of the Prime Number Theorem. Space does not permit us to include this application, but the basic importance of Hadamard's factorization theorem will be quite evident.

3.1. Jensen's Formula. If $f(z)$ is an analytic function, then $\log |f(z)|$ is harmonic except at the zeros of $f(z)$. Therefore, if $f(z)$ is analytic and free from zeros in $|z| \leqq \rho$,

$$\log |f(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(\rho e^{i\theta})| \, d\theta, \tag{43}$$

and $\log |f(z)|$ can be expressed by Poisson's formula.

The equation (43) remains valid if $f(z)$ has zeros on the circle $|z| = \rho$. The simplest proof is by dividing $f(z)$ with one factor $z - \rho e^{i\theta_0}$ for each zero. It is sufficient to show that

$$\log \rho = \frac{1}{2\pi} \int_0^{2\pi} \log |\rho e^{i\theta} - \rho e^{i\theta_0}| \, d\theta$$

or

$$\int_0^{2\pi} \log |e^{i\theta} - e^{i\theta_0}| \, d\theta = 0.$$

This integral is evidently independent of θ_0, and we have only to show that

$$\int_0^{2\pi} \log |1 - e^{i\theta}| \, d\theta = 0.$$

But this is a consequence of the formula

$$\int_0^{\pi} \log \sin x \, dx = -\pi \log 2$$

proved in Chap. 4, Sec. 5.3 (cf. Chap. 4, Sec. 6.4, Ex. 5).

We will now investigate what becomes of (43) in the presence of zeros in the interior $|z| < \rho$. Denote these zeros by $a_1, a_2, \ldots, a_n$, multiple zeros being repeated, and assume first that $z = 0$ is not a zero. Then the function

$$F(z) = f(z) \prod_{i=1}^{n} \frac{\rho^2 - \bar{a}_i z}{\rho(z - a_i)}$$

is free from zeros in the disk, and $|F(z)| = |f(z)|$ on $|z| = \rho$. Consequently we obtain

$$\log |F(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(\rho e^{i\theta})| \, d\theta$$

and, substituting the value of $F(0)$,

$$\log |f(0)| = -\sum_{i=1}^{n} \log \left(\frac{\rho}{|a_i|}\right) + \frac{1}{2\pi} \int_0^{2\pi} \log |f(\rho e^{i\theta})| \, d\theta. \tag{44}$$

This is known as *Jensen's formula.* Its importance lies in the fact that it relates the modulus $|f(z)|$ on a circle to the moduli of the zeros.

If $f(0) = 0$, the formula is somewhat more complicated. Writing $f(z) = cz^h + \cdots$ we apply (44) to $f(z)(\rho/z)^h$ and find that the left-hand member must be replaced by $\log |c| + h \log \rho$.

There is a similar generalization of Poisson's formula. All that is needed is to apply the ordinary Poisson formula to $\log |F(z)|$. We obtain

$$\log |f(z)| = -\sum_{i=1}^{n} \log \left| \frac{\rho^2 - \bar{a}_i z}{\rho(z - a_i)} \right| + \frac{1}{2\pi} \int_0^{2\pi} \operatorname{Re} \frac{\rho e^{i\theta} + z}{\rho e^{i\theta} - z} \log |f(\rho e^{i\theta})| \, d\theta, \tag{45}$$

provided that $f(z) \neq 0$. Equation (45) is frequently referred to as the *Poisson-Jensen formula.*

Strictly speaking the proof is valid only if $f \neq 0$ on $|z| = \rho$. But (44) shows that the integral on the right is a continuous function of ρ, and from there it is easy to infer that the integral in (45) is likewise continuous. In the general case (45) can therefore be derived by letting ρ approach a limit.

The Jensen and Poisson-Jensen formulas have important applications in the theory of entire functions.

3.2. Hadamard's Theorem. Let $f(z)$ be an entire function, and denote its zeros by a_n; for the sake of simplicity we will assume that $f(0) \neq 0$. We recall that the genus of an entire function (Sec. 2.3) is the smallest integer h such that $f(z)$ can be represented in the form

$$f(z) = e^{g(z)} \prod_n \left(1 - \frac{z}{a_n}\right) e^{z/a_n + \frac{1}{2}(z/a_n)^2 + \cdots + (1/h)(z/a_n)^h} \tag{46}$$

where $g(z)$ is a polynomial of degree $\leq h$. If there is no such representation, the genus is infinite.

Denote by $M(r)$ the maximum of $|f(z)|$ on $|z| = r$. The *order* of the entire function $f(z)$ is defined by

$$\lambda = \varlimsup_{r \to \infty} \frac{\log \log M(r)}{\log r}.$$

According to this definition λ is the smallest number such that

$$M(r) \leq e^{r^{\lambda + \epsilon}} \tag{47}$$

for any given $\varepsilon > 0$ as soon as r is sufficiently large.

