# THE CHROMATIC NUMBER OF THE PLANE IS AT LEAST 5

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ABSTRACT. We present a family of finite unit-distance graphs in the plane that are not 4-colourable, thereby improving the lower bound of the Hadwiger-Nelson problem. The smallest such graph that we have so far discovered has 1581 vertices.

# 1. Introduction

How many colours are needed to colour the plane so that no two points at distance exactly 1 from each other are the same colour? This quantity, termed the chromatic number of the plane or CNP, was first discussed (though not in print) by Nelson in 1950 (see [Soi]). Since that year it has been known that at least four and at most seven colours are needed. The lower bound was also noted by Nelson (see [Soi]) and arises because there exist 4-chromatic finite graphs that can be drawn in the plane with each edge being a straight line of unit length, the smallest of which is the 7-vertex Moser spindle [MM] (see Figure 7, left panel). The upper bound arises because, as first observed by Isbell also in 1950 (see [Soi]), congruent regular hexagons tiling the plane can be assigned seven colours in a pattern that separates all same-coloured pairs of tiles by more than their diameter. The question of the chromatic number of the plane is termed the Hadwiger-Nelson problem, because of the contributions of Nelson just mentioned and because the 7-colouring of the hexagonal tiling was first discussed (though in another context) by Hadwiger in 1945 [Had]. The rich history of this problem and related ones is wonderfully documented in [Soi]. Since 1950, no improvement has been made to either bound.

We now give a brief overview of our construction. The graphs described in this report are obtained as follows:

- (1) We note that the 7-vertex, 12-edge unit-distance graph H consisting of the centre and vertices of a regular hexagon of side-length 1 can be coloured with at most four colours in four essentially distinct ways (that is, up to rotation, reflection and colour transposition). Two of these colourings contain a monochromatic triple of vertices and two do not.
- (2) We construct a unit-distance graph L that contains 52 copies of H and show that, in all 4-colourings of L, at least one copy of H contains a monochromatic triple.
- (3) We construct a unit-distance graph M that contains a copy of H and show that there is no 4-colouring of M in which that H contains a monochromatic triple. Thus, the unit-distance graph N created by arranging 52 copies of M so that their counterparts of H form a copy of L is not 4-colourable. This completes our demonstration that the CNP is at least 5.

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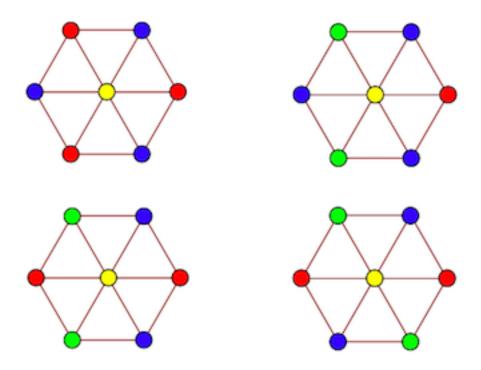


FIGURE 1. The essentially distinct ways to colour H with at most four colours.

(4) We identify smaller non-4-colourable unit-distance graphs, first by identifying vertices in N whose deletion preserves the absence of a 4-colouring, and then by more elaborate methods.

# 2. The 4-colourings of H

Figure 1 shows the four essentially distinct (i.e., up to rotation, reflection and colour transposition) colourings of H with at most four colours. The top two colourings possess, and the bottom two lack, a triple of vertices all the same colour.

# 3. Construction and colourings of L

- 3.1. A 31-vertex graph J assembled from 13 copies of H. The graph J, shown in Figure 2, contains a copy of H in its centre, six copies centred at distance 1 from its centre, and six copies centred at distance  $\sqrt{3}$  from its centre.
- 3.2. The 4-colourings of J in which no copy of H contains a monochromatic triple. Figure 3 shows six essentially distinct such colourings up to rotation, reflection, colour transposition and the colours of the vertices coloured black in Figure 3 (which will not concern us hereafter). That these are the only such colourings can be checked by grouping the possibilities according to whether the central copy of H has two (the top row in Figure 3) or no (bottom row) monochromatic pairs of vertices at distance  $\sqrt{3}$  and, in the case where it has none, whether all the copies of H centred at distance 1 from the centre also have none (bottom left colouring)

