2. PARTIAL FRACTIONS AND FACTORIZATION

A rational function has two standard representations, one by partial fractions and the other by factorization of the numerator and the denominator. The present section is devoted to similar representations of arbitrary meromorphic functions.

2.1. Partial Fractions. If the function $f(z)$ is meromorphic in a region Ω, there corresponds to each pole b_ν a singular part of $f(z)$ consisting of the part of the Laurent development which contains the negative powers of $z - b_\nu$; it reduces to a polynomial $P_\nu(1/(z - b_\nu))$. It is tempting to subtract all singular parts in order to obtain a representation

$$f(z) = \sum_\nu P_\nu\left(\frac{1}{z - b_\nu}\right) + g(z) \tag{4}$$

where $g(z)$ would be analytic in Ω. However, the sum on the right-hand side is in general infinite, and there is no guarantee that the series will converge. Nevertheless, there are many cases in which the series converges, and what is more, it is frequently possible to determine $g(z)$ explicitly from general considerations. In such cases the result is very rewarding; we obtain a simple expansion which is likely to be very helpful.

If the series in (4) does not converge, the method needs to be modified. It is clear that nothing essential is lost if we subtract an analytic function $p_\nu(z)$ from each singular part P_ν. By judicious choice of the functions p_ν the series $\sum_\nu (P_\nu - p_\nu)$ can be made convergent. It is even possible to take the $p_\nu(z)$ to be polynomials.

We shall not prove the most general theorem to this effect. In the case where Ω is the whole plane we shall, however, prove that every meromorphic function has a development in partial fractions and, moreover, that the singular parts can be described arbitrarily. The theorem and its generalization to arbitrary regions are due to Mittag-Leffler.

Theorem 4. *Let $\{b_\nu\}$ be a sequence of complex numbers with $\lim_{\nu\to\infty} b_\nu = \infty$, and let $P_\nu(\zeta)$ be polynomials without constant term. Then there are functions which are meromorphic in the whole plane with poles at the points b_ν and the corresponding singular parts $P_\nu(1/(z - b_\nu))$. Moreover, the most general meromorphic function of this kind can be written in the form*

$$f(z) = \sum_\nu \left[P_\nu\left(\frac{1}{z - b_\nu}\right) - p_\nu(z)\right] + g(z) \tag{5}$$

where the $p_\nu(z)$ are suitably chosen polynomials and $g(z)$ is analytic in the whole plane.

We may suppose that no b_ν is zero. The function $P_\nu(1/(z - b_\nu))$ is analytic for $|z| < |b_\nu|$ and can thus be expanded in a Taylor series about the origin. We choose for $p_\nu(z)$ a partial sum of this series, ending, say, with the term of degree n_ν. The difference $P_\nu - p_\nu$ can be estimated by use of the explicit expression for the remainder given in Chap. 4, Sec. 3.1. If the maximum of $|P_\nu|$ for $|z| \leqq |b_\nu|/2$ is denoted by M_ν, we obtain

$$\left|P_\nu\left(\frac{1}{z - b}\right) - p_\nu(z)\right| \leqq 2M\left(\frac{2|z|}{|b_\nu|}\right)^{n_\nu+1} \tag{6}$$

for all $|z| \leqq |b_\nu|/4$. By this estimate it is clear that the series in the right-hand member of (5) can be made absolutely convergent in the whole plane, except at the poles, by choosing the n_ν sufficiently large. For instance, if we choose n_ν so large that $2^{n_\nu} \geqq M_\nu 2^\nu$, the estimate (6) will show that the general term is majorized by $2^{-\nu}$ for all sufficiently large ν.

Moreover, the estimate holds uniformly in any closed disk $|z| \leqq R$, so that the convergence is actually uniform in that disk provided we omit the terms with $|b_\nu| \leqq R$. By Weierstrass's theorem the remaining series represents an analytic function in $|z| \leqq R$, and it follows that the full series is meromorphic in the whole plane with the singular parts $P_\nu(1/(z - b_\nu))$. The rest of the theorem is trivial.

As a first example we consider the function $\pi^2/\sin^2 \pi z$, which has double poles at the points $z = n$ for integral n. The singular part at the origin is $1/z^2$, and since $\sin^2 \pi(z - n) = \sin^2 \pi z$, the singular part at $z = n$ is $1/(z - n)^2$. The series

$$\sum_{n=-\infty}^{+\infty} \frac{1}{(z - n)^2} \tag{7}$$

is convergent for $z \neq n$, as seen by comparison with the familiar series $\sum_1^\infty 1/n^2$. It is uniformly convergent on any compact set after omission of the terms which become infinite on the set. For this reason we can write

$$\frac{\pi^2}{\sin^2 \pi z} = \sum_{n=-\infty}^{+\infty} \frac{1}{(z - n)^2} + g(z) \tag{8}$$

where $g(z)$ is analytic in the whole plane. We contend that $g(z)$ is identically zero.

To prove this we observe that the function $\pi^2/\sin^2 \pi z$ and the series (7) are both periodic with the period 1. Therefore the function $g(z)$ has the same period. For $z = x + iy$ we have (Chap. 2, Sec. 3.2, Ex. 4)

$$|\sin \pi z|^2 = \cosh^2 \pi y - \cos^2 \pi x$$

and hence $\pi^2/\sin^2 \pi z$ tends uniformly to 0 as $|y| \to \infty$. But it is easy to see that the function (7) has the same property. Indeed, the convergence is uniform for $|y| \geqq 1$, say, and the limit for $|y| \to \infty$ can thus be obtained by taking the limit in each term. We conclude that $g(z)$ tends uniformly to 0 for $|y| \to \infty$. This is sufficient to infer that $|g(z)|$ is bounded in a period strip $0 \leqq x \leqq 1$, and because of the periodicity $|g(z)|$ will be bounded in the whole plane. By Liouville's theorem $g(z)$ must reduce to a constant, and since the limit is 0 the constant must vanish. We have thus proved the identity

$$\frac{\pi^2}{\sin^2 \pi z} = \sum_{-\infty}^{\infty} \frac{1}{(z-n)^2}. \tag{9}$$

