Tuesday, 30 July 2024 9:29 PM

Date : 06|08|2024
(In class)

$\sum_{n=0}^{\infty} a_n x^n$
$\Downarrow$
$\boxed{\sum_{n=0}^{\infty} a_n z^n = f(z)}$
$\Downarrow$
$|z-z_0|<R$

Taylor's Th^m :-

If $f(z)$ is an analytic function in the domain D, then for any point $z \in D$, we can represent
$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n,$$
where $a_n = \frac{f^n(z_0)}{n!}$

① We can use binomial expansion, $\sin z$, e^z, $\cos z$, $\log(1+z)$ ---

$\frac{1}{1-z} = 1+z+z^2+z^3+ ----$; $|z|<1$

$e^z = 1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+----$, $z\in$

$\sin z = z-\frac{z^3}{3!}+\frac{z^5}{5!} ---- z\in$

$\cos z =$ ——

$\log(1+z) = z-\frac{z^2}{2}+\frac{z^3}{3}-\frac{z^4}{4}+ --$ $|z|<1$

① R.O.C for analytic function $f(z)$ is the distance between the point z_0 & the nearest singularity of $f(z)$

$\frac{1}{(z-1)(z-2)} = \sum_{n=0}^{\infty} a_n (z-1/2)^n$

Problem :- Expand $f(z) = 1/z$ about $z=1$ in Taylor series & obtain its ROC.

Sol^n :- $f(z) = \frac{1}{z}$

$a_n = \frac{f^n(z_0)}{n!}$

$f'(z) = -\frac{1}{z^2}$

$f''(z) = \frac{(-1)(-2)}{z^3}$

$f'''(z) = \frac{(-1)(-2)(-3)}{z^4}$

$f^n(z) = \frac{(-1)^n n!}{z^{n+1}}$

at $z = z_0 = 1$

$a_n = \frac{f^n(z_0)}{n!}$

$= \frac{(-1)^n n!}{1^{n+1}. n!}$

$= (-1)^n$

$\frac{1}{z} = \sum_{n=0}^{\infty} a_n (z-1)^n$

$= \sum_{n=0}^{\infty} (-1)^n (z-1)^n$

$= 1-(z-1)+(z-1)^2-(z-1)^3+ ....$

other method
$f(z) = \frac{1}{z}$

$z-1 = t$ (say)

$\left(\frac{1}{t+1}\right) = (1+t)^{-1}$

$= 1-t+t^2-t^3+ ---$; $|t|<1$

$= 1-(z-1)+(z-1)^2-(z-1)^3+ ---$

$\sum_{n=0}^{\infty} a_n (z-z_0)^n$ $|z-1|<1$
$\downarrow$
$\boxed{\sum_{n=0}^{\infty} a_n t^n}$

Prob: Expand $f(z) = \frac{2i}{4+iz}$ about $z=-3i$ in Taylor series and obtain its ROC.

Sol^n :- $f(z) = \frac{2i}{4+iz}$

$= \frac{2}{-4i+z}$

Put $z+3i = t$

$= \frac{2}{-4i+t-3i}$

$= \frac{2}{t-7i}$

$= \frac{2}{-7i\left(1-\frac{t}{7i}\right)}$

$= \frac{2}{-7i}\left(1-\frac{t}{7i}\right)^{-1}$

$= \frac{-2i}{7i^2}\left(1+\frac{t}{7i}+\frac{t^2}{7^2 i^2}+\frac{t^3}{7^3 i^3}+ ---\right)$

$= \frac{2i}{7}\left(\sum_{n=0}^{\infty} \frac{1}{7^n i^n} t^n\right)$

$= \frac{2i}{7}\left(\sum_{n=0}^{\infty} \frac{1}{7^n i^n} (z+3i)^n\right)$

when $\left|\frac{t}{7i}\right|<1$

$|t|<|7i|$

$|t|<7$

$\boxed{|z+3i|<7}$

Prob: Expand $f(z) = \frac{1}{(1-z)^3}$ about $z=0$ in Taylor series & obtain its Roc.

