Asset definition: worth in form of other than currency.

Types of asset:

1. Risk free asset : bank deposit government Bonds, institutions Bonds
2. Risky asset : stock, gold, foreign currency more generally any asset whose future price is not known today.

$S(t)$ represent the price of one share of the stock at time t (not known to investor) and $S(0)$ price of one share of the stock (known to investor) today that is at $t = 0$.

And at $t = 1$ (say one year from now) the stock price becomes $S(t)$ or $S(1)$.

Value of share price in future, say after one year is unknown and is uncertain: It may go up as well as go down.

Rate of return or Return

The difference $S(1) - S(0)$ as a fraction of the initial value $S(0)$ represents the rate of return

$$k_s = \frac{S(1) - S(0)}{S(0)}$$

Similarly $A(0)$ & $A(t)$ represents the price of one Bond risk free at time $t = 0$ and any time $t = 1$ respectively. Here $A(0)$ & $A(t = 1)$ both are known and payment is guaranteed by the bond issue institution. Return on bonds is defined in similar way as that of stock

$$k_A = \frac{A(1) - A(0)}{A(0)}$$

Now to build a mathematical model of a financial securities we take some assumptions (to compromise between complex real-world and limitation and simplification of mathematical model)

Assumption 1 - Randomness

The future value of stock price $S(t)$ is a random variable with at least two different values, while $A(t)$ the future Bond prices known number.

Assumption 2 - Positivity of the price

All stocks and Bond prices are strictly positive that is

$A(t) > 0 \; \& \; S(t) > 0 \; for \; t = 0,1$

Total wealth of an investor holding $x - \text{stock } shares \; and \; y \; - \; bond$ at time $t = 0,1$ is

$V_P(0) = x\,S(0) + y\,A(0)$ and $V_P(1) = x\,S(1) + y\,A(1)\; x, y \in R$

The pair(x, y) is called portfolio. The return on the portfolio

$$K_V = \frac{V(1) - V(0)}{V(0)}$$

Assumption 3- Divisibility, Liquidity and Short Selling

An investor may hold any number $x \,\&\, y$ of stock share and bonds , ie,

$$x, y \in R \qquad \ldots(1)$$

The fact that one can hold a fraction of a share or bond is referred to as *divisibility*.

Since by equation(1) $x \;\&\; y$ unbounded, that means any asset can be bought or sold on demand at market price in arbitrary quantities - this is *liquidity*.

If the number of securities of a particular kind held in a Portfolio is positive, investor is said to have *long position* otherwise we say that a *short position* is taken, or that asset is shorted.

Assumption 4- Solvency

The wealth of an investor must be non-negative at all time

$$V(t) \geq 0\,, \;\; for \; t = 0,1.$$

A portfolio satisfying the condition is called *admissible*.

Assumption 5- Discrete unit price

The future price $S(1)$ of a share of stock is a random variable taking only finitely many values.

Assumption 6- No - Arbitrage Principle

The market does not allow for risk free profits with no initial investment.

That is, there is no admissible portfolio with initial value

$$V(0) = 0 \quad such\ that \quad V(1) > 0$$

with non zero probability. This means that no investor can lock in profit without risk and with no initial endowment.

If a Portfolio violating this principle did exist, then an arbitrage opportunity was available.

(Note: Arbitrage opportunities really exist in practice. Even if they do exist, gains are extremely small as compared to volume of transactions. They are also short-lived.)

(Note: The assumption of no arbitrage principle is the main tool for financial mathematics.)

Risk and return

Instead of using historical data for calculating return and risk, we may use forecasted data. The price of a share of a particular company in future depends on many factor, viz; performance of the company, economic scenario, and other factors like National and international politics, etc. Under this condition the price of a share may go up or down in future. Going up results into profit and going down in the loss. The period of holding share may go through various phases.

e.g; Assume four (equally likely) possible states of economic condition and performance: high growth, expansion, stagnation and decline. Suppose today price of one share of a company is Rs. 261.25 and under above 4-states (in a year) expected to be Rs.305.5, Rs.285.5, Rs.263.75, and Rs.243.5 the return for each phase is given as

$$k_s = \frac{S(1) - S(0)}{S(0)}$$

but it is different for different economic states

Economic state	share price at t=o in Rs.	share price at t=1 in Rs.	return K_s in %
high growth	261.25	305.5	16.9%
expansion	261.25	285.5	9.3%
stagnation	261.25	263.25	0.8%
decline	261.25	243.5	-6.8%

Now the total return anticipated is varying from -6.8% to 16.9%.

Since the Expected rate of return is given

$$R = E(R_i) = \sum_{i=1}^{n} p_i R_i \text{ }, p_i \rightarrow probability\ of\ phase\ 'i',\ R_i$$

$$\rightarrow return\ in\ phase\ 'i'$$

Hence the average rate of return $= \frac{\{16.9+9.3+0.8+(-6.8)\}}{4} \approx 4.85\%$

Since the phases were taken equally likely hence $p_i = 1/4$

(But the probability of return in various phases is on the basis of judgement of investor, it will be different for different phase.)

Now how much is the dispersion, while possible return is varying in the range -6.8% to 16.9%.

It is explained by variance are standard deviation.

$$\sigma^2 = \sum_{i=1}^{n} [R_i - E(R_i)]^2 p_i$$

In case of present problem $\sigma^2 = 81.06 \Longrightarrow \sigma \approx 9\,\%$.