From this equation a related identity can be obtained by integration. The left-hand member is the derivative of $-\pi \cot \pi z$, and the terms on the right are derivatives of $-1/(z-n)$. The series with the general term $1/(z-n)$ diverges, and a partial sum of the Taylor series must be subtracted from all the terms with $n \neq 0$. As it happens it is sufficient to subtract the constant terms, for the series

$$\sum_{n \neq 0} \left(\frac{1}{z-n} + \frac{1}{n} \right) = \sum_{n \neq 0} \frac{z}{n(z-n)}$$

is comparable with $\sum_1^\infty 1/n^2$ and hence convergent. The convergence is uniform on every compact set, provided that we omit the terms which become infinite. For this reason termwise differentiation is permissible, and we obtain

$$\pi \cot \pi z = \frac{1}{z} + \sum_{n \neq 0} \left(\frac{1}{z-n} + \frac{1}{n} \right) \tag{10}$$

except for an additive constant. If the terms corresponding to n and $-n$ are bracketed together, (10) can be written in the equivalent forms

$$\pi \cot \pi z = \lim_{m \to \infty} \sum_{n=-m}^{m} \frac{1}{z-n} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}. \tag{11}$$

With this way of writing it becomes evident that both members of the equation are odd functions of z, and for this reason the integration constant must vanish. The equations (10) and (11) are thus correctly stated.

Let us now reverse the procedure and try to evaluate the analogous sum

$$\lim_{m\to\infty} \sum_{-m}^{m} \frac{(-1)^n}{z-n} = \frac{1}{z} + \sum_{1}^{\infty} (-1)^n \frac{2z}{z^2-n^2} \tag{12}$$

which evidently represents a meromorphic function. It is very natural to separate the odd and even terms and write

$$\sum_{-(2k+1)}^{2k+1} \frac{(-1)^n}{z-n} = \sum_{n=-k}^{k} \frac{1}{z-2n} - \sum_{n=-k-1}^{k} \frac{1}{z-1-2n}.$$

By comparison with (11) we find that the limit is

$$\frac{\pi}{2}\cot\frac{\pi z}{2} - \frac{\pi}{2}\cot\frac{\pi(z-1)}{2} = \frac{\pi}{\sin \pi z},$$

and we have proved that

$$\frac{\pi}{\sin \pi z} = \lim_{m\to\infty} \sum_{-m}^{m} (-1)^n \frac{1}{z-n}. \tag{13}$$

EXERCISES

1. Comparing coefficients in the Laurent developments of $\cot \pi z$ and of its expression as a sum of partial fractions, find the values of

$$\sum_{1}^{\infty} \frac{1}{n^2}, \quad \sum_{1}^{\infty} \frac{1}{n^4}, \quad \sum_{1}^{\infty} \frac{1}{n^6}.$$

Give a complete justification of the steps that are needed.

2. Express

$$\sum_{-\infty}^{\infty} \frac{1}{z^3-n^3}$$

in closed form.

3. Use (13) to find the partial fraction development of $1/\cos \pi z$, and show that it leads to $\pi/4 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$.

4. What is the value of

$$\sum_{-\infty}^{\infty} \frac{1}{(z+n)^2+a^2}?$$

5. Using the same method as in Ex. 1, show that

$$\sum_{1}^{\infty} \frac{1}{n^{2k}} = 2^{2k-1} \frac{B_k}{(2k)!} \pi^{2k}.$$

(See Sec. 1.3, Ex. 4, for the definition of B_k.)

2.2. Infinite Products. An infinite product of complex numbers

$$p_1p_2 \cdots p_n \cdots = \prod_{n=1}^{\infty} p_n \tag{14}$$

is evaluated by taking the limit of the partial products $P_n = p_1p_2 \cdots p_n$. It is said to converge to the value $P = \lim_{n\to\infty} P_n$ if this limit exists and is different from zero. There are good reasons for excluding the value zero. For one thing, if the value $P = 0$ were permitted, any infinite product with one factor 0 would converge, and the convergence would not depend on the whole sequence of factors. On the other hand, in certain connections this convention is too radical. In fact, we wish to express a function as an infinite product, and this must be possible even if the function has zeros. For this reason we make the following agreement: The infinite product (14) is said to converge if and only if at most a finite number of the factors are zero, and if the partial products formed by the nonvanishing factors tend to a finite limit which is different from zero.

In a convergent product the general factor p_n tends to 1; this is clear by writing $p_n = P_n/P_{n-1}$, the zero factors being omitted. In view of this fact it is preferable to write all infinite products in the form

$$\prod_{n=1}^{\infty} (1 + a_n) \tag{15}$$

so that $a_n \to 0$ is a necessary condition for convergence.

If no factor is zero, it is natural to compare the product (15) with the infinite series

$$\sum_{n=1}^{\infty} \log (1 + a_n). \tag{16}$$

Since the a_n are complex we must agree on a definite branch of the logarithms, and we decide to choose the principal branch in each term. Denote the partial sums of (16) by S_n. Then $P_n = e^{S_n}$, and if $S_n \to S$ it follows that P_n tends to the limit $P = e^S$ which is $\neq 0$. In other words, the convergence of (16) is a sufficient condition for the convergence of (15).

In order to prove that the condition is also necessary, suppose that $P_n \to P \neq 0$. It is not true, in general, that the series (16), formed with the principal values, converges to the principal value of $\log P$; what we wish to show is that it converges to some value of $\log P$. For greater clarity we shall temporarily adopt the usage of denoting the principal value of the logarithm by Log and its imaginary part by Arg.

