}^\infty(M)$. Which subsequently makes T_pM a sub-vector space of the dual vector space of $\mathcal{C}^\infty(M)$. ($X_{\gamma,p}$ is a particular choice of a linear map, more specifically the derivative with respect to the parameter, and not **all** possible linear maps. This is why T_pM is not the whole dual vector space of $\mathcal{C}^\infty(M)$)

However, if we take the extra step and turn the $\mathcal{C}^\infty(M)$ from a vector space to an algebra (by defining an appropriate operation) then we can show that $X_{\gamma,p}$ is actually a derivation of the algebra.

More specifically we will define a product on $\mathcal{C}^\infty(M)$ by

$$\begin{aligned}\bullet\colon \mathcal{C}^\infty(M)\times\mathcal{C}^\infty(M) &\to \mathcal{C}^\infty(M)\\ (f,g) &\mapsto f\bullet g,\end{aligned}$$

where $f\bullet g$ is defined pointwise. Then $(\mathcal{C}^\infty(M),+,\cdot,\bullet)$ is an associative, unital and commutative algebra over $\mathbb{R}$.

Now that we have an algebra, let us remind ourselves what a derivation is and also try to combine the definition with our case:

Definition 5.22 (Derivation (On A Manifold))**.** *Let M be a manifold and let $p\in U\subseteq M$, where U is open. A* derivation on U at p *is an $\mathbb{R}$-linear map $D\colon \mathcal{C}^\infty(U)\xrightarrow{\sim}\mathbb{R}$ satisfying the Leibniz rule*

$$D(fg)=D(f)g(p)+f(p)D(g).$$

The usual derivative operator is a derivation on $\mathcal{C}^\infty(\mathbb{R})$, the algebra of smooth real functions, since it is linear and satisfies the Leibniz rule. (The second derivative operator, however, is not a derivation on $\mathcal{C}^\infty(\mathbb{R})$, since it does not satisfy the Leibniz rule. This shows that the composition of derivations need not be a derivation.) Hence, we managed to show that indeed $X_{\gamma,p}$ is actually a derivation of the algebra of smooth real functions on M.

5.4.1 Co-Ordinate Induced Basis For The Tangent Space

The following is a crucially important result about tangent spaces.

Theorem 5.3. *Let M be a manifold and let $p\in M$. Then*

$$\dim T_pM=\dim M.$$

Remark 5.4. Note carefully that, despite us using the same symbol, the two "dimensions" appearing in the statement of the theorem are, at least on the surface, entirely unrelated. Indeed, recall that $\dim M$ is defined in terms of charts (U,x), with $x\colon U\to x(U)\subseteq\mathbb{R}^{\dim M}$, while $\dim T_pM=|\mathcal{B}|$, where $\mathcal{B}$ is a Hamel basis for the vector space T_pM. The idea behind the proof is to construct a basis of T_pM from a chart on M.

Proof. W.l.o.g., let (U,x) be a chart *centred* at p, i.e. $x(p)=0\in\mathbb{R}^{\dim M}$. Define $(\dim M)$-many curves $\gamma_{(a)}\colon\mathbb{R}\to U$ through p by requiring $(x^b\circ\gamma_{(a)})(t)=\delta^b_a t$, i.e.

$$\begin{aligned}\gamma_{(a)}(0)&:=p\\ \gamma_{(a)}(t)&:=x^{-1}\circ(0,\dots,0,t,0,\dots,0)\end{aligned}$$

where the t is in the a^{th} position, with $1\le a\le\dim M$. Let us calculate the action of the tangent vector $X_{\gamma_{(a)},p}\in T_pM$ on an arbitrary function $f\in\mathcal{C}^\infty(U)$:

$$\begin{aligned}X_{\gamma_{(a)},p}(f)&:=(f\circ\gamma_{(a)})'(0)\\ &=(f\circ\mathrm{id}_U\circ\gamma_{(a)})'(0)\\ &=(f\circ x^{-1}\circ x\circ\gamma_{(a)})'(0)\\ &=[\partial_b(f\circ x^{-1})(x(p))]\,(x^b\circ\gamma_{(a)})'(0)\\ &=[\partial_b(f\circ x^{-1})(x(p))]\,(\delta^b_a t)'(0)\\ &=[\partial_b(f\circ x^{-1})(x(p))]\,\delta^b_a\\ &=\partial_a(f\circ x^{-1})(x(p))\end{aligned}$$

We introduce a special notation for this last line, namely:

$$\partial_a(f \circ x^{-1})(x(p)) := \left(\frac{\partial}{\partial x^a}\right)_p (f)$$

Remark 5.5. While the symbol $\left(\frac{\partial}{\partial x^a}\right)_p$ has nothing to do with the idea of partial differentiation with respect to the variable x^a (since x refers to the chart map and no differentiation has been defined there), it is notationally consistent with it, in the following sense.

Let $M = \mathbb{R}^d$, $(U, x) = (\mathbb{R}^d, \mathrm{id}_{\mathbb{R}^d})$ and let $\left(\frac{\partial}{\partial x^a}\right)_p \in T_p\mathbb{R}^d$. If $f \in \mathcal{C}^\infty(\mathbb{R}^d)$, then

$$\left(\frac{\partial}{\partial x^a}\right)_p (f) = \partial_a(f \circ x^{-1})(x(p)) = \partial_a f(p),$$

since $x = x^{-1} = \mathrm{id}_{\mathbb{R}^d}$. Moreover, we have $\mathrm{proj}_a = x^a$. Thus, we can think of $x^1, \ldots, x^d$ as the independent variables of f, and we can then write

$$\left(\frac{\partial}{\partial x^a}\right)_p (f) = \frac{\partial f}{\partial x^a}(p).$$

Hence, up to this point we showed that:

$$X_{\gamma_{(a)},p}(f) = \left(\frac{\partial}{\partial x^a}\right)_p (f)$$

Or by removing the action on the function, simply:

$$X_{\gamma_{(a)},p} = \left(\frac{\partial}{\partial x^a}\right)_p$$

We now claim that

$$\mathcal{B} = \left\{ \left(\frac{\partial}{\partial x^a}\right)_p \in T_pM \;\middle|\; 1 \leq a \leq \dim M \right\}$$

is a basis of T_pM. First, we show that $\mathcal{B}$ spans T_pM.

Let $X \in T_pM$. Then, by definition, there exists some smooth curve σ through p such that $X = X_{\sigma,p}$. For any $f \in \mathcal{C}^\infty(U)$, we have

$$\begin{aligned} X(f) &= X_{\sigma,p}(f) \\ &:= (f \circ \sigma)'(0) \\ &= (f \circ x^{-1} \circ x \circ \sigma)'(0) \\ &= [\partial_b(f \circ x^{-1})(x(p))]\,(x^b \circ \sigma)'(0) \\ &= (x^b \circ \sigma)'(0) \left(\frac{\partial}{\partial x^b}\right)_p (f). \end{aligned}$$

Since $(x^b \circ \sigma)'(0) =: X^b \in \mathbb{R}$, we have:

$$X = X^b \left(\frac{\partial}{\partial x^b}\right)_p,$$

i.e. any $X \in T_pM$ is a linear combination of elements from $\mathcal{B}$.

To show linear independence, suppose that

$$\lambda^a \left(\frac{\partial}{\partial x^a}\right)_p = 0,$$

for some scalars λ^a. Note that this is an operator equation, and the zero on the right hand side is the zero operator $0 \in T_pM$.

Recall that, given the chart (U, x), the coordinate maps $x^b \colon U \to \mathbb{R}$ are smooth, i.e. $x^b \in \mathcal{C}^\infty(U)$. Thus, we can feed them into the left hand side to obtain

$$\begin{aligned}
0 &= \lambda^a \left(\frac{\partial}{\partial x^a}\right)_p (x^b) \\
&= \lambda^a \, \partial_a (x^b \circ x^{-1})(x(p)) \\
&= \lambda^a \, \partial_a (\mathrm{proj}_b)(x(p)) \\
&= \lambda^a \delta^b_a \\
&= \lambda^b
\end{aligned}$$

i.e. $\lambda^b = 0$ for all $1 \leq b \leq \dim M$. So $\mathcal{B}$ is indeed a basis of T_pM, and since by construction $|\mathcal{B}| = \dim M$, the proof is complete. □

Remark 5.6. While it is possible to define infinite-dimensional manifolds, in this course we will only consider finite-dimensional ones. Hence $\dim T_pM = \dim M$ will always be finite in this course.

Remark 5.7. Note that the basis that we have constructed in the proof is *not* chart-independent. Indeed, each different chart will induce a different tangent space basis, and we distinguish between them by keeping the chart map in the notation for the basis elements.

This is not a cause of concern for our proof however, since every basis of a vector space must have the same cardinality, and hence it suffices to find one basis to determine the dimension.

Definition 5.23 (Co-Ordinate Induced Basis)**.** *Let $X \in T_pM$ be a tangent vector and let (U, x) be a chart containing p. Then the basis $\{\left(\frac{\partial}{\partial x^a}\right)_p\}$ created by the usage of the chart is called a **co-ordinate induced basis**. In this basis an element $X \in T_pM$ can be expressed as:*

$$X = X^a \left(\frac{\partial}{\partial x^a}\right)_p,$$

*where the real numbers $X^1, \dots, X^{\dim M}$ are called the **vector components** of X with respect to the co-ordinate induced basis by the chart (U, x).*

5.4.2 Change Of Vector Components Under A Change Of Chart

One of the most heavily used concepts is the transformation of the components of a vector under different co-ordinate systems (i.e under a chart transition map that subsequently changes the co-ordinate induced basis). Let's find out the rule.

Let $X \in T_pM$ and let (U, x) and (V, y) be two charts containing p. Then X can be expressed in any of the two charts as:

$${X^a}_{(y)} \left(\frac{\partial}{\partial y^a}\right)_p = X = {X^a}_{(x)} \left(\frac{\partial}{\partial x^a}\right)_p$$

Let us act with X on some smooth function f of $\mathcal{C}^\infty(M)$ by using first the components of (U, x) chart:

$$\begin{aligned}
X(f) &= {X^a}_{(x)} \left(\frac{\partial}{\partial x^a}\right)_p (f) \\
&= {X^a}_{(x)} \partial_a (f \circ x^{-1})(x(p)) \\
&= {X^a}_{(x)} \partial_a (f \circ y^{-1} \circ y \circ x^{-1})(x(p)) \\
&= {X^a}_{(x)} \partial_a (y^b \circ x^{-1})(x(p)) \, \partial_b (f \circ y^{-1})(y(p)) \\
&= {X^a}_{(x)} \frac{\partial y^b}{\partial x^a} \left(\frac{\partial}{\partial y^b}\right)_p (f)
\end{aligned}$$

Similarly, let us now act with X on the smooth function f of $\mathcal{C}^{\infty}(M)$ by using the components of (V, y) chart:

$$X(f) = X^a{}_{(y)} \left(\frac{\partial}{\partial y^a} \right)_p (f)$$

These expressions are, of course, equal to each other so by suppressing now the action on the function f, we obtain:

$$X^a{}_{(x)} \frac{\partial y^b}{\partial x^a} \left(\frac{\partial}{\partial y^b} \right)_p = X^b{}_{(y)} \left(\frac{\partial}{\partial y^b} \right)_p$$

$$X^a{}_{(x)} \frac{\partial y^b}{\partial x^a} \left(\frac{\partial}{\partial y^b} \right)_p - X^b{}_{(y)} \left(\frac{\partial}{\partial y^b} \right)_p = 0$$

$$\left(X^a{}_{(x)} \frac{\partial y^b}{\partial x^a} - X^b{}_{(y)} \right) \left(\frac{\partial}{\partial y^b} \right)_p = 0$$

Finally, since the base vectors of $\{ \left(\frac{\partial}{\partial y^a} \right)_p \}$ are linearly independent the only way for this equation to be zero is for the coefficients to be zero hence:

$$X^a{}_{(x)} \frac{\partial y^b}{\partial x^a} - X^b{}_{(y)} = 0$$

Of finally by solving w.r.t $X^b{}_{(y)}$ and renaming the indices:

$$X^a{}_{(y)} = \frac{\partial y^a}{\partial x^b} X^b{}_{(x)}$$

This equation shows as how the components of a vector transform under a chart transition map, i.e under the change of charts, i.e from one co-ordinate induced basis to another. Of course the formula agrees completely with the transformations of vector components under the change of basis that we showed in previous chapter: $\widetilde{v}^b = A^b{}_a v^a$.

The function $y^a = y^a(x^1, \dots, x^{\dim M})$ expresses the new co-ordinates in terms of the old ones, and $A^b{}_a$ is the *Jacobian* matrix of this map, evaluated at $x(p)$. Note that no matter how non-linear the transformations of the co-ordinates are, the vectors always transform in a linear fashion. In a way, "vectors do not care about the non-linearity of co-ordinate transformations".

5.5 Cotangent Spaces

Since the tangent space is a vector space, we can do all the constructions we saw previously in the abstract vector space setting.

Definition 5.24 (Cotagent Space)**.** *Let M be a manifold and $p \in M$. The **cotangent space** to M at p is*

$$T_p^* M := (T_p M)^*$$

Since $\dim T_p M$ is finite, we have $T_p M \cong_{\text{vec}} T_p^* M$.

And of course, once we have the cotangent space, we can define the tensor spaces.

Definition 5.25 (Tensor Space)**.** *Let M be a manifold and $p \in M$. The **tensor space** $(T^r_s)_p M$ is defined as*

$$(T^r_s)_p M := T^r_s(T_p M) = \underbrace{T_p M \otimes \cdots \otimes T_p M}_{r \text{ copies}} \otimes \underbrace{T_p^* M \otimes \cdots \otimes T_p^* M}_{s \text{ copies}}.$$

5.5.1 Dual Basis For The Cotangent Space

Now let's give a very important definition that will help us to formalize elements, and subsequently a basis, for the cotangent space.

Definition 5.26 (Gradient)**.** *Let M be a manifold and let $f\colon M \to \mathbb{R}$ be smooth. The **gradient of** f **at** $p \in M$ is the $\mathbb{R}$-linear map*

$$\begin{aligned} \mathrm{d}_p\colon \mathcal{C}^\infty(M) &\xrightarrow{\sim} T_p^*M \\ f &\mapsto \mathrm{d}_p f, \end{aligned}$$

with $p \in U \subseteq M$, defined by

$$\mathrm{d}_p f(X) := X(f)$$

Remark 5.8. Note that since d_p is a map from $\mathcal{C}^\infty(M) \xrightarrow{\sim} T_p^*M$ that means that when it acts on a function of $\mathcal{C}^\infty(M)$ the final result $\mathrm{d}_p f$ is an element of T_p^*M hence a covector. By its turn, as an element of the dual space of T_pM it maps elements of T_pM to the real numbers (that's the definition of the dual space of a vector space). Hence the expression $\mathrm{d}_p f(X)$ must end up to a real number, which is indeed what $X(f)$ is. By writing $\mathrm{d}_p f(X) := X(f)$, we have committed a slight (but nonetheless real) abuse of notation, since $\mathrm{d}_p f(X) \in T_{f(p)}\mathbb{R}$ takes in a function and return a real number, but $X(f)$ is already a real number! However by doing so we can now talk about $\mathrm{d}_p f$ without providing the vector that it acts on. In other words we can talk about covectors without the need of their actions on vectors.

Remark 5.9. The gradient of a function is a covector and **not** a vector.

Recall that if (U, x) is a chart on M, then the co-ordinate maps $x^a\colon U \to x(U) \subseteq \mathbb{R}^{\dim M}$ are smooth functions on U hence they belong to $\mathcal{C}^\infty(M)$. We can thus apply the gradient operator d_p (with $p \in U$) to each of them to obtain $(\dim M)$-many elements of T_p^*M.

Proposition 5.4. *Let (U, x) be a chart on M, with $p \in U$. The set $\mathcal{B} = \{\mathrm{d}_p x^a \mid 1 \leq a \leq \dim M\}$ forms the dual basis of T_p^*M.*

Proof. By simply acting on $\left(\frac{\partial}{\partial x^a}\right)_p$ with $\mathrm{d}_p x^a$ (in our notation, we have $(\mathrm{d}x^a)_p = \mathrm{d}_p x^a$) we obtain:

$$\begin{aligned} \mathrm{d}_p x^a \left(\left(\frac{\partial}{\partial x^b} \right)_p \right) &= \left(\frac{\partial}{\partial x^b} \right)_p (x^a) && \text{(definition of } \mathrm{d}_p x^a\text{)} \\ &= \partial_b (x^a \circ x^{-1})(x(p)) && \text{(definition of } \left(\tfrac{\partial}{\partial x^b}\right)_p\text{)} \\ &= \partial_b (\mathrm{proj}_a)(x(p)) \\ &= \delta^a_b \end{aligned}$$

Therefore, $\mathcal{B}$ is, in fact, the dual basis to $\{\left(\frac{\partial}{\partial x^a}\right)_p\}$. □

5.5.2 Change Of Covector Components Under A Change Of Chart

Once again, as we did in the vector case with the vector components, one needs to find the transformation of the components of a covector under different co-ordinate systems. We will follow exactly the same procedure.

Let $\omega \in T_p^*M$ and let (U, x) and (V, y) be two charts containing p. Then ω can be expressed in any of the two charts by using the dual basis as:

$$\omega_{(y)a}(dy^a)_p = \omega = \omega_{(x)a}(dx^a)_p$$

By repeating the same process as we did for the vectors it is very easy to show that covectors components transform as

$$\omega_{(y)a} = \left(\frac{\partial x^b}{\partial y^a} \right)_p \omega_{(x)b}$$

5.6 Push-Forward And Pull-Back

Definition 5.27 (Push-Forward). *Let $\phi\colon M \to N$ be a smooth map between smooth manifolds. The **push-forward** (or **derivative**) of ϕ at $p \in M$ is the linear map $(\phi_*)_p$:*

$$\begin{aligned}(\phi_*)_p\colon T_pM &\xrightarrow{\sim} T_{\phi(p)}N\\ X &\mapsto (\phi_*)_p(X)\end{aligned}$$

where $(\phi_*)_p(X)$ is defined as

$$\begin{aligned}(\phi_*)_p(X)\colon \mathcal{C}^\infty(N) &\xrightarrow{\sim} \mathbb{R}\\ f &\mapsto (\phi_*)_p(X)f := X(f \circ \phi).\end{aligned}$$

In other words, since $(\phi_*)_p$ is a map from one tangent space to another this means that it acts on a tangent vector and produces another one, hence $(\phi_*)_p(X)$ is again a tangent vector (but on N). As a tangent vector it can act on a smooth function (again on N) and produce a real number, hence the action of a push-forward on a function is simply the one we wrote above.

