~~Tuesday, 3 September 2024 9:37 AM~~

(10 Sep 2024)

(IN CLASS)

UNIT-2

[Made by: OM KUMAR 2K22/MC/106]

Singularities (Revisited)

Let $f(z)$ has the following Laurent Series Expansion

$$f(z) = \underbrace{\sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}}_{\text{Principal part}} + \underbrace{\sum_{n=0}^{\infty} a_n (z-z_0)^n}_{\text{Analytic Part}}$$

valid in $r<|z-z_0|<R$

(i) If all b_n's are zero, that is $f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots$

then z_0 is a removable singularity.

(ii) If $b_{k+1}=0$ $\forall$ $k \geq n$, i.e.,

$$f(z) = \frac{b_k}{(z-z_0)^k} + \frac{b_{k-1}}{(z-z_0)^{k-1}} + \dots + \frac{b_1}{z-z_0} + \sum_{n=0}^{\infty} a_n (z-z_0)^n,$$

then z_0 will be a pole of order k.

(iii) If principal part contains infinite terms, then z_0 will be an essential singularity.

Meromorphic functions: A function is said to be meromorphic in $\mathbb{C}$ if at any point it is either regular or has isolated singularity (poles).

for ex: $\frac{\cos z}{\sin z}$, $\frac{1}{(z-1)(z-3)}$, $\tan z$, $\operatorname{cosec} z$

(i) The ratio of two entire functions is meromorphic

(ii) sum, product or Linear combination of meromorphic fun is meromorphic.

(iii) Entire functions are meromorphic but not conversily.

Partial fraction :-

* Let $\langle b_n \rangle$ be a seq^n tending to ∞ and function $f(z)$ have poles at b_n. For each n, associate a rational function (ratio of polynomials) of the form

$$P_n\left(\frac{1}{z-b_n}\right) = \frac{a_1^{(n)}}{(z-b_n)} + \frac{a_2^{(n)}}{(z-b_n)^2} + \dots + \frac{a_{k_n}^{(n)}}{(z-b_n)^{k_n}} \longrightarrow (A)$$

where a_i's are complex constants & k_n denotes the order of pole at $z = b_n$.

Def^n: Let $f(z)$ be a meromorphic function defined on a domain D in $\mathbb{C}$. A partial fraction decomposition of $f(z)$ is a representation of the term

$$f(z) = g(z) + \sum_{n=1}^{\infty} P_n\left(\frac{1}{z-b_n}\right),$$

where $g(z)$ is analytic in D.

Th^m: (Mittag Laffler's Th^m): Let $\langle b_n \rangle$ be a seq^n tending to ∞, and $P_n\left(\frac{1}{z-b_n}\right)$ be a poly^n in $\left(\frac{1}{z-b_n}\right)$ of the form (A). Then there exist a meromorphic Function $f(z)$ that has poles at the points b_n with principal part $P_n\left(\frac{1}{z-b_n}\right)$, and is analytic otherwise.

Proof:- Since $\langle b_n \rangle \longrightarrow \infty$, so, we can assume that

$$0<|b_1| \leq |b_2| \leq |b_3| \leq \dots$$

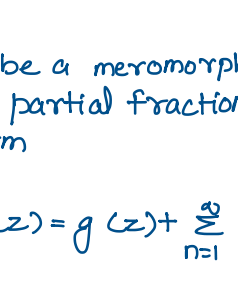
Now,

$P_n\left(\frac{1}{z-b_n}\right)$ will be analytic in $|z|<|b_n|$, Hence we can expand $P_n\left(\frac{1}{z-b_n}\right)$ in Maclaurin's series form

$$P_n\left(\frac{1}{z-b_n}\right) = \underbrace{\sum_{k=0}^{\infty} a_k^{(n)} z^k}_{|z|<b_n} \text{ ——— ①}$$

Note: $\boxed{\sum_{n=0}^{\infty} a_n z^n \longrightarrow f(z) \quad |z|<R \quad \underbrace{|z| \leq r < R}_{\downarrow} \text{ Uniformly Cn.}}$

This series uniformly converges in the domain $|z| \leq \frac{|b_n|}{2}$ (say)



Thus $P_n\left(\frac{1}{z-b_n}\right)$ can be approximated in $|z| \leq \frac{|b_n|}{2}$ by a polynomial or partial sum

$$Q_n(z) = \sum_{k=0}^{n_k} a_k^{(n)} z^k$$

$$= a_0^{(n)} + a_1^{(n)} z + a_2^{(n)} z^2 + \dots + a_{n_k}^{(n)} z^k$$

We can write that

$$\left| P_n\left(\frac{1}{z-b_n}\right) - Q_n(z) \right| < \frac{1}{2^n} \text{ for sufficiently large } n.$$

Now we have to show that

$$f(z) = \sum_{n=1}^{\infty} \left(P_n\left(\frac{1}{z-b_n}\right) - Q_n(z) \right) \text{ ——(i)}$$

is meromorphic function. To prove this is sufficed to show that the series in (i) converges uniformly on an arbitrary compact subset $|z| \leq R$ of $\mathbb{C}$ that excludes the points $|b_n| \leq R$. Choosing N such that $|b_n| \geq 2R$, from (i)

$$\sum_{n=N}^{\infty} \left| P_n\left(\frac{1}{z-b_n}\right) - Q_n(z) \right| < \sum \frac{1}{2^n} < \infty$$

By Weierstrass M test the series

$$\sum_{n=N}^{\infty} P_n\left(\frac{1}{z-b_n}\right) - Q_n(z) \rightrightarrows g(z), \text{ which is}$$

analytic in $|z| \leq R$.

Weierstrass M test for $\overset{\text{Uniformly Convergent}}{\text{U.C}}$

If $\sum f_n(z)$ is a series of f^n. and $\exists$ $\langle a_n \rangle$ of positive real nos. such that

(i) $|f_n(z)| \leq a_n$

(ii) $\sum a_n$ is converges then

$\boxed{\sum f_n(z) \rightrightarrows f(z)}$

for $|z| \leq R$,

$$\sum_{n=0}^{N-1} \left[P_n\left(\frac{1}{z-b_n}\right) - Q_n(z) \right] \text{ is an analytic function with no singularity}$$

except the prescribed poles. Since R is arbitrary, $f(z)$ is meromorphic in $\mathbb{C}$.

Date :-(12/09/2024)

Infinite Product:- An infinite product of complex numbers is an expression of the form

$$p_1 p_2 \dots p_n \dots = \prod_{n=1}^{\infty} p_n$$

* It is evaluated by taking the limit of partial product

$$P_n = p_1 p_2 \dots p_n$$

Convergence & Divergence:

* An infinite product $\prod_{n=1}^{\infty} p_n$ is said to be converges iff there is an N such that $p_n \neq 0$ for all $k \geq N$ & $\lim_{k\to\infty} \prod_{k=N} p_k$ exist & is non-zero.

* An infinite product that does not converge is said to be divergent.

* If convergent condition holds but finitely many p_k's are zero, then the infinite product $\prod_{n=1}^{\infty} p_n$ converges to zero. In this case the value of the product is set as 0.

* If $p_n \neq 0$ for all $n \geq 1$, and

$$\lim_{n\to\infty} \prod_{k=1}^{n} p_k = 0,$$

then we say that the infinite product diverges to zero.

* The partial product

$$P_n = p_1 p_2 \dots p_n.$$

Then $P_{n+1} = p_1 p_2 \dots p_n p_{n+1}$

If $\lim_{n\to\infty} P_n$ exist such that $\lim_{n\to\infty} P_n = P$. Then

$$p_{n+1} = \frac{P_{n+1}}{P_n}$$

$$\lim_{n\to\infty} p_{n+1} = \frac{\lim_{n\to\infty} P_{n+1}}{\lim_{n\to\infty} P_n} = \frac{P}{P}$$

$$\lim_{n\to\infty} p_{n+1} = 1$$

Thm:- (Necessary Condition) If the Infinite product $\prod_{n=1}^{\infty} (1+a_n)$ converges, then $\lim_{n\to\infty} a_n \to 0$, & $a_n = -1$ for at most finitely many n.

Ex:- $\prod_{k=1}^{\infty} \left(1+\frac{1}{k}\right)$

$$P_n = 2/1 \cdot 3/2 \cdot 4/3 \dots \frac{n+1}{n} = n+1$$

$$\lim_{n\to\infty} P_n \to \infty$$

Ex:- $\prod_{n=1}^{\infty} (1-1/n)$

$$P_n = \frac{1}{2} \cdot \frac{-2}{3} \cdot \frac{3}{4} \cdot \frac{4}{5} \dots \frac{n}{n+1}$$

$$P_n = \frac{1}{n+1}$$

$$\lim_{n\to\infty} P_n \to 0$$

The infinite product $\prod_{n=2}^{\infty} \left(1-\frac{1}{n}\right)$ diverges to zero.