Because $P_n/P \to 1$ it is clear that $\operatorname{Log}(P_n/P) \to 0$ for $n \to \infty$. There exists an integer h_n such that $\operatorname{Log}(P_n/P) = S_n - \operatorname{Log} P + h_n \cdot 2\pi i$. We pass to the differences to obtain $(h_{n+1} - h_n)2\pi i = \operatorname{Log}(P_{n+1}/P) - \operatorname{Log}(P_n/P) - \operatorname{Log}(1 + a_n)$ and hence $(h_{n+1} - h_n)2\pi = \operatorname{Arg}(P_{n+1}/P) - \operatorname{Arg}(P_n/P) - \operatorname{Arg}(1 + a_n)$. By definition, $|\operatorname{Arg}(1 + a_n)| \leqq \pi$, and we know that $\operatorname{Arg}(P_{n+1}/P) - \operatorname{Arg}(P_n/P) \to 0$. For large n this is incompatible with the previous equation unless $h_{n+1} = h_n$. Hence h_n is ultimately equal to a fixed integer h, and it follows from $\operatorname{Log}(P_n/P) = S_n - \operatorname{Log} P + h \cdot 2\pi i$ that $S_n \to \operatorname{Log} P - h \cdot 2\pi i$. We have proved:

Theorem 5. *The infinite product* $\prod_1^\infty (1 + a_n)$ *with* $1 + a_n \neq 0$ *converges simultaneously with the series* $\sum_1^\infty \log(1 + a_n)$ *whose terms represent the values of the principal branch of the logarithm.*

The question of convergence of a product can thus be reduced to the more familiar question concerning the convergence of a series. It can be further reduced by observing that the series (16) converges absolutely at the same time as the simpler series $\Sigma\, |a_n|$. This is an immediate consequence of the fact that

$$\lim_{z \to 0} \frac{\log(1 + z)}{z} = 1.$$

If either the series (16) or $\sum_1^\infty |a_n|$ converges, we have $a_n \to 0$, and for a given $\varepsilon > 0$ the double inequality

$$(1 - \varepsilon)|a_n| < |\log(1 + a_n)| < (1 + \varepsilon)|a_n|$$

will hold for all sufficiently large n. It follows immediately that the two series are in fact simultaneously absolutely convergent.

An infinite product is said to be absolutely convergent if and only if the corresponding series (16) converges absolutely. With this terminology we can state our result in the following terms:

Theorem 6. *A necessary and sufficient condition for the absolute convergence of the product* $\prod_1^\infty (1 + a_n)$ *is the convergence of the series* $\sum_1^\infty |a_n|$.

In the last theorem the emphasis is on absolute convergence. By simple examples it can be shown that the convergence of $\sum_1^\infty a_n$ is neither sufficient nor necessary for the convergence of the product $\prod_1^\infty (1 + a_n)$.

It is clear what to understand by a uniformly convergent infinite product whose factors are functions of a variable. The presence of zeros may cause some slight difficulties which can usually be avoided by considering only sets on which at most a finite number of the factors can vanish. If these factors are omitted, it is sufficient to study the uniform convergence of the remaining product. Theorems 5 and 6 have obvious counterparts for uniform convergence. If we examine the proofs, we find that all estimates can be made uniform, and the conclusions lead to uniform convergence, at least on compact sets.

EXERCISES

1. Show that

$$\prod_{n=2}^\infty \left(1 - \frac{1}{n^2}\right) = \frac{1}{2}.$$

2. Prove that for $|z| < 1$

$$(1 + z)(1 + z^2)(1 + z^4)(1 + z^8) \ldots = \frac{1}{1 - z}.$$

3. Prove that

$$\prod_1^\infty \left(1 + \frac{z}{n}\right) e^{-z/n}$$

converges absolutely and uniformly on every compact set.

4. Prove that the value of an absolutely convergent product does not change if the factors are reordered.

5. Show that the function

$$\theta(z) = \prod_1^\infty (1 + h^{2n-1}e^z)(1 + h^{2n-1}e^{-z})$$

where $|h| < 1$ is analytic in the whole plane and satisfies the functional equation

$$\theta(z + 2 \log h) = h^{-1}e^{-z}\, \theta(z).$$

2.3. *Canonical Products.* A function which is analytic in the whole plane is said to be *entire*, or *integral.* The simplest entire functions which are not polynomials are e^z, $\sin z$, and $\cos z$.

If $g(z)$ is an entire function, then $f(z) = e^{g(z)}$ is entire and $\neq 0$. Conversely, if $f(z)$ is any entire function which is never zero, let us show that $f(z)$ is of the form $e^{g(z)}$. To this end we observe that the function $f'(z)/f(z)$, being analytic in the whole plane, is the derivative of an entire function $g(z)$. From this fact we infer, by computation, that $f(z)e^{-g(z)}$ has the derivative zero, and hence $f(z)$ is a constant multiple of $e^{g(z)}$; the constant can be absorbed in $g(z)$.

By this method we can also find the most general entire function with a finite number of zeros. Assume that $f(z)$ has m zeros at the origin (m may be zero), and denote the other zeros by $a_1, a_2, \ldots, a_N$, multiple zeros being repeated. It is then plain that we can write

$$f(z) = z^m e^{g(z)} \prod_1^N \left(1 - \frac{z}{a_n}\right).$$

If there are infinitely many zeros, we can try to obtain a similar representation by means of an infinite product. The obvious generalization would be

$$f(z) = z^m e^{g(z)} \prod_1^\infty \left(1 - \frac{z}{a_n}\right). \tag{17}$$

This representation is valid if the infinite product converges uniformly on every compact set. In fact, if this is so the product represents an entire function with zeros at the same points (except for the origin) and with the same multiplicities as $f(z)$. It follows that the quotient can be written in the form $z^m e^{g(z)}$.

The product in (17) converges absolutely if and only if $\sum\limits_1^\infty 1/|a_n|$ is convergent, and in this case the convergence is also uniform in every closed disk $|z| \leqq R$. It is only under this special condition that we can obtain a representation of the form (17).

In the general case convergence-producing factors must be introduced. We consider an arbitrary sequence of complex numbers $a_n \neq 0$ with $\lim\limits_{n\to\infty} a_n = \infty$, and prove the existence of polynomials $p_n(z)$ such that

$$\prod_1^\infty \left(1 - \frac{z}{a_n}\right) e^{p_n(z)} \tag{18}$$

converges to an entire function. The product converges together with the series with the general term

$$r_n(z) = \log\left(1 - \frac{z}{a_n}\right) + p_n(z)$$

where the branch of the logarithm shall be chosen so that the imaginary part of $r_n(z)$ lies between $-\pi$ and π (inclusive).