Note that one has to define a push-forward $(\phi_*)_p$ for every point p of M. Although we have only one map ϕ we have many push-forward maps $(\phi_*)_p$.

Proposition 5.5. *Let $\phi\colon M \to N$ be smooth. The tangent vector $X_{\gamma,p} \in T_pM$ is pushed forward to the tangent vector $X_{\phi\circ\gamma,\phi(p)} \in T_{\phi(p)}N$, i.e.*

$$(\phi_*)_p(X_{\gamma,p}) = X_{\phi\circ\gamma,\phi(p)}.$$

Proof. Let $f \in \mathcal{C}^\infty(V)$, with (V, x) a chart on N and $\phi(p) \in V$. By applying the definitions, we have

$$\begin{aligned}(\phi_*)_p(X_{\gamma,p})(f) &= (X_{\gamma,p})(f \circ \phi) && \text{(definition of } (\phi_*)_p)\\ &= ((f \circ \phi) \circ \gamma)'(0) && \text{(definition of } X_{\gamma,p})\\ &= (f \circ (\phi \circ \gamma))'(0) && \text{(associativity of } \circ)\\ &= X_{\phi\circ\gamma,\phi(p)}(f) && \text{(definition of } X_{\phi\circ\gamma,\phi(p)})\end{aligned}$$

Since f was arbitrary, we have $(\phi_*)_p(X_{\gamma,p}) = X_{\phi\circ\gamma,\phi(p)}$. □

Related to the push-forward, there is the notion of pull-back of a smooth map.

Definition 5.28 (Pull-Back). *Let $\phi\colon M \to N$ be a smooth map between smooth manifolds. The **pull-back** of ϕ at $p \in M$ is the linear map:*

$$\begin{aligned}(\phi^*)_p\colon T^*_{\phi(p)}N &\xrightarrow{\sim} T^*_pM\\ \omega &\mapsto (\phi^*)_p(\omega),\end{aligned}$$

where $(\phi^*)_p(\omega)$ is defined as

$$\begin{aligned}(\phi^*)_p(\omega)\colon T_pM &\xrightarrow{\sim} \mathbb{R}\\ X &\mapsto (\phi^*)_p(\omega)(X) := \omega((\phi_*)_p(X)).\end{aligned}$$

In words, if ω is a covector on N, its pull-back $(\phi^*)_p(\omega)$ is a covector on M. It acts on tangent vectors on M by first pushing them forward to tangent vectors on N, and then applying ω to them to produce a real number.

Diagrammatically, what we've defined so far is the following

$$\begin{tikzcd} \mathcal{C}^\infty(M) \arrow[d, "X"] & \mathcal{C}^\infty(N) \arrow[l, "-\circ\phi"'] \arrow[dl, "(\phi_*)_p(X)"] \\ \mathbb{R} & \end{tikzcd} \qquad \begin{tikzcd} T_pM \arrow[r, "(\phi_*)_p"] \arrow[dr, "(\phi^*)_p(\omega)"'] & T_{\phi(p)}N \arrow[d, "\omega"] \\ & \mathbb{R} \end{tikzcd}$$

Remark 5.10. It is quite easy to show that everything we have defined in this section is, in fact, linear.

Remark 5.11. We have seen that, given a smooth $\phi\colon M \to N$, we can push a vector $X \in T_pM$ forward to a vector $(\phi_*)_p(X) \in T_{\phi(p)}N$, and pull a covector $\omega \in T^*_{\phi(p)}N$ back to a covector $(\phi^*)_p(\omega) \in T^*_pM$. In other words both push-forward and pull-back work only in the direction of their definition. However, if $\phi\colon M \to N$ is a diffeomorphism (and only then), we can also pull a vector $Y \in T_{\phi(p)}N$ back to a vector $(\phi^*)_p(Y) \in T_pM$, and push a covector $\eta \in T^*_pM$ forward to a covector $(\phi_*)_p(\eta) \in T^*_{\phi(p)}N$, by using ϕ^{-1} as follows:

$$(\phi^*)_p(Y) := ((\phi^{-1})_*)_{\phi(p)}(Y)$$
$$(\phi_*)_p(\eta) := ((\phi^{-1})^*)_{\phi(p)}(\eta).$$

In general, we should keep in mind that:

Vectors are pushed forward,
covectors are pulled back.

5.7 Immersions And Embeddings

We will now consider the question of under which circumstances a smooth manifold can "sit" in $\mathbb{R}^d$, for some $d \in \mathbb{N}$. There are, in fact, two notions of sitting inside another manifold, called immersion and embedding.

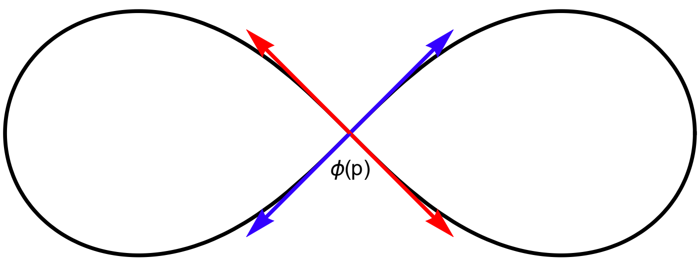
Definition 5.29 (Immersion)**.** *A smooth map $\phi\colon M \to N$ is said to be an **immersion** of M into N if the derivative*

$$(\phi_*)_p\colon T_pM \xrightarrow{\sim} T_{\phi(p)}N$$

is injective, for all $p \in M$. In that case, the manifold M is said to be an immersed submanifold *of N.*

From the theory of linear algebra, we immediately deduce that, for $\phi\colon M \to N$ to be an immersion, we must have $\dim M \leq \dim N$. A closely related notion is that of a *submersion*, where we require each $(\phi_*)_p$ to be surjective, and thus we must have $\dim M \geq \dim N$. However, we will not need this here.

Example 5.9. Consider the map $\phi\colon S^1 \to \mathbb{R}^2$ whose image is reproduced below.



The map ϕ is not injective, i.e. there are $p, q \in S^1$, with $p \neq q$ and $\phi(p) = \phi(q)$. Of course, this means that $T_{\phi(p)}\mathbb{R}^2 = T_{\phi(q)}\mathbb{R}^2$. However, the maps $(\phi_*)_p$ and $(\phi_*)_q$ are both injective, with their images being represented by the blue and red arrows, respectively. Hence, the map ϕ is immersion.

Definition 5.30 (Embedding)**.** *A smooth map $\phi\colon M \to N$ is said to be a **embedding** of M into N if*

- *$\phi\colon M \to N$ is an immersion;*
- *$M \cong_{\text{top}} \phi(M) \subseteq N$, where $\phi(M)$ carries the subset topology inherited from N.*

In that case the manifold M is said to be an embedded submanifold *of N.*

Remark 5.12. If a continuous map between topological spaces satisfies the second condition above, then it is called a *topological embedding.* Therefore, an embedding is a topological embedding which is also an immersion (as opposed to simply being a topological embedding).

In the early days of differential geometry there were two approaches to study manifolds. One was the extrinsic view, within which manifolds are defined as special subsets of $\mathbb{R}^d$, and the other was the intrinsic view, which is the view that we have adopted here.

Whitney's theorem, which we will state without proof, states that these two approaches are, in fact, equivalent.

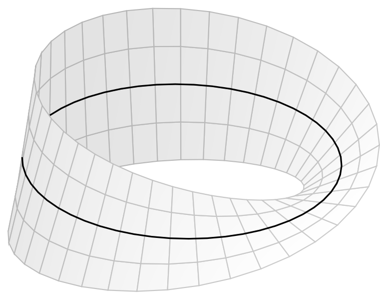
Theorem 5.4 (Whitney)**.** *Any smooth manifold M can be*

- *embedded in $\mathbb{R}^{2\dim M}$;*
- *immersed in $\mathbb{R}^{2\dim M-1}$.*

Example 5.10. The Klein bottle can be embedded in $\mathbb{R}^4$ but not in $\mathbb{R}^3$. It can, however, be immersed in $\mathbb{R}^3$.

5.8 Topological Bundles

While topological products are very useful, very often one intuitively thinks of the product of two manifolds as attaching a copy of the second manifold to each point of the first. However, not all interesting manifolds can be understood as products of manifolds. A classic example of this is the *Möbius strip*.



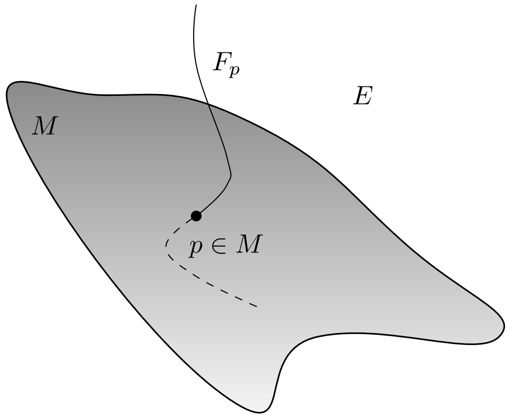
It looks locally like the finite cylinder $S^1 \times [0,1]$, which we can picture as the circle S^1 (the thicker line in figure) with the finite interval $[0,1]$ attached to each of its points in a "smooth" way. The Möbius strip has a "twist", which makes it globally different from the cylinder.

Definition 5.31 (Topological Bundles)**.** *A topological* ***bundle*** *(of topological manifolds) is a triple (E,π,M) where E and M are topological manifolds called the* total space *and the* base space *respectively, and π is a continuous, surjective map $\pi\colon E \to M$ called the* projection map*.*

We will often denote the bundle (E,π,M) by $E \xrightarrow{\pi} M$.

Definition 5.32 (Fiber)**.** *Let $E \xrightarrow{\pi} M$ be a bundle and let $p \in M$. Then, $F_p := \operatorname{preim}_\pi(\{p\})$ is called the* ***fiber*** *at the point p.*

Intuitively, the fiber at the point $p \in M$ is a set of points in E (represented below as a line) attached to the point p. The projection map sends all the points in the fiber F_p to the point p.



Example 5.11. A trivial example of a bundle is the *product bundle*. Let M and N be manifolds. Then, the triple $(M \times N, \pi, M)$, where:

$$\begin{aligned} \pi \colon M \times N &\to M \\ (p, q) &\mapsto p \end{aligned}$$

is a bundle since (one can easily check) π is a continuous and surjective map. Similarly, $(M \times N, \pi, N)$ with the appropriate π, is also a bundle.

Example 5.12. In a bundle, different points of the base manifold may have (topologically) different fibers. For example, consider the bundle $E \xrightarrow{\pi} \mathbb{R}$ where:

$$F_p := \mathrm{preim}_\pi(\{p\}) \cong_{\mathrm{top}} \begin{cases} S^1 & \text{if } p < 0 \\ \{p\} & \text{if } p = 0 \\ [0,1] & \text{if } p > 0 \end{cases}$$

Definition 5.33 (Fiber Bundle)**.** *Let $E \xrightarrow{\pi} M$ be a bundle and let F be a manifold. Then, $E \xrightarrow{\pi} M$ is called a **fiber bundle**, with (typical) fiber F, if:*

$$\forall\, p \in M : \mathrm{preim}_\pi(\{p\}) \cong_{\mathrm{top}} F.$$

A fiber bundle is often represented diagrammatically as:

$$\begin{tikzcd} F \arrow[r] & E \arrow[d, "\pi"] \\ & M \end{tikzcd}$$

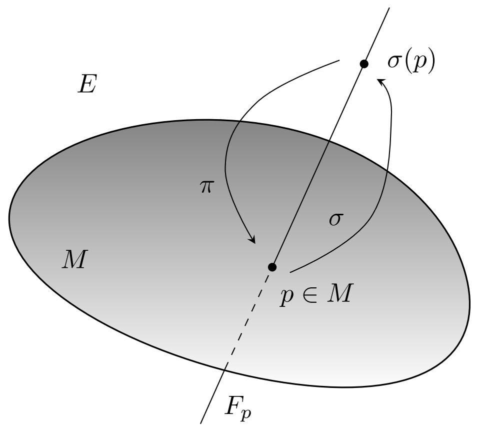
Example 5.13. The bundle $M \times N \xrightarrow{\pi} M$ is a fiber bundle with fiber $F := N$.

Example 5.14. The Möbius strip is a fiber bundle $E \xrightarrow{\pi} S^1$, with fiber $F := [0,1]$, where $E \neq S^1 \times [0,1]$, i.e. the Möbius strip is not a product bundle.

Example 5.15. A $\mathbb{C}$-line bundle over M is the fiber bundle (E, π, M) with fiber $\mathbb{C}$. Note that the product bundle $(M \times \mathbb{C}, \pi, M)$ is a $\mathbb{C}$-line bundle over M, but a $\mathbb{C}$-line bundle over M need not be a product bundle.

Definition 5.34 (Section)**.** *Let $E \xrightarrow{\pi} M$ be a bundle. A map $\sigma \colon M \to E$ is called a **section** of the bundle if $\pi \circ \sigma = \mathrm{id}_M$.*

Intuitively, a section is a map σ which sends each point $p \in M$ to *some* point $\sigma(p)$ in its fiber F_p, so that the projection map π takes $\sigma(p) \in F_p \subseteq E$ back to the point $p \in M$.



Example 5.16. Let $(M \times F, \pi, M)$ be a product bundle. Then, a section of this bundle is a map:

$$\begin{aligned} \sigma\colon M &\to M \times F \\ p &\mapsto (p, s(p)) \end{aligned}$$

where $s\colon M \to F$ is any map.

Definition 5.35 (Sub-Bundle). *A **sub-bundle** of a bundle (E, π, M) is a triple (E', π', M') where $E' \subseteq E$ and $M' \subseteq M$ are submanifolds and $\pi' := \pi|_{E'}$.*

Definition 5.36 (Restricted Bundle). *Let (E, π, M) be a bundle and let $N \subseteq M$ be a submanifold. The **restricted bundle** (to N) is the triple (E, π', N) where:*

$$\pi' := \pi|_{\mathrm{preim}_\pi(N)}$$

Definition 5.37 (Bundle Morphism). *Let $E \xrightarrow{\pi} M$ and $E' \xrightarrow{\pi'} M'$ be bundles and let $u\colon E \to E'$ and $v\colon M \to M'$ be maps. Then (u, v) is called a* bundle morphism *if the following diagram commutes:*

$$\begin{tikzcd} E \arrow[r, "u"] \arrow[d, "\pi"] & E' \arrow[d, "\pi'"] \\ M \arrow[r, "v"] & M' \end{tikzcd}$$

i.e. if $\pi' \circ u = v \circ \pi$.

If (u, v) and (u, v') are both bundle morphisms, then $v = v'$. That is, given u, if there exists v such that (u, v) is a bundle morphism, then v is unique.

Definition 5.38 (Isomorphic Bundles). *Two bundles $E \xrightarrow{\pi} M$ and $E' \xrightarrow{\pi'} M'$ are said to be **isomorphic (as bundles)** if there exist bundle morphisms (u, v) and (u^{-1}, v^{-1}) satisfying:*

$$\begin{tikzcd} E \arrow[r, shift left, "u"] \arrow[d, "\pi"'] & E' \arrow[l, shift left, "u^{-1}"] \arrow[d, "\pi'"] \\ M \arrow[r, shift left, "v"] & M' \arrow[l, shift left, "v^{-1}"] \end{tikzcd}$$

Such a (u, v) is called a bundle isomorphism *and we write $E \xrightarrow{\pi} M \cong_{\mathrm{bdl}} E' \xrightarrow{\pi'} M'$.*

Bundle isomorphisms are the structure-preserving maps for bundles.

Definition 5.39 (Locally Isomorphic Bundles)**.** *A bundle $E \xrightarrow{\pi} M$ is said to be* ***locally isomorphic (as a bundle)*** *to a bundle $E' \xrightarrow{\pi'} M'$ if for all $p \in M$ there exists a neighbourhood $U(p)$ such that the restricted bundle:*

$$\operatorname{preim}_\pi(U(p)) \xrightarrow{\pi|_{\operatorname{preim}_\pi(U(p))}} U(p)$$

is isomorphic to the bundle $E' \xrightarrow{\pi'} M'$.

Definition 5.40 (Trivial / Locally Trivial Bundle)**.** *A bundle $E \xrightarrow{\pi} M$ is said to be:*

i) ***trivial*** *if it is isomorphic to a product bundle;*

ii) ***locally trivial*** *if it is locally isomorphic to a product bundle.*

Example 5.17. The cylinder C is trivial as a bundle, and hence also locally trivial.

Example 5.18. The Möbious strip is not trivial but it is locally trivial.

From now on, we will mostly consider locally trivial bundles.

Remark 5.13. In quantum mechanics, what is usually called a "wave function" is not a function at all, but rather a section of a $\mathbb{C}$-line bundle over physical space. However, if we assume that the $\mathbb{C}$-line bundle under consideration is locally trivial, then each section of the bundle can be represented (locally) by a map from the base space to the total space and hence it is appropriate to use the term "wave *function*".

Definition 5.41 (Pull-Back Bundle)**.** *Let $E \xrightarrow{\pi} M$ be a bundle and let $f\colon M' \to M$ be a map from some manifold M'. The* ***pull-back bundle of*** $E \xrightarrow{\pi} M$ induced by f *is defined as $E' \xrightarrow{\pi'} M'$, where:*

$$E' := \{(m', e) \in M' \times E \mid f(m') = \pi(e)\}$$

and $\pi'(m', e) := m'$.

