6. HARMONIC FUNCTIONS

The real and imaginary parts of an analytic function are conjugate harmonic functions. Therefore, all theorems on analytic functions are also theorems on pairs of conjugate harmonic functions. However, harmonic functions are important in their own right, and their treatment is not always simplified by the use of complex methods. This is particularly true when the conjugate harmonic function is not single-valued.

We assemble in this section some facts about harmonic functions that are intimately connected with Cauchy's theorem. The more delicate properties of harmonic functions are postponed to a later chapter.

6.1. Definition and Basic Properties. A real-valued function $u(z)$ or $u(x,y)$, defined and single-valued in a region Ω, is said to be *harmonic* in Ω, or a *potential function*, if it is continuous together with its partial derivatives of the first two orders and satisfies *Laplace's equation*

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \tag{54}$$

We shall see later that the regularity conditions can be weakened, but this is a point of relatively minor importance.

The sum of two harmonic functions and a constant multiple of a harmonic function are again harmonic; this is due to the linear character of Laplace's equation. The simplest harmonic functions are the linear functions $ax + by$. In polar coordinates (r,θ) equation (54) takes the form

$$r\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{\partial^2 u}{\partial \theta^2} = 0.\dagger$$

This shows that $\log r$ is a harmonic function and that any harmonic function which depends only on r must be of the form $a \log r + b$. The argument θ is harmonic whenever it can be uniquely defined.

If u is harmonic in Ω, then

$$f(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \tag{55}$$

is analytic, for writing $U = \frac{\partial u}{\partial x}$, $V = -\frac{\partial u}{\partial y}$ we have

$$\frac{\partial U}{\partial x} = \frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2} = \frac{\partial V}{\partial y}$$
$$\frac{\partial U}{\partial y} = \frac{\partial^2 u}{\partial x\,\partial y} = -\frac{\partial V}{\partial x}.$$

† This form cannot be used for $r = C$.

This, it should be remembered, is the most natural way of passing from harmonic to analytic functions.

From (55) we pass to the differential

$$f\,dz = \left(\frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy\right) + i\left(-\frac{\partial u}{\partial y}\,dx + \frac{\partial u}{\partial x}\,dy\right). \tag{56}$$

In this expression the real part is the differential of u,

$$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy.$$

If u has a conjugate harmonic function v, then the imaginary part can be written as

$$dv = \frac{\partial v}{\partial x}\,dx + \frac{\partial v}{\partial y}\,dy = -\frac{\partial u}{\partial y}\,dx + \frac{\partial u}{\partial x}\,dy.$$

In general, however, there is no single-valued conjugate function, and in these circumstances it is better not to use the notation dv. Instead we write

$$^*du = -\frac{\partial u}{\partial y}\,dx + \frac{\partial u}{\partial x}\,dy$$

and call $^*du$ the *conjugate differential of* du. We have by (56)

$$f\,dz = du + i\,{}^*du. \tag{57}$$

By Cauchy's theorem the integral of $f\,dz$ vanishes along any cycle which is homologous to zero in Ω. On the other hand, the integral of the exact differential du vanishes along all cycles. It follows by (57) that

$$\int_\gamma {}^*du = \int_\gamma -\frac{\partial u}{\partial y}\,dx + \frac{\partial u}{\partial x}\,dy = 0 \tag{58}$$

for all cycles γ which are homologous to zero in Ω.

The integral in (58) has an important interpretation which cannot be left unmentioned. If γ is a regular curve with the equation $z = z(t)$, the direction of the tangent is determined by the angle $\alpha = \arg z'(t)$, and we can write $dx = |dz| \cos \alpha$, $dy = |dz| \sin \alpha$. The normal which points to the right of the tangent has the direction $\beta = \alpha - \pi/2$, and thus $\cos \alpha = -\sin \beta$, $\sin \alpha = \cos \beta$. The expression

$$\frac{\partial u}{\partial n} = \frac{\partial u}{\partial x}\cos\beta + \frac{\partial u}{\partial y}\sin\beta$$

is a directional derivative of u, the right-hand *normal derivative* with respect to the curve γ. We obtain $^*du = (\partial u/\partial n)\,|dz|$, and (58) can be written in the form

$$\int_\gamma \frac{\partial u}{\partial n}\,|dz| = 0. \tag{59}$$

This is the classical notation. Its main advantage is that $\partial u/\partial n$ actually represents a rate of change in the direction perpendicular to γ. For instance, if γ is the circle $|z| = r$, described in the positive sense, $\partial u/\partial n$ can be replaced by the partial derivative $\partial u/\partial r$. It has the disadvantage that (59) is not expressed as an ordinary line integral, but as an integral with respect to arc length. For this reason the classical notation is less natural in connection with homology theory, and we prefer to use the notation $*du$.