Sol^n: $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$; $|z|<1$

$\frac{+1}{(1-z)} = \sum_{n=1}^{\infty} n z^{n-1}$; $|z|<1$

$\frac{2}{(1-z)^3} = \sum_{n=2}^{\infty} n(n-1) z^{n-1}$

$\frac{1}{(1-z)^3} = \sum_{n=2}^{\infty} \frac{n(n-1)}{2} z^{n-2}$

Prob: $f(z) = \log z$; $z_0 = -1+i$

Put $z+1-i = t$

$F(z) = \log[t-1+i]$

$= \log[(i-1)+t]$

$= \log\left[(i-1)\left(1+\frac{t}{i-1}\right)\right]$

$= \log(i-1)+\log\left(1+\frac{t}{i-1}\right)$; $\left|\frac{t}{i-1}\right|<1$

$= \log(i-1)+\left(\frac{t}{i-1}-\frac{t^2}{(i-1)^2}+\frac{t^3}{(i-1)^3} ---\right)$

$= \log(i-1)+\left(\frac{z-(-1+i)}{i-1}-\frac{[z-(-1+i)]^2}{(i-1)^2}+ ---\right]$

$|t|<|i-1|$

$|x+iy| = \sqrt{x^2+y^2}$

$\boxed{|z-(-1+i)|<\sqrt{2}}$

Prob :- $f(z) = \frac{1}{z^2+(1+2i)z+2i}$ about $z=0$
Sol^n : H.W

Laurent Series: The series expansion of the form
$$\underbrace{\sum_{n=0}^{\infty} a_n (z-z_0)^n}_{\text{Analytic Part}} + \underbrace{\sum_{n=1}^{\infty} b_n (z-z_0)^{-n}}_{\text{principal Part}}$$
is defined as Laurent Series.

① suppose $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ is convergent in $|z-z_0|<R_1=R$
& $\sum_{n=0}^{\infty} b_n (z-z_0)^{-n}$ is convergent in $\frac{1}{|z-z_0|}<R_2$
or $|z-z_0|>\frac{1}{R_2}=r$

② If $r<R$, then this complete series will be convergent $r<|z-z_0|<R$

③ If $r>R$, series will not converged.

Prob: $\sum_{n=0}^{\infty} \left(\frac{z}{3}\right)^n + \sum_{n=1}^{\infty} \left(\frac{1}{z}\right)^n$

$\left|\frac{z}{3}\right|<1 \Rightarrow |z|<3$; $|1/z|<1 \Rightarrow |z|>1$

$\Rightarrow \boxed{1<|z|<3}$ $|z-z_0|=1$

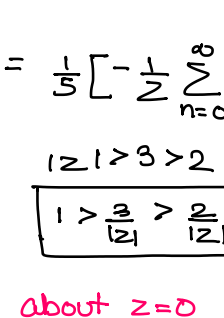
Series will converge

$= \frac{1}{1-z/3} + \frac{1/z}{1-\frac{1}{z}}$

$= \frac{3}{3-z} + \frac{1}{z-1}$

Laurent's Theorem:

A function $f(z)$ analytic in the annulus $r<|z-z_0|<R$ & on these bounding circles, can be expressed as a unique Laurent series expansion given by
$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{a_n}{(z-z_0)^n}$$
$$a_n = \frac{n!}{2\pi i}\int_c \frac{f(z)}{(z-z_0)^{n+1}}$$



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Laurent Series :-

$F(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} b_n (z-z_0)^{-n}$ $r<|z|<R$

Prob: Find all Taylor & Laurent Series expansion of $f(z) = \frac{1}{6-z-z^2}$

Sol^n :- $f(z) = \frac{1}{5}\left[\frac{1}{2-z}+\frac{1}{z+3}\right]$

(i) $|z|<2$ (ii) $2<|z|<3$ (iii) $|z|>3$

Sol(i) :- $\frac{1}{5}\left[\frac{1}{2-z}+\frac{1}{z+3}\right]$

$= \frac{1}{5}\left[\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}+\frac{1}{3}\left(1+\frac{z}{3}\right)^{-1}\right]$ $\frac{|z|}{2}<1$, $|z|<2<3$, $\frac{|z|}{3}<1$