If $E' \xrightarrow{\pi'} M'$ is the pull-back bundle of $E \xrightarrow{\pi} M$ induced by f, then one can easily construct a bundle morphism by defining:

$$\begin{aligned} u\colon \quad E' &\to E \\ (m', e) &\mapsto e \end{aligned}$$

This corresponds to the diagram:

$$\begin{tikzcd} E' \arrow[r, "u"] \arrow[d, "\pi'"] & E \arrow[d, "\pi"] \\ M' \arrow[r, "f"] & M \end{tikzcd}$$

Remark 5.14. Sections on a bundle pull back to the pull-back bundle. Indeed, let $E' \xrightarrow{\pi'} M'$ be the pull-back bundle of $E \xrightarrow{\pi} M$ induced by f.

$$\begin{tikzcd} E' \arrow[d, shift left, "\pi'"] & E \arrow[d, shift left, "\pi"] \\ M' \arrow[u, shift left, "\sigma'"] \arrow[r, "f"] \arrow[ur, "\sigma\circ f"] & M \arrow[u, shift left, "\sigma"] \end{tikzcd}$$

If σ is a section of $E \xrightarrow{\pi} M$, then $\sigma \circ f$ determines a map from M' to E which sends each $m' \in M'$ to $\sigma(f(m')) \in E$. However, since σ is a section, we have:

$$\pi(\sigma(f(m')) = (\pi \circ \sigma \circ f)(m') = (\mathrm{id}_M \circ f)(m') = f(m')$$

and hence $(m', (\sigma \circ f)(m')) \in E'$ by definition of E'. Moreover:

$$\pi'(m', (\sigma \circ f)(m')) = m'$$

and hence the map:

$$\begin{aligned} \sigma' \colon M' &\to E' \\ m' &\mapsto (m', (\sigma \circ f)(m')) \end{aligned}$$

satisfies $\pi' \circ \sigma' = \mathrm{id}_{M'}$ and it is thus a section on the pull-back bundle $E' \xrightarrow{\pi'} M'$.

The reason of introducing the concept of a topological bundle, is because we need it in order to construct the so called "tangent bundle".

5.9 The Tangent Bundle

Up to this point, we have defined everything on the level of a point on the manifold. However, since we are interested in describing quantities as a whole in the entire manifold, we would like to define a vector field on a manifold M as a "smooth" map that assigns to each $p \in M$ a tangent vector in T_pM. However, since this would then be a "map" to a different space at each point, it is unclear how to define its smoothness.

The simplest solution is to merge all the tangent spaces into a unique set and equip it with a smooth structure, so that we can then define a vector field as a smooth map between smooth manifolds.

Definition 5.42 (Tangent Bundle)**.** *Given a smooth manifold M, the* ***tangent bundle*** *of M is the disjoint union of all the tangent spaces to M, i.e.*

$$TM := \dot{\bigcup}_{p \in M} T_pM,$$

equipped with the canonical projection map

$$\begin{aligned} \pi \colon TM &\to M \\ X &\mapsto p, \end{aligned}$$

where p is the unique $p \in M$ such that $X \in T_pM$.

Since TM is simpy a set (and not a smooth manifold), up to here what we have is a set bundle. In order for this set bundle to turn to a topological bundle as we defined it previously, we need to equip TM with the structure of a smooth manifold. We can achieve this by constructing a smooth atlas for TM from a smooth atlas on M, as follows.

Let $\mathscr{A}_M$ be a smooth atlas on M and let $(U, x) \in \mathscr{A}_M$. If $X \in \mathrm{preim}_\pi(U) \subseteq TM$, then $X \in T_{\pi(X)}M$, by definition of π. Moreover, since $\pi(X) \in U$, we can expand X in terms of the basis induced by the chart (U, x):

$$X = X^a \left(\frac{\partial}{\partial x^a} \right)_{\pi(X)},$$

where $X^1, \ldots, X^{\dim M} \in \mathbb{R}$. We can then define the map

$$\begin{aligned} \xi \colon \mathrm{preim}_\pi(U) &\to x(U) \times \mathbb{R}^{\dim M} \cong_{\mathrm{set}} \mathbb{R}^{2 \dim M} \\ X &\mapsto (x(\pi(X)), X^1, \ldots, X^{\dim M}). \end{aligned}$$

Assuming that TM is equipped with a suitable topology, for instance the initial topology (i.e. the coarsest topology on TM that makes π continuous), we claim that the pair $(\mathrm{preim}_\pi(U), \xi)$ is a chart on TM and

$$\mathscr{A}_{TM} := \{(\mathrm{preim}_\pi(U), \xi) \mid (U, x) \in \mathscr{A}_M\}$$

is a smooth atlas on TM. Note that, from its definition, it is clear that ξ is a bijection. We will not show that $(\mathrm{preim}_\pi(U), \xi)$ is a chart here, but we will show that $\mathscr{A}_{TM}$ is a smooth atlas.

Proposition 5.6. *Any two charts $(\mathrm{preim}_\pi(U), \xi), (\mathrm{preim}_\pi(\widetilde{U}), \widetilde{\xi}) \in \mathscr{A}_{TM}$ are $\mathcal{C}^\infty$-compatible.*

Proof. Let (U, x) and $(\widetilde{U}, \widetilde{x})$ be the two charts on M giving rise to $(\mathrm{preim}_\pi(U), \xi)$ and $(\mathrm{preim}_\pi(\widetilde{U}), \widetilde{\xi})$, respectively. We need to show that the map

$$\widetilde{\xi} \circ \xi^{-1} \colon x(U \cap \widetilde{U}) \times \mathbb{R}^{\dim M} \to \widetilde{x}(U \cap \widetilde{U}) \times \mathbb{R}^{\dim M}$$

is smooth, as a map between open subsets of $\mathbb{R}^{2\dim M}$. Recall that such a map is smooth if, and only if, it is smooth componentwise. On the first $\dim M$ components, $\widetilde{\xi} \circ \xi^{-1}$ acts as

$$\begin{aligned} \widetilde{x} \circ x^{-1} \colon x(U \cap \widetilde{U}) &\to \widetilde{x}(U \cap \widetilde{U}) \\ x(p) &\mapsto \widetilde{x}(p), \end{aligned}$$

while on the remaining $\dim M$ components it acts as the change of vector components we met previously, i.e.

$$X^a \mapsto \widetilde{X}^a = \partial_b(y^a \circ x^{-1})(x(p))\, X^b.$$

Hence, we have

$$\begin{aligned} \widetilde{\xi} \circ \xi^{-1} \colon \qquad x(U \cap \widetilde{U}) \times \mathbb{R}^{\dim M} &\to \widetilde{x}(U \cap \widetilde{U}) \times \mathbb{R}^{\dim M} \\ (x(\pi(X)), X^1, \dots, X^{\dim M}) &\mapsto (\widetilde{x}(\pi(X)), \widetilde{X}^1, \dots, \widetilde{X}^{\dim M}), \end{aligned}$$

which is smooth in each component, and hence smooth. □

The tangent bundle of a smooth manifold M is therefore itself a smooth manifold of dimension $2\dim M$, and the projection $\pi \colon TM \to M$ is smooth with respect to this structure.

Now by using the smooth manifold M as the base space, the smooth manifold TM as the total space, and the smooth projection π we can define the topological tangent bundle as the triple:

$$TM \xrightarrow{\pi} M$$

Similarly, one can construct the *cotangent bundle* T^*M to M by defining

$$T^*M := \dot{\bigcup}_{p \in M} T_p^*M$$

and going through the above again, using the dual basis $\{(\mathrm{d}x^a)_p\}$ instead of $\{\big(\frac{\partial}{\partial x^a}\big)_p\}$.

5.10 Vector, Covector And Tensor Fields

Now that we have defined the tangent and cotangent bundles, we are ready to define fields.

Definition 5.43 (Vector Field)**.** *Let M be a smooth manifold, and let $TM \xrightarrow{\pi} M$ be its tangent bundle. A **vector field** σ on M is a smooth section of the tangent bundle, i.e. a smooth map $\sigma \colon M \to TM$ such that $\pi \circ \sigma = \mathrm{id}_M$.*

$$\begin{tikzcd} TM \arrow[d, shift left, "\pi"] \\ M \arrow[u, shift left, "\sigma"] \end{tikzcd}$$

Definition 5.44 ($\Gamma(TM)$)**.** *We denote the set of all vector fields on M by $\Gamma(TM)$, i.e.*

$$\Gamma(TM) := \{\sigma \colon M \to TM \mid \sigma \textit{ is smooth and } \pi \circ \sigma = \mathrm{id}_M\}.$$

This is, in fact, the standard notation for the set of all sections on a bundle.

Remark 5.15. An equivalent definition is that a vector field σ on M is a derivation on the algebra $\mathcal{C}^\infty(M)$, i.e. an $\mathbb{R}$-linear map

$$\sigma \colon \mathcal{C}^\infty(M) \xrightarrow{\sim} \mathcal{C}^\infty(M)$$

satisfying the Leibniz rule (with respect to pointwise multiplication on $\mathcal{C}^\infty(M)$)

$$\sigma(fg) = g\,\sigma(f) + f\,\sigma(g).$$

This definition is better suited for some purposes, and later on we will switch from one to the other without making any notational distinction between them.

We can equip the set $\Gamma(TM)$ with the following operations. The first is our, by now familiar, pointwise addition:

$$\begin{aligned} \oplus\colon \Gamma(TM) \times \Gamma(TM) &\to \Gamma(TM) \\ (\sigma, \tau) &\mapsto \sigma \oplus \tau, \end{aligned}$$

where

$$\begin{aligned} \sigma \oplus \tau\colon M &\to \Gamma(TM) \\ p &\mapsto (\sigma \oplus \tau)(p) := \sigma(p) + \tau(p). \end{aligned}$$

Note that the $+$ on the right hand side above is the addition in T_pM.

More interestingly, we can define a multiplication operation not by a simple number (i.e an element of $\mathbb{R}$) but with a whole function (i.e an element of $\mathcal{C}^\infty(M)$) as follows:

$$\begin{aligned} \odot\colon \mathcal{C}^\infty(M) \times \Gamma(TM) &\to \Gamma(TM) \\ (f, \sigma) &\mapsto f \odot \sigma, \end{aligned}$$

where

$$\begin{aligned} f \odot \sigma\colon M &\to \Gamma(TM) \\ p &\mapsto (f \odot \sigma)(p) := f(p)\sigma(p). \end{aligned}$$

Note that since $f \in \mathcal{C}^\infty(M)$, we have $f(p) \in \mathbb{R}$ and hence the multiplication above is the scalar multiplication on T_pM.

Remark 5.16. Of course, we could have defined $\odot$ simply as pointwise *global* scaling, using the reals $\mathbb{R}$ instead of the real functions $\mathcal{C}^\infty(M)$. Then, since $(\mathbb{R}, +, \cdot)$ is an algebraic field, we would then have the obvious $\mathbb{R}$-vector space structure on $\Gamma(TM)$. There are two reasons why we don't do that:

- Since the vector field acts on the whole manifold M (it assigns a value $f(p)$ on every point p of the manifold) we want to be able to assign different values to different points. Otherwise we would only be able to assign the same value to every point (i.e having a constant vector field)
- A basis for the corresponding vector space would be necessarily uncountably infinite, and hence it would not provide a very useful decomposition for our vector fields. Instead, the operation $\odot$ that we have defined allows for *local* scaling, i.e. we can scale a vector field by a different value at each point, and a much more useful decomposition of vector fields.

The question now is, mathematically speaking, what exactly the triple $(\Gamma(TM), \oplus, \odot)$ is. Its nature of course depends on what the triple $(\mathcal{C}^\infty(M), +, \cdot)$ is. Let's recall that the triple $(\mathcal{C}^\infty(M), +, \cdot)$ can be viewed in two different ways:

- $(\mathcal{C}^\infty(M), +, \cdot)$, where $\cdot$ is scalar multiplication (by a real number), is an $\mathbb{R}$-vector space.
- $(\mathcal{C}^\infty(M), +, \bullet)$, where $\bullet$ is pointwise multiplication of maps, is a commutative, unital ring, but not a division ring since not every function has an inverse at every point (i.e at all points that a function is zero, we cannot define an inverse since we would divide by zero).

The first view is of no use since if the triple is seen as a vector space over the real numbers, there is nothing else we can do. However, if we consider the second view. i.e the triple $(\mathcal{C}^\infty(M), +, \bullet)$, where $\bullet$ is pointwise function multiplication as a ring, then the triple $(\Gamma(TM), \oplus, \odot)$ built on top of this ring satisfies

- $(\Gamma(TM), \oplus)$ is an abelian group, with $0 \in \Gamma(TM)$ being the section that maps each $p \in M$ to the zero tangent vector in T_pM;
- $\Gamma(TM) \setminus \{0\}$ satisfies:
  i) $\forall f \in \mathcal{C}^\infty(M) : \forall \sigma, \tau \in \Gamma(TM) \setminus \{0\} : f \odot (\sigma \oplus \tau) = (f \odot \sigma) \oplus (f \odot \tau)$;
  ii) $\forall f, g \in \mathcal{C}^\infty(M) : \forall \sigma \in \Gamma(TM) \setminus \{0\} : (f + g) \odot \sigma = (f \odot \sigma) \oplus (g \odot \sigma)$;
  iii) $\forall f, g \in \mathcal{C}^\infty(M) : \sigma \in \Gamma(TM) \setminus \{0\} : (f \bullet g) \odot \sigma = f \odot (g \odot \sigma)$;
  iv) $\forall \sigma \in \Gamma(TM) \setminus \{0\} : 1 \odot \sigma = \sigma$,

  where $1 \in \mathcal{C}^\infty(M)$ maps every $p \in M$ to $1 \in \mathbb{R}$.

which are precisely the axioms for a vector space! Hence given that the triple $(\mathcal{C}^\infty(M), +, \bullet)$ is a ring, that turns the triple $(\Gamma(TM), \oplus, \odot)$ to a $\mathcal{C}^\infty(M)$-module.

And this of course is of crucial importance since as we showed in previous chapters, if a ring R is not a division ring, then a R-module does not need to have a basis. And since as we already said $(\mathcal{C}^\infty(M), +, \bullet)$ is not a division ring, the vector fields as $\mathcal{C}^\infty(M)$-modules do not need to have a basis! And this is a shame, since if they would have a basis (let's say X_i) we would be able to write a vector field σ as:

$$\sigma = \sigma^i X_i$$

where σ^i would be functions acting as components of the vector field!

In a similar manner one can construct a covector field through the use of the cotangent bundle, and from there to define the set of all covector fields $\Gamma(T^*M)$ and subsequently a triple $(\Gamma(T^*M), \oplus, \odot)$.

Finally using $\Gamma(TM)$ and $\Gamma(T^*M)$ we can define a tensor field.

Definition 5.45 (Tensor Field)**.** *Let M be a smooth manifold. A smooth (r, s)* ***tensor field*** *τ on M is a $\mathcal{C}^\infty(M)$-multilinear map*

$$\tau\colon \underbrace{\Gamma(T^*M) \times \cdots \times \Gamma(T^*M)}_{r \text{ copies}} \times \underbrace{\Gamma(TM) \times \cdots \times \Gamma(TM)}_{s \text{ copies}} \to \mathcal{C}^\infty(M).$$

The equivalence of this to the bundle definition is due to the pointwise nature of tensors. For instance, a covector field $\omega \in \Gamma(T^*M)$ can act on a vector field $X \in \Gamma(TM)$ to yield a smooth function $\omega(X) \in \mathcal{C}^\infty(M)$ by

$$(\omega(X))(p) := \omega(p)(X(p)).$$

Then, we see that for any $f \in \mathcal{C}^\infty(M)$, we have

$$(\omega(fX))(p) = \omega(p)(f(p)X(p)) = f(p)\omega(p)(X(p)) =: (f\omega(X))(p)$$

and hence, the map $\omega\colon \Gamma(TM) \xrightarrow{\sim} \mathcal{C}^\infty(M)$ is $\mathcal{C}^\infty(M)$-linear.

Similarly, the set $\Gamma(T^r_sM)$ of all (r, s) smooth tensor fields on M can be made into a $\mathcal{C}^\infty(M)$-module, with module operations defined pointwise.

We can also define the tensor product of tensor fields

$$\begin{aligned} \otimes\colon \Gamma(T^p_qM) \times \Gamma(T^r_sM) &\to \Gamma(T^{p+r}_{q+s}M) \\ (\tau, \sigma) &\mapsto \tau \otimes \sigma \end{aligned}$$

analogously to what we had with tensors on a vector space, i.e.

$$\begin{aligned} &(\tau \otimes \sigma)(\omega_1, \ldots, \omega_p, \omega_{p+1}, \ldots, \omega_{p+r}, X_1, \ldots, X_q, X_{q+1}, \ldots, X_{q+s}) \\ &\qquad := \tau(\omega_1, \ldots, \omega_p, X_1, \ldots, X_q)\, \sigma(\omega_{p+1}, \ldots, \omega_{p+r}, X_{q+1}, \ldots, X_{q+s}), \end{aligned}$$

with $\omega_i \in \Gamma(T^*M)$ and $X_i \in \Gamma(TM)$.

Therefore, we can think of tensor fields on M either as sections of some tensor bundle on M, that is, as maps assigning to each $p \in M$ a tensor ($\mathbb{R}$-multilinear map) on the vector space T_pM, or as a $\mathcal{C}^\infty(M)$-multilinear map as above. We will always try to pick the most useful or easier to understand, based on the context.

To summarize, fields are the generalization of the definitions of vectors, covectors and tensors at a specific point p of M, to every possible point p of manifold M, hence to the whole manifold M. In a similar way we can generalize the concept of the gradient of f at $p \in M$ in the gradient of f at M.

Recall the definition of the gradient operator at a point $p \in M$. We can extend that definition to define the ($\mathbb{R}$-linear) operator:

$$\begin{aligned} \mathrm{d} \colon \mathcal{C}^\infty(M) &\xrightarrow{\sim} \Gamma(T^*M) \\ f &\mapsto \mathrm{d}f \end{aligned}$$

where, of course, $\mathrm{d}f \colon p \mapsto \mathrm{d}_p f$. Alternatively, we can think of $\mathrm{d}f$ as the $\mathbb{R}$-linear map

$$\begin{aligned} \mathrm{d}f \colon \Gamma(TM) &\xrightarrow{\sim} \mathcal{C}^\infty(M) \\ X &\mapsto \mathrm{d}f(X) = X(f). \end{aligned}$$

Remark 5.17. Locally on some chart (U, x) on M, the covector field $\mathrm{d}f$ can be expressed as

$$\mathrm{d}f = \lambda_a \, \mathrm{d}x^a$$

for some smooth functions $\lambda_i \in \mathcal{C}^\infty(U)$. To determine what they are, we simply apply both sides to the vector fields induced by the chart. We have

$$\mathrm{d}f\left(\frac{\partial}{\partial x^b}\right) = \frac{\partial}{\partial x^b}(f) = \partial_b f$$

and

$$\lambda_a \, \mathrm{d}x^a \left(\frac{\partial}{\partial x^b}\right) = \lambda_a \frac{\partial}{\partial x^b}(x^a) = \lambda_a \, \delta^a_b = \lambda_b.$$

Hence, the local expression of $\mathrm{d}f$ on (U, x) is

$$\mathrm{d}f = \partial_a f \, \mathrm{d}x^a.$$

Note that the operator d satisfies the Leibniz rule

$$\mathrm{d}(fg) = g \, \mathrm{d}f + f \, \mathrm{d}g.$$

Finally, we want to generalize the concepts of push-forward and pull-back from a point to the whole manifold (a.k.a from a vector/covector to a vector/covector field). For a good reason we will first start with the pull-back.