(N & S cond^n)

Thm:- For a_n complex, $a_n \neq -1$, the product $\prod_{n=1}^{\infty} (1+a_n)$ converges iff $\sum_{n=1}^{\infty} \log(1+a_n)$ converges.

Proof:- Let

$$P_n = \prod_{k=1}^{n} (1+a_k)$$

$$S_n = \sum_{k=1}^{n} \log(1+a_k)$$

$$P_n = e^{S_n} = e^{\log(1+a_1) + \log(1+a_2) + \dots + \log(1+a_n)}$$

$$= e^{\log(1+a_1)} \cdot e^{\log(1+a_2)} \dots e^{\log(1+a_n)}$$

Let us suppose that $\sum_{n=1}^{\infty} \log(1+a_n)$ converges.

Therefore,

$$\lim_{n\to\infty} S_n = S \text{ exist}$$

Thus

$$\lim_{n\to\infty} P_n = \lim_{n\to\infty} e^{S_n}$$

$$= e^{\lim_{n\to\infty} S_n}$$

$$= e^S \neq 0$$

Therefore, $\prod_{n=1}^{\infty} P_n$ converges.

Conversely, let $\lim_{n\to\infty} P_n = P$

$$\lim_{n\to\infty} \frac{P_n}{P} = 1$$

$$\lim_{n\to\infty} \operatorname{Log}\left(\frac{P_n}{P}\right) \to 0$$

$$\operatorname{Log}\left(\frac{P_n}{P}\right) = \log P_n - \log P + i\, 2\pi h_n, \qquad [\log z = \log|z| + i \arg z]$$

where h_n is any integer

$$(h_{n+1} - h_n) 2\pi i = \left(\log \frac{P_{n+1}}{P} - \log P_{n+1} + \cancel{\log P}\right) - \left(\log \frac{P_n}{P} - \log P_n + \cancel{\log P}\right)$$

$$(h_{n+1} - h_n) 2\pi i = \log\left(\frac{P_{n+1}}{P}\right) - \log\left(\frac{P_n}{P}\right) - \log P_{n+1} + \log P_n$$

$$(h_{n+1} - h_n) 2\pi = \operatorname{Arg}\left(\frac{P_{n+1}}{P}\right) - \operatorname{Arg}\left(\frac{P_n}{P}\right) - \operatorname{Arg}(1+a_{n+1})$$

$$|(h_{n+1} - h_n)| 2\pi < \left|\operatorname{Arg}\left(\frac{P_{n+1}}{P}\right)\right| + \left|\operatorname{Arg} \frac{P_n}{P}\right| + |\operatorname{Arg}(1+a_{n+1})|$$

as $n \to \infty$; $\arg\left(\frac{P_{n+1}}{P}\right)$ & $\arg\left(\frac{P_n}{P}\right) > 0$ as $\frac{P_n}{P} \to 1$

& $-\pi < \operatorname{Arg}(1+a_{n+1}) \leq \pi$.

How h_n comes?

$\boxed{P_n = e^{i\theta_n}}$

$\boxed{P = e^{i\theta}}$

$\arg\left(\frac{P_n}{P}\right) = \theta_n - \theta$

$\lim_{n\to\infty} \theta_n \to \theta$

$\theta_n = \theta + 2\pi \cdot h_n$

$\boxed{\theta_n - \theta = 2\pi h_n}$

Thus,

$$|h_{n+1} - h_n| 2\pi < \pi$$

$$|h_{n+1} - h_n| < 1/2$$

This implies that $h_{n+1} = h_n$.

Hence h_n is equal to a fixed integer h and it follows from

$$\log\left(\frac{P_n}{P}\right) = \underset{\overset{\shortparallel}{\log(P_n)}}{S_n} - \operatorname{Log} P + h \cdot 2\pi i$$

that $\boxed{S_n \to \operatorname{Log} P + h \cdot 2\pi i \quad \text{as } n \to \infty}$

(03/10/2024)

Canonical Product :- Let f be an entire function with a zero of multiplicity $m \geq 0$ at $z=0$. Let $\{a_n\}$ be the seq^n of zeros of f, $a_n \neq 0$, arranged so that a zero of multiplicity k is repeated in this seq^n k times. Also assume that $|a_1| \leq |a_2| \leq \dots$. If $\{p_n\}$ is a seq^n of integers such that $\sum_{n=1}^{\infty} \left(\frac{r}{|a_n|}\right)^{p_n+1} < \infty$ for every $r>0$

then

$$P(z) = \prod_{n=1}^{\infty} E_{p_n}(z/a_n) \quad \text{(Canonical Product)}$$

converges uniformly on every compact subset in $\mathbb{C}$, where

$$E_p(z) = (1-z) e^{z + z^2/2 + \dots + z^p/p}$$

&

$$E_0(z) = (1-z)$$

Rank of a Function: Let f be an entire function with zeros $\{a_1, a_2, \dots\}$ repeated according to multiplicity & arranged so that $|a_1| \leq |a_2| \leq |a_3| \dots$. Then f is of finite rank if $\exists$ an integer p such that

$$\sum \frac{1}{|a_n|^{p+1}} < \infty$$

* If p is the smallest integer such that this occurs, then f is said to be of rank p.

* If f has finite zeros then rank $p=0$

* f is said to be of infinite rank if it is not of finite rank.

* Let f be an entire function of rank p with zeros a_n, then

The product

$$P(z) = \prod_{n=1}^{\infty} E_p(z/a_n)$$

is said to be in standard form.

Genus:- An entire function f has finite genus if f has finite rank and if

$$f(z) = z^m e^{g(z)} P(z),$$

where $P(z)$ is canonical product in standard form and g is poly^n of degree q then,

$$\text{genus } \mu = \max(p, q)$$

Order of an entire f^n:- An entire function is said to be of finite order if $\exists$ a positive Integer λ, independent of r, such that its maximum value $M(r)$ on the circle $|z|=r$, satisfies the inequality

$$\log|f(z)| = \log M(r) < r^\lambda$$

for all sufficiently large value of r.

$\lambda \leq \mu+1$

* If no such λ exists, then the entire function is said to be of infinite order.

* The lower bound of all values of λ is called the order of $f(z)$.

Thm (Lemma 1):- If f is an entire function of finite genus μ, then f is of finite order $\boxed{\lambda \leq \mu+1}$

(Lemma 2):- Let $f(z)$ be an entire function of finite order λ & $n(r)$ be the no. of zeros of $f(z)$ in $|z| \leq r$, not including possible zeros at the origin. Then

$$n(r) < r^{\lambda+\epsilon} \text{ for large value of } r \text{ \& } \epsilon > 0$$

(Lemma 3):- Let f be a non-constant entire function of order λ with $\underline{f(0)=1}$ & let $\{a_n\}$ be the zeros of f counted acc. to multiplicity & arranged so that $|a_1| \leq |a_2| \leq \dots$. If an integer

$$p > \lambda - 1, \text{ then}$$

$$\frac{d^p}{dz^p}\left[\frac{f'(z)}{f(z)}\right] = -p! \sum_{n=1}^{\infty} \frac{1}{(a_n - z)^{p+1}}$$

Hadmard's Factorization Th^m :-

If f is an entire function of Finite order λ, then f has finite genus

$$\mu \leq \lambda.$$

Proof:- Let p be any positive integer such that

$$p \leq \lambda < p+1$$

We can assume $f(0)=1$ with no loss of generality

Suppose $f(z)$ has zero of order m at $z=0$. Then

$$f(z) = z^m g(z)$$

Now,

$$\log|f(z) z^{-m}| = \log M(r) - m \log r$$

$$\leq r^{\lambda+\epsilon} - r^m < r^{\lambda+\epsilon}$$

Since, $f(z)$ and $f(z) z^{-m}$ has same order λ, so we can consider $f(0)=1$

Next, we have to prove that

$$\sum \frac{1}{|a_n|^{p+1}} < \infty$$

Let $n(|a_k|)$ or $n(r)$ denote the no. of zeros in $|a_k| \leq r$. Then

$$K < n|a_k| \leq |a_k|^{\lambda+\epsilon}$$

$$K \leq |a_k|^{\lambda+\epsilon}$$

$$\exists\ |a_k| \geq K^{1/\lambda+\epsilon}$$

$$|a_k|^{p+1} \geq K^{(p+1)/\lambda+\epsilon}$$

$$\frac{1}{|a_k|^{(p+1)}} \leq \frac{1}{K^{(p+1)/\lambda+\epsilon}}$$

since $p \leq \lambda < p+1$

$$p+1 > \lambda$$

If we choose $\epsilon > 0$ such that

$$p+1 > \lambda + \epsilon$$

$$\frac{p+1}{\lambda+\epsilon} > 1$$

Then $\sum \frac{1}{|a_n|^{p+1}} < \infty$ (By the convergence of $\sum \frac{1}{K^{(p+1)/\lambda+\epsilon}}$)