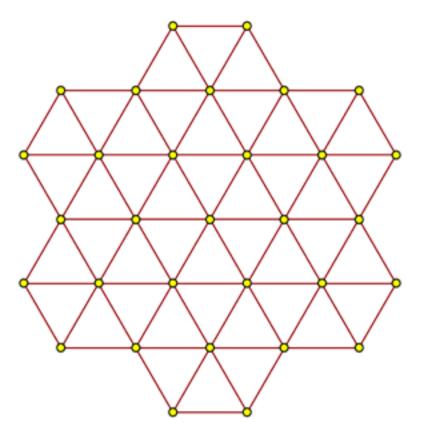


FIGURE 2. The graph J, containing 31 vertices and 13 copies of H.

or some have two (last two colourings). These last two colourings are the reason we need the vertices coloured black; though those vertices can be coloured in many ways, it turns out that if they were deleted then there would be additional 4-colourings of the remaining graph in which none of the seven remaining copies of H contained a monochromatic triple, and some of those new colourings lack a key property shared by all those in Figure 3, which we now discuss.

The aspect of the colourings in Figure 3 that will be our focus below is that they feature only three essentially distinct colourings of the vertices at distance 2 from the centre, which we shall hereafter term the linking vertices. Either (a) the linking vertices are all the same colour as the centre (left-hand cases in Figure 3), or (b) four consecutive ones (when enumerated going clockwise around the centre) are the same colour as the centre and the other two are a second colour (middle cases), or (c) two opposite ones are the same colour as the centre and all the other four are a second colour (right-hand cases). Hereafter we refer to an opposite pair of linking vertices as a linking diagonal.

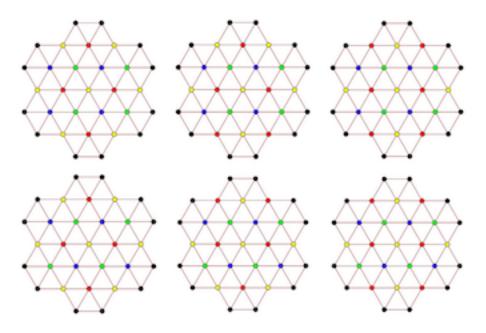


FIGURE 3. The essentially distinct colourings of J in which no copy of H contains a monochromatic triple.

3.3. A 61-vertex graph K assembled from two copies of J. We now construct K by arranging two copies of J with their centres coincident and with their relative rotation chosen so that corresponding linking vertices are at distance 1 from each other; see Figure 4.

We note that, in any 4-colouring of K in which none of the 26 copies of H contains a monochromatic triple, both copies of J must have their linking vertices coloured according to option (c) above. Further, in option (c) each of the three linking diagonals of J is monochromatic. Thus, in all 4-colourings of K where no copy of H contains a monochromatic triple, all six linking diagonals are monochromatic.

3.4. A 121-vertex graph L assembled from two copies of K. Finally, we construct L by arranging two copies of K with their counterparts of one linking vertex coincident and their counterparts of the opposite linking vertex distance 1 apart (see Figure 5). These linking vertices are labelled.

The property noted above for K trivially implies that in no 4-colouring of L do all of its 52 constituent copies of H lack a monochromatic triple.

## 4. Construction and colourings of M

In seeking graphs that can serve as M in our construction, we focus on graphs that contain a high density of Moser spindles. The motivation for exploring such graphs is that a spindle contains two pairs of vertices distance  $\sqrt{3}$  apart, and these pairs cannot both be monochromatic. Intuitively, therefore, a graph containing a high density of interlocking spindles might be constrained to have its monochromatic  $\sqrt{3}$ -apart vertex pairs distributed rather uniformly (in some sense) in any

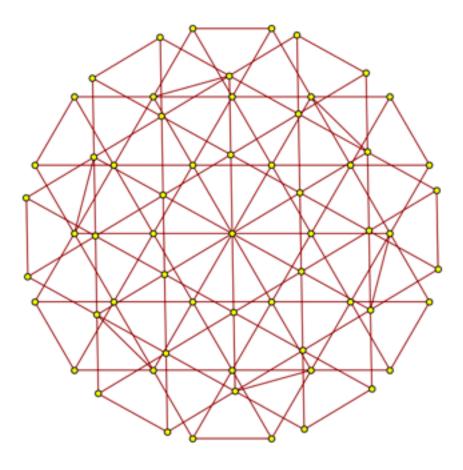


FIGURE 4. The graph K, containing 61 vertices and 26 copies of H.

