Submitted by:
Affan Arif
2K22/MC/08

MC319 Assignment-I

① Weierstrass's theorem for the sequence of functions states that if $\langle f_n(z)\rangle$ be a sequence of functions that are analytic in a domain D and it converges uniformly to $f(z)$, then the limit function is also analytic in D and the sequence of derivatives $\langle f_n'(z)\rangle$ converge uniformly to $f'(z)$.

Proof:

Since, $f_n(z)$ is analytic in D $\forall\, n \in N$,

Hence, By Morera's theorem,

$$\int_\gamma f_n(z)\,dz = 0$$, where γ is any closed curve in D.

Now, $$\lim_{n\to\infty} \int_\gamma f_n(z)\,dz = 0$$

$$\Rightarrow \int_\gamma \lim_{n\to\infty} f_n(z)\,dz = 0 \quad \text{[Due to uniform Convergence]}$$

$$\Rightarrow \int_\gamma f(z)\,dz = 0$$

Hence, again by Morera's theorem,
$f(z)$ is analytic in D.

Now,

$$|f_n'(z) - f'(z)| = \left| \frac{1}{2\pi i}\int_\gamma \frac{f_n(\zeta)}{(\zeta - z)^2}\,d\zeta - \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta-z)^2}\,d\zeta \right|$$

$$= \frac{1}{2\pi}\left| \int_\gamma \frac{f_n(\zeta) - f(\zeta)}{(\zeta - z)^2}\,d\zeta \right|$$

$$= \frac{1}{2\pi} \int \frac{|f_n(\zeta) - f(\zeta)|}{|\zeta - z|^2} |d\zeta|$$

As $f_n(z)$ uniformly converges to $f(z)$, So $|f_n(z) - f(z)|$ can be arbitrarily small

and hence
$|f_n'(z) - f'(z)| \longrightarrow 0$ for sufficiently large n.
Therefore, $f_n'(z)$ converges uniformly to $f'(z)$.

② $f(z) = e^z + e^{1/z^2}, \quad |z| > 0$
The Laurent series expansion is given as,

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} + \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{1}{z^2}\right)^n$$

$$f(z) = 2 + z + \frac{1}{z^2} + \frac{z^2}{2} + \frac{1}{2z^4} + \cdots$$

③ $f(z) = \dfrac{1}{(z+1)(z^2+2)}$

$$f(z) = \frac{1}{(z+1)(z-\sqrt{2}i)(z+\sqrt{2}i)} = \frac{1}{3(z+1)} - \frac{(\sqrt{2}+i)}{6\sqrt{2}(z-\sqrt{2}i)} - \frac{(\sqrt{2}-i)}{(z+\sqrt{2}i)}$$

a) $1 < |z| < \sqrt{2} \Rightarrow |z| < \sqrt{2}$ and $|z| > 1 \Rightarrow \left|\frac{1}{z}\right| < 1$
$\hookrightarrow \left|\frac{z}{\sqrt{2}}\right| < 1$

So, $f(z) = \dfrac{1}{3z}\left[\dfrac{1}{1-\left(-\frac{1}{z}\right)}\right] - \dfrac{(\sqrt{2}+i)}{6\sqrt{2}}\left[\dfrac{1}{\left(1-\frac{z}{\sqrt{2}i}\right)}\right]\dfrac{1}{-\sqrt{2}i}$

$$-\frac{(\sqrt{2}-i)}{6\sqrt{2}}\left[\frac{1}{1-\left(\frac{-z}{\sqrt{2}i}\right)}\right]\frac{1}{\sqrt{2}i}$$

$$= \frac{1}{3z}\sum_{n=0}^{\infty}\left(\frac{-1}{z}\right)^n + \frac{(1-\sqrt{2}i)}{12}\sum_{n=0}^{\infty}\left(\frac{z}{\sqrt{2}i}\right)^n$$

$$- \frac{(1+\sqrt{2}i)}{12}\sum_{n=0}^{\infty}\left(\frac{-z}{\sqrt{2}i}\right)^n$$

(b) $|z| > \sqrt{2} \Rightarrow 1 > \left|\frac{\sqrt{2}}{z}\right|$ and $1 > \left|\frac{1}{z}\right|$

$$f(z) = \frac{1}{32}\left[\frac{1}{1-\left(\frac{-1}{z}\right)}\right] - \frac{(\sqrt{2}+i)}{6\sqrt{2}}\left[\frac{1}{z\left(1-\frac{\sqrt{2}i}{z}\right)}\right] - \frac{(\sqrt{2}-i)}{6\sqrt{2}}\left[\frac{1}{z\left(1-\left(\frac{-\sqrt{2}i}{z}\right)\right)}\right]$$

$$= \frac{1}{32}\sum_{n=0}^{\infty}\left(\frac{-1}{z}\right)^n - \frac{(\sqrt{2}+i)}{6\sqrt{2}\,z}\sum_{n=0}^{\infty}\left(\frac{\sqrt{2}i}{z}\right)^n - \frac{(\sqrt{2}-i)}{6\sqrt{2}\,z}\sum_{n=0}^{\infty}\left(\frac{-\sqrt{2}i}{z}\right)^n$$

4. $u(x,y) = x^2 - y^2$

$u_x = 2x$, $u_{xx} = 2$ So, $u_{xx} + u_{yy} = 0$

$u_y = -2y$, $u_{yy} = -2$

Hence, there exists an analytic function $f(z)$ whose real part is u and its imaginary part, the harmonic conjugate, say v.

