5 SERIES AND PRODUCT DEVELOPMENTS

Very general theorems have their natural place in the theory of analytic functions, but it must also be kept in mind that the whole theory originated from a desire to be able to manipulate explicit analytic expressions. Such expressions take the form of infinite series, infinite products, and other limits. In this chapter we deal partly with the rules that govern such limits, partly with quite explicit representations of elementary transcendental functions and other specific functions.

1. POWER SERIES EXPANSIONS

In a preliminary way we have considered power series in Chap. 2, mainly for the purpose of defining the exponential and trigonometric functions. Without use of integration we were not able to prove that every analytic function has a power series expansion. This question will now be resolved in the affirmative, essentially as an application of Cauchy's theorem.

The first subsection deals with more general properties of sequences of analytic functions.

1.1. Weierstrass's Theorem. The central theorem concerning the convergence of analytic functions asserts that the limit of a uniformly convergent sequence of analytic functions is an analytic function. The precise assumptions must be carefully stated, and they should not be too restrictive.

We are considering a sequence $\{f_n(z)\}$ where each $f_n(z)$ is defined and analytic in a region Ω_n. The limit function $f(z)$ must also be considered in some region Ω, and clearly, if $f(z)$ is to be defined in Ω, each point of Ω must belong to all Ω_n for n greater than a certain n_0. In the general case n_0 will not be the same for all points of Ω, and for this reason it would not make sense to require that the convergence be uniform in Ω. In fact, in the most typical case the regions Ω_n form an increasing sequence, $\Omega_1 \subset \Omega_2 \subset \cdots \subset \Omega_n \subset \cdots$, and Ω is the union of the Ω_n. In these circumstances no single function $f_n(z)$ is defined in all of Ω; yet the limit $f(z)$ may exist at all points of Ω, although the convergence cannot be uniform.

As a very simple example take $f_n(z) = z/(2z^n + 1)$ and let Ω_n be the disk $|z| < 2^{-1/n}$. It is practically evident that $\lim_{n \to \infty} f_n(z) = z$ in the disk $|z| < 1$ which we choose as our region Ω. In order to study the uniformity of the convergence we form the difference

$$f_n(z) - z = -2z^{n+1}/(2z^n + 1).$$

For any given value of z we can make $|z^n| < \varepsilon/4$ by taking $n > \log(4/\varepsilon)/\log(1/|z|)$. If $\varepsilon < 1$ we have then $2|z|^{n+1} < \varepsilon/2$ and $|1 + 2z^n| > \frac{1}{2}$ so that $|f_n(z) - z| < \varepsilon$. It follows that the convergence is uniform in any closed disk $|z| \leqq r < 1$, or on any subset of such a closed disk.

With another formulation, in the preceding example the sequence $\{f_n(z)\}$ tends to the limit function $f(z)$ uniformly on every compact subset of the region Ω. In fact, on a compact set $|z|$ has a maximum $r < 1$ and the set is thus contained in the closed disk $|z| \leqq r$. This is the typical situation. We shall find that we can frequently prove uniform convergence on every compact subset of Ω; on the other hand, this is the natural condition in the theorem that we are going to prove.

Theorem 1. *Suppose that $f_n(z)$ is analytic in the region Ω_n, and that the sequence $\{f_n(z)\}$ converges to a limit function $f(z)$ in a region Ω, uniformly on every compact subset of Ω. Then $f(z)$ is analytic in Ω. Moreover, $f_n'(z)$ converges uniformly to $f'(z)$ on every compact subset of Ω.*

The analyticity of $f(z)$ follows most easily by use of Morera's theorem (Chap. 4, Sec. 2.3). Let $|z - a| \leqq r$ be a closed disk contained in Ω; the assumption implies that this disk lies in Ω_n for all n greater than a certain n_0.† If γ is any closed curve contained in $|z - a| < r$, we have

$$\int_\gamma f_n(z)\,dz = 0$$

† In fact, the regions Ω_n form an open covering of $|z - a| \leqq r$. The disk is compact and hence has a finite subcovering. This means that it is contained in a fixed Ω_{n_0}.

for $n > n_0$, by Cauchy's theorem. Because of the uniform convergence on γ we obtain

$$\int_\gamma f(z)\,dz = \lim_{n\to\infty} \int_\gamma f_n(z)\,dz = 0,$$

and by Morera's theorem it follows that $f(z)$ is analytic in $|z - a| < r$. Consequently $f(z)$ is analytic in the whole region Ω.

