# Unit-3

* $\sigma$-field : F (σ algebra) is a family of subset of $\Omega$ (sample space) which satisfies following properties :

① $\phi \in F$

② if $A \in F$ then $A^c \in F$

③ if $A_1, A_2, \ldots$ are in F and is countable sequence, then
$\bigcup_{i \in N} A_i \;/\; \sum_{i}^{n} A_i \in F$

eg A Coin is tossed 3 times

$\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

$A_1 = \{HHH, HHT, HTH, HTT\}$ (first coin is head)

$A_2 = \{THH, THT, TTH, TTT\}$ ( " " " tail)

$F_1 = \{\phi, \Omega\}$

$F_2 = \{\phi, \Omega, A_1, A_2\}$

$A_{11} = \{HHH, HHT\}$

$A_{12} = \{HTH, HTT\}$

$F_3 = \{\phi, \Omega, A_{11}, A_{12}, A_2, A_{12} \cup A_2, A_{11} \cup A_2, A_{11} \cup A_{12}\}$

$A_{111} = \{HHH\}$

$A_{112} = \{HHT\}$

$F_4 = \{\phi, \Omega, A_{111}, A_{112}, A_{12}, A_2, (A_{112} \cup A_{12} \cup A_2 = A_{111}^c),$

$(A_{111} \cup A_{12} \cup A_2 = A_{112}^c),$

$(A_{111} \cup A_{112} \cup A_{12} = A_2^c)\}$

confirm $\rightarrow$ $(A_{111} \cup A_{112} \cup A_2 = A_{12}^c)$

Q// A die is thrown once ~~such that it contains 3~~ ~~disjoint sets~~ with a-field on 3 disjoint subsets, How many a-fields can we generate.

Ans:

# Stochastic Process

$\Rightarrow$ let $(\Omega, F, P)$ where $P$ is the probability measure defined on $F$ be a given probability space, a collection of random variables $\{X_t, t \in T\}$

$T$ = index set/{time index} defined on the probability space $(\Omega, F, P)$

is called a Stochastic process.

The Stochastic process is function of two variables which are independent.

$$\{X_t(\omega),\ t \in T,\ \omega \in \Omega\}$$

$$X : T \times \Omega \longrightarrow R$$

$X(\cdot, \omega) \longrightarrow$ t is varying, fixed $\omega$, the trajectory is called sample path

$X(t, \cdot)$

↳ fixed t in a random variable.

let $\{X_t,\ t \in T\}$ be a given stochastic process

the set $\{t \in T\}$ is called a parameter space.

the collection of all possible values of over index set

$X_t\ \forall\ t \in T$ is called state space.

→ Discrete time Discrete Space

→ Continuous time Discrete Space

→ Discrete time Continuous Space

→ Cont. time, " "

whenever ~~discrete~~ State space/parameter space is finite or countably infinite then it is said to have discrete nature when it is represented on ~~number~~ real line ~~oth~~

it is continuous.

→ independent increament

For all $n$ and $t_1 < t_2, \dots < t_n$

the random variable

$X(t_2) - X(t_1),\ X(t_3) - X(t_2),\ \dots,\ X(t_n) - X(t_{n-1})$

are independent R.V. then the process is said to have independent increament.

⟹ Strict Sense Stationary process (Strong Stationary process)

The stochastic process $\{X(t) : t \geq 0\}$ is a SSSP if for arbitary $0 < t_1 < t_2 \dots < t_n$ the finite dimensional ~~radom~~ random vector

$(X(t_1), X(t_2), \dots, X(t_n))$ &

$(X(t_1 + h); X(t_2 + h), \dots, X(t_n + h))$ have

same joint distribution for all $h > 0$

and all $t_1, t_2, \dots, t_n$

$\Longrightarrow$ Wide sense stationary process

The SP $\{X(t) ; t \geq 0\}$ is WSSP if it satisfies the following conditions.