By Weierstrass Product th^m,

$$f(z) = e^{g(z)} P(z)$$

$$\log f(z) = \log(e^{g(z)} \cdot P(z))$$

$$= \log e^{g(z)} + \log P(z)$$

$$= g(z) + \log P(z)$$

$$\frac{f'(z)}{f(z)} = g'(z) + \frac{P'(z)}{P(z)}$$

differ with respect p times.

$$\frac{d^p}{dz^p}\left[\frac{f'(z)}{f(z)}\right] = g^{(p+1)}(z) + \frac{d^p}{dz^p}\left[\frac{P'(z)}{P(z)}\right]$$

By Lemma I,

$$\frac{d^p}{dz^p}\left[\frac{f'(z)}{f(z)}\right] = -p! \sum_{n=1}^{\infty} \frac{1}{(a_n - z)^{p+1}}$$

Again, we can show that

$$\frac{d^p}{dz^p}(P(z)) = p! \sum_{n=1}^{\infty} \frac{1}{(a_n - z)^{p+1}}$$

Hence $g^{(p+1)}(z) = 0$

$\Rightarrow g(z)$ is poly^n of degree at most p.

We have proved

rank $= p$

degree $= p$

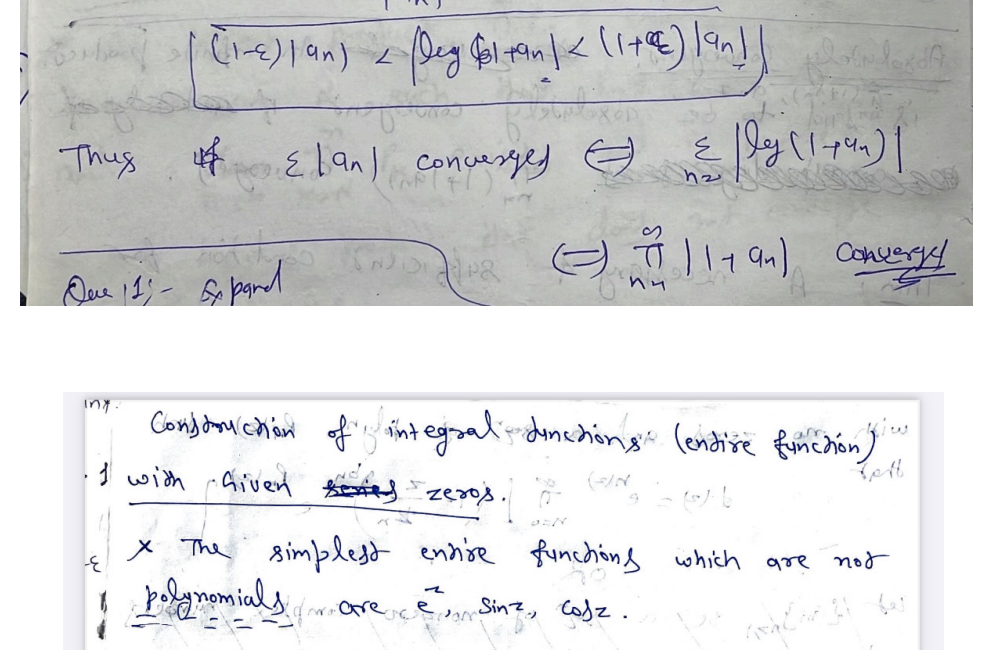
Therefore

genus $\mu \leq p \leq \lambda$

$\boxed{\mu \leq \lambda}$

degree

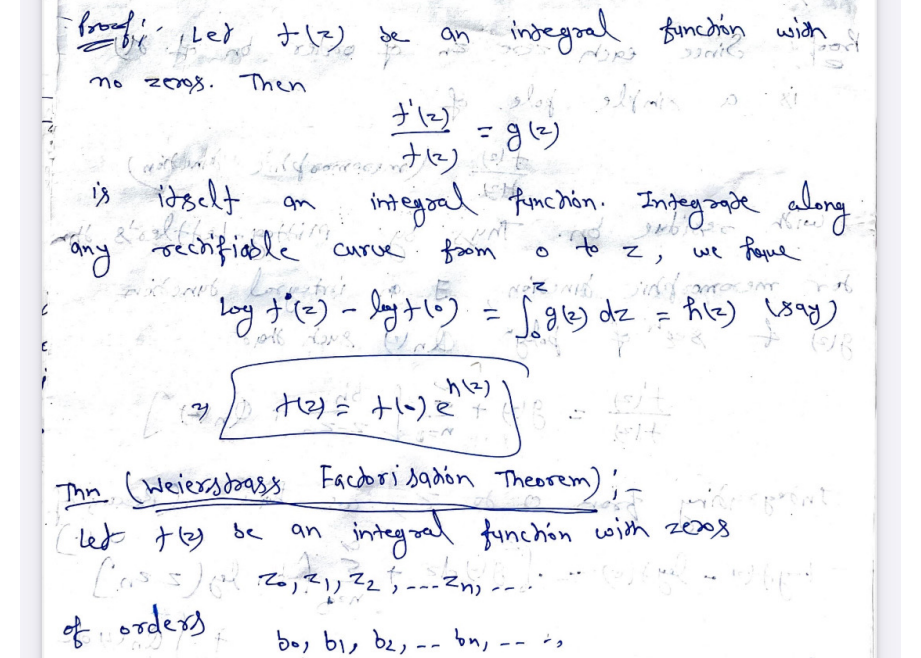
Notes Provided by sir:-



Absolutely Convergent: ~~[illegible]~~ An infinite product $\prod_{n=1}^{\infty}(1+a_n)$, $a_n \neq -1$, is said to be absolutely convergent if ~~[illegible]~~ $\prod_{n=1}^{\infty}(1+|a_n|)$ converges.

Thm: A necessary & sufficient condition for the absolute convergence ~~for the absolute~~ convergence of the product $\prod_{n=1}^{\infty}(1+a_n)$ is the convergence of the series $\sum_{n=1}^{\infty} |a_n|$.

~~[illegible]~~ - * Infinite product converges absolutely iff the corresponding series $\sum |\log(1+a_n)|$ converges.



Proof:- If either the series $\sum \log(1+a_n)$ or $\sum |a_n|$ converges, then $\lim_{n\to\infty} a_n \to 0$. ㉘

