CLASSTIME Pg. No.
Date / /

UNIT : 3

* **Uniformly Bounded:** A family of function F is said to be Uniformly bounded i.e.

$F = \{ f_n : n \in \mathbb{N} \}$

if for every $f \in F$ and x belongs to the domain.

$|f_n(x)| \leq M \quad \forall n \in \mathbb{N}$

* **Locally Uniformly Bounded:** A family of functions is said to be locally Uniformly bounded on a set A if to each $z \in A$, there corresponds a nbd in which F is Uniformly bounded.

eg: $f_n(z) = \dfrac{1}{1-z^n}$; $|z| < 1$

eg: $f_n(z) = nz$; $|z| < 1$

Sol-(1) $f(z) = \dfrac{1}{1-z^n}$ is Uniformly bounded in every compact subset $|z| \leq R < 1$.

$\Rightarrow |z| \leq R$

$\Rightarrow -|z| \geq -R$

$\Rightarrow -|z|^n \geq -(R)^n$

$\Rightarrow 1-|z|^n \geq 1-(R)^n$

$\Rightarrow \dfrac{1}{1-|z|^n} \leq \dfrac{1}{1-R^n}$

$\Rightarrow f(z) \leq \dfrac{1}{1-R^n}$

Sol2:

$|z| < 1$

$n|z| < n$

$n|z| \leq n$

$f(z) \leq n$

but as $n \to \infty$, $f(z)$ is not bounded. n is not finite as we are considering family of functions.

* **Point-wise Bounded:** A family of functions is said to be pointwise bounded on a Domain D if for every $z \in D$ $\exists$ a constant depending on n such

$$|f_n(z)| \leq m(z) \quad \text{for all } n.$$

* **Equicontinuity:** A family of functions F is said to be equicontinuous if for every $\varepsilon > 0$ $\exists$ $\delta > 0$ free from z & n such that $\forall f_n$

$$|f_n(z_1) - f_n(z_2)| < \varepsilon \quad \text{whenever } |z_1 - z_2| < \delta$$

Each member of a equicontinuous family ($\delta = \delta(\varepsilon)$) is uniformly continuous not conversely.

eg:- $f_n(z) = nz$

$$|f_n(z_1) - f_n(z_2)| = |nz_1 - nz_2|$$
$$= n|z_1 - z_2| < \varepsilon$$

when $|z_1 - z_2| < \delta = \frac{\varepsilon}{n}$

Hence, each member is uniformly convergent.

Equicontinuity $\rightarrow$ No, cause δ depends upon 'n' too.

* **Normal family:** A family of function is said to be normal in a domain D if every seqⁿ in F has a sequence $\{f_{n_k}\}$ that converges uniformly on each compact subset of D.

$a_n = \frac{1}{n} \Rightarrow \langle 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \ldots \rangle$

$a_{2n+1} \Rightarrow \frac{1}{2n+1} \Rightarrow \langle 1, \frac{1}{3}, \frac{1}{5} \ldots \rangle$

eg:- $a_n = a_n(-1)^n$

$$a_n = \begin{cases} 1, & n = \text{even} \\ -1, & n = \text{odd} \end{cases}$$

$a_{2n} = 1, 1, 1, \ldots$

$a_{2n+1} = -1, -1, -1, \ldots$

CLASSTIME Pg. No.
Date / /

* **Arzela – Ascoli's Theorem:** Suppose that F is a pointwise bounded equicontinuous collection of complex functions on a metric space X containing a countable dense subset E.
Every seq^n $\{f_n\}$ in F has a subseq^n that converges uniformly on every compact subset of X.

Proof :- Let $E = \{z_1, z_2, z_3 \ldots\ldots\}$ be the countable set in X.
Let $\langle f_n \rangle$ be a seq^n in F defined on each z. Since F is pointwise bounded equicontinuous collection therefore each $f_n(x_i)$ in F will be bounded. ~~Again~~
Again by B.W Theorem, it will have a convergent subseq^n

$f_n(x_1)$
$\downarrow$
$f_{n_1}(x_1) \rightarrow$ convergent subseq.
$\downarrow$
$f_{n_1}(x_2)$
$\downarrow$
$f_{n_2}(x_2) \rightarrow$ convergent subseq
$\downarrow$
$f_{n_2}(x_3)$
$\downarrow$
$f_{n_3}(x_3) \rightarrow$ convergent subseq.

$\langle f_{n_1}(x_1), f_{n_2}(x_2), f_{n_3}(x_3) \rangle$

By diagonal process, we can find a convergent subsequence which converges at every point of set E.