4-colouring. Since such graphs typically also contain regular hexagons of sidelength 1, one might be optimistic that they could contain some such hexagon that does not contain a monochromatic triple in any 4-colouring of the overall graph, since such a triple is always an equilateral triangle of edge  $\sqrt{3}$  and thus constitutes a locally high density, i.e. a departure from the aforementioned uniformity, of monochromatic  $\sqrt{3}$ -apart vertex pairs.

- 4.1. Graphs with high edge density and spindle density. In seeking graphs with high spindle density, we begin by noting two attractive features of the 9-vertex unit-distance graph T that is obtained by adding two particular vertices to a spindle (see Figure 6, left). These vertices, P and Q:
  - (1) form an equilateral triangle with the tip X of the spindle
  - (2) lie on (the extension of) the line forming the base YZ of the spindle

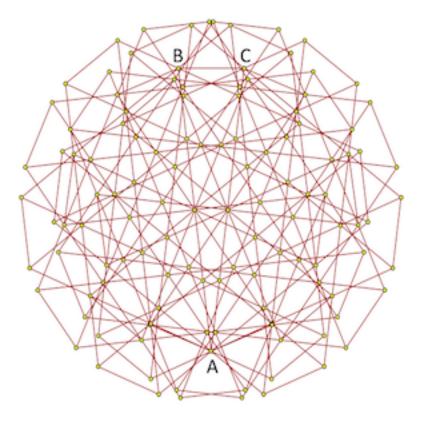
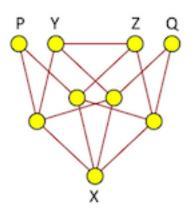


FIGURE 5. The graph L, containing 121 vertices and 52 copies of H.

We can thus construct a 15-vertex unit-distance graph U (Figure 6, right) that contains three spindles and possesses rotational and reflectional triangular symmetry. This symmetry suggests that graphs formed by combining translations and 60-degree rotations of U might have particulatly high edge and spindle density.

4.2. Construction of a graph that serves as M. The speculation just mentioned turns out to be true; for example, we have found a 97-vertex graph containing 78 spindles (not shown). Therefore, a custom program (outlined in the next section) was written to test graphs of this form for possession of a 4-colouring in which the central H contains a monochromatic triple. Disappointingly, we could not find a graph of this form that enforces sufficient uniformity of the distribution of  $\sqrt{3}$ -apart monochromatic vertex pairs to deliver the property we require for M, even though we checked examples with well over 1000 vertices.

However, this approach can be extended. Graphs arising from the construction described thus far can have spindles in only six different orientations, with edges falling into just three equivalence classes based on triangular symmetry; see Figure 7 (left). A natural elaboration is to add such classes based on the relative orientations of spindles that share a lot of vertices, such as in Figure 7 (middle). The maximum possible degree of a vertex in a graph constructed from these vectors is thereby increased from 18 to 30; see Figure 7 (right), which we denote as graph V.



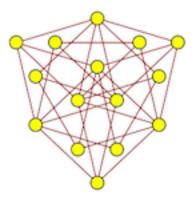
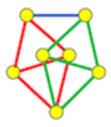


FIGURE 6. The graphs T and U.





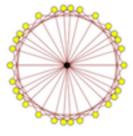


FIGURE 7. The vector classes present in one (left), or three tightly linked (middle), Moser spindles; the graph V (right).

Happily, this turns out to suffice. The 301-vertex graph W shown in Figure 8 (left) is built by centring copies of V at all vertices of V and then deleting all vertices that lie further than  $\sqrt{3}$  from its centre. The 1345-vertex graph shown in Figure 8 (right) is then built by centring copies of W at all seven vertices of H. Our program did not find any 4-colouring of this latter graph in which the central H contains a monochromatic triple, so it can serve as our M. Arranging 52 copies of it so that their counterparts of H form a copy of L gives a 20425-vertex unit-distance graph N which, therefore, is not 4-colourable. (We do not show a picture of N, because it is visually impenetrable and we have anyway discovered smaller examples, discussed below. Even W and M are too large for their pictures to be explanatory; we include them just in case it is interesting to see the general shape.)