So, $f(z) = u + iv$

Using Cauchy-Riemann equations,

$u_x = v_y$ and $u_y = -v_x$

So, $v_y = 2x$ ~~$\Rightarrow \int v_y\,dy = \int 2x\,dy$~~

and $v_x = 2y$

Using total differential,

$$dv = v_x\,dx + v_y\,dy$$

$$dv = 2(y\,dx + x\,dy) = 2\,d(xy)$$

So, $v = 2xy + C$, where $C \in \mathbb{R}$

So, $f(z) = (x^2-y^2) + i(2xy+c)$
$= x^2 + (iy)^2 + 2x(iy) + ic$
$= (x+iy)^2 + ic$
$\Rightarrow f(z) = z^2 + ic$, where $c \in \mathbb{R}$

5. Let u and v be two harmonic functions in a domain D.

Now, let $w = u + v$

As, u & v are harmonic, So I^{st} and II^{nd} order partial derivatives exists for u & v and hence for w as well.

Also, u and v satisfy the laplace equation,

So, $u_{xx} + u_{yy} = 0$ —(1)
$v_{xx} + v_{yy} = 0$ —(2)

$(1)+(2) \Rightarrow (u_{xx} + v_{xx}) + (u_{yy} + v_{yy}) = 0$

$$\Rightarrow \frac{\partial^2}{\partial x^2}(u+v) + \frac{\partial^2}{\partial y^2}(u+v) = 0$$

$$\Rightarrow \frac{\partial^2}{\partial x^2}(w) + \frac{\partial^2}{\partial y^2}(w) = 0 \Rightarrow w_{xx} + w_{yy} = 0$$

Hence, w satisfies the laplace equation.

Hence, proved that w is a harmonic function in D as well.

⑥ The mean-value theorem for harmonic functions states that for a harmonic function $u(z)$ in a domain containing the disk $|z-z_0| \leq R$,

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + Re^{i\theta})\, d\theta$$

Proof:

Let $f(z)$ be an analytic function in $|z-z_0| \leq R$ whose real part is u.

By Mean-value theorem for analytic functions,

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + Re^{i\theta})\, d\theta$$

$$\Rightarrow \quad u(z_0) + i\,v(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0+Re^{i\theta})\, d\theta + \frac{i}{2\pi}\int_0^{2\pi} v(z_0+Re^{i\theta})\, d\theta$$

Comparing the real parts, we get

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + Re^{i\theta})\, d\theta$$

⑦ The maximum principle for harmonic functions states that a non-constant harmonic function cannot attain its maximum or minimum value in a domain.

Proof:

Let u be a non-constant harmonic function in a domain D.

Now, there exists an analytic function whose real part is u, say $g(z)$.

Now,

let $f(z) = e^{g(z)}$ be an ~~any~~ analytic function.

So, $|f(z)| = |e^{g(z)}| = |e^{u}|$

Without loss of generality, let $z_0 \in D$ be the maxima of u.

Then, at z_0, $|f(z)|$ is also maximum.

Hence, $|f(z)| \leq |f(z_0)|$ (say k)

Hence, $f(z)$ is a bounded analytic function.

So, $f(z)$ must be a constant function.

Hence, u must also be constant but that contradicts our hypothesis.

Hence, proved.

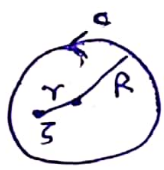
(8) Poisson integral formula for harmonic function states that for a harmonic function u in a domain D, containing the disk $|z| \leq R$,

for $\zeta = re^{i\theta}$, $(r < R)$

$$u(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{(R^2 - r^2)}{R^2 - 2rR\cos(\theta - \phi) + r^2} \right) u(Re^{i\phi})\, d\phi$$

Proof:

Let $f(z)$ be an analytic function in domain D containing $|z| \leq R$ & $\zeta = re^{i\theta}$ be any point inside it.