For a given R we consider only the terms with $|a_n| > R$. In the disk $|z| \leqq R$ the principal branch of $\log(1 - z/a_n)$ can be developed in a Taylor series

$$\log\left(1 - \frac{z}{a_n}\right) = -\frac{z}{a_n} - \frac{1}{2}\left(\frac{z}{a_n}\right)^2 - \frac{1}{3}\left(\frac{z}{a_n}\right)^3 - \cdots.$$

We reverse the signs and choose $p_n(z)$ as a partial sum

$$p_n(z) = \frac{z}{a_n} + \frac{1}{2}\left(\frac{z}{a_n}\right)^2 + \cdots + \frac{1}{m_n}\left(\frac{z}{a_n}\right)^{m_n}.$$

Then $r_n(z)$ has the representation

$$r_n(z) = -\frac{1}{m_n + 1}\left(\frac{z}{a_n}\right)^{m_n+1} - \frac{1}{m_n + 2}\left(\frac{z}{a_n}\right)^{m_n+2} - \cdots$$

and we obtain easily the estimate

$$|r_n(z)| \leqq \frac{1}{m_n + 1}\left(\frac{R}{|a_n|}\right)^{m_n+1}\left(1 - \frac{R}{|a_n|}\right)^{-1}. \tag{19}$$

Suppose now that the series

$$\sum_{n=1}^{\infty} \frac{1}{m_n + 1}\left(\frac{R}{|a_n|}\right)^{m_n+1} \tag{20}$$

converges. By the estimate (19) it follows first that $r_n(z) \to 0$, and hence $r_n(z)$ has an imaginary part between $-\pi$ and π as soon as n is sufficiently large. Moreover, the comparison shows that the series $\Sigma r_n(z)$ is absolutely and uniformly convergent for $|z| \leqq R$, and thus the product (18) represents an analytic function in $|z| < R$. For the sake of the reasoning we had to exclude the values $|a_n| \leqq R$, but it is clear that the uniform convergence of (18) is not affected when the corresponding factors are again taken into account.

It remains only to show that the series (20) can be made convergent for all R. But this is obvious, for if we take $m_n = n$ it is clear that (20) has a majorant geometric series with ratio < 1 for any fixed value of R.

Theorem 7. *There exists an entire function with arbitrarily prescribed zeros a_n provided that, in the case of infinitely many zeros, $a_n \to \infty$. Every entire function with these and no other zeros can be written in the form*

$$f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) e^{\frac{z}{a_n} + \frac{1}{2}\left(\frac{z}{a_n}\right)^2 + \cdots + \frac{1}{m_n}\left(\frac{z}{a_n}\right)^{m_n}} \tag{21}$$

where the product is taken over all $a_n \neq 0$, the m_n are certain integers, and $g(z)$ is an entire function.

This theorem is due to Weierstrass. It has the following important corollary:

Corollary. *Every function which is meromorphic in the whole plane is the quotient of two entire functions.*

In fact, if $F(z)$ is meromorphic in the whole plane, we can find an entire function $g(z)$ with the poles of $F(z)$ for zeros. The product $F(z)g(z)$ is then an entire function $f(z)$, and we obtain $F(z) = f(z)/g(z)$.

The representation (21) becomes considerably more interesting if it is possible to choose all the m_n equal to each other. The preceding proof has shown that the product

$$\prod_1^\infty \left(1 - \frac{z}{a_n}\right) e^{\frac{z}{a_n} + \frac{1}{2}\left(\frac{z}{a_n}\right)^2 + \cdots + \frac{1}{h}\left(\frac{z}{a_n}\right)^h} \tag{22}$$

converges and represents an entire function provided that the series $\sum\limits_{n=1}^\infty (R/|a_n|)^{h+1}/(h+1)$ converges for all R, that is to say provided that $\Sigma 1/|a_n|^{h+1} < \infty$. Assume that h is the smallest integer for which this series converges; the expression (22) is then called the *canonical product* associated with the sequence $\{a_n\}$, and h is the *genus* of the canonical product.

Whenever possible we use the canonical product in the representation (21), which is thereby uniquely determined. If in this representation $g(z)$ reduces to a polynomial, the function $f(z)$ is said to be of finite genus, and the genus of $f(z)$ is by definition equal to the degree of this polynomial or to the genus of the canonical product, whichever is the larger. For instance, an entire function of genus zero is of the form

$$Cz^m \prod_1^\infty \left(1 - \frac{z}{a_n}\right)$$

with $\Sigma 1/|a_n| < \infty$. The canonical representation of an entire function of genus 1 is either of the form

$$Cz^m e^{\alpha z} \prod_1^\infty \left(1 - \frac{z}{a_n}\right) e^{z/a_n}$$

with $\Sigma 1/|a_n|^2 < \infty$, $\Sigma 1/|a_n| = \infty$, or of the form

$$Cz^m e^{\alpha z} \prod_1^\infty \left(1 - \frac{z}{a_n}\right)$$

with $\Sigma 1/|a_n| < \infty$, $\alpha \neq 0$.

As an application we consider the product representation of $\sin \pi z$. The zeros are the integers $z = \pm n$. Since $\Sigma 1/n$ diverges and $\Sigma 1/n^2$ converges, we must take $h = 1$ and obtain a representation of the form

$$\sin \pi z = z e^{g(z)} \prod_{n \neq 0} \left(1 - \frac{z}{n}\right) e^{z/n}.$$

In order to determine $g(z)$ we form the logarithmic derivatives on both sides. We find

$$\pi \cot \pi z = \frac{1}{z} + g'(z) + \sum_{n \neq 0} \left(\frac{1}{z - n} + \frac{1}{n}\right)$$

where the procedure is easy to justify by uniform convergence on any compact set which does not contain the points $z = n$. By comparison with the previous formula (10) we conclude that $g'(z) = 0$. Hence $g(z)$ is a constant, and since $\lim_{z \to 0} \sin \pi z/z = \pi$ we must have $e^{g(z)} = \pi$, and thus

$$\sin \pi z = \pi z \prod_{n \neq 0} \left(1 - \frac{z}{n}\right) e^{z/n}. \tag{23}$$

In this representation the factors corresponding to n and $-n$ can be bracketed together, and we obtain the simple form

$$\sin \pi z = \pi z \prod_{1}^{\infty} \left(1 - \frac{z^2}{n^2}\right). \tag{24}$$

It follows from (23) that $\sin \pi z$ is an entire function of genus 1.

