6 CONFORMAL MAPPING. DIRICHLET'S PROBLEM

In the geometrically oriented part of the theory of analytic functions the problem of conformal mapping plays a dominating role. Existence and uniqueness theorems permit us to define important analytic functions without resorting to analytic expressions, and geometric properties of the regions that are being mapped lead to analytic properties of the mapping function.

The Riemann mapping theorem deals with the mapping of one simply connected region onto another. We shall give a proof that leans on the theory of normal families. To handle the more difficult case of multiply connected regions we shall have to solve the Dirichlet problem, which is the boundary-value problem for the Laplace equation.

1. THE RIEMANN MAPPING THEOREM

We shall prove that the unit disk can be mapped conformally onto any simply connected region in the plane, other than the plane itself. This will imply that any two such regions can be mapped conformally onto each other, for we can use the unit disk as an intermediary step. The theorem is applied to polygonal regions, and in this case an explicit form for the mapping function is derived.

1.1. Statement and Proof. Although the mapping theorem was formulated by Riemann, its first successful proof was due to

P. Koebe.† The proof we shall present is a shorter variant of the original proof.

Theorem 1. *Given any simply connected region Ω which is not the whole plane, and a point $z_0 \in \Omega$, there exists a unique analytic function $f(z)$ in Ω, normalized by the conditions $f(z_0) = 0$, $f'(z_0) > 0$, such that $f(z)$ defines a one-to-one mapping of Ω onto the disk $|w| < 1$.*

The uniqueness is easily proved, for if f_1 and f_2 are two such functions, then $f_1[f_2^{-1}(w)]$ defines a one-to-one mapping of $|w| < 1$ onto itself. We know that such a mapping is given by a linear transformation S (Chap. 4, Sec. 3.4, Ex. 5). The conditions $S(0) = 0$, $S'(0) > 0$ imply $S(w) = w$; hence $f_1 = f_2$.

An analytic function $g(z)$ in Ω is said to be *univalent* if $g(z_1) = g(z_2)$ only for $z_1 = z_2$, in other words, if the mapping by g is one to one (the German word *schlicht*, which lacks an adequate translation, is also in common use). For the existence proof we consider the family $\mathfrak{F}$ formed by all functions g with the following properties: (i) g is analytic and univalent in Ω, (ii) $|g(z)| \leqq 1$ in Ω, (iii) $g(z_0) = 0$ and $g'(z_0) > 0$. We contend that f is the function in $\mathfrak{F}$ for which the derivative $f'(z_0)$ is a maximum. The proof will consist of three parts: (1) it is shown that the family $\mathfrak{F}$ is not empty; (2) there exists an f with maximal derivative; (3) this f has the desired properties.

To prove that $\mathfrak{F}$ is not empty we note that there exists, by assumption, a point $a \neq \infty$ not in Ω. Since Ω is simply connected, it is possible to define a single-valued branch of $\sqrt{z-a}$ in Ω; denote it by $h(z)$. This function does not take the same value twice, nor does it take opposite values. The image of Ω under the mapping h covers a disk $|w - h(z_0)| < \rho$, and therefore it does not meet the disk $|w + h(z_0)| < \rho$. In other words, $|h(z) + h(z_0)| \geqq \rho$ for $z \in \Omega$, and in particular $2|h(z_0)| \geqq \rho$. It can now be verified that the function

$$g_0(z) = \frac{\rho}{4}\frac{|h'(z_0)|}{|h(z_0)|^2}\cdot\frac{h(z_0)}{h'(z_0)}\cdot\frac{h(z)-h(z_0)}{h(z)+h(z_0)}$$

belongs to the family $\mathfrak{F}$. Indeed, because it is obtained from the univalent function h by means of a linear transformation, it is itself univalent. Moreover, $g_0(z_0) = 0$ and $g_0'(z_0) = (\rho/8)|h'(z_0)|/|h(z_0)|^2 > 0$. Finally, the estimate

$$\left|\frac{h(z)-h(z_0)}{h(z)+h(z_0)}\right| = |h(z_0)|\cdot\left|\frac{1}{h(z_0)} - \frac{2}{h(z)+h(z_0)}\right| \leqq \frac{4|h(z_0)|}{\rho}$$

shows that $|g_0(z)| \leqq 1$ in Ω.

† A related theorem from which the mapping theorem can be derived had been proved earlier by W. F. Osgood, but did not attract the attention it deserves.

