and elliptic functions was discovered, but not published, by Gauss; it was rediscovered by Abel and Jacobi.

EXERCISES

1. Prove that formula (7) gives $F(\infty) = iK'$.

2. Show that $K = K'$ if and only if $k = (\sqrt{2} - 1)^2$.

3. Show that $f(z)$, $f(z + K)$, and $f(z + iK')$ are odd functions of z while $f(z + K/2)$ and $f(z + K/2 + iK')$ are even.

2.4. The Triangle Functions of Schwarz. The upper half plane is mapped on a triangle with angles $\alpha_1\pi$, $\alpha_2\pi$, $\alpha_3\pi$ by

$$F(w) = \int_0^w w^{\alpha_1 - 1}(w - 1)^{\alpha_2 - 1}\,dw.$$

There are no accessory parameters, as we have already noted.

The inverse function $f(z)$ can again be extended to neighboring triangles by reflection over the sides. This process is particularly interesting when it leads, as in the case of a rectangle, to a meromorphic function. In order that this be so it is necessary that repeated reflections across sides with a common end point should ultimately lead back to the original triangle in an even number of steps. In other words, the angles must be of the form π/n_1, π/n_2, π/n_3 with integral denominators. Elementary reasoning shows that the condition

$$\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} = 1$$

is fulfilled only by the triples (3,3,3), (2,4,4), and (2,3,6). They correspond to an equilateral triangle, an isosceles right triangle, and half an equilateral triangle.

In each case it is easy to verify that the reflected images of the triangle fill out the plane, without overlapping and without gaps. This shows that the mapping functions are indeed restrictions of meromorphic functions, known as the *Schwarz triangle functions*.

The reader is urged to draw a picture of the triangle net in each of the three cases. He will then observe that each triangle function has a pair of periods with nonreal ratio, and is thus an elliptic function. As an exercise, the reader should determine how many triangles there are in a parallelogram spanned by the periods.

3. A CLOSER LOOK AT HARMONIC FUNCTIONS

We have already discussed the basic properties of harmonic functions in Chap. 4, Sec. 6. At that time it was expedient to use a rather crude definition, namely one that requires all second-order derivatives to be continuous. This was sufficient to prove the mean-value property from which we could in turn derive the Poisson representation and the reflection principle. We shall now show that a more satisfactory theory is obtained if we make the mean-value property rather than the Laplace equation our starting point.

In this connection we shall also derive an important theorem on monotone sequences of harmonic functions, usually referred to as *Harnack's principle.*

3.1. Functions with the Mean-value Property. Let $u(z)$ be a real-valued continuous function in a region Ω. We say that u satisfies the mean-value property if

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta})\, d\theta \tag{8}$$

when the disk $|z - z_0| \leq r$ is contained in Ω. We showed in Chap. 4 that the mean-value property implies the maximum principle. Actually, closer examination of the proof shows that it is sufficient to assume that (8) holds for sufficiently small r, $r < r_0$, where we may even allow r_0 to depend on z_0. We repeat the conclusion: a continuous function with this property cannot have a relative maximum (or minimum) without reducing to a constant.

We have shown earlier that every harmonic function satisfies the mean-value condition, and we shall now prove the following converse:

Theorem 6. *A continuous function $u(z)$ which satisfies condition* (8) *is necessarily harmonic.*

Again, the condition need be satisfied only for sufficiently small r. If u satisfies (8), so does the difference between u and any harmonic function. Suppose that the disk $|z - z_0| \leq \rho$ is contained in Ω, the region where u is defined. By use of Poisson's formula (Chap. 4, Sec. 6.3) we can construct a function $v(z)$ which is harmonic for $|z - z_0| < \rho$, continuous and equal to $u(z)$ on $|z - z_0| = \rho$. The maximum and minimum principle, applied to $u - v$, implies that $u(z) = v(z)$ in the whole disk, and consequently $u(z)$ is harmonic.

The implication of Theorem 6 is that we may, if we choose, define a harmonic function to be a continuous function with the mean-value property. Such a function has automatically continuous derivatives of all orders, and it satisfies Laplace's equation.

An analogous reasoning shows that even without the condition (8) the assumptions about the derivatives can be relaxed to a considerable degree. Suppose merely that $u(z)$ is continuous and that the derivatives $\partial^2u/\partial x^2$, $\partial^2u/\partial y^2$ exist and satisfy $\Delta u = 0$. With the same notations as above we show that the function

$$V = u - v + \varepsilon(x - x_0)^2,$$

$\varepsilon > 0$, must obey the maximum principle. Indeed, if V had a maximum the rules of the calculus would yield $\partial^2V/\partial x^2 \leqq 0$, $\partial^2V/\partial y^2 \leqq 0$, and hence $\Delta V \leqq 0$ at that point. On the other hand,

$$\Delta V = \Delta u - \Delta v + 2\varepsilon = 2\varepsilon > 0.$$

The contradiction shows that the maximum principle obtains. We can thus conclude that $u - v + \varepsilon(x - x_0)^2 \leqq \varepsilon\rho^2$ in the disk $|z - z_0| \leqq \rho$. Letting ε tend to zero we find $u \leqq v$, and the opposite inequality can be proved in the same way. Hence u is harmonic.†

