FINANCIAL ENGINEERING

(MC306)

ASSIGNMENT - 1

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(A1) Bond Prices : $B(0) = ₹100$

$B(1) = ₹110$

Stock Prices : $S(0) = ₹80$

$$S(1) = \begin{cases} ₹100 & p = 0.8 \\ ₹60 & 1-p = 0.2 \end{cases}$$

Initial Wealth $V(0) = ₹100000$

Stock Investment : $V_S(0) = \frac{3}{5} \times 100000 = ₹60000$

Bond Investment : $V_b(0) = \frac{2}{5} \times 100000 = ₹40000$

No. of stocks : $n_S = \frac{V_S(0)}{S(0)} = 750$

No. of bonds : $n_b = \frac{V_S(0)}{B(0)} = 400$

So, our portfolio is $(750, 400)$.

Now, to find wealth at $t=1$.

(case 1) $S(1) = ₹100 \quad (p = 0.8)$

$n_S S(1) = 750 \times 100 = ₹75000$

$n_b B(1) = 400 \times 110 = ₹44000$

$V(1) = 75000 + 44000 = ₹119000$

(case 2) $S(1) = ₹60 \quad (1-p = 0.2)$

$n_S S(1) = 750 \times 60 = ₹45000$

$n_b B(1) = 400 \times 110 = ₹44000$

$V(1) = 45000 + 44000 = ₹89000$

$$\text{Return} = \frac{V(1) - V(0)}{V(0)} = \begin{cases} \frac{119000 - 100000}{100000} = 19\% & p = 0.8 \\ \frac{89000 - 100000}{100000} = -11\% & 1-p = 0.2 \end{cases}$$

$$R = E(R_i) = \sum_i R_i p_i = 0.8(0.19) + 0.2(-0.11)$$
$$= 0.13 = 13\%$$

$$\sigma = \sqrt{\sum_i p_i (R_i - R)^2} = \sqrt{0.8(0.19 - 0.13)^2 + 0.2(-0.11 - 0.13)^2}$$
$$= \sqrt{0.0144}$$
$$= 0.12 = 12\%$$

Hence, expected return of the portfolio is 13% and risk (s.d.) is 12%.

(A2) $B(0) = ₹90$; $B(1) = ₹100$;

$S(0) = ₹25$;

$$S(1) = \begin{cases} ₹30 & p \\ ₹20 & 1-p \end{cases} \qquad 0<p<1$$

Given portfolio is $(\overset{x}{10}, \overset{y}{15})$.

$$V(0) = x\,S(0) + y\,B(0)$$
$$= 10 \times 25 + 15 \times 90$$
$$= ₹1600$$

Now, to calculate $V(1)$.

$$V(1) = x\,S(1) + y\,B(1)$$

$$V(1) = \begin{cases} 10 \times 30 + 15 \times 100 & p \\ 10 \times 20 + 15 \times 100 & 1-p \end{cases}$$

$$V(1) = \begin{cases} ₹1800 & \text{with prob. } p \\ ₹1700 & \text{with prob. } 1-p \end{cases}$$

$$K_y = \frac{V(1) - V(0)}{V(0)} = \begin{cases} \dfrac{1800-1600}{1600} & p \\ \dfrac{1700-1600}{1600} & 1-p \end{cases}$$

$$= \begin{cases} 0.125 = 12.5\% & p \\ 0.0625 = 6.25\% & 1-p \end{cases}$$

Expected Return $(R) = E(R_i) = \sum_i k_i p_i$

$$= 0.125p + 0.0625(1-p)$$
$$= 0.0625(p+1)$$

So its value depends on p.

For eg. if $p = 0.8$ then,

$$R = 0.0625(0.8+1)$$
$$= 0.1125 = 11.25\%$$

(A3) $B(0) = ₹100$; $B(1) = ₹110$;

$S(0) = ₹80$;

$$S(1) = \begin{cases} ₹100 & p = 0.8 \\ ₹60 & 1-p = 0.2 \end{cases}$$

Given $V(0) = ₹10000$

Investment in Stock $V_S(0) = \frac{10000}{2} = ₹5000$

Investment in Bond $V_b(0) = \frac{10000}{2} = ₹5000$

No. of Stocks $n_S = \frac{5000}{80} = 62.5$

No. of Bonds $n_b = \frac{5000}{100} = 50$

So initial portfolio is $(62.5, 50)$.

$$V(1) = \begin{cases} n_S S(1) + n_b B(1) & p = 0.8 \\ n_S S(1) + n_b B(1) & 1-p = 0.2 \end{cases}$$

$$= \begin{cases} 62.5 \times 100 + 50 \times 110 & p = 0.8 \\ 62.5 \times 60 + 50 \times 110 & 1-p = 0.2 \end{cases}$$

$$= \begin{cases} ₹11,750 & p = 0.8 \\ ₹9,250 & 1-p = 0.2 \end{cases}$$

$$K_v = \frac{V(1) - V(0)}{V(0)} = \begin{cases} \frac{11750 - 10000}{10000} = 0.175 = 17.5\% & p = 0.8 \\ \frac{9250 - 10000}{10000} = -0.075 = -7.5\% & 1-p = 0.2 \end{cases}$$

$$R = E(R_i) = \sum_i K_i p_i$$

$$= 0.8(17.5) + 0.2(-7.5)$$

$$= 0.125 = 12.5\%$$

$$\sigma = \sqrt{\sum_i p_i (R_i - R)^2}$$

$$= \sqrt{0.8(0.175 - 0.125)^2 + 0.2(-0.075 - 0.125)^2}$$

$$= 0.1 = 10\%$$

Hence, expected return is 12.5 % and risk (s.d.) is 10 %.