$$\lim_{n\to\infty} \frac{|\log(1+a_n)|}{|a_n|} \to 1$$

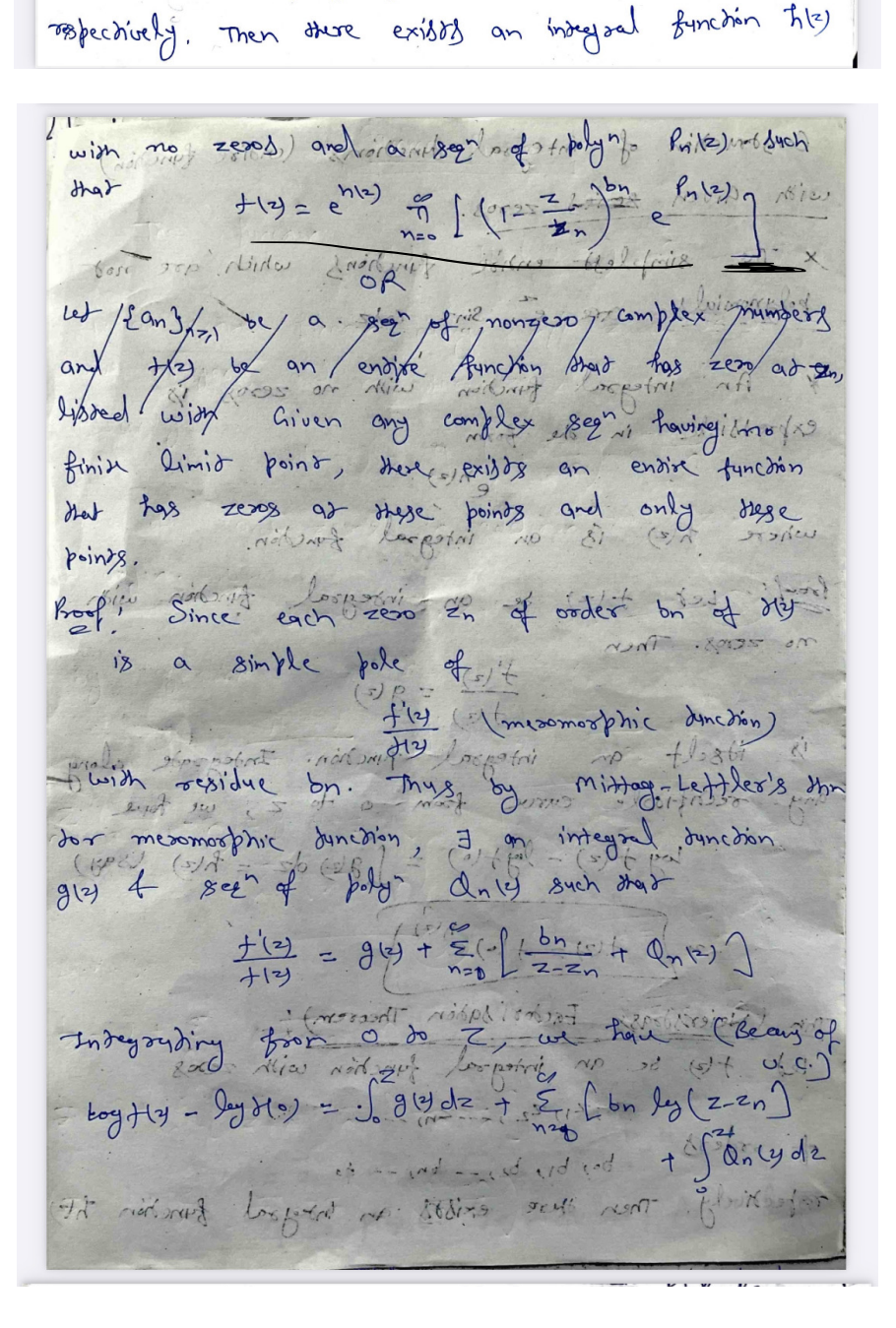
$$\left|\frac{\log(1+a_n)}{a_n}\right| \in (1-\epsilon, 1+\epsilon) \text{ for all } n > N$$

$$1-\epsilon < \frac{|\log(1+a_n)|}{|a_n|} < 1+\epsilon$$

$$(1-\epsilon)|a_n| < |\log(1+a_n)| < (1+\epsilon)|a_n|$$

Thus if $\sum |a_n|$ converges $\Leftrightarrow \sum_{n=1} |\log(1+a_n)|$ $\Leftrightarrow \prod_{n=1}^{\infty} |1+a_n|$ converges

Ques 1:- [illegible]



Construction of integral functions (entire function) with Given ~~[illegible]~~ zeros.

* The simplest entire functions which are not polynomials are e^z, $\sin z$, $\cos z$.

* An integral function with no zeros is expressible in the form $e^{h(z)}$ where $h(z)$ is an integral function.

Proof: Let $f(z)$ be an integral function with no zeros. Then

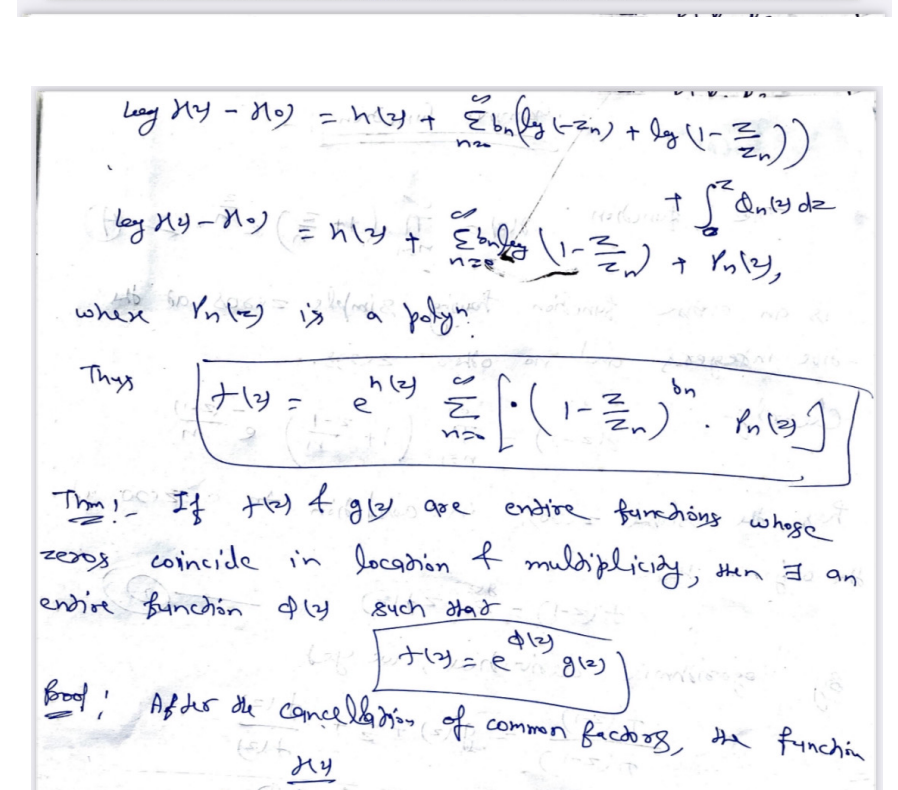
$$\frac{f'(z)}{f(z)} = g(z)$$

is itself an integral function. Integrate along any rectifiable curve from 0 to z, we have

$$\log f(z) - \log f(0) = \int_0^z g(z)\, dz = h(z) \text{ (say)}$$

$$\Rightarrow \boxed{f(z) = f(0) e^{h(z)}}$$

Thm (Weierstrass Factorisation Theorem): Let $f(z)$ be an integral function with zeros $z_0, z_1, z_2, \dots, z_n, \dots$ of orders $b_0, b_1, b_2, \dots, b_n, \dots$ respectively. Then there exists an integral function $h(z)$



with no zeros and a seq^n of polyn^n $P_n(z)$ such that

$$f(z) = e^{h(z)} \prod_{n=1}^{\infty} \left[\left(1-\frac{z}{z_n}\right)^{b_n} e^{P_n(z)}\right]$$

OR

Let $\{a_n\}$ be a seq^n of nonzero complex numbers and $f(z)$ be an entire function that has zero at a_n ~~[illegible]~~. Given any complex seq^n having no finite limit point, there exists an entire function that has zeros at these points and only these points.

Proof: Since each zero z_n of order b_n of $f(z)$ is a simple pole of $\frac{f'(z)}{f(z)}$ (meromorphic function) with residue b_n. Thus, by Mittag-Leffler's thm for meromorphic function, $\exists$ an integral function $g(z)$ & seq^n of poly^n $Q_n(z)$ such that

$$\frac{f'(z)}{f(z)} = g(z) + \sum_{n=1}^{\infty} \left[\frac{b_n}{z-z_n} + Q_n(z)\right]$$

