9|Nov|24

Periodic f^n — A f^n $f: \mathbb{C} \to \mathbb{C}$ is said to be periodic if $\exists$ a complex no. ω such that

$$f(z+\omega) = f(z) \ \forall z \in \mathbb{C}$$

(attains a value ∞ly many times)

Eg- $\sin z$, $\cos z$, e^z

$$e^{z+2\pi i} = e^z \cdot e^{2\pi i}$$

Solve $e^z = a+bi$

Q- $e^x \cos y = a$ $\quad e^x \sin y = b$

(1^2+2^2)

$y = \tan^{-1}\frac{a}{b}$

$$\left[y = \tan^{-1}\left(\frac{b}{a}\right) + 2n\pi \right] \quad e^{2x} = a^2+b^2$$

$$\left[x = \frac{1}{2}\log(a^2+b^2) \right]$$

Q- $e^z = 1$

$e^x \cos y = 1$ $\quad e^x \sin y =$

$0 + 2\pi n i$

Q- $(\cos z = 2)$ $\frac{e^{iz}+e^{-iz}}{2} = 2$

$(e^{iz}+e^{-iz} = 4)$

$(x + \frac{1}{x} = 4)$

$$e^{iz} = -b+ai$$

$$e^{2iz} + 1 = 4e^{iz}$$

$$e^{2iz} - 4e^{iz} + 1 = 0$$

$$e^{iz} = \frac{4 \pm \sqrt{16-4.1}}{2}$$

$$e^{iz} = 2 \pm 2\sqrt{3}$$

$$e^{-y}\,(\cos x + i\sin x) = 2 \pm 2\sqrt{3}$$

$$\cancel{e^{-y}} \sin x = 0$$

$$x = \underline{\underline{0 + 2n\pi}}$$

$$e^{-y} = 2 \pm 2\sqrt{3}$$

$$-y = \log(2 \pm 2\sqrt{3})$$

$$y = \cancel{-} \log\left(\frac{1}{2 \pm 2\sqrt{3}}\right)$$

$$\left(y = \log\left(\frac{1}{2+2\sqrt{3}}\right)\right)$$

$\Rightarrow$ Simply periodic f^n:-

- Integral period
- Linear comb^n of period

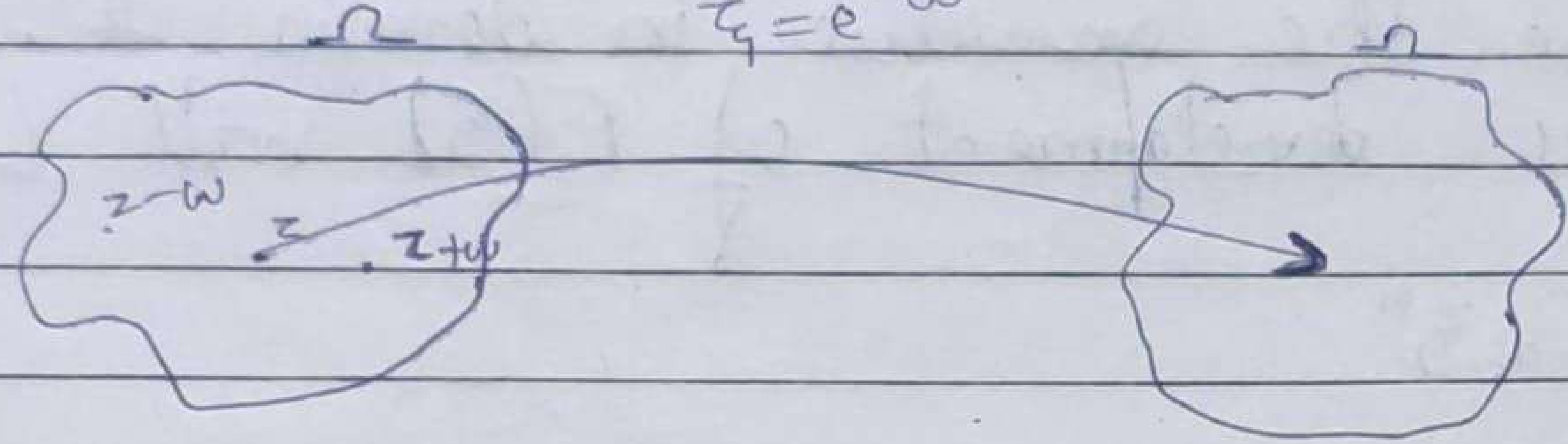
Representation of Exponentials :-

The simplest periodic function with period ω is $e^{\frac{2\pi i z}{\omega}}$

$$e^{\frac{2\pi i (z+\omega)}{\omega}} = e^{\frac{2\pi i z}{\omega}} \cdot e^{2\pi i}$$

$$= e^{\frac{2\pi i z}{\omega}}$$

$$\zeta = e^{\frac{2\pi i z}{\omega}}$$



Case (i) $\Omega = \mathbb{C}$

$e^{\left|\frac{2\pi i z}{\omega}\right|}$ covers the whole plane except point 0.

