Stochastic Process (SP)

Definition: $\sigma - field$

A $\sigma - field$ F (or $\sigma - algebra$) is a family of subset of Ω (sample space) which satisfy the following properties,

i. $\phi \in F$
ii. If $A \in F$ the $A^c \in F$
iii. If $A_1, A_2 \dots .$ are in F and is a countable sequence, then $\cup_i A_i \in F$

e.g; A fair coin is tossed 3-times generating a sample space

$\Omega = \{\text{HHH, HHT, HTH, HTT, THH, TTH, THT, TTT}\}$

Let $A_1 = \{first\ toss\ head\} = \{\text{HHH, HHT, HTH, HTT}\}$

& $A_2 = \{first\ toss\ tail\} = \{\text{THH, TTH, THT, TTT}\}$

then , $F = \{\phi, \Omega, A_1, A_2\}$ is a $\sigma - field$.

Definition: Stochastic Process (SP)

Let $(\Omega, \text{F}, \text{P})$ where P is probability measure defined on F be a given probability space. A collection of random variables (r.v.s) $\{X_t, t \in T\}$, T is Index set defined on the probability space $(\Omega, \text{F}, \text{P})$ is called a Stochastic Process (SP).

$$X_t = X_t(\omega),\ where\ \omega \in R$$

Hence , $\{X_t, t \in T\} = \{X_t(\omega),\ \omega \in \Omega,\ t \in T\}$ ....(1)

It is clear from representation that a Stochastic Process (SP) is function of two variables t, ω which are independent .

$X: T \times \Omega \to R$ ....(2)

The mapping X gives rise to 2- mappings

i. $X(\cdot, \omega) \to fixed\ \omega$ (trajectory is called sample path)
ii. $X(t, \cdot) \to fixed\ t$ (is a random varriable)

Parameter Space and State Space

Let $\{X_t, t \in T\}$ be a given a Stochastic Process (SP). The set $\{t \in T\}$ is called the parameter space or index set. The collection of all possible values of X_t for $\forall\, t \in T$ is called state space denoted by S.

This gives rise to four situations

i. discrete-time, discrete state
ii. discrete time, continuous state
iii. continuous time, discrete state
iv. continuous time, continuous state

Whenever is state space or parameter space is finite or countably infinite then it is said to have discrete nature.

eg. If $t \in \{0, \pm 1, \pm 2, \dots\}\}$ is discrete.

And when t *or* ω takes values on real line (whole) are partially (in an interval) then it is a continuous situation.

i. Continuous time discrete space Stochastic process (SP)

Total number of share $\{X_t, t \in [0, \infty)\}$ held by an investor at any time t.

or Number of cars passing through a signal in one cycle.

ii. Continuous time continuous space Stochastic Process (SP)

The price of a stock (particular item) at any time t.

or Variation of humidity in an AC room between two cut off of AC.

iii. Discrete time continuous space Stochastic Process (SP)

The value of one US Dollar in Rupees at the end of day in a month.

or Temperature recorded of a city at 7.0 am every day in a month.

iv. Discrete time discrete state Stochastic Process (SP)

Total number of share held by an investor at end of day in a month

Definition: Independent increment

If for all $'n'$ and $t_1 < t_2 \ldots . < t_n$ the random variables (r.v.s) $X(t_2) - X(t_1), X(t_3) - X(t_2), \ldots . X(t_n) - X(t_{n-1})$ are independent random variables (r.v.s) then the process is said to have independent increment.

Definition: Strict Sense Stationary Stochastic Process (SP)

(also called is strong stationary Stochastic Process (SP))

The Stochastic Process (SP) $\{X_t, t \geq 0\}$ is called *Strict Sense Stationary Stochastic Process* if for arbitrary $0 < t_1 < \cdots . < t_n$ the finite dimensional random vectors $\{X(t_1), X(t_2), \ldots . . X(t_n)\}$ and $\{X(t_1 + h), X(t_2 + h), \ldots . . X(t_n + h)\}$ have the same joint distribution for all $h > 0$ *and all* $0 < t_1 < \cdots . < t_n$.

Definition: Wide Sense Stationary Stochastic Process (SP)

The Stochastic Process (SP) $\{X_t, t \geq 0\}$ is *Wide Sense Stationary Stochastic Process* if it satisfies the following ,

i. $E(X_t) = \mu(t)$ is independent of t .
ii. $Cov(X_t, X_s)$ depends only on $|t - s|$ for all t, s .
iii. $E(X_t{}^2) < \infty$ (finite second order moment)

A wide sense stationary Stochastic Process is also called covariance stationary or weak stationary or second order stationary Stochastic Process (SP).

Example: $X_t = A\cos\,\omega t + B\sin\,\omega t$, where $A, \& B$ are un-correlated random variables with

expectation '0' and variance 1. ω is a positive constant.

Sol:

i. $E(X_t) = E(A\cos\,\omega t + B\sin\,\omega t) = \cos\,\omega t\, E(A) + \sin\,\omega t\, E(B) = 0$
ii. $Cov(X_t, X_s) = E(X_t . X_s) - E(X_t)E(X_s) = E(X_t . X_s)$ $\quad (\because\ E(X_t) = 0\ )$

$= \cos\,\omega t \cos\,\omega s\, E(A^2) + \sin\,\omega t \sin\,\omega s\, E(B^2) + (\cos\,\omega t \sin\,\omega s +$
$\sin\,\omega t \cos\,\omega s) E(AB)$

$= \cos\,\omega(t - s)$ $\quad (\because\ E(A^2) = E(B^2) = 1, \&\ E(AB) = 0\ \ A\&B\ are\ uncorrelated\ )$

iii. $E(X_t{}^2) = Var(X_t) = 1$

Hence the given stochastic process is wide sense stationary.

***Definition*: Markov property**

A given Stochastic Process (SP) $\{X_t, t \in T\}$ is said to have Markov property if for all *'n'* and for all $0 < t_1 < \cdots . < t_n < t$ the CDF satisfies

$$P\{X_t \leq x / X(0) = x_0, X(t_1) = x_1 \ldots . X(t_n) = x_n\} = P\{X_t \leq x / X(t_n) = x_n\}$$

That is, future predictions depend only on the current state of the Stochastic Process (SP) and does not depend on the past information.

***Definition*: Random Walk**

If each trial has more than two possible outcomes, $Y_i, i = 1,2, \ldots . . n$ be the set of independent discrete random variables (r.v.s)

$$S_n = \sum_{i=1}^{n} Y_i$$

Then $\{S_n, n = 0,1,2 \ldots . .\}$ $where$ $S_0 = 0$ is called *Random Walk.*

***Definition*: Symmetric Random Walk**

consider a random experiment of tossing a fair coin finitely many times. Let the successive outcomes be denoted by $\omega = (\omega_1, \omega_2 \ldots .)$ (eg. (H,T,H,H,T,T…) or (,T,T,T,H,H,T..) we now define for $j = 1,2, ..$

$$X_j = \begin{cases} 1 & if\ \omega_j = H \\ -1 & if\ \omega_j = T \end{cases},$$

and $P(X_j = 1) = P(\omega_j = H) = 0.5$, $P(X_j = -1) = P(\omega_j = T) = 0.5$

Set $S_0 = 0$. Let

$$S_k = \sum_{j=1}^{k} X_j \quad , \quad k = 1,2, \ldots \ldots \ldots$$

Then $\{S_n, n = 0,1,2 \ldots . .\}$ is known as a *symmetric random walk* .

Theorem: Let $\{S_k , k = 0,1,2 \ldots\ldots\}$ be a asymmetric random walk. Then

i. For each k , $E(S_k) = 0 \ and \ Var(S_k) = k$
ii. it has independent increment
iii. It has stationary increment
iv. It is a Markov process

Proof:

i. For $j = 1,2, \ldots$, $E(X_j) = o \ and \ Var(X_j) = 1$

Therefore

$$E(S_k) = E\left(\sum_{j=1}^{k} X_j\right) = \sum_{j=1}^{k} E(X_j) = 0$$

$$Var(S_k) = \sum_{j=1}^{k} Var(X_j) = k$$

ii. We choose an arbitrary positive integer $'n'$ and then choose non negative $0 = k_0 < k_1 < \cdots . < k_n$ integers then

$$S_{k_i+1} - S_{k_i} = \sum_{j=k_i+1}^{k_{i+1}} X_j$$

Since X_j are independent and identically distributed (i.i.d.) random variables (r.v.s) having Bernoulli distribution .

