26.15 The metric tensor

Any particular curvilinear coordinate system is completely characterised at each point in space by the nine quantities

$$g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j, \tag{26.56}$$

which, as we will show, are the covariant components of a symmetric second-order tensor $\mathbf{g}$ called the *metric tensor*.

Since an infinitesimal vector displacement can be written as $d\mathbf{r} = du^i \mathbf{e}_i$, we find that the square of the infinitesimal arc length $(ds)^2$ can be written in terms of the metric tensor as

$$(ds)^2 = d\mathbf{r} \cdot d\mathbf{r} = du^i \mathbf{e}_i \cdot du^j \mathbf{e}_j = g_{ij} du^i du^j. \tag{26.57}$$

It may further be shown that the volume element dV is given by

$$dV = \sqrt{g}\, du^1 du^2 du^3, \tag{26.58}$$

where g is the determinant of the matrix $[g_{ij}]$, which has the covariant components of the metric tensor as its elements.

If we compare equations (26.57) and (26.58) with the analogous ones in section 10.10 then we see that in the special case where the coordinate system is orthogonal (so that $\mathbf{e}_i \cdot \mathbf{e}_j = 0$ for $i \neq j$) the metric tensor can be written in terms of the coordinate-system scale factors $h_i, i = 1, 2, 3$ as

$$g_{ij} = \begin{cases} h_i^2 & i = j, \\ 0 & i \neq j. \end{cases}$$

Its determinant is then given by $g = h_1^2 h_2^2 h_3^2$.

►*Calculate the elements g_{ij} of the metric tensor for cylindrical polar coordinates. Hence find the square of the infinitesimal arc length $(ds)^2$ and the volume dV for this coordinate system.*

As discussed in section 10.9, in cylindrical polar coordinates $(u^1, u^2, u^3) = (\rho, \phi, z)$ and so the position vector $\boldsymbol{r}$ of any point P may be written

$$\boldsymbol{r} = \rho\cos\phi\,\boldsymbol{i} + \rho\sin\phi\,\boldsymbol{j} + z\,\boldsymbol{k}.$$

From this we obtain the (covariant) basis vectors:

$$\begin{aligned}\boldsymbol{e}_1 = \frac{\partial \boldsymbol{r}}{\partial \rho} &= \cos\phi\,\boldsymbol{i} + \sin\phi\,\boldsymbol{j};\\ \boldsymbol{e}_2 = \frac{\partial \boldsymbol{r}}{\partial \phi} &= -\rho\sin\phi\,\boldsymbol{i} + \rho\cos\phi\,\boldsymbol{j};\\ \boldsymbol{e}_3 = \frac{\partial \boldsymbol{r}}{\partial z} &= \boldsymbol{k}.\end{aligned} \tag{26.59}$$

Thus the components of the metric tensor $[g_{ij}] = [\boldsymbol{e}_i \cdot \boldsymbol{e}_j]$ are found to be

$$\mathsf{G} = [g_{ij}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{26.60}$$

from which we see that, as expected for an orthogonal coordinate system, the metric tensor is diagonal, the diagonal elements being equal to the squares of the scale factors of the coordinate system.

From (26.57), the square of the infinitesimal arc length in this coordinate system is given by

$$(ds)^2 = g_{ij}\,du^i\,du^j = (d\rho)^2 + \rho^2(d\phi)^2 + (dz)^2,$$

and, using (26.58), the volume element is found to be

$$dV = \sqrt{g}\;du^1du^2du^3 = \rho\;d\rho\;d\phi\;dz.$$

These expressions are identical to those derived in section 10.9. ◄

We may also express the scalar product of two vectors in terms of the metric tensor:

$$\boldsymbol{a}\cdot\boldsymbol{b} = a^i\boldsymbol{e}_i \cdot b^j\boldsymbol{e}_j = g_{ij}a^ib^j, \tag{26.61}$$

where we have used the contravariant components of the two vectors. Similarly, using the covariant components, we can write the same scalar product as

$$\boldsymbol{a}\cdot\boldsymbol{b} = a_i\boldsymbol{e}^i \cdot b_j\boldsymbol{e}^j = g^{ij}a_ib_j, \tag{26.62}$$