Case (ii) $a' < \left|\frac{2\pi i z}{\omega}\right| < b'$

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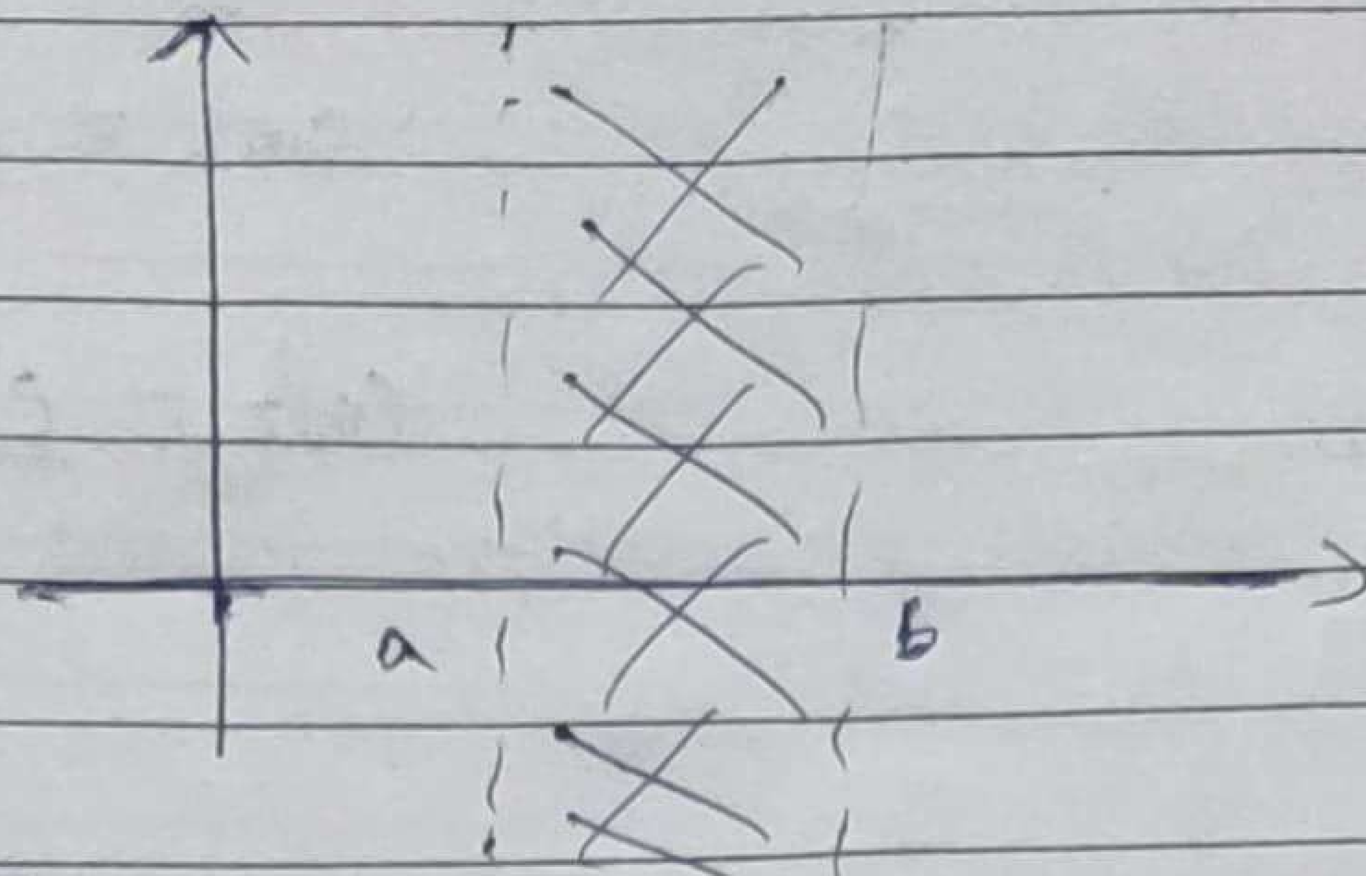

$$|\xi| = e^{\left|\frac{2\pi i z}{\omega}\right|}$$

$$\boxed{e^{a'} < |\xi| < e^{b'}}$$

$$\xi = e^{\frac{2\pi i z}{\omega}}$$

$$|\xi| = e^{\left|\frac{2\pi i z}{\omega}\right|}$$

<u>Fourier Development</u>:-

* Suppose $f(z)$ is meramorphic function in Ω with period ω. Then there exist a unique function F in Ω' such that

$$\boxed{f(z) = F(\xi) = F\left(e^{\frac{2\pi i z}{\omega}}\right)}$$

Moreover, F will be meramorphic.