$S_{k_1} - S_{k_0}, S_{k_2} - S_{k_1}, \ldots . S_{k_n} - S_{k_{n-1}}$ are mutually independent variables. Hence the Stochastic Process (SP) $\{S_n , n = 0,1,2 \ldots\ldots\}$ has independent increment property.

iii. Choose non-negative integers $k_1 < k_2$ then

$$S_{k_2} - S_{k_1} = \sum_{j=k_1+1}^{k_2} X_j$$

Since X_j are i.i.d. random variables (r.v.s) having Barnoulli distribution , $S_{k_2} - S_{k_1}$ has the same distribution of $S_{k_2-k_1} - S_0$ hence the Stochastic Process (SP) $\{S_n , n = 0,1,2 \ldots\ldots\}$ has the stationary increment property.

iv. We have for $k = 1,2,\ldots$

$S_k = S_{k-1} + X_k$

Now $\quad P\{S_k \le x_k / S_{k-1} = x_{k-1}, S_{k-2} = x_{k-2}, \ldots S_1 = x_1\}$

$$= \frac{P\{S_k \le x_k,\ S_{k-1} = x_{k-1}, S_{k-2} = x_{k-2}, \ldots S_1 = x_1\}}{P\{S_{k-1} = x_{k-1}\}\{P\{S_{k-2} = x_{k-2}\}, \ldots P\{S_1 = x_1\}}$$

$$= \frac{P\{S_k \le x_k /\ S_{k-1} = x_{k-1}\} \ldots \ldots P\{S_2 = x_2 / S_1 = x_1\} P\{S_1 = x_1\}}{P\{S_{k-1} = x_{k-1} / S_{k-2} = x_{k-2}\}\{P\{S_{k-2} = x_{k-2} / S_{k-3} = x_{k-3}\}, \ldots P\{S_1 = x_1\}}$$

$$= P\{S_k \le x_k /\ S_{k-1} = x_{k-1}\}$$

Hence $\{S_n, n = 0,1,2 \ldots \ldots\}$ is a Markov process.

***Definition*: Poisson process**

A Stochastic Process (SP) $\{S(t), t \ge 0\}$ is said to be a *Poison Process* with intensity or rate (parameter) $\lambda > 0$ if it satisfies the following properties,

i. It is start from zero ie. $X(0) = 0$.
ii. For all $'n'$ and for all $0 \le t_0 < t_1 < \cdots . < t_n$ increments $S(t_i) - S(t_{i-1})$, $i = 1,2, \ldots n$ are independent and stationary
iii. For $0 < s < t$, $S(t) - S(s)$ is a poison distributed random variable with parameter $\lambda(t - s)$ ie,

$$P\{S(t) - S(s) = n\} = \frac{e^{-\lambda(t-s)}\{\lambda(t-s)\}^n}{n!}, \quad n = 0,1,2, \ldots.$$

§§ By virtue of property (ii) the Stochastic Process (SP) $\{S(t), t \ge 0\}$ satisfies the Markov property and hence it is a Markov process.

Definition: **Brownian Motion (BM) or Wiener Process**

A Stochastic Process (SP) $\{W(t), t \geq 0\}$ is said to be a Brownian Motion (BM) if it satisfies the following properties

i. $W(0) = 0$ ie, it starts from zero.

ii. $for\ t > 0$ the sample path of $W(t)$ is continuous.

iii. $W(t), t \geq 0$ has independent and stationary increment.

iv. $for\ 0 \leq s < t < \infty$, $W(t) - W(s)$ is normally distributed random variable with mean '0' and variance $(t - s)$.

The path is always continuous but it is nowhere differentiable that is it is not possible to define unique tangent line at any point on a curve for this we can use the convergence of second order moment we shall show that

$\lim_{\Delta t \to 0} Var\{\frac{W(t_o+\Delta t) - W(t_0)}{\Delta t}\}$ does not exist for any point t_o.

Since we know that $W(t_o + \Delta t) - W(t_0)$ it has normal distribution with mean'0' and variance $(t - s)$ that is $\sim N(0, \Delta t)$ .Hence

$\lim_{\Delta t \to 0} Var\{\frac{W(t_o+\Delta t) - W(t_0)}{\Delta t}\} = \lim_{\Delta t \to 0} \frac{\Delta t}{(\Delta t)^2} = \lim_{\Delta t \to 0} \frac{1}{\Delta t}$. Which does not exist.

§§ The W.P. is not wide sense stationary because for $s < t$ the $Cov\{W(t), W(s)\}$ is not a function of $(t - s)$.

For $Cov\{W(t), W(s)\} = E\left[\{W(t) - E\left(W(t)\right)\}\{W(s) - E\left(W(s)\right)\}\right]$

$$= E\{W(s)\, W(t)\} = E\left[\{W(t) - W(s) + W(s)\}\, W(s)\right]$$

$$= E\left[W(t) - W(s)\right] E\{W(s)\} + E\{W(s)\, W(s)\}$$

$$= 0 + s$$

Hence , $Cov\{W(t), W(s)\} = \min\{s, t\}$

§§ Given $W(t)$ the future $W(t + h)$ having $h > 0$ only depends on the increment $W(t + h) - W(t)$ which is independent of the past. Hence $\{W(t), t \geq 0\}$ is a Markov process.

***Definition*: Brownian Motion (BM) with drift μ and volatility σ**

A Stochastic Process (SP) $\{ X(t), t \geq 0 \}$ is said to be a Brownian Motion (BM) with drift μ and volatility σ if $\{ X(t) = \mu t + \sigma W(t)\}$ where

i. $W(t)$ is a standard Brownian Motion (BM)

ii. $-\infty < \mu < \infty$ is a constant

iii. $\sigma > 0$ is a constant

This is a generalization of standard Brownian Motion (BM) in this process

$E\{ X(t) \} = \mu t$, and

$$Cov\{ X(t), X(s) \} = \sigma^2 Cov\{ W(t), W(s) \} = \sigma^2 \min\{ s, t \} \quad , \quad s, t > 0$$

***Definition*: Geometric Brownian Motion (GBM)**

A Stochastic Process (SP) $\{ X(t), t \geq 0 \}$ is said to be a Geometric Brownian Motion (GBM) if

$X(t) = X(0)\, e^{w(t)}$ where $W(t)$ is a standard Brownian Motion (BM).

§§ for any $h > 0$, we have

$$X(t+h) = X(0)e^{w(t+h)} = X(0)e^{w(t+h)-w(t)+w(t)}$$

$$= X(0)e^{w(t)}e^{w(t+h)-w(t)} = X(t)e^{w(t+h)-w(t)}$$

§§ We know that Brownian Motion (BM) has independent increments. Hence given $X(t)$ the future $X(t+h)$ only depends on the future increment of the Brownian Motion (BM). Thus future is independent of the past and therefore the Markov property is satisfied hence $\{ X(t), t \geq 0 \}$ is a Markov process.

GBM To model stock price

Let the stock price $S(t)$ at time 't' is given by $S(t) = S(0)e^{H(t)}$, $S(0)$ is initial price and $H(t) = \mu t + \sigma W(t)$ is Brownian Motion (BM) with drift.

In this case $H(t)$ represents a continuously compounded rate of interest of the stock price over the period $[0, t]$. Hence

$$ln(S(t)) = ln(S(0)) + H(t)$$

Therefore $ln\,(S(t))$ has normal distribution with mean $\mu t + ln(S(0))$ and variance $\sigma^2 t$.

(If a random variable X has property that $ln(X)$ has normal distribution then the random variable X is said to have *lognormal distribution*. Accordingly $\left(\frac{S(t)}{S(0)}\right)$ is log normally distributed random variable.)

§§ If $S(t) = S(0)e^{H(t)}$ at any time t' where $S(0)$ is initial price and $H(t) = \mu t + \sigma W(t)$ is a Brownian Motion (BM) with drift μ and volatility σ then

i. $E\{\, S(t)\,\} = S(0)\exp\{\, (\mu + \frac{\sigma^2}{2})t\}$

ii. $Var\{S(t)\} = \left[S(0)\exp\{\, (\mu + \frac{\sigma^2}{2})t\}\right]^2 \{\exp(\,\sigma^2 t) - 1\}$

Proof: Since for every 't' $W(t)$ is normally distributed with mean '0' and variance 't' ,

$H(t) = \mu t + \sigma W(t)$ normally distributed with mean μt and variance $\sigma^2 t$.