(i) $E(X(t)) = \mu(t)$ is independent of $t$

(ii) $Cov(X(t), X(s))$ depends only on the time diff. $|t-s|$ for all $t$ and $s$

(iii) $E(X^2(t)) < \infty$ (finite second order moment)

A WSSP is also called covariance stationary or weak stationary or second order stationary

Q $X(t) = A \cos \omega t + B \sin \omega t$ where $A$ and $B$ are uncorrelated R.V with

$$\begin{cases} E(A) = E(B) = 0 \\ Var(A) = Var(B) = 1 \end{cases}$$

$\omega$ is +ve constant

A $E(X(t)) = \cos\omega t \, E(A) + \sin \omega t \, E(B)$

$E(X(t)) = 0$ ✓ independent of $t$

$$Cov(X_t, X_s) = E(X_t . X_s)$$

$$= E\{(A \cos \omega t + B \sin \omega t). (A \cos \omega s + B \sin \omega s)\}$$

$$= \cos\omega t \cos \omega s \, E(\overset{1}{A^2}) + \sin \omega t \sin \omega s \, E(\overset{1}{B^2})$$

$$+ \sin(\omega t + \omega s) E(\overset{0}{AB})$$

$= \cos(\omega t - \omega s)$ , $t > s$

$= \cos(\omega(t-s))$

↳ func. of $t-s$

(iii) $E(X^2) - \cancel{E(X)}^2 = Var(X)$

$\downarrow$

$0$

$E(X^2) = Var(X)$

$E(X^2(t)) = Var(X(t)) < \infty$

## Conditional Expectation

- conditional distribution of X given

$$F(X = x \mid Y = y) = P\{X \le x \mid Y = y\}$$

⇒ Conditional expression of X

$$E(X \mid Y = y) = \int_{-\infty}^{\infty} x \, dF(X = x \mid Y = y)$$

If X and Y are R. V, then ~~E(X)~~

$$E(X) = \sum_{y} E(X \mid Y = y) \cdot P(Y = y)$$ {proof from Notes}

Q. 2 refill's for ball pen are selected at random from a box containing 3 blue, 2 red and 3 green refills

$x$ = no. of blue refill selected

$y$ = no. of red " "

(a) Find joint prob. distribution.

~~(b) The prob $\{(x,y) \in A\}$~~

(b) $P\{(X,Y);(X,Y) \in A\}$ where $A: X+Y \leq 1$

(c) ~~Expected val.~~ $E(X \mid Y=1)$

(d) Conditional dist. of $X$ given $Y=1$

Ans. (a) $R_X = \{0, 1, 2\}$

$R_Y = \{0, 1, 2\}$

| X \ Y | 0 | 1 | 2 | h(x) |
|---|---|---|---|---|
| 0 | 3/28 | 6/28 | 1/28 | 10/28 |
| 1 | 9/28 | 6/28 | 0 | 15/28 |
| 2 | 3/28 | 0 | 0 | 3/28 |
| k(y) | 15/28 | 12/28 | 1/28 | 1 |