An alternative and more explicit proof is based on the integral formula

$$f_n(z) = \frac{1}{2\pi i}\int_C \frac{f_n(\zeta)\,d\zeta}{\zeta - z},$$

where C is the circle $|\zeta - a| = r$ and $|z - a| < r$. Letting n tend to ∞ we obtain by uniform convergence

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)\,d\zeta}{\zeta - z}.$$

and this formula shows that $f(z)$ is analytic in the disk. Starting from the formula

$$f_n'(z) = \frac{1}{2\pi i}\int_C \frac{f_n(\zeta)\,d\zeta}{(\zeta - z)^2}$$

the same reasoning yields

$$\lim_{n\to\infty} f_n'(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)\,d\zeta}{(\zeta - z)^2} = f'(z),$$

and simple estimates show that the convergence is uniform for $|z - a| \leqq \rho < r$. Any compact subset of Ω can be covered by a finite number of such closed disks, and therefore the convergence is uniform on every compact subset. The theorem is proved, and by repeated applications it follows that $f_n^{(k)}(z)$ converges uniformly to $f^{(k)}(z)$ on every compact subset of Ω.

Theorem 1 is due to Weierstrass, in an equivalent formulation. Its application to series whose terms are analytic functions is particularly important. The theorem can then be expressed as follows:

If a series with analytic terms,

$$f(z) = f_1(z) + f_2(z) + \cdots + f_n(z) + \cdots,$$

converges uniformly on every compact subset of a region Ω, then the sum $f(z)$ is analytic in Ω, and the series can be differentiated term by term.

The task of proving uniform convergence on a compact point set A can be facilitated by use of the maximum principle. In fact, with the notations of Theorem 1, the difference $|f_m(z) - f_n(z)|$ attains its maximum in A on the boundary of A. For this reason uniform convergence on the boundary of A implies uniform convergence on A. For instance, if the functions $f_n(z)$ are analytic in the disk $|z| < 1$, and if it can be shown that the sequence converges uniformly on each circle $|z| = r_m$ where $\lim_{m \to \infty} r_m = 1$, then Weierstrass's theorem applies and we can conclude that the limit function is analytic.

The following theorem is due to A. Hurwitz:

Theorem 2. *If the functions $f_n(z)$ are analytic and $\neq 0$ in a region Ω, and if $f_n(z)$ converges to $f(z)$, uniformly on every compact subset of Ω, then $f(z)$ is either identically zero or never equal to zero in Ω.*

Suppose that $f(z)$ is not identically zero. The zeros of $f(z)$ are in any case isolated. For any point $z_0 \in \Omega$ there is therefore a number $r > 0$ such that $f(z)$ is defined and $\neq 0$ for $0 < |z - z_0| \leqq r$. In particular, $|f(z)|$ has a positive minimum on the circle $|z - z_0| = r$, which we denote by C. It follows that $1/f_n(z)$ converges uniformly to $1/f(z)$ on C. Since it is also true that $f_n'(z) \to f'(z)$, uniformly on C, we may conclude that

$$\lim_{n \to \infty} \frac{1}{2\pi i} \int_C \frac{f_n'(z)}{f_n(z)}\, dz = \frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)}\, dz.$$

But the integrals on the left are all zero, for they give the number of roots of the equation $f_n(z) = 0$ inside of C. The integral on the right is therefore zero, and consequently $f(z_0) \neq 0$ by the same interpretation of the integral. Since z_0 was arbitrary, the theorem follows.

