Gamma function

The function
$$f(z) = \prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}} \quad (1)$$
is an entire function having simple zeros at the -tive integers and no other zeros.

Clearly
$$f(z-1) = \prod_{n=1}^{\infty}\left(1+\frac{z-1}{n}\right)e^{-\frac{(z-1)}{n}}$$
has the same zeros in addition to a zero at the origin. Here
$$f(z-1) = e^{g(z)}\, z f(z) \quad (2)$$

By Logarithmic derivative, we get
$$\frac{f'(z-1)}{f(z-1)} = g'(z) + \frac{1}{z} + \frac{f'(z)}{f(z)}$$
$$\sum_{n=1}^{\infty}\left(\frac{1}{z+n-1}-\frac{1}{n}\right) = g'(z)+\frac{1}{z}+\sum_{n=1}^{\infty}\left(\frac{1}{z+n}-\frac{1}{n}\right) \quad (3)$$

from (1)
$$\log f(z) = \sum_{n=1}^{\infty}\left(\log\left(1+\frac{z}{n}\right)-\frac{z}{n}\right)$$
$$\frac{f'(z)}{f(z)} = \sum_{n=1}^{\infty}\left(\frac{1}{1+\frac{z}{n}}\left(\frac{1}{n}\right)-\frac{1}{n}\right)$$
$$= \sum_{n=1}^{\infty}\left(\frac{1}{z+n}-\frac{1}{n}\right)$$

L.H.S.
$$\left(\frac{1}{z}-1\right)+\sum_{n=2}^{\infty}\left(\frac{1}{z+n-1}-\frac{1}{n}\right)$$
$$=\left(\frac{1}{z}-1\right)+\sum_{n=1}^{\infty}\left[\left(\frac{1}{z+n}-\frac{1}{n+1}\right)-\frac{1}{n}+\frac{1}{n}\right]$$
$$=\left(\frac{1}{z}-1\right)+\sum_{n=1}^{\infty}\left[\left(\frac{1}{z+n}-\frac{1}{n}\right)+\left(\frac{n+1-n}{n(n+1)}\right)\right]$$

$$=\left(\frac{1}{z}-1\right)+\sum_{n=1}^{\infty}\left(\frac{1}{z+n}-\frac{1}{n}\right)+\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)$$

$$=\frac{1}{z}+\sum_{n=1}^{\infty}\left(\frac{1}{z+n}-\frac{1}{n}\right)$$

$$\boxed{\left(1-\cancel{\tfrac{1}{2}}\right)+\left(\cancel{\tfrac{1}{2}}-\cancel{\tfrac{1}{3}}\right)+\cdots\left(\cancel{\tfrac{1}{n}}-\frac{1}{n+1}\right) \quad 1-\frac{1}{n+1}=1}$$

Now from (3)

$$\frac{1}{z}+\sum_{n=1}^{\infty}\left(\frac{1}{z+n}-\frac{1}{n}\right)=g'(z)+\frac{1}{z}+\sum_{n=1}^{\infty}\left(\frac{1}{z+n}-\frac{1}{n}\right)$$

$$\boxed{g'(z)\equiv 0} \quad \& \quad \boxed{g(z)=\gamma} \text{ (constant)}$$

To Determine the value γ :-

Put $z=1$

$$f(0)=e^{g(1)}\cdot f(1)$$

$$1=e^{\gamma}\prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)e^{-\frac{1}{n}}$$

$$\boxed{e^{-\gamma}=\prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)e^{-\frac{1}{n}}}$$

$$\boxed{\gamma=\sum_{n=1}^{\infty}\left[\frac{1}{n}-\log\left(1+\frac{1}{n}\right)\right]}$$

$$\boxed{\gamma=\lim_{n\to\infty}\left[1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n}-\log(n+1)\right]}$$

γ is known as Euler's constant. ($\gamma \simeq 0.577$)

The function

$$\Gamma(z) = \frac{1}{e^{\gamma z} z f(z)}$$

$$= \frac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty} \left(1+\frac{z}{n}\right)^{-1} e^{z/n}$$

is known as Gamma function.