Hence

$M_{H(t)}(\theta) = E\left(e^{\theta H(t)}\right) = \exp\left(\mu t\theta + \frac{1}{2}\sigma^2 t\theta^2\right)$ (mgf of Normal dist.) ……….(1)

i. $E\big(S(t)\big) = E\left(S(0)e^{H(t)}\right) = S(0)E\left(e^{H(t)}\right)$

Putting $\theta = 1$ in equation (1) we get

$E\left(e^{H(t)}\right) = \exp\left(\mu t + \frac{1}{2}\sigma^2 t\right)$

Hence , $E\big(S(t)\big) = S(0)\exp\left(\mu t + \frac{1}{2}\sigma^2 t\right)$

ii. $Var(S(t)) = E\left(\big(S(t)\big)^2\right) - \Big(E\big(S(t)\big)\Big)^2$

$= E\left(S^2(0)e^{2H(t)}\right) - \left(S(0)\exp\{\, (\mu + \frac{\sigma^2}{2})t\}\right)^2$

Putting $\theta = 2$ in equation (1) we get

$= S^2(0)\,\exp\{(2\mu t + 2\sigma^2 t)\} - S^2(0)\,\exp\{(2\mu t + \sigma^2 t)\}$

$= S^2(0)\,\exp\{(2\mu t + \sigma^2 t)\}\,[\exp(\,\sigma^2 t) - 1]$

$= \left[S(0)\exp\{\, (\mu + \frac{\sigma^2}{2})t\}\right]^2 \{\exp(\,\sigma^2 t) - 1\}$

Filtration

Definition: Filtration in Discrete Time

Let Ω be the sample space and $F_0 = \{\phi, \Omega\}$. Then a filtration in discrete time is an increasing sequence of $F_0 \subset F_1 \subset F_2 \subset \cdots$ $of \ \sigma - field$, one per time instant.

e.g.: $\Omega = \{\mathrm{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}\}$

$$E_H = head\ in\ first\ toss \qquad \& \qquad E_T = tail\ in\ first\ toss$$

$F_0 = \{\phi, \Omega\}$, $F_1 = \{\phi, \{E_H\}, \{E_T\}, \Omega\}$,

$F_2 = \{\phi, \{E_H\}, \{E_T\}, \{E_{HH}\}, \{E_{TT}\}, \{E_{HH}{}^c\}, \{E_{TT}{}^c\}, \{E_{HH} \cup E_{TT}\}\Omega\}$, then $F_0 \subset F_1 \subset F_2$

Definition: Filtration in continuous time

Let Ω be the sample space. Let T be a fixed positive number and assume that for $t \in [0, T]$, there is $\sigma - field$ F_t . Assume further that, if $s \le t$ then every set of F_s is in F_t . Then the collection of $\sigma - field$ $\{F_t : t \ge 0\}$ is called filtration in continuous time .

Thus a collection of $\sigma - field$ $\{F_t : t \ge 0\}$ is called filtration in continuous time if $F_s \subset F_t$ for all $0 \le s \le t$.

§§ Filtration is used to model the flow of information over time.

e.g.: we think X_t as the price of an asset at time t and F_t as the information obtained by watching all prices in the market upto time t .

Definition: $\sigma - field$ generated by a Stochastic Process (SP)

Let $\{Y_t : t \in T\}$ with the given stochastic process. Then $\sigma - field$ generated by the Stochastic Process (SP) $\{Y_t : t \in T\}$ is the smallest $\sigma - field$ containing all sets of the form

$\{w : the\ sample\ path\ (Y_t : t \in T) belongs\ to\ C\}$ for all suitable sets C of on function on T.

Definition: $\sigma - field$ generated by Brownian Motion (BM)

Let $W = \{W(s): 0 < s \leq t\}$ with the given Brownian Motion (BM) on $[0, t]$ then $\sigma - field$ generated by all sets of the form

$A_{t_2, t_3 \ldots t_n} = \{\omega \in \Omega: \mathrm{W}(t_1, \omega), \mathrm{W}(t_2, \omega) \ldots . \mathrm{W}(t_n, \omega) \in C\}$

For any n-dimensional Borel Set , and for any choice of $t_i \in [0, t]$, $i \geq 1$ is called $\sigma - field$ generated by Brownian Motion (BM) 'W'.

Definition: X_t is F_t - measurable

Let F_t be a $\sigma - field$ of subsets of Ω . Then a random variable X_t is F_t – measurable if every set in $\sigma(X_t)$ is also in F_t that is a random variable is F_t – measurable iff the information in F_t is sufficient to determine the value of X_t .

Definition: Adapted process

A sequence of random variables (r.v.s) X_1, X_2 .... are said to be adopted to a filtration $F_1, F_2, ..$ if X_n is F_n – measurable for each $n = 1,2, ...$

Or

A discrete time Stochastic Process (SP) $\{X_0, X_1, X_2 \ldots .\}$ is said to be adopted to a given filtration $\{F_n; n = 0,1, ..\}$ if the $\sigma - field$ generated by X_n is a subset of F_n ie., $(X_n) \subset F_n$

.

(same is for continuous SP)

Definition: Natural filtration

Natural filtration corresponding to a process is the smallest filtration to which it is adapted.

Martingales

***Definition*: Discrete-time Martingale**

Let $(\Omega, \mathrm{F}, \mathrm{P})$ be a probability space. Let $\{X_n; n = 0,1,..\}$ be a Stochastic Process (SP) and $\{F_n; n = 0,1,..\}$ be the filtration. The Stochastic Process (SP) $\{X_n; n = 0,1,..\}$ is said to be a Martingale corresponding to the filtration $\{F_n; n = 0,1,..\}$ if it satisfies the following conditions,

1. For every n , $E(X_n)$ exists
2. Each X_n is F_n – measurable
3. For every n , $E(X_{n+1}/F_n) = X_n$

§§ Using the property of conditional expectation

$$E\big(E(X/Y)\big) = E(X)$$

In the definition of Martingale we observe that if $\{X_n\}$ is a Martingale then

$E(X_{n+1}) = X_n$, for every . it implies that $E(X_n) = c$ (constant) , therefore, if for some $n > 0$, $E(X_n) < \infty$ and the increments $X_{n+1} - X_n$ of the Martingale $\{X_n\}$ are bounded then $E(X_n) = E(X_0)$

Sub-Martingale and Super-Martingale

The 3rd condition of the def of Martingale is

if $E(X_{n+1}/F_n) \geq X_n$ then $\{X_n; n = 0,1,..\}$ is a sub-Martingale .

While if $E(X_{n+1}/F_n) \leq X_n$ then $\{X_n; n = 0,1,..\}$ is a super-Martingale

Ex.(8.6.2)

Let X_1, X_2 .... be a sequence of i.i.d. random variables (r.v.s)each taking values +1 &-1 with equal probabilities. Let us define $S_0 = 0$ and

$$S_n = \sum_{i=1}^{n} X_i \ , \ n = 1,2, \dots \ .$$

This discrete-time stochastic Process (SP) is a symmetric random walk. Prove that $\{S_n; n = 0,1,..\}$ is a Martingale with respect to $\{X_n; n = 1,2,..\}$.

Solution: we have

$E(|S_n|) \leq E(|X_1|) + E(|X_2|) \dots + E(|X_n|) < \infty$

Also

$E(S_{n+1} \ / \ X_1, X_2 .. X_n) = E(S_n + X_{n+1} \ / \ X_1, X_2 .. X_n)$

$= E(S_n \ / \ X_1, X_2 .. X_n) + E(X_{n+1} \ / \ X_1, X_2 .. X_n)$

$= S_n + E(X_{n+1})$ $\quad (\because \ X_i \ are \ independent)$

$= S_n$ $\quad (\because \ E(X_{n+1}) = 0 \ )$

Hence $\{S_n; n = 0,1,..\}$ is a Martingale w.r.t $\{X_n; n = 1,2,..\}$.

