Source coding with the DPCM method

Introduction

DPCM (Differential Pulse Code Modulation) coding can be seen as a generalization of Delta coding where the signal that is quantized and sent to the receiver is the difference between the current sample (of time n) and a linear prediction of it. That is, in DPCM coding, we compute, at each time instant, a prediction of the value of the current sample based on the values of previous samples that have already been coded and then compute the error of that prediction. The prediction error signal is then encoded using one or more binary digits per sample.

DPCM encoding

The encoder and decoder of a DPCM system are shown in Figure 2. In order to encode the value of the current sample we first calculate a prediction of the value based on encoded values of previous samples. The prediction of the signal $x(n)$ is denoted as y^'(n). In Figure we observe a memory device (both at the transmitter and the receiver) that keeps the reconstructed values of the previous samples stored on the basis of which the prediction of the current sample value will be computed. Our goal is to minimize the dispersion of the error signal $y(n)$ = $x(n)$ - y^'(n) so that it has a small dynamic range and can be satisfactorily described by a small number of binary digits.

The process of quantization of the error signal $y(n)$ leads to the signal y^(n) which is sent to the receiver. In particular, the quantizer in Figure 2 should limit the dynamic range of the prediction error between a minimum and a maximum value by setting the samples that are outside the dynamic range to the corresponding extreme acceptable value. The quantizer then calculates the quantization step Δ, the centers of each region, calculates which region the input sample belongs to, and outputs the encoded sample $\hat{y}(n)$1 as the output. This sample will be used as a pointer to the vector $centers$ to get the quantized sample as $centers(\hat{y}(n))$. Use the uniform binary digit quantizer N implemented earlier to quantize the prediction error which has a smaller dynamic range compared to the input signal.

<u>The quantizer in particular will quantize each sample of the prediction error separately:</u>

$$\boldsymbol{\hat{y}(n) = my\_quantizer(y(n), N, min\_value, max\_value)}$$

$y(n)$: the current sample of the prediction error as input to the quantizer

max_value: the maximum acceptable value of the prediction error

min_vaalue: the minimum acceptable value of the prediction error

$\hat{y}(n)$: the quantized sample of the current sample of the forecast error

At the receiver, the signal $y\hat{}(n)$ is combined with the signal $y\hat{}'(n)$ (the prediction of $x(n)$). Since the previously reconstructed values as well as the prediction method used by the transmitter are known at the receiver, it follows that the transmitter and the receiver are able to compute exactly the same values for the prediction $y\hat{}'(n)$. As in the case of delta coding, the transmitter includes as part of the transmitter the receiver device which computes the reconstruction $x\hat{}(n)$. These values are used by the transmitter to compute the prediction, rather than the actual values $x(n)$, in order to fully mimic the receiver device which of course does not know the actual values. By using the reconstructed values to compute the prediction and then the prediction error, we ensure (as in the case of delta coding) that we do not have an accumulation of quantization error.

In the simple case where we rely only on the prediction of the previous sample, the equations that describe the operation of the DPCM system in Figure 2 are as follows:

$$y(n) = x(n) - \hat{y}'(n-1)$$

$$\hat{y}(n) = Q\big(y(n)\big)$$

$$\hat{y}'(n) = \hat{y}(n) + \hat{y}'(n-1)$$

where Q(-) is the input-output function of the graded (uniform) quantizer used. From the above relations we obtain for the quantization error the expression:

$$y_Q(n) = \hat{x}(n) - x(n) = \hat{y}(n) - y(n)$$

We note that if we set $y\hat{}'(n)$ = 0 in the above equations, i.e., a DPCM system that does not use prediction, then this system is equivalent to a simple PCM coding system