* It represents an analytic function at all points excepts the negative integers & zero, where it has simple poles

* From (2)

$$f(z) = e^{\gamma} (z+1) f(z+1)$$

So $$\Gamma(z+1) = \frac{1}{e^{\gamma(z+1)} (z+1) f(z+1)}$$

$$= \frac{1}{e^{\gamma z} f(z)}$$

$$\Gamma(z+1) \quad \boxed{\Gamma(z+1) = z\,\Gamma(z)}$$

6 When $(z=n)$ is a +tive integer

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1) = \cdots = n(n-1) \cdots 3 \cdot 2 \cdot \Gamma(1)$$

By γ $$\Gamma(1) = \frac{1}{e^{\gamma} \cdot f(1)}$$

$$= \frac{1}{e^{\gamma} \cdot e^{-\gamma}} = 1$$

Hence $\boxed{\Gamma n+1 = n!}$

Thm :- The gamma function is analytic in $\mathbb{C}$ except at the simple poles $0, -1, -2, \ldots$. Also $\Gamma z+1 = \Gamma z$ & $\Gamma n+1 = n!$ for $n \in \mathbb{N}$.

Moreover
$$\frac{1}{\Gamma(z)} = e^{\gamma z} z \prod_{n=1}^{\infty} \left(1+\frac{z}{n}\right) e^{-z/n}$$
is an entire function.

Relationship b/w Sin z & Gamma function :-

$$\sin \pi z = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$$

$$\sin \pi z = \pi z \, H(z) H(-z) \; ; \text{ where } H(z) = \prod_{n=1}^{\infty} \left(1+\frac{z}{n}\right) e^{-z/n}$$

$$\Rightarrow \sin \pi z = \pi z \cdot \frac{1}{(-z^2) \cdot \Gamma z \, \Gamma -z} \qquad H(z) = \frac{1}{z e^{\gamma z} \Gamma(z)}$$

$$\boxed{\sin \pi z = \frac{\pi}{-z \, \Gamma -z \, \Gamma z}}$$

Since $-z \, \Gamma -z = \Gamma(1-z)$

$$\sin \pi z = \frac{\pi}{\Gamma z \, \Gamma(1-z)}$$

$$\boxed{\frac{\pi}{\sin \pi z} = \Gamma z \, \Gamma(1-z)} \quad \text{for } \boxed{z \neq n \in \mathbb{Z}}$$

* Put $z=\frac{1}{2}$; $\left(\Gamma\frac{1}{2}\right)^2 = \frac{\pi}{1}$

$$\boxed{\Gamma\frac{1}{2} = \sqrt{\pi}}$$

* $\Gamma\frac{(2n+1)}{2} = \left(\frac{2n-1}{2}\right)\Gamma\left(\frac{2n-1}{2}\right) = \frac{(2n+1)}{2}\frac{2n-1}{2}\Gamma\left(\frac{(2n-3)}{2}\right)$

$$= \frac{(2n+1)(2n-1)\cdots 3\cdot 1}{2\cdot 2\cdots 2}\Gamma\frac{1}{2}$$

$$= \frac{(2n+1)\cdots 3\cdot 1}{2^n}\sqrt{\pi}.$$

*

$$\frac{\pi}{\sin(\pi(n+3/2))} = \Gamma\left(\frac{2n+3}{2}\right)\Gamma\left(-\frac{(2n+1)}{2}\right)$$

$$\Gamma\left(-\frac{(2n+1)}{2}\right) = \frac{\pi}{\sin\pi\left(\frac{2n+3}{2}\right)\Gamma\left(\frac{2n+3}{2}\right)}$$

$$= \frac{(-1)^{(n+1)}\pi}{\Gamma\left(\frac{2n+3}{2}\right)} = \frac{(-1)^{n+1}\,2^{n+1}\sqrt{\pi}}{(2n+1)\cdots 3.1}$$

