~~Tuesday, 3 September 2024 9:37 AM~~
(10 sep 2024)

(IN CLASS)

UNIT-2

[Made by: OM KUMAR 2K22/MC/106]

Singularities (Revisited)

Let $f(z)$ has the following Laurent Series Expansion

$$f(z) = \underbrace{\sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}}_{\text{Principal part}} + \underbrace{\sum_{n=0}^{\infty} a_n (z-z_0)^n}_{\text{Analytic Part}}$$

valin in $r<|z-z_0|<R$

(i) If all b_n's are zero, that is $f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$

then z_0 is a removable singularity.

(ii) If $b_{k+1}=0 \ \forall \ k \geq n$, i.e,

$$f(z) = \frac{b_k}{(z-z_0)^k} + \frac{b_{k-1}}{(z-z_0)^{k-1}} + \cdots + \frac{b_1}{z-z_0} + \sum_{n=0}^{\infty} a_n (z-z_0)^n,$$

then z_0 will be a pole of order k.

(iii) If principal part contains infinite terms, then z_0 will be an essential singularity.

Meromorphic functions: A function is said to be meromorphic in $\mathbb{C}$ if at any point it is either regular or has isolated singularity (poles).

for ex: $\frac{\cos z}{\sin z}, \frac{1}{(z-1)(z-3)}, \tan z, \operatorname{cosec} z$

(i) The ratio of two entire functions is meromorphic
(ii) sum, product or Linear combination of meromorphic fun is meromorphic.
(iii) Entire functions are meromorphic but not conversily.

Partial fraction :-

* Let $\langle b_n \rangle$ be a seq^n tending to ∞ and function $f(z)$ have poles at b_n. For each n, associate a rational function (ratio of polynomials) of the form

$$P_n\left(\frac{1}{z-b_n}\right) = \frac{a_1^{(n)}}{(z-b_n)} + \frac{a_2^{(n)}}{(z-b_n)^2} + \cdots + \frac{a_{k_n}^{(n)}}{(z-b_n)^{k_n}} \longrightarrow (A)$$

where a_i's are complex constants & k_n denotes the order of pole at $z = b_n$.

Def^n: Let $f(z)$ be a meromorphic function defined on a domain D in $\mathbb{C}$. A partial fraction decomposition of $f(z)$ is a representation of the term

$$f(z) = g(z) + \sum_{n=1}^{\infty} P_n\left(\frac{1}{z-b_n}\right),$$

where $g(z)$ is analytic in D.

Th^m: (Mittag Laffler's Th^m): Let $\langle b_n \rangle$ be a seq^n tending to ∞, and $P_n\left(\frac{1}{z-b_n}\right)$ be a poly^n in $\left(\frac{1}{z-b_n}\right)$ of the form (A). Then there exist a meromorphic Function $f(z)$ that has poles at the points b_n with principal part $P_n\left(\frac{1}{z-b_n}\right)$, and is analytic otherwise.

Proof:- Since $\langle b_n \rangle \longrightarrow \infty$, so, we can assume that

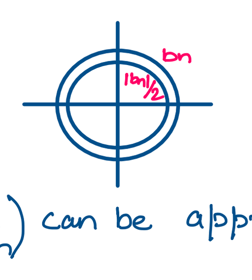
$$0 < |b_1| \leq |b_2| \leq |b_3| \leq \cdots$$

Now,
$P_n\left(\frac{1}{z-b_n}\right)$ will be analytic in $|z|<|b_n|$, Hence we can expand $P_n\left(\frac{1}{z-b_n}\right)$ in Maclaurin's series form

$$P_n\left(\frac{1}{z-b_n}\right) = \underbrace{\sum_{k=0}^{\infty} a_k^{(n)} z^k}_{|z|<b_n} \quad \text{—— (i)}$$

Note: $\boxed{\sum_{n-}^{\infty} a_n z^n \longrightarrow f(z) \quad |z|<R \quad \underbrace{|z| \leq r < R}_{\downarrow \text{ Uniformly Co.}}}$

This series uniformly converges in the domain $|z| \leq \frac{|b_n|}{2}$ (say)