3.2. Harnack's Principle. We recall that Poisson's formula (Chap. 4, Sec. 6.3) permits us to express a harmonic function through its values on a circle. To fit our present needs we write it in the form

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\rho^2 - r^2}{|\rho e^{i\theta} - z|^2}\, u(\rho e^{i\theta})\, d\theta \tag{9}$$

where $|z| = r < \rho$ and u is assumed to be harmonic in $|z| \leqq \rho$ (or harmonic for $|z| < \rho$, continuous for $|z| \leqq \rho$). Together with the second of the elementary inequalities

$$\frac{\rho - r}{\rho + r} \leqq \frac{\rho^2 - r^2}{|\rho e^{i\theta} - z|^2} \leqq \frac{\rho + r}{\rho - r} \tag{10}$$

formula (9) yields the estimate

$$|u(z)| \leqq \frac{1}{2\pi}\frac{\rho + r}{\rho - r}\int_0^{2\pi} |u(\rho e^{i\theta})|\, d\theta.$$

If it is known that $u(\rho e^{i\theta}) \geqq 0$ we can use the first inequality (10) as well, and obtain a double estimate

$$\frac{1}{2\pi}\frac{\rho - r}{\rho + r}\int_0^{2\pi} u\, d\theta \leqq u(z) \leqq \frac{1}{2\pi}\frac{\rho + r}{\rho - r}\int_0^{2\pi} u\, d\theta.$$

But the arithmetic mean of $u(\rho e^{i\theta})$ equals $u(0)$, and we end up with the following upper and lower bounds:

$$\frac{\rho - r}{\rho + r}\, u(0) \leqq u(z) \leqq \frac{\rho + r}{\rho - r}\, u(0). \tag{11}$$

† This proof is due to C. Carathéodory.

This is *Harnack's inequality;* we emphasize that it is valid only for positive harmonic functions.

The main application of (11) is to series with positive terms or, equivalently, increasing sequences of harmonic functions. It leads to a powerful and simple theorem known as *Harnack's principle:*

Theorem 7. *Consider a sequence of functions $u_n(z)$, each defined and harmonic in a certain region Ω_n. Let Ω be a region such that every point in Ω has a neighborhood contained in all but a finite number of the Ω_n, and assume moreover that in this neighborhood $u_n(z) \leqq u_{n+1}(z)$ as soon as n is sufficiently large. Then there are only two possibilities: either $u_n(z)$ tends uniformly to $+\infty$ on every compact subset of Ω, or $u_n(z)$ tends to a harmonic limit function $u(z)$ in Ω, uniformly on compact sets.*

The simplest situation occurs when the functions $u_n(z)$ are harmonic and form a nondecreasing sequence in Ω. There are, however, applications for which this case is not sufficiently general.

For the proof, suppose first that $\lim_{n\to\infty} u_n(z_0) = \infty$ for at least one point $z_0 \in \Omega$. By assumption there exist r and m such that the functions $u_n(z)$ are harmonic and form a nondecreasing sequence for $|z - z_0| < r$ and $n \geqq m$. If the left-hand inequality (11) is applied to the nonnegative functions $u_n - u_m$, it follows that $u_n(z)$ tends uniformly to ∞ in the disk $|z - z_0| \leqq r/2$. On the other hand, if $\lim_{n\to\infty} u_n(z_0) < \infty$, application of the right-hand inequality shows in the same way that $u_n(z)$ is bounded on $|z - z_0| \leqq r/2$. Therefore the sets on which $\lim u_n(z)$ is, respectively, finite or infinite are both open, and since Ω is connected, one of the sets must be empty. As soon as the limit is infinite at a single point, it is hence identically infinite. The uniformity follows by the usual compactness argument.

In the opposite case the limit function $u(z)$ is finite everywhere. With the same notations as above $u_{n+p}(z) - u_n(z) \leqq 3(u_{n+p}(z_0) - u_n(z_0))$ for $|z - z_0| \leqq r/2$ and $n + p \geqq n \geqq m$. Hence convergence at z_0 implies uniform convergence in a neighborhood of z_0, and use of the Heine-Borel property shows that the convergence is uniform on every compact set. The harmonicity of the limit function can be inferred from the fact that $u(z)$ can be represented by Poisson's formula.

EXERCISES

1. If E is a compact set in a region Ω, prove that there exists a constant M, depending only on E and Ω, such that every positive harmonic function $u(z)$ in Ω satisfies $u(z_2) \leqq Mu(z_1)$ for any two points $z_1, z_2 \in E$.