$= \frac{1}{5}\left[\frac{1}{2}\left(\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n\right)+\frac{1}{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{3}\right)^n\right]$

Sol(ii): $2<|z| \Rightarrow \frac{2}{|z|}<1$
$|z|<3 \Rightarrow \frac{|z|}{3}<1$

$= \frac{1}{5}\left[-\frac{1}{z}\left(1-\frac{2}{z}\right)^{-1}+\frac{1}{3}\left(1+\frac{z}{3}\right)^{-1}\right]$

$= \frac{1}{5}\left[-\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n+\frac{1}{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{3}\right)^n\right]$

Sol(iii) : $|z|>3>2$

$\boxed{1>\frac{3}{|z|}>\frac{2}{|z|}}$

Problem: $e^{z+1/z}$ about $z=0$ (H.W)

Problem: $f(z) = \frac{7z-2}{z(z+1)(z-2)}$ in the domain $1<|z+1|<3$

Sol^n :- $z+1 = t$

$f(z) = \frac{7(t-1)-2}{(t-1)(t)(t-3)}$

$= \frac{-3}{t}+\frac{1}{t-1}+\frac{2}{t-3}$

Harmonic Functions

Definition: A real valued function $u(x,y)$ defined & single valued in a domain D, is said to be Harmonic in D, if it has cond^n first & second order partial derivative that satisfy
$$\boxed{\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0}$$

$f(z) = u+iv$
$F'(z) = u_x(x,y)+iv_x(x,y)$
$= U(x,y)+iV(x,y)$
$f''(z) = U_x(x,y)+iV_x(x,y)$
$= u_{xx}(x,y)+iv_{xx}(x,y)$

Thm :- If u is harmonic on a simply connected domain D, then there exist an analytic function on D where real part is u.

Proof :- Let $u(x,y)$ be a harmonic function. Then u_x, u_y, u_{xx}, & u_{yy} exist.

Let us define $g(z) = u_x - iu_y$

$g(z) = U(x,y)+iV(x,y)$

$\frac{\partial U}{\partial x} = u_{xx}$ $\frac{\partial U}{\partial y} = u_{xy}$

$\frac{\partial V}{\partial x} = -u_{xy}$ $\frac{\partial V}{\partial y} = -u_{yy}$

$\boxed{\begin{array}{l} F(z)=u+iv \\ f'(z)=u_x+iv_y \end{array}}$

$U_x - V_y = u_{xx}+u_{yy}$
$= 0$

Now, $V_y + V_x = u_{xy} - u_{xy}$
$= 0$

Since U & V satisfy C-R eq^n & Cond^n, $g(z) = u_x + iu_y$ is an analytic f^n.

* Since g is an analytic function, then for $z_0 \in D$
$$F(z) = \int_{z_0}^{z} g(t)\,dt$$
is analytic.

$F'(z) = g(z)$

$F'(z) = u_x(x,y) - iu_y(x,y)$

$\frac{\partial}{\partial x}[\text{Re}\, F(z)] + i\frac{\partial}{\partial y}[\text{Im}\, f(z)] = \frac{\partial}{\partial x}(u(x,y)) - i\frac{\partial}{\partial y}(u(x,y))$ $f(z)=u+iv$, $f'(z)=u_x+iv_x$

Compare the real & imaginary parts

$\frac{\partial}{\partial x}[\text{Re}\, f(z)] = \frac{\partial}{\partial x}[u(x,y)]$

$\text{Re}\, F(z) = u(x,y)+c$

$\boxed{\text{Re}\ F(z)-c = u(x,y)}$

Thm :- If u is harmonic on a simply connected Domain D, then there exist an analytic function whose imaginary part equals u.

Proof :- Since u is a harmonic function, therefore the exist (∃) an analytic function $h(z)$ such that $\text{Re}\, h(z) = u$

$h(z) = u+iv$

$ih(z) = iu - v$

$h(z)$ is analytic so $f(z) = ih(z)$ will again an analytic function

$f(z) = iv - v$

Hence analytic function $f(z)$ has imag. part u.