In a simply connected region the integral of $*du$ vanishes over all cycles, and u has a single-valued conjugate function v which is determined up to an additive constant. In the multiply connected case the conjugate function has *periods*

$$\int_{\gamma_i} *du = \int_{\gamma_i} \frac{\partial u}{\partial n}\,|dz|$$

corresponding to the cycles in a homology basis.

There is an important generalization of (58) which deals with a pair of harmonic functions. If u_1 and u_2 are harmonic in Ω, we claim that

$$\int_\gamma u_1\,*du_2 - u_2\,*du_1 = 0 \tag{60}$$

for every cycle γ which is homologous to zero in Ω. According to Theorem 16, Sec. 4.6, it is sufficient to prove (60) for $\gamma = \partial R$, where R is a rectangle contained in Ω. In R, u_1 and u_2 have single-valued conjugate functions v_1, v_2 and we can write

$$u_1\,*du_2 - u_2\,*du_1 = u_1\,dv_2 - u_2\,dv_1 = u_1\,dv_2 + v_1\,du_2 - d(u_2v_1).$$

Here $d(u_2v_1)$ is an exact differential, and $u_1dv_2 + v_1du_2$ is the imaginary part of

$$(u_1 + iv_1)(du_2 + i\,dv_2).$$

The last differential can be written in the form $F_1f_2\,dz$ where $F_1(z)$ and $f_2(z)$ are analytic on R. The integral of $F_1f_2\,dz$ vanishes by Cauchy's theorem, and so does therefore the integral of its imaginary part. We conclude that (60) holds for $\gamma = \partial R$, and we have proved:

Theorem 19. *If u_1 and u_2 are harmonic in a region Ω, then*

$$\int_\gamma u_1\,*du_2 - u_2\,*du_1 = 0 \tag{60}$$

for every cycle γ which is homologous to zero in Ω.

For $u_1 = 1$, $u_2 = u$ the formula reduces to (58). In the classical notation (60) would be written as

$$\int_\gamma \left(u_1 \frac{\partial u_2}{\partial n} - u_2 \frac{\partial u_1}{\partial n} \right) |dz| = 0.$$

6.2. The Mean-value Property. Let us apply Theorem 19 with $u_1 = \log r$ and u_2 equal to a function u, harmonic in $|z| < \rho$. For Ω we choose the punctured disk $0 < |z| < \rho$, and for γ we take the cycle $C_1 - C_2$ where C_i is a circle $|z| = r_i < \rho$ described in the positive sense. On a circle $|z| = r$ we have $*du = r(\partial u/\partial r)\, d\theta$ and hence (60) yields

$$\log r_1 \int_{C_1} r_1 \frac{\partial u}{\partial r}\, d\theta - \int_{C_1} u\, d\theta = \log r_2 \int_{C_2} r_2 \frac{\partial u}{\partial r}\, d\theta - \int_{C_2} u\, d\theta.$$

In other words, the expression

$$\int_{|z|=r} u\, d\theta - \log r \int_{|z|=r} r \frac{\partial u}{\partial r}\, d\theta$$

is constant, and this is true even if u is only known to be harmonic in an annulus. By (58) we find in the same way that

$$\int_{|z|=r} r \frac{\partial u}{\partial r}\, d\theta$$

is constant in the case of an annulus and zero if u is harmonic in the whole disk. Combining these results we obtain:

Theorem 20. *The arithmetic mean of a harmonic function over concentric circles $|z| = r$ is a linear function of* $\log r$,

$$\frac{1}{2\pi} \int_{|z|=r} u\, d\theta = \alpha \log r + \beta, \tag{61}$$

and if u is harmonic in a disk $\alpha = 0$ and the arithmetic mean is constant.