§§ Suppose F_k is the $\sigma - field$ of information corresponding to the first k random variables (r.v.s) X_k , we have for non negative integers $k < n$, $E(S_n/F_k) = S_k$

Ex.(8.6.3)

Consider a symmetric random walk $\{S_n; n = 0,1,..\}$ which is a Martingale with respect to filtration $\{F_n; n = 0,1,..\}$ where $F_0 = \{\phi, \Omega\}$ $F_0 = \{\phi, \Omega\}$ and $F_n = \sigma(X_1, X_2 .. X_n), n \geq 1$ is the $\sigma - field$ of information corresponding to the n random variable $X_1, X_2 .. X_n$. Verify if $\{{S_n}^2; n = 0,1,..\}$ is a Martingale with respect to filtration $\{F_n; n = 0,1,..\}$

Solution: For each $n = 1,2,..$, ${S_n}^2$ F_n – measurable. Also

$E({S_n}^2) = \sum_{i=1}^{n} E({X_i}^2) < \infty$ $\quad$ (since $E(X_i, X_j) = 0 \ for \ i \neq j$)

Now, $E({S_{n+1}}^2 / F_n) = E[(S_{n+1} - S_n + S_n)^2 / F_n)]$

$=E[(S_{n+1} - S_n)^2 / F_n)] + 2\,E[S_n(S_{n+1} - S_n)/F_n] + E\,[{S_n}^2 / F_n]$

$= E[(X_{n+1})^2 / F_n)] + 2\,E[S_n\,X_{n+1}/F_n] + E\,[{S_n}^2 / F_n]$

X_{n+1} is independent of F_n and ${S_n}^2$ is F_n - measurable , we have

$E({S_{n+1}}^2 / F_n) = E[(X_{n+1})^2 / F_n)] + 2\,S_n\,E[\,X_{n+1}/F_n] + {S_n}^2$

$= 1 + {S_n}^2$

Since ,

$$E({S_{n+1}}^2 / F_n) \neq {S_n}^2 \quad \text{hence} \quad \{{S_n}^2; n = 0,1,..\} \text{ is not a Martingale.}$$

However

$$E({S_{n+1}}^2 / F_n) > {S_n}^2 \quad \text{hence} \quad \{{S_n}^2; n = 0,1,..\} \text{ is a sub-Martingale.}$$

Ex.(8.6.5)

Let a person start with Rs.1. A fair coin is tossed infinitely many times. A person gets Rs.2 if it turns up head and gets nothing if it is a tail in n[th] toss. Let Y_n be his / her fortune at the end of the n[th] trial. Prove that Y_n is a Martingale.

Solution:

Let $X_1, X_2 ...$ be a sequence of i.i.d. random variables (r.v.s) each defined as

$$X_n = \begin{cases} 2 & \textit{with probability } 0.5 \\ 0 & \textit{with probability } 0.5 \end{cases}$$

according to problem

$$Y_n = X_1, X_2 \dots . X_n \qquad , \quad n = 1,2, \dots.$$

let F_n be $\sigma - field$ generated by $X_1, X_2 .. X_n$. We have

$$0 \leq Y_n \leq 2^n \qquad and \qquad E(X_i) = 1$$

Now, $\quad E(Y_{n+1}/F_n) = E(Y_n X_{n+1}/F_n) = Y_n E(X_{n+1}/F_n)$

$= Y_n E(X_{n+1})$ since X_{n+1} is independent of F_n .

$= Y_n$ since $E(X_{n+1}) = 1$

hence $\{Y_n; n = 1,2,..\}$ is a Martingale.

Ex.(8.6.6)

Consider a binomial lattice model. let S_n be the stock prices at period 'n' and

$$S_{n+1} = \begin{cases} uS_n & \textit{with probability } p \\ dS_n & \textit{with probability } 1 - p \end{cases}$$

define a related process R_n as $R_n = \ln(S_n) - \mathrm{n}[\,\mathrm{p} \ln u + (1 - \mathrm{p}) \ln d\,]$

prove that $\{\ln S_n\,; n = 1,2,..\}$ is not a Martingale whereas is a Martingale

where as $\{R_n; n = 1,2,..\}$ is a Martingale with respect to $\{S_n; n = 1,2,..\}$. Also prove that the discounted stock price $S_0, S_1 e^{-r}, S_2 e^{-2r}, \ldots$ is a Martingale only if

$$p = \frac{e^r - d}{u - d} \qquad \text{Where 'r' is the nominal interest rate.}$$

Solution:

In this binomial lattice model $\{S_0, S_1, S_2, ..\}$ with the natural filtration $\{F_0, F_1, F_2, ..\}$ we have

$$P(S_{n+1} = uS_n/F_n) = 1 - P(S_{n+1} = dS_n/F_n) = p \qquad \ldots\ldots\ldots\ldots(1)$$

hence

$$E\left(\frac{S_{n+1}}{F_n}\right) = p.uS_n + (1-p).dS_n = S_n\{p.u + (1-p).d\} \qquad \ldots\ldots\ldots\ldots(2)$$

we consider the variable $\ln(S_n)$ in and observe that

$$E\{\ln\left(\frac{S_n}{S_{n-1}}\right) / S_{n-1}, S_{n-2}, ..S_0\} = p.\ln u + (1-p).\ln d$$

therefore , $E\{\ln(S_{n+1}) / S_{n-1}, S_{n-2}, ..S_0\} = lnS_n + p.\ln u + (1-p).\ln d$ ……..(3)

hence $\{\ln S_n\,; n = 1,2,..\}$ is not a Martingale. Depending upon the value of u & d it can be Sab Martingale are super Martingale.

Consider

$$E\{R_n/ R_{n-1}, R_{n-2}, ..R_0\} = E[\{\ln(S_n) - \mathrm{n}[\mathrm{p}.\ln u + (1-\mathrm{p}).\ln d\}/ R_{n-1}, R_{n-2}, ..R_0]$$

since the information of $R_{n-1}, R_{n-2}, ..R_0$ is yield by history of $S_{n-1}, S_{n-2}, ..S_0$ and vice versa, so we get

$$E\{R_n/ R_{n-1}, R_{n-2}, ..R_0\} = \ln(S_{n-1}) - (\mathrm{n}-1)[\mathrm{p}.\ln u + (1-\mathrm{p}).\ln d \qquad \ldots\ldots\ldots (4)$$

$$= R_{n-1}$$

therefore $\{R_n; n = 1,2,..\}$ is a Martingale.

Consider the discounted process $(S_0, S_1 e^{-r}, S_2 e^{-2r}, \ldots)$ where 'r' is the interest rate. We have

$$E\left(e^{-(n+1)r}\, S_{n+1}/F_n\right) = p.u.e^{-(n+1)r}S_n + (1-p).d.e^{-(n+1)r}S_n$$

$$= e^{-nr} S_n \ e^{-r} \{ p.u + (1-p).d \} \qquad \dots\dots\dots\dots\dots(5)$$

Therefore the discounted process will be Martingale if the RHS of equation (5) is $e^{-nr} S_n$ ie.

$$p.u + (1-p).d = e^r$$

Or
$$p = \frac{e^r - d}{u-d}$$

Ex.(8.6.7)

Prove that under RNPM Q the discounted stock price process $\{(1+r)^{-k} S_k;\ k = 1,2,..\}$ is a Martingale.

Solution: Let Ω be the sequence of heads - H and tail - T such that the stock price goes up by factor u and goes down by factor d with the occurrence of H or T respectively.

Let S_0 be a fix number. Define

$$S_k(\omega) = u^j d^{k-j} S_0$$

where the first k element $\omega \in \Omega$ has 'j' occurrence of H and '$k-j$' occurrence of T .

Let F_n be the $\sigma - field$ generated by $S_0, S_1, \dots S_k$ and

$\bar{p} = \frac{(1+r)-d}{u-d}$, & $\bar{q} = \frac{u-(1+r)}{u-d}$, also $Q(\omega) = \bar{p}^j \bar{q}^{n-j}$

ie, for any sequence of 'n' movements $H \to j \ times \quad and \ T \to n-j \ times$ appears

now the random variable

$$\mathbb{Y}_k = S_{k+1}/S_k \qquad \dots\dots\dots\dots\dots\dots\dots\dots(A)$$

(defined like this) are i.i.d. with probability mass function as

$$P(\mathbb{Y} = u) = \bar{p} \ , \qquad P(\mathbb{Y} = d) = \bar{q}$$

Since $\mathbb{Y}_k$ are i.i.d. random variable therefore they satisfy

$P\,(\mathbb{Y}_1 = Y_1, \dots \mathbb{Y}_n = Y_n) = P(\mathbb{Y}_1 = Y_1).P(\mathbb{Y}_2 = Y_2) \dots . P(\mathbb{Y}_n = Y_n)$

$$= \bar{p}^j \bar{q}^{n-j}$$

also from equation (A) we get that $\mathbb{Y}_k$ is the factor by which the stock price is moving. The random variable $\mathbb{Y}_k$ are independent of F_k .

$$E_Q\{ (1+r)^{-(k+1)} S_{k+1}/F_k\} = (1+r)^{-(k+1)} E_Q\{S_{k+1}/F_k\}$$

$$= (1+r)^{-(k+1)}\ S_k E_Q\{\tfrac{S_{k+1}}{S_k}/F_k\}$$

$$= (1+r)^{-k}\ S_k\ (1+r)^{-1} E_Q\{\tfrac{S_{k+1}}{S_k}\} \quad \text{since } \mathbb{Y}_k \text{ are independent of } F_k$$

$$= (1+r)^{-k}\ S_k\ (1+r)^{-1}\ \{\bar{p}.u + \bar{q}.d\}$$

$$= (1+r)^{-k}\ S_k \qquad \text{since} \quad 1+r = \{\bar{p}.u + \bar{q}.d\}$$

the process $\{(1+r)^{-k} S_k;\ k = 1,2,..\}$ is a Martingale.