EXERCISES

1. Suppose that $a_n \to \infty$ and that the A_n are arbitrary complex numbers. Show that there exists an entire function $f(z)$ which satisfies $f(a_n) = A_n$.

Hint: Let $g(z)$ be a function with simple zeros at the a_n. Show that

$$\sum_{1}^{\infty} g(z) \frac{e^{\gamma_n(z - a_n)}}{z - a_n} \cdot \frac{A_n}{g'(a_n)}$$

converges for some choice of the numbers γ_n.

2. Prove that

$$\sin \pi(z + \alpha) = e^{\pi z \cot \pi \alpha} \prod_{-\infty}^{\infty} \left(1 + \frac{z}{n + \alpha}\right) e^{-z/(n+\alpha)}$$

whenever α is not an integer. *Hint:* Denote the factor in front of the canonical product by $g(z)$ and determine $g'(z)/g(z)$.

3. What is the genus of $\cos \sqrt{z}$?

4. If $f(z)$ is of genus h, how large and how small can the genus of $f(z^2)$ be?

5. Show that if $f(z)$ is of genus 0 or 1 with real zeros, and if $f(z)$ is real for real z, then all zeros of $f'(z)$ are real. *Hint:* Consider $\operatorname{Im} f'(z)/f(z)$.

2.4. The Gamma Function. The function $\sin \pi z$ has all the integers for zeros, and it is the simplest function with this property. We shall now introduce functions which have only the positive or only the negative integers for zeros. The simplest function with, for instance, the negative integers for zeros is the corresponding canonical product

$$G(z) = \prod_1^\infty \left(1 + \frac{z}{n}\right) e^{-z/n}. \tag{25}$$

It is evident that $G(-z)$ has then the positive integers for zeros, and by comparison with the product representation (23) of $\sin \pi z$ we find at once

$$zG(z)G(-z) = \frac{\sin \pi z}{\pi}. \tag{26}$$

Because of the manner in which $G(z)$ has been constructed, it is bound to have other simple properties. We observe that $G(z-1)$ has the same zeros as $G(z)$, and in addition a zero at the origin. It is therefore clear that we can write

$$G(z-1) = ze^{\gamma(z)}G(z),$$

where $\gamma(z)$ is an entire function. In order to determine $\gamma(z)$ we take the logarithmic derivatives on both sides. This gives the equation

$$\sum_{n=1}^\infty \left(\frac{1}{z-1+n} - \frac{1}{n}\right) = \frac{1}{z} + \gamma'(z) + \sum_{n=1}^\infty \left(\frac{1}{z+n} - \frac{1}{n}\right). \tag{27}$$

In the series to the left we can replace n by $n+1$. By this change we obtain

$$\begin{aligned}\sum_{n=1}^\infty \left(\frac{1}{z-1+n} - \frac{1}{n}\right) &= \frac{1}{z} - 1 + \sum_{n=1}^\infty \left(\frac{1}{z+n} - \frac{1}{n+1}\right) \\ &= \frac{1}{z} - 1 + \sum_{n=1}^\infty \left(\frac{1}{z+n} - \frac{1}{n}\right) + \sum_{n=1}^\infty \left(\frac{1}{n} - \frac{1}{n+1}\right).\end{aligned}$$

The last series has the sum 1, and hence equation (27) reduces to $\gamma'(z) = 0$.

Thus $\gamma(z)$ is a constant, which we denote by γ, and $G(z)$ has the reproductive property $G(z-1) = e^{\gamma}zG(z)$. It is somewhat simpler to consider the function $H(z) = G(z)e^{\gamma z}$ which evidently satisfies the functional equation $H(z-1) = zH(z)$.

The value of γ is easily determined. Taking $z = 1$ we have

$$1 = G(0) = e^{\gamma}G(1),$$

and hence

$$e^{-\gamma} = \prod_{n=1}^{\infty} \left(1 + \frac{1}{n}\right) e^{-1/n}.$$

Here the nth partial product can be written in the form

$$(n+1)e^{-(1+\frac{1}{2}+\frac{1}{3}+\cdots+1/n)},$$

and we obtain

$$\gamma = \lim_{n\to\infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n\right).$$

The constant γ is called Euler's constant; its approximate value is .57722.

If $H(z)$ satisfies $H(z-1) = zH(z)$, then $\Gamma(z) = 1/[zH(z)]$ satisfies $\Gamma(z-1) = \Gamma(z)/(z-1)$, or

$$\Gamma(z+1) = z\Gamma(z). \tag{28}$$

This is found to be a more useful relation, and for this reason it has become customary to implement the restricted stock of elementary functions by inclusion of $\Gamma(z)$ under the name of *Euler's gamma function.*

Our definition leads to the explicit representation

$$\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} e^{z/n}, \tag{29}$$

and the formula (26) takes the form

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}. \tag{30}$$

We observe that $\Gamma(z)$ is a meromorphic function with poles at $z = 0, -1, -2, \ldots$ but *without zeros.*

We have $\Gamma(1) = 1$, and by the functional equation we find $\Gamma(2) = 1$, $\Gamma(3) = 1 \cdot 2$, $\Gamma(4) = 1 \cdot 2 \cdot 3$ and generally $\Gamma(n) = (n-1)!$. The Γ-function can thus be considered as a generalization of the factorial. From (30) we conclude that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

Other properties are most easily found by considering the second derivative of $\log \Gamma(z)$ for which we find, by (29), the very simple expression

$$\frac{d}{dz}\left(\frac{\Gamma'(z)}{\Gamma(z)}\right) = \sum_{n=0}^{\infty} \frac{1}{(z+n)^2}. \tag{31}$$