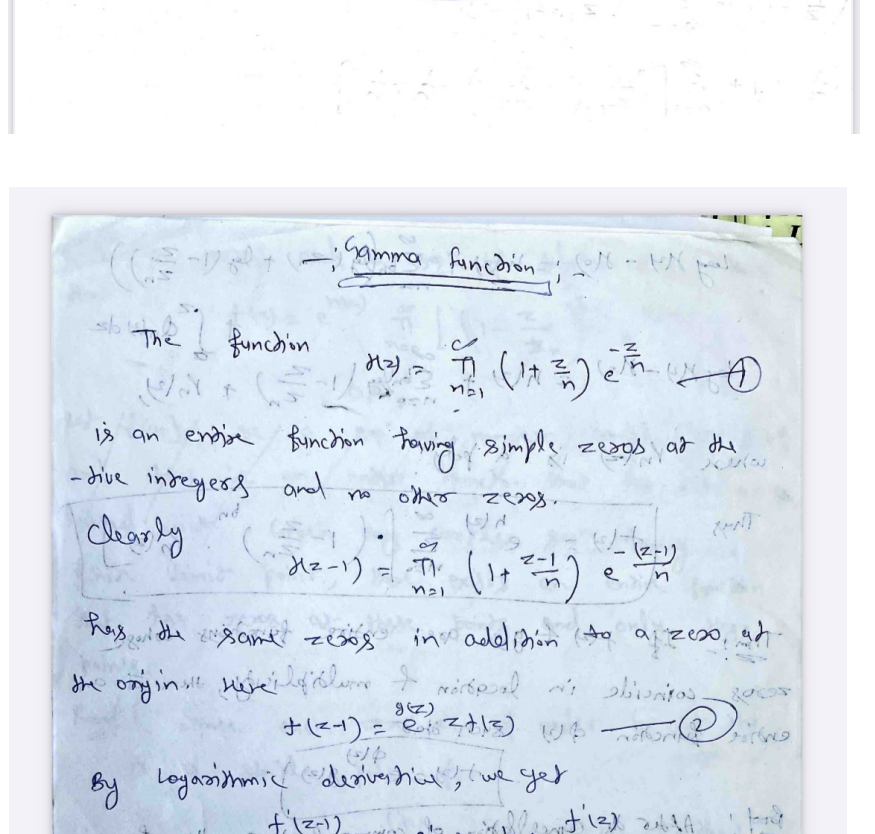
Integrating from 0 to z, we have [illegible]

$$\log f(z) - \log f(0) = \int_0^z g(z)\, dz + \sum_{n=1}^{\infty} \left[b_n \log(z-z_n)\right] + \int_0^z Q_n(z)\, dz$$

$e^{h(z)}$

$n=0$

$\left(1-\frac{z}{z_n}\right)$



$$\log f(z) - f(0) = h(z) + \sum_{n=1}^{\infty} b_n \left(\log(-z_n) + \log\left(1-\frac{z}{z_n}\right)\right) + \int_0^z Q_n(z)\, dz$$

$$\log f(z) - f(0) = h(z) + \sum_{n=1}^{\infty} b_n \log\left(1-\frac{z}{z_n}\right) + P_n(z),$$

where $P_n(z)$ is a polyn^n

Thus

$$\boxed{f(z) = e^{h(z)} \prod_{n=1}^{\infty} \left[\left(1-\frac{z}{z_n}\right)^{b_n} \cdot P_n(z)\right]}$$

Thm:- If $f(z)$ & $g(z)$ are entire functions whose zeros coincide in location & multiplicity, then $\exists$ an entire function $\phi(z)$ such that

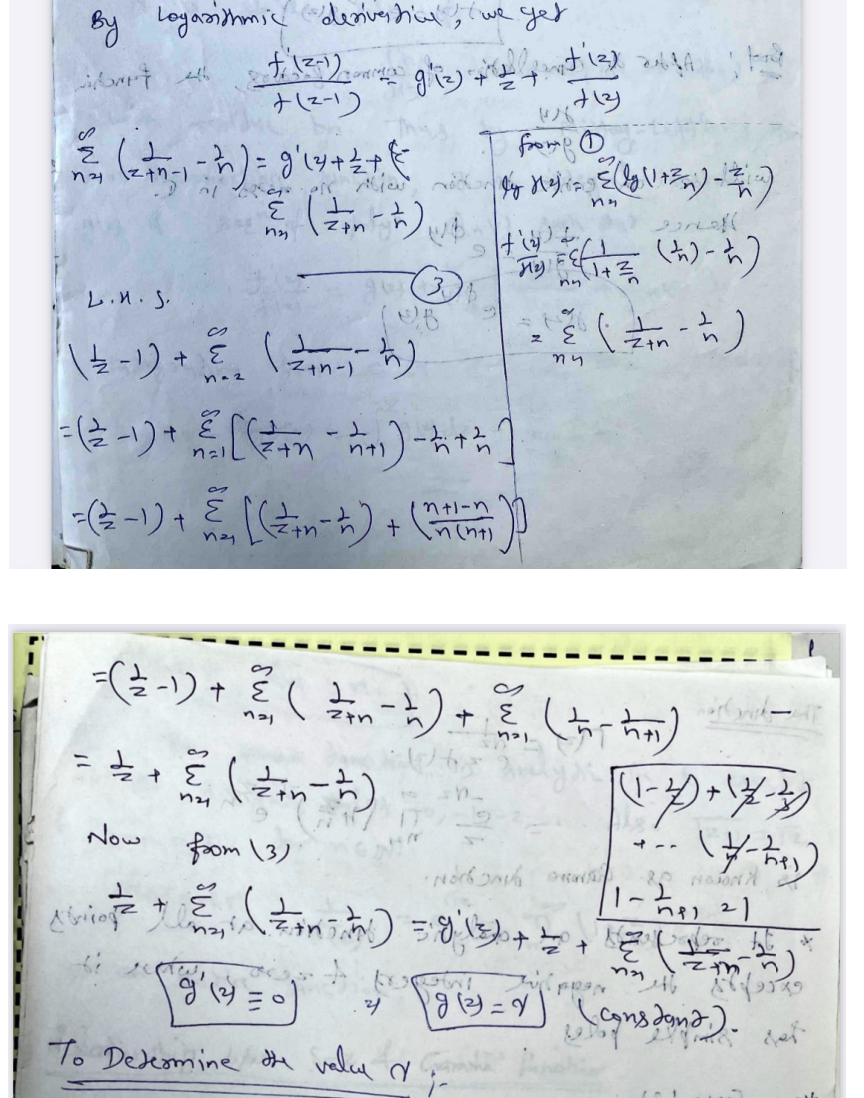
$$\boxed{f(z) = e^{\phi(z)} g(z)}$$

Proof: After the cancellation of common factors, the function $\frac{f(z)}{g(z)}$ will be an entire function with no zeros in $\mathbb{C}$.

Hence

$$\frac{f(z)}{g(z)} = e^{\phi(z)}$$

$$\boxed{f(z) = e^{\phi(z)} g(z)}$$



Gamma function

The function

$$f(z) = \prod_{n=1}^{\infty} \left(1+\frac{z}{n}\right) e^{-\frac{z}{n}} \text{ ——①}$$

is an entire function having simple zeros at the -tive integers and no other zeros.

Clearly

$$f(z-1) = \prod_{n=1}^{\infty} \left(1+\frac{z-1}{n}\right) e^{-\frac{(z-1)}{n}}$$

has the same zeros in addition to a zero at the origin. Hence

$$f(z-1) = e^{g(z)} z f(z) \text{ ——②}$$

By Logarithmic derivative, we get

$$\frac{f'(z-1)}{f(z-1)} = g'(z) + \frac{1}{z} + \frac{f'(z)}{f(z)}$$

$$\sum_{n=1}^{\infty} \left(\frac{1}{z+n-1} - \frac{1}{n}\right) = g'(z) + \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{z+n} - \frac{1}{n}\right) \text{ ——③}$$

From ①: $\log f(z) = \sum_{n=1}^{\infty} \left(\log\left(1+\frac{z}{n}\right) - \frac{z}{n}\right)$

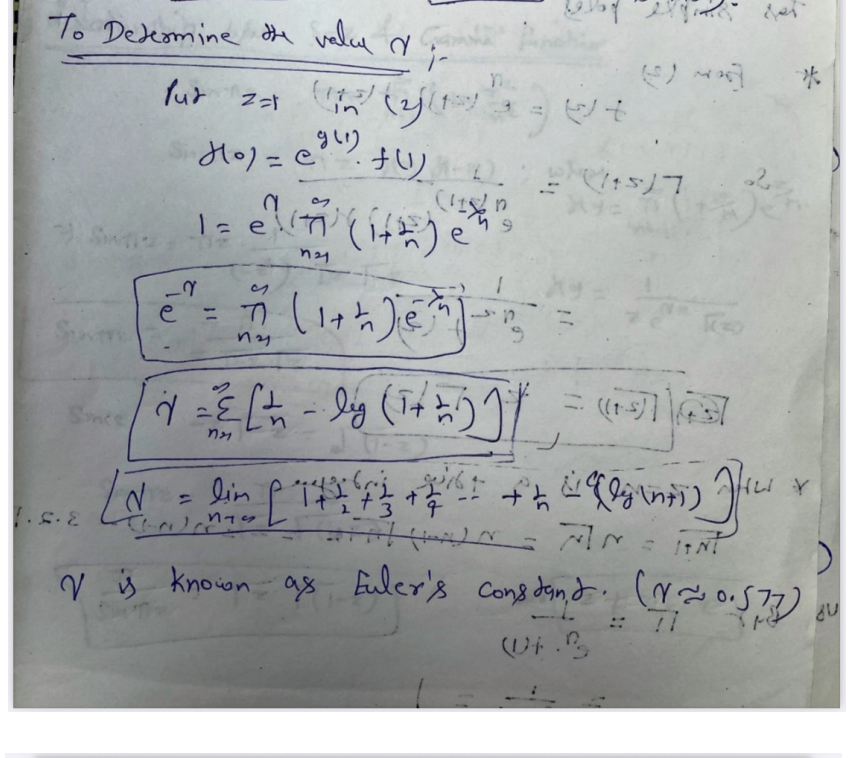
$\frac{f'(z)}{f(z)} = \sum_{n=1}^{\infty} \left(\frac{1}{1+\frac{z}{n}} \left(\frac{1}{n}\right) - \frac{1}{n}\right) = \sum_{n=1}^{\infty} \left(\frac{1}{z+n} - \frac{1}{n}\right)$