Wealth process

Let Δ_k be the number of shares of a stock held between time k *and* $k+1$. We assume that Δ_k is F_k - measurable and X_0 is the amount of money we have started with time $t = 0$ now $\Delta_k . S_{k+1}$ will be the worth of the stock at time $k+1$ where S_{k+1} is the price of the stock.

Now the amount of cash we hold between k *and* $k+1$ is X_k minus the amount held in stock ie, $X_k - \Delta_k . S_k$.

hence the worth of this amount at time $k+1$ is

$$(1+r)(X_k - \Delta_k . S_k)$$

Therefore, the amount of money we have at time $k+1$ is

$$X_{k+1} = \Delta_k . S_{k+1} + (1+r)(X_k - \Delta_k . S_k) \qquad \text{.................. (A)}$$

When $r = 0$, it reduces to $X_{k+1} = \Delta_k\ (S_{k+1} - S_k) + X_k$.

Thus , $X_{k+1} = X_0 + \sum_{i=0}^{k} \Delta_i\ (S_{i+1} - S_i)$

The Stochastic Process (SP) $\{X_k : k = 0,1,2 \ldots\}$ is called the wealth process.

§§ Discounted wealth process is a Martingale under RNPM Q .

$E_Q[X_{k+1} - X_k/F_k] = E_Q[\Delta_k\,(S_{k+1} - S_k)/F_k]$

$=\Delta_k\ E_Q[(S_{k+1} - S_k)/F_k]$ since Δ_k is F_k – measurable

$=\Delta_k\ [E_Q(S_{k+1}/F_k) - E_Q(S_k/F_k]$

$=\Delta_k\,(S_k - S_k)$ since $\{S_k; k = 0,1,2,..\}$ is a Martingale.

$=$ 0 (earlier proved that discounted stock price process is a Martingale)

Now noting that > 0 , from equation (A) we have

$E_Q[\{\,(1+r)^{-(k+1)}X_{k+1} - (1+r)^{-k}X_k\}/F_k]$

$= E_Q[\Delta_k\ \{(1+r)^{-(k+1)}S_{k+1} - (1+r)^{-k}S_k\}/F_k]$

$=\Delta_k\ E_Q[\{\,(1+r)^{-(k+1)}S_{k+1} - (1+r)^{-k}S_k\}/F_k]$

$=\Delta_k\ [E_Q\,\{\,(1+r)^{-(k+1)}S_{k+1}/F_k\} - E_Q\{(1+r)^{-k}S_k/F_k\}]$

$=\Delta_k\,[\,(1+r)^{-k}S_k - \ (1+r)^{-k}S_k]$

$= 0$

(since $\{(1+r)^{-k}S_k;\ k = 1,2,..\}$ is a Martingale & for $r = 0$ reduces to S_k)

Or we have

$$E_Q[\,(1+r)^{-(k+1)}X_{k+1}/F_k] = (1+r)^{-k}X_k$$

Hence discounted wealth process $\{(1+r)^{-k}X_k;\ k = 1,2,..\}$ is a Martingale.

***Definition*: Continuous time Martingale**

Let $(\Omega, \mathrm{F}, \mathrm{P})$ be a probability space. Let $\{X_t, t \geq 0\}$ be a Stochastic Process (SP) and $\{F_t, t \geq 0\}$ be a filtration. The Stochastic Process (SP) $\{X_t, t \geq 0\}$ is said to be a Martingale corresponding to the filtration $\{F_t, t \geq 0\}$ if it satisfies the following conditions

1. For every t, $E(X(t))$ exists.
2. Each $X(t)$ is $F_t - measurable$.
3. For every $0 < s < t$ $E(X(t)/F_s) = X(s)$.

e.g. Prove that $\{W(t), t \geq 0\}$ is a Martingale

Solution: for $0 < s < t$

$$E(W(t)/F_s) = E[\{W(t) - W(s) + W(s)\}/F_s)]$$

$$= E[\{W(t) - W(s)\}/F_s)] + E[W(s)/F_s)] = W(s)$$

[since $\{W(t) - W(s)\}$ is normally distributed with mean 0]

Hence $\{W(t), t \geq 0\}$ is a Martingale .

Prob. Show that $\exp(W(t) - \frac{t}{2})$ is a Martingale .

Sol: Let $0 < s < t$. Since $\{W(t) - W(s)\}$ is independent of F_s and $W(s)$ is $F_s - measurable$. We have

$$E(e^{W(t)}/F_s) = E(e^{W(t)-W(s)+W(s)}/F_s)$$

$$= e^{W(s)}E(e^{W(t)-W(s)}/F_s)$$

$$= e^{W(s)}E(e^{W(t)-W(s)})$$ since $\{W(t) - W(s)\}$ has independent increment

Since $\{W(t) - W(s)\}$ has normal distribution with mean 0 and variance $(t - s)$, we have $E(e^{W(t)-W(s)}) = e^{(t-s)/2}$

[using mgf for normal distribution $E(e^{tX}) = \exp(\mu t + \sigma^2 t^2/2)$]

hence , $E(e^{W(t)}/F_s) = e^{W(s)}e^{(t-s)/2}$

Therefore for $0 < s < t$,

$$E(e^{W(t)-t/2}/F_s) = e^{-t/2}(e^{W(t)}/F_s) = e^{-t/2}e^{W(s)}e^{(t-s)/2} = e^{W(s)-s/2}$$

Hence it is a Martingale.

Stochastic calculus

Let the time interval be $[0,T]$ and its one of the partition is

$$\pi = \{\, 0 = t_0 < t_1 < \cdots . < t_n = T \,\} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Now Π is collection of all such partition ie, $\pi \in \Pi$. Now the norm of π is given as

$$\|\pi\| = \max_{0\le k\le n-1}(t_{k+1} - t_k) \quad \text{where } \pi \text{ is an arbitrary partition .} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

The quadratic variation for Brownian Motion (BM) $\{W(t), t \ge 0\}$ over the interval $[0,T]$ is denoted by $[W,W](T)$ and is given by

$$[W,W](T) = \lim_{\|\pi\|\to 0} \sum_{i=0}^{n-1} \{W(t_{i+1}) - W(t_i)\}^2 = \lim_{\|\pi\|\to 0} Q_\pi \qquad \ldots\ldots\ldots\ldots..(3)$$

Where

$$Q_\pi = \sum_{i=0}^{n-1} \{W(t_{i+1}) - W(t_i)\}^2 \qquad \ldots\ldots\ldots\ldots\ldots..(4)$$

(as $n \to \infty$, $\|\pi\| \to 0$)

Definition: Let $\{X_n, n \ge 1\}$ and X be random variables (r.v.s) defined on a common space $(\Omega, \mathrm{F}, \mathrm{P})$ we say that X_n converges to X in mean square sense if

$$\lim_{n\to\infty} E(|X_n - X|)^2 = 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots(5)$$

Theorem: Let Q_π is defined as in equation (4) then

1) $E(Q_\pi) = T$
2) $Var\,(Q_\pi) \le 2\,\|\pi\|\,T$

Proof:

1) We have from equation (4)

$$E(Q_\pi) = \sum_{i=0}^{n-1} E\{W(t_{i+1}) - W(t_i)\}^2$$

Since $W(t_{i+1}) - W(t_i)$ is normally distributed with mean 0 and variance $(t_{i+1} - t_i)$ for fixed i .
Hence,

$$E(Q_\pi) = \sum_{i=0}^{n-1} (t_{i+1} - t_i) = T \qquad \ldots\ldots\ldots\ldots\ldots\ldots(6)$$

2)

$$Var(Q_\pi) = \sum_{i=0}^{n-1} Var\,\{W(t_{i+1}) - W(t_i)\}^2 \qquad \ldots\ldots\ldots\ldots\ldots\ldots.(7)$$

But

$Var\{W(t_{i+1}) - W(t_i)\}^2$

$= E[\,\{W(t_{i+1}) - W(t_i)\}^4] - 2E[\,\{W(t_{i+1}) - W(t_i)\}^2(t_{i+1} - t_i)] + (t_{i+1} - t_i)^2$

...(8)

(using the formula $VarX = E(X - E(X))^2$)

Since the fourth order moment of Normal Distribution with mean 0 and variance $(t_{i+1} - t_i)$ is

$$3(t_{i+1} - t_i)^2$$

Substituting in equation (8) we get

$$Var\{W(t_{i+1}) - W(t_i)\}^2 = 3(t_{i+1} - t_i)^2 - 2(t_{i+1} - t_i)^2 + (t_{i+1} - t_i)^2$$

$$= 2(t_{i+1} - t_i)^2 \qquad \ldots\ldots\ldots\ldots\ldots(9)$$

Putting in (7) we have

$$Var(Q_\pi) = \sum_{i=0}^{n-1} 2(t_{i+1} - t_i)^2 \le 2\,\|\pi\| \sum_{i=0}^{n-1} (t_{i+1} - t_i) = 2\,\|\pi\|\,T \qquad \ldots\ldots\ldots\ldots.(10)$$

§§ With above discussion we have for a Brownian Motion (BM) $\{\,W(t), t \ge 0\,\}$, since

$$Var(Q_\pi) = E\left[\left(Q_\pi - E(Q_\pi)\right)^2\right] = E[(Q_\pi - T)^2]$$

Using (10)

$$\lim_{\|\pi\|\to 0} E[(Q_\pi - T)^2] = \lim_{\|\pi\|\to 0} Var(Q_\pi) = \lim_{\|\pi\|\to 0} 2\,\|\pi\|\,T$$

Therefore , $[W, W](T) = \lim_{\|\pi\|\to 0} (Q_\pi) = T$

Hence we write $[W, W](T) = T$ almost surely.