In the latter case $\beta = u(0)$, by continuity, and changing to a new origin we find

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta})\, d\theta. \tag{62}$$

It is clear that (62) could also have been derived from the corresponding formula for analytic functions, Sec. 3.4, (34). It leads directly to the *maximum principle* for harmonic functions:

Theorem 21. *A nonconstant harmonic function has neither a maximum nor a minimum in its region of definition. Consequently, the maximum and the minimum on a closed bounded set E are taken on the boundary of E.*

The proof is the same as for the maximum principle of analytic functions and will not be repeated. It applies also to the minimum for the reason that $-u$ is harmonic together with u. In the case of analytic functions the corresponding procedure would have been to apply the maximum principle to $1/f(z)$ which is illegitimate unless $f(z) \neq 0$. Observe that the maximum principle for analytic functions follows from the maximum principle for harmonic functions by applying the latter to $\log |f(z)|$ which is harmonic when $f(z) \neq 0$.

EXERCISES

1. If u is harmonic and bounded in $0 < |z| < \rho$, show that the origin is a removable singularity in the sense that u becomes harmonic in $|z| < \rho$ when $u(0)$ is properly defined.

2. Suppose that $f(z)$ is analytic in the annulus $r_1 < |z| < r_2$ and continuous on the closed annulus. If $M(r)$ denotes the maximum of $|f(z)|$ for $|z| = r$, show that

$$M(r) \leqq M(r_1)^\alpha M(r_2)^{1-\alpha}$$

where $\alpha = \log (r_2/r) \colon \log (r_2/r_1)$ (Hadamard's three-circle theorem). Discuss cases of equality. *Hint:* Apply the maximum principle to a linear combination of $\log |f(z)|$ and $\log |z|$.

6.3. Poisson's Formula. The maximum principle has the following important consequence: If $u(z)$ is continuous on a closed bounded set E and harmonic on the interior of E, then it is uniquely determined by its values on the boundary of E. Indeed, if u_1 and u_2 are two such functions with the same boundary values, then $u_1 - u_2$ is harmonic with the boundary values 0. By the maximum and minimum principle the difference $u_1 - u_2$ must then be identically zero on E.

There arises the problem of finding u when its boundary values are given. At this point we shall solve the problem only in the simplest case, namely for a closed disk.

Formula (62) determines the value of u at the center of the disk. But this is all we need, for there exists a linear transformation which carries any point to the center. To be explicit, suppose that $u(z)$ is harmonic in the closed disk $|z| \leqq R$. The linear transformation

$$z = S(\zeta) = \frac{R(R\zeta + a)}{R + \bar{a}\zeta}$$

maps $|\zeta| \leqq 1$ onto $|z| \leqq R$ with $\zeta = 0$ corresponding to $z = a$. The function $u(S(\zeta))$ is harmonic in $|\zeta| \leqq 1$, and by (62) we obtain

$$u(a) = \frac{1}{2\pi} \int_{|\zeta|=1} u(S(\zeta))\, d \arg \zeta.$$

From

$$\zeta = \frac{R(z - a)}{R^2 - \bar{a}z}$$

we compute

$$d \arg \zeta = -i\frac{d\zeta}{\zeta} = -i\left(\frac{1}{z - a} + \frac{\bar{a}}{R^2 - \bar{a}z}\right) dz = \left(\frac{z}{z - a} + \frac{\bar{a}z}{R^2 - \bar{a}z}\right) d\theta.$$

On substituting $R^2 = z\bar{z}$ the coefficient of $d\theta$ in the last expression can be rewritten as

$$\frac{z}{z - a} + \frac{\bar{a}}{\bar{z} - \bar{a}} = \frac{R^2 - |a|^2}{|z - a|^2}$$

or, equivalently, as

$$\frac{1}{2}\left(\frac{z + a}{z - a} + \frac{\bar{z} + \bar{a}}{\bar{z} - \bar{a}}\right) = \operatorname{Re} \frac{z + a}{z - a}.$$