EXERCISES

1. Using Taylor's theorem applied to a branch of $\log\,(1 + z/n)$, prove that

$$\lim_{n \to \infty} \left(1 + \frac{z}{n}\right)^n = e^z$$

uniformly on all compact sets.

2. Show that the series

$$\zeta(z) = \sum_{n=1}^{\infty} n^{-z}$$

converges for $\operatorname{Re} z > 1$, and represent its derivative in series form.

3. Prove that

$$(1 - 2^{1-z})\zeta(z) = 1^{-z} - 2^{-z} + 3^{-z} - \cdots$$

and that the latter series represents an analytic function for $\operatorname{Re} z > 0$.

4. As a generalization of Theorem 2, prove that if the $f_n(z)$ have at most m zeros in Ω, then $f(z)$ is either identically zero or has at most m zeros.

5. Prove that

$$\sum_{n=1}^{\infty} \frac{nz^n}{1 - z^n} = \sum_{n=1}^{\infty} \frac{z^n}{(1 - z^n)^2}$$

for $|z| < 1$. (Develop in a double series and reverse the order of summation.)

1.2. The Taylor Series. We show now that every analytic function can be developed in a convergent Taylor series. This is an almost immediate consequence of the finite Taylor development given in Chap. 4, Sec. 3.1, Theorem 8, together with the corresponding representation of the remainder term. According to this theorem, if $f(z)$ is analytic in a region Ω containing z_0, we can write

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z - z_0) + \cdots + \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n + f_{n+1}(z)(z - z_0)^{n+1}$$

with

$$f_{n+1}(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)\,d\zeta}{(\zeta - z_0)^{n+1}(\zeta - z)}.$$

In the last formula C is any circle $|z - z_0| = \rho$ such that the closed disk $|z - z_0| \leqq \rho$ is contained in Ω.

If M denotes the maximum of $|f(z)|$ on C, we obtain at once the estimate

$$|f_{n+1}(z)(z - z_0)^{n+1}| \leqq \frac{M|z - z_0|^{n+1}}{\rho^n(\rho - |z - z_0|)}$$

We conclude that the remainder term tends uniformly to zero in every disk $|z - z_0| \leqq r < \rho$. On the other hand, ρ can be chosen arbitrarily close to the shortest distance from z_0 to the boundary of Ω. We have proved:

Theorem 3. *If $f(z)$ is analytic in the region Ω, containing z_0, then the representation*

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z - z_0) + \cdots + \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n + \cdots$$

is valid in the largest open disk of center z_0 contained in Ω.

The radius of convergence of the Taylor series is thus at least equal to the shortest distance from z_0 to the boundary of Ω. It may well be larger, but if it is there is no guarantee that the series still represents $f(z)$ at all points which are simultaneously in Ω and in the circle of convergence.

We recall that the developments

$$e^z = 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!} + \cdots$$
$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots$$
$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots$$

served as definitions of the functions they represent. Of course, as we have remarked before, every convergent power series is its own Taylor series. We gave earlier a direct proof that power series can be differentiated term by term. This is also a direct consequence of Weierstrass's theorem.

If we want to represent a fractional power of z or $\log z$ through a power series, we must first of all choose a well-defined branch, and secondly we have to choose a center $z_0 \neq 0$. It amounts to the same thing if we develop the function $(1 + z)^\mu$ or $\log (1 + z)$ about the origin, choosing the branch which is respectively equal to 1 or 0 at the origin. Since this branch is single-valued and analytic in $|z| < 1$, the radius of convergence is at least 1. It is elementary to compute the coefficients, and we obtain

$$(1 + z)^\mu = 1 + \mu z + \binom{\mu}{2} z^2 + \cdots + \binom{\mu}{n} z^n + \cdots$$
$$\log (1 + z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \frac{z^5}{5} - \cdots$$

where the binomial coefficients are defined by

$$\binom{\mu}{n} = \frac{\mu(\mu - 1) \cdots (\mu - n + 1)}{1 \cdot 2 \cdots n}.$$

If the logarithmic series had a radius of convergence greater than 1, then $\log (1 + z)$ would be bounded for $|z| < 1$. Since this is not the case, the radius of convergence must be exactly 1. Similarly, if the binomial series were convergent in a circle of radius >1, the function $(1 + z)^\mu$ and all its derivatives would be bounded in $|z| < 1$. Unless μ is a positive integer, one of the derivatives will be a negative power of $1 + z$, and hence unbounded. Thus the radius of convergence is precisely 1 except in the trivial case in which the binomial series reduces to a polynomial.