§§ Now we know that for any given Brownian Motion (BM) $\{W(t), t \geq 0\}$

$[W,W](T) = T$ ie., $\lim_{\|\pi\| \to 0} \sum_{i=0}^{n-1} \{W(t_{i+1}) - W(t_i)\}^2 = T$ ....(11)

Also for $0 < T_1 < T_2$

$[W,W](T_2) - [W,W](T_1) = T_2 - T_1$

We say Brownian Motion (BM) accumulates $(T_2 - T_1)$ quadratic variation over the interval $[T_1 , T_2]$.

Since it is true for every interval we inform that Brownian Motion (BM) accumulates quadratic variation at rate one per unit time this can be informally written as

$dW(t) \cdot dW(t) = dt$ ....(12)

Theorem: Let $\{W(t), t \geq 0\}$ be the given Brownian Motion (BM) and $\pi = \{ 0 = t_0 < t_1 < \cdots . < t_n = T \}$ be a partition of [0,T] , then

i. $\lim_{\|\pi\| \to 0} \sum_{i=0}^{n-1} \{(W(t_{i+1}) - W(t_i))(t_{i+1} - t_i)\} = 0$ ....(13)

ii. $\lim_{\|\pi\| \to 0} \sum_{i=0}^{n-1} \{t_{i+1} - t_i\}^2 = 0$ ....(14)

Proof: We observe that

i. $\left|\left(W(t_{i+1}) - W(t_i)\right)(t_{i+1} - t_i)\right| \leq \max_{0 \leq i \leq n-1} |W(t_{i+1}) - W(t_i)|(t_{i+1} - t_i)$

Therefore

$$\left|\sum_{i=0}^{n-1} \{(W(t_{i+1}) - W(t_i))(t_{i+1} - t_i)\}\right| \leq \max_{0 \leq i \leq n-1} |W(t_{i+1}) - W(t_i)| T$$

Since $W(t)$ is continuous hens RHS tends to zero for $\|\pi\| \to 0$

$$\lim_{\|\pi\| \to 0} \sum_{i=0}^{n-1} \{(W(t_{i+1}) - W(t_i))(t_{i+1} - t_i)\} = 0$$

ii.

$$\sum_{i=0}^{n-1} \{t_{i+1} - t_i\}^2 \leq \max_{0 \leq i \leq n-1} (t_{i+1} - t_i) \sum_{i=o}^{n-1} (t_{i+1} - t_i) \leq \|\pi\| T$$

as $\|\pi\| \to 0$,

$$\lim_{\|\pi\| \to 0} \sum_{i=0}^{n-1} \{t_{i+1} - t_i\}^2 = 0$$

§§ Using analogy of equation (12) we can write from (13) & (14)

$dW(t) \cdot dt = 0$,

$dt \cdot dt = 0$ ………………..(12A)

Definition: **Stochastic Integral**

Let $\{X(t), t \geq 0\}$ be a Stochastic Process (SP) which is adopted to the natural filtration $\{F_t, t \geq 0\}$ of Wiener process (BM) $\{W(t), t \geq 0\}$,

That is X_t is $F_t - measurable$.

Next we consider the partition $\pi \ of \ [0, T]$ where $\pi = \{ 0 = t_0 < t_1 < \cdots . < t_n = T \}$ and form the sum under limit $\|\pi\| \to 0$ of

$$\lim_{\|\pi\| \to 0} \sum_{i=0}^{n-1} \{X(t_i) \cdot (W(t_{i+1}) - W(t_i))\} = \int_0^T X(s)\, dW(s) = I(T) \qquad \dots \dots \dots . (15)$$

In defining Stochastic Integral the convergence used is mean square convergence hence the equation (15) is stochastic integral or Ito integral of the Stochastic Process (SP) $\{X(t), t \geq 0\}$ with respect to Brownian Motion (BM) $\{W(t), t \geq 0\}$.

Properties of ITO integrals

The stochastic integral I(t) , $0 < t \leq T$ satisfies the following properties

i. $E(I(t)) = 0$

ii. $E \left\{\int_0^t X(s)\, dW(s)\right\}^2 = E \left\{\int_0^t X^2(s) dW(s)\right\}$, (Ito isometry)

iii. Let $\{X^1(t), t \geq 0\} \& \{X^2(t), t \geq 0\}$ be two stochastic process having stochastic integral w.r.t. BM $\{W(t), t \geq 0\}$. Let $\alpha \,\&\, \beta$ be the constant, then

   $\int_0^t [\alpha X^1(s) + \beta X^2(s)] dW(s) = \alpha \int_0^t X^1(s) dW(s) + \beta \int_0^t X^2(s)\, dW(s)$ (linearity of Ito Int)

iv. $\int_0^t X(s)\, dW(s) = \int_0^{t_1} X(s)\, dW(s) \int_{t_1}^t X(s)\, dW(s) \quad for \ 0 < t_1 < t$

v. The process I(t) has a continuous sample path.

vi. For each t , I(t) is $F_t - measurable$.

vii. $[I, I](t) = \int_0^t X^2(s) ds$

viii. The process $\int_0^t X(s)\, dW(s)$, $t \in [0, T]$ is a Martingale with respect to natural Brownian filtration $F_t,\ 0 \leq t \leq T$.

Prob: Find the value of integral $\int_0^T W(s)\, dW(s)$

Sol: Let $\pi = \{ 0 = t_0 < t_1 < \cdots . < t_n = T \}$ be an arbitrary partition of $[0, T]$.

we have , $\int_0^T W(s)\, dW(s) = \lim\limits_{\|\pi\| \to 0} \sum_{i=0}^{n-1} \{W(t_i) \cdot (W(t_{i+1}) - W(t_i))\}$ ....(1)

but for each $'i'$, $W(t_i)$ & $W(t_{i+1}) - W(t_i)$ are independent random variables (r.v.s)and are having normal distribution. Now

$$Q_\pi = \sum_{i=0}^{n-1} \{W(t_{i+1}) - W(t_i)\}^2 = \sum_{i=0}^{n-1} [W^2(t_{i+1}) - W^2(t_i) - 2W(t_i)(W(t_{i+1}) - W(t_i))]$$

$$= W^2(T) - W^2(0) - 2 \sum_{i=0}^{n-1} [W(t_i) \cdot (W(t_{i+1}) - W(t_i))]$$

ie. $\sum_{i=0}^{n-1} [W(t_i) \cdot (W(t_{i+1}) - W(t_i))] = \frac{1}{2} [W^2(T) - W^2(0) - Q_\pi]$

taking limit as $\|\pi\| \to 0$ using equation (1)

$$\int_0^T W(s)\, dW(s) = \frac{1}{2} [W^2(T) - T]$$

Prob: Evaluate $\int_0^t W(1)\, dW(s)$, $0 \le t \le 1$.

Sol: Since $W(1)$ is not adopted to filtration $\sigma\{W(s), 0 \le s \le t\}$, $0 \le t \le 1$ because it depends on future events hence this Ito integral does not exist.

Ito - Doeblin Formula for Brownian Motion (BM) : First Version

Let f be at least twice continuously differentiable function of 't' and $\{W(t), t \ge 0\}$ be a wiener process. Then

$$df(W(t)) = f'(W(t)) \cdot dW(t) + \frac{1}{2} f''(W(t)) \cdot dt \qquad \ldots\ldots\ldots (1)$$

Or equivalently

$$f(W(t)) = f(W(0)) + \int_0^t f'(W(t)\, dW(t) + \frac{1}{2} \int_0^t f''(W(t))\, dt \qquad \ldots\ldots\ldots (2)$$

The first integral is Ito integral and second integral is Reimann integral.