Thus $P_n\left(\frac{1}{z-b_n}\right)$ can be approximated in $|z| \leq \frac{|b_n|}{2}$ by a polynomial or partial sum

$$Q_n(z) = \sum_{k=0}^{n_k} a_k^{(n)} z^k$$
$$= a_0^{(n)} + a_1^{(n)} z + a_2^{(n)} z^2 + \cdots + a_{n_k}^{(n)} z^k$$

We can write that

$$\left| P_n\left(\frac{1}{z-b_n}\right) - Q_n(z) \right| < \frac{1}{2^n} \text{ for sufficiently large } n.$$

Now we have to show that

$$f(z) = \sum_{n=1}^{\infty} \left(P_n\left(\frac{1}{z-b_n}\right) - Q_n(z) \right) \quad \text{—— (i)}$$

is meromorphic function. To prove this is sufficed to show that the series in (i) converges uniformly on an arbitrary compact subset $|z| \leq R$ of $\mathbb{C}$ that excludes the points $|b_n| \leq R$. Choosing N such that $|b_n| \geq 2R$, from (i)

$$\sum_{n=N}^{\infty} \left| P_n\left(\frac{1}{z-b_n}\right) - Q_n(z) \right| < \sum \frac{1}{2^n} < \infty$$

By Weierstrass M test the series

$$\sum_{n=N}^{\infty} P_n\left(\frac{1}{z-b_n}\right) - Q_n(z) \rightrightarrows g(z), \text{ which is}$$

analytic in $|z| \leq R$.

Weierstrass M test for U.C (Uniformly Convergent)
If $\sum f_n(z)$ is a series of f^n. and $\exists$ $\langle a_n \rangle$ of positive real nos. such that
(i) $|f_n(z)| \leq a_n$
(ii) $\sum a_n$ is converges then
$\boxed{\sum f_n(z) \rightrightarrows f(z)}$

for $|z| \leq R$,

$$\sum_{n=0}^{N-1} \left[P_n\left(\frac{1}{z-b_n}\right) - Q_n(z) \right] \text{ is an analytic function with no singularity}$$

except the prescribed poles. Since R is arbitrary, $f(z)$ is meromorphic in $\mathbb{C}$.

Date :-(12/09/2024)

Infinite Product:- An infinite product of complex numbers is an expression of the form

$$p_1 p_2 \cdots p_n \cdots = \prod_{n=1}^{\infty} p_n$$

* It is evaluated by taking the limit of partial product

$$P_n = p_1 p_2 \cdots p_n$$

Convergence & Divergence:

* An infinite product $\prod_{n=1}^{\infty} p_n$ is said to be converges iff there is an N such that $p_n \neq 0$ for all $k \geq N$ & $\lim_{k\to\infty} \prod_{k=N}^{\infty} p_k$ exist & is non-zero.

* An infinite product that does not converge is said to be divergent.

* If convergent condition holds but finitely many p_k's are zero, then the infinite product $\prod_{n=1}^{\infty} p_n$ converges to zero. In this case the value of the product is set as 0.

* If $p_n \neq 0$ for all $n \geq 1$, and
$$\lim_{n\to\infty} \prod_{k=1}^{n} p_k = 0,$$
then we say that the infinite product diverges to zero.

* The partial product
$$P_n = p_1 p_2 \cdots p_n.$$
Then $P_{n+1} = p_1 p_2 \cdots p_n p_{n+1}$
If $\lim_{n\to\infty} P_n$ exist such that $\lim_{n\to\infty} P_n = P$. Then
$$p_{n+1} = \frac{P_{n+1}}{P_n}$$
$$\lim_{n\to\infty} p_{n+1} = \frac{\lim_{n\to\infty} P_{n+1}}{\lim_{n\to\infty} P_n} = \frac{P}{P}$$
$$\lim_{n\to\infty} p_{n+1} = 1$$

Thm:- (Necessary Condition) If the Infinite product $\prod_{n=1}^{\infty}(1+a_n)$ converges, then $\lim_{n\to\infty} a_n \to 0$, & $a_n = -1$ for at most finitely many n.

Ex:- $\prod_{k=1}^{\infty}\left(1+\frac{1}{k}\right)$

$$P_n = 2/1 \cdot 3/2 \cdot 4/3 \cdots \frac{n+1}{n} = n+1$$
$$\lim_{n\to\infty} P_n \to \infty$$

Ex:- $\prod_{n=1}^{\infty}(1-1/n)$

$$P_n = \frac{1}{2} \cdot \frac{-2}{3} \cdot \frac{3}{4} \cdot \frac{4}{5} \cdots \frac{n}{n+1}$$
$$P_n = \frac{1}{n+1}$$
$$\lim_{n\to\infty} P_n \to 0$$

The infinite product $\prod_{n=2}^{\infty}\left(1-\frac{1}{n}\right)$ diverges to zero.