The series developments of the cyclometric functions arc tan z and arc sin z are most easily obtained by consideration of the derived series. From the expansion

$$\frac{1}{1+z^2} = 1 - z^2 + z^4 - z^6 + \cdots$$

we obtain by integration

$$\text{arc tan } z = z - \frac{z^3}{3} + \frac{z^5}{5} - \frac{z^7}{7} + \cdots$$

where the branch is uniquely determined as

$$\text{arc tan } z = \int_0^z \frac{dz}{1+z^2}$$

for any path inside the unit circle. For justification we can either rely on uniform convergence or apply Theorem 1. The radius of convergence cannot be greater than that of the derived series, and hence it is exactly 1.

If $\sqrt{1-z^2}$ is the branch with a positive real part, we have

$$\frac{1}{\sqrt{1-z^2}} = 1 + \frac{1}{2}z^2 + \frac{1 \cdot 3}{2 \cdot 4}z^4 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}z^6 + \cdots$$

for $|z| < 1$, and through integration we obtain

$$\text{arc sin } z = z + \frac{1}{2}\frac{z^3}{3} + \frac{1 \cdot 3}{2 \cdot 4}\frac{z^5}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\frac{z^7}{7} + \cdots.$$

The series represents the principal branch of arc sin z with a real part between $-\pi/2$ and $\pi/2$.

For combinations of elementary functions it is mostly not possible to find a general law for the coefficients. In order to find the first few coefficients we need not, however, calculate the successive derivatives. There are simple techniques which allow us to compute, with a reasonable amount of labor, all the coefficients that we are likely to need.

It is convenient to introduce the notation $[z^n]$ for any function which is analytic and has a zero of at least order n at the origin; less precisely, $[z^n]$ denotes a function which "contains the factor z^n." With this notation any function which is analytic at the origin can be written in the form

$$f(z) = a_0 + a_1 z + \cdots + a_n z^n + [z^{n+1}],$$

where the coefficients are uniquely determined and equal to the Taylor coefficients of $f(z)$. Thus, in order to find the first n coefficients of the Taylor expansion, it is sufficient to determine a polynomial $P_n(z)$ such that $f(z) - P_n(z)$ has a zero of at least order $n + 1$ at the origin. The degree of $P_n(z)$ does not matter; it is true in any case that the coefficients of z^m, $m \leqq n$, are the Taylor coefficients of $f(z)$.

For instance, suppose that

$$\begin{aligned} f(z) &= a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n + \cdots \\ g(z) &= b_0 + b_1 z + b_2 z^2 + \cdots + b_n z^n + \cdots . \end{aligned}$$

With an abbreviated notation we write

$$f(z) = P_n(z) + [z^{n+1}]; \qquad g(z) = Q_n(z) + [z^{n+1}].$$

It is then clear that $f(z)g(z) = P_n(z)Q_n(z) + [z^{n+1}]$, and the coefficients of the terms of degree $\leqq n$ in P_nQ_n are the Taylor coefficients of the product $f(z)g(z)$. Explicitly we obtain

$$\begin{aligned} f(z)g(z) = a_0b_0 + (a_0b_1 + a_1b_0)z + \cdots \\ + (a_0b_n + a_1b_{n-1} + \cdots + a_nb_0)z^n + \cdots . \end{aligned}$$

In deriving this expansion we have not even mentioned the question of convergence, but since the development is identical with the Taylor development of $f(z)g(z)$, it follows by Theorem 3 that the radius of convergence is at least equal to the smaller of the radii of convergence of the given series $f(z)$ and $g(z)$. In the practical computation of P_nQ_n it is of course not necessary to determine the terms of degree higher than n.