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Ex: Let $u(x,y) = \sin x \cosh y + \cos x \sinh y + x^2 - y^2 + 2xy$

Clearly $u(x,y)$ is a harmonic functions

since $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$

$g'(y) = u_x - iu_y$

$g'(y) = (\cos x \cosh y - \sin x \sinh y + 2x + 2y) - i(\sin x \sinh y + \cos x \cosh y - 2y + 2x)$

Note: $\left\{\begin{array}{l}\sin ix = i \sinh x \\ \cos ix = \cosh x\end{array}\right\}$

$g'(x) = \cos x \cos iy - \frac{\sin x \sin iy (i)}{i \cdot i} + 2x + 2y - i \sin x \frac{\sin iy}{i}$
$- i\cos x \cos iy + 2iy - 2ix$

$= \cos x \cos iy + i \sin x \sin iy + 2x + 2y - \sin x \sin y$
$- i\cos x \cos iy + 2iy - 2ix$

$= \cos(x+iy) - i(\cos(x+iy)) + 2x + 2y + 2iy - 2ix$

$= (1-i)(\cos z) + 2(x+iy) - 2i(x+iy)$ $(\because z = x+iy)$

$= (1-i)\cos z + 2(1-i)z$

$g'(z) = (1-i)(\cos z + 2z)$

$\boxed{g(z) = (1-i)(\sin z + z^2) + c}$

A non constant entire function cannot be bounden in C.
$\downarrow$
Analog for harmonic functions.

Thm 2: A function harmonic & bounded in C must be a constant.

Proof :- Let u be a harmonic function. So, the exist an analytic (g) function whose real part is u.

Next define $f(z) = e^{g(z)}$

$|f(z)| = e^{|g(z)|}$

$= e^{|u|}$

Since u is bounded function, so $f(z)$ will also be a banded function. Hence by Liouville's thm, $f(z)$ is a constant function. Hence $u(x,y)$ is also a constant function.

(H.W)
Thm :- If the real or img. part of an entire function is bounded above or below then the function is constant.

Mean Value Property : Suppose $u(z)$ of harmonic in a domain containing the distance $|z-z_0| \leq R$. Then
$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + Re^{i\theta})\,d\theta$$

Proof: Let $f(z)$ be an analytic function in $|z-z_0| \leq R$ whose real part is u.

So, by Mean value th^m for analytic function
$$f(z_0) = \frac{1}{2\pi}\int_0^{\pi} f(z_0 + Re^{i\theta})\,d\theta$$

$u(z) + iv(z_0)$

$= \frac{1}{2\pi}\int_0^{2\pi}[u(z_0+Re^{i\theta}) + iv(z_0+Re^{i\theta})]\,d\theta$

By de Comparison of real & img parts

$\boxed{u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + Re^{i\theta})\,d\theta}$

Uniqueness th^m (Analytic functions)

Let $f(z)$ is analytic in a domain D and $\langle z_n \rangle$ is a seq^n of distinct points in D converging to $z_0 \in D$

If $f(z_n) = 0$ for each n, then
$\boxed{f(z)=0}$ $\forall\, z \in D$

Thm (Harmonic Function) : If $u(z)$ is harmonic in D and constant in the nbd of some point z_0 in D, then $u(z)$ is constant throughout D.

Proof :- Let $A = \{z \in D ; u(z) = c\}$ be the set of all points in D in the nbd of z_0 for which u is constant

Clearly, A is non-empty & open set in D.

Let $\langle z_n \rangle$ be a seq^n in A and $\langle z_n \rangle$ converges to some point $z_0 \in D$

Since, u is a constant function therefore

$\lim_{n\to\infty} u(z_n) = u(z_0)$

$\boxed{C = u(z_0)}$

Hence A is closed set.

Now, we have proved that A is open & closed set in domain D (connected domain), Hence

$\boxed{A = D}$

Thm : (Maximum Principle for Harmonic Function)

A nonconstant harmonic f^n cannot attain its maximum or minimum in a domain.