4.3. Testing 4-colourability of edge-dense, spindle-dense graphs. A vertex count of 1345 exceeds the size of general graphs whose chromatic number can be determined computationally in reasonable time by simplistic search, and a count of 20425 may not be tractable even with state-of-the-art algorithms and hardware. It was thus necessary to develop a custom program to test graphs for possession of the property required for our graph M.

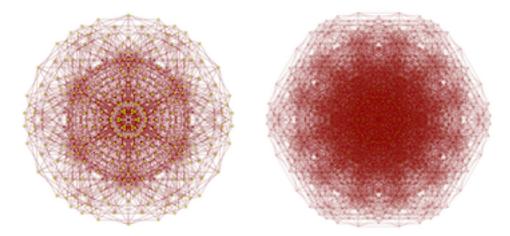


FIGURE 8. The graphs W (left) and M (right).

Because we are only asking whether a specific number of colours is or is not sufficient, and also because of the high density of edges and spindles in our target graphs, the required test turns out to be computationally far cheaper than a general determination of chromatic number of comparable-sized graphs. It can be performed rapidly by a simple depth-first search optimised only slightly, as follows:

- (1) Since our question is whether there is any 4-colouring of M in which the central H contains a monochromatic triple, we begin by specifying the colourings of the vertices of that central H, which we term the initialising vertices. Since M has the same symmetry as H, we need only check the two essentially distinct triple-containing colourings of those vertices.
- (2) We order the remaining vertices according to a hierarchy (most significant first) of parameters (all decreasing): how many spindles they are part of, their degree, and how many unit triangles they are part of.
- (3) We colour the next not-yet-coloured vertex (initially vertex 8) with the first colour that we have not already tried for it (initially colour 1).
- (4) We check each not-yet-coloured neighbour (if any) of the just-coloured vertex to see how many colours are still permissible for it. If any such vertex already has neighbours of all four colours, we will need to backtrack (see step 6 below). If any has neighbours of exactly three colours, we assign the remaining (forced) colour to it.
- (5) If we do not need to backtrack, but we did colour some vertices in step 4, we repeat step 4 for each such vertex.
- (6) If we need to backtrack, we uncolour everything that we just coloured in the most recent iteration of steps 3 and (any resulting iterations of) 4.
- (7) If we just did an uncolouring and the just-uncoloured vertex that was coloured at step 3 has no colours that have not yet been tried, we repeat step 6 for the next-most-recent iteration of steps 3 and 4 unless we have already backtracked all the way to the vertices of H. Otherwise, if there are still some uncoloured vertices we return to step 3.
- (8) We terminate when we get here, i.e. when either all vertices are coloured or we have backtracked all the way down to the vertices of H.

This algorithm was implemented in Mathematica 11 on a standard MacBook Air and terminated in only a few minutes for our candidate M, without finding a 4-colouring starting from either of the triple-containing colourings of its central H. Essentially, the speed arises because the fixing of only 20 or so colours at step 3 typically lets almost all remaining colours be forced at step 4.

## 5. Identification of smaller solutions

One may naturally wonder how large the smallest non-4-colourable unit-distance graph is. Moreover, if substantially smaller graphsare found, they may be verifiable independently by standard algorithms such as SAT solvers. Accordingly, in this section we outline some promising lines of attack for identifying non-4-colourable unit-distance graphs with fewer than 20425 vertices.

The most direct way to find such graphs is to seek a succession of small simplifications of N. Although our colouring program is extremely fast for graphs like M, unfortunately it is not fast enough to be practical for the more sparsely-interlinked combinations of M that constitute N. Thus, we now list a selection of approaches to shrinking N that are subtler than simply deleting individual vertices from it and seeking a 4-colouring.