In the case of a quotient $f(z)/g(z)$ the same method can be applied, provided that $g(0) = b_0 \neq 0$. By use of ordinary long division, continued until the remainder contains the factor z^{n+1}, we can determine a polynomial R_n such that $P_n = Q_nR_n + [z^{n+1}]$. Then $f - R_ng = [z^{n+1}]$, and since $g(0) \neq 0$ we find that $f/g = R_n + [z^{n+1}]$. The coefficients of R_n are the Taylor coefficients of $f(z)/g(z)$. They can be determined explicitly in determinant form, but the expressions are too complicated to be of essential help.

It is also important that we know how to form the development of a composite function $f(g(z))$. In this case, if $g(z)$ is developed around z_0, the expansion of $f(w)$ must be in powers of $w - g(z_0)$. To simplify, let us assume that $z_0 = 0$ and $g(0) = 0$. We can then set

$$f(w) = a_0 + a_1 w + \cdots + a_n w^n + \cdots$$

and $g(z) = b_1 z + b_2 z^2 + \cdots + b_n z^n + \cdots$. Using the same notations as before we write $f(w) = P_n(w) + [w^{n+1}]$ and $g(z) = Q_n(z) + [z^{n+1}]$ with $Q_n(0) = 0$. Substituting $w = g(z)$ we have to observe that

$$P_n(Q_n + [z^{n+1}]) = P_n(Q_n(z)) + [z^{n+1}]$$

and that any expression of the form $[w^{n+1}]$ becomes a $[z^{n+1}]$. Thus we obtain $f(g(z)) = P_n(Q_n(z)) + [z^{n+1}]$, and the Taylor coefficients of $f(g(z))$ are the coefficients of $P_n(Q_n(z))$ for powers $\leqq n$.

Finally, we must be able to expand the inverse function of an analytic function $w = g(z)$. Here we may suppose that $g(0) = 0$, and we are looking for the branch of the inverse function $z = g^{-1}(w)$ which is analytic in a neighborhood of the origin and vanishes for $w = 0$. For the existence of the inverse function it is necessary and sufficient that $g'(0) \neq 0$; hence we assume that

$$g(z) = a_1 z + a_2 z^2 + \cdots = Q_n(z) + [z^{n+1}]$$

with $a_1 \neq 0$. Our problem is to determine a polynomial $P_n(w)$ such that $P_n(Q_n(z)) = z + [z^{n+1}]$. In fact, under the assumption $a_1 \neq 0$ the notations $[z^{n+1}]$ and $[w^{n+1}]$ are interchangeable, and from $z = P_n(Q_n(z)) + [z^{n+1}]$ we obtain $z = P_n(g(z) + [z^{n+1}]) + [z^{n+1}] = P_n(w) + [w^{n+1}]$. Hence $P_n(w)$ determines the coefficients of $g^{-1}(w)$.

In order to prove the existence of a polynomial P_n we proceed by induction. Clearly, we can take $P_1(w) = w/a_1$. If P_{n-1} is given, we set $P_n = P_{n-1} + b_n w^n$ and obtain

$$\begin{aligned} P_n(Q_n(z)) &= P_{n-1}(Q_n(z)) + b_n a_1^n z^n + [z^{n+1}] \\ &= P_{n-1}(Q_{n-1}(z) + a_n z^n) + b_n a_1^n z^n + [z^{n+1}] \\ &= P_{n-1}(Q_{n-1}(z)) + P'_{n-1}(Q_{n-1}(z)) a_n z^n + b_n a_1^n z^n + [z^{n+1}]. \end{aligned}$$

In the last member the first two terms form a known polynomial of the form $z + c_n z^n + [z^{n+1}]$, and we have only to take $b_n = -c_n a_1^{-n}$.