Proof :- Let $u(z)$ be a non-constant harmonic function in domain D. Since u is harmonic therefore ∃ an analytic function, say $g(z)$ such that $g(z) = u+iv$

Consider $f(z) = e^{g(z)}$

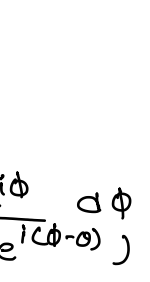
$|f(z)| = e^{u(z)}$

Let $z_0 \in D$ be the maximum point of $u(z)$. Then $|f(z)|$ attain its maximum value as z_0, which implies that $f(z)$ is a constant f^n. Hence, $u(z)$ will also be a constant function, which contradicts our hypothesis.

Cauchy's integral formula :-
$$f(z_0) = \frac{1}{2\pi i}\int_c \frac{f(z)}{z-z_0}\,dz$$

Inverse Point/ Symmetric Point/ Conjugate Point

Let r be a circle with center C & radius r. Two points P & Q are symmetric if
(i) C, P, Q are colinear
(ii) P & Q are on the same side of C.
(iii) $CP \cdot CQ = r^2$



* Let $|z-z_0| = r$ be a circle. z_1 & z_2 are colinear iff
$$z_2 = z_0 + \frac{r^2}{\bar{z}_1 - \bar{z}_0}$$

Thm :- Poisson Integral formula for Analytic function :-

Suppose $f(z)$ is analytic in a domain containing the disk $|z| \leq R$. Then
$$f(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta-\phi) + r^2} f(Re^{i\phi})\,d\phi,$$
where $z_1 = re^{i\theta}$ is any point of the domain $|z|<R$.

Proof :- Let $f(z)$ be analytic function in the domain containing the disk $|z| \leq R$ and $z_1 = re^{i\theta}$ be any point inside the disk $|z|<R$. Then by Cauchy integral formula $\frac{1}{2\pi i}\int_c \frac{f(z)}{z-z_1}\,dz = f(z_1)$ — ①

Now, inverse point of z_1 is $\frac{R^2}{\bar{z}_1}$, which lies outside the circle.

$\frac{1}{2\pi i}\int_c \left(\frac{f(z)}{z-\frac{R^2}{\bar{z}}}\right)dz = 0$ — ②

② − ①

$f(z_1) = \frac{1}{2\pi i}\int_c \frac{(R^2 - z_1\bar{z}_1)f(z)}{(z-z_1)(R^2 - z\bar{z}_1)}\,dz$

Now, $z_1 = re^{i\theta}$ & $z = Re^{i\theta}$

$dz = iRe^{i\theta}\,d\theta$

$f(re^{i\theta}) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{(R^2-r^2) + (Re^{i\phi})\,iRe^{i\phi}}{(Re^{i\phi} - re^{i\theta})(R^2 - R \cdot re^{i(\phi-\theta)})}\,d\phi$

$\boxed{f(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2-r^2)}{R^2 - 2Rr\cos(\theta-\phi)+r^2} \cdot f(Re^{i\phi})\,d\phi}$

Thm: Poisson Integral formula for Harmonic f^n

Suppose $u(z)$ is harmonic in a domain containing the disk $|z| \leq R$. Then for $\zeta = re^{i\theta}$, $r<R$,
$$u(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \left(\frac{(R^2-r^2)}{R^2 - 2rR\cos(\theta-\phi)+r^2}\right) u(Re^{i\phi})\,d\phi$$

Poisson Kernal : The expression (with $z = Re^{i\phi}$, $\zeta = re^{i\theta}$)
$$P(\zeta, z) = \frac{|z|^2 - |\zeta|^2}{|z-\zeta|^2} = \text{Re}\left(\frac{z+\zeta}{z-\zeta}\right)$$
$$= \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta-\phi)+r^2}$$
is known as Poisson kernal for the disk $|z|<R$.

* So we can write
$$f(\zeta) = \frac{1}{2\pi i}\int_{|z|=R} P(\zeta, z) f(z)\,d\phi$$

* $u(\zeta) = \frac{1}{2\pi i}\int_{|z|=R} P(\zeta, z)\, u(z)\,d\phi)$

* A harmonic function in $|z|<R$ and contains for $|z| \leq R$, has the property that its value inside the disk can be determined by its value on the boundary.