where we have defined the nine quantities $g^{ij} = \boldsymbol{e}^i \cdot \boldsymbol{e}^j$. As we shall show, they form the contravariant components of the metric tensor $\boldsymbol{g}$ and are, in general, different from the quantities g_{ij}. Finally, we could express the scalar product in terms of the contravariant components of one vector and the covariant components of the other,

$$\boldsymbol{a}\cdot\boldsymbol{b} = a_i\boldsymbol{e}^i \cdot b^j\boldsymbol{e}_j = a_ib^j\delta^i_j = a_ib^i, \tag{26.63}$$

where we have used the reciprocity relation (26.54). Similarly, we could write

$$\boldsymbol{a} \cdot \boldsymbol{b} = a^i \boldsymbol{e}_i \cdot b_j \boldsymbol{e}^j = a^i b_j \delta_i^j = a^i b_i. \qquad (26.64)$$

By comparing the four alternative expressions (26.61)–(26.64) for the scalar product of two vectors we can deduce one of the most useful properties of the quantities g_{ij} and g^{ij}. Since $g_{ij}a^i b^j = a^i b_i$ holds for any arbitrary vector components a^i, it follows that

$$g_{ij} b^j = b_i,$$

which illustrates the fact that the covariant components g_{ij} of the metric tensor can be used to *lower* an *index*. In other words, it provides a means of obtaining the covariant components of a vector from its contravariant components. By a similar argument, we have

$$g^{ij} b_j = b^i,$$

so that the contravariant components g^{ij} can be used to perform the reverse operation of *raising* an *index*.

It is straightforward to show that the contravariant and covariant basis vectors, $\boldsymbol{e}^i$ and $\boldsymbol{e}_i$ respectively, are related in the same way as other vectors, i.e. by

$$\boldsymbol{e}^i = g^{ij} \boldsymbol{e}_j \qquad \textit{and} \qquad \boldsymbol{e}_i = g_{ij} \boldsymbol{e}^j.$$

We also note that, since $\boldsymbol{e}_i$ and $\boldsymbol{e}^i$ are reciprocal systems of vectors in three-dimensional space (see chapter 7), we may write

$$\boldsymbol{e}^i = \frac{\boldsymbol{e}_j \times \boldsymbol{e}_k}{\boldsymbol{e}_i \cdot (\boldsymbol{e}_j \times \boldsymbol{e}_k)},$$

for the combination of subscripts $i, j, k = 1, 2, 3$ and its cyclic permutations. A similar expression holds for $\boldsymbol{e}_i$ in terms of the $\boldsymbol{e}^i$-basis. Moreover, it may be shown that $|\boldsymbol{e}_1 \cdot (\boldsymbol{e}_2 \times \boldsymbol{e}_3)| = \sqrt{g}$.

►*Show that the matrix $[g^{ij}]$ is the inverse of the matrix $[g_{ij}]$. Hence calculate the contravariant components g^{ij} of the metric tensor in cylindrical polar coordinates.*

Using the index-lowering and index-raising properties of g_{ij} and g^{ij} on an arbitrary vector $\boldsymbol{a}$, we find

$$\delta_k^i a^k = a^i = g^{ij} a_j = g^{ij} g_{jk} a^k.$$

But, since $\boldsymbol{a}$ is arbitrary, we must have

$$g^{ij} g_{jk} = \delta_k^j. \qquad (26.65)$$

Denoting the matrix $[g_{ij}]$ by G and $[g^{ij}]$ by $\hat{G}$, equation (26.65) can be written in matrix form as $\hat{G}G = I$, where I is the unit matrix. Hence G and $\hat{G}$ are inverse matrices of each other.

Thus, by inverting the matrix G in (26.60), we find that the elements g^{ij} are given in cylindrical polar coordinates by

$$\hat{G} = [g^{ij}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\rho^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \blacktriangleleft$$

So far we have not considered the components of the metric tensor $g^i_{\ j}$ with one subscript and one superscript. By analogy with (26.56), these mixed components are given by

$$g^i_{\ j} = \boldsymbol{e}^i \cdot \boldsymbol{j}_j = \delta^j_i,$$

and so the components of $g^i_{\ j}$ are identical to those of $\delta^i_{\ j}$. We may therefore consider the $\delta^i_{\ j}$ to be the mixed components of the metric tensor $\boldsymbol{g}$.