For instance, it is plain that $\Gamma(z)\,\Gamma(z + \frac{1}{2})$ and $\Gamma(2z)$ have the same poles, and by use of (31) we find indeed that

$$\frac{d}{dz}\left(\frac{\Gamma'(z)}{\Gamma(z)}\right) + \frac{d}{dz}\left(\frac{\Gamma'(z+\frac{1}{2})}{\Gamma(z+\frac{1}{2})}\right) = \sum_{n=0}^{\infty} \frac{1}{(z+n)^2} + \sum_{n=0}^{\infty} \frac{1}{(z+n+\frac{1}{2})^2}$$
$$= 4\left[\sum_{n=0}^{\infty} \frac{1}{(2z+2n)^2} + \sum_{n=0}^{\infty} \frac{1}{(2z+2n+1)^2}\right] = 4\sum_{m=0}^{\infty} \frac{1}{(2z+m)^2}$$
$$= 2\frac{d}{dz}\left(\frac{\Gamma'(2z)}{\Gamma(2z)}\right).$$

By integration we obtain

$$\Gamma(z)\Gamma(z+\tfrac{1}{2}) = e^{az+b}\Gamma(2z),$$

where the constants a and b have yet to be determined. Substituting $z = \frac{1}{2}$ and $z = 1$ we make use of the known values $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, $\Gamma(1) = 1$, $\Gamma(1\frac{1}{2}) = \frac{1}{2}\Gamma(\frac{1}{2}) = \frac{1}{2}\sqrt{\pi}$, $\Gamma(2) = 1$ and are led to the relations

$$\tfrac{1}{2}a + b = \tfrac{1}{2}\log \pi, \qquad a + b = \tfrac{1}{2}\log \pi - \log 2.$$

It follows that

$$a = -2\log 2 \qquad \text{and} \qquad b = \tfrac{1}{2}\log \pi + \log 2;$$

the final result is thus

$$\sqrt{\pi}\,\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma(z+\tfrac{1}{2})$$

which is known as Legendre's duplication formula.

EXERCISES

1. Prove the formula of Gauss:

$$(2\pi)^{\frac{n-1}{2}}\,\Gamma(z) = n^{z-\frac{1}{2}}\,\Gamma\left(\frac{z}{n}\right)\Gamma\left(\frac{z+1}{n}\right)\cdots\Gamma\left(\frac{z+n-1}{n}\right).$$

2. Show that

$$\Gamma\left(\frac{1}{6}\right) = 2^{-\frac{1}{3}}\left(\frac{3}{\pi}\right)^{\frac{1}{2}}\Gamma\left(\frac{1}{3}\right)^2.$$

3. What are the residues of $\Gamma(z)$ at the poles $z = -n$?

2.5. Stirling's Formula. In most connections where the Γ function can be applied, it is of utmost importance to have some information on the behavior of $\Gamma(z)$ for very large values of z. Fortunately, it is possible to calculate $\Gamma(z)$ with great precision and very little effort by means of a classical formula which goes under the name of Stirling's formula. There are many proofs of this formula. We choose to derive it by use of the residue calculus, following mainly the presentation of Lindelöf in his classical book on the calculus of residues. This is a very simple and above all a very instructive proof inasmuch as it gives us an opportunity to use residues in less trivial cases than previously.

The starting point is the formula (31) for the second derivative of $\log \Gamma(z)$, and our immediate task is to express the partial sum

$$\frac{1}{z^2} + \frac{1}{(z+1)^2} + \frac{1}{(z+2)^2} + \cdots + \frac{1}{(z+n)^2}$$

as a convenient line integral. To this end we need a function with the residues $1/(z+\nu)^2$ at the integral points ν; a good choice is

$$\Phi(\zeta) = \frac{\pi \cot \pi\zeta}{(z+\zeta)^2}.$$

Here ζ is the variable while z enters only as a parameter, which in the first part of the derivation will be kept at a fixed value $z = x + iy$ with $x > 0$.

We apply the residue formula to the rectangle whose vertical sides lie on $\xi = 0$ and $\xi = n + \frac{1}{2}$ and with horizontal sides $\eta = \pm Y$, with the intention of letting first Y and then n tend to ∞. This contour, which we denote by K, passes through the pole at 0, but we know that the formula remains valid provided that we take the principal value of the integral and include one-half of the residue at the origin. Hence we obtain

$$\text{pr.v.}\ \frac{1}{2\pi i}\int_K \Phi(\zeta)\,d\zeta = -\frac{1}{2z^2} + \sum_{\nu=0}^{n} \frac{1}{(z+\nu)^2}.$$

On the horizontal sides of the rectangle $\cot \pi\zeta$ tends uniformly to $\pm i$ for $Y \to \infty$. Since the factor $1/(z+\zeta)^2$ tends to zero, the corresponding integrals have the limit zero. We are now left with two integrals over infinite vertical lines. On each line $\xi = n + \frac{1}{2}$, $\cot \pi\zeta$ is bounded, and because of the periodicity the bound is independent of n. The integral over the line $\xi = n + \frac{1}{2}$ is thus less than a constant times

$$\int_{\xi=n+\frac{1}{2}} \frac{d\eta}{|\zeta+z|^2}$$

This integral can be evaluated, for on the line of integration

$$\bar{\zeta} = 2n + 1 - \zeta,$$

and we obtain by residues

$$\frac{1}{i}\int \frac{d\zeta}{|\zeta + z|^2} = \frac{1}{i}\int \frac{d\zeta}{(\zeta + z)(2n + 1 - \zeta + \bar{z})} = \frac{2\pi}{2n + 1 + 2x}.$$

The limit for $n \to \infty$ is thus zero.

Finally, the principal value of the integral over the imaginary axis from $-i\infty$ to $+i\infty$ can be written in the form

$$\frac{1}{2}\int_0^\infty \cot \pi i\eta \left[\frac{1}{(i\eta + z)^2} - \frac{1}{(i\eta - z)^2}\right] d\eta = -\int_0^\infty \coth \pi\eta \cdot \frac{2\eta z}{(\eta^2 + z^2)^2}\, d\eta.$$

The sign has to be reversed, and we obtain the formula

$$\frac{d}{dz}\left(\frac{\Gamma'(z)}{\Gamma(z)}\right) = \frac{1}{2z^2} + \int_0^\infty \coth \pi\eta \cdot \frac{2\eta z}{(\eta^2 + z^2)^2}\, d\eta. \tag{32}$$

It is preferable to write

$$\coth \pi\eta = 1 + \frac{2}{e^{2\pi\eta} - 1}$$

and observe that the integral obtained from the term 1 has the value $1/z$. We can thus rewrite (32) in the form

$$\frac{d}{dz}\left(\frac{\Gamma'(z)}{\Gamma(z)}\right) = \frac{1}{z} + \frac{1}{2z^2} + \int_0^\infty \frac{4\eta z}{(\eta^2 + z^2)^2} \cdot \frac{d\eta}{e^{2\pi\eta} - 1} \tag{33}$$

where the integral is now very strongly convergent.