We obtain the two forms

$$u(a) = \frac{1}{2\pi} \int_{|z|=R} \frac{R^2 - |a|^2}{|z - a|^2}\, u(z)\, d\theta = \frac{1}{2\pi} \int_{|z|=R} \operatorname{Re} \frac{z + a}{z - a}\, u(z)\, d\theta \tag{63}$$

of *Poisson's formula*. In polar coordinates,

$$u(re^{i\varphi}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2rR\cos(\theta - \varphi) + r^2}\, u(Re^{i\theta})\, d\theta.$$

In the derivation we have assumed that $u(z)$ is harmonic in the closed disk. However, the result remains true under the weaker condition that $u(z)$ is harmonic in the open disk and continuous in the closed disk. Indeed, if $0 < r < 1$, then $u(rz)$ is harmonic in the closed disk, and we obtain

$$u(ra) = \frac{1}{2\pi} \int_{|z|=R} \frac{R^2 - |a|^2}{|z - a|^2}\, u(rz)\, d\theta.$$

Now all we need to do is to let r tend to 1. Because $u(z)$ is uniformly continuous on $|z| \leqq R$ it is true that $u(rz) \to u(z)$ uniformly for $|z| = R$, and we conclude that (63) remains valid.

We shall formulate the result as a theorem:

Theorem 22. *Suppose that $u(z)$ is harmonic for $|z| < R$, continuous for $|z| \leqq R$. Then*

$$u(a) = \frac{1}{2\pi} \int_{|z|=R} \frac{R^2 - |a|^2}{|z - a|^2} u(z)\, d\theta \tag{64}$$

for all $|a| < R$.

The theorem leads at once to an explicit expression for the conjugate function of u. Indeed, formula (63) gives

$$u(z) = \operatorname{Re} \left[\frac{1}{2\pi i} \int_{|\zeta|=R} \frac{\zeta + z}{\zeta - z} u(\zeta) \frac{d\zeta}{\zeta} \right]. \tag{65}$$

The bracketed expression is an analytic function of z for $|z| < R$. It follows that $u(z)$ is the real part of

$$f(z) = \frac{1}{2\pi i} \int_{|\zeta|=R} \frac{\zeta + z}{\zeta - z} u(\zeta) \frac{d\zeta}{\zeta} + iC \tag{66}$$

where C is an arbitrary real constant. This formula is known as Schwarz's formula.

As a special case of (64), note that $u = 1$ yields

$$\int_{|z|=R} \frac{R^2 - |z|^2}{|z - a|^2}\, d\theta = 2\pi \tag{67}$$

for all $|a| < R$.

6.4. Schwarz's Theorem. Theorem 22 serves to express a given harmonic function through its values on a circle. But the right-hand side of formula (64) has a meaning as soon as u is defined on $|z| = R$, provided it is sufficiently regular, for instance piecewise continuous. As in (65) the integral can again be written as the real part of an analytic function, and consequently it is a harmonic function. The question is, does it have the boundary values $u(z)$ on $|z| = R$?

There is reason to clarify the notations. Choosing $R = 1$ we define, for any piecewise continuous function $U(\theta)$ in $0 \leqq \theta \leqq 2\pi$,

$$P_U(z) = \frac{1}{2\pi}\int_0^{2\pi} \operatorname{Re}\frac{e^{i\theta}+z}{e^{i\theta}-z}\,U(\theta)\,d\theta$$

and call this the *Poisson integral* of U. Observe that $P_U(z)$ is not only a function of z, but also a function of the function U; as such it is called a *functional.* The functional is *linear* inasmuch as

$$P_{U+V} = P_U + P_V$$

and

$$P_{cU} = cP_U$$

for constant c. Moreover, $U \geqq 0$ implies $P_U(z) \geqq 0$; because of this property P_U is said to be a *positive* linear functional.

We deduce from (67) that $P_c = c$. From this property, together with the linear and positive character of the functional, it follows that any inequality $m \leqq U \leqq M$ implies $m \leqq P_U \leqq M$.

The question of boundary values is settled by the following fundamental theorem that was first proved by H. A. Schwarz:

Theorem 23. *The function $P_U(z)$ is harmonic for $|z| < 1$, and*

(68) $$\lim_{z\to e^{i\theta_0}} P_U(z) = U(\theta_0)$$

provided that U is continuous at θ_0.

