COMPLEX ANALYSIS

(MA-C-S)

Recommended Books:

1. Complex Variables and Complexities, Ahlfors, Theorem
2. Complex Analysis, Ahlfors
3. Complex Variables, Ahlfors
4. Complex Analysis, Ahlfors
5. Functions and Spaces, Bonsall
6. Complex Analysis, Bonsall
7. Complex Analysis, Bonsall

Uniformly Bounded :- A family of functions F is said to be uniformly bounded if $\exists\, M \in \mathbb{R}$ such that

$|f(z)| \le M$ for every $f \in F$ and z belonging to the compact.

$|f(z)| \le M \quad \forall f \in F$

Locally uniformly bounded :- A family of functions is said to be locally uniformly bounded on a set S if on each point, there corresponds a δ that is which F is uniformly bounded

Ex:- $f(z) = \frac{1}{1-z}$; $|z| < 1$

G_n :- $f(z) = n z$; $|z| < 1$

Sol:- $f(z) = \frac{1}{1-z}$ is uniformly bounded in every compact subset $|z| \le r < 1$

$\Rightarrow |f(z)| \le \frac{1}{1-|z|}$

$\Rightarrow |f(z)| \le \frac{1}{1-r}$

$\Rightarrow |f(z)| \le M$

Sol:- G_n $|z| < 1$

$|f(z)| = |nz| \le n$

$\Rightarrow |f(z)| \le n$

$\Rightarrow |f(z)| \le M$

Hence, each member is G_n equicontinuity holds, cause it depends upon n not z

Normal family :- A family of functions is said to be normal in a domain D if every seq $\{f_n\} \in D$ has a subseq $\{f_{n_k}\}$ that converge uniformly on each compact subset of D.

Ex: $f_n(z) = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}$

$f_n(z) = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}$

Ex: $f_n(z) = z^n$

Sol:- $f_n(z) = z^n$

$\Rightarrow |f_n(z)| = |z^n|$

$\Rightarrow |f_n(z)| = |z|^n$

$\Rightarrow |f_n(z)| \le 1$

$\Rightarrow |f_n(z)| \le M$

Equicontinuity :- A family of functions F is said to be equicontinuous if for every $\epsilon > 0$ $\exists\, \delta > 0$ $\ni$ $\forall f \in F$, each that $|f(z_1) - f(z_2)| < \epsilon$ whenever $|z_1 - z_2| < \delta$ in a mean manner of a equicontinuous family. $\{f_n\} \in D$ is uniformly uniform, not continuous

Ex:- $f_n(z) = n z$

$|f_n(z_1) - f_n(z_2)| = |n z_1 - n z_2|$

$= n|z_1 - z_2| < \epsilon$

where $|z_1 - z_2| < \frac{\epsilon}{n}$

Hence, each member is G_n

Equicontinuity holds, cause it depends upon n not z

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$\Rightarrow |f_n| \le 1$

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Arzela - Ascoli Thm :- Suppose that F is a pointwise bounded equicontinuous collection of complex functions on a mutually open/closing a compact domain subset D.

Along the f $\{f_n\} \in F$ has a subseq that converge uniformly on every compact subset of D.

Ex: $f_n = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}$

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Proof :- Let $E = \{z_1, z_2, z_3, \ldots\}$ be the countable set in X

Let $x \in E$ be a seq $\in D$. F is defined on each E. Since F is pointwise bounded equicontinuous collection therefore each $f(z_n) \in E$ will be bounded. Again by B-W theorem, F will have a convergent subseq.

$f_1(z_1)$

$f_2(z_1)$ convergent subseq

$f_3(z_1)$

$f_4(z_1)$ is convergent subseq

$f_5(z_1)$

$f_6(z_1)$ is convergent subseq

$\Rightarrow f_1, f_2(z_1), f_3, f_4(z_1)$

By diagonal process, we can find a conv subseq which converge at every point of the set E

Th point Uniformly Convergent

Let $\{f_n\}$ be a compact subset in X being F is equicontinuous in for every $\epsilon > 0$ $|f(z_1) - f(z_2)| < \epsilon$ whenever $|z_1 - z_2| < \delta$. Now consider the set X with finite open balls $B_\delta = B(x, \delta)$. Let $x \in X$ B_δ for $x \in X$ and for each x, $\exists\, \delta_x$ $\ni$ $|f(x) - f(y)| < \epsilon$ for each x.