For practical purposes the development of the inverse function is found by successive substitutions. To illustrate the method we determine the expansion of tan w from the series

$$w = \text{arc tan } z = z - \frac{z^3}{3} + \frac{z^5}{5} - \cdots .$$

If we want the development to include fifth powers, we write

$$z = w + \frac{z^3}{3} - \frac{z^5}{5} + [z^7]$$

and substitute this expression in the terms to the right. With appropriate remainders we obtain

$$\begin{aligned} z &= w + \frac{1}{3}\left(w + \frac{z^3}{3} + [w^5]\right)^3 - \frac{1}{5}(w + [w^3])^5 + [w^7] \\ &= w + \frac{1}{3}w^3 + \frac{1}{3}w^2 z^3 - \frac{1}{5}w^5 + [w^7] \\ &= w + \frac{1}{3}w^3 + \frac{1}{3}w^2(w + [w^3])^3 - \frac{1}{5}w^5 + [w^7] = w + \frac{1}{3}w^3 + \frac{2}{15}w^5 + [w^7]. \end{aligned}$$

Thus the development of tan w begins with the terms

$$\tan w = w + \frac{1}{3}w^3 + \frac{2}{15}w^5 + \cdots .$$

EXERCISES

1. Develop $1/(1 + z^2)$ in powers of $z - a$, a being a real number. Find the general coefficient and for $a = 1$ reduce to simplest form.

2. The Legendre polynomials are defined as the coefficients $P_n(\alpha)$ in the development

$$(1 - 2\alpha z + z^2)^{-\frac{1}{2}} = 1 + P_1(\alpha)z + P_2(\alpha)z^2 + \cdots .$$

Find P_1, P_2, P_3, and P_4.

3. Develop $\log(\sin z/z)$ in powers of z up to the term z^6.

4. What is the coefficient of z^7 in the Taylor development of $\tan z$?

5. The Fibonacci numbers are defined by $c_0 = 0$, $c_1 = 1$,

$$c_n = c_{n-1} + c_{n-2}.$$

Show that the c_n are Taylor coefficients of a rational function, and determine a closed expression for c_n.

1.3. The Laurent Series. A series of the form

$$b_0 + b_1z^{-1} + b_2z^{-2} + \cdots + b_nz^{-n} + \cdots \tag{1}$$

can be considered as an ordinary power series in the variable $1/z$. It will therefore converge outside of some circle $|z| = R$, except in the extreme case $R = \infty$; the convergence is uniform in every region $|z| \geqq \rho > R$, and hence the series represents an analytic function in the region $|z| > R$. If the series (1) is combined with an ordinary power series, we get a more general series of the form

$$\sum_{n=-\infty}^{+\infty} a_nz^n. \tag{2}$$

It will be termed convergent only if the parts consisting of nonnegative powers and negative powers are separately convergent. Since the first part converges in a disk $|z| < R_2$ and the second series in a region $|z| > R_1$, there is a common region of convergence only if $R_1 < R_2$, and (2) represents an analytic function in the annulus $R_1 < |z| < R_2$.

Conversely, we may start from an analytic function $f(z)$ whose region of definition contains an annulus $R_1 < |z| < R_2$, or more generally an annulus $R_1 < |z - a| < R_2$. We shall show that such a function can always be developed in a general power series of the form

$$f(z) = \sum_{n=-\infty}^{+\infty} A_n(z-a)^n.$$

The proof is extremely simple. All we have to show is that $f(z)$ can be written as a sum $f_1(z) + f_2(z)$ where $f_1(z)$ is analytic for $|z - a| < R_2$ and $f_2(z)$ is analytic for $|z - a| > R_1$ with a removable singularity at ∞. Under these circumstances $f_1(z)$ can be developed in nonnegative powers of $z - a$, and $f_2(z)$ can be developed in nonnegative powers of $1/(z - a)$.