We have already remarked that P_U is harmonic. To study the boundary behavior, let C_1 and C_2 be complementary arcs of the unit circle, and denote by U_1 the function which coincides with U on C_1 and vanishes on C_2, by U_2 the corresponding function for C_2. Clearly, $P_U = P_{U_1} + P_{U_2}$.

Since P_{U_1} can be regarded as a line integral over C_1 it is, by the same reasoning as before, harmonic everywhere except on the closed arc C_1. The expression

$$\operatorname{Re}\frac{e^{i\theta}+z}{e^{i\theta}-z} = \frac{1-|z|^2}{|e^{i\theta}-z|^2}$$

vanishes on $|z| = 1$ for $z \neq e^{i\theta}$. It follows that P_{U_1} is zero on the open arc C_2, and since it is continuous $P_{U_1}(z) \to 0$ as $z \to e^{i\theta} \in C_2$.

In proving (68) we may suppose that $U(\theta_0) = 0$, for if this is not the case we need only replace U by $U - U(\theta_0)$. Given $\varepsilon > 0$ we can find C_1 and C_2 such that $e^{i\theta_0}$ is an interior point of C_2 and $|U(\theta)| < \varepsilon/2$ for $e^{i\theta} \in C_2$. Under this condition $|U_2(\theta)| < \varepsilon/2$ for all θ, and hence $|P_{U_2}(z)| < \varepsilon/2$ for all $|z| < 1$. On the other hand, since U_1 is continuous and vanishes

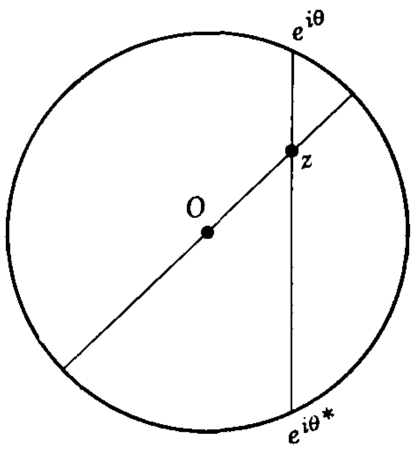

FIG. 4-15

at $e^{i\theta_0}$ there exists a δ such that $|P_{U_1}(z)| < \varepsilon/2$ for $|z - e^{i\theta_0}| < \delta$. It follows that $|P_U(z)| \leqq |P_{U_1}| + |P_{U_2}| < \varepsilon$ as soon as $|z| < 1$ and $|z - e^{i\theta_0}| < \delta$, which is precisely what we had to prove.

There is an interesting geometric interpretation of Poisson's formula, also due to Schwarz. Given a fixed z inside the unit circle we determine for each $e^{i\theta}$ the point $e^{i\theta^*}$ which is such that $e^{i\theta}$, z and $e^{i\theta^*}$ are in a straight line (Fig. 4-15). It is clear geometrically, or by simple calculation, that

$$1 - |z|^2 = |e^{i\theta} - z|\,|e^{i\theta^*} - z|. \tag{69}$$

But the ratio $(e^{i\theta} - z)/(e^{i\theta^*} - z)$ is negative, so we must have

$$1 - |z|^2 = -(e^{i\theta} - z)(e^{-i\theta^*} - \bar{z}).$$

We regard θ^* as a function of θ and differentiate. Since z is constant we obtain

$$\frac{e^{i\theta}\,d\theta}{e^{i\theta} - z} = \frac{e^{-i\theta^*}\,d\theta^*}{e^{-i\theta^*} - \bar{z}}$$

and, on taking absolute values,

$$\frac{d\theta^*}{d\theta} = \left|\frac{e^{i\theta^*} - z}{e^{i\theta} - z}\right|. \tag{70}$$

It follows by (69) and (70) that

$$\frac{1 - |z|^2}{|e^{i\theta} - z|^2} = \frac{d\theta^*}{d\theta}$$

and hence

$$P_U(z) = \frac{1}{2\pi}\int_0^{2\pi} U(\theta)\,d\theta^* = \frac{1}{2\pi}\int_0^{2\pi} U(\theta^*)\,d\theta.$$

In other words, to find $P_U(z)$, replace each value of $U(\theta)$ by the value at the point opposite to z, and take the average over the circle.