For z restricted to the right half plane this formula can be integrated. We find

$$\frac{\Gamma'(z)}{\Gamma(z)} = C + \log z - \frac{1}{2z} - \int_0^\infty \frac{2\eta}{\eta^2 + z^2} \cdot \frac{d\eta}{e^{2\pi\eta} - 1}, \tag{34}$$

where $\log z$ is the principal branch and C is an integration constant. The integration of the last term needs some justification. We have to make sure that the integral in (34) can be differentiated under the sign of integration; this is so because the integral in (33) converges uniformly when z is restricted to any compact set in the half plane $x > 0$.

We wish to integrate (34) once more. This would obviously introduce arc tan (z/η) in the integral, and although a single-valued branch could be defined we prefer to avoid the use of multiple-valued functions. That is possible if we first transform the integral in (34) by partial integration. We obtain

$$\int_0^\infty \frac{2\eta}{\eta^2 + z^2} \cdot \frac{d\eta}{e^{2\pi\eta} - 1} = \frac{1}{\pi} \int_0^\infty \frac{z^2 - \eta^2}{(\eta^2 + z^2)^2} \log (1 - e^{-2\pi\eta})\, d\eta$$

where the logarithm is of course real. Now we can integrate with respect to z and obtain

$$(35) \quad \log \Gamma(z)$$
$$= C' + Cz + \left(z - \frac{1}{2}\right) \log z + \frac{1}{\pi} \int_0^\infty \frac{z}{\eta^2 + z^2} \log \frac{1}{1 - e^{-2\pi\eta}}\, d\eta$$

where C' is a new integration constant and for convenience $C - 1$ has been replaced by C. The formula means that there exists, in the right half plane, a single-valued branch of $\log \Gamma(z)$ whose value is given by the right-hand member of the equation. By proper choice of C' we obtain the branch of $\log \Gamma(z)$ which is real for real z.

It remains to determine the constants C and C'. To this end we must first study the behavior of the integral in (35) which we denote by

$$(36) \qquad J(z) = \frac{1}{\pi} \int_0^\infty \frac{z}{\eta^2 + z^2} \log \frac{1}{1 - e^{-2\pi\eta}}\, d\eta.$$

It is practically evident that $J(z) \to 0$ for $z \to \infty$ provided that z keeps away from the imaginary axis. Suppose for instance that z is restricted to the half plane $x \geqq c > 0$. Breaking the integral into two parts we write

$$J(z) = \int_0^{\frac{|z|}{2}} + \int_{\frac{|z|}{2}}^\infty = J_1 + J_2.$$

In the first integral $|\eta^2 + z^2| \geqq |z|^2 - |z/2|^2 = 3|z|^2/4$, and hence

$$|J_1| \leqq \frac{4}{3\pi|z|} \int_0^\infty \log \frac{1}{1 - e^{-2\pi\eta}}\, d\eta.$$

In the second integral $|\eta^2 + z^2| = |z - i\eta| \cdot |z + i\eta| > c|z|$, and we find

$$|J_2| < \frac{1}{\pi c} \int_{\frac{|z|}{2}}^\infty \log \frac{1}{1 - e^{-2\pi\eta}}\, d\eta.$$

Since the integral of $\log (1 - e^{-2\pi\eta})$ is obviously convergent, we conclude that J_1 and J_2 tend to 0 as $z \to \infty$.

The value of C is found by substituting (35) in the functional equation $\Gamma(z + 1) = z\Gamma(z)$ or $\log \Gamma(z + 1) = \log z + \log \Gamma(z)$; if we restrict z to positive values, there is no hesitancy about the branch of the logarithm. The substitution yields

$$C' + Cz + C + (z + \tfrac{1}{2}) \log (z + 1) + J(z + 1)$$
$$= C' + Cz + (z + \tfrac{1}{2}) \log z + J(z),$$

and this reduces to

$$C = -\left(z + \frac{1}{2}\right) \log \left(1 + \frac{1}{z}\right) + J(z) - J(z + 1).$$

Letting $z \to \infty$ we find that $C = -1$.

Next we apply (35) to the equation $\Gamma(z)\Gamma(1 - z) = \pi/\sin \pi z$, choosing $z = \frac{1}{2} + iy$. We obtain

$$2C' - 1 + iy \log (\tfrac{1}{2} + iy) - iy \log (\tfrac{1}{2} - iy) + J(\tfrac{1}{2} + iy) + J(\tfrac{1}{2} - iy)$$
$$= \log \pi - \log \cosh \pi y.$$

This equation, in which the logarithms are to have their principal values, is so far proved only up to a constant multiple of $2\pi i$. But for $y = 0$ the equation is correct as it stands because (35) determines the real value of $\log \Gamma(\frac{1}{2})$; hence it holds for all y. As $y \to \infty$ we known that $J(\frac{1}{2} + iy)$ and $J(\frac{1}{2} - iy)$ tend to 0. Developing the logarithm in a Taylor series we find

$$iy \log \frac{\frac{1}{2} + iy}{\frac{1}{2} - iy} = iy \left(\pi i + \log \frac{1 + \dfrac{1}{2iy}}{1 - \dfrac{1}{2iy}} \right) = -\pi y + 1 + \varepsilon_1(y)$$

while in the right-hand member

$$\log \cosh \pi y = \pi y - \log 2 + \varepsilon_2(y)$$

with $\varepsilon_1(y)$ and $\varepsilon_2(y)$ tending to 0. These considerations yield the value $C' = \frac{1}{2} \log 2\pi$. We have thus proved Stirling's formula in the form

$$\log \Gamma(z) = \tfrac{1}{2} \log 2\pi - z + (z - \tfrac{1}{2}) \log z + J(z) \tag{37}$$

or equivalently

$$\Gamma(z) = \sqrt{2\pi}\, z^{z - \frac{1}{2}} e^{-z} e^{J(z)} \tag{38}$$

with the representation (36) of the remainder valid in the right half plane. We know that $J(z)$ tends to 0 when $z \to \infty$ in a half plane $x \geqq c > 0$.