(b) $A = \{(0,0),(0,1),(1,0)\}$

$$P((x,y) \in A) = P(0,0) + P(0,1) + P(1,0)$$

$$= \frac{18}{28}$$

(d) $P(X \mid Y=1) = f(X \mid Y=1)$ mass function

$$= \frac{P(X,Y)}{P(Y)} = \frac{P(X,Y=1)}{P(Y=1)} = \frac{28}{12} P(x,1)$$

$$f(x \mid y=1) = \frac{28}{12} P(x, 1)$$

$$f(0 \mid y=1) = \frac{28}{12} P(0,1) = \frac{6}{12} = \frac{1}{2}$$

$$f(1 \mid y=1) = \frac{28}{12} P(1,1) = \frac{6}{12} = \frac{1}{2}$$

$$f(2 \mid y=1) = \frac{28}{12} P(2,1) = 0$$

(c) $E(X \mid y=1) = \sum_x x\, f(x \mid y=1)$

$$= 0 \cdot f(0 \mid y=1) + 1 f(1 \mid y=1) + 2 f(2 \mid y=1)$$

$$\boxed{E(X \mid y=1) = \frac{1}{2}}$$

**Thm.** Let a stochastic process $\{S_K\}$ ; $K = 0, 1, 2, \ldots n$ be a symmetric random walk. Then,

(i) for each K, $E(S_K) = 0$ ; $Var(S_K) = K$

(ii) it has independent increament

(iii) it has stationary "

(iv) It is a markov process

<u>Proof</u>

(i) $$S_k = \sum_{j=1}^{k} X_j$$

$$E(S_k) = E\left(\sum X_j\right) = \sum E(X_j)$$

$$\because E(X_j) = 0 \;;\; Var(X_j) = 1$$

$$\boxed{E(S_k) = 0}$$

$$Var(S_k) = Var\left(\sum_{j=1}^{n} X_j\right)$$

$$= \sum_{j=1}^{k} Var\, X_j$$

$$= k$$

(ii) we choose an arbitary integer $n$, and then choose non-negative integers

~~then~~ $\cancel{S_{k_{i+1}} - S_{k_i} = \sum_{S_{k_i}+1}^{k_{i+1}}}$

Then
$$S_{k_{i+1}} - S_{k_i} = \sum_{k_i+1}^{k_{i+1}} X_j$$

Since $X_j$ are iid R.V having bernoulli distribution

$\therefore S_{k_1} - S_{k_0}, S_{k_2} - S_{k_1}, \ldots, S_{k_n} - S_{k_{n-1}}$ are mutually independent

Hence the SP $\{ S_k ; k = 0, 1, 2, \ldots n \}$

has independent increament.

Class Notes

Proof : Since for every $t$

$$S(t) = S(0)\, e^{H(t)} \quad - (A)$$

$$H(t) = \mu(t) + \sigma H(T)$$

is normally distributed

$$H(t) \sim N(\mu t, \sigma^2 t)$$

$$E(S(t)) = E(S(0) e^{H(t)})$$

$$= S(0)\, E(e^{H(t)})$$

$$= S(0)\, e^{(\mu + \frac{1}{2}\sigma^2)}$$

$$= S(0) \exp\left[\left(\mu + \sigma^2/2\right)t\right]$$

$$Var(S(t)) = E(S(t)^2) - [E(S(t))]^2$$

$$= E(S^2(0)\, e^{2H(t)}) - E[S(t)]^2$$

$$= S^2(0)\, E(e^{2H(t)}) - \text{"}$$

$$\boxed{E(e^{\theta H(t)}) = e^{(\theta \mu t + \frac{1}{2}\sigma^2 t \theta^2)}}$$

$$= S^2(0)\, e^{2\mu t + 2\sigma^2 t} - S^2(0)\, e^{(2\mu t + \sigma^2 t)}$$

$$Var(S(t)) = \left[S(0)\, e^{(\mu + \sigma^2/2)t}\right]^2 \left(\exp(\sigma^2 t) - 1\right)$$

## Filtration :

Let $\omega$ be the sample space and $f_0 = \{\phi, \Omega\}$ then a filtration in discrete time is an increasing sequence ~~of sigma-fi~~ $f_0 \subset f_1 \subset f_2 \subset \dots \subset f_n$ of sigma field per time instance.

Eg : $\Omega = \{HHH, HHT, THH, TTH, TTT, THT, HTT, HTH\}$

$f_0 = \{\phi, \Omega\}$

$f_1 = \{\phi, E_H, E_T, \Omega\}$

$f_2 = \{\phi, E_{H_1}, E_{H_2}, E_T, E_H, E_{H_1} \vee E_T, E_{H_2} \vee E_T, \Omega\}$

check from book

$f_0 \subset f_1 \subset f_2$

Let $w$ be the sample space and $T$ be a fixed positive number. and assume that for $t \in [0, T]$ there is sigma field $f_t$. Assume further that if $s \leq t$ then every set of $f_s$ is in $f_t$. then the collection of sigma field $\{f_t ; 0 \leq t \leq T\}$

is called a filteration in continuous time.