The derivatives $g'(z_0)$, $g \in \mathfrak{F}$, have a least upper bound B which a priori could be infinite. There is a sequence of functions $g_n \in \mathfrak{F}$ such that $g_n'(z_0) \to B$. By Chap. 5, Theorem 12 the family $\mathfrak{F}$ is normal. Hence there exists a subsequence $\{g_{n_k}\}$ which tends to an analytic limit function f, uniformly on compact sets. It is clear that $|f(z)| \leqq 1$ in Ω, $f(z_0) = 0$ and $f'(z_0) = B$ (this proves that $B < +\infty$). If we can show that f is univalent, it will follow that f is in $\mathfrak{F}$ and has a maximal derivative at z_0.

In the first place f is not a constant, for $f'(z_0) = B > 0$. Choose a point $z_1 \in \Omega$, and consider the functions $g_1(z) = g(z) - g(z_1)$, $g \in \mathfrak{F}$. They are all $\neq 0$ in the region obtained by omitting z_1 from Ω. By Hurwitz's theorem (Chap. 5, Theorem 2) every limit function is either identically zero or never zero. But $f(z) - f(z_1)$ is a limit function, and it is not identically zero. Hence $f(z) \neq f(z_1)$ for $z \neq z_1$, and since z_1 was arbitrary we have proved that f is univalent.

It remains to show that f takes every value w with $|w| < 1$. Suppose it were true that $f(z) \neq w_0$ for some w_0, $|w_0| < 1$. Then, since Ω is simply connected, it is possible to define a single-valued branch of

$$F(z) = \sqrt{\frac{f(z) - w_0}{1 - \bar{w}_0 f(z)}} \tag{1}$$

(Recall that all closed curves in a simply connected region are homologous to 0. If $\varphi(z) \neq 0$ in Ω we can define $\log \varphi(z)$ by integration of $\varphi'(z)/\varphi(z)$, and $\sqrt{\varphi(z)} = \exp\left(\frac{1}{2}\log \varphi(z)\right)$.)

It is clear that F is univalent and that $|F| \leqq 1$. To normalize it we form

$$G(z) = \frac{|F'(z_0)|}{F'(z_0)} \cdot \frac{F(z) - F(z_0)}{1 - \overline{F(z_0)}F(z)} \tag{2}$$

which vanishes and has a positive derivative at z_0. For its value we find, after brief computation,

$$G'(z_0) = \frac{|F'(z_0)|}{1 - |F(z_0)|^2} = \frac{1 + |w_0|}{2\sqrt{|w_0|}}\, B > B.$$

This is a contradiction, and we conclude that $f(z)$ assumes all values w, $|w| < 1$. The proof is now complete.

At first glance it may seem like pure luck that our computation yields $G'(z_0) > f'(z_0)$. This is not quite so, for the formulas (1) and (2) permit us to express $f(z)$ as a single-valued analytic function of $W = G(z)$ which maps $|W| < 1$ into itself. The inequality $|f'(z_0)| < |G'(z_0)|$ is therefore a consequence of Schwarz's lemma.

The purely topological content of Theorem 1 is important by itself. We know now that any simply connected region can be mapped topologically onto a disk (for the whole plane a very simple mapping can be constructed), and hence any two simply connected regions are topologically equivalent.

EXERCISES

1. If z_0 is real and Ω is symmetric with respect to the real axis, prove by the uniqueness that f satisfies the symmetry relation $f(\bar{z}) = \overline{f(z)}$.

2. What is the corresponding conclusion if Ω is symmetric with respect to the point z_0?

1.2. Boundary Behavior. We are assuming that $f(z)$ defines a conformal mapping of a region Ω onto another region Ω'. What happens when z approaches the boundary? There are cases where the boundary behavior can be foretold with great precision. For instance, if Ω and Ω' are Jordan regions,† then f can be extended to a topological mapping of the closure of Ω onto the closure of Ω'. Unfortunately, considerations of space do not permit us to include a proof of this important theorem (the proof would require a considerable amount of preparation).

What we can and shall prove is a very modest theorem of purely topological content. Let us first make it clear what we mean when we say that z approaches the boundary of Ω. There are two cases: we may consider a sequence $\{z_n\}$ of points in Ω, or we may consider an arc $z(t)$, $0 \leqq t < 1$, such that all $z(t)$ are in Ω. We shall say that the sequence or the arc tends to the boundary if the points z_n or $z(t)$ will ultimately stay away from any point in Ω. In other words, if $z \in \Omega$ there shall exist an $\varepsilon > 0$ and an n_0 or a t_0 such that $|z_n - z| \geqq \varepsilon$ for $n > n_0$, or such that $|z(t) - z| \geqq \varepsilon$ for all $t > t_0$.