(N & S cond^n)
Thm:- For a_n complex, $a_n \neq -1$, the product $\prod_{n=1}^{\infty}(1+a_n)$ converges iff $\sum_{n=1}^{\infty} \log(1+a_n)$ converges.

Proof:- Let
$$P_n = \prod_{k=1}^{n}(1+a_k)$$
$$S_n = \sum_{k=1}^{n} \operatorname{Log}(1+a_k)$$
$$P_n = e^{S_n} = e^{\log(1+a_1) + \log(1+a_2) + \cdots + \log(1+a_n)}$$
$$= e^{\log(1+a_1)} \cdot e^{\log(1+a_2)} \cdots e^{\log(1+a_n)}$$

Let us suppose that $\sum_{n=1}^{\infty} \log(1+a_n)$ converges.

Therefore,
$$\lim_{n\to s} S_n = S \text{ exist}$$

Thus
$$\lim_{n\to\infty} P_n = \lim_{n\to\infty} e^{S_n}$$
$$= e^{\lim_{n\to\infty} S_n}$$
$$= e^S \neq 0$$

Therefore, $\prod_{n=1}^{\infty} P_n$ converges.

Conversly, let $\lim_{n\to\infty} P_n = P$

$$\lim_{n\to\infty} \frac{P_n}{P} = 1$$
$$\lim_{n\to\infty} \operatorname{Log}\left(\frac{P_n}{P}\right) \to 0$$
$$\operatorname{Log}\left(\frac{P_n}{P}\right) = \log P_n - \log P + i\, 2\pi h_n, \quad [\log z = \log|z| + i \arg z]$$

where h_n is any integer

How h_n comes?
$\boxed{P_n = e^{i\theta_n}}$
$\boxed{P = e^{i\theta}}$
$\arg\left(\frac{P_n}{P}\right) = \theta_n - \theta$
$\lim_{n\to\infty} \theta_n \to \theta$
$\theta_n = \theta + 2\pi \cdot h_n$
$\boxed{\theta_n - \theta = 2\pi h_n}$

$$(h_{n+1} - h_n) 2\pi i = \left(\log \frac{P_{n+1}}{P} - \log P_n + \cancel{\log P}\right) - \left(\log \frac{P_n}{P} - \log P_n + \cancel{\log P}\right)$$

$$(h_{n+1} - h_n) 2\pi i = \log\left(\frac{P_{n+1}}{P}\right) - \log\left(\frac{P_n}{P}\right) - \log P_{n+1} + \log P_n$$

$$(h_{n+1} - h_n) 2\pi = \operatorname{Arg}\left(\frac{P_{n+1}}{P}\right) - \operatorname{Arg}\left(\frac{P_n}{P}\right) - \operatorname{Arg}(1+a_{n+1})$$

$$|(h_{n+1} - h_n)| 2\pi < \left|\operatorname{Arg}\left(\frac{P_{n+1}}{P}\right)\right| + \left|\operatorname{Arg} \frac{P_n}{P}\right| + \left|\operatorname{Arg}(1+a_{n+1})\right|$$

as $n\to\infty$; $\arg\left(\frac{P_{n+1}}{P}\right)$ & $\arg\left(\frac{P_n}{P}\right) > 0$ as $\frac{P_n}{P} \to 1$

& $-\pi < \operatorname{Arg}(1+a_{n+1}) \leq \pi$.

Thus,
$$|h_{n+1} - h_n| 2\pi < \pi$$
$$|h_{n+1} - h_n| < 1/2$$

This implies that $h_{n+1} = h_n$.

Hence h_n is equal to a fixed integer h and it follows from

$$\operatorname{Log}\left(\frac{P_n}{P}\right) = \underbrace{S_n}_{= \log(P_n)} - \operatorname{Log} P + h \cdot 2\pi i$$

that
$$\boxed{S_n \to \operatorname{Log} P + h \cdot 2\pi i \quad \text{as } n\to\infty}$$