# Sigma field generated by SP

let $\{Y(t), t \in T\}$ be a given SP then sigma field generated by this SP is the smallest sigma field containing all sets in the form $\{\omega :$ the sample path $Y(t), t \in T,$

belong to $C\}$.

for all ~~sets~~ sutiable sets $C$ of the functions

are $T$.

$X_t$ is $F_t$-measurable let $F_t$ is a sigma field (a field) then the random variable if every $X_t$ is $F_t$-measurable

$\Rightarrow$ A R.V is $F_t$-measurable if and only if the info in $F_t$ is sufficient to determine the value of $X_t$.

$\Rightarrow$ Adopted Process

A sequence of RV $X_1, X_2$ ------ are said to be adopted to a filtration $F_1, F_2, ---$ if $X_n$ is $F_n$-measurable for each $n = 1, 2, ----$

# Martingales

field

Let $(\Omega, F, P)$ be a probability space and let $\{X_n, n = 0, 1, 2, \ldots\}$ be a SP and let $\{F_n, n = 0, 1, 2, \ldots\}$ be the filteration.

So the SP $\{X_n, n = 0, \ldots\}$ is said to be a martingale corresponding to the filteration if it satisfies the following condition for every $n$.

(i) $E(X_n)$ exists. (and finite)

(ii) Each $X_n$ is $F_n$-measurable $\forall n$

(iii) $\forall n$, $E(X_{n+1} / F_n)$ must be equal to $X_n$

$$E(E(X|Y)) = E(X)$$

- if $E(X_{n+1} | F_n) > X_n$ then it is Sub-martingale

if $E(X_{n+1} | F_n) < X_n \Rightarrow$ Super-martingle

Ex Let $X_1, X_2, \ldots$ be a seq. of iid R.V each taking values $+1$ and $-1$ with equal probability.

Let us define $S_0 = 0$, $S_n = \sum_{i=1}^{n} X_i$, $n = 1, 2, \ldots$

This ~~discrete~~ SP is simple random walk

$\{S_n$

Sol$^n$

$$E(S_{n+1} | X_n) = E(S_n + X_{n+1} | X_n X_{n-1} \ldots X_1)$$

$$= S_n + E(X_{n+1} | X_n \ldots)$$

$$= S_n + E(\cancel{X_{n+1}})^{0}$$

$\{X_n$ is iid$\}$

$\therefore \boxed{E(S_{n+1} | X_n) = S_n}$

$\therefore S_n$ is martinagle

Q. Consider a Symmetric walk $\{S_n, n=0,1,2,\dots\}$ which is a martingale with respect to filteration $\{f_n, n=0,1,2,\dots\}$ where $f_0 = \{\phi, \Omega\}$ and $f_n = \sigma\{X_1, X_2, \dots, X_n\}; n \geq 1$

verify if $\{S_n^2, n=0,1,2,\dots\}$ is a martingale wrt filteration $f_n$

Sol$^n$ $S_n^2 = (X_1 + X_2 + X_3 + \dots + X_n)^2$

$$E(S_n^2) = E(X_1^2 + X_2^2 + \dots)$$
$$= E(X_1^2) + E(X_2^2) + \dots$$
$$+ 2\sum E(X_i X_j), \quad i \neq j$$

$E(S_n^2) \Rightarrow$ exists $\qquad S_3^2 = \{1, 4, 9\}$

$$E(S_{n+1}^2 \mid f_n) = E((S_n + X_{n+1})^2 \mid f_n)$$
$$= E(S_n^2 \mid f_n) + 2E(S_n X_{n+1} \mid f_n)$$
$$+ E(X_{n+1}^2 \mid f_n)$$
$$= S_n^2 + 2 S_n E(X_{n+1})$$
$$+ E(X_{n+1}^2)$$

$$E(S_{n+1}^2 | F_n) = 1 + S_n^2 \neq S_n^2$$

not martinagle

01/04/24

Ito - doeblin formula for B.M (first

Q $\{N(T); T \geq 0\}$ is a poisson process with parameter $\lambda$. Decide the nature if martingle or not.