Schwarz's Thm: Let F be a constant function of a real variable defined on the unit circle $|z|=1$. Then the real valued function
$$u(\zeta) = \frac{1}{2\pi}\int_0^{2\pi} P(\zeta, e^{i\phi}) F(e^{i\phi})\,d\phi \quad (|\zeta|<1)$$
is harmonic in the disk $|z|<1$, and for each fixed t, $0 \leq t \leq 2\pi$,
$$\lim_{z\to e^{it}} u(\zeta) = \lim_{\substack{r\to 1 \\ \theta\to t}} u(re^{i\theta}) = f(e^{it})$$

(05|09|2024) The expression (with $z = Re^{i\phi}$ & $\zeta = re^{i\theta}$) $(r<R)$

$\boxed{P(\zeta, z) = \text{Re}\left(\frac{z+\zeta}{z-\zeta}\right)}$
ζ (zeta)

Schwarz's th^m :- Let f be a constant function of a real variable defined on the unit circle $|z|=1$. Then the real valued function
$$u(\zeta) = \frac{1}{2\pi}\int_0^{2\pi} P(\zeta, e^{i\phi}) F(e^{i\phi})\,d\phi$$
is harmonic in the disk $|z|<1$.

Proof :- $u(\zeta) = \text{Re}\left[\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{e^{i\phi}+\zeta}{e^{i\phi}-\zeta}\right) F(e^{i\phi})\,d\phi\right]$

$= \text{Re}\left[\frac{1}{2\pi}\int_{|z|=1}\left(\frac{z+\zeta}{z-\zeta}\right) F(z)\frac{dz}{iz}\right]$ $z = e^{i\phi}$, $dz = ie^{i\phi}\,d\phi$

$= \text{Re}\left[\frac{1}{2\pi i}\int_{|z|=1}\left(\frac{z+\zeta}{z(z-\zeta)}\right) f(z)\,dz\right]$

$= \text{Re}\left[\frac{1}{2\pi i}\left\{\int_{|z|=1}\left(\frac{2}{z-\zeta}-\frac{1}{z}\right) f(z)\,dz\right\}\right]$

$u(\zeta) = \text{Re}\left[\frac{1}{2\pi i}\left\{\int_{|z|=1}\frac{2f(z)}{z-\zeta}\,dz - \int_{|z|}\frac{f(z)}{z-0}\,dz\right\}\right]$

$= \text{Re}\left[\frac{1}{2\pi i}\left\{2\pi i f(\zeta) - 2\pi i f(0)\right\}\right]$

$\boxed{u(\zeta) = \text{Re}\left[f(\zeta) - f(0)\right]}$

Since, $f(\zeta)$ is an analytic function Therefore $u(\zeta)$ is harmonic f^n in $|z|<1$.

Sequence of functions :-

Let $D \subseteq \mathbb{C}$ and let for every $n \in \mathbb{N}$,
$f_n : D \to \mathbb{C}$
is a function. Then $\langle f_n(z) \rangle$ is a sequence of functions defined on D.

Ex. (i) $f_n(z) = z^n$; $|z|<1$
(ii) $f_n(z) = \frac{1}{1+nz}$; $|z|<1$
(iii) $\sin nz$, $\cos nz$, $\frac{nz}{1+n^2z^2}$

Q.(ii) $f_n(z) = \frac{1}{1+nz}$; $|z|<1$

for $z=0$

$f_n(z) = 1$

$\lim_{n\to\infty} f_n(z) = 1$ for $z=0$

for $z \neq 0$

$f_n(z) = \frac{1}{1+nz}$

$\boxed{\lim_{n\to\infty} f_n(z) = 0}$

$\boxed{\lim_{n\to\infty} f_n(z) \neq 0 \ \forall\, |z|<1}$

Pointwise Convergence: A sequence of functions $\langle f_n(z) \rangle$ is said to be pointwise converge to $f(z)$ if for every $\varepsilon>0$, there exist an integer $N(\varepsilon, z)$ such that
$$|f_n(z) - f(z)| < \varepsilon \quad \forall\, n \geq N(\varepsilon, z)$$

* It is denoted by $f_n(z) \to f(z)$

Uniform Convergence :- A seq^n of functions $\langle f_n(z) \rangle$ is said to be uniformly converge to $f(z)$ if for every $\varepsilon>0$, there exist an integer $N(\varepsilon)$ such that
$$|f_n(z) - f(z)| < \varepsilon \quad \forall\, n \geq N(\varepsilon)$$