(A4) $B(0) = ₹90$; $B(1) = ₹100$;

$S(0) = ₹25$;

$$S(1) = \begin{cases} ₹30 & p \\ ₹20 & 1-p \end{cases} \qquad 0 < p < 1$$

Given $$V(1) = \begin{cases} ₹1160 & p \\ ₹1040 & 1-p \end{cases}$$

Let portfolio be (x, y).
We know that,

$$V(1) = x\,S(1) + y\,B(1)$$

$$= \begin{cases} 30x + 100y & p \\ 20x + 100y & 1-p \end{cases}$$

Comparing with given $V(1)$,

$$30x + 100y = 1160 \text{ ——— } (1)$$

$$20x + 100y = 1040 \text{ ——— } (2)$$

Subtracting (2) from (1),

$$10x = 120 \Rightarrow x = 12$$

$$\Rightarrow y = \frac{1160 - 360}{100} = 8$$

Hence, initial portfolio is $(12, 8)$.

$$\therefore V(0) = x\,S(0) + y\,B(0)$$

$$= 12 \times 25 + 8 \times 90$$

$$= ₹1020$$

Value of portfolio $(12, 8)$ at $t=0$ is ₹1020.

(A5) $B(0) = ₹\,100$; $B(1) = ₹110$; $S(0) = ₹80$;

$$S(1) = \begin{cases} ₹100 & p = 0.8 \\ ₹60 & 1-p = 0.2 \end{cases}$$

$K = ₹90$; $T = 1$ year ;

For call option

(i) Let initial portfolio be (x, y).

$$C(0) = x\,S(0) + y\,B(0). \quad \text{—— (A)}$$

At $t = 1$,

$$V_p(1) = C(1) = x\,S(1) + y\,A(1) = \begin{cases} 100x + 110y = 100 - 90 = 10 & \text{—— (1)} \\ 60x + 110y = 0 & \text{—— (2)} \end{cases}$$

Solving (1) and (2), (1) − (2),

$$40x = 10 \Rightarrow x = 1/4$$

$$y = \frac{10 - 25}{110} = -3/22$$

Hence, $C(0) = \frac{1}{4}(80) - \frac{3}{22}(100)$ (From (A))

$= ₹$ 6.36 [redacted]

For put option

Now for put option,

$$P(0) = x\,S(0) + y\,B(0) \quad \text{—— (B)}$$

At $t = 1$,

$$V_p(1) = P(1) = x\,S(1) + y\,A(1) = \begin{cases} 100x + 110y = 0 & \text{—— (3)} \\ 60x + 110y = 90 - 60 = 30 & \text{—— (4)} \end{cases}$$

Solving (3) and (4), (3) − (4),

$$40x = -30 \Rightarrow x = -3/4$$

$$y = \frac{30 + 45}{110} = 15/22$$

Hence, $P(0) = -\frac{3}{4}(80) + \frac{15}{22}(100)$ (From (B))

$= ₹\,8.18$

Hence $C(0) = ₹$ [redacted] 6.36 and $P(0) = ₹\,8.18$.

(ii) ₹900 invested equally in stock, call and put.

₹300 invested in stock.

Hence, no. of stocks $n_S = 300/80 = 3.75$

So initial portfolio is $(3.75, 0)$.

No. of call options $n_c = \dfrac{300}{6.36} = 47.16$

No. of put options $n_p = \dfrac{300}{8.18} = 36.67$

Full initial portfolio is $(3.75, 0, 47.16, 36.67)$.

Now, to calculate final wealth $V(1) = n_S S(1) + n_b B(1) + n_c C(1) + n_p P(1)$

(Case 1) $S(1) = ₹\,100$

$$V(1) = 3.75 \times 100 + 47.16 \times 10$$
$$= ₹\,846.60$$ (only call options exercised)

(Case 2) $S(1) = ₹\,60$

$$V(1) = 3.75 \times 60 + 36.67 \times 30$$
$$= ₹\,1325.10$$ (only put options exercised)

Hence, $V(1) = \begin{cases} ₹846.60 & p = 0.8 \\ ₹1325.10 & 1-p = 0.2 \end{cases}$

(A6) $B(0) = ₹100$; $B(1) = ₹110$; $S(0) = ₹100$

$$S(1) = \begin{cases} ₹120 & p = 0.75 \\ ₹80 & 1-p = 0.25 \end{cases}$$

$K = ₹100$; $T = 1$ unit ;

We need to find $C(0)$ and $P(0)$.