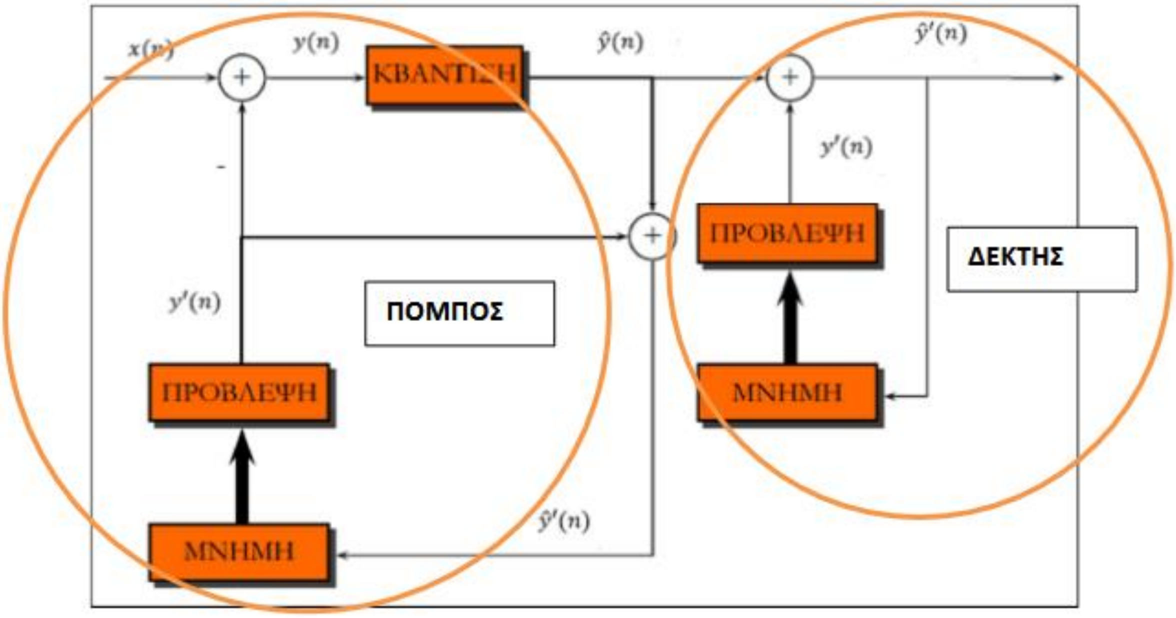

Figure 2. DPCM coding

Calculation of the Prediction Filter

In a general DPCM system, the sample prediction $x(n)$ is given as a linear combination of p previous values y^'$(n$ - $i)$ which have already been encoded and then reconstructed, i.e.

$$\hat{y}'(n) = \sum_{i=1}^{p} a_i \hat{y}'(n-i) = a_i\big(\hat{y}(n-j) + \hat{y}'(n-j-1)\big)$$

Our goal is to compute the coefficients ai with the criterion of minimizing the mean squared error between each current input sample and its prediction:

$$MSE = E[e^2(n)] = E\left[x(n) - \sum_{i=1}^{p} a_i \hat{y}'(n-i)\right]$$

The above criterion, however, is difficult to minimize since MSE depends on the coefficients and the quantizer we use. Therefore, it is a non-linear minimization problem. To get around this difficulty, we replace y^'$(n$ - $i)$ in the above relation with $x(n$ - $i)$ assuming that since the latter is the quantized version of the former we do not make a large error.

Thus, we can minimize the expression:

$$\widehat{MSE} = E[e^2(n)] = E\left[\left(x(n) - \sum_{i=1}^{p} a_i x(n-i)\right)^2\right]$$

$$\widehat{MSE} = E\left[\left(x(n)\right)^2 - 2x(n)\sum_{i=1}^{p} a_i x(n-i) + \left(\sum_{i=1}^{p} a_i x(n-i)\right)^2\right]$$

$$= E\left[\left(x(n)\right)^2\right] - 2E\left[x(n)\sum_{i=1}^{p} a_i x(n-i)\right] + E\left[\left(\sum_{i=1}^{p} a_i x(n-i)\right)^2\right]$$

$$= E\left[\left(x(n)\right)^2\right] - 2\sum_{k=1}^{p} a_i E[x(n)x(n-i)] + E\left[\sum_{i=1}^{p} a_i x(n-i) \sum_{j=1}^{p} a_j x(n-j)\right]$$