The genus and the order are closely related, as seen by the following theorem:

Theorem 8. *The genus and the order of an entire function satisfy the double inequality* $h \leqq \lambda \leqq h + 1$.

Assume first that $f(z)$ is of finite genus h. The exponential factor in (46) is quite obviously of order $\leqq h$, and the order of a product cannot exceed the orders of both factors. Hence it is sufficient to show that the canonical product is of order $\leqq h + 1$. The convergence of the canonical product implies $\sum_n |a_n|^{-h-1} < \infty$; this is the essential hypothesis.

We denote the canonical product by $P(z)$ and write the individual factors as $E_h(z/a_n)$ where

$$E_h(u) = (1 - u)e^{u+\frac{1}{2}u^2+\cdots+(1/h)u^h}$$

with the understanding that $E_0(u) = 1 - u$. We will show that

$$\log |E_h(u)| \leqq (2h + 1)|u|^{h+1} \tag{48}$$

for all u.

If $|u| < 1$ we have by power-series development

$$\log |E_h(u)| \leqq \frac{|u|^{h+1}}{h+1} + \frac{|u|^{h+2}}{h+2} + \cdots \leqq \frac{1}{h+1}\frac{|u|^{h+1}}{1-|u|}$$

and thus

$$(1 - |u|) \log |E_h(u)| \leqq |u|^{h+1}. \tag{49}$$

For arbitrary u and $h \geqq 1$ it is also clear that

$$\log |E_h(u)| \leqq \log |E_{h-1}(u)| + |u|^h. \tag{50}$$

The truth of (48) is seen by induction. For $h = 0$ we need merely note that $\log |1 - u| \leqq \log (1 + |u|) \leqq |u|$. We assume (48) with $h - 1$ in the place of h, that is to say

$$\log |E_{h-1}(u)| \leqq (2h - 1)|u|^h. \tag{51}$$

It follows from (50) and (51) that $\log |E_h(u)| \leqq 2h|u|^h$, and if $|u| \geqq 1$, this implies (48). But if $|u| < 1$ we can also use (49), and together with (50) and (51) we obtain

$$\log |E_h(u)| \leqq |u| \log |E_{h-1}(u)| + 2|u|^{h+1} \leqq (2h + 1)|u|^{h+1}.$$

This completes the induction.

The estimate (48) gives at once

$$\log |P(z)| = \sum_n \log \left| E_h\left(\frac{z}{a_n}\right)\right| \leqq (2h+1)|z|^{h+1} \sum_n |a_n|^{-h-1}$$

and it follows that $P(z)$ is at most of order $h + 1$.

For the opposite inequality assume that $f(z)$ is of finite order λ and let h be the largest integer $\leqq \lambda$. Then $h + 1 > \lambda$, and we have to prove, first of all, that $\sum_n |a_n|^{-h-1}$ converges. It is for this proof that Jensen's formula is needed.

Let us denote by $\nu(\rho)$ the number of zeros a_n with $|a_n| \leqq \rho$. In order to find an upper bound for $\nu(\rho)$ we apply (44) with 2ρ in the place of ρ and omit the terms $\log (2\rho/|a_n|)$ with $|a_n| \geqq \rho$. We find

$$\nu(\rho) \log 2 \leqq \frac{1}{2\pi} \int_0^{2\pi} \log |f(2\rho e^{i\theta})|\, d\theta - \log |f(0)|. \tag{52}$$

In view of (47) it follows that $\lim\limits_{\rho\to\infty} \nu(\rho)\rho^{-\lambda-\varepsilon} = 0$ for every $\varepsilon > 0$.

We assume now that the zeros a_n are ordered according to absolute values: $|a_1| \leqq |a_2| \leqq \cdots \leqq |a_n| \leqq \cdots$. Then it is clear that $n \leqq \nu(|a_n|)$, and from a certain n on we must have, for instance,

$$n \leqq \nu(|a_n|) < |a_n|^{\lambda+\varepsilon}.$$

According to this inequality the series $\sum_n |a_n|^{-h-1}$ has the majorant

$$\sum_n n^{-\frac{h+1}{\lambda+\varepsilon}},$$

and if we choose ε so that $\lambda + \varepsilon < h + 1$ the majorant converges. We have thus proved that $f(z)$ can be written in the form (46) where so far $g(z)$ is only known to be entire.