Pob: Evaluate $\int_0^T W(t)\,dW(t)$ using Ito-Doeblin formula version one .

Sol: According to Ito Doeblin formula in equation (2)

If we take $f(x) = \frac{x^2}{2}$, we get $f'(x) = x$, $f''(x) = 1$ & $\int_0^t x\,dx = \frac{t^2}{2}$ where $x = W(t)$

We get ,

$$\frac{W^2(T)}{2} - 0 = \int_0^T W(t)\,dW(t) + \frac{1}{2}\int_0^T 1\,dt \qquad \text{(since } W(0) = 0\text{)}$$

$$\int_0^T W(s)\,dW(s) = \frac{1}{2}[W^2(T) - T]$$

Ito - Doeblin formula for Brownian Motion (BM) : Second Version

Let $f(t,x)$ have continuous partial derivatives of at least second order and $\{W(t), t \geq 0\}$ is a given Wiener process (W.P.). Then

$$df(t,W(t)) = f_t(t,W(t))dt + f_x(t,W(t))dW(t) + \tfrac{1}{2}f_{xx}(t,W(t))dt \qquad \ldots\ldots\ldots(3)$$

where $x = W(t)$, or equivalently

$$f(t,W(t)) - f(0,W(0))$$
$$= \int_0^t f_t(u,W(u))du + \frac{1}{2}\int_0^t f_{xx}(u,W(u))du + \int_0^t f_x(u,W(u))dW(u) \quad \ldots(4)$$

Version one and 2nd can be justified by considering Taylor's expansion of function of one variable or function of two variable respectively.

Pob: Evaluate $\int_0^T W(t)\,dW(t)$ using Ito-Doeblin formula version two .

Sol: Considering $(t,x) = \frac{x^2}{2}$, we get $f_t(t,x) = 0,\ f_x(t,x) = x,\ f_{xx}(t,x) = 1$

where $x = W(t)$ and substituting in equation (4)

$$\frac{W^2(T)}{2} - f(0,W(0)) = \int_0^T 0 + \frac{1}{2}dt + \int_0^T W(u)\,dW(u)$$

$$\int_0^T W(u)\,dW(u) = \frac{W^2(T) - T}{2}$$

Stochastic Differential Equation

Consider an IVP $\frac{dx(t)}{dt} = f(t, x(t))$, $t \in [0, T]$ & $x(0) = x_0$

Where $f: [0, T] \times R \to R$ is continuous function. This ODE possess a solution

$x(t) = x_0 + \int_0^t f(s, x(s))ds$ Provided Lipschitz condition is met by function 'f' . ie.,

$\exists\ constant\ k > 0\ \ s.t.\ |f(t, x) - f(t, y)| \le k|x - y|\ \ for\ \forall\ t \in [0, T]\ \&\ x, y \in R$

Or the solution can be obtained using standard Picard's method .

e.g., A circuit containing L-R is $L\frac{dI}{dt} + RI = aI_0 \sin \omega t$

or $\frac{L}{R}\frac{dV}{dt} + V = aI_0 \sin \omega t$ $\Rightarrow \frac{dV}{dt} = \frac{R}{L}[aI_0 \sin \omega t - V]$

Now suppose $x_0\ or\ f$ is random then solution is not unique rather it will depend up on the value $\omega \epsilon \Omega$ (sample space) .

ie, $\{X(t, \omega(t)), \omega \epsilon \Omega\ \&\ t \in [0, T]\}$, which becomes an Stochastic Process (SP) and such d.e. is called *Random Differential Equation* .

Adding an uncertainty by way of differential of Brownian Motion (BM) we get

$$\frac{dX(t)}{dt} = b(t, X(t)) + \sigma(t, X(t)) \cdot \frac{dW(t)}{dt}\ ,\quad 0 \le t \le T \qquad \text{.............(1)}$$

Where $b: [0, T] \times R \to R\ \ and\ \ \sigma: [0, T] \times R \to R$ are two given function. Equation (1) can be symbolically written as

$$dX(t) = b(t, X(t))dt + \sigma(t, X(t)) \cdot dW(t) \qquad \text{....................(2)}$$

Equation (2) is *Stochastic Differential Equation*. It can equivalent be written as

$$X(t) = X(0) + \int_0^t b(s, X(s))ds + \int_0^t \sigma(s, X(s))dW(s)\ ,\quad 0 \le t \le T \qquad \text{.............(3)}$$

Equation (3) is called *Stochastic Integral Equation* .

Strong Solution

The strong solution of SDE given by equation (2) is a Stochastic Process (SP) $\{X(t), t \in [0, T]\}$ which satisfies the following ,

i. $\{X(t), t \in [0, T]\}$ is adapted to the Brownian Motion (BM) ie, at time 't' it is a function of $W(s), s \leq t$.

ii. The integral given in equation (3) is well defined and satisfied by $\{X(t), t \in [0, T]\}$.

iii. $\{X(t), t \in [0, T]\}$ is a function of underlying BM sample path and of the coefficient $b(t, x)$ & $\sigma(t, x)$.

Thus strong solution is an explicit function 'f' such that

$X(t) = f\big(t, W(s)\big)$, $s \leq t$.

Since strong solution is based on the path of underlying BM therefore solution $\{X(t), t \in [0, T]\}$ is called unique strong solution if for any given other solution

$\{Y(t), t \in [0, T]\}$, $P\{X(t) = Y(t)\} = 1$ $for\ all$ $t \in [0, T]$

Weak Solution

For a week solution, the path behaviour is not essential. That means we are only interested in distribution of $X(t)$, which can determine expectation, variance and covariance of the process.

Diffusion

A solution of SDE (strong or weak) is called diffusion .

§§ Putting $b(t, x) = 0$ & $\sigma(t, x) = 1$ in equation (2) we see that BM is also a diffusion process.

Existence theorem

Let $E(X^2(0)) < \infty$ and $X(0)$ be independent of $\{W(t), t \geq 0\}$.

Let for all $t \in [0, T]$ and $x, y \in R$, $b(t, x)$ & $\sigma(t, x)$ be continuous and satisfy Lipschitz condition with respect to second variable ie,

$|b(t, x) - b(t, y)| + |\sigma(t, x) - \sigma(t, y)| \leq k|x - y|$ $\forall\, t \in [0, T]$, k is a constant .

Then the SDE has unique strong solution $\{X(t), t \in [0, T]\}$.

Definition: **Ito process**

Let $\{W(t), t \geq 0\}$ be a BM and let $\{F_t, t \geq 0\}$ be the associated natural filtration. And Ito process is a Stochastic Process (SP) $\{X(t), t \geq 0\}$ of the form

$$X(t) = X(0) + \int_0^t \Delta(u)dW(u) + \int_0^t \Theta(u)ds$$

where $\{X(0)$ is non-random, $\Delta(u)$ & $\Theta(u)$ are adopted process.

The SDE form of the Ito process is $\{X(t), t \geq 0\}$ is

$$dX(t) = \Delta(t)dW(t) + \Theta(t)dt$$

SDE of GBM

Let $S(t)$ be the stock price at time 't'. Let $-\infty < \mu < \infty$ be the constant growth rate of the stock and $\sigma > 0$ be the volatility. Considered the SDE

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t)\,,\ S(0)\ is\ known. \qquad \text{……….(1)}$$

We wish to find the strong solution of $S(t)$ if it exist.

Now the condition of existence theorem is verified since μ & σ are constant .