To prove Uniformly Convergent :-

Let $K \subset X$ be a compact subset in X. Since F is equicontinuous so for every $\varepsilon > 0$
$|f_n(z_1) - f_n(z_2)| < \varepsilon$ whenever $|z_1 - z_2| < \delta$

* Now cover the set K with finite open balls B_i, $1 \le i \le m$

Let $z_i \in E \cap B_i$ for $1 \le i \le m$ & for each z_i,

$$\lim_{n\to\infty} f_n(z_i) \text{ exist}$$

By Cauchy Criteria,
$$|f_m(z_i) - f_n(z_i)| < \varepsilon \quad \forall\ m, n \ge K.$$

* Let $z \in K$, then $z \in B_i$ for some i & $|z - z_i| < \delta$

$$\Rightarrow |f_m(z) - f_n(z)| = |f_m(z) - f_m(z_i) + f_m(z_i) - f_n(z_i) + f_n(z_i) - f_n(z)|$$
$$\Rightarrow |f_m(z) - f_m(z_i)| + |f_n(z) - f_n(z_i)| + |f_m(z_i) - f_n(z_i)|$$
$$\Rightarrow \varepsilon + \varepsilon + \varepsilon$$
$$\Rightarrow 3\varepsilon \quad \forall\ m, n \ge K.$$

By Cauchy Criteria,
$f_n(z)$ is uniformly convergent.

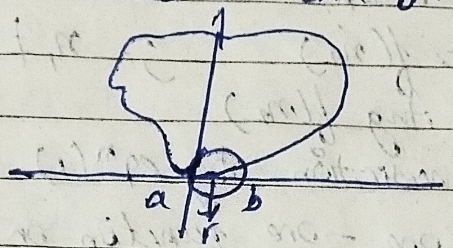
* Univalent functions :- A functions $f : D \to \mathbb{C}$ is said to be Univalent if it does not take any value twice i.e

$$f(z_1) = f(z_2) \Rightarrow z_1 = z_2$$

For eg:
$f(z) = 1 + z$
$f(z) = \sin z \quad ; |z| < 1$
$f(z) = e^z \quad ; |z| < 1$

CLASSTIME Pg. No.
Date / /

* Free Boundary Arc :- α is said to be free boundary Arc if every point of α has a nbd whose intersection with the whole boundary $\partial\Omega$ is the same as its intersection with γ.



Theorem 2 :- Suppose that the boundary of a simply connected region Ω contains a line segment γ as a free boundary ~~etc~~ arc. Then the function $f(z)$ which maps Ω onto the unit ~~arc~~ disk can be extended to a function which is analytic & one-to-one on $\Omega \cup \gamma$. The image of γ is an arc γ' on the unit circle.

Proof :- By Reimann mapping thm, $\exists$ a one-one & onto function from Ω to Δ. Let us consider a nbd as γ & that z_0 is not in this nbd. So, we can define an analytic branch of

$$\log f(z) = \log |f(z)| + i \operatorname{Arg} f(z).$$

By reflection principle, the function $\log f(z)$ will be analytic on the free boundary arc & by Thm 1, the image of γ will be some γ' on the boundary of unit disk.

One-one :- Suppose $f(z)$ is not one-one on the boundary arc γ.

$f'(x) = 0$ for some x on γ.

$$\frac{\delta}{\delta y} \log |f(z)| < 0$$

By C-R eqn

$$\frac{\delta}{\delta y} \log |f(z)| = -\frac{\delta}{\delta x} \operatorname{Arg} f(z) < 0.$$

$\frac{\delta}{\delta n}$ Arg $f(z) > 0$ —(1)

Since $f(z)$ is not one-one.
$\therefore \exists$ n_1 & n_2 such that
$$f(n_1) = f(n_2) \quad ; \quad n_1 \neq n_2$$
$$\text{Arg } f(n_1) = \text{Arg } f(n_2)$$
which is a contradiction by eq^n (1).
Hence, $f(z)$ is a one-one function on γ.

* Real Analytic function: A real function $Q(t)$ of a real variable t, defined on an interval $a < t < b$, is said to be real analytic, if for every point in the interval. The Taylor series expansion of $Q(t)$ about t_0,
$$Q(t) = Q(t_0) + t\,Q'(t_0) + \frac{t^2}{2!} Q''(t_0) + \text{——}$$
converges in some nbd of t_0, i.e. in $(t_0-\varepsilon, t_0+\varepsilon)$, $\varepsilon > 0$

* By Abel's thm the series is also convergent for complex values of $x \in |n-n_0| < \varepsilon$ & it represents an analytic function in this disk.

