7 ELLIPTIC FUNCTIONS

1. SIMPLY PERIODIC FUNCTIONS

A function $f(z)$ is said to be *periodic* with period $\omega \neq 0$ if

$$f(z + \omega) = f(z)$$

for all z. For instance, e^z has the period $2\pi i$, and $\sin z$ and $\cos z$ have the period 2π. To be more precise, we are interested only in analytic or meromorphic functions $f(z)$, and they shall be considered in a region Ω which is mapped onto itself by the translation $z \to z + \omega$.

If ω is a period, so are all integral multiples $n\omega$. There may be other periods as well, but for the present we focus our attention exclusively on the periods $n\omega$. From this point of view we shall call $f(z)$ a *simply periodic function* with period ω. In particular, it is irrelevant whether ω is itself a multiple of another period.

1.1. *Representation by Exponentials.*

The simplest function with period ω is the exponential $e^{2\pi iz/\omega}$. It is a fundamental fact that any function with period ω can be expressed in terms of this particular function.

Let Ω be a region with the property that $z \in \Omega$ implies $z + \omega \in \Omega$ and $z - \omega \in \Omega$. We define Ω' in the ζ-plane to be the image of Ω under the mapping $\zeta = e^{2\pi iz/\omega}$; it is obviously a region. For instance, if Ω is the whole plane, then Ω' is the plane punctured at 0. If Ω is a parallel strip, defined by $a < \text{Im}\,(2\pi z/\omega) < b$, then Ω' is the annulus $e^{-b} < |\zeta| < e^{-a}$.

Suppose that $f(z)$ is meromorphic in Ω and has the period ω. Then there exists a unique function F in Ω' such that

$$f(z) = F(e^{2\pi iz/\omega}). \tag{1}$$

Indeed, to determine $F(\zeta)$ we write $\zeta = e^{2\pi iz/\omega}$; z is unique up to an additive multiple of ω, and this multiple does not influence the value $f(z)$. It is evident that F is meromorphic. Conversely, if F is meromorphic in Ω', then (1) defines a meromorphic function f with period ω.

1.2. The Fourier Development. Assume that Ω' contains an annulus $r_1 < |\zeta| < r_2$ in which F has no poles. In this annulus F has a Laurent development

$$F(\zeta) = \sum_{n=-\infty}^{\infty} c_n\zeta^n,$$

and we obtain

$$f(z) = \sum_{-\infty}^{\infty} c_n e^{2\pi inz/\omega}.$$

This is the complex Fourier development of $f(z)$, valid in the parallel strip that corresponds to the given annulus.

The coefficients (cf. Chap. 5, Sec. 1.3) are given by

$$c_n = \frac{1}{2\pi i}\int_{|\zeta|=r} F(\zeta)\zeta^{-n-1}\,d\zeta, \qquad (r_1 < r < r_2),$$

and by change of variable this becomes

$$c_n = \frac{1}{\omega}\int_a^{a+\omega} f(z)e^{-2\pi inz/\omega}\,dz.$$

Here a is an arbitrary point in the parallel strip, and the integration is along any path from a to $a + \omega$ which remains within the strip. If $f(z)$ is analytic in the whole plane, the same Fourier development is valid everywhere.

1.3. Functions of Finite Order. When Ω is the whole plane $F(\zeta)$ has isolated singularities at $\zeta = 0$ and $\zeta = \infty$. If both these singularities are inessential, that is, either removable singularities or poles, then F is a rational function. We say in this case that f has finite order, equal to the order of F.

We recall that a rational function assumes every complex value, including ∞, the same number of times, provided that we observe the usual multiplicity convention. We obtain a similar result for simply periodic functions of finite order if we agree not to distinguish between z

and $z + \omega$. For convenient terminology, let us say that $z + n\omega$ is equivalent to z. If f is of order m we find that every complex value $c \neq F(0)$ and $F(\infty)$ is assumed at m inequivalent points, with due count of multiplicities. We observe further that $f(z) \to F(0)$ when $\operatorname{Im}(z/\omega) \to -\infty$ and $f(z) \to F(\infty)$ when $\operatorname{Im}(z/\omega) \to \infty$. If we are willing to agree that these values are also "assumed" (with proper multiplicity), we can maintain that all complex values are assumed exactly m times.

For another interpretation we may consider the period strip, defined by $0 \leqq \operatorname{Im}(z/\omega) < 2\pi$. Since this strip contains only one representative from each equivalence class we find that $f(z)$ assumes each complex value m times in the period strip, except that the values $F(0)$ and $F(\infty)$ require a special convention.

2. DOUBLY PERIODIC FUNCTIONS

The terms *elliptic function* and *doubly periodic function* are interchangeable; we have already met examples of such functions in connection with the conformal mapping of rectangles and certain triangles (Chap. 6, Sec. 2). Elliptic functions have been the object of very extensive study, partly because of their function theoretic properties and partly because of their importance in algebra and number theory. Our introduction to the topic covers only the most elementary aspects.

2.1. The Period Module. Let $f(z)$ be meromorphic in the whole plane. We shall examine the set M of all its periods. If ω is a period, so are all integral multiples $n\omega$, and if ω_1 and ω_2 belong to M, so does $\omega_1 + \omega_2$; as a consequence, all linear combinations $n_1\omega_1 + n_2\omega_2$ are in M. In algebra, a set with these properties is called a module (more precisely: a module over the integers), and we shall call M the *period module* of f.

Apart from the trivial case of a constant function, M has also a topological property: all its points are isolated. In fact, since $f(\omega) = f(0)$ for all $\omega \in M$ the existence of a finite accumulation point would immediately imply that f is constant. A module with isolated points is said to be *discrete*.

Our first step is to determine all discrete modules.

Theorem 1. *A discrete module consists either of zero alone, of the integral multiples $n\omega$ of a single complex number $\omega \neq 0$, or of all linear combinations $n_1\omega + n_2\omega_2$ with integral coefficients of two numbers ω_1, ω_2 with nonreal ratio ω_2/ω_1.*

As soon as M contains a number $\omega \neq 0$ it also contains one, call it ω_1, whose absolute value is a minimum. Indeed, if r is large enough the