It remains to prove that $g(z)$ is a polynomial of degree $\leqq h$. For this purpose it is easiest to use the Poisson-Jensen formula. If the operation $(\partial/\partial x) - i(\partial/\partial y)$ is applied to both sides of the identity (45), we obtain

$$\frac{f'(z)}{f(z)} = \sum_1^{\nu(\rho)} (z - a_n)^{-1} + \sum_1^{\nu(\rho)} \bar{a}_n(\rho^2 - \bar{a}_n z)^{-1} + \frac{1}{2\pi} \int_0^{2\pi} 2\rho e^{i\theta}(\rho e^{i\theta} - z)^{-2} \log |f(\rho e^{i\theta})|\, d\theta.$$

On differentiating h times with respect to z this yields

$$D^{(h)} \frac{f'(z)}{f(z)} = -h! \sum_1^{\nu(\rho)} (a_n - z)^{-h-1} + h! \sum_1^{\nu(\rho)} \bar{a}_n^{h+1}(\rho^2 - \bar{a}_n z)^{-h-1} \tag{53}$$

$$+ (h+1)! \frac{1}{2\pi} \int_0^{2\pi} 2\rho e^{i\theta}(\rho e^{i\theta} - z)^{-h-2} \log |f(\rho e^{i\theta})| \, d\theta.$$

It is our intention to let ρ tend to ∞. In order to estimate the integral in (53) we observe first that

$$\int_0^{2\pi} \rho e^{i\theta}(\rho e^{i\theta} - z)^{-h-2} \, d\theta = 0.$$

Therefore nothing changes if we subtract $\log M(\rho)$ from $\log |f|$. If $\rho > 2|z|$ it follows that the last term in (53) has a modulus at most equal to

$$(h+1)! 2^{h+3} \rho^{-h-1} \frac{1}{2\pi} \int_0^{2\pi} \log \frac{M(\rho)}{|f(\rho e^{i\theta})|} \, d\theta,$$

for $\log M(\rho)/|f(\rho e^{i\theta})| \geqq 0$. But

$$\frac{1}{2\pi} \int_0^{2\pi} \log |f| \, d\theta \geqq \log |f(0)|$$

by Jensen's formula, and $\rho^{-h-1} \log M(\rho) \to 0$ since $\lambda < h + 1$. We conclude that the integral in (53) tends to 0.

As for the second sum in (53), the same preliminary inequality $\rho > 2|z|$ together with $|a_n| \leqq \rho$ makes each term absolutely less than $(2/\rho)^{h+1}$, and the whole sum has modulus at most $2^{h+1}\nu(\rho)\rho^{-h-1}$. We have already proved that this tends to 0. Therefore we obtain

$$D^{(h)} \frac{f'(z)}{f(z)} = -h! \sum_{n=1}^{\infty} (a_n - z)^{-h-1}. \tag{54}$$

Writing $f(z) = e^{g(z)}P(z)$ we find

$$g^{(h+1)}(z) = D^{(h)} \frac{f'}{f} - D^{(h)} \frac{P'}{P}.$$

However, by Weierstrass's theorem the quantity $D^{(h)}(P'/P)$ can be found by separate differentiation of each factor, and in this way we obtain precisely the right-hand member of (54). Consequently, $g^{(h+1)}(z)$ is identically zero, and $g(z)$ must be a polynomial of degree $\leqq h$. We have proved Theorem 8.

The theorem is a factorization theorem for entire functions of finite order λ. If λ is not an integer, the genus h, and thereby the form of the product, is uniquely determined. If the order is integral, there is an ambiguity.

The following impressive corollary shows the strength of Hadamard's theorem:

Corollary. *An entire function of fractional order assumes every finite value infinitely many times.*

It is clear that f and $f - a$ have the same order for any constant a. Therefore we need only show that f has infinitely many zeros. If f has only a finite number of zeros we can divide by a polynomial and obtain a function of the same order without zeros. By the theorem it must be of the form e^g where g is a polynomial. But it is evident that the order of e^g is exactly the degree of g, and hence an integer. The contradiction proves the corollary.

EXERCISES

1. The characterization of the genus given in the first paragraph of Sec. 3.2 is not literally the same as the definition in Sec. 2.3. Supply the reasoning necessary to see that the conditions are equivalent.

2. Assume that $f(z)$ has genus zero so that

$$f(z) = z^m \prod_n \left(1 - \frac{z}{a_n}\right).$$

Compare $f(z)$ with

$$g(z) = z^m \prod_n \left(1 - \frac{z}{|a_n|}\right)$$

and show that the maximum modulus $\max_{|z|=r} |f(z)|$ is $\leqq$ the maximum modulus of g, and that the minimum modulus of f is $\geqq$ the minimum modulus of g.

4. THE RIEMANN ZETA FUNCTION

The series $\sum_{n=1}^{\infty} n^{-\sigma}$ converges uniformly for all real σ greater than or equal to a fixed $\sigma_0 > 1$. It is a majorant of the series

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \qquad (s = \sigma + it), \tag{55}$$

which therefore represents an analytic function of s in the half plane $\text{Re}\, s > 1$ (see Sec. 1.1, Ex. 2; the notation $s = \sigma + it$ is traditional in this context).

The function $\zeta(s)$ is known as *Riemann's ζ-function.* It plays a central role in the applications of complex analysis to number theory. It would lead us too far astray to develop even a few of these applications in this book, but we can and will acquaint the reader with some of the more elementary properties of the ζ-function.