$\Rightarrow |f(x) - f(y)| < \epsilon$

By Cauchy criterion,

$|f(x) - f(y)| < \epsilon$

$\Rightarrow |f(x) - f(y)| < \epsilon$

Let such, then select for $x \in X$ $|z - x| < \delta$.

$|f(x) - f(z)| < |f(x) - f(x_1)| + |f(x_1) - f(x_2)| + \cdots + |f(x_n) - f(z)|$

$\Rightarrow |f(x) - f(z)| < |f(x) - f(x_1)| + |f(x_1) - f(x_2)| + \cdots + |f(x_n) - f(z)|$

$\Rightarrow \epsilon + \epsilon + \epsilon + \epsilon$

$\Rightarrow 5\epsilon \quad \because \forall\, n, n \in X$

By Cauchy criterion,

$f(z)$ is uniform by convergence

Exam (Complex Variables)

(MA-C-S)

Thm :- Let f be a holomorphic mapping of a region in into a region Ω'. If $f(z_0)$ finite is the boundary of Ω then $f(z_0)$ finite is the boundary of Ω'.

Definition Domain :- Let Ω' be the part in the plane that lies above a Ω approaching point z_0 as $z \to z_0$ the limit of sup exist Ω'.

Suppose that f is onto Ω Ω' $\Rightarrow$ analytic in Ω is true and value on the real axis then $f(z)$ has an analytic continuation to the whole domain Ω, which satisfies

$$f(z) = \overline{f(z)}$$

Proof Boundary Pts :- A is said to be from boundary pts of empty part of A the a that contain intersection with the same belonging to A is the same as the intersection with V.

Thm :- Suppose that the boundary of a simply connected region Ω contains a line segment α as a line boundary etc. Then the function $f(z)$ which maps Ω onto the unit disk can be extended to a function which is analytic in one-to-one on $\Omega \cup \alpha$. The image of α is an arc γ' on the circle.

Proof :- By Riemann mapping thm, $\exists$ a conformal with function from Ω to Ω' $\Rightarrow$ a conformal is real on α or that is on α the map Ω'. So we can define an analytic branch of

$$\log f(z) = \log|f(z)| + i\arg f(z)$$

By the reflection principle, the function $\log f(z)$ will be analytic on the line bounding part of Ω that is, the image of α will be arc γ on the boundary of unit circle.

Cauchy :- Suppose $f(z)$ is not continuous on the boundary arc γ'. $f(z)$ is not continuous on γ'

$\lim_{z \to \gamma'} f(z) \ne 0$

$\lim_{z \to \gamma'} f(z) \ne 0$

$\lim_{z \to \gamma'} |f(z)| = \lim_{z \to \gamma'} f(z) < 0$

$\frac{1}{f(z)}$ $\lim_{z \to \gamma'} \frac{1}{f(z)} = 0$

where $f(z)$ is not one-one

$\Rightarrow z_1 = z_2$ such that

$f(z_1) = f(z_2) \quad z_1 \ne z_2$

$\lim_{z \to \gamma'} f(z_1) = \lim_{z \to \gamma'} f(z_2)$

which is a contradiction to $\lim_{z \to \gamma'} f(z)$

Hence, $f(z)$ is a one-one function on γ'.

Real Analytic Function :- A real function $f(x)$ of a real variable is defined on an interval (a, b), is said to be real analytic if for every point in an interval, the Taylor series expansion of $f(x)$ around x_0,

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots$$

converges to some nbd of x_0 $\forall\, x \in (a, b)$ and $x_0 \in (a, b)$

- By Taylor's thm the series is also convergent for complex values $z \in (a, b) \times (a, b)$ and is representation analytic function in the strip.
- $f(x)$ can be said to be analytic function in a region in symmetric to the real axis which contains the segment (a, b)
- In some circumstances, the seq $f(x)$ becomes an analytic fnc.
- **Regular Analytic Fnc** :- $f(x)$ is said to be regular if $f(z) \ne 0$ $\forall\, z \in (a, b)$
- **Simple Analytic Fnc** :- $f(x)$ is said to be simple analytic fnc if $f(z_1) = f(z_2)$ only when $z_1 = z_2$ $\in (a, b)$

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