L.H.S.

$$\left(\frac{1}{z}-1\right) + \sum_{n=2}^{\infty} \left(\frac{1}{z+n-1} - \frac{1}{n}\right)$$

$$= \left(\frac{1}{z}-1\right) + \sum_{n=1}^{\infty} \left[\left(\frac{1}{z+n} - \frac{1}{n+1}\right) - \frac{1}{n} + \frac{1}{n}\right]$$

$$= \left(\frac{1}{z}-1\right) + \sum_{n=1}^{\infty} \left[\left(\frac{1}{z+n} - \frac{1}{n}\right) + \left(\frac{n+1-n}{n(n+1)}\right)\right]$$



$$= \left(\frac{1}{z}-1\right) + \sum_{n=1}^{\infty} \left(\frac{1}{z+n} - \frac{1}{n}\right) + \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{z+n} - \frac{1}{n}\right)$$

$\left(1-\frac{1}{2}\right) + \left(\frac{1}{2}-\frac{1}{3}\right) + \dots \left(\frac{1}{n}-\frac{1}{n+1}\right)$

$1 - \frac{1}{n+1} = 1$

Now from (3)

$$\frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{z+n} - \frac{1}{n}\right) = g'(z) + \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{z+n} - \frac{1}{n}\right)$$

$\boxed{g'(z) = 0}$ $\Rightarrow$ $\boxed{g(z) = \gamma}$ (constant)

To Determine the value γ:

Put $z=1$ in (2)

$$f(0) = e^{g(1)} f(1)$$

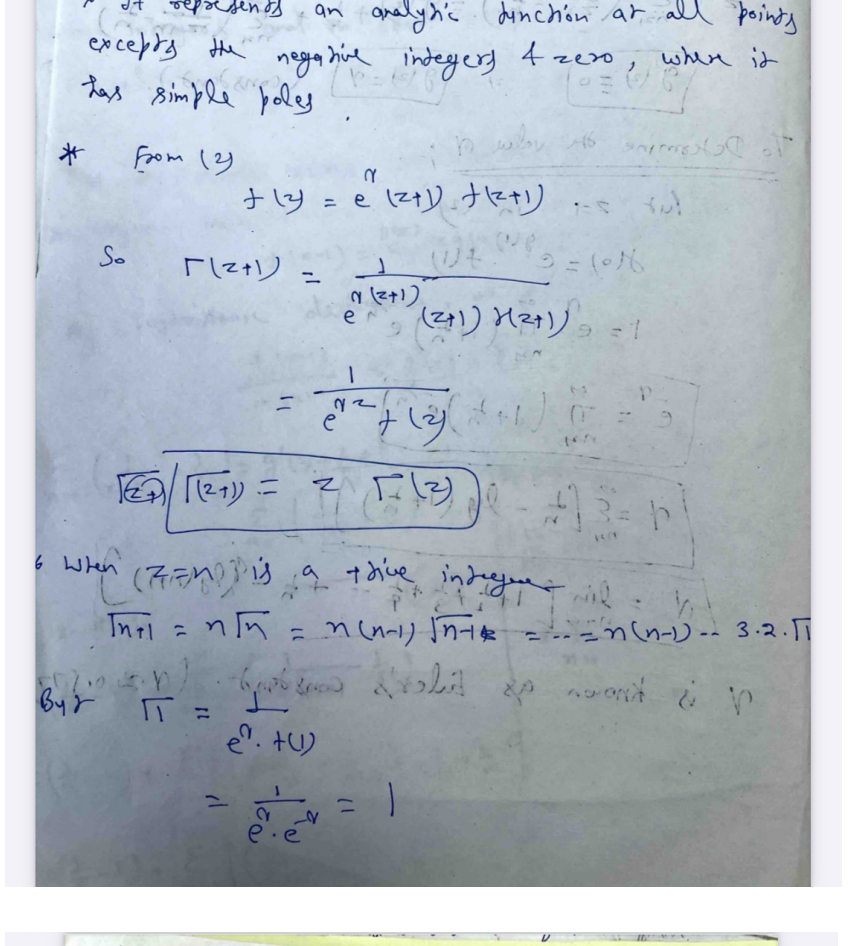
$$1 = e^{\gamma} \prod_{n=1}^{\infty} \left(1+\frac{1}{n}\right) e^{-\frac{1}{n}}$$

$$\boxed{e^{-\gamma} = \prod_{n=1}^{\infty} \left(1+\frac{1}{n}\right) e^{-\frac{1}{n}}}$$

$$\boxed{\gamma = \sum_{n=1}^{\infty} \left[\frac{1}{n} - \log\left(1+\frac{1}{n}\right)\right]}$$

$$\boxed{\gamma = \lim_{n\to\infty} \left[1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} - \log(n+1)\right]}$$

γ is known as Euler's constant. ($\gamma \approx 0.577$)



The function

$$\Gamma(z) = \frac{1}{e^{\gamma z} f(z)}$$

$$= \frac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty} \left(1+\frac{z}{n}\right)^{-1} e^{\frac{z}{n}}$$

is known as Gamma function.

* It represents an analytic function at all points excepts the negative integers & zero, when it has simple poles

* From (2)

$$f(z) = e^{\gamma} (z+1) f(z+1)$$

So

$$\Gamma(z+1) = \frac{1}{e^{\gamma(z+1)} (z+1) f(z+1)}$$

$$= \frac{1}{e^{\gamma z} f(z)}$$

$$\boxed{\Gamma(z+1) = z\,\Gamma(z)}$$

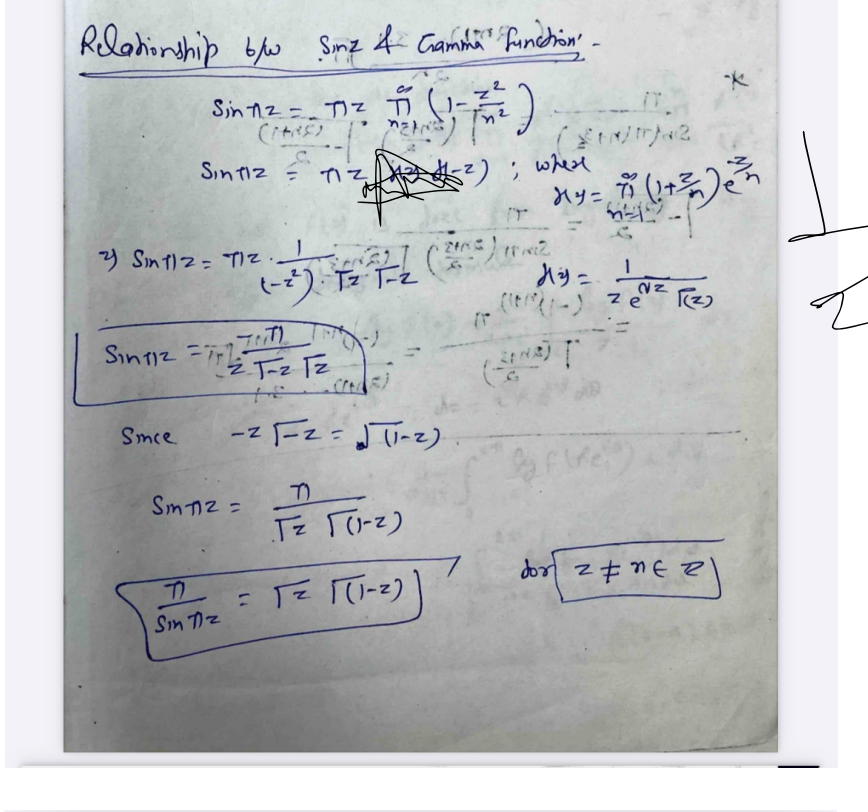
* When $(z=n)$ is a +tive integer

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1) = \dots = n(n-1) \dots 3 \cdot 2 \cdot \Gamma(1)$$

But

$$\Gamma(1) = \frac{1}{e^{\gamma} f(1)}$$

$$= \frac{1}{e^{\gamma} \cdot e^{-\gamma}} = 1$$



Hence $\boxed{\Gamma(n+1) = n!}$

Thm:- The gamma function is analytic in $\mathbb{C}$ except at the simple poles $0, -1, -2, \dots$. Also $\Gamma(z+1) = z\Gamma(z)$ & $\Gamma(n+1) = n!$ for $n \in \mathbb{N}$.