* It is denoted by $f_n(z) \rightrightarrows f(z)$

Ex :- $f_n(z) = \frac{1}{nz}$; $0<|z|<1$

$|f_n(z) - 0| = \left|\frac{1}{nz}\right|$

$= \frac{1}{n|z|} < \varepsilon$

whenever $n > \frac{1}{\varepsilon|z|}$

If we chose any integer greater then $\frac{1}{\varepsilon|z|}$, then $|f_n(z) - 0| < \varepsilon$ for all $n > N \geq N_0 = \frac{1}{\varepsilon|z|}$

Ex: $f_n(z) = \frac{1}{1+nz}$; $|z| \geq 2$.

Sol^n: $|f_n(z) - 0| = \frac{1}{|1+nz|}$

$(\because |z_1+z_2| \geq |z_1| - |z_2|)$

$\left(\frac{1}{|z_1+z_2|} \leq \frac{1}{|z_1|-|z_2|}\right)$

$\frac{1}{|1+nz|} \leq \left(\frac{1}{n|z|-1}\right) \leq \frac{1}{2n-1}$

$|f_n(z) - 0| < \frac{1}{2n-1} < \varepsilon$

whenever $\varepsilon > N = \frac{1}{2n-1}$

If we choose any integer greater then $\frac{1}{2n-1}$! then

$|f_n(z) - 0| < \varepsilon$ for all $n > N \geq N_0 = \frac{1}{2n-1}$

Hence $\boxed{f_n(z) \rightrightarrows 0}$

Weierstrass's th^m :- Let $\langle f_n(z) \rangle$ be a seq^n of functions that are analytic in $D \subseteq \mathbb{C}$. If the seq^n $\langle f_n(z) \rangle$ converges uniformly to $f(z)$ on D, then the limit function is also analytic in D & the seq^n of derivatives

$\boxed{f_n'(z) \rightrightarrows f'(z)}$

Thm: Let $D \subset \mathbb{C}$ & $\langle f_n(z) \rangle$ be a seq^n of functions defind on D.
Let
$M_n = \sup_{z \in D} |f_n(z) - f(z)|$
Then
$f_n(z) \rightrightarrows f(z)$ iff $\boxed{\lim_{n\to\infty} M_n = 0}$

Morera's th^m : If a cond^n function defined on a domain $D \subseteq$ and for every closed curve γ in D

$\int_\gamma f(z)\,dz = 0$, then $f(z)$ is analytic in D.

(Only use)

Proof : Since $\langle f_n(z) \rangle$ is a seq^n of analytic function, by Morera's thm, for any closed curve γ in D, we have
$$\int_\gamma f_n(z)\,dz = 0$$

$\lim_{n\to\infty}\int_\gamma f_n(z)\,dz = 0$

Since $f_n(z) \rightrightarrows \gamma$, therefore

$\int_\gamma \lim_{n\to\infty} f_n(z)\,dz = 0$

$\int_\gamma f(z)\,dz = 0$

Again by Morera's th^m, $f(z)$ is an analytic function.

$|f_n'(z) - f'(z)| = \left|\frac{1}{2\pi i}\int_\gamma \frac{f_n(\zeta)}{(\zeta-z)^2}\,d\zeta - \frac{1}{2\pi i}\int_\gamma \frac{d(\zeta)}{(\zeta-z)^2}\,d\zeta\right|$

$= \frac{1}{2\pi}\left|\int_\gamma \frac{f_n(z) - f(z)}{(\zeta-z)^2}\,d\zeta\right|$

$= \frac{1}{2\pi}\int_\gamma \frac{|f_n(z) - f(z)|}{|\zeta - z|^2}\,|d\zeta|$

Since $f_n(z) \rightrightarrows f(z)$, therefore $|f_n(z) - f(z)|$ can be arbitrarily small, hence

$|f_n'(z) - f'(z)| \to 0$ for sufficiently large n. therefore

$\boxed{f_n'(z) \rightrightarrows f'(z)}$

(UNIT 1 over)