In this situation, the disks of center z and radius ε (which may depend on z) form an open covering of Ω. It follows that any compact subset $K \subset \Omega$ is covered by a finite number of these disks. If we consider the largest of the corresponding n_0 or t_0 we find that z_n or $z(t)$ cannot belong to K for $n > n_0$ or $t > t_0$. Colloquially speaking, for any compact set $K \subset \Omega$ there exists a tail end of the sequence or of the arc which does not meet K. Conversely, this implies the original condition, for if $z \in \Omega$ is given we may choose K to be a closed disk with center z that is contained in Ω. If the radius of the disk is ρ the original statement holds for any $\varepsilon < \rho$.

After these preliminary considerations the theorem we shall prove is almost trivial:

† It is known, although not so easy to prove, that a Jordan curve (Chap. 3 Sec. 2.1) divides the plane into exactly two regions, one bounded and one unbounded. The bounded region is called a Jordan region.

Theorem 2. *Let f be a topological mapping of a region Ω onto a region Ω'. If $\{z_n\}$ or $z(t)$ tends to the boundary of Ω, then $\{f(z_n)\}$ or $f(z(t))$ tends to the boundary of Ω'.*

Indeed, let K be a compact set in Ω'. Then $f^{-1}(K)$ is a compact set in Ω, and there exists n_0 (or t_0) such that z_n (or $z(t)$) is not in $f^{-1}(K)$ for $n > n_0$ (or $t > t_0$). But then $f(z_n)$ [or $f(z(t))$] is not in K.

Although the theorem is topological, it is the application to conformal mappings that is of greatest interest to us.

1.3. Use of the Reflection Principle. Stronger statements become possible if we have more information. We are mainly interested in simply connected regions and may therefore assume that one of the regions is a disk. With the same notation as in Sec. 1.1, let $f(z)$ define a conformal mapping of the region Ω onto the unit disk with the normalization $f(z_0) = 0$ (the normalization by the derivative is irrelevant). We shall derive additional information by use of the reflection principle (Chap. 4, Theorem 26).

Let us assume that the boundary of Ω contains a segment γ of a straight line. Because rotations are unimportant we may as well suppose that γ lies on the real axis; let it be the interval $a < x < b$. The assumption involves a significant simplification only if the rest of the boundary stays away from γ. For this reason we shall strengthen the hypothesis and require that every point of γ has a neighborhood whose intersection with the whole boundary $\partial\Omega$ is the same as its intersection with γ. We say then that γ is a *free boundary arc.*

By this assumption every point on γ is the center of a disk whose intersection with $\partial\Omega$ is its real diameter. It is clear that each of the half disks determined by this diameter is entirely in or entirely outside of Ω, and at least one must be inside. If only one is inside we call the point a one-sided boundary point, and if both are inside it is a two-sided boundary point. Because γ is connected all its points will be of the same kind, and we speak of a one-sided or a two-sided boundary arc.

Theorem 3. *Suppose that the boundary of a simply connected region Ω contains a line segment γ as a one-sided free boundary arc. Then the function $f(z)$ which maps Ω onto the unit disk can be extended to a function which is analytic and one to one on $\Omega \cup \gamma$. The image of γ is an arc γ' on the unit circle.*

For two-sided arcs the same will be true with obvious modifications.

For the proof we consider a disk around $x_0 \in \gamma$ which is so small that the half disk in Ω does not contain the point z_0 with $f(z_0) = 0$. Then

$\log f(z)$ has a single-valued branch in the half disk, and its real part tends to 0 as z approaches the diameter, for we know by Theorem 2 that $|f(z)|$ tends to 1. It follows by the reflection principle that $\log f(z)$ has an analytic extension to the whole disk. Therefore $\log f(z)$, and consequently $f(z)$, is analytic at x_0. The extensions to overlapping disks must coincide and define a function which is analytic on $\Omega \cup \gamma$.

We note further that $f'(z) \neq 0$ on γ. Indeed, $f'(x_0) = 0$ would imply that $f(x_0)$ were a multiple value, in which case the two subarcs of γ that meet at x_0 would be mapped on arcs that form an angle π/n with $n \geqq 2$; this is clearly impossible. If, for instance, the upper half disks are in Ω, then

$$\partial \log |f|/\partial y = -\partial \arg f/\partial x < 0$$

on γ, and $\arg f$ moves constantly in the same direction. This proves that the mapping is one to one on γ.

The theorem can be generalized to regions with free boundary arcs on a circle. With obvious modifications the theorem is also true for two-sided boundary arcs.