Ans

$[w(t) - w(s)]$

$F_n \longrightarrow$ filtration

1) For every $n$, $E(X_n)$ exists
2) Each $X_n$ is $F_n$ - measureable
3) For every $n$, $E(X_{n+1}/F_n) = X_n$

1) $X_n = X(t)$

~~$E((X(t))^2)$~~ ~~need~~ ~~R+Q is~~

$X_t$

1)

2) Each $X(t)$ is $F_t$ - measureable

3) For every $0 < s < t$ $E(X(t)/F_s) = X(s)$

# Let $f(x,t)$ ~~is a function of x & t~~ have continuous partial derivative of atleast second order. s.t. $\{w(t), ~~w(0)~~ ; t \geq 0\}$ be a process then, ~~of~~ Itô

$$df(t, w(t)) = f_t(t, w(t))\,dt + f_x(t, w(t))\,d(w(t)) + \frac{1}{2} f_{xx}(t, w(t))\,dt \quad -(3)$$

where $x = w(t)$

$$\int_0^T df(t, w(t)) = \int_0^T f_t(t, w(t))\,dt + \int_0^T f_x(t, w(t))\,d\,w(t) + \frac{1}{2}\int_0^T f_{xx}(t, w(t))\,dt \quad -(4)$$

$\Downarrow$

$f(T, w(T)) - f(0, w(0)) =$ " "

$f(t, x) = x^2/2$, $x = w(t)$

$f_t = 0$ $\qquad f_{xx} = 1$

$f_x = x$

$$\frac{w^2(T)}{2} - 0 = \int_0^T 0 \, dt + \int_0^T w(s)\, d(w(s)) + \frac{1}{2}\int_0^T 1 . dt$$

$$\boxed{\int_0^T w(s)\, d(w(s)) = \frac{w^2(T) - T}{2}}$$

# Stochastic differential equation

$$\frac{d.X(t)}{dt} = f(t, X(t)) , \quad t \in [0, T]$$
$$X(0) = x_0$$

$$f : [0, T] \times R \longrightarrow R$$

and $f$ is a continuous func.

then this diff. eq$^n$ has a solution :

$$X(t) = X(0) + \int_0^t f(s, X(s))\, ds$$

⊛ provided the lipseltz's condition by the function

$f$ : $\exists\, K > 0$ s.t

$$|f(t, x) - f(t, y)| < K|x - y|$$
$$\forall\, t \in [0, T] \ \ x, y \in R$$

Now suppose $X(0)$ or $b$ is random then solution is not unique, rather it will depend upon $\omega \in \Omega$ and it will be reffered random differential eqn.

# Adding an uncertainity

$$\frac{dX(t)}{dt} = b(t, X(t)) + a(t, X(t)) \frac{d(w(t))}{dt} \quad -(1)$$

$$(0 \leq t \leq T)$$

$$dX(t) = b(t, X(t))\,dt + a(t, X(t))\,dW(t) \quad -(2)$$

$$b(t, X(t)) : [0, T] \times R \longrightarrow R$$
$$a(t, X(t)) : [0, T] \times R \rightarrow R$$

$b$ and $a$ are two given function

eq (2) is stochastic diff. eqn

$$\int_0^T dX(\cancel{t}) = \int_0^T b(\cancel{t}, w(\cancel{t}))\,d\cancel{t} + \int_0^T a(\cancel{t}, w(\cancel{t}))\,dw(\cancel{t})$$