- 5.1. Shrinking M. Most simply, N can potentially be shrunk by identifying vertices in M whose deletion does not introduce any 4-colourings in which its central copy of H contains a monochromatic triple.
- 5.2. Shrinking S. Define S to be the graph formed by arranging 13 copies of M so that their central copies of H form a copy of J. The property of J that we use in building L refers only to its linking vertices (and its central vertex), not to the copies of H from which J is built. Thus, we can delete vertices from S if no new 4-colouring of the remaining graph arises that incorporates a new colouring of its linking vertices, even if some such new colourings do give some constituent copy of H a monochromatic triple. Since N is built from four copies of S, this translates into a potential shrinking of N.
- 5.3. Relaxing the constraint on J and thus on S. Further, it is not strictly required that the linking vertices of J (hence of S) conform to one of the three colouring schemes from Figure 3. In order to ensure that all the linking diagonals of K are monochromatic, it suffices that any of the following hold for J and S:
  - (1) At least five of the linking vertices are the same colour as the centre
  - (2) Exactly four are, and they do not form a rectangle
  - (3) Exactly two are, and all three linking diagonals are monochromatic
- 5.4. Relaxing the construction of N. N is constructed by joining four copies of S in a chain, so those four copies fall into two pairs with potentially different constraints. Let N be built from two copies each of Sa and Sb, such that the vertices labelled in Figure 5 are linking diagonals of the copies of Sa rather than Sb. Two relaxations then arise:
  - (1) Only one, not all three, of the linking diagonals of Sa need be monochromatic (though we must specify which one);

- (2) The possible 4-colourings of the linking vertices of Sa and Sb must be mutually incompatible when the specified linking diagonal of Sa is not monochromatic, but that is all. Thus, for example, it is enough that in Sa either the specified linking diagonal is monochromatic or at least one vertex of each linking diagonal is the same colour as the centre, so long as in Sb at least one linking diagonal has both vertices the same colour as the centre. Both of these constraints are weaker than that given above for S.
- 5.5. Adding new vertices that enable the deletion of others. S (whether generically or as Sa and Sb) contains many sets X of four or more vertices that lie on a circle of unit radius whose centre is not a vertex of S. Suppose that (perhaps after shrinking it) the deletion of any vertex of S allows the remaining graph to be 4-coloured, but that for some such set S there is a set S of at least two vertices of S whose simultaneous deletion allows the remaining graph to be 4-coloured but only in ways that assign all four colours to members of S. Then S (hence S) can be shrunk by adding a single vertex joined to all the members of S (and located at the centre of the circle on which they lie) and deleting all members of S. In principle this approach can of course also be applied directly to S itself.

### 6. Status of shrinking N

We have so far shrunk N by a factor of nearly 13, our current record being the 1581-vertex graph G that (again more for artistic than expository reasons) is shown in Figure 9. It was verified using the same program described above, applied to its constituent Sa and Sb with the permitted 4-colourings of linking vertices that were given in section 5.4(2) above. Sb retains the same symmetry as H so the initialising vertices and colourings used for it were as before; Sa has only reflectional symmetry about the x-axis so we used (0,0), (-2,0) and (2,0) with colourings 1,1,2 or 1,2,1 or 1,2,3. To construct it, we create Sb by placing vertices at

