Sol^n 1. Given $\Omega = \{a, b, c, d\}$

Construction of $F_1 \subset F_2 \subset F_3 \subset F_4$

So, let us assume,

$F_1 = \{\phi, \Omega\}$

$F_2 = \{\phi, \{a\}, \{b, c, d\}, \Omega\}$

$F_3 = \{\phi, \{a\}, \{b, c, d\}, \{b\}, \{a, c, d\}, \{c, d\}, \{a, b\}, \Omega\}$

$F_4 = \{\phi, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{b, c\}, \{c, d\}$
$\{a, c\}, \{a, d\}, \{d, b\}, \{a, b, c\}, \{a, b, d\}$
$\{b, c, d\}, \{a, c, d\}, \Omega\}$

As it is clear from above derivation,

$F_1 \subset F_2 \subset F_3 \subset F_4$ (required condition)

Hence, we found F_1, F_2, F_3, F_4 which satisfy required conditions.

Sol^n 2. Given Wiener processes $W(s)$ & $W(t)$, $0 \le s \le t$

Let $X = W(s) + W(t)$

$E(X) = E(W(s) + W(t))$ $\quad\quad (E(A+B) = E(A) + E(B))$

$= E(W(s)) + E(W(t))$

$= 0 + 0 = 0$ ——— (i)

Covariance, $Cov(W(s), W(t))$

$= Cov(W(s), W(s) + W(t) - W(s))$

$= Cov(W(s), W(s)) + Cov(W(s), W(t) - W(s))$

$= s + Cov(W(s), W(t) - W(s))$

$= s + 0 = s$ ——— (ii)

Variance, $Var(X) = Var(W(s) + W(t))$

$= Var(W(s)) + Var(W(t)) + 2\,Cov(W(s), W(t))$

from eq^n (ii) we get,,

$= s + t + 2s = t + 3s$ ——— (iii)

Hence, from eq^n (i) & (iii) we get

$X = W(s) + W(t) \sim N(0, t + 3s)$

ie. normally distributed

Solⁿ 3 Given X & Y are i.i.d random variables uniformly distributed on $(-\pi, \pi)$

$$Z(t) = \cos(tX + Y), \; t \geq 0$$

$$E(X) = E(Y) = 0$$

$$Var(X) = Var(Y) = \frac{\pi^2}{3}$$

As we know the expansion,

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

$$E(Z(t)) = E[\cos(tX+Y)]$$

$$= E\left[1 - \frac{(tX+Y)^2}{2!} + \frac{(tX+Y)^4}{4!} - \ldots\right]$$

$$= 1 - \frac{1}{2} E[X^2t^2 + Y^2 + 2YXt] + \ldots$$

$$= 1 - \frac{1}{2}[t^2 E(X^2) + E(Y^2)] + \ldots$$

$$= 1 - \frac{\pi^2}{6}(1+t^2) + \ldots$$

As $E(Z(t))$ depends on t^2

$\therefore$ $Z(t)$ is not wide sense stationary process.

Sol^{n}4. Z is normally distributed with $Z \sim N(0,1)$

Given stochastic process $X = \sqrt{t}\, Z$

Given $E(Z) = 0$, $Var(Z) = 1 = E(Z^2)$

Now, $X = \sqrt{t}\, Z$

$E(X) = E(\sqrt{t}\, Z) = \sqrt{t}\, E(Z)$ (as $\sqrt{t}$ is constant)

$= \sqrt{t}\,(0) = 0$ ______ (i)

$Var(X) = Var(\sqrt{t}\, Z) = (\sqrt{t})^2\, Var(Z)$

$= t(1) = t$ ______ (ii)

from (i) & (ii) it is clear that

$X(t) \sim N(0, t)$

& now lets take $s > 0$ & hence,

$E[X(t+s) - X(t)] = E(X(t+s)) - E(X(t)) = 0$ — (iii)

$Var[X(t+s) - X(t)] = Var(X(t+s)) - Var(X(t))$

$= t + s - t = s$ ______ (iv)

from (iii) & (iv) we conclude that

$X(t)$ is a Brownian motion

Hence, proved.

Solⁿ 5. Given $\{W(t), t \geq 0\}$ be a Wiener process

Let $0 < s < t$. Since $W(t) - W(s)$ is independent of F_s

& $W(s)$ is F_s measurable.

$$E(e^{\sigma W(t)}/F_s) = E\left[e^{\sigma(W(t)-W(s))} e^{\sigma W(s)}/F_s\right]$$

$$= e^{\sigma W(s)} E\left[e^{\sigma(W(t)-W(s))}/F_s\right]$$

$$= e^{\sigma W(s)} E\left[e^{\sigma(W(t)-W(s))}\right]$$

$$= e^{\sigma W(s)} e^{\sigma^2 \frac{(t-s)}{2}} \quad \text{———} (i)$$

Now, we use $\exp\{\sigma W(t) - \frac{\sigma^2}{2} t\}$

$$E\left[e^{\sigma W(t) - \frac{\sigma^2}{2}t}/F_s\right] = e^{-\frac{\sigma^2}{2}t} E\left[e^{\sigma W(t)}/F_s\right] \quad (\text{using } (i))$$

$$= e^{-\frac{\sigma^2}{2}t} e^{\sigma^2\frac{(t-s)}{2}} e^{\sigma W(s)}$$

$$= e^{\frac{\sigma^2}{2}(t-s) - \frac{\sigma^2}{2}t} e^{\sigma W(s)}$$

$$= e^{\sigma W(s) - \frac{\sigma^2}{2}s}$$

Hence, we proved that $\exp\{\sigma W(t) - \frac{\sigma^2}{2} t\}$ is a martingale.