$\boxed{\cancel{0 \leq t \leq T}}$

$X(T) - X(0) =$ " "

$- (3)$

# Diffussion process

is a solution of stochastic diff. eq$^n$ (Strong or weak) is called diffusion

# existence theorem

Let $E(X^2(0)) < \infty$ and $X(0)$ be independent of $\{w(t), t \geq 0\}$ for $\forall$ $t \in [0, T]$ and $x, y \in \mathbb{R}$. $b(t, x), a(t, x)$ be continuous function and satisfies the lipchitz condition wrt 2$^{nd}$ variable

$$|b(t, x) - b(t, y)| + |a(t, x) - a(t, y)|$$

$$\leq k|x - y| +$$

$$\forall \ t \in [0, T]$$

and $k$ is a constant. then the stochastic diff. eq$^n$ has a unique strong solution $\{X(t);\ T \geq t \geq 0\}$

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$$dX(t) = b(t, X(t))dt + a(t, X(t))\,d(w(t))$$

$$X(t) - X(0) = \int_0^t b(u, X(u))\,du + \int_0^t a(u, X(u))\,d\,w(u)$$

**Q** $\sin(w(t))$, $df(t, w(t)) = f_t\,dt + f_x\,dw(t) + \frac{1}{2} f_{xx}\,dt$

**A** Let $f(t, x) = \sin x$

when $x = w(t)$

$f_t = 0$

$f_x = \cos x$

$f_{xx} = -\sin x$

Substituting in eco-doublate formula,

$$d\sin(w(t)) = -\frac{1}{2}\sin(w(t))\,dt + \cos(w(t)) + d(w(t))$$

**Q** SDE : $w^3(t)$

$f(t,x) = x^3$, $x = w(t)$

$f_t = 0$

$f_x = 3x^2$

$f_{xx} = 6x$

$$d\,w^3(t) = 3w(t)\,dt + 3w^3(t) + d(w(t))$$

# SDE of GBM

let $S(t)$ be the stock price at time $= t$ and $\mu$ $(-\infty < \mu < \infty)$ is the constant growth rate and $\sigma > 0$ be the volatility.

then SDE

$$d\,S(t) = \mu S(t)\,dt + \sigma S(t)\,d(W(t)) \quad \text{①}$$

$S(0)$ is known

$$\ast\; S(t) = S(0)\,e^{H}$$

$$H = \mu t + \sigma(W(t))$$

$$\ln(S(t)) = \ln(S(0)) + \mu(t) + \sigma W(t)$$

$$\frac{d(S(t))}{S(t)} = 0 + \mu\,dt + \sigma\,W(t)$$

$$d(S(t)) = \mu S(t)\,dt + \sigma S(t)\,d(W(t))$$

let $S(t) = f(t, W(t))$

$$df(t, W(t)) = \left(f_t + \frac{1}{2} f_{xx}\right)dt + f_x\,d(W(t)) \quad \text{②}$$

$$f_t + \frac{1}{2} f_{xx} = \mu S(t) = \mu f \quad \text{③}$$

$$f_x = \sigma S(t) = \sigma f \quad \text{④}$$

$$\Rightarrow f(t,x) = A(t)\,e^{ax} \quad (5)$$

$$\left(A'(t) + \frac{1}{2}a^2 A(t)\right)e^{ax} = \mu A(t)\,e^{ax}$$

$$A'(t)\,e^{ax} = \left(\mu - \frac{a^2}{2}\right)A(t)\,e^{ax}$$

$$A(t) = \underbrace{c}_{\text{arbitary constant}}\,e^{(\mu - a^2/2)t} \quad (6)$$