1.4. Analytic Arcs. A real or complex function $\varphi(t)$ of a real variable t, defined on an interval $a < t < b$, is said to be *real analytic* (or analytic in the real sense) if, for every t_0 in the interval, the Taylor development $\varphi(t) = \varphi(t_0) + \varphi'(t_0)(t - t_0) + \frac{1}{2}\varphi'(t_0)(t - t_0)^2 + \cdots$ converges in some interval $(t_0 - \rho, t_0 + \rho)$, $\rho > 0$. But if this is so we know by Abel's theorem that the series is also convergent for complex values of t, as long as $|t - t_0| < \rho$, and that it represents an analytic function in that disk. In overlapping disks the functions are the same, for they coincide on a segment of the real axis. We conclude that $\varphi(t)$ can be defined as an analytic function in a region Δ, symmetric to the real axis, which contains the segment (a,b).

In these circumstances we say that $\varphi(t)$ determines an *analytic arc*. It is *regular* if $\varphi'(t) \neq 0$, and it is a simple arc if $\varphi(t_1) = \varphi(t_2)$ only when $t_1 = t_2$.

We shall assume that the boundary of Ω contains a regular, simple, analytic arc γ, and that it is a free one-sided arc. The definition could be modeled on the previous one, but to avoid long explanations we shall assume offhand that there exists a region Δ, symmetric to the interval (a,b), with the property that $\varphi(t) \in \Omega$ when t lies in the upper half of Δ, and that $\varphi(t)$ lies outside Ω for t in the lower half.

If $f(z)$ is the mapping function with $f(z_0) = 0$, and if we take care that $\varphi(t) \neq z_0$ in Δ, then the reflection principle tells us that $\log f(\varphi(t))$, and hence $f(\varphi(t))$, has an analytic extension from the upper to the lower half of Δ. For a real $t_0 \in (a,b)$ we know further that $\varphi'(t_0) \neq 0$. Therefore φ has an analytic inverse φ^{-1} in a neighborhood of $\varphi(t_0)$, and it follows by composition that $f(z)$ is analytic in that neighborhood.

Theorem 4. *If the boundary of Ω contains a free one-sided analytic arc γ, then the mapping function has an analytic extension to $\Omega \cup \gamma$, and γ is mapped on an arc of the unit circle.*

We trust the reader to make the last statement more precise and to complete the proof.

2. CONFORMAL MAPPING OF POLYGONS

When Ω is a polygon, the mapping problem has an almost explicit solution. Indeed, we shall find that the mapping function can be expressed through a formula in which only certain parameters have values that depend on the specific shape of the polygon.

2.1. The Behavior at an Angle. We assume that Ω is a bounded simply connected region whose boundary is a closed polygonal line without self-intersections. Let the consecutive vertices be $z_1, \ldots, z_n$ in positive cyclic order (we set $z_{n+1} = z_1$). The *angle* at z_k is given by the value of $\arg(z_{k-1} - z_k)/(z_{k+1} - z_k)$ between 0 and 2π. We shall denote it by $\alpha_k\pi$, $0 < \alpha_k < 2$. It is also convenient to introduce the *outer angles* $\beta_k\pi = (1 - \alpha_k)\pi$, $-1 < \beta_k < 1$. Observe that $\beta_1 + \cdots + \beta_n = 2$. The polygon is convex if and only if all $\beta_k > 0$.

We know by Theorem 3 that the mapping function $f(z)$ can be extended by continuity to any side of the polygon (that is, to the open line segment between two consecutive vertices), and that each side is mapped in a one-to-one way onto an arc of the unit circle. We wish to show that these arcs are disjoint and leave no gap between them.

To see this we consider a circular sector S_k which is the intersection of Ω with a sufficiently small disk about z_k. A single-valued branch of $\zeta = (z - z_k)^{1/\alpha_k}$ maps S_k onto a half disk S'_k. A suitable branch of $z_k + \zeta^{\alpha_k}$ has its values in Ω, and we may consider the function $g(\zeta) = f(z_k + \zeta^{\alpha_k})$ in S'_k. It follows by Theorem 2 that $|g(\zeta)| \to 1$ as ζ approaches the diameter. The reflection principle applies, and we conclude that $g(\zeta)$ has an analytic continuation to the whole disk. In particular, this implies that $f(z)$ has a limit $w_k = e^{i\theta_k}$ as $z \to z_k$, and we find that the arcs that correspond to the sides meeting at z_k do indeed have a common end point. Since $\arg f(z)$ must increase as z traces the boundary in positive direction, the arcs do not overlap, at least not in a neighborhood of w_k. If we take into account that f maps the boundary