Jensen's formula :- If $f(z)$ is analytic within and on the circle C such that $|z|=R$, and if $f(z)$ has zeros at the points $a_j \neq 0$ $(j=1,2,\dots,m)$ and poles at $b_j \neq 0$, $(j=1,2,\dots n)$ inside C, multiple zeros & poles being repeated, then

$$\frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|\,d\theta = \log|f(0)| + \sum_{r=1}^{m} \log\frac{R}{|a_r|} - \sum_{r=1}^{n} \log\frac{R}{|b_r|}$$

Proof :- Consider the function

$$F(z) = f(z) \prod_{r=1}^{m} \frac{R^2 - z\bar{a}_r}{R(z-a_r)} \cdot \prod_{s=1}^{n} \frac{R(z-b_s)}{(R^2 - z\bar{b}_s)} \qquad \text{A}$$

Clearly $F(z)$ is free from singularity. By Cauchy's integral formula

$$\log F(0) = \frac{1}{2\pi i}\int_C \frac{\log F(z)}{z-0}\,dz$$

Since on C $z = Re^{i\theta}$; $dz = iRe^{i\theta}d\theta$; $dz = iz\,d\theta$

$$\log F(0) = \frac{1}{2\pi i}\int_0^{2\pi} \log F(Re^{i\theta})\, i\, d\theta$$

$$\log F(0) = \frac{1}{2\pi}\int_0^{2\pi} \log F(Re^{i\theta})\,d\theta \qquad (1)$$

$$\log|F(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|F(Re^{i\theta})|\,d\theta \qquad (1)^*$$

~~On C~~ Now,

$$\cancel{R^2(z-a)} \quad |R(z-a)|^2 = R^2(z-a)(\bar{z}-\bar{a})$$

$$= R^2[z\bar{z} - (\bar{a}z + a\bar{z}) + a\bar{a}]$$

$$= R^2[R^2 - (\bar{a}z + a\bar{z}) + a\bar{a}]$$

$$= (R^2 - \bar{a} z)(R^2 - a\bar{z})$$

$$|R(z-a)|^2 = (R^2 - \bar{a} z)(\overline{R^2 - \bar{a} z})$$

$$|R(z-a)|^2 = |R^2 - \bar{a} z|^2$$

$$\left|\frac{R(z-a)}{R^2 - \bar{a} z}\right| = 1 \quad \text{——— (2)}$$

Similarly

$$\left|\frac{R(z-b)}{R^2 - \bar{b} z}\right| = 1 \quad \text{or} \quad \left|\frac{R^2 - \bar{b} z}{R(z-b)}\right| = 1 \quad \text{——— (3)}$$

By (2) & (3)

$$\left|\frac{R(z-a)}{R^2 - \bar{a} z}\right| = \left|\frac{R^2 - \bar{b} z}{R(z-b)}\right| = 1 \quad \text{on}$$

Then by (A)

$$|F(z)| = |f(z)| \quad \text{on} \quad |z| = R$$

or

$$|F(Re^{i\theta})| = |f(re^{i\theta})| \quad \text{——— (8)}$$

Put $z=0$ in (A), we get

$$F(0) = f(0) \prod_{r=1}^{m} \frac{R^2}{R(-a_r)} \prod_{s=1}^{n} \frac{R(-b_s)}{R^2}$$

$$|F(0)| = |f(0)| \prod_{r=1}^{m} \frac{R}{|a_r|} \prod_{s=1}^{n} \frac{|b_s|}{R}$$

Taking log and using (P)

(1) $$\frac{1}{2\pi}\int_0^{2\pi} \log|F(Re^{i\theta})|\, d\theta = \log|f(0)| + \sum_{r=1}^{m} \log\frac{R}{|a_r|} + \sum_{s=1}^{n} \log\frac{|b_s|}{R}$$

by (1)

$$\frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|\, d\theta = \log|f(0)| + \sum_{r=1}^{m} \log\frac{R}{|a_r|} + \sum_{s=1}^{n} \log\frac{|b_s|}{R}$$