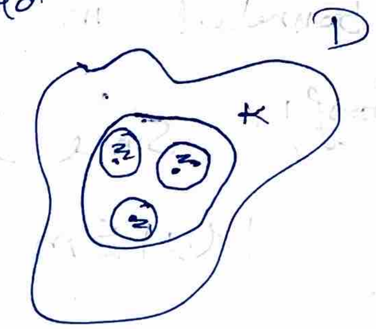
Thm:- A family F of functions is locally uniformly bounded in a domain D iff F is uniformly bounded on each compact subset of D.

Proof:- Let F be a locally uniformly bounded family and let K is a compact subset in D.

Since F is a locally uniformly bdd therefore for each $z_0 \in K$, F will be uniformly bounded in some nbd of z_0. Since K is a compact set,



by Heine-Borel thm, there exists a finite subcover of K. That is, there are finitely many $z_i \in K$ and $\varepsilon_i > 0$ such that

$$K \subset \bigcup_{i=1}^{n} N(z_i, \varepsilon_i),$$

where $|f(z)| \le M_i$ for all $f \in F$

and for all $z \in |z - z_i| < \varepsilon_i$

$\therefore$ $z \in N(z_i, \varepsilon_i)$.

If we define $M = \max\{f_1, f_2, \ldots, f_n\}$

Then F is uniformly bounded on K.

Conversely, since the closure of a nbd is a compact set. Here if F is uniformly bound on each compact set it will uniform bdd on each nbd in D. i.e. locally ~~bo~~ uniformly bdd.

Thm2: Suppose $\mathcal{F}$ is a family of locally uniformly bounded analytic functions in a domain D. Then the family $\mathcal{F}^{(n)}$, consisting of the nth derivatives of all functions in $\mathcal{F}$, is also locally uniformly bounded in D.

Proof: Since $\mathcal{F}$ is locally uniformly bdd, therefore

$$|f(z)| \le m \quad \text{for all } f \in \mathcal{F} \text{ and}$$
$$\text{for all } z \in |z-z_0| \le r \subseteq D.$$

Then for z in the smaller disk

$$|z-z_0| \le r/2$$

Cauchy's Integral formula gives

$$f'(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta-z)^2}\,d\zeta$$

$$f'(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^2}\,dz$$

$$f'(\zeta) = \int_C \frac{f(z)}{(z-\zeta)^2}\,dz$$

or $\quad f'(\zeta) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-\zeta)^2}\,dz$

$$|z-\zeta| = |z-\zeta-z_0+z_0| = |(z-z_0)-(\zeta-z_0)|$$
$$\ge |z-z_0| - |\zeta-z_0|$$
$$\ge r - r/2 = r/2$$

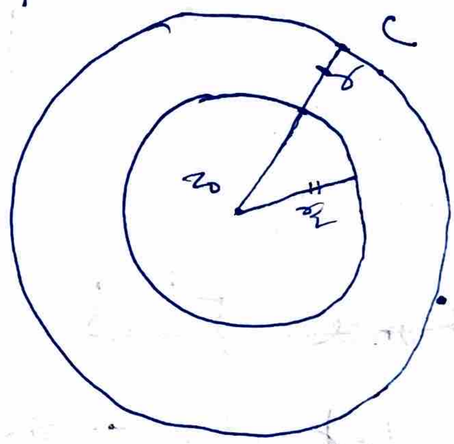
$$|f'(z)| \le \frac{1}{2\pi}\int_C \frac{|f(\zeta)|}{|\zeta-z|^2}\,d\zeta$$

$$\le \frac{1}{2\pi}\frac{m}{(r/2)^2}\int_C |d\zeta|$$

Proof Contd:

$$f'(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)^2}\, d\zeta \qquad \forall\, z \in |z - z_0| \leq \frac{r}{2}$$

$$\begin{aligned}
|\zeta - z| &= |\zeta - z - z_0 + z_0| \\
&= |(\zeta - z_0) - (z - z_0)| \\
&\geq |\zeta - z_0| - |z - z_0| \\
&\geq r - \frac{r}{2} \\
&\geq \frac{r}{2}
\end{aligned}$$



$$\begin{aligned}
|f'(z)| &\leq \frac{1}{2\pi}\int_C \frac{|f(\zeta)|}{|\zeta - z|^2}\,|d\zeta| \\
&\leq \frac{1}{\cancel{2\pi}}\, \frac{M \cdot 4}{r^{\cancel{2}}}\, \cancel{2\pi} \cancel{r} \\
&= \frac{4M}{r}
\end{aligned}$$

$$|f'(z)| \leq \frac{4M}{r} \qquad \text{for all } z \in |z - z_0| \leq \frac{r}{2}$$

This shows that f' is locally uniformly bounded at z_0.

Thm 3:- If $\mathcal{F}$ is a locally uniformly bounded family of analytic functions in a domain D, then $\mathcal{F}$ is equicontinuous on compact subsets of D

Proof: let K is a ~~closed disk contained~~ in D. compact set contained in D. That is there are finitely many ~~subcover of k~~ $z_i \in K$ and $\varepsilon_i > 0$ such that $K \subseteq \bigcup_{i=1}^{n} N(z_i, \varepsilon_i)$.

Since F is locally uniformly bounded for ~~all $f \in$~~. therefore F' is also locally uniformly bounded. Therefore

$$|f'(z)| \leq M_i \quad \text{for all } z \in N(z_i, \varepsilon_i)$$
$$z \in |z - z_i| < \varepsilon_i.$$

~~Since F' is locally uniformly bdd~~

Let $M = \max_{i \geq} \{M_1, M_2, \ldots, M_n\}$.