To avoid confusion, for this part we will denote a covector as W and a covector field as ω. Recall that given a smooth map $\phi \colon M \to N$ the definition of a pull-back for a covector was

$$\begin{aligned} (\phi^*)_p \colon T^*_{\phi(p)}N &\xrightarrow{\sim} T^*_pM \\ W &\mapsto (\phi^*)_p(W), \end{aligned}$$

where $(\phi^*)_p(W)$ is defined as

$$\begin{aligned} (\phi^*)_p(W) \colon T_pM &\xrightarrow{\sim} \mathbb{R} \\ X &\mapsto (\phi^*)_p(W)(X) := W((\phi_*)_p(X)). \end{aligned}$$

Now we can simply extend the definition of a pull-back for a covector at point p denoted $(\phi^*)_p$, to this of a pull-back for a covector field on a manifold M denoted ϕ^*, by simpy acting with $(\phi^*)_p$ at every point

p of the manifold M

$$\begin{aligned} \phi^*\colon T^*N &\to T^*M \\ \omega &\mapsto \phi^*(\omega) \end{aligned}$$

where now ω is a covector field and not just a covector, and $\phi^*(\omega)$ is defined for every point p of M as

$$\phi^*(\omega)(p) := (\phi^*)_p(W)$$

where W is the the corresponding covector that the covector field σ produces at point p. It is a common thing to this point to drop the p from the pull-back of covectors and simply write:

$$(\phi^*\omega)|_p := \phi^*(W|_p)$$

which actually means that the pull-back of a covector field evaluated at point p is equal to the pull-back of the covector $W|_p$ generated by the covector field ω at point p (a.k.a $W|_p = \omega(p)$) . From now on we will be using this equation to switch from covector field to covector equations, although practically it's the same thing from a different perspective.

While the pull-back can be extended from covectros to covector fields without problems, the push-forward of a vector cannot be generalized to the push-forward of a vector field unless the underlying map ϕ is a diffeomorphism between the manifold M and N. Let's see why.

Let's start again with the pull-back that we have already defined. Observe that the pull back of a covector field includes the notion of the pull-back of a covector at a point p. Now, the map ϕ, as a map, maps every single point of its domain M to a single point of its target N. Hence the whole target M is hit by the map, but the whole target N is not (recall that the part of N that is hit by the map is called the image of ϕ). This means that in the case of a pull-back (after we have defined the tangent vectors in both M and N) every single point of the image of ϕ on N will have a corresponding point back on M hence the definition of the pull-back of a covector field will be well-defined.

On the other hand, in the case of a push forward we get two problems coming from the fact that, in general, the map ϕ may not be neither surjective nor injective. First of all if the map ϕ is not surjective that means that the image of ϕ is not equal to the entire domain M ($\mathrm{im}_\phi(M) \neq N$), hence from a vector field defined on M we will never be able to define a vector field on N that lies outside the image of ϕ. Moreover, if ϕ is not injective, that means that distinct elements of the domain M are mapped to the same element in the target N hence it might be the case that the push-forward will create many different vectors for one given point p on N which will make it ill-defined.

Of course, if the map ϕ is both surjective and injective, hence bijective, hence has an inverse, then none of this problems exist any more, since then the case is similar to the case of pull-backs (both directions of the map behave similarly). But recall that a bijection between topological spaces is called a "homeomorphism", and moreover if the map is smooth (which in the case of smooth manifolds by definition always is) then the smooth "homeomorphism" is called a"diffeomorphism".

So we ended up to our initial conclusion that the push-forward of a vector can be generalized to the push-forward of a vector field only if the underlying map ϕ is a diffeomorphism between the manifold M and N.

Then we can simply follow the same procedure as with the pull-back and define the push-forward of a vector field by simpy acting with the push-forward at every point $\phi(p)$ on the manifold N. Recall that the push-forward of ϕ at $p \in M$ is the linear map:

$$\begin{aligned} (\phi_*)_p\colon T_pM &\xrightarrow{\sim} T_{\phi(p)}N \\ X &\mapsto (\phi_*)_p(X) \end{aligned}$$

where $(\phi_*)_p(X)$ is defined as

$$\begin{aligned} (\phi_*)_p(X)\colon \mathcal{C}^\infty(N) &\xrightarrow{\sim} \mathbb{R} \\ f &\mapsto (\phi_*)_p(X)f := X(f\circ\phi). \end{aligned}$$

Now we can simply extend this definition to a push-forward for a vector field by simpy acting with the push-forward at every point p of the manifold M

$$\begin{aligned} \phi_*\colon TM &\to TN \\ \sigma &\mapsto \phi_*(\sigma) \end{aligned}$$

where $\phi_*(\sigma)$ is defined for every point $\phi(p)$ of N as

$$\phi_*(\sigma)(\phi(p)) := (\phi_*)_p(X)$$

As with covectors, it is a common thing to this point to drop the p from the equation and simply write:

$$(\phi_*\sigma)|_{\phi(p)} := \phi_*(X|_p)$$

which actually means that the push-forward of a vector field evaluated at point $\phi(p)$ is equal to the push-forward of the vector $X|_p$ generated by the vector field σ at point p (a.k.a $X|_p = X(p)$). From now on we will be using this equation to switch from vector field to vector equations, although practically it's the same thing from a different perspective.

5.11 Differential Forms

Definition 5.46 (Differential Form)**.** *Let M be a smooth manifold. A **(differential) n-form** on M is a $(0, n)$ smooth tensor field ω which is totally antisymmetric, i.e.*

$$\omega(X_1, \dots, X_n) = \operatorname{sgn}(\pi)\, \omega(X_{\pi(1)}, \dots, X_{\pi(n)}),$$

for any $\pi \in S_n$, with $X_i \in \Gamma(TM)$. We call n the degree *of the form.*

Alternatively, we can define a differential form as a smooth section of the appropriate bundle on M, i.e. as a map assigning to each $p \in M$ an n-form on the vector space T_pM.

Of course, by definition, differential forms are nothing more but a very specific kind of tensors, hence it's a subset of the tensor space.

Example 5.19. The electromagnetic field strength F is a differential 2-form built from the electric and magnetic fields, which are also taken to be forms. We will define these later in some detail.

Definition 5.47 ($\Omega^n(M)$)**.** *We denote by $\Omega^n(M)$ the set of all differential n-forms on M, which then becomes a $\mathcal{C}^\infty(M)$-module by defining the addition and multiplication operations pointwise.*

We have $\Omega^0(M) \equiv \mathcal{C}^\infty(M)$ since they are $(0, 0)$ tensors a.k.a functions and $\Omega^1(M) \equiv \Gamma(T^0_1M) \equiv \Gamma(T^*M)$ sincey they are $(0, 1)$ tensors a.k.a covectors.

We can specialise the pull-back of tensors to differential forms.

Definition 5.48 (Pull-Back On Differential Forms)**.** *Let $\phi\colon M \to N$ be a smooth map and let $\omega \in \Omega^n(N)$. Then we define the **pull-back** $\Phi^*(\omega) \in \Omega^n(M)$ of ω as*

$$\begin{aligned} \Phi^*(\omega)\colon M &\to T^*M \\ p &\mapsto \Phi^*(\omega)(p), \end{aligned}$$

where

$$\Phi^*(\omega)(p)(X_1, \dots, X_n) := \omega(\phi(p))\big(\phi_*(X_1), \dots, \phi_*(X_n)\big),$$

for $X_i \in T_pM$.

The map $\Phi^*\colon \Omega^n(N) \to \Omega^n(M)$ is $\mathbb{R}$-linear, and its action on $\Omega^0(M)$ is simply

$$\begin{aligned} \Phi^*\colon \Omega^0(M) &\to \Omega^0(M) \\ f &\mapsto \Phi^*(f) := f \circ \phi. \end{aligned}$$

This works for any smooth map ϕ, and it leads to a slight modification of our mantra:

Vectors are pushed forward,
forms are pulled back.

The tensor product $\otimes$ does not interact well with forms, since the tensor product of two forms is not necessarily a form (it might be, for example, a symmetric $(0, n)$ tensor which, be definition, is not a form). Hence, we define the following.

Definition 5.49 (Wedge Product). *Let M be a smooth manifold. We define the* ***wedge*** *(or* exterior*)* product *of forms as the map*

$$\begin{aligned} \wedge \colon \Omega^n(M) \times \Omega^m(M) &\to \Omega^{n+m}(M) \\ (\omega, \sigma) &\mapsto \omega \wedge \sigma, \end{aligned}$$

where

$$(\omega \wedge \sigma)(X_1, \ldots, X_{n+m}) := \frac{1}{n!\, m!} \sum_{\pi \in S_{n+m}} \operatorname{sgn}(\pi)(\omega \otimes \sigma)(X_{\pi(1)}, \ldots, X_{\pi(n+m)})$$

and $X_1, \ldots, X_{n+m} \in \Gamma(TM)$. By convention, for any $f, g \in \Omega^0(M)$ and $\omega \in \Omega^n(M)$, we set

$$f \wedge g := fg \qquad \text{and} \qquad f \wedge \omega = \omega \wedge f = f\omega.$$

Example 5.20. Suppose that $\omega, \sigma \in \Omega^1(M)$. Then, for any $X, Y \in \Gamma(TM)$

$$\begin{aligned} (\omega \wedge \sigma)(X, Y) &= (\omega \otimes \sigma)(X, Y) - (\omega \otimes \sigma)(Y, X) \\ &= (\omega \otimes \sigma)(X, Y) - \omega(Y)\sigma(X) \\ &= (\omega \otimes \sigma)(X, Y) - (\sigma \otimes \omega)(X, Y) \\ &= (\omega \otimes \sigma - \sigma \otimes \omega)(X, Y). \end{aligned}$$

Hence

$$\omega \wedge \sigma = \omega \otimes \sigma - \sigma \otimes \omega.$$

The wedge product is bilinear over $\mathcal{C}^\infty(M)$, that is

$$(f\omega_1 + \omega_2) \wedge \sigma = f\, \omega_1 \wedge \sigma + \omega_2 \wedge \sigma,$$

for all $f \in \mathcal{C}^\infty(M)$, $\omega_1, \omega_2 \in \Omega^n(M)$ and $\sigma \in \Omega^m(M)$, and similarly for the second argument.

Remark 5.18. If (U, x) is a chart on M, then every n-form $\omega \in \Omega^n(U)$ can be expressed locally on U as

$$\omega = \omega_{a_1 \cdots a_n} \, \mathrm{d}x^{a_1} \wedge \cdots \wedge \mathrm{d}x^{a_n},$$

where $\omega_{a_1 \cdots a_n} \in \mathcal{C}^\infty(U)$ and $1 \leq a_1 < \cdots < a_n \leq \dim M$. The $\mathrm{d}x^{a_i}$ appearing above are the covector fields (1-forms)

$$\mathrm{d}x^{a_i} \colon p \mapsto \mathrm{d}_p x^{a_i}.$$

The pull-back distributes over the wedge product.

Theorem 5.5. *Let $\phi \colon M \to N$ be smooth, $\omega \in \Omega^n(N)$ and $\sigma \in \Omega^m(N)$. Then, we have*

$$\Phi^*(\omega \wedge \sigma) = \Phi^*(\omega) \wedge \Phi^*(\sigma).$$

Proof. Let $p \in M$ and $X_1, \dots, X_{n+m} \in T_pM$. Then we have

$$\begin{aligned}
\big(\Phi^*(\omega) \wedge \Phi^*(\sigma)\big)(p)(X_1, \dots, X_{n+m}) \\
&= \frac{1}{n!\,m!} \sum_{\pi \in S_{n+m}} \operatorname{sgn}(\pi)\big(\Phi^*(\omega) \otimes \Phi^*(\sigma)\big)(p)(X_{\pi(1)}, \dots, X_{\pi(n+m)}) \\
&= \frac{1}{n!\,m!} \sum_{\pi \in S_{n+m}} \operatorname{sgn}(\pi)\Phi^*(\omega)(p)(X_{\pi(1)}, \dots, X_{\pi(n)}) \\
&\qquad\qquad\qquad\qquad \Phi^*(\sigma)(p)(X_{\pi(n+1)}, \dots, X_{\pi(n+m)}) \\
&= \frac{1}{n!\,m!} \sum_{\pi \in S_{n+m}} \operatorname{sgn}(\pi)\omega(\phi(p))\big(\phi_*(X_{\pi(1)}), \dots, \phi_*(X_{\pi(n)})\big) \\
&\qquad\qquad\qquad\qquad \sigma(\phi(p))\big(\phi_*(X_{\pi(n+1)}), \dots, \phi_*(X_{\pi(n+m)})\big) \\
&= \frac{1}{n!\,m!} \sum_{\pi \in S_{n+m}} \operatorname{sgn}(\pi)(\omega \otimes \sigma)(\phi(p))\big(\phi_*(X_{\pi(1)}), \dots, \phi_*(X_{\pi(n+m)})\big) \\
&= (\omega \wedge \sigma)(\phi(p))\big(\phi_*(X_1), \dots, \phi_*(X_{n+m})\big) \\
&= \Phi^*(\omega \wedge \sigma)(p)(X_1, \dots, X_{n+m}).
\end{aligned}$$

Since $p \in M$ was arbitrary, the statement follows. □

5.11.1 The Grassmann Algebra

Note that the wedge product takes two differential forms and produces a differential form of a different type. It would be much nicer to have a space which is closed under the action of $\wedge$. In fact, such a space exists and it is called the Grassmann algebra of M.

Definition 5.50 (Grassmann Algebra)**.** *Let M be a smooth manifold. Define the $\mathcal{C}^\infty(M)$-module*

$$\operatorname{Gr}(M) \equiv \Omega(M) := \bigoplus_{n=0}^{\dim M} \Omega^n(M).$$

*The **Grassmann algebra** on M is the algebra $(\Omega(M), +, \cdot, \wedge)$, where*

$$\wedge \colon \Omega(M) \times \Omega(M) \to \Omega(M)$$

is the linear continuation of the previously defined $\wedge \colon \Omega^n(M) \times \Omega^m(M) \to \Omega^{n+m}(M)$.

Recall that the direct sum of modules has the Cartesian product of the modules as underlying set and module operations defined componentwise. Also, note that by "algebra" here we really mean "algebra over a module".

Example 5.21. Let $\psi = \omega + \sigma$, where $\omega \in \Omega^1(M)$ and $\sigma \in \Omega^3(M)$. Of course, this "+" is neither the addition on $\Omega^1(M)$ nor the one on $\Omega^3(M)$, but rather that on $\Omega(M)$ and, in fact, $\psi \in \Omega(M)$.

Let $\varphi \in \Omega^n(M)$, for some n. Then

$$\varphi \wedge \psi = \varphi \wedge (\omega + \sigma) = \varphi \wedge \omega + \varphi \wedge \sigma,$$

where $\varphi \wedge \omega \in \Omega^{n+1}(M)$, $\varphi \wedge \sigma \in \Omega^{n+3}(M)$, and $\varphi \wedge \psi \in \Omega(M)$.

Example 5.22. There is a lot of talk about *Grassmann numbers*, particularly in supersymmetry. One often hears that these are "numbers that do not commute, but anticommute". Of course, objects cannot be commutative or anticommutative by themselves. These qualifiers only apply to operations on the objects. In fact, the Grassmann numbers are just the elements of a Grassmann algebra.

The following result is about the anticommutative behaviour of $\wedge$.

Theorem 5.6. *Let $\omega \in \Omega^n(M)$ and $\sigma \in \Omega^m(M)$. Then*

$$\omega \wedge \sigma = (-1)^{nm}\, \sigma \wedge \omega.$$

We say that $\wedge$ is *graded commutative*, that is, it satisfies a version of anticommutativity which depends on the degrees of the forms.

Proof. First note that if $\omega, \sigma \in \Omega^1(M)$, then

$$\omega \wedge \sigma = \omega \otimes \sigma - \sigma \otimes \omega = -\,\sigma \wedge \omega.$$

Recall that is $\omega \in \Omega^n(M)$ and $\sigma \in \Omega^m(M)$, then locally on a chart (U, x) we can write

$$\begin{aligned}\omega &= \omega_{a_1 \cdots a_n} \mathrm{d}x^{a_1} \wedge \cdots \wedge \mathrm{d}x^{a_n}\\ \sigma &= \sigma_{b_1 \cdots b_m} \mathrm{d}x^{b_1} \wedge \cdots \wedge \mathrm{d}x^{b_m}\end{aligned}$$

with $1 \leq a_1 < \cdots < a_n \leq \dim M$ and similarly for the b_i. The coefficients $\omega_{a_1 \cdots a_n}$ and $\sigma_{b_1 \cdots b_m}$ are smooth functions in $\mathcal{C}^\infty(U)$. Since $\mathrm{d}x^{a_i}, \mathrm{d}x^{b_j} \in \Omega^1(M)$, we have

$$\begin{aligned}\omega \wedge \sigma &= \omega_{a_1 \cdots a_n} \sigma_{b_1 \cdots b_m}\, \mathrm{d}x^{a_1} \wedge \cdots \wedge \mathrm{d}x^{a_n} \wedge \mathrm{d}x^{b_1} \wedge \cdots \wedge \mathrm{d}x^{b_m}\\ &= (-1)^n\, \omega_{a_1 \cdots a_n} \sigma_{b_1 \cdots b_m}\, \mathrm{d}x^{b_1} \wedge \mathrm{d}x^{a_1} \wedge \cdots \wedge \mathrm{d}x^{a_n} \wedge \mathrm{d}x^{b_2} \wedge \cdots \wedge \mathrm{d}x^{b_m}\\ &= (-1)^{2n}\, \omega_{a_1 \cdots a_n} \sigma_{b_1 \cdots b_m}\, \mathrm{d}x^{b_1} \wedge \mathrm{d}x^{b_2} \wedge \mathrm{d}x^{a_1} \wedge \cdots \wedge \mathrm{d}x^{a_n} \wedge \mathrm{d}x^{b_3} \wedge \cdots \wedge \mathrm{d}x^{b_m}\\ &\;\;\vdots\\ &= (-1)^{nm}\, \omega_{a_1 \cdots a_n} \sigma_{b_1 \cdots b_m}\, \mathrm{d}x^{b_1} \wedge \cdots \wedge \mathrm{d}x^{b_m} \wedge \mathrm{d}x^{a_1} \wedge \cdots \wedge \mathrm{d}x^{a_n}\\ &= (-1)^{nm}\, \sigma \wedge \omega\end{aligned}$$

since we have swapped 1-forms nm-many times. □

Remark 5.19. We should stress that this is only true when ω and σ are pure degree forms, rather than linear combinations of forms of different degrees. Indeed, if $\varphi, \psi \in \Omega(M)$, a formula like

$$\varphi \wedge \psi = \cdots \psi \wedge \varphi$$

does not make sense in principle, because the different parts of φ and ψ can have different commutation behaviours.