Moreover

$$\frac{1}{\Gamma(z)} = e^{\gamma z} z \prod_{n=1}^{\infty} \left(1+\frac{z}{n}\right) e^{-\frac{z}{n}}$$

is an entire function.

Relationship b/w $\sin z$ & Gamma function:-

$$\sin \pi z = \pi z \prod_{n=1}^{\infty} \left(1-\frac{z^2}{n^2}\right)$$

$$\sin \pi z = \pi z \, [\text{illegible}] \text{ ; where } f(z) = \prod_{n=1}^{\infty} \left(1+\frac{z}{n}\right) e^{-\frac{z}{n}}$$

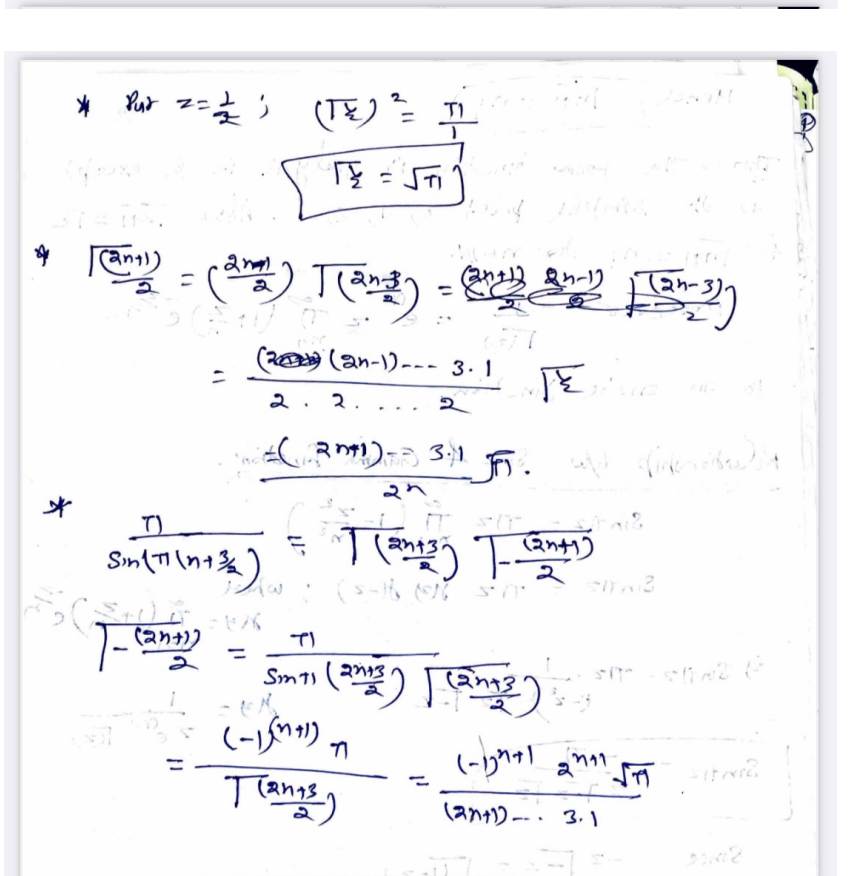
$$\Rightarrow \sin \pi z = \pi z \cdot \frac{1}{(-z^2)\, \Gamma(z)\, \Gamma(-z)} \qquad f(z) = \frac{1}{z e^{\gamma z} \Gamma(z)}$$

$$\boxed{\sin \pi z = \frac{-\pi}{z\, \Gamma(-z)\, \Gamma(z)}}$$

Since $-z\,\Gamma(-z) = \Gamma(1-z)$

$$\sin \pi z = \frac{\pi}{\Gamma(z)\, \Gamma(1-z)}$$

$$\boxed{\frac{\pi}{\sin \pi z} = \Gamma(z)\, \Gamma(1-z)} \quad \text{for } \boxed{z \neq n \in \mathbb{Z}}$$



* Put $z = \frac{1}{2}$; $(\Gamma(\frac{1}{2}))^2 = \frac{\pi}{1}$

$$\boxed{\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}}$$

* $\Gamma\left(\frac{2n+1}{2}\right) = \left(\frac{2n-1}{2}\right) \Gamma\left(\frac{2n-1}{2}\right) = \frac{(2n-1)}{2} \frac{(2n-3)}{2} \Gamma\left(\frac{2n-3}{2}\right)$

$$= \frac{(2n-1)(2n-3) \dots 3 \cdot 1}{2 \cdot 2 \dots 2} \Gamma\left(\tfrac{1}{2}\right)$$

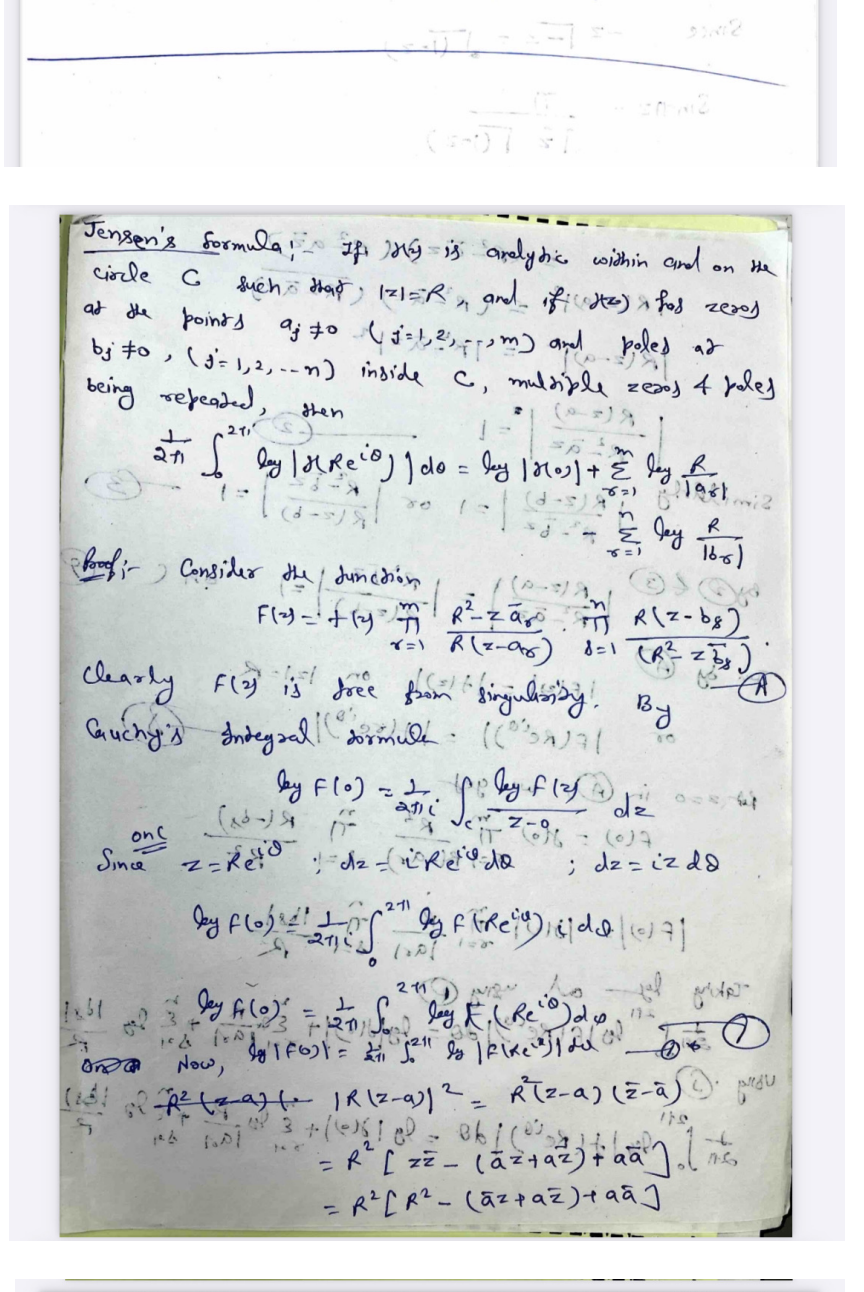
$$= \frac{(2n-1) \dots 3 \cdot 1}{2^n} \sqrt{\pi}.$$