$$= E\left[\left(x(n)\right)^2\right] - 2\sum_{k=1}^{p} a_i E[x(n)x(n-i)] + E\left[\sum_{i=1}^{p}\sum_{j=1}^{p} a_i a_j x(n-i)\, x(n-j)\right]$$

$$= E\left[\left(x(n)\right)^2\right] - 2\sum_{k=1}^{p} a_i E[x(n)x(n-i)] + \sum_{i=1}^{p}\sum_{j=1}^{p} a_i a_j E[x(n-i)x(n-j)]$$

$$= R_x(0) - 2\sum_{k=1}^{p} a_i R_x(i) + \sum_{i=1}^{p}\sum_{j=1}^{p} a_i a_j R_x(i-j)$$

By deriving the previous relation of the error with respect to each of the coefficients ai of the prediction filter and setting the derivative equal to zero, we obtain a set of linear equations for the filter coefficients, i.e,

$$\frac{\partial\, \widehat{MSE}}{\partial a_i} = 0 \Longrightarrow$$

$$\sum_{k=1}^{p} a_i R_x(i-j) = R_x(i), 1 \le i, j \le p$$

$$\boldsymbol{R} * \boldsymbol{a} = \boldsymbol{r} \text{ ή } \boldsymbol{a} = \boldsymbol{R}^{-1} * \boldsymbol{r}$$

$\boldsymbol{R}$: the autocorrelation matrix of dimension $p \times p$ whose (i, j) element is $Rx(i - j)$

$\boldsymbol{r}$: autocorrelation vector of dimension $p \times 1$ whose element i is $Rx(i)$

$\boldsymbol{a}$: dimension vector $p \times 1$ with the coefficients of the prediction filter

Rx: the autocorrelation function of the random process $x(n)$. By solving the above set of Yule-Walker equations, we find the optimal coefficients of the prediction filter. In addition, for this stationary point to correspond to the minimum of the function it is sufficient that the matrix $\boldsymbol{R}$ is positively definite.

The stochastic quantities appearing in the above expression can be statistically estimated for an input sequence $x(n)$ of dimension $1 \times N$ based on the relations:

$$\hat{R}_x(i) = \frac{1}{N-p} \sum_{n=p+1}^{N} x(n)x(n-i) \,, 1 \leq i \leq p$$

$$\hat{R}_x(i-j) = \frac{1}{N-p} \sum_{n=p+1}^{N} x(n-j)x(n-i) \,, 1 \leq i,j \leq p$$

The prediction filter is calculated at the transmitter, and then the prediction filter coefficients are quantized and sent to the receiver. At this point it is worth noting that the transmitter should also use the quantized values of the prediction filter coefficients so that the transmitter and receiver operate in agreement. To calculate the quantized values of the coefficients, use the uniform quantizer you will construct, setting $\boldsymbol{N} = \mathbf{8}$ bits and dynamic range [-2 2].

Input source

The source you are asked to encode/decode is a 20,000-sample signal.The samples of the source you will be experimenting with have good predictability, i.e. a current sample can be predicted (in the statistical sense) with a small prediction error by combining previous values of the same signal. The samples of the source you will experiment with are stored in the file named source.mat. To retrieve the input data it is sufficient to type

load source.mat

Questions - Part A2

In the experiments you will perform, have the dynamic range of the quantizer be between the values max_value = 3.5 , min_value = -3.5 .

1. Implement the above DPCM coding/decoding scheme.

2. Choose two values of $p \geq 5$ and for N = 1,2,3 bits plot the original signal and the prediction error y on the same graph. Comment on the results. What do you observe?

3. Evaluate its performance with a graph showing the mean squared prediction error with respect to N and for different values of p. Specifically, for a number of binary digits N = 1,2,3bits which the uniform quantizer uses to encode the prediction signal and for a

predictor order p = 5:10. In addition, for each p, record in your report and annotate the values of the predictor coefficients.

4. For N = 1,2,3bits plot the original and reconstructed signal at the receiver for p = 5,10 and comment on the reconstruction results with respect to the quantization bits.