(i) To find $C(0)$.

Let portfolio be (x, y).

- At $t=1$,

$$V_p(1) = C(1) = x\,S(1) + y\,B(1) = \begin{cases} 120x + 110y = 20 \text{ --- (1)} & p = 0.75 \\ 80x + 110y = 0 \text{ --- (2)} & 1-p = 0.25 \end{cases}$$

(1) − (2),

$40x = 20 \Rightarrow x = 1/2$

$\Rightarrow y = -4/11$

$\Rightarrow C(0) = \frac{1}{2}(100) - \frac{4}{11}(100) = ₹13.64$

(ii) To find $P(0)$.

Let portfolio be (x, y).

At $t=1$,

$$V_p(1) = P(1) = x\,S(1) + y\,B(1) = \begin{cases} 120x + 110y = 0 \text{ --- (3)} & p = 0.75 \\ 80x + 110y = 20 \text{ --- (4)} & 1-p = 0.25 \end{cases}$$

(1) − (2),

$40x = -20 \Rightarrow x = -1/2$

$\Rightarrow y = 6/11$

$\Rightarrow P(0) = -\frac{1}{2}(100) + \frac{6}{11}(100) = ₹4.55$

Now $V(0) = ₹1000$ divided equally between stocks and options.

No. of stocks $n_s = \frac{500}{100} = 5$

No. of call options $n_c = \frac{250}{13.64} = 18.33$

No. of put options $n_p = \frac{250}{4.55} = 54.95$

So initial portfolio is, $(5, 0, 18.33, 54.95)$

Now to find final wealth :-

$$V(1) = n_s S(1) + n_c C(1) + n_p P(1) = \begin{cases} 5 \times 120 + 18.33 \times 20 = ₹966.60 & p = 0.75 \\ 5 \times 80 + 54.95 \times 20 = ₹1099.00 & 1-p = 0.25 \end{cases}$$

(A7) To check if an arbitrage oportunity is present, we need to show $V(1) > 0$ when $V(0) = 0$.

At $t=0$, do the following :-

(i) Borrow ₹800 at 4% interest rate.

(ii) Buy 10 pounds.

(iii) Invest in risk-free at 6% interest rate.

(iv) Enter into short forward position for $F(0,1) = ₹79$.

$$V(0) = (80 \times 10) - 800 = ₹0 \quad \text{——— (1)}$$

Now, at $t=1$, do the following :-

(i) Pay back $800(1+0.04) = ₹832$.

(ii) Collect $10(1+0.06) = 10.6$ pounds.

(iii) Close forward position by selling 10 pounds for $10.6 \times 79 = ₹837.4$.

$$V(1) = ₹837.4 - ₹832 = ₹5.4 \quad \text{——— (2)}$$

From (1) and (2),

$V(1) > 0$ when $V(0) = 0$.

This proves that an arbitrage opportunity has arisen.

(A8) $B(0) = ₹100$; $B(1) = ₹110$; $S(0) = ₹50$; $F = ?$;

We know $F(0,1) = \frac{S(0)}{d(0,1)}$ where $d(0,1)$ is discount factor.

Let r be rate of interest.

$$d(0,1) = \frac{1}{(1+r)}$$

Now, $B(0) = \frac{B(1)}{1+r}$

$$\Rightarrow d(0,1) = \frac{B(0)}{B(1)} = \frac{10}{11}$$

$$\Rightarrow F(0,1) = \frac{S(0)}{d(0,1)}$$

$$= 50 \times \frac{11}{10}$$

$$= ₹55$$

Hence, forward price F is ₹55.

(A9) When the investor writes a put option, they receive ₹4 upfront as the option premium.

The investor only incurs a loss if the put option is exercised, i.e., when $S(T) < E$.

Otherwise they simply keep the premium.

So, the condition for the investor's gain is :-

$S(T) \geq E$.

Here premium = ₹4 and $E = ₹30$.

$S(1) \geq 30$.

Now, accounting for the premium, the writer would be in profit if

$S(1) \geq 30 - 4 = 26$

$S(1)$ should be greater than or equal to ₹26.

(A10) Current Price of Silver = ₹5000 / 100g

Storage Cost = ₹0.5 /g /yr

$= ₹0.125 / g / \tfrac{1}{4} yr$

Constant Interest Rate = 9 % compounded quarterly.

$T = 6$ months = 2 periods = $\tfrac{1}{2}$ yr

Forward Price of 1 Kg Silver = $F(0, \tfrac{1}{2})$

$$S(0) = 5000 \times \frac{1000}{100} = ₹\,50{,}000$$

Storage Costs = ₹125 + ₹125 $\quad$ (0.125 x 1000 = 125 per period)

$$\text{So } F(0, \tfrac{1}{2}) = (S(0) + c) e^{rT} + c e^{rT/2}$$

$$= (50000 + 125) e^{0.09(0.5)} + 125 e^{0.09(0.25)}$$

$$= ₹\ 52{,}560$$

So Forward price is ₹52,560.