Construction of integral functions (entire function) with given zeros.

* The simplest entire functions which are not polynomials are e^z, $\sin z$, $\cos z$.

* An integral function with no zeros is expressible in the form

$$e^{h(z)}$$

where $h(z)$ is an integral function.

Proof: Let $f(z)$ be an integral function with no zeros. Then

$$\frac{f'(z)}{f(z)} = g(z)$$

is itself an integral function. Integrate along any rectifiable curve from 0 to z, we have

$$\operatorname{Log} f(z) - \log f(0) = \int_0^z g(z)\, dz = h(z) \quad (\text{say})$$

$$\Rightarrow \boxed{f(z) = f(0)\, e^{h(z)}}$$

Thm (Weierstrass Factorisation Theorem):

Let $f(z)$ be an integral function with zeros

$$z_0, z_1, z_2, \dots z_n, \dots$$

of orders

$$b_0, b_1, b_2, \dots b_n, \dots,$$

respectively. Then there exists an integral function $h(z)$

with no zeros,) and a seq^n of polyn^s $P_n(z)$ such that
$$f(z) = e^{h(z)} \prod_{n=0}^{\infty} \left[\left(1-\frac{z}{z_n}\right)^{b_n} e^{P_n(z)} \right]$$

OR

~~Let $\{a_n\}_{n\geq1}$ be a seq^n of nonzero complex numbers and $f(z)$ be an entire function that has zero at a_n, listed with~~ Given any complex seq^n having no finite limit point, there exists an entire function that has zeros at these points and only these points.

Proof: Since each zero z_n of order b_n of $f(z)$ is a simple pole of
$$\frac{f'(z)}{f(z)} \quad (\text{meromorphic function})$$
with residue b_n. Thus by Mittag-Leffler's thm for meromorphic function, $\exists$ an integral function $g(z)$ & seq^n of polyn^s $Q_n(z)$ such that
$$\frac{f'(z)}{f(z)} = g(z) + \sum_{n=0}^{\infty} \left[\frac{b_n}{z-z_n} + Q_n(z) \right]$$

Integrating from 0 to z, we have (Because of U.C.)
$$\log f(z) - \log f(0) = \int_0^z g(z)\,dz + \sum_{n=0}^{\infty} \left[b_n \log(z-z_n) + \int_0^z Q_n(z)\,dz \right]$$

$$\log f(z) - f(0) = h(z) + \sum_{n=0}^{\infty} b_n\left(\log(-z_n) + \log\left(1-\frac{z}{z_n}\right)\right) + \int_0^z Q_n(z)\,dz$$

$$\log f(z) - f(0) = h(z) + \sum_{n=0}^{\infty} b_n \log\left(1-\frac{z}{z_n}\right) + P_n(z),$$

where $P_n(z)$ is a polyn.

Thus

$$\boxed{f(z) = e^{h(z)} \sum_{n=0}^{\infty}\left[\left(1-\frac{z}{z_n}\right)^{b_n} \cdot P_n(z)\right]}$$

Thm:- If $f(z)$ & $g(z)$ are entire functions whose zeros coincide in location & multiplicity, then $\exists$ an entire function $\phi(z)$ such that

$$\boxed{f(z) = e^{\phi(z)} g(z)}$$

Proof: After the cancellation of common factors, the function

$$\frac{f(z)}{g(z)}$$

will be an entire function with no zeros in $\mathbb{C}$.

Hence

$$\frac{f(z)}{g(z)} = e^{\phi(z)}$$

$$\boxed{f(z) = e^{\phi(z)} g(z)}$$