* $Q(t)$ can be defined as an analytic function in a region Δ, symmetric to the real axis which contains the segment (a,b).

* In those circumstances, we say $Q(t)$ determines an analytic arc.

* Regular Analytic Arc: $Q(t)$ is said to be regular if
$$Q'(t) \neq 0 \quad \forall\ t \in (a,b)$$

* Simple Analytic Arc: $Q(t)$ is said to be simple analytic arc if
$Q(t_1) = Q(t_2)$ only when $t_1 = t_2$ in (a,b).

CLASSTIME Pg. No.
Date / /

Theorem:- A family F of functions is locally uniformly bounded in Domain D iff F is uniformly bounded on each compact subset of D.

Proof:- Let F be a locally uniformly bounded family & let K is a compact subset in D.
Since F is a locally uniformly ~~bdd~~ bounded therefore for each $z_0 \in K$, F will be uniformly bounded in some nbd of z_0. Since K is a compact set
By Heine-Borel theorem, there exists a finite subset of K.
That is, there are finitely many

$z_i \in K$. & $\varepsilon_i > 0$ Such that

$$K \subset \bigcup_{i=1}^{n} N(z_i, \varepsilon_i),$$

where $|f(z)| \leq m_i$ for all $f \in F$.
& for all $z \in |z - z_i| < \varepsilon_i$

$$z \in N(z_i, \varepsilon_i)$$

If we define $m = \max\{F_1, F_2, \ldots F_n\}$
Then F is uniformly bounded on K.

Conversely, since the closure of a nbd is a compact set.
Hence if F is uniformly bound on each compact set it will uniformly bdd on each nbd in D & locally uniformly bounded.

Thm. 2:- Suppose F is a family of locally uniformly bounded analytic functions in a Domain D. Then the family $F^{(n)}$, consisting of the n^{th} derivatives of all function in F, is also locally uniformly bound in D.

CLASSTIME Pg. No.
Date / /

Proof:- Since F is locally uniformly bounded, therefore
$|f(z)| \le m$ for all $f \in F$,
& $\forall$ ~~for all~~ $z \in |z - z_0| \le r \subseteq D$

Then, for z in the smaller disk
$|z - z_0| \le \frac{r}{2}$

Cauchy's integral formula gives
$$f'(z) = \frac{1}{2\pi i} \int_C \frac{f(z_c)}{(z_c - z)^2} \, dz_c \quad \forall z \in |z - z_0|$$

$$\begin{aligned}
|z_c - z| &= |z_c - z - z_0 + z_0| \\
&= |(z_c - z_0) - (z - z_0)| \\
&\ge |z_c - z_0| - |z - z_0| \\
&\ge r - \frac{r}{2} \\
&\ge \frac{r}{2}
\end{aligned}$$

$$|f'(z)| \le \frac{1}{2\pi} \int \frac{|f(z_c)|}{|z_c - z|^2} \, |dz_c|$$

$$\le \frac{1}{2\pi} \frac{4m}{r^2} \cdot 2\pi r$$

$|f'(z)| \le \frac{4m}{r}$ for all $z \in |z - z_0|$

This shows that f' is locally uniformly bounded at z_0

Theorem: If F is a locally uniformly bounded family of analytic functions in a Domain D. Then F is equicontinuous on compact subset of D.

CLASSTIME Pg. No.
Date / /

Proof: Let K is a compact set in D. That is there are finitely many $z_i \in K$ & $\varepsilon_i > 0$ such that
$$K \subset \bigcup_{i=1}^{n} N(z_i, \varepsilon_i).$$

Since F is locally uniformly bounded so therefore F' is also locally uniformly bounded.
Therefore,

$$|f'(z)| \leq m_i \quad \forall \quad z \in N(z_i, \varepsilon_i)$$
$$z \in |z - z_i| < \varepsilon_i.$$

Let $m = \max\{m_1, m_2, \ldots, m_n\}$
Then
For $z_0^*, z_1^* \in K$,
We have,
$$|f(z_1^*) - f(z_0^*)| = \left| \int_{z_0^*}^{z_1^*} f'(z)\, dz \right|$$
$$\leq \int_{z_0}^{z_1} |f'(z)| |dz|$$
$$\leq m |z_1^* - z_0^*|$$

$$|f(z_1^*) - f(z_0^*)| < \varepsilon \quad \text{where } |z_1^* - z_0| < \frac{\varepsilon}{m}.$$