Subsituting value of $A(t)$ from (6) and (5)

$$f(t,x) = c \cdot e^{(\mu - a^2/2)t} \cdot e^{ax}$$

at $t = 0$, $\boxed{c = f(0, w(0)) = S(0)}$

↳ put $x = w(t)$

$$f(t,x) = S(0)\,e^{(\mu - \frac{a^2}{2})t + ax}$$

$$\boxed{S(t) = f(t, w(t)) = S(0)\,e^{(\mu - a^2/2)t + a\,w(t)}}$$

(1) $E(S(t)) = S(0)\,e^{\mu t}$

(2) $E(S^2(t)) = S^2(0)\,e^{(2\mu + a^2)t}$

(3) $Var(S(t)) = E(S^2(t)) + [E(S(t))]^2$

$$= S^2(0)\,e^{2\mu t}\left[e^{a^2 t} - 1\right]$$

Proof

$$E(S(t)) = E\left(S(0) \cdot e^{(\mu - \sigma^2/2)t + \sigma W(t)}\right)$$

$$= S(0)\, E\left(e^{\underbrace{(\mu - \sigma^2/2)t + \sigma W(t)}_{X}}\right)$$

$$X \sim N\left((\mu - \sigma^2/2)t, \sigma^2 t\right)$$

$$E(e^X) = e^{(\mu - \cancel{\sigma^2/2})t + \cancel{\frac{1}{2}\sigma^2 t}}$$

$$= e^{\mu t}$$

$$\boxed{E(S(t)) = S(0)\, e^{\mu t}}$$

Q $\{S(t), t \geq 0\}$ follows GBM with

drift $(\mu) = 0.12$ per year

and $\sigma = 0.24$ per annum

$S(0) = $ Rs 40

what is the prob. that an european call option having 4 yrs to exercise with strike price Rs 42 will be exercised.

Ans.

$\mu = 0.12$ $\quad T = 4$ yrs

$\sigma = 0.24$ $\quad K = $ Rs 42

$S(0) = $ Rs 40

$$P(S(t) > 42) = P(S(4) > 42)$$

$$P\left(\frac{S(4)}{S(0)} > \frac{42}{S(0)}\right)$$

$$= P\left(\ln\left(\frac{S(4)}{S(0)}\right) > \ln\left(\frac{42}{40}\right)\right)$$

$\mu t$

$X \sim N(\mu, \sigma^2)$

$$= P\left(\frac{\ln(S(4)/S(0)) - 0.48}{0.48} > \frac{\ln(42/40) - 0.48}{0.48}\right) \quad (X-\mu)/\sigma$$

$\sigma\sqrt{t}$

$$= P(Z > -0.8983)$$

**Q** A stock price is currently \$ 50.
$\mu = 18\%$ per annum (expected return)
$\sigma = 30\%$. What is prob. distribution for stock price in 2 years ($T = 2yr$).
Calculate the mean and standard deviation of the distribution and determine 95% confidence interval.

if not mentioned then SDE

Sol$^n$ As nothing is mentioned it is SDE

SDE follows log-normal distribution

so $\ln(S(t))$ follows normal dist.

$$\ln S(t) \sim N\left(\left(\mu - \frac{\sigma^2}{2}\right)T + \ln S(0),\ \sigma\sqrt{T}\right)$$

$$\sim N\left(\ln S(0) + \left(0.18 - \frac{0.09}{2}\right)2,\ 0.3 \times \sqrt{2}\right)$$

$$\sim N(4.18,\ 0.42)$$

$$\frac{x-\mu}{\sigma} = Z_\alpha$$

$$x \in (\mu \pm Z_\alpha \sigma)$$

$\mu \pm 1.96\sigma \to 95\%$

$$E(S(t)) = S(0)\, e^{\mu t}$$

$$= S(0)\, e^{0.18 \times 2}$$

$$= 71.67$$

$$S.D = \sqrt{Var(S(t))} = S(0)\, e^{\mu t}\left(e^{\sigma^2 T} - 1\right)^{1/2}$$

$$S.D = 31.83$$

$$x \in (9.28,\ 134.06)$$

for 95%

$$\ln(S(2)) \in (4.18 - 1.96 \times 0.42,\ 4.18 + 1.96 \times 0.$$

$$\ln(S(2)) \in (3.36, 5)$$

$$S(2) \in (e^{3.36}, e^{5})$$

$$S(2) \in (28.79, 148.41)$$

this will become $S(0)$ if we calculate after 1 year from now.