To find the representation $f(z) = f_1(z) + f_2(z)$ define $f_1(z)$ by

$$f_1(z) = \frac{1}{2\pi i} \int_{|\zeta - a| = r} \frac{f(\zeta)\,d\zeta}{\zeta - z}$$

for $|z - a| < r < R_2$ and $f_2(z)$ by

$$f_2(z) = -\frac{1}{2\pi i} \int_{|\zeta - a| = r} \frac{f(\zeta)\,d\zeta}{\zeta - z}$$

for $R_1 < r < |z - a|$. In both integrals the value of r is irrelevant as long as the inequality is fulfilled, for it is an immediate consequence of Cauchy's theorem that the value of the integral does not change with r provided that the circle does not pass over the point z. For this reason $f_1(z)$ and $f_2(z)$ are uniquely defined and represent analytic functions in $|z - a| < R_2$ and $|z - a| > R_1$ respectively. Moreover, by Cauchy's integral theorem $f(z) = f_1(z) + f_2(z)$.

The Taylor development of $f_1(z)$ is

$$f_1(z) = \sum_{n=0}^{\infty} A_n(z - a)^n$$

with

$$A_n = \frac{1}{2\pi i} \int_{|\zeta - a| = r} \frac{f(\zeta)\,d\zeta}{(\zeta - a)^{n+1}}. \tag{3}$$

In order to find the development of $f_2(z)$ we perform the transformation $\zeta = a + 1/\zeta'$, $z = a + 1/z'$. This transformation carries $|\zeta - a| = r$ into $|\zeta'| = 1/r$ with negative orientation, and by simple calculations we obtain

$$f_2\left(a + \frac{1}{z'}\right) = \frac{1}{2\pi i} \int_{|\zeta'| = \frac{1}{r}} \frac{z'}{\zeta'} \frac{f\left(a + \frac{1}{\zeta'}\right) d\zeta'}{\zeta' - z'} = \sum_{n=1}^{\infty} B_n z'^n$$

with

$$B_n = \frac{1}{2\pi i} \int\limits_{|\zeta'|=\frac{1}{r}} \frac{f\left(a + \frac{1}{\zeta'}\right) d\zeta'}{\zeta'^{n+1}} = \frac{1}{2\pi i} \int\limits_{|\zeta - a| = r} f(\zeta)(\zeta - a)^{n-1}\, d\zeta.$$

This formula shows that we can write

$$f(z) = \sum_{n=-\infty}^{+\infty} A_n(z-a)^n$$

where all the coefficients A_n are determined by (3). Observe that the integral in (3) is independent of r as long as $R_1 < r < R_2$.

If $R_1 = 0$ the point a is an isolated singularity and $A_{-1} = B_1$ is the residue at a, for $f(z) - A_{-1}(z-a)^{-1}$ is the derivative of a single-valued function in $0 < |z - a| < R_2$.

EXERCISES

1. Prove that the Laurent development is unique.

2. Let Ω be a doubly connected region whose complement consists of the components E_1, E_2. Prove that every analytic function $f(z)$ in Ω can be written in the form $f_1(z) + f_2(z)$ where $f_1(z)$ is analytic outside of E_1 and $f_2(z)$ is analytic outside of E_2. (The precise proof requires a construction like the one in Chap. 4, Sec. 4.5.)

3. The expression

$$\{f,z\} = \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2$$

is called the *Schwarzian derivative* of f. If f has a multiple zero or pole, find the leading term in the Laurent development of $\{f,z\}$. *Answer:* If $f(z) = a(z - z_0)^m + \cdots$, then $\{f,z\} = \frac{1}{2}(1 - m^2)(z - z_0)^{-2} + \cdots$.

4. Show that the Laurent development of $(e^z - 1)^{-1}$ at the origin is of the form

$$\frac{1}{z} - \frac{1}{2} + \sum_{1}^{\infty} (-1)^{k-1} \frac{B_k}{(2k)!} z^{2k-1}$$

where the numbers B_k are known as the Bernoulli numbers. Calculate B_1, B_2, B_3. (By Sec. 2.1, Ex. 5, the B_k are all positive.)

5. Express the Taylor development of $\tan z$ and the Laurent development of $\cot z$ in terms of the Bernoulli numbers.