5.11.2 The Exterior Derivative

Recall the "extended" definition of the gradient operator of a function d on the whole manifold M:

$$\begin{aligned}\mathrm{d}\colon \mathcal{C}^\infty(M) &\xrightarrow{\sim} \Gamma(T^*M)\\ f &\mapsto \mathrm{d}f\end{aligned}$$

Since $\Omega^0(M) \equiv \mathcal{C}^\infty(M)$ and $\Omega^1(M) \equiv \Gamma(T^0_1 M) \equiv \Gamma(T^*M)$, we can also understand this as an operator that takes in 0-forms and outputs 1-forms

$$\mathrm{d}\colon \Omega^0(M) \xrightarrow{\sim} \Omega^1(M).$$

This can then be extended to an operator which acts on any n-form. For this definition, we need to remind ourselves of the definition of commutator we gave in the algebra section of the notes. More precisely, if M is a smooth manifold and $X, Y \in \Gamma(TM)$ then the commutator (or Lie bracket) of X and Y is defined as

$$\begin{aligned}[X, Y]\colon \mathcal{C}^\infty(M) &\xrightarrow{\sim} \mathcal{C}^\infty(M)\\ f &\mapsto [X, Y](f) := X(Y(f)) - Y(X(f)),\end{aligned}$$

where we are using the definition of vector fields as $\mathbb{R}$-linear maps $\mathcal{C}^\infty(M) \xrightarrow{\sim} \mathcal{C}^\infty(M)$.

Using the commutator we can now extend the gradient as follows:

Definition 5.51 (Exterior Derivative)**.** *The* ***exterior derivative*** *on M is the $\mathbb{R}$-linear operator*

$$\begin{aligned}\mathrm{d}\colon \Omega^n(M) &\xrightarrow{\sim} \Omega^{n+1}(M)\\ \omega &\mapsto \mathrm{d}\omega\end{aligned}$$

with $\mathrm{d}\omega$ *being defined as*

$$\begin{aligned}\mathrm{d}\omega(X_1,\dots,X_{n+1}) := {}&\sum_{i=1}^{n+1}(-1)^{i+1}\,X_i\big(\omega(X_1,\dots,\widehat{X_i},\dots,X_{n+1})\big)\\ &+\sum_{i<j}(-1)^{i+j}\,\omega\big([X_i,X_j],X_1,\dots,\widehat{X_i},\dots,\widehat{X_j},\dots,X_{n+1}\big),\end{aligned}$$

where $X_i \in \Gamma(TM)$ *and the hat denotes omissions.*

Remark 5.20. Note that the operator d is only well-defined when it acts on forms. In order to define a derivative operator on general tensors we will need to add extra structure to our differentiable manifold.

Example 5.23. In the case $n = 1$, the form $\mathrm{d}\omega \in \Omega^2(M)$ is given by

$$\mathrm{d}\omega(X,Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y]).$$

Let us check that this is indeed a 2-form, i.e. an antisymmetric, $\mathcal{C}^\infty(M)$-multilinear map

$$\mathrm{d}\omega\colon \Gamma(TM)\times\Gamma(TM)\to\mathcal{C}^\infty(M).$$

By using the antisymmetry of the Lie bracket, we immediately get

$$\mathrm{d}\omega(X,Y) = -\,\mathrm{d}\omega(Y,X).$$

Moreover, thanks to this identity, it suffices to check $\mathcal{C}^\infty(M)$-linearity in the first argument only. Additivity is easily checked

$$\begin{aligned}\mathrm{d}\omega(X_1+X_2,Y) &= (X_1+X_2)(\omega(Y)) - Y(\omega(X_1+X_2)) - \omega([X_1+X_2,Y])\\ &= X_1(\omega(Y)) + X_2(\omega(Y)) - Y(\omega(X_1)+\omega(X_2)) - \omega([X_1,Y]+[X_2,Y])\\ &= X_1(\omega(Y)) + X_2(\omega(Y)) - Y(\omega(X_1)) - Y(\omega(X_2)) - \omega([X_1,Y]) - \omega([X_2,Y])\\ &= \mathrm{d}\omega(X_1,Y) + \mathrm{d}\omega(X_2,Y).\end{aligned}$$

For $\mathcal{C}^\infty(M)$-scaling, first we calculate $[fX,Y]$. Let $g\in\mathcal{C}^\infty(M)$. Then

$$\begin{aligned}[fX,Y](g) &= fX(Y(g)) - Y(fX(g))\\ &= fX(Y(g)) - fY(X(g)) - Y(f)X(g)\\ &= f(X(Y(g)) - Y(X(g))) - Y(f)X(g)\\ &= f[X,Y](g) - Y(f)X(g)\\ &= (f[X,Y] - Y(f)X)(g).\end{aligned}$$

Therefore

$$[fX,Y] = f[X,Y] - Y(f)X.$$

Hence, we can calculate

$$\begin{aligned}\mathrm{d}\omega(fX,Y) &= fX(\omega(Y)) - Y(\omega(fX)) - \omega([fX,Y])\\ &= fX(\omega(Y)) - Y(f\omega(X)) - \omega(f[X,Y] - Y(f)X)\\ &= fX(\omega(Y)) - fY(\omega(X)) - Y(f)\omega(X) - f\omega([X,Y]) + \omega(Y(f)X)\\ &= fX(\omega(Y)) - fY(\omega(X)) - \not{[}gray]Y(f)\omega(X) - f\omega([X,Y]) + \not{[}gray]Y(f)\omega(X)\\ &= f\,\mathrm{d}\omega(X,Y),\end{aligned}$$

which is what we wanted.

The exterior derivative satisfies a graded version of the Leibniz rule with respect to the wedge product.

Theorem 5.7. *Let* $\omega\in\Omega^n(M)$ *and* $\sigma\in\Omega^m(M)$. *Then*

$$\mathrm{d}(\omega\wedge\sigma) = \mathrm{d}\omega\wedge\sigma + (-1)^n\,\omega\wedge\mathrm{d}\sigma.$$

Proof. We will work in local coordinates. Let (U, x) be a chart on M and write

$$\begin{aligned}\omega &= \omega_{a_1\cdots a_n}\mathrm{d}x^{a_1}\wedge\cdots\wedge\mathrm{d}x^{a_n} =: \omega_A\mathrm{d}x^A\\ \sigma &= \sigma_{b_1\cdots b_m}\mathrm{d}x^{b_1}\wedge\cdots\wedge\mathrm{d}x^{b_m} =: \sigma_B\mathrm{d}x^B.\end{aligned}$$

Locally, the exterior derivative operator d acts as

$$\mathrm{d}\omega = \mathrm{d}\omega_A\wedge\mathrm{d}x^A.$$

Hence

$$\begin{aligned}\mathrm{d}(\omega\wedge\sigma) &= \mathrm{d}(\omega_A\sigma_B\,\mathrm{d}x^A\wedge\mathrm{d}x^B)\\ &= \mathrm{d}(\omega_A\sigma_B)\wedge\mathrm{d}x^A\wedge\mathrm{d}x^B\\ &= (\sigma_B\mathrm{d}\omega_A+\omega_A\mathrm{d}\sigma_B)\wedge\mathrm{d}x^A\wedge\mathrm{d}x^B\\ &= \sigma_B\mathrm{d}\omega_A\wedge\mathrm{d}x^A\wedge\mathrm{d}x^B+\omega_A\mathrm{d}\sigma_B\wedge\mathrm{d}x^A\wedge\mathrm{d}x^B\\ &= \sigma_B\mathrm{d}\omega_A\wedge\mathrm{d}x^A\wedge\mathrm{d}x^B+(-1)^n\omega_A\mathrm{d}x^A\wedge\mathrm{d}\sigma_B\wedge\mathrm{d}x^B\\ &= \sigma_B\mathrm{d}\omega\wedge\mathrm{d}x^B+(-1)^n\omega_A\mathrm{d}x^A\wedge\mathrm{d}\sigma\\ &= \mathrm{d}\omega\wedge\sigma+(-1)^n\,\omega\wedge\mathrm{d}\sigma\end{aligned}$$

since we have "anticommuted" the 1-form $\mathrm{d}\sigma_B$ through the n-form $\mathrm{d}x^A$, picking up n minus signs in the process. □

An important property of the exterior derivative is the following.

Theorem 5.8. *Let $\phi\colon M\to N$ be smooth. For any $\omega\in\Omega^n(N)$, we have*

$$\Phi^*(\mathrm{d}\omega) = \mathrm{d}(\Phi^*(\omega)).$$

Remark 5.21. Informally, we can write this result as $\Phi^*\mathrm{d} = \mathrm{d}\Phi^*$, and say that the exterior derivative "commutes" with the pull-back.

However, you should bear in mind that the two d's appearing in the statement are two different operators. On the left hand side, it is $\mathrm{d}\colon\Omega^n(N)\to\Omega^{n+1}(N)$, while it is $\mathrm{d}\colon\Omega^n(M)\to\Omega^{n+1}(M)$ on the right hand side.

Remark 5.22. Of course, we could also combine the operators d into a single operator acting on the Grassmann algebra on M

$$\mathrm{d}\colon\Omega(M)\to\Omega(M)$$

by linear continuation.

5.11.3 De Rham Cohomology

Definition 5.52 (Closed / Exact Forms)**.** *Let M be a smooth manifold and let $\omega\in\Omega^n(M)$. We say that ω is*

- ***closed*** *if* $\mathrm{d}\omega = 0$*;*
- ***exact*** *if* $\exists\,\sigma\in\Omega^{n-1}(M):\omega=\mathrm{d}\sigma$.

The question of whether every closed form is exact and vice versa, i.e. whether the implications

$$(\mathrm{d}\omega = 0)\Leftrightarrow(\exists\,\sigma:\omega=\mathrm{d}\sigma)$$

hold in general, belongs to the branch of mathematics called cohomology theory, to which we will now provide an introduction.

The answer for the $\Leftarrow$ direction is affirmative thanks to the following result.

Theorem 5.9. *Let M be a smooth manifold. The operator*

$$\mathrm{d}^2\equiv\mathrm{d}\circ\mathrm{d}\colon\Omega^n(M)\to\Omega^{n+2}(M)$$

is identically zero, i.e. $\mathrm{d}^2 = 0$.

Proof. This can be shown directly using the definition of d. Here, we will instead show it by working in local coordinates.

Recall that, locally on a chart (U, x), we can write any form $\omega \in \Omega^n(M)$ as

$$\omega = \omega_{a_1 \cdots a_n} \mathrm{d}x^{a_1} \wedge \cdots \wedge \mathrm{d}x^{a_n}.$$

Then, we have

$$\begin{aligned} \mathrm{d}\omega &= \mathrm{d}\omega_{a_1 \cdots a_n} \wedge \mathrm{d}x^{a_1} \wedge \cdots \wedge \mathrm{d}x^{a_n} \\ &= \partial_b \omega_{a_1 \cdots a_n} \mathrm{d}x^b \wedge \mathrm{d}x^{a_1} \wedge \cdots \wedge \mathrm{d}x^{a_n}, \end{aligned}$$

and hence

$$\mathrm{d}^2\omega = \partial_c \partial_b \omega_{a_1 \cdots a_n} \mathrm{d}x^c \wedge \mathrm{d}x^b \wedge \mathrm{d}x^{a_1} \wedge \cdots \wedge \mathrm{d}x^{a_n}.$$

We can perform a little "trick" in the last equation and write it as twice the half expression

$$\mathrm{d}^2\omega = \frac{1}{2}\partial_c \partial_b \omega_{a_1 \cdots a_n} \mathrm{d}x^c \wedge \mathrm{d}x^b \wedge \mathrm{d}x^{a_1} \wedge \cdots \wedge \mathrm{d}x^{a_n} + \frac{1}{2}\partial_c \partial_b \omega_{a_1 \cdots a_n} \mathrm{d}x^c \wedge \mathrm{d}x^b \wedge \mathrm{d}x^{a_1} \wedge \cdots \wedge \mathrm{d}x^{a_n}$$

Now we can inter-switch the c and b dummy indices in the second half part (we can do it since they are just dummy indices) and we get

$$\mathrm{d}^2\omega = \frac{1}{2}\partial_c \partial_b \omega_{a_1 \cdots a_n} \mathrm{d}x^c \wedge \mathrm{d}x^b \wedge \mathrm{d}x^{a_1} \wedge \cdots \wedge \mathrm{d}x^{a_n} + \frac{1}{2}\partial_b \partial_c \omega_{a_1 \cdots a_n} \mathrm{d}x^b \wedge \mathrm{d}x^c \wedge \mathrm{d}x^{a_1} \wedge \cdots \wedge \mathrm{d}x^{a_n}$$

Since $\mathrm{d}x^b \wedge \mathrm{d}x^c = -\,\mathrm{d}x^c \wedge \mathrm{d}x^b$, and moreover, by Schwarz's theorem, we have $\partial_c \partial_b \omega_{a_1 \cdots a_n} = \partial_b \partial_c \omega_{a_1 \cdots a_n}$ we get

$$\mathrm{d}^2\omega = \frac{1}{2}\partial_c \partial_b \omega_{a_1 \cdots a_n} \mathrm{d}x^c \wedge \mathrm{d}x^b \wedge \mathrm{d}x^{a_1} \wedge \cdots \wedge \mathrm{d}x^{a_n} - \frac{1}{2}\partial_c \partial_b \omega_{a_1 \cdots a_n} \mathrm{d}x^c \wedge \mathrm{d}x^b \wedge \mathrm{d}x^{a_1} \wedge \cdots \wedge \mathrm{d}x^{a_n}$$

Hence

$$\mathrm{d}^2\omega = 0$$

Since this holds for any ω, we have $\mathrm{d}^2 = 0$. □

Corollary 5.1. *Every exact form is closed.*

We can extend the action of d to the zero vector space $0 := \{0\}$ by mapping the zero in 0 to the zero function in $\Omega^0(M)$. In this way, we obtain the chain of $\mathbb{R}$-linear maps

$$0 \xrightarrow{\mathrm{d}} \Omega^0(M) \xrightarrow{\mathrm{d}} \Omega^1(M) \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \Omega^n(M) \xrightarrow{\mathrm{d}} \Omega^{n+1}(M) \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \Omega^{\dim M}(M) \xrightarrow{\mathrm{d}} 0,$$

where we now think of the spaces $\Omega^n(M)$ as $\mathbb{R}$-vector spaces.

Recall from linear algebra section in the notes that, given a linear map $\phi \colon V \to W$, one can define the subspace of V

$$\ker(\phi) := \{v \in V \mid \phi(v) = 0\},$$

called the *kernel* of ϕ, and the subspace of W

$$\mathrm{im}(\phi) := \{\phi(v) \mid v \in V\},$$

called the *image* of ϕ.

Going back to our chain of maps, the equation $\mathrm{d}^2 = 0$ is equivalent to

$$\mathrm{im}(\mathrm{d} \colon \Omega^n(M) \to \Omega^{n+1}(M)) \subseteq \ker(\mathrm{d} \colon \Omega^{n+1}(M) \to \Omega^{n+2}(M))$$

for all $0 \leq n \leq \dim M - 2$. Moreover, we have

$$\omega \in \Omega^n(M) \text{ is closed } \Leftrightarrow \ \omega \in \ker(\mathrm{d}\colon \Omega^n(M) \to \Omega^{n+1}(M))$$
$$\omega \in \Omega^n(M) \text{ is exact } \Leftrightarrow \ \omega \in \operatorname{im}(\mathrm{d}\colon \Omega^{n-1}(M) \to \Omega^n(M)).$$

The traditional notation for the spaces on the right hand side above is

$$Z^n := \ker(\mathrm{d}\colon \Omega^n(M) \to \Omega^{n+1}(M)),$$
$$B^n := \operatorname{im}(\mathrm{d}\colon \Omega^{n-1}(M) \to \Omega^n(M)),$$

so that Z^n is the space of closed n-forms and B^n is the space of exact n-forms.

Our original question can be restated as: does $Z^n = B^n$ for all n? We have already seen that $\mathrm{d}^2 = 0$ implies that $B^n \subseteq Z^n$ for all n (B^n is, in fact, a vector subspace of Z^n). Unfortunately the equality does not hold in general, but we do have the following result.

Lemma 5.1 (Poincaré)**.** *Let $M \subseteq \mathbb{R}^d$ be a simply connected domain. Then*

$$Z^n = B^n\,, \qquad \forall\, n > 0.$$

In the cases where $Z^n \neq B^n$, we would like to quantify by how much the closed n-forms fail to be exact. The answer is provided by the cohomology group.