Then for $z^0, z' \in K$, we have

$$|f(z') - f(z_0)| = \left| \int_{z^0}^{z'} f'(z)\, dz \right|$$
$$\leq \int_{z^0}^{z'} |f'(z)||dz|$$
$$\leq M |z' - z^0|$$

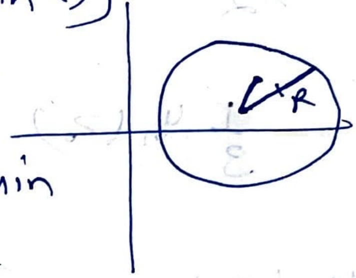
$\delta = \frac{\varepsilon}{M}$.

$|f(z') - f(z^0)| < \varepsilon$ when $|z' - z| < \delta$, $\delta = \frac{\varepsilon}{M}$.

Harnack's Inequality: Suppose $u(z)$ is harmonic in the disk $\Delta(z_0;R) = \{z; |z-z_0|<R\}$, with $u(z) \geq 0$ for all $z \in \Delta(z_0;R)$. Then for every z in this disk, we have

$$u(z_0)\left(\frac{R-|z-z_0|}{R+|z-z_0|}\right) \leq u(z) \leq u(z_0)\,\frac{R+|z-z_0|}{R-|z-z_0|}$$

Harnack's Principle:- Suppose $\{u_n(z)\}$ is a seq^n of real-valued harmonic fu^n defined in a domain D, and that $u_{n+1}(z) \geq u_n(z)$ for each $z \in D$ and $n \in \mathbb{N}$. If $\{u_n(z)\}$ converges for at least one point in D, then $\{u_n(z)\}$ converges for all points in D. Furthermore, the convergence is uniform on compact subsets of D, and the limit function is harmonic throughout D.



Proof:- let us suppose that

$$u_n(z) \geq 0 \quad \text{for all } z \in D,\ n \in \mathbb{N}.$$

If it is not so, we can consider the seq^n

$$\{u_n(z) - u_1(z)\}.$$

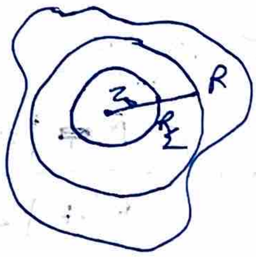
Since $\{u_n(z)\}$ is a monotonically increasing seq^n, therefore either $u_n(z)$ converges or

$$u_n(z) \to \infty.$$

let $A = \{z \in D : u_n(z) \to \infty\}$ & $B = \{z \in D ; u_n(z) \text{ converges}\}$.

let $z_0 \in D$ and $|z - z_0| \le R$ is contained in D.

Then for all $z \in |z - z_0| \le R/2$,



Harnack's inequality gives

$$\frac{1}{3} \quad \frac{R - R/2}{R + R/2} u_n(z_0) \le u_n(z) \le \frac{R + R/2}{R - R/2} u_n(z_0)$$

$$\Rightarrow \quad \frac{1}{3} u_n(z_0) \le u_n(z) \le 3\, u_n(z_0) \quad \text{—— (1)}$$

If $u_n(z_0) \to \infty$ then from (1)

$u_n(z) \to \infty$ for all $z \in |z - z_0| \le R/2$.

If $u_n(z_0)$ converges, then from (1)

$u_n(z)$ also converges for all

$z \in |z - z_0| \le R/2$.

Therefore A and B are both open sets with $A \cup B = D$. But since D is connected it can't be written as a union of two open & disjoint sets. Hence either

$A = \phi$ or $B = \phi$.

But by hypothesis, B is non-empty. Hence

$$B = D.$$

and $\{u_n(z)\}$ converges for all $z \in D$.

__Uniformly convergence:-__

Now, we shall prove the uniform convergence of $\{u_n(z)\}$.

Applying Harnack's inequality to $(u_{n+p}(z) - u_n(z))$,

we get

$$u_{n+p}(z) - u_n(z) \leq 3\{u_{n+p}(z_0) - u_n(z_0)\} \quad \text{--- (2)}$$

for $|z - z_0| \leq R/2$ and $p = 1, 2, 3 \ldots$

By Cauchy-criteria, $\exists$ $N(\epsilon)$ such that

$$|u_{n+p}(z_0) - u_n(z_0)| < \cancel{\epsilon}\,(\epsilon/3) \text{ for all } n > N(\epsilon).$$

Hence from (2), we have

$$|u_{n+p}(z) - u_n(z)| < \epsilon \text{ for all } n > N(\epsilon),$$

~~in~~ in the nbd of z_0 i.e. $\forall z \in |z - z_0| \leq R/2$.

Since z_0 was arbitrary point in D, there corresponds a nbd to every point in D for which the convergence of $\{u_n(z)\}$ is uniform.

~~Co~~ Uniform Convergence on Compact Subset:-

Let K be a compact subset in D. for each point of K construct a nbd in which $\{u_n(z)\}$ converges uniformly. By Heine-Borel thm, finitely many such nbd covers K. But a seq^n converging uniformly on ~~their~~ finitely many different sets must converges uniformly on their union.

Therefore $\{u_n(z)\}$ converges uniformly on K.

As we know, the harmonic property preserve under uniform convergence, the limit function $u_n(z) \rightrightarrows u(z)$ converges throughout D.