Let us assume that $S(t) = f(t, W(t))$

Using second version of Ito-Doeblin formula we get set

$$df\big(t, W(t)\big) = f_t\big(t, W(t)\big)dt + f_x\big(t, W(t)\big)dW(t) + \frac{1}{2}f_{xx}\big(t, W(t)\big)dt\ , \ \text{where } x = W(t)$$

…….(2)

Comparing with given SDE (1) we get

$$\tfrac{1}{2}f_{xx} + f_t = \mu S(t) = \mu f \qquad \text{…….(3)}$$

$$f_x = \sigma S(t) = \sigma f \qquad \text{……(4)}$$

Solving equation (4) we get

$$f(t,x) = k(t)e^{\sigma x} \quad for\ some\ function\ \ k(t) \qquad \text{…..(5)}$$

From equation (5) we have

$$f_t = k'(t)e^{\sigma x} \qquad \& \qquad f_{xx} = \sigma^2 k(t) e^{\sigma x} \qquad \ldots\ldots\ldots\ldots(6)$$

substituting in equation (3) we get

$$\left[\frac{1}{2}\sigma^2 k(t) + k'(t)\right] e^{\sigma x} = \mu k(t) e^{\sigma x}$$

or

$$k'(t)e^{\sigma x} = \left(\mu - \frac{\sigma^2}{2}\right) k(t) e^{\sigma x} \qquad \ldots\ldots\ldots\ldots(7)$$

Solving equations (7) gives

$$k(t) = S(0) e^{\left(\mu - \frac{\sigma^2}{2}\right)t} \qquad \ldots\ldots\ldots\ldots(8)$$

[$since\ \ k(t) = Ae^{\left(\mu - \frac{\sigma^2}{2}\right)t}, at\ t = 0\,, A = k(0)\ \ , using\ (5)\quad S(t) = f(t, W(t))$
$= k(t)e^{\sigma W(t)} \ \rightarrow\ k(0) = S(0)$]

Hence the required solution is, form equation (5)

$$S(t) = f(t, W(t)) = S(0) e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W(t)}$$

We observe that for fixed 't' , $S(t)$ follows lognormal distribution.

§§ $E(S(t)) = S(0)e^{\mu t}$

§§ $E\left(S^2(t)\right) = S^2(0)e^{(2\mu + \sigma^2)t}$

§§ $Var(S(t) = E\left(S^2(t)\right) - [E(S(t))]^2 = S^2(0)e^{2\mu t}\left(e^{\sigma^2 t} - 1\right)$

Discounted Portfolio Process

let the stock having price $S(t)$ per unit follows a generalized GBM with constant mean return μ and a constant volatility $\sigma > 0$. The price is governed by SDE

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t)\ , \quad t \in [0, T] \qquad \ldots\ldots\ldots\ldots(1)$$

Also let $\beta(t)$ be the price of risk free asset which satisfy the ordinary d.e.

$$d\beta(t) = r \cdot \beta(t) \cdot dt \qquad \ldots\ldots\ldots\ldots(2)$$

where $'r'$ is constant risk free interest rate .

Suppose at time $'t'$ we take a Portfolio consisting of $a(t)$ shares of stock and $b(t)$ shares of risk free asset. Let $V(t)$ be the value of this portfolio at $'t'$,that is

$$V(t) = a(t)\, S(t) + b(t)\, \beta(t)\,, \quad t \in [0,T] \qquad \ldots\ldots\ldots\ldots..(3)$$

Then,

$$dV(t) = a(t)\, dS(t) + b(t)\, d\beta(t) \qquad \ldots\ldots\ldots\ldots.(4)$$

The discounted price of one share of stock is

$$\tilde{S}(t) = e^{-rt} S(t)\ , \quad t \in [0,T] \qquad \ldots\ldots\ldots\ldots(5)$$

Applying Ito-Doeblin formula of second variant ,

$$d\tilde{S}(t) = -re^{-rt} S(t)dt + e^{-rt} dS(t) \qquad \ldots\ldots\ldots(6)$$

$$= -re^{-rt} S(t)dt + e^{-rt}[\, \mu S(t)dt + \sigma S(t)dW(t)\,] \qquad \text{using equation (1)}$$

$$= \tilde{S}(t)\, [\, (\mu - r)\, dt + \sigma dW(t)\,]$$

$$= \sigma\, \tilde{S}(t)\, d\widetilde{W}(t)\ , \ where\ \ \widetilde{W}(t) = \frac{(\mu - r)}{\sigma}\, t + W(t),\ \ t \in [0,T] \qquad \ldots\ldots\ldots\ldots(7)$$

Now $(\mu - r)$ is called risk premium .

Therefore $\frac{(\mu - r)}{\sigma}$ is the risk premium per unit of risk and is called the market price of risk .

Feynman-Kac Theorem (*R. Feynman & M. Kac*)

It establishes a link between parabolic p.d.e. and stochastic process .

Let the stochastic process $\{X(t), 0 \leq t \leq T\}$ satisfy the following SDE

$$dX(t) = \mu(t, X(t))dt + \sigma(t, X(t)) \cdot dW(t)$$

Where $\mu(t, X(t))$ & $\sigma(t, X(t))$ are functions on $[0, T] \times R \rightarrow R$ called drift and diffusion function respectively . Also $X(0) = x$ $for\ some$ $x \in R$. Then the solution of the following p.d.e.

$$g_t(t, x) + \mu(t, x)g_x(t, x) + \tfrac{1}{2}\sigma^2(t, x)g_{xx}(t, x) - r\, g(t, x) = 0 \qquad \text{......(8)}$$

Subject to the boundary condition

$$g(T, X(T)) = x = h(x) \ , \quad x \in R$$

Is a function $g: [0, T] \times R \rightarrow R$ given by

$$g(t, x) = E\left[e^{-r(T-t)} \cdot h(X(T)/(X(t) = x))\right] \qquad \text{.......(9)}$$

We define an operator as following (called generator of the process)

$$\mathcal{A} = \mu(t, x(t))\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2(t, x(t))\frac{\partial^2}{\partial x^2}$$

§§ Feynman-Kac theorem implies both way, ie, if pde is given then solution is known and if a solution satisfying the boundary condition, then the pde whose solution is this is known.

Then equation (8) can be written as

$$\frac{\partial g}{\partial t} + \mathcal{A}g - rg = 0$$

Derivation of Black- Scholes formula for a derivative security

Let the stock price $S(t)$ be driven by the process

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t)$$

Using equations (7) where $\widetilde{W}(t) = \frac{(\mu - r)}{\sigma}\, t + W(t)$ the risk neutral process is given by

$$dS(t) = rS(t)dt + \sigma S(t)d\widetilde{W}(t) \qquad \text{............(10)}$$

Suppose a derivative is written on this stock. Let $V(t, S(t))$ be the price of this security at any $t \in [0, T]$ and $V(T, S(T))$ be its pay off on maturity.

Here $V: [0, T] \times R_+ \rightarrow R_+$, R_+ is non-negative real number .

Using Ito-Lemma we have

$$dV(t) = dV(t, S(t)) = V_t dt + V_x dS(t) + \frac{1}{2}V_{xx}dS(t)dS(t)$$

Here $x = S(t)$

$$= \left[\frac{\partial V}{\partial t} + r \cdot S(t)\frac{\partial V}{\partial x} + \frac{1}{2}\sigma^2 \cdot S^2(t)\frac{\partial^2 V}{\partial x^2}\right] dt + \sigma \cdot S(t)\frac{\partial V}{\partial x} d\widetilde{W}(t) \qquad ....(11)$$

$$\left[dS(t)dS(t) = \sigma^2 S^2(t) d\widetilde{W}(t) d\widetilde{W}(t) = \sigma^2 S^2(t)dt \qquad squarring\ (10)\right]$$

Suppose the derivative security can be hedged .We replicate the portfolio taking $a(t)$ shares of stock and $b(t)$ shares of risk free asset whose price is governed by *ode*

$$d\beta(t) = r \cdot \beta(t) \cdot dt$$

Then we have

$$dV(t) = a(t)\, dS(t) + b(t) \cdot r \cdot \beta(t)\, dt \qquad \text{(using equation (2 \&3))} \qquad ....(12)$$

$$= a(t)\left[\, rS(t)dt + \sigma S(t) d\widetilde{W}(t)\,\right] + r \cdot \beta(t) \cdot b(t)dt$$

comparing equation (11) and equation (12)

$$\frac{\partial V}{\partial x} = a(t), \quad \& \quad \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2(t)\frac{\partial^2 V}{\partial x^2} = r \cdot \beta(t) \cdot b(t) \qquad ....(13)$$

now using equation (13) in equation (3)

$$b(t)\,\beta(t) = V(t) - a(t)\,S(t) = V(t) - S(t)\frac{\partial V}{\partial x}$$

putting the value of $b(t)\beta(t)$ in equation (13) we get

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2(t)\frac{\partial^2 V}{\partial x^2} = \left[V(t) - S(t)\frac{\partial V}{\partial x}\right].r$$

$$\frac{\partial V}{\partial t} + r.S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2(t)\frac{\partial^2 V}{\partial S^2} - r.V = 0$$

which is ***Black-Scholes*** pde for derivative price.

The generator of the process is given by

$$\mathcal{A} = r.S\frac{\partial}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2}{\partial S^2}$$

By Feynman-Kac theorem the time ‘t’ value of the derivative is the solution

$$V\big(t, S(t)\big) = e^{-r(T-t)} E_{\tilde{P}}[h(S(t))/F_t]$$

Where $\tilde{P}$ is risk neutral probability measure, and $h(S(t))$ is pay-off of the derivative security at maturity.