**Q** A S.P $S(t)$

{Solved in book}

$$dS(t) = a S(t)\,dt + b S(t)\,dw(t)$$

when a and b are constant

(i) $\sqrt{S(t)}$

(ii) $\ln(S(t))$

**Ans.** $f(t, x) = \sqrt{S(t)}$

$$\sqrt{S(t)} = \sqrt{x}$$

$$x = S(t)$$

$$f_t = 0 \quad (2)$$

$$f_x = \frac{1}{2\sqrt{x}} \qquad f_{xx} = \frac{-1}{4x^{3/2}} \quad -(3)$$

$$df(t,x) = f_t\, dt + f_x\, dS(t) + \frac{1}{2} f_{xx} dS(t)\, dS(t) \quad -④$$

Substitute in ④ from ② and ③,

$$df(t,x) = 0 + \frac{1}{2\sqrt{S(t)}} \left(a S(t)\, dt + b S(t)\, dW(t)\right)$$

$$+ \left(-\frac{1}{4 (S(t))^{3/2}}\right) b^2 S^2(t)\, dt$$

$$= dt \left[ \frac{a}{2} \frac{S(t)}{\sqrt{S(t)}} - \frac{b^2}{4} \frac{S^2(t)}{S(t)^{3/2}} \right]$$

$$+ \frac{b}{2} \sqrt{S(t)}\, d(W(t))$$

$$d\left(\sqrt{S(t)}\right) = \frac{\sqrt{S(t)}}{2} \left(a - b^2/2\right) dt + \frac{b}{2}$$

missing

\# $d(S(t)) = \mu S(t)\, dt + \sigma S(t)\, dW(t)$

$t \in [0, T]$

$B(t)$ be a risk free asset

$$dB(t) = B(t) \cdot r\, dt$$

$$B(t) = B(0)\, e^{rt}$$

when $r$ is the risk-free rate

At time $t$ we take a portfolio which contains $a(t)$ shares and $b(t)$ risk free assets.

$$V_p(t) = a(t) \cdot S(t) + b(t)\, \beta(t) \quad -③$$

$$dV_p(t) = a(t)\, dS(t) + b(t)\, d\beta(t) \quad -④$$

$$t \in [0, T]$$

discounted stock price of one ~~stock~~ share of stock is

$$\tilde{S}(t) = e^{-rt} S(t)$$

Applying ITO doblet formular of version 2

$$d(\tilde{S}(t)) = -re^{-rt} S(t)\, dt + e^{-rt}\, d(S(t))$$

$$= -re^{-rt} S(t)\, dt + e^{-rt}\left[\mu S(t)\, dt + \sigma S(t)\, dW(t)\right]$$

$$= e^{-rt} S(t)(\mu - r)\, dt + e^{-rt} \sigma S(t)\, dW(t)$$

$$= \tilde{S}(t)\left[(\mu - r)\, dt + \sigma\, dW(t)\right]$$

$$= \sigma \tilde{S}(t)\, d\tilde{w}(t) \quad ; \quad d\tilde{w}(t) = \left(\frac{\mu - r}{\sigma}\right)dt + dw(t)$$

$$\Rightarrow \quad \tilde{w}(t) = \frac{\mu - r}{\sigma}\, t + w(t)$$

$(\mu - r)$ is called risk premium

$\frac{\mu - r}{\sigma}$ is risk premium per unit of risk and is called <u>Market price of risk</u>.