* Let $r_1 < |\xi| < r_2$ be the annulus in domain Ω'. Then the Laurent Series development of $F(\xi)$ will be

$$F(\xi) = \sum_{n=-\infty}^{\infty} C_n \xi^n$$

where $C_n = \dfrac{1}{2\pi i} \displaystyle\int_{|\xi|=r} \dfrac{F(\xi)}{\xi^{n+1}} d\xi \; ; \quad r_1 < r < r_2$

$$\cancel{\xi = \frac{2\pi i z}{\omega}} \qquad \xi = e^{\frac{2\pi i z}{\omega}}$$

$$d\xi = e^{\frac{2\pi i z}{\omega}} \cdot \frac{2\pi i}{\omega} dz$$

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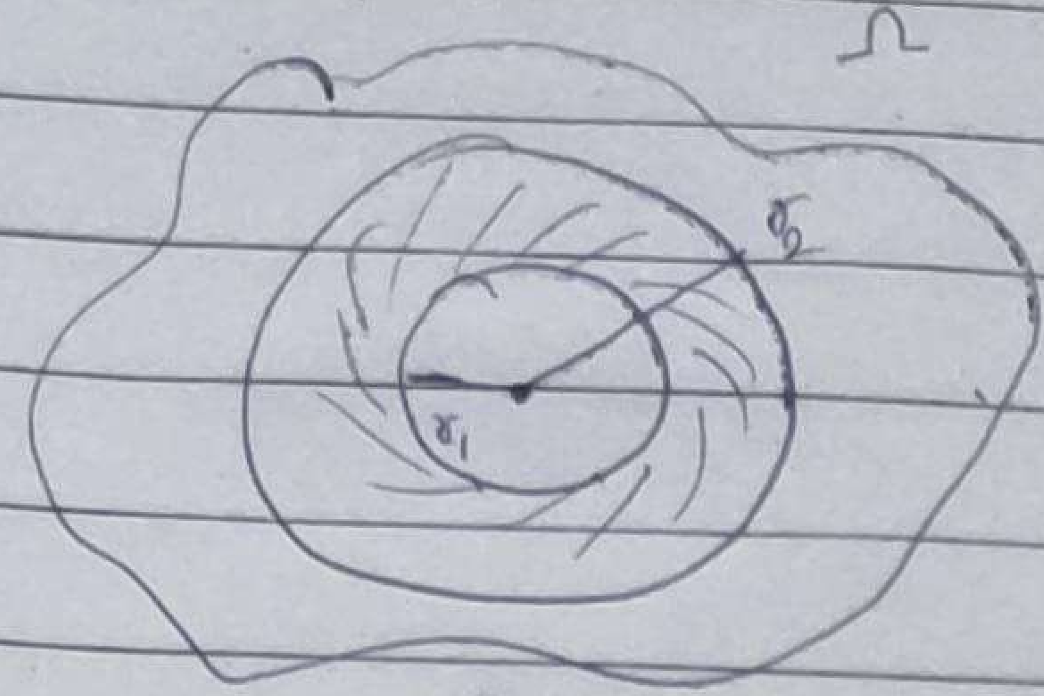

c_n

$$f(z) = \cancel{F(\xi)} = \frac{1}{2\pi i}\int_a^{a+\omega} \frac{f(z)}{e^{\frac{2\pi i(n+1)z}{\omega}}}\, e^{\frac{2\pi i z}{\omega}}\, dz \cdot \frac{2\pi i}{\omega}$$

$$\cancel{f(z) = \frac{1}{\omega}\int}$$

$$\boxed{C_n = \frac{1}{\omega}\int_a^{a+\omega} \frac{f(z)}{e^{\frac{2\pi i n z}{\omega}}}\, dz}$$

From ①, $\boxed{f(z) = F(\xi) = \sum_{n=0}^{\infty} C_n\, e^{\frac{2\pi n i z}{\omega}}}$ — ②

This is called Fourier series development of $f(z)$.