EXERCISES

1. Assume that $U(\xi)$ is piecewise continuous and bounded for all real ξ. Show that

$$P_U(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y}{(x-\xi)^2 + y^2}\, U(\xi)\, d\xi$$

represents a harmonic function in the upper half plane with boundary values $U(\xi)$ at points of continuity (Poisson's integral for the half plane).

2. Prove that a function which is harmonic and bounded in the upper half plane, continuous on the real axis, can be represented as a Poisson integral (Ex. 1).

Remark. The point at ∞ presents an added difficulty, for we cannot immediately apply the maximum and minimum principle to $u - P_u$. A good try would be to apply the maximum principle to $u - P_u - \varepsilon y$ for $\varepsilon > 0$, with the idea of letting ε tend to 0. This almost works, for the function tends to 0 for $y \to 0$ and to $-\infty$ for $y \to \infty$, but we lack control when $|x| \to \infty$. Show that the reasoning can be carried out successfully by application to $u - P_u - \varepsilon \operatorname{Im}(\sqrt{iz})$.

3. In Ex. 1, assume that U has a jump at 0, for instance $U(+0) = 0$, $U(-0) = 1$. Show that $P_U(z) - \dfrac{1}{\pi}\arg z$ tends to 0 as $z \to 0$. Generalize to arbitrary jumps and to the case of the circle.

4. If C_1 and C_2 are complementary arcs on the unit circle, set $U = 1$ on C_1, $U = 0$ on C_2. Find $P_U(z)$ explicitly and show that $2\pi P_U(z)$ equals the length of the arc, opposite to C_1, cut off by the straight lines through z and the end points of C_1.

5. Show that the mean-value formula (62) remains valid for $u = \log|1 + z|$, $z_0 = 0$, $r = 1$, and use this fact to compute

$$\int_0^{\pi} \log \sin \theta\, d\theta.$$

6. If $f(z)$ is analytic in the whole plane and if $z^{-1} \operatorname{Re} f(z) \to 0$ when $z \to \infty$, show that f is a constant. *Hint:* Use (66).

7. If $f(z)$ is analytic in a neighborhood of ∞ and if $z^{-1} \operatorname{Re} f(z) \to 0$ when $z \to \infty$, show that $\lim_{z\to\infty} f(z)$ exists. (In other words, the isolated singularity at ∞ is removable.)

Hint: Show first, by use of Cauchy's integral formula, that $f = f_1 + f_2$ where $f_1(z) \to 0$ for $z \to \infty$ and $f_2(z)$ is analytic in the whole plane.

*8. If $u(z)$ is harmonic for $0 < |z| < \rho$ and $\lim_{z \to 0} zu(z) = 0$, prove that u can be written in the form $u(z) = \alpha \log |z| + u_0(z)$ where α is a constant and u_0 is harmonic in $|z| < \rho$.

Hint: Choose α as in (61). Then show that u_0 is the real part of an analytic function $f_0(z)$ and use the preceding exercise to conclude that the singularity at 0 is removable.

6.5. The Reflection Principle. An elementary aspect of the *symmetry principle*, or *reflection principle*, has been discussed already in connection with linear transformations (Chap. 3, Sec. 3.3). There are many more general variants first formulated by H. A. Schwarz.

The principle of reflection is based on the observation that if $u(z)$ is a harmonic function, then $u(\bar{z})$ is likewise harmonic, and if $f(z)$ is an analytic function, then $\overline{f(\bar{z})}$ is also analytic. More precisely, if $u(z)$ is harmonic and $f(z)$ analytic in a region then $u(\bar{z})$ is harmonic and $\overline{f(\bar{z})}$ analytic as functions of z in the region Ω^* obtained by reflecting Ω in the real axis; that is, $z \in \Omega^*$ if and only if $\bar{z} \in \Omega$. The proofs of these statements consist in trivial verifications.