In the expression for $J(z)$ we can develop the integrand in powers of $1/z$ and obtain

$$J(z) = \frac{C_1}{z} + \frac{C_2}{z^3} + \cdots + \frac{C_k}{z^{2k-1}} + J_k(z)$$

with

$$(39) \qquad C_\nu = (-1)^{\nu-1}\frac{1}{\pi}\int_0^\infty \eta^{2\nu-2}\log\frac{1}{1-e^{-2\pi\eta}}\,d\eta$$

and

$$J_k(z) = \frac{(-1)^k}{z^{2k+1}}\frac{1}{\pi}\int_0^\infty \frac{\eta^{2k}}{1+(\eta/z)^2}\log\frac{1}{1-e^{-2\pi\eta}}\,d\eta.$$

It can be proved (for instance by means of residues) that the coefficients C_ν are connected with the Bernoulli numbers (cf. Ex. 4, Sec. 1.3) by

$$(40) \qquad C_\nu = (-1)^{\nu-1}\frac{1}{(2\nu-1)2\nu}B_\nu.$$

Thus the development of $J(z)$ takes the form

$$(41) \quad J(z) = \frac{B_1}{1\cdot 2}\frac{1}{z} - \frac{B_2}{3\cdot 4}\cdot\frac{1}{z^3} + \cdots + (-1)^{k-1}\frac{B_k}{(2k-1)2k}\frac{1}{z^{2k-1}} + J_k(z).$$

The reader is warned not to confuse this with a Laurent development. The function $J(z)$ is not defined in a neighborhood of ∞ and, therefore, does not have a Laurent development; moreover, if $k \to \infty$, the series obtained from (41) does not converge. What we can say is that for a fixed k the expression $J_k(z)z^{2k}$ tends to 0 for $z \to \infty$ (in a half plane $x \geqq c > 0$). This fact characterizes (41) as an *asymptotic development*. Such developments are very valuable when z is large in comparison with k, but for fixed z there is no advantage in letting k become very large.

Stirling's formula can be used to prove that

$$(42) \qquad \Gamma(z) = \int_0^\infty e^{-t}t^{z-1}\,dt$$

whenever the integral converges, that is to say for $x > 0$. Until the identity has been proved, let the integral in (42) be denoted by $F(z)$. Integrating by parts we find at once that

$$F(z+1) = \int_0^\infty e^{-t}t^z\,dt = z\int_0^\infty e^{-t}t^{z-1}\,dt = zF(z).$$

Hence $F(z)$ satisfies the same functional equation as $\Gamma(z)$, and we find that $F(z)/\Gamma(z) = F(z+1)/\Gamma(z+1)$. In other words $F(z)/\Gamma(z)$ is periodic with the period 1. This shows, incidentally, that $F(z)$ can be defined in the whole plane although the integral representation is valid only in a half plane.

In order to prove that $F(z)/\Gamma(z)$ is constant we have to estimate $|F/\Gamma|$ in a period strip, for instance in the strip $1 \leqq x \leqq 2$. In the first place we have by (42)

$$|F(z)| \leqq \int_0^\infty e^{-t}t^{x-1}\,dt = F(x),$$

and hence $F(z)$ is bounded in the strip. Next, we use Stirling's formula to find a lower bound of $|\Gamma(z)|$ for large y. From (37) we obtain

$$\log|\Gamma(z)| = \tfrac{1}{2}\log 2\pi - x + (x - \tfrac{1}{2})\log|z| - y\arg z + \operatorname{Re} J(z).$$

Only the term $-y \arg z$ becomes negatively infinite, being comparable to $-\pi|y|/2$. Thus $|F/\Gamma|$ does not grow much more rapidly than $e^{\pi|y|/2}$.

For an arbitrary function this would not suffice to conclude that the function must be constant, but for a function of period 1 it is more than enough. In fact, it is clear that F/Γ can be expressed as a single-valued function of the variable $\zeta = e^{2\pi iz}$; to every value of $\zeta \neq 0$ there correspond infinitely many values of z which differ by multiples of 1, and thus a single value of F/Γ. The function has isolated singularities at $\zeta = 0$ and $\zeta = \infty$, and our estimate shows that $|F/\Gamma|$ grows at most like $|\zeta|^{-\frac{1}{4}}$ for $\zeta \to 0$ and $|\zeta|^{\frac{1}{4}}$ for $\zeta \to \infty$. It follows that both singularities are removable, and hence F/Γ must reduce to a constant. Finally, the fact that $F(1) = \Gamma(1) = 1$ shows that $F(z) = \Gamma(z)$.

EXERCISES

1. Prove the development (41).

2. For real $x > 0$ prove that

$$\Gamma(x) = \sqrt{2\pi}\, x^{x-\frac{1}{2}} e^{-x} e^{\theta(x)/12x}$$

with $0 < \theta(x) < 1$.

3. The formula (42) permits us to evaluate the *probability integral*

$$\int_0^\infty e^{-t^2}\,dt = \tfrac{1}{2}\int_0^\infty e^{-x}x^{-\frac{1}{2}}\,dx = \tfrac{1}{2}\Gamma(\tfrac{1}{2}) = \tfrac{1}{2}\sqrt{\pi}.$$

Use this result together with Cauchy's theorem to compute the *Fresnel integrals*

$$\int_0^\infty \sin(x^2)\,dx, \qquad \int_0^\infty \cos(x^2)\,dx.$$

Answer: Both are equal to $\frac{1}{2}\sqrt{\pi/2}$.

3. ENTIRE FUNCTIONS

In Sec. 2.3 we have already considered the representation of entire functions as infinite products, and, in special cases, as canonical products. In this section we study the connection between the product representation and the rate of growth of the function. Such questions were first investigated by Hadamard who applied the results to his celebrated proof