Definition 5.53 (de Rham Cohomology Group)**.** *Let M be a smooth manifold. The n-th* ***de Rham cohomology group*** *on M is the quotient $\mathbb{R}$-vector space*

$$H^n(M) := Z^n/B^n.$$

You can think of the above quotient as $Z^n/\!\sim$, where $\sim$ is the equivalence relation

$$\omega \sim \sigma \ :\Leftrightarrow\ \omega - \sigma \in B^n.$$

The answer to our question as it is addressed in cohomology theory is: every exact n-form on M is also closed and vice versa if, only if,

$$H^n(M) \cong_{\mathrm{vec}} 0.$$

Of course, rather than an actual answer, this is yet another restatement of the question. However, if we are able to determine the spaces $H^n(M)$, then we do get an answer.

A crucial theorem by de Rham states (in more technical terms) that $H^n(M)$ only depends on the global topology of M. In other words, the cohomology groups are topological invariants. This is remarkable because $H^n(M)$ is defined in terms of exterior derivatives, which have everything to do with the local differentiable structure of M, and a given topological space can be equipped with several inequivalent differentiable structures.

Example 5.24. Let M be any smooth manifold. We have

$$H^0(M) \cong_{\mathrm{vec}} \mathbb{R}^{(\#\text{ of connected components of } M)}$$

since the closed 0-forms are just the locally constant smooth functions on M. As an immediate consequence, we have

$$H^0(\mathbb{R}) \cong_{\mathrm{vec}} H^0(S^1) \cong_{\mathrm{vec}} \mathbb{R}.$$

Example 5.25. By Poincaré lemma, we have

$$H^n(M) \cong_{\mathrm{vec}} 0$$

for any simply connected $M \subseteq \mathbb{R}^d$.

5.12 Application - Part 1: $\mathrm{SL}(2,\mathbb{C})$

In this final chapter we will go through an application containing (almost) everything we have mentioned so far. More specifically, we we will examine in detail the special linear group of degree 2 over $\mathbb{C}$, also known as the relativistic spin group.

$\mathrm{SL}(2,\mathbb{C})$ **As A Set**

We define the following subset of $\mathbb{C}^4 := \mathbb{C}\times\mathbb{C}\times\mathbb{C}\times\mathbb{C}$

$$\mathrm{SL}(2,\mathbb{C}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{C}^4 \;\middle|\; ad - bc = 1 \right\},$$

where the array is just an alternative notation for a quadruple (a,b,c,d). It's this extra constraint $ad-bc=1$ that removes one degree of freedom and makes it a subset and not the whole $\mathbb{C}^4$.

$\mathrm{SL}(2,\mathbb{C})$ **As A Group**

We define an operation

$$\begin{aligned} \bullet \colon \mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C}) &\to \mathrm{SL}(2,\mathbb{C}) \\ \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} e & f \\ g & h \end{pmatrix}\right) &\mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \bullet \begin{pmatrix} e & f \\ g & h \end{pmatrix}, \end{aligned}$$

where

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \bullet \begin{pmatrix} e & f \\ g & h \end{pmatrix} := \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}.$$

Formally, this operation is the same as matrix multiplication. We can check directly that the result of applying $\bullet$ lands back in $\mathrm{SL}(2,\mathbb{C})$, or simply recall that the determinant of a product is the product of the determinants. Moreover, the operation $\bullet$

i) is associative (straightforward but tedious to check);

ii) has an identity element, namely $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C})$;

iii) admits inverses: for each $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C})$, we have $\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C})$ and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \bullet \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \bullet \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence, we have $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

Therefore, the pair $(\mathrm{SL}(2,\mathbb{C}),\bullet)$ is a (non-commutative) group.

$\mathrm{SL}(2,\mathbb{C})$ **As A Topological Space**

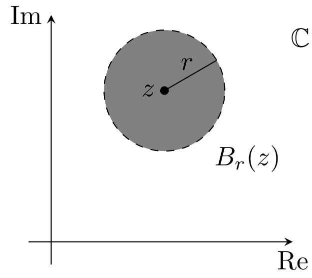
Recall that if N is a subset of M and $\mathcal{O}$ is a topology on M, then we can equip N with the subset topology inherited from M

$$\mathcal{O}|_N := \{U \cap N \mid U \in \mathcal{O}\}.$$

We begin by establishing a topology on $\mathbb{C}$ as follows. Let

$$B_r(z) := \{y \in \mathbb{C} \mid |z-y| < r\}$$

be the open ball of radius $r>0$ and centre $z\in\mathbb{C}$.



Define $\mathcal{O}_{\mathbb{C}}$ implicitly by

$$U \in \mathcal{O}_{\mathbb{C}} \;:\Leftrightarrow\; \forall\, z \in U : \exists\, r > 0 : B_r(z) \subseteq U.$$

Then, the pair $(\mathbb{C}, \mathcal{O}_{\mathbb{C}})$ is a topological space. In fact, we have

$$(\mathbb{C}, \mathcal{O}_{\mathbb{C}}) \cong_{\text{top}} (\mathbb{R}^2, \mathcal{O}_{\text{std}}).$$

We can then equip $\mathbb{C}^4$ with the product topology so that we can finally define

$$\mathcal{O} := (\mathcal{O}_{\mathbb{C}})|_{\mathrm{SL}(2,\mathbb{C})},$$

so that the pair $(\mathrm{SL}(2,\mathbb{C}), \mathcal{O})$ is a topological space. In fact, it is a connected topological space, and we will need this property later on.

$\mathrm{SL}(2,\mathbb{C})$ **As A Topological Manifold**

Recall that a topological space $(M, \mathcal{O})$ is a complex topological manifold if each point $p \in M$ has an open neighbourhood $U(p)$ which is homeomorphic to an open subset of $\mathbb{C}^d$. Equivalently, there must exist a $\mathcal{C}^0$-atlas, i.e. a collection $\mathscr{A}$ of charts (U_α, x_α), where the U_α are open and cover M and each x is a homeomorphism onto a subset of $\mathbb{C}^d$.

Let U be the set

$$U := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C}) \;\middle|\; a \neq 0 \right\}$$

and define the map

$$\begin{aligned} x \colon \quad & U \to x(U) \subseteq \mathbb{C}^* \times \mathbb{C} \times \mathbb{C} \\ & \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (a, b, c), \end{aligned}$$

where $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

As we said in the beginning of this chapter, $\mathrm{SL}(2,\mathbb{C})$ (as a set) is a subset of $\mathbb{C}^4 := \mathbb{C} \times \mathbb{C} \times \mathbb{C} \times \mathbb{C}$ due to the constraint $ad - bc = 1$ that removes one degree of freedom. This is why we map it (locally) to $\mathbb{C}^3$ and not $\mathbb{C}^4$. Because given the mapping (a, b, c) one can reconstruct d as $d = \frac{1+bc}{a}$.

With a little more work on this direction, one can show that U is an open subset of $(\mathrm{SL}(2,\mathbb{C}), \mathcal{O})$ and x is a homeomorphism with inverse

$$\begin{aligned} x^{-1} \colon \quad & x(U) \to U \\ & (a, b, c) \mapsto \begin{pmatrix} a & b \\ c & \frac{1+bc}{a} \end{pmatrix}. \end{aligned}$$

This is the reason why we excluded the case $a = 0$ when we defined the set U of the chart (U, x), since if we hadn't, we wouldn't be able to divide with a and the map x wouldn't have an inverse. However, this makes the chart (U, x) to not cover the whole $\mathrm{SL}(2,\mathbb{C})$ since U as a set takes care only the elements of $\mathrm{SL}(2,\mathbb{C})$ with $a \neq 0$. Hence we need at least one more chart. We thus define the set

$$V := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C}) \;\middle|\; b \neq 0 \right\}$$

and the map

$$\begin{aligned} y\colon \qquad & V \to x(V) \subseteq \mathbb{C} \times \mathbb{C}^* \times \mathbb{C} \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \mapsto (a, b, d). \end{aligned}$$

Similarly to the above, V is open and y is a homeomorphism with inverse

$$\begin{aligned} y^{-1}\colon \quad & x(V) \to V \\ (a,b,d) & \mapsto \begin{pmatrix} a & b \\ \frac{ad-1}{b} & d \end{pmatrix}. \end{aligned}$$

An element of $\mathrm{SL}(2,\mathbb{C})$ cannot have both a and b equal to zero, for otherwise $ad - bc = 0 \neq 1$. Hence $\mathscr{A}_{\text{top}} := \{(U,x),(V,y)\}$ is an atlas, and since every atlas is automatically a $\mathcal{C}^0$-atlas, the triple $(\mathrm{SL}(2,\mathbb{C}), \mathcal{O}, \mathscr{A}_{\text{top}})$ is a 3-dimensional, complex, topological manifold.

$\mathrm{SL}(2,\mathbb{C})$ As A Complex Differentiable Manifold

Recall that to obtain a $\mathcal{C}^1$-differentiable manifold from a topological manifold with atlas $\mathscr{A}$, we have to check that every transition map between charts in $\mathscr{A}$ is differentiable in the usual sense.

In our case, we have the atlas $\mathscr{A}_{\text{top}} := \{(U,x),(V,y)\}$. We evaluate

$$(y \circ x^{-1})(a,b,c) = y(\begin{pmatrix} a & b \\ c & \frac{1+bc}{a} \end{pmatrix}) = (a, b, \tfrac{1+bc}{a}).$$

Hence we have the transition map

$$\begin{aligned} y \circ x^{-1}\colon x(U \cap V) & \to y(U \cap V) \\ (a,b,c) & \mapsto (a, b, \tfrac{1+bc}{a}). \end{aligned}$$

Similarly, we have

$$(x \circ y^{-1})(a,b,d) = y(\begin{pmatrix} a & b \\ \frac{ad-1}{b} & d \end{pmatrix}) = (a, b, \tfrac{ad-1}{b}).$$

Hence, the other transition map is

$$\begin{aligned} x \circ y^{-1}\colon y(U \cap V) & \to x(U \cap V) \\ (a,b,c) & \mapsto (a, b, \tfrac{ad-1}{b}). \end{aligned}$$

Since $a \neq 0$ and $b \neq 0$, the transition maps are complex differentiable.

Therefore, the atlas $\mathscr{A}_{\text{top}}$ is a differentiable atlas. By defining $\mathscr{A}$ to be the maximal differentiable atlas containing $\mathscr{A}_{\text{top}}$, we have that $(\mathrm{SL}(2,\mathbb{C}), \mathcal{O}, \mathscr{A})$ is a 3-dimensional, complex differentiable manifold.

Chapter 6

Lie Groups

6.1 Lie Groups

Definition 6.1 (Lie Group)**.** *A **Lie group** is a group $(G, \bullet)$, where G is a smooth manifold and the maps*

$$\begin{aligned} \mu\colon\ & G \times G \to G \\ & (g_1, g_2) \mapsto g_1 \bullet g_2 \end{aligned}$$

and

$$\begin{aligned} i\colon\ & G \to G \\ & g \mapsto g^{-1} \end{aligned}$$

are both smooth. Note that $G \times G$ inherits a smooth atlas from the smooth atlas of G.

Definition 6.2 (Dimension Of Lie Group)**.** *The **dimension** of a Lie group $(G, \bullet)$ is the dimension of G as a manifold.*

Example 6.1.

a) Consider $(\mathbb{R}^n, +)$, where $\mathbb{R}^n$ is understood as a smooth n-dimensional manifold. This is a commutative (or abelian) Lie group (since $\bullet$ is commutative), often called the n-dimensional translation group.

b) Let $S^1 := \{z \in \mathbb{C} \mid |z| = 1\}$ and let $\cdot$ be the usual multiplication of complex numbers. Then $(S^1, \cdot)$ is a commutative Lie group usually denoted $\mathrm{U}(1)$.

c) Let $\mathrm{GL}(n, \mathbb{R}) = \{\phi\colon \mathbb{R}^n \xrightarrow{\sim} \mathbb{R}^n \mid \det\phi \neq 0\}$. This set can be endowed with the structure of a smooth n^2-dimensional manifold, by noting that there is a bijection between linear maps $\phi\colon \mathbb{R}^n \xrightarrow{\sim} \mathbb{R}^n$ and $\mathbb{R}^{2n}$. The condition $\det\phi \neq 0$ is a so-called *open condition*, meaning that $\mathrm{GL}(n, \mathbb{R})$ can be identified with an open subset of $\mathbb{R}^{2n}$, from which it then inherits a smooth structure.

   Then, $(\mathrm{GL}(n, \mathbb{R}), \circ)$ is a Lie group called the *general linear group*.

Definition 6.3 (Lie Group Homomorphism)**.** *Let $(G, \bullet)$ and $(H, \circ)$ be Lie groups. A map $\phi\colon G \to H$ is **Lie group homomorphism** if it is a group homomorphism and a smooth map.*

Definition 6.4 (Lie Group Isomorphism)**.** *A **Lie group isomorphism** is a Lie group homomorphism which is also a diffeomorphism of the underlying manifolds.*

6.2 The Left Translation Map

To every element of a Lie group there is associated a special map. Note that everything we will do here can be done equivalently by using right translation maps.

Definition 6.5 (Left Translation)**.** *Let $(G, \bullet)$ be a Lie group and let $g \in G$. The map*

$$\begin{aligned} \ell_g\colon\ & G \to G \\ & h \mapsto \ell_g(h) := g \bullet h \equiv gh \end{aligned}$$

*is called the **left translation** by g.*

One might thing that this is an overkill of notation since we already had the operation between two element from the group structure. However the left translation map is different, since we first have to fix an element of the group g (hence the index in ℓ_g) and then apply this element to the whole group (a.k.a to each element of the group).

If there is no danger of confusion, we usually suppress the $\bullet$ notation.

Proposition 6.1. *Let G be a Lie group. For any $g \in G$, the left translation map $\ell_g \colon G \to G$ is a diffeomorphism.*

Proof. Let $h, h' \in G$. Then, we have

$$\ell_g(h) = \ell_g(h') \;\Leftrightarrow\; gh = gh' \;\Leftrightarrow\; h = h'.$$

Moreover, for any $h \in G$, we have $g^{-1}h \in G$ and

$$\ell_g(g^{-1}h) = gg^{-1}h = h.$$

Therefore, ℓ_g is a bijection on G.

Note that

$$\ell_g = \mu(g, -)$$

and since $\mu \colon G \times G \to G$ is smooth by definition, so is ℓ_g.

The inverse map is $(\ell_g)^{-1} = \ell_{g^{-1}}$, since

$$\ell_{g^{-1}} \circ \ell_g = \ell_g \circ \ell_{g^{-1}} = \mathrm{id}_G\,.$$

Then, for the same reason as above with g replaced by g^{-1}, the inverse map $(\ell_g)^{-1}$ is also smooth. Hence, the map ℓ_g is indeed a diffeomorphism. $\square$

Note that, ℓ_g is not an isomorphism of groups, i.e.

$$\ell_g(hh') \neq \ell_g(h)\,\ell_g(h')$$

in general. However that does not stop ℓ_g from being a diffeomorphism of the underlying manifolds.

Since a lie group is a topological manifold, on top of being a group, this means that at any point g we can define the tangent space, and by following the analysis we did in the previous chapter to define fields on G. Recall from the previous chapter than once we have a diffeomorphism ϕ between two manifolds M and N, we can define the push-forward of a vector field X as

$$(\phi_* X)|_{\phi(p)} := \phi_*(X|_p)$$

where $X|_p$ is the vector created by the field X on point p.

Coming in our case, we just showed that the map $\ell_g \colon G \to G$ is a diffeomorphism so we can push-forward any vector field X on G to another vector field (again on G since the maps is between the same manifold). So in our case $\phi_*(X) = (\ell_g)_*(X)$ and for any point h in G: $\ell_g(h) = gh$ so the push-forward equation reads

$$(\ell_{g*} X)|_{gh} := (\ell_g)_*(X|_h)$$

6.3 The Lie Algebra Of A Lie Group

In Lie theory, we are typically not interested in general vector fields, but rather on special class of vector fields which are invariant under the induced push-forward of the left translation maps ℓ_g.

Definition 6.6 (Left Invariant Vector Fields)**.** *Let G be a Lie group. A vector field $X \in \Gamma(TG)$ is said to be **left-invariant** if*

$$\forall\, g \in G:\ (l_g)_*(X) = X.$$

Equivalently, we can require this to hold pointwise

$$\forall\, g, h \in G:\ (\ell_g)_*(X|_h) = X|_{gh}.$$

We can manipulate a bit the pointwise formulation to yield another reformulation. Since both sides are vectors we can let them act on a function f

$$(\ell_g)_*(X|_h)f = X|_{gh}f$$

By using the definition of a push-forward of a vector $(\phi_*)_p(X)f := X(f \circ \phi)$ the left part of the equation reads

$$(\ell_g)_*(X|_h)f = X|_h(f \circ \ell_g) = \big(X(f \circ \ell_g)\big)|_h$$

The right part can be manipulated as follows

$$X|_{gh}f = (Xf)|_{gh} = \Big((Xf) \circ \ell_g\Big)|_h$$

By substituting both final expressions back to the original one and discarding the point h since they must be true for any h we obtain the last reformulation of the push-forward

$$X(f \circ \ell_g) = X(f) \circ \ell_g.$$

Definition 6.7 ($\mathcal{L}(G)$ (As A Set))**.** *We denote the set of all left-invariant vector fields on G as $\mathcal{L}(G)$.*

Of course,

$$\mathcal{L}(G) \subseteq \Gamma(TG)$$

but, in fact, more is true. Recall that we equipped $(\Gamma(TG)$ with two operations and we showed that $(\Gamma(TG), +, \cdot)$ is in fact a $\mathcal{C}^\infty(G)$-submodule. One can check that $\mathcal{L}(G)$ is closed under

$$\begin{aligned} +&\colon \mathcal{L}(G) \times \mathcal{L}(G) \to \mathcal{L}(G)\\ \cdot&\colon \mathcal{C}^\infty(G) \times \mathcal{L}(G) \to \mathcal{L}(G), \end{aligned}$$

therefore, $\mathcal{L}(G)$ is a $\mathcal{C}^\infty(G)$-submodule of $\Gamma(TG)$.

However, as we said, $(\Gamma(TG), +, \cdot)$ can also be seen as an $\mathbb{R}$-vector space. Up to now, we have refrained from thinking of $\Gamma(TG)$ as an $\mathbb{R}$-vector space since it is infinite-dimensional and, even worse, a basis is in general uncountable. A priori, this could be true for $\mathcal{L}(G)$ as well, but we will see that the situation is, in fact, much nicer as $\mathcal{L}(G)$ will turn out to be a finite-dimensional vector space over $\mathbb{R}$ (as an $\mathbb{R}$-vector subspace of $\Gamma(TG)$).