Sol^n 6. We have to find stochastic differential of $W^2(t)$

Consider 1st version of the Ito-Doeblin formula,,

$$df(W(t)) = f'(W(t))\, dW(t) + \frac{1}{2} f''(W(t))\, dt \quad \text{—— (i)}$$

Let us consider $f(x) = \frac{x^2}{2}$ & we get

$f'(x) = x$

$f''(x) = 1$

where x is $W(t)$

substituting values of f, f', f'' in (i) we get,,

$$\frac{dW^2(t)}{2} = W(t)\, dW(t) + \frac{dt}{2}$$

$$dW^2(t) = 2W(t)\, dW(t) + dt$$

which is required stochastic differential

Solⁿ 7. Given SDE,

$$dS = rS\,dt + \sigma S\,dW \quad \text{——— (i)}$$

Current price $S(0)$, $r = 0.04$, $\sigma = 0.10$

Strike price, $K = 1.25\,S(0)$

Expiration time, $T = 0.5$

Using 2^{nd} version of Ito-Doublin formula,

Solving i) using this & substitution, we get,

$$S(t) = S(0)\exp\left\{\left(r - \frac{\sigma^2}{2}\right)t + \sigma W(t)\right\}$$

Probability that call option expires in the money,

$$P(S(t) > K) \quad \text{——— (ii)}$$

$$S(t) = S(0)\exp\left\{\left(r - \frac{\sigma^2}{2}\right)t + \sigma W(t)\right\} > K$$

solving inequality we get,

$$W(t) > -\frac{\ln\left(\frac{S(0)}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)t}{\sigma}$$

Substituting values we get,

$$W(t) > -\frac{\ln\left(\frac{4}{5}\right) + \left(\frac{4}{100} - \frac{1}{200}\right)\frac{1}{2}}{1/10} = \frac{\ln\left(5/4\right) - 7/400}{1/10}$$

$$W(t) > 2.056435 \approx 2.06$$

ie. from (ii) we get,

$$P(S(t) > K) = P(W(t) > 2.06) = P(W(t) \leq -2.06)$$

$$= \Phi(-2.06) = \underline{\underline{0.0197}}$$

which is required probability.

Solⁿ 8. Consider 1st version of Ito - Doeblin formula,,

$$df(W(t)) = f'(W(t))\,dW(t) + \frac{1}{2} f''(W(t))\,dt \quad \text{——— (i)}$$

Let us consider $f(x) = \frac{x^3}{3}$ & we get

$f'(x) = x^2$, $f''(x) = 2x$

where x is $W(t)$

substituting values of f, f', f'' in (i) we get,,

$$\int_0^T \frac{dW^3(t)}{3} = \int_0^T W^2(t)\,dW(t) + \int_0^T W(t)\,dt$$

$$\int_0^T W^2(t)\,dW(t) = \frac{W^3(T)}{3} - \int_0^T W(t)\,dt$$

Now we have to find value of $\int_0^T W(t)\,dt$,,

$$E\left[\int_0^T W(t)\,dt\right] = \int_0^T E(W(t))\,dt = 0$$

$$Var\left[\int_0^T W(t)\,dt\right] = \int_0^T (T-t)^2\,dt = \frac{T^3}{3}$$

ie. from here we get,

$$\int_0^T W(t)\,dt \sim N(0, T^3/3)$$

ie. $\int_0^T W(t)\,dt$ is a random variable normally distributed with mean 0 & variance $\frac{T^3}{3}$

Sol^{n} 9. We have to find stochastic differentials of $\sin(W(t))$

Consider 1^{st} version of the Ito-Doeblin formula,,

$$df(W(t)) = f'(W(t))\, dW(t) + \frac{1}{2} f''(W(t))\, dt \quad \text{—— (i)}$$

Let us consider $f(x) = \sin x$ & we get

$f'(x) = \cos x$

$f''(x) = -\sin x$

where x is $W(t)$

substitituting values of f, f', f'' in (i) we get,,

$$d(\sin(W(t))) = \cos(W(t))\, dW(t) + \frac{1}{2}(-\sin(W(t)))\, dt$$

$$d(\sin(W(t))) = \cos(W(t))\, dW(t) - \frac{1}{2} \sin(W(t))\, dt$$

which is required stochastic differential

Sol^n 10. Given stock price, $S(0) = Rs\ 50$

Time (in years), $T = 2$ years

$\mu = 18\% = 0.18$, $\sigma = 30\% = 0.3$

We use probability distribution of stock price in 2 years using log normal distribution

$$\ln S(T) = \phi\left(\ln S(0) + \left(\mu - \frac{\sigma^2}{2}\right)T, \sigma\sqrt{T}\right)$$

$$= \phi\left(\ln 50 + \left(0.18 - \frac{0.09}{2}\right)2, \overset{0.3}{\text{[redacted]}}\sqrt{2}\right)$$

$$\ln S(T) = \phi(4.18, 0.\overset{42}{\text{[redacted]}})$$

Mean, $E(S(T)) = S(0)e^{\mu T} = 50\,e^{0.18\times 2} = Rs\ 71.67$

Std. deviation, $\sigma(S(T)) = S(0)e^{\mu T}\sqrt{e^{\sigma^2 T} - 1}$

$$= 50\,e^{0.18\times 2}\sqrt{e^{0.09\times 2} - 1}$$

$$= Rs.\ 31.83$$

95% confidence interval for $\ln S(T)$ are:

By normal table for critical value at $\frac{0.05}{2} = 0.025$ is 1.96 then we get,

$\Rightarrow 4.18 \pm (1.96 \times 0.42)$

$\Rightarrow 3.35, 5.01$

Corresponding 95% confidence interval for $S(T)$ are

$e^{3.35}$ and $e^{5.01}$

$\Rightarrow 28.52$ and 150.44