Consider the case of a symmetric region: $\Omega^* = \Omega$. Because Ω is connected it must intersect the real axis along at least one open interval. Assume now that $f(z)$ is analytic in Ω and real on at least one interval of the real axis. Since $f(z) - \overline{f(\bar{z})}$ is analytic and vanishes on an interval it must be identically zero, and we conclude that $f(z) = \overline{f(\bar{z})}$ in Ω. With the notation $f = u + iv$ we have thus $u(z) = u(\bar{z})$, $v(z) = -v(\bar{z})$.

This is important, but it is a rather weak result, for we are assuming that $f(z)$ is already known to be analytic in all of Ω. Let us denote the intersection of Ω with the upper half plane by Ω^+, and the intersection of Ω with the real axis by σ. Suppose that $f(z)$ is defined on $\Omega^+ \cup \sigma$, analytic in Ω^+, continuous and real on σ. Under these conditions we want to show that $f(z)$ is the restriction to Ω^+ of a function which is analytic in all of Ω and satisfies the symmetry condition $f(z) = \overline{f(\bar{z})}$. In other words, part of our theorem asserts that $f(z)$ has an *analytic continuation* to Ω.

Even in this formulation the assumptions are too strong. Indeed, the main thing is that the imaginary part $v(z)$ vanishes on σ, and nothing at all need to be assumed about the real part. In the definitive statement of the reflection principle the emphasis should therefore be on harmonic functions.

Theorem 24. *Let Ω^+ be the part in the upper half plane of a symmetric region Ω, and let σ be the part of the real axis in Ω. Suppose that $v(x)$ is continuous in $\Omega^+ \cup \sigma$, harmonic in Ω^+, and zero on σ. Then v has a harmonic extension to Ω which satisfies the symmetry relation $v(\bar{z}) = -v(z)$.*

In the same situation, if v is the imaginary part of an analytic function $f(z)$ in Ω^+, then $f(z)$ has an analytic extension which satisfies $f(z) = \overline{f(\bar{z})}$.

For the proof we construct the function $V(z)$ which is equal to $v(z)$ in Ω^+, 0 on σ, and equal to $-v(\bar{z})$ in the mirror image of Ω^+. We have to show that V is harmonic on σ. For a point $x_0 \in \sigma$ consider a disk with center x_0 contained in Ω, and let P_V denote the Poisson integral with respect to this disk formed with the boundary values V. The difference $V - P_V$ is harmonic in the upper half of the disk. It vanishes on the half circle, by Theorem 23, and also on the diameter, because V tends to zero by definition and P_V vanishes by obvious symmetry. The maximum and minimum principle implies that $V = P_V$ in the upper half disk, and the same proof can be repeated for the lower half. We conclude that V is harmonic in the whole disk, and in particular at x_0.

For the remaining part of the theorem, let us again consider a disk with center on σ. We have already extended v to the whole disk, and v has a conjugate harmonic function $-u_0$ in the same disk which we may normalize so that $u_0 = \operatorname{Re} f(z)$ in the upper half. Consider

$$U_0(z) = u_0(z) - u_0(\bar{z}).$$

On the real diameter it is clear that $\partial U_0/\partial x = 0$ and also

$$\frac{\partial U_0}{\partial y} = 2\frac{\partial u_0}{\partial y} = -2\frac{\partial v}{\partial x} = 0.$$

It follows that the analytic function $\partial U_0/\partial x - i\, \partial U_0/\partial y$ vanishes on the real axis, and hence identically. Therefore U_0 is a constant, and this constant is evidently zero. We have proved that $u_0(z) = u_0(\bar{z})$.

The construction can be repeated for arbitrary disks. It is clear that the u_0 coincide in overlapping disks. The definition can be extended to all of Ω, and the theorem follows.

The theorem has obvious generalizations. The domain Ω can be taken to be symmetric with respect to a circle C rather than with respect to a straight line, and when z tends to C it may be assumed that $f(z)$ approaches another circle C'. Under such conditions $f(z)$ has an analytic continuation which maps symmetric points with respect to C onto symmetric points with respect to C'.

EXERCISES

1. If $f(z)$ is analytic in the whole plane and real on the real axis, purely imaginary on the imaginary axis, show that $f(z)$ is odd.

2. Show that every function f which is analytic in a symmetric region Ω can be written in the form $f_1 + if_2$ where f_1, f_2 are analytic in Ω and