Theorem 6.1. *Let G be a Lie group with identity element $e \in G$. Then $\mathcal{L}(G) \cong_{\mathrm{vec}} T_eG$.*

Proof. We will construct a linear isomorphism $j\colon T_eG \xrightarrow{\sim} \mathcal{L}(G)$. Define

$$\begin{aligned} j\colon T_eG &\to \Gamma(TG)\\ A &\mapsto j(A), \end{aligned}$$

where $j(A)$ is define as

$$\begin{aligned} j(A)\colon G &\to TG\\ g &\mapsto j(A)|_g := (\ell_g)_*(A). \end{aligned}$$

Now we have to prove that this is actually a linear isomorphism, and we will do it in steps.

i) First, we show that for any $A \in T_eG$, $j(A)$ is a smooth vector field on G. It suffices to check that for any $f \in \mathcal{C}^\infty(G)$, we have $j(A)(f) \in \mathcal{C}^\infty(G)$. Indeed

$$\begin{aligned}(j(A)(f))(g) &= j(A)|_g(f) \\ &:= (\ell_g)_*(A)(f) \\ &= A(f \circ \ell_g) \\ &= (f \circ \ell_g \circ \gamma)'(0),\end{aligned}$$

where γ is a curve through $e \in G$ whose tangent vector at e is A. The map

$$\begin{aligned}\varphi \colon \mathbb{R} \times G &\to \mathbb{R} \\ (t, g) &\mapsto \varphi(t, g) := (f \circ \ell_g \circ \gamma)(t) \\ &\qquad\qquad\;\;\, = f(g\gamma(t))\end{aligned}$$

is a composition of smooth maps, hence it is smooth. Then

$$(j(A)(f))(g) = (\partial_1 \varphi)(0, g)$$

depends smoothly on g and thus $j(A)(f) \in \mathcal{C}^\infty(G)$.

ii) We proved that $j(A)$ is indeed a smooth vector field, however now we need to prove that it is a left invariant vector field since it is an element of $\Gamma(TG)$. Let $g, h \in G$. Then, for every $A \in T_eG$, we have

$$\begin{aligned}(\ell_g)_*(j(A)|_h) &:= (\ell_g)_*((\ell_h)_*(A)) \\ &= (\ell_{gh})_*(A) \\ &= j(A)|_{gh},\end{aligned}$$

so $j(A) \in \mathcal{L}(G)$. Hence, the map j is really $j \colon T_eG \to \mathcal{L}(G)$.

iii) We also need to check the linearity. Let $A, B \in T_eG$ and $\lambda \in \mathbb{R}$. Then, for any $g \in G$

$$\begin{aligned}j(\lambda A + B)|_g &= (\ell_g)_*(\lambda A + B) \\ &= \lambda(\ell_g)_*(A) + (\ell_g)_*(B) \\ &= \lambda j(A)|_g + j(B)|_g,\end{aligned}$$

since the push-forward is an $\mathbb{R}$-linear map. Hence, we have $j \colon T_eG \xrightarrow{\sim} \mathcal{L}(G)$.

iv) We also need to check that the map is injective. Let $A, B \in T_eG$. Then

$$\begin{aligned}j(A) = j(B) &\Leftrightarrow \forall\, g \in G : j(A)|_g = j(B)|_g \\ &\Rightarrow j(A)|_e = j(B)|_e \\ &\Leftrightarrow (\ell_e)_*(A) = (\ell_e)_*(B) \\ &\Leftrightarrow A = B,\end{aligned}$$

since $(\ell_e)_* = \mathrm{id}_{TG}$. Hence, the map j is injective.

v) Finally we nee to check that the map is surjective. Let $X \in \mathcal{L}(G)$. Define $A^X := X|_e \in T_eG$. Then, we have

$$j(A^X)|_g = (\ell_g)_*(A^X) = (\ell_g)_*(X|_e) = X_{ge} = X_g,$$

since X is left-invariant. Hence $X = j(A^X)$ and thus j is surjective.

Therefore, $j \colon T_eG \xrightarrow{\sim} \mathcal{L}(G)$ is indeed a linear isomorphism. □

Corollary 6.1. *The space $\mathcal{L}(G)$ is finite-dimensional and $\dim \mathcal{L}(G) = \dim G$.*

We can go one step further now. Recall from the Lie algebra chapter in the notes, that a Lie algebra over an algebraic field K is a vector space over K equipped with a Lie bracket $[-, -]$, i.e. a K-bilinear, antisymmetric map which satisfies the Jacobi identity.

Considering $\Gamma(TM)$ as an infinite-dimensional R-vector space, for two vector fields $X, Y \in \Gamma(TM)$, we can define their Lie bracket, or commutator, as

$$[X,Y](f) := X(Y(f)) - Y(X(f))$$

for any $f \in \mathcal{C}^\infty(M)$. Now we can check that indeed $[X,Y] \in \Gamma(TM)$, and that the bracket is $\mathbb{R}$-bilinear, antisymmetric and satisfies the Jacobi identity. Thus, $(\Gamma(TM), +, \cdot, [-,-])$ is an infinite-dimensional Lie algebra over $\mathbb{R}$. We suppress the $+$ and $\cdot$ when they are clear from the context.

By inheriting the commutator to the finite dimensional R-vector space $\mathcal{L}(G)$, we can then turn $(\mathcal{L}(G), +, \cdot, [-,-])$ to a subalgebra of $\Gamma(TG)$ (of course we need to show that $\mathcal{L}(G)$ is closed under the commutator).

Theorem 6.2. *Let G be a Lie group. Then $\mathcal{L}(G)$ is a Lie subalgebra of $\Gamma(TG)$.*

Proof. A Lie subalgebra of a Lie algebra is simply a vector subspace which is closed under the action of the Lie bracket. Therefore, we only need to check that

$$\forall\, X, Y \in \mathcal{L}(G) : \ [X,Y] \in \mathcal{L}(G).$$

Let $X, Y \in \mathcal{L}(G)$. For any $g \in G$ and $f \in \mathcal{C}^\infty(G)$, we have

$$\begin{aligned}
[X,Y](f \circ \ell_g) &:= X(Y(f \circ \ell_g)) - Y(X(f \circ \ell_g)) \\
&= X(Y(f) \circ \ell_g) - Y(X(f) \circ \ell_g) \\
&= X(Y(f)) \circ \ell_g - Y(X(f)) \circ \ell_g \\
&= \big(X(Y(f)) - Y(X(f))\big) \circ \ell_g \\
&= [X,Y](f) \circ \ell_g.
\end{aligned}$$

Hence, $[X,Y]$ is left-invariant. □

Definition 6.8 ($\mathcal{L}(G)$ (As An Algebra))**.** *Let G be a Lie group. The* ***associated Lie algebra*** *of G is $\mathcal{L}(G)$.*

Notice that we began with $\mathcal{L}(G)$ as a set of all left invariant vector fields of G, which is a subset of $\Gamma(TG)$, then we inherited the $+$ and $\cdot$ of $\Gamma(TG)$ to $\mathcal{L}(G)$ and we showed that it is also a submodule and a subvector space of $\Gamma(TG)$, and finally we inherited the Lie bracket from $\Gamma(TG)$ and we showed that it is also a subalgebra of $\Gamma(TG)$. From now on when we will be referring to $\mathcal{L}(G)$, we will mean its algebra structure.

Given the nature of $\mathcal{L}(G)$, it is a rather complicated object, since its elements are vector fields, hence we would like to work with T_eG instead, whose elements are tangent vectors. We have already shown that $\mathcal{L}(G)$ and T_eG are isomorphic as vector spaces, but we will now see that the identification of $\mathcal{L}(G)$ and T_eG goes beyond the level of linear isomorphism as vector spaces, as they are isomorphic as Lie algebras. Indeed, we can use the bracket on $L(G)$ to define a bracket on T_eG such that they be isomorphic as Lie algebras. First, let us define the isomorphism of Lie algebras.

Definition 6.9 (Lie Algebra Homomorphism)**.** *Let $(L_1, [-,-]_{L_1})$ and $(L_2, [-,-]_{L_2})$ be Lie algebras over the same field. A linear map $\phi \colon L_1 \xrightarrow{\sim} L_2$ is a* ***Lie algebra homomorphism*** *if*

$$\forall\, x, y \in L_1 : \ \phi([x,y]_{L_1}) = [\phi(x), \phi(y)]_{L_2}.$$

Definition 6.10 (Lie Algebra Isomorphism)**.** *A bijective Lie algebra homomorphism, is called a* ***Lie algebra isomorphism*** *and we write $L_1 \cong_{\text{Lie alg}} L_2$.*

By using the bracket $[-,-]_{\mathcal{L}(G)}$ on $\mathcal{L}(G)$ we can define, for any $A, B \in T_eG$

$$[A,B]_{T_eG} := j^{-1}\big([j(A), j(B)]_{\mathcal{L}(G)}\big),$$

where $j^{-1}(X) = X|_e$. Equipped with these brackets, we have

$$\mathcal{L}(G) \cong_{\text{Lie alg}} T_eG.$$

Hence, given a Lie group we have seen how we can construct a Lie algebra as the space of left-invariant vector fields and this algebra is isomorphic to the algebra of tangent vectors at the identity. We will later explore the opposite direction, i.e. given a Lie algebra, we will see how to construct a Lie group whose associated Lie algebra is the one we started from.

6.4 Application - Part 2: $\mathrm{SL}(2,\mathbb{C})$

In the first part of of the application in the previous chapter, we defined the set $\mathrm{SL}(2,\mathbb{C})$ as a subset of $\mathbb{C}^4 := \mathbb{C}\times\mathbb{C}\times\mathbb{C}\times\mathbb{C}$. Then we showed that:

- $\mathrm{SL}(2,\mathbb{C})$ can be made into a group
- $\mathrm{SL}(2,\mathbb{C})$ can be made into a topological space
- $\mathrm{SL}(2,\mathbb{C})$ can be made into a topological manifold
- $\mathrm{SL}(2,\mathbb{C})$ can be made into a complex differentiable manifold

Hence we have left with $\mathrm{SL}(2,\mathbb{C})$ as a 3-dimensional, complex differentiable manifold.

$\mathrm{SL}(2,\mathbb{C})$ As A Lie Group

We equipped $\mathrm{SL}(2,\mathbb{C})$ with both a group and a manifold structure. In order to obtain a Lie group structure, we have to check that these two structures are compatible, that is, we have to show that the two maps

$$\begin{aligned}\mu\colon \mathrm{SL}(2,\mathbb{C})\times\mathrm{SL}(2,\mathbb{C}) &\to \mathrm{SL}(2,\mathbb{C})\\ \left(\begin{pmatrix} a & b\\ c & d\end{pmatrix}, \begin{pmatrix} e & f\\ g & h\end{pmatrix}\right) &\mapsto \begin{pmatrix} a & b\\ c & d\end{pmatrix}\bullet\begin{pmatrix} e & f\\ g & h\end{pmatrix}\end{aligned}$$

and

$$\begin{aligned}i\colon \mathrm{SL}(2,\mathbb{C}) &\to \mathrm{SL}(2,\mathbb{C})\\ \begin{pmatrix} a & b\\ c & d\end{pmatrix} &\mapsto \begin{pmatrix} a & b\\ c & d\end{pmatrix}^{-1}\end{aligned}$$

are differentiable with respect to the differentiable structure on $\mathrm{SL}(2,\mathbb{C})$. For instance, for the inverse map i, we have to show that the map $y\circ i\circ x^{-1}$ is differentiable in the usual for any pair of charts $(U,x),(V,y)\in\mathscr{A}$.

$$\begin{tikzcd} U\subseteq \mathrm{SL}(2,\mathbb{C}) \arrow[r, "i"] \arrow[d, "x"] & V\subseteq \mathrm{SL}(2,\mathbb{C}) \arrow[d, "y"] \\ x(U)\subseteq\mathbb{C}^3 \arrow[r, "y\circ i\circ x^{-1}"] & y(V)\subseteq\mathbb{C}^3 \end{tikzcd}$$

where we remind that

$$\begin{aligned}x^{-1}\colon\quad x(U) &\to U\\ (a,b,c) &\mapsto \begin{pmatrix} a & b\\ c & \frac{1+bc}{a}\end{pmatrix}.\end{aligned}$$

and

$$\begin{aligned}y\colon\qquad V &\to x(V)\subseteq\mathbb{C}\times\mathbb{C}^*\times\mathbb{C}\\ \begin{pmatrix} a & b\\ c & d\end{pmatrix} &\mapsto (a,b,d).\end{aligned}$$

Since $\mathrm{SL}(2,\mathbb{C})$ is connected, the differentiability of the transition maps in $\mathscr{A}$ implies that if $y\circ i\circ x^{-1}$ is differentiable for any two given charts, then it is differentiable for all charts in $\mathscr{A}$. Hence, we can simply let (U, x) and (V, y) be the two charts on $\mathrm{SL}(2, \mathbb{C})$ defined above. Then, we have

$$(y \circ i \circ x^{-1})(a, b, c) = (y \circ i)(\begin{pmatrix} a & b \\ c & \frac{1+bc}{a} \end{pmatrix}) = y(\begin{pmatrix} \frac{1+bc}{a} & -b \\ -c & a \end{pmatrix}) = (\tfrac{1+bc}{a}, -b, a)$$

which is certainly complex differentiable as a map between open subsets of $\mathbb{C}^3$ (recall that $a \neq 0$ on $x(U)$).

Checking that μ is complex differentiable is slightly more involved, since we first have to equip $\mathrm{SL}(2, \mathbb{C}) \times \mathrm{SL}(2, \mathbb{C})$ with a suitable "product differentiable structure" and then proceed as above. Once that is done, we can finally conclude that $((\mathrm{SL}(2, \mathbb{C}), \mathcal{O}, \mathscr{A}), \bullet)$ is a 3-dimensional complex Lie group.

The Lie Algebra Of $\mathrm{SL}(2, \mathbb{C})$

Recall that to every Lie group G, there is an associated Lie algebra $\mathcal{L}(G)$, where

$$\mathcal{L}(G) := \{X \in \Gamma(TG) \mid \forall\, g, h \in G : (\ell_g)_*(X|_h) = X|_{gh}\},$$

which we then proved to be isomorphic to the Lie algebra T_eG with Lie bracket

$$[A, B]_{T_eG} := j^{-1}([j(A), j(B)]_{\mathcal{L}(G)})$$

induced by the Lie bracket on $\mathcal{L}(G)$ via the isomorphism j

$$j(A)|_g := (\ell_g)_*(A).$$

In the case of $\mathrm{SL}(2, \mathbb{C})$, the left translation map by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is

$$\begin{aligned} \ell_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)} \colon \mathrm{SL}(2, \mathbb{C}) &\to \mathrm{SL}(2, \mathbb{C}) \\ \begin{pmatrix} e & f \\ g & h \end{pmatrix} &\mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \bullet \begin{pmatrix} e & f \\ g & h \end{pmatrix} \end{aligned}$$

By using the standard notation $\mathfrak{sl}(2, \mathbb{C}) \equiv \mathcal{L}(\mathrm{SL}(2, \mathbb{C}))$, we have

$$\mathfrak{sl}(2, \mathbb{C}) \cong_{\text{Lie alg}} T_{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)} \mathrm{SL}(2, \mathbb{C}).$$

We would now like to explicitly determine the Lie bracket on $T_{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)} \mathrm{SL}(2, \mathbb{C})$, and hence determine its structure constants.

Recall that if (U, x) is a chart on a manifold M and $p \in U$, then the chart (U, x) induces a basis of the tangent space T_pM. We shall use our previously defined chart (U, x) on $\mathrm{SL}(2, \mathbb{C})$, where $U := \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2, \mathbb{C}) \mid a \neq 0\}$ and

$$\begin{aligned} x \colon \quad U &\to x(U) \subseteq \mathbb{C}^3 \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\mapsto (a, b, c). \end{aligned}$$

Note that the d appearing here is completely redundant, since the membership condition of $\mathrm{SL}(2, \mathbb{C})$ forces $d = \frac{1+bc}{a}$. However, we will keep writing the d to avoid having a fraction in a matrix in a subscript.

The chart (U, x) contains the identity $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ (we must include the identity since we are interested in the tangent space at the identity $T_{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)} \mathrm{SL}(2, \mathbb{C})$) and hence we get an induced co-ordinate basis

$$\left\{ \left(\frac{\partial}{\partial x^i} \right)_{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)} \in T_{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)} \mathrm{SL}(2, \mathbb{C}) \;\middle|\; 1 \leq i \leq 3 \right\}$$

so that any $A \in T_{\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)} \mathrm{SL}(2,\mathbb{C})$ can be written as

$$A = \alpha \left(\frac{\partial}{\partial x^1}\right)_{\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)} + \beta \left(\frac{\partial}{\partial x^2}\right)_{\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)} + \gamma \left(\frac{\partial}{\partial x^3}\right)_{\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)},$$

for some $\alpha, \beta, \gamma \in \mathbb{C}$. Since the Lie bracket is bilinear, its action on these basis vectors uniquely extends to the whole of $T_{\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)} \mathrm{SL}(2,\mathbb{C})$ by linear continuation. Hence, we simply have to determine the action of the Lie bracket of $\mathfrak{sl}(2,\mathbb{C})$ on the images under the isomorphism j of these basis vectors.