Now, By Cauchy integral formula,

$$f(\zeta) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - \zeta}\, dz \quad -(1)$$

Now, inverse point of ζ is $\frac{R^2}{\bar{\zeta}}$, which lies outside the circle, so

$$\frac{1}{2\pi i}\int_C \frac{f(z)}{\left(z-\frac{R^2}{\bar{\zeta}}\right)}dz = 0 \quad -(2)$$

Now, (2) − (1) $\Rightarrow f(\zeta) = \frac{1}{2\pi i}\int_C \frac{(R^2-\zeta\bar{\zeta})f(z)}{(z-\zeta)(R^2-z\bar{\zeta})}dz$

Putting, $\zeta = re^{i\theta}$ and $z = Re^{i\phi}$
$dz = iRe^{i\phi}d\phi$

$$\text{So, } f(re^{i\theta}) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{(R^2-r^2)f(Re^{i\phi})\,iRe^{i\phi}}{(Re^{i\phi}-re^{i\theta})(R^2-R.r.e^{i(\phi-\theta)})}d\phi$$

$$\Rightarrow f(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2-r^2)}{R^2-2rR\cos(\theta-\phi)+r^2} f(Re^{i\phi})d\phi$$

Now, comparing the real parts on both sides,

$$u(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2-r^2)}{R^2-2rR\cos(\theta-\phi)+r^2} u(Re^{i\phi})d\phi$$

where u is a harmonic function.

(9) $u(x,y)$ is a harmonic function in a domain D.
So, u_x, u_y, u_{xx} & u_{yy} exist &

$$u_{xx} + u_{yy} = 0$$

Let $g(z) = u_x - iu_y = U + iV$

comparing real and imaginary parts and differentiating w.r.t x & y

$$\frac{\partial U}{\partial x} = u_{xx} \;-(1), \quad \frac{\partial U}{\partial y} = u_{xy} \;-(2), \quad \frac{\partial V}{\partial x} = -u_{xy} \;-(3), \quad \frac{\partial V}{\partial y} = -u_{yy} \;-(4)$$

Now, ① − ④ $\Rightarrow U_x - V_y = u_{xx} + u_{yy} = 0$

$\Rightarrow U_x = V_y$

and ② + ③ $\Rightarrow U_y + V_x = u_{xy} - u_{xy} = 0$

$\Rightarrow U_y = -V_x$

Hence, the function $g(z)$ satisfies the Cauchy-Riemann equation in domain D.

Hence, $g(z)$ is an analytic function in D.

Now, for ~~any~~ some $z_0 \in D$,

$f(z) = \int_{z_0}^{z} g(t)\,dt$ is also analytic

$\Rightarrow f'(z) = g(t)$

Comparing real parts, we get

$\frac{\partial}{\partial x}\left[Re(f(z))\right] = u_x$

$\Rightarrow Re(f(z)) = u + c$

$\Rightarrow Re(\underbrace{f(z) - c}_{F(z)}) = u \Rightarrow Re(F(z)) = u$

Hence, there exists a function $F(z)$, whose real part is the harmonic function $u(x,y)$.

(10) Schwarz's theorem states that if f be a constant real function defined on the unit circle $|z|=1$, then the real valued function,

$$u(\zeta) = \frac{1}{2\pi}\int_0^{2\pi} P(\zeta, e^{i\phi})\, f(e^{i\phi})\, d\phi$$

is harmonic in the disk $|z|<1$.

where $P(\zeta, z) = \cancel{Re\left(\frac{\zeta+z}{\zeta-z}\right)}\ Re\left(\frac{z+\zeta}{z-\zeta}\right)$

is the Poisson kernel.

(for disk $|z|<1$ here)

Proof:

$$u(\zeta) = Re\left[\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{e^{i\phi}+\zeta}{e^{i\phi}-\zeta}\right) f(e^{i\phi})\, d\phi\right]$$

let $z = e^{i\phi} \Rightarrow dz = ie^{i\phi}d\phi$

$\Rightarrow d\phi = \frac{dz}{iz}$

So, $$u(\zeta) = Re\left[\frac{1}{2\pi}\int_{|z|=1}\left(\frac{z+\zeta}{z-\zeta}\right) f(z)\,\frac{dz}{iz}\right]$$

$$= Re\left[\frac{1}{2\pi i}\left(\int_{|z|=1}\frac{2f(z)}{z-\zeta}\,dz - \int_{|z|=1}\frac{f(z)}{z-0}\,dz\right)\right]$$

$$= Re\left[\frac{1}{2\pi i}\left(4\pi i f(\zeta) - 2\pi i f(0)\right)\right]$$

$$\Rightarrow u(\zeta) = Re[\underbrace{2f(\zeta) - f(0)}_{F(\zeta)}] \Rightarrow u(\zeta) = Re(F(\zeta))$$

As $f(\zeta)$ is analytic, so $F(\zeta)$ is also analytic and $u(\zeta)$ is the real part of $F(\zeta)$,

Hence $u(\zeta)$ is a harmonic function. Proved