*

$$\frac{\pi}{\sin(\pi(n+\frac{3}{2}))} = \Gamma\left(\frac{2n+3}{2}\right) \Gamma\left(-\frac{(2n+1)}{2}\right)$$

$$\Gamma\left(-\frac{(2n+1)}{2}\right) = \frac{\pi}{\sin \pi\left(\frac{2n+3}{2}\right) \Gamma\left(\frac{2n+3}{2}\right)}$$

$$= \frac{(-1)^{(n+1)} \pi}{\Gamma\left(\frac{2n+3}{2}\right)} = \frac{(-1)^{n+1}\, 2^{n+1} \sqrt{\pi}}{(2n+1) \dots 3 \cdot 1}$$



Jensen's formula: If $f(z)$ is analytic within and on the circle C such that $|z|=R$, and if $f(z)$ has zeros at the points $a_j \neq 0$ $(j = 1, 2, \dots, m)$ and poles at $b_j \neq 0$, $(j = 1, 2, \dots, n)$ inside C, multiple zeros & poles being repeated, then

$$\frac{1}{2\pi} \int_0^{2\pi} \log|f(Re^{i\theta})|\, d\theta = \log|f(0)| + \sum_{r=1}^{m} \log \frac{R}{|a_r|} - \sum_{r=1}^{n} \log \frac{R}{|b_r|}$$

Proof: Consider the function

$$F(z) = f(z) \prod_{r=1}^{m} \frac{R^2 - z\bar{a}_r}{R(z-a_r)} \prod_{s=1}^{n} \frac{R(z-b_s)}{(R^2 - z\bar{b}_s)} \text{ ——Ⓐ}$$

Clearly $F(z)$ is free from singularity. By Cauchy's Integral formula

$$\log F(0) = \frac{1}{2\pi i} \int_C \frac{\log F(z)}{z-0}\, dz$$

Since on C $z = Re^{i\theta}$; $dz = iRe^{i\theta} d\theta$; $dz = iz\, d\theta$

$$\log F(0) = \frac{1}{2\pi i} \int_0^{2\pi} \log F(Re^{i\theta})\, i\, d\theta$$

$$\log F(0) = \frac{1}{2\pi} \int_0^{2\pi} \log F(Re^{i\theta})\, d\theta$$

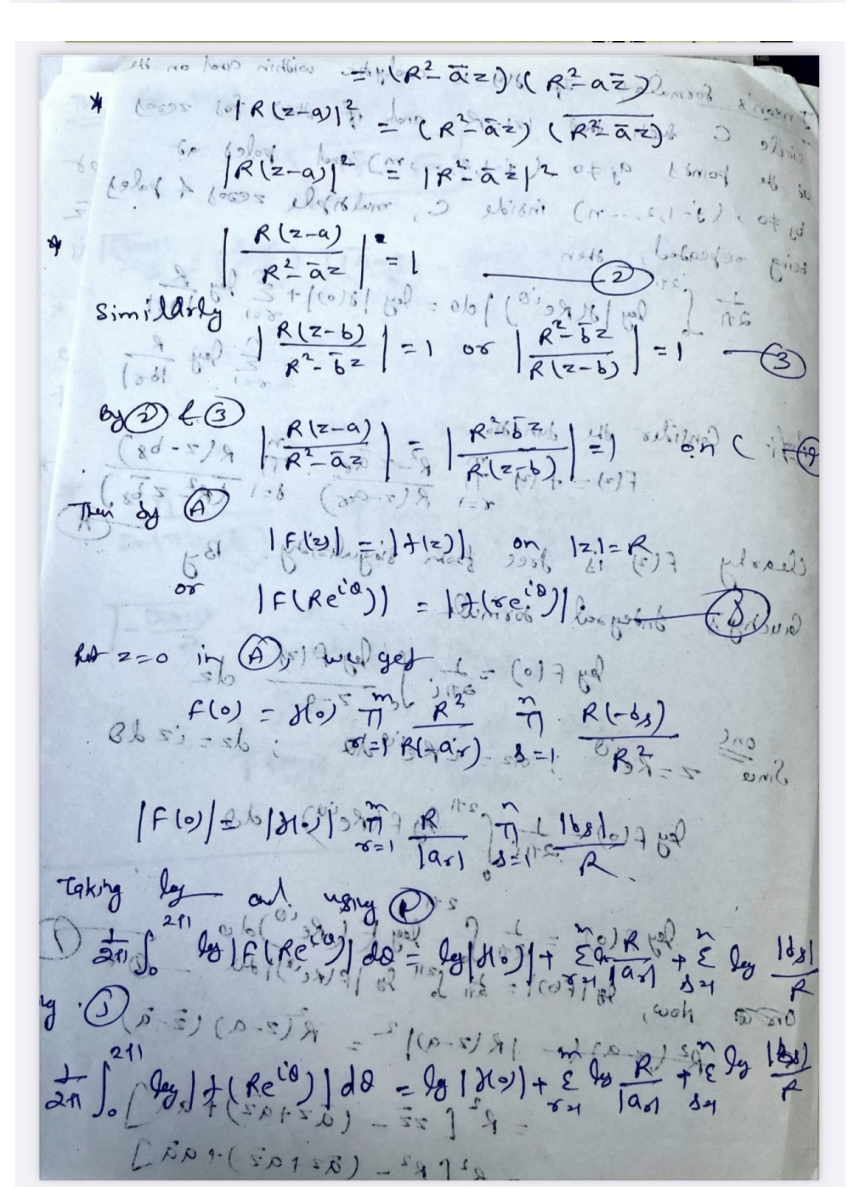
$$\log|F(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log|F(Re^{i\theta})|\, d\theta \text{ ——①}$$

Now,

$$|R(z-a)|^2 = R^2 (z-a)(\bar{z}-\bar{a})$$

$$= R^2 [z\bar{z} - (\bar{a}z + a\bar{z}) + a\bar{a}]$$

$$= R^2 [R^2 - (\bar{a}z + a\bar{z}) + a\bar{a}]$$



$$|R(z-a)|^2 = (R^2 - \bar{a}z)(R^2 - a\bar{z})$$

$$|R(z-a)|^2 = |R^2 - \bar{a}z|^2$$

*

$$\left|\frac{R(z-a)}{R^2 - \bar{a}z}\right| = 1 \text{ ——②}$$

Similarly

$$\left|\frac{R(z-b)}{R^2 - \bar{b}z}\right| = 1 \text{ or } \left|\frac{R^2 - \bar{b}z}{R(z-b)}\right| = 1 \text{ ——③}$$

By ② & ③

$$\left|\frac{R(z-a)}{R^2 - \bar{a}z}\right| = \left|\frac{R^2 - \bar{b}z}{R(z-b)}\right| = 1 \text{ on } C$$

Thus by Ⓐ

$$|F(z)| = |f(z)| \text{ on } |z| = R$$

or

$$|F(Re^{i\theta})| = |f(Re^{i\theta})| \text{ ——④}$$

Put $z=0$ in Ⓐ, we get

$$F(0) = f(0) \prod_{r=1}^{m} \frac{R^2}{R(-a_r)} \prod_{s=1}^{n} \frac{R(-b_s)}{R^2}$$

$$|F(0)| = |f(0)| \prod_{r=1}^{m} \frac{R}{|a_r|} \prod_{s=1}^{n} \frac{|b_s|}{R}$$

Taking log and using ④

$$\frac{1}{2\pi} \int_0^{2\pi} \log|F(Re^{i\theta})|\, d\theta = \log|f(0)| + \sum_{r=1}^{m} \log \frac{R}{|a_r|} + \sum_{s=1}^{n} \log \frac{|b_s|}{R}$$

By ①

$$\frac{1}{2\pi} \int_0^{2\pi} \log|f(Re^{i\theta})|\, d\theta = \log|f(0)| + \sum_{r=1}^{m} \log \frac{R}{|a_r|} + \sum_{s=1}^{n} \log \frac{|b_s|}{R}$$