Let us now determine the image of these co-ordinate induced basis elements under the isomorphism j. The object

$$j\left(\left(\frac{\partial}{\partial x^i}\right)_{\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)}\right) \in \mathfrak{sl}(2,\mathbb{C})$$

is a left-invariant vector field on $\mathrm{SL}(2,\mathbb{C})$. It assigns to each point $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right) \in U \subseteq \mathrm{SL}(2,\mathbb{C})$ the tangent vector

$$j\left(\left(\frac{\partial}{\partial x^i}\right)_{\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)}\right)\bigg|_{\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)} := \left(\ell_{\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)}\right)_* \left(\frac{\partial}{\partial x^i}\right)_{\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)} \in T_{\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)} \mathrm{SL}(2,\mathbb{C}).$$

This tangent vector is a $\mathbb{C}$-linear map $\mathcal{C}^\infty(\mathrm{SL}(2,\mathbb{C})) \xrightarrow{\sim} \mathbb{C}$, where $\mathcal{C}^\infty(\mathrm{SL}(2,\mathbb{C}))$ is the $\mathbb{C}$-vector space (in fact, the $\mathbb{C}$-algebra) of smooth complex-valued functions on $\mathrm{SL}(2,\mathbb{C})$ although, to be precise, since we are working in a chart we should only consider functions defined on U. For (the restriction to U of) any $f \in \mathcal{C}^\infty(\mathrm{SL}(2,\mathbb{C}))$ we have, explicitly,

$$\begin{aligned}\left(\ell_{\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)}\right)_* \left(\frac{\partial}{\partial x^i}\right)_{\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)} (f) &= \left(\frac{\partial}{\partial x^i}\right)_{\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)} \left(f \circ \ell_{\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)}\right) \\ &= \partial_i\left(f \circ \ell_{\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)} \circ x^{-1}\right)\left(x\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)\right),\end{aligned}$$

where the argument of ∂_i in the last line is a map $x(U) \subseteq \mathbb{C}^3 \to \mathbb{C}$, hence ∂_i is simply the operation of complex differentiation with respect to the i-th (out of the 3) complex variable of the map $f \circ \ell_{\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)} \circ x^{-1}$, which is then to be evaluated at $x\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right) \in \mathbb{C}^3$. By inserting an identity in the composition, we have

$$\begin{aligned} &= \partial_i\left(f \circ \mathrm{id}_U \circ \ell_{\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)} \circ x^{-1}\right)\left(x\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)\right) \\ &= \partial_i\left(f \circ (x^{-1} \circ x) \circ \ell_{\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)} \circ x^{-1}\right)\left(x\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)\right) \\ &= \partial_i\left((f \circ x^{-1}) \circ (x \circ \ell_{\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)} \circ x^{-1})\right)\left(x\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)\right),\end{aligned}$$

where $f \circ x^{-1} \colon x(U) \subseteq \mathbb{C}^3 \to \mathbb{C}$ and $(x \circ \ell_{\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)} \circ x^{-1}) \colon x(U) \subseteq \mathbb{C}^3 \to x(U) \subseteq \mathbb{C}^3$ and hence, we can use the multi-dimensional chain rule to obtain

$$= \left(\partial_m (f \circ x^{-1})\left((x \circ \ell_{\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)} \circ x^{-1})(x\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right))\right)\right)\left(\partial_i (x^m \circ \ell_{\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)} \circ x^{-1})(x\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right))\right),$$

with the summation going from $m = 1$ to $m = 3$. The first factor is simply

$$\begin{aligned}\partial_m (f \circ x^{-1})\left((x \circ \ell_{\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)})\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)\right) &= \partial_m (f \circ x^{-1})\left(x\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\right) \\ &=: \left(\frac{\partial}{\partial x^m}\right)_{\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)} (f).\end{aligned}$$

To see what the second factor is, we first consider the map $x^m \circ \ell_{\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)} \circ x^{-1}$. This map acts on the triple $(e, f, g) \in x(U)$ as

$$\begin{aligned}(x^m \circ \ell_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)} \circ x^{-1})(e,f,g) &= (x^m \circ \ell_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)})\begin{pmatrix} e & f \\ g & \frac{1+fg}{e} \end{pmatrix} \\ &= x^m(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \bullet \begin{pmatrix} e & f \\ g & \frac{1+fg}{e} \end{pmatrix}) \\ &= x^m(\begin{pmatrix} ae+bg & af + \frac{b(1+fg)}{e} \\ ce+dg & cf + \frac{d(1+fg)}{e} \end{pmatrix}),\end{aligned}$$

and since $x^m := \mathrm{proj}_m \circ x$, with $m \in \{1,2,3\}$, we have

$$(x^m \circ \ell_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)} \circ x^{-1})(e,f,g) = \mathrm{proj}_m(ae+bg, af + \tfrac{b(1+fg)}{e}, ce+dg),$$

the map proj_m simply picks the m-th component of the triple. We now have to apply ∂_i to this map, with $i \in \{1,2,3\}$, i.e. we have to differentiate with respect to each of the three complex variables e, f, and g. We can write the result as

$$\partial_i(x^m \circ \ell_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)} \circ x^{-1})(e,f,g) = D(e,f,g)^m{}_i,$$

where m labels the rows and i the columns of the matrix

$$D(e,f,g) = \begin{pmatrix} a & 0 & b \\ -\frac{b(1+fg)}{e^2} & a + \frac{bg}{e} & \frac{bf}{e} \\ c & 0 & d \end{pmatrix}.$$

Finally, by evaluating this at $(e,f,g) = x\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) = (1,0,0)$, we obtain

$$\partial_i(x^m \circ \ell_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)} \circ x^{-1})(x\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)) = D^m{}_i,$$

where, by recalling that $d = \frac{1+bc}{a}$,

$$D := D(1,0,0) = \begin{pmatrix} a & 0 & b \\ -b & a & 0 \\ c & 0 & \frac{1+bc}{a} \end{pmatrix}.$$

Putting the two factors back together yields

$$\left(\ell_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)}\right)_* \left(\frac{\partial}{\partial x^i}\right)_{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)}(f) = D^m{}_i \left(\frac{\partial}{\partial x^m}\right)_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)}(f).$$

Since this holds for an arbitrary $f \in \mathcal{C}^\infty(\mathrm{SL}(2,\mathbb{C}))$, we have

$$j\left(\left(\frac{\partial}{\partial x^i}\right)_{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)}\right)\Bigg|_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)} := \left(\ell_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)}\right)_* \left(\frac{\partial}{\partial x^i}\right)_{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)} = D^m{}_i \left(\frac{\partial}{\partial x^m}\right)_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)},$$

and since the point $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in U \subseteq \mathrm{SL}(2,\mathbb{C})$ is also arbitrary, we have

$$j\left(\left(\frac{\partial}{\partial x^i}\right)_{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)}\right) = D^m{}_i \frac{\partial}{\partial x^m} \in \mathfrak{sl}(2,\mathbb{C}),$$

where D is now the corresponding matrix of co-ordinate functions

$$D := \begin{pmatrix} x^1 & 0 & x^2 \\ -x^2 & x^1 & 0 \\ x^3 & 0 & \frac{1+x^2x^3}{x^1} \end{pmatrix}.$$

Note that while the three vector fields

$$
\begin{aligned}
\frac{\partial}{\partial x^m} \colon \mathrm{SL}(2,\mathbb{C}) &\to T\,\mathrm{SL}(2,\mathbb{C}) \\
\begin{pmatrix} a & b \\ c & d \end{pmatrix} &\mapsto \left(\frac{\partial}{\partial x^m}\right)_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)}
\end{aligned}
$$

are not individually left-invariant, their linear combination with coefficients $D^m{}_i$ is indeed left-invariant. Recall that these vector fields

i) are $\mathbb{C}$-linear maps

$$
\begin{aligned}
\frac{\partial}{\partial x^m} \colon \mathcal{C}^\infty(\mathrm{SL}(2,\mathbb{C})) &\xrightarrow{\sim} \mathcal{C}^\infty(\mathrm{SL}(2,\mathbb{C})) \\
f &\mapsto \partial_m(f \circ x^{-1}) \circ x;
\end{aligned}
$$

ii) satisfy the Leibniz rule

$$
\frac{\partial}{\partial x^m}(fg) = f\frac{\partial}{\partial x^m}(g) + g\frac{\partial}{\partial x^m}(f);
$$

iii) act on the coordinate functions $x^i \in \mathcal{C}^\infty(\mathrm{SL}(2,\mathbb{C}))$ as

$$
\frac{\partial}{\partial x^m}(x^i) = \partial_m(x^i \circ x^{-1}) \circ x = \partial_m(\mathrm{proj}_i \circ x \circ x^{-1}) \circ x = \delta^i_m \circ x = \delta^i_m,
$$

since the composition of a constant function with any composable function is just the constant function.

Hence, we have an expansion of the images of the basis of $T_{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)}\mathrm{SL}(2,\mathbb{C})$ under j:

$$
\begin{aligned}
j\left(\left(\frac{\partial}{\partial x^1}\right)_{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)}\right) &= x^1\frac{\partial}{\partial x^1} - x^2\frac{\partial}{\partial x^2} + x^3\frac{\partial}{\partial x^3} \\
j\left(\left(\frac{\partial}{\partial x^2}\right)_{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)}\right) &= x^1\frac{\partial}{\partial x^2} \\
j\left(\left(\frac{\partial}{\partial x^3}\right)_{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)}\right) &= x^2\frac{\partial}{\partial x^1} + \tfrac{1+x^2x^3}{x^1}\frac{\partial}{\partial x^3}.
\end{aligned}
$$

We now have to calculate the bracket (in $\mathfrak{sl}(2,\mathbb{C})$) of every pair of these. We can also do them all at once, which is a good exercise in index gymnastics. We have

$$
\left[j\left(\left(\frac{\partial}{\partial x^i}\right)_{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)}\right), j\left(\left(\frac{\partial}{\partial x^k}\right)_{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)}\right)\right] = \left[D^m{}_i\frac{\partial}{\partial x^m}, D^n{}_k\frac{\partial}{\partial x^n}\right].
$$

Letting this act on an arbitrary $f \in \mathcal{C}^\infty(\mathrm{SL}(2,\mathbb{C}))$, by definition

$$
\left[D^m{}_i\frac{\partial}{\partial x^m}, D^n{}_k\frac{\partial}{\partial x^n}\right](f) := D^m{}_i\frac{\partial}{\partial x^m}\Big(D^n{}_k\frac{\partial}{\partial x^n}(f)\Big) - D^n{}_k\frac{\partial}{\partial x^n}\Big(D^m{}_i\frac{\partial}{\partial x^m}(f)\Big).
$$

The first term gives

$$
\begin{aligned}
D^m{}_i\frac{\partial}{\partial x^m}\Big(D^n{}_k\frac{\partial}{\partial x^n}(f)\Big) &= D^m{}_i\frac{\partial}{\partial x^m}(D^n{}_k\,\partial_n(f\circ x^{-1})\circ x) \\
&= D^m{}_i\frac{\partial}{\partial x^m}(D^n{}_k)\,(\partial_n(f\circ x^{-1})\circ x) + D^m{}_iD^n{}_k\frac{\partial}{\partial x^m}(\partial_n(f\circ x^{-1})\circ x) \\
&= D^m{}_i\frac{\partial}{\partial x^m}(D^n{}_k)\,(\partial_n(f\circ x^{-1})\circ x) + D^m{}_iD^n{}_k\,\partial_m(\partial_n(f\circ x^{-1})\circ x\circ x^{-1})\circ x \\
&= D^m{}_i\frac{\partial}{\partial x^m}(D^n{}_k)\,(\partial_n(f\circ x^{-1})\circ x) + D^m{}_iD^n{}_k\,\partial_m\partial_n(f\circ x^{-1})\circ x.
\end{aligned}
$$

Similarly, we have

$$D^n{}_k \frac{\partial}{\partial x^n}\Big(D^m{}_i \frac{\partial}{\partial x^m}(f)\Big) = D^n{}_k \frac{\partial}{\partial x^n}(D^m{}_i)\,(\partial_m(f\circ x^{-1})\circ x) + D^n{}_k D^m{}_i\,\partial_n\partial_m(f\circ x^{-1})\circ x.$$

Hence, recalling that $\partial_m\partial_n = \partial_n\partial_m$ by Schwarz's theorem, we have

$$\begin{aligned}
\left[D^m{}_i \frac{\partial}{\partial x^m}, D^n{}_k \frac{\partial}{\partial x^n}\right](f) &= D^m{}_i \frac{\partial}{\partial x^m}(D^n{}_k)\,(\partial_n(f\circ x^{-1})\circ x) + \text{[gray]}D^m{}_i D^n{}_k\,\partial_m\partial_n(f\circ x^{-1})\circ x\\
&\quad - D^n{}_k \frac{\partial}{\partial x^n}(D^m{}_i)\,(\partial_m(f\circ x^{-1})\circ x) - \text{[gray]}D^n{}_k D^m{}_i\,\partial_n\partial_m(f\circ x^{-1})\circ x\\
&= \Big(D^m{}_i \frac{\partial}{\partial x^m}(D^n{}_k) - D^m{}_k \frac{\partial}{\partial x^m}(D^n{}_i)\Big)\partial_n(f\circ x^{-1})\circ x\\
&= \Big(D^m{}_i \frac{\partial}{\partial x^m}(D^n{}_k) - D^m{}_k \frac{\partial}{\partial x^m}(D^n{}_i)\Big)\frac{\partial}{\partial x^n}(f),
\end{aligned}$$

where we relabelled some dummy indices. Since the $f \in \mathcal{C}^\infty(\mathrm{SL}(2,\mathbb{C}))$ was arbitrary,

$$\left[D^m{}_i \frac{\partial}{\partial x^m}, D^n{}_k \frac{\partial}{\partial x^n}\right] = \Big(D^m{}_i \frac{\partial}{\partial x^m}(D^n{}_k) - D^m{}_k \frac{\partial}{\partial x^m}(D^n{}_i)\Big)\frac{\partial}{\partial x^n}.$$

We can now evaluate this explicitly. For $i = 1$ and $k = 2$, we have

$$\begin{aligned}
\left[D^m{}_1 \frac{\partial}{\partial x^m}, D^n{}_2 \frac{\partial}{\partial x^n}\right] &= \Big(\text{[gray]}D^m{}_1 \frac{\partial}{\partial x^m}(D^1{}_2) - D^m{}_2 \frac{\partial}{\partial x^m}(D^1{}_1)\Big)\frac{\partial}{\partial x^1}\\
&\quad + \Big(D^m{}_1 \frac{\partial}{\partial x^m}(D^2{}_2) - D^m{}_2 \frac{\partial}{\partial x^m}(D^2{}_1)\Big)\frac{\partial}{\partial x^2}\\
&\quad + \Big(\text{[gray]}D^m{}_1 \frac{\partial}{\partial x^m}(D^3{}_2) - D^m{}_2 \frac{\partial}{\partial x^m}(D^3{}_1)\Big)\frac{\partial}{\partial x^3}\\
&= -D^1{}_2 \frac{\partial}{\partial x^1} + (D^1{}_1 + D^2{}_2)\frac{\partial}{\partial x^2} - D^3{}_2 \frac{\partial}{\partial x^3}\\
&= 2x^1 \frac{\partial}{\partial x^2}.
\end{aligned}$$

Similarly, we compute

$$\begin{aligned}
\left[D^m{}_1 \frac{\partial}{\partial x^m}, D^n{}_3 \frac{\partial}{\partial x^n}\right] &= \Big(D^m{}_1 \frac{\partial}{\partial x^m}(D^1{}_3) - D^m{}_3 \frac{\partial}{\partial x^m}(D^1{}_1)\Big)\frac{\partial}{\partial x^1}\\
&\quad + \Big(\text{[gray]}D^m{}_1 \frac{\partial}{\partial x^m}(D^2{}_3) - D^m{}_3 \frac{\partial}{\partial x^m}(D^2{}_1)\Big)\frac{\partial}{\partial x^2}\\
&\quad + \Big(D^m{}_1 \frac{\partial}{\partial x^m}(D^3{}_3) - D^m{}_3 \frac{\partial}{\partial x^m}(D^3{}_1)\Big)\frac{\partial}{\partial x^3}\\
&= -2x^2 \frac{\partial}{\partial x^1} - 2(\tfrac{1+x^2x^3}{x^1})\frac{\partial}{\partial x^3}
\end{aligned}$$

and

$$\begin{aligned}
\left[D^m{}_2 \frac{\partial}{\partial x^m}, D^n{}_3 \frac{\partial}{\partial x^n}\right] &= \Big(D^m{}_2 \frac{\partial}{\partial x^m}(D^1{}_3) - \text{[gray]}D^m{}_3 \frac{\partial}{\partial x^m}(D^1{}_2)\Big)\frac{\partial}{\partial x^1}\\
&\quad + \Big(\text{[gray]}D^m{}_2 \frac{\partial}{\partial x^m}(D^2{}_3) - D^m{}_3 \frac{\partial}{\partial x^m}(D^2{}_2)\Big)\frac{\partial}{\partial x^2}\\
&\quad + \Big(D^m{}_2 \frac{\partial}{\partial x^m}(D^3{}_3) - \text{[gray]}D^m{}_3 \frac{\partial}{\partial x^m}(D^3{}_2)\Big)\frac{\partial}{\partial x^3}\\
&= (D^2{}_1 - D^1{}_3)\frac{\partial}{\partial x^1} + D^2{}_3 \frac{\partial}{\partial x^2} - D^3{}_2 \frac{\partial}{\partial x^3}\\
&= x^1 \frac{\partial}{\partial x^1} - x^2 \frac{\partial}{\partial x^2} + x^3 \frac{\partial}{\partial x^3},
\end{aligned}$$

where the differentiation rules that we have used come from the definition of the vector field $\frac{\partial}{\partial x^m}$, the Leibniz rule, and the action on co-ordinate functions.
By applying j^{-1}, which is just evaluation at the identity, to these vector fields, we finally see that the induced Lie bracket on $T_{\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)}\mathrm{SL}(2,\mathbb{C})$ satisfies

$$\left[\left(\frac{\partial}{\partial x^1}\right)_{\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)}, \left(\frac{\partial}{\partial x^2}\right)_{\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)}\right] = 2\left(\frac{\partial}{\partial x^2}\right)_{\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)}$$

$$\left[\left(\frac{\partial}{\partial x^1}\right)_{\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)}, \left(\frac{\partial}{\partial x^3}\right)_{\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)}\right] = -2\left(\frac{\partial}{\partial x^3}\right)_{\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)}$$

$$\left[\left(\frac{\partial}{\partial x^2}\right)_{\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)}, \left(\frac{\partial}{\partial x^3}\right)_{\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)}\right] = \left(\frac{\partial}{\partial x^1}\right)_{\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)}.$$

Hence, the structure constants of $T_{\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)}\mathrm{SL}(2,\mathbb{C})$ with respect to the co-ordinate basis are

$$C^2{}_{12} = 2, \qquad C^3{}_{13} = -2, \qquad C^1{}_{23} = 1,$$

with all other being either zero or related to these by anti-symmetry.