$$(0,0), (1/3,0), (1,0), (2,0), ((\sqrt{33}-3)/6,0), (1/2,1/\sqrt{12}), (1,1/\sqrt{3}), (3/2,\sqrt{3}/2), \\ (7/6,\sqrt{11}/6), (1/6,(\sqrt{12}-\sqrt{11})/6), (5/6,(\sqrt{12}-\sqrt{11})/6), \\ (2/3,(\sqrt{11}-\sqrt{3})/6), (2/3,(3\sqrt{3}-\sqrt{11})/6), ((\sqrt{33}/6,1/\sqrt{12}), \\ ((\sqrt{33}+3)/6,1/\sqrt{3}), ((\sqrt{33}+1)/6,(3\sqrt{3}-\sqrt{11})/6), ((\sqrt{33}-1)/6,(3\sqrt{3}-\sqrt{11})/6), \\ ((\sqrt{33}+1)/6,(\sqrt{11}-\sqrt{3})/6), ((\sqrt{33}-1)/6,(\sqrt{11}-\sqrt{3})/6), \\ ((\sqrt{33}-2)/6,(2\sqrt{3}-\sqrt{11})/6), ((\sqrt{33}-4)/6,(2\sqrt{3}-\sqrt{11})/6), \\ ((\sqrt{33}+13)/12,(\sqrt{11}-\sqrt{3})/12), ((\sqrt{33}+11)/12,(\sqrt{3}+\sqrt{11})/12), \\ ((\sqrt{33}+9)/12,(\sqrt{11}-\sqrt{3})/4), ((\sqrt{33}+9)/12,(3\sqrt{3}+\sqrt{11})/12), \\ ((\sqrt{33}+7)/12,(\sqrt{3}+\sqrt{11})/12), ((\sqrt{33}+7)/12,(3\sqrt{3}-\sqrt{11})/12), \\ ((\sqrt{33}+5)/12,(5\sqrt{3}-\sqrt{11})/12), ((\sqrt{33}+5)/12,(\sqrt{11}-\sqrt{3})/12), \\ ((\sqrt{33}+3)/12,(3\sqrt{11}-5\sqrt{3})/12), ((\sqrt{33}+3)/12,(\sqrt{11}-\sqrt{3})/12), \\ ((\sqrt{33}+3)/12,(3\sqrt{3}-\sqrt{11})/12), ((\sqrt{33}+1)/12,(\sqrt{11}-\sqrt{3})/12), \\ ((\sqrt{33}-1)/12,(3\sqrt{3}-\sqrt{11})/12), ((\sqrt{33}-3)/12,(\sqrt{11}-\sqrt{3})/12), \\ ((15-\sqrt{33})/12,(3\sqrt{3}-\sqrt{11})/12), ((15-\sqrt{33})/12,(\sqrt{11}-\sqrt{3})/12), \\ ((13-\sqrt{33})/12,(3\sqrt{3}-\sqrt{11})/12), ((11-\sqrt{33})/12,(\sqrt{11}-\sqrt{3})/12), \\ ((13-\sqrt{33})/12,(3\sqrt{3}-\sqrt{11})/12), ((11-\sqrt{33})/12,(\sqrt{11}-\sqrt{3})/12), \\ ((11-\sqrt{33})/12,(3\sqrt{3}-\sqrt{11})/12), ((11-\sqrt{33})/12,(\sqrt{11}-\sqrt{3})/12), \\ ((11-\sqrt{33})/12,(\sqrt{11}-\sqrt{3})/12), ((11-\sqrt{33})/12,(\sqrt{11}-\sqrt{3})/12), \\ ((11-\sqrt{33})/12,(\sqrt{11}-\sqrt{3})/12), ((11-\sqrt{33})/12,(\sqrt{11}-\sqrt{3})/12), \\ ((11-\sqrt{33})/12,(\sqrt{11}-\sqrt{33})/12,(\sqrt{11}-\sqrt{33})/12,(\sqrt{11}-\sqrt{33})/12), \\ ((11-\sqrt{33})/12,(\sqrt{11}-\sqrt{33})/12,(\sqrt{11}-\sqrt{33})/12,(\sqrt{11}-\sqrt{33})/12, \\ ((11-\sqrt{33})/12,(\sqrt{11}-\sqrt{33})/12,(\sqrt{11}-$$

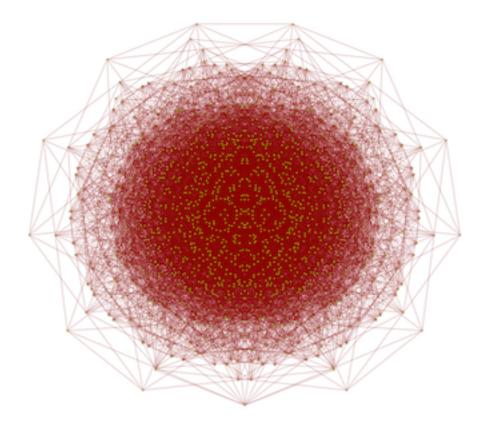


FIGURE 9. The 1581-vertex, non-4-colourable unit-distance graph G.

and at all points obtained by rotating those vectors around the origin by multiples of 60 degrees and/or by negating their y-coordinates, giving a total of 397 vertices. Sa is then created by deleting the vertices (1/3,0) and (-1/3,0) from Sb, and the full graph is assembled from two copies each of Sa and Sb as before, using (-2,0) and (2,0) as the specified linking diagonal of Sa. Happily, G has turned out to be within the reach of standard SAT solvers, with which others have now confirmed its chromatic number to be 5 without the need to resort to checking weaker properties of subgraphs.

This attempt to identify smaller examples has thus far been rather cursory, so we think it highly likely that smaller examples than G exist.

### 7. Acknowledgements

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