Should one invest in the company? No! Because the dispersion is more than return. One should always look for higher expected return and lower standard deviation.

Example 2.

Let $A(0) = \$\,100$ *and* $A(1) = \$\,110$, and, but $S(0) = \$\,80$ and

$$S(1) = \begin{cases} \$100 & \textit{with probability } 0.8 \\ 60 & \textit{with probability } 0.2 \end{cases}$$

Suppose that you have $10, 000 to invest in a portfolio. You decide to buy x = 50 shares, which fixes the risk-free investment at y = 60. Then

$$V\,(1) = \begin{cases} \$11,600 & \textit{if stock goes up} \\ \$9,600 & \textit{if stock goes down} \end{cases}$$

$$k_V = \begin{cases} 0.16 & \textit{if stock goes up} \\ -0.04 & \textit{if stock goes down} \end{cases}$$

The *expected return*, that is, the mathematical expectation of the return on the portfolio is

$$E(k_V) = 0.16 \times 0.8 - 0.04 \times 0.2 = 0.12,$$

that is, 12%. The *risk* of this investment is defined to be the standard deviation of the random variable K_V

$$\sigma = \sqrt{(0.16 - 0.12)^2 \times 0.8 + (-0.04 - 0.12)^2 \times 0.2} = 0.08 \quad ie, \quad 8\%$$

Let us compare this with investments in just one type of security. If x = 0, then y = 100, that is, the whole amount is invested risk-free. In this case the return is known with certainty to be $k_A = 0.1$ that is, 10% and the risk as measured by the standard deviation is zero, $\sigma_A = 0$

On the other hand, if x = 125 and y = 0, the entire amount being invested in stock, then

$$V\,(1) = \begin{cases} \$12500 & \textit{if stock goes up} \\ \$7500 & \textit{if stock goes down} \end{cases}$$

$E(K_V) = 0.15$ with $\sigma_V = 0.20$, that is, 15% and 20%, respectively.

Therefore higher return is associated with higher risk.

Zero coupon Bond

It is one of the risk free asset, in which payment of money is guaranteed to Investor by the bond issuing agency (government are Financial Institutions) for example issued by Bank-FD, Post-office – NSC, Commercial institutions - debentures.

The zero coupon Bond involves a single payment. The issuing agency promises to pay a fixed amount of money F, called face value, in exchange of the bond, on a given day T, called maturity date.

Usually the face value is a round figure, e.g; 100.

Given the Interest rate the present value of bond is easily calculated

$$V(0) = (1+r)^{-1}F \qquad \text{(for 1-year bond)} \qquad \ldots(1)$$

e.g; if face value of a bond is Rs.100 with r being 12% annual compounding for one year. Present value of the bond would be

$$V(0) = (1+0.12)^{-1}100 \cong Rs. 89.29$$

From equ (1) , we have $r = \frac{F}{V(0)} - 1$

This gives the *implied* annual compounding rate.

Now for simplicity, we consider the face value of bonds equal to 1, ie; $F = 1$ (any currency). It is called unit bond.

In practice bonds are freely traded in the market and their price is governed by market forces.

Typically, a bond can be sold at any time prior to maturity at the market price. This price at time 't' is denoted by $B(t,T)$. In particular $B(0,T)$ is the current time $'t = 0'$ price, $B(T,T) = 1$ is equal to face value.

These prices determined interest rates.

The value of bond at any time 't' prior to maturity is determined by discounting the face value, ie;

$B(t,T) = V(t) = e^{-(T-t)r}F$ provided the continuously compounded interest is 'r'.

(Note: When trading of bonds takes place then it is 'r' which is manipulated)

$for\ t = 0\ ,\quad V(0) = e^{-rT}F$

Now for $\quad F = 1,\ B(t,T) = V(t) = e^{-(T-t)r}$

For periodic compounding with frequency 'm' we have

$$B(t,T) = \left(1 + \frac{r}{m}\right)^{-(T-t)m}$$

For annual compounding $m = 1$ hence $\quad B(t,T) = (1 + r)^{-(T-t)}$.

§§ $\quad B(0,T) = e^{-rT}$ is discount factor and

$\quad (B(0,T))^{-1} = e^{rT}$ is growth factor

Coupon bonds

Bonds promising a sequence of payments are called coupon bonds. This payments consists of face value due at maturity, and coupons paid regularly (annually, half yearly, quarterly). The last coupon is paid at maturity. Presentation of constant interest rate allows us to compute the price of the coupon Bond by discounting all the future payments.

If F if is the face value, c is the coupon paid annually, and time of maturity be $n - years$ then price of the bond

$$V(0) = ce^{-r} + ce^{-2r} + \cdots + (F + c)e^{-nr} \qquad \ldots(1)$$

Where 'r' is the interest rate with continuous compounding.

After one year, once the first coupon is cashed the bond becomes $(n - 1)$ year Bond worth

$$V(1) = ce^{-r} + ce^{-2r} + \cdots + (F + c)e^{-(n-1)r} \qquad \ldots(2)$$

Comparing equation (1) & (2) we have total wealth at time 1 is

$$V(1) + c = V(0)e^{r}$$

One step Binomial model

If (x, y) be the portfolio, then

$$V(0) = xS(0) + yA(0)$$

Now the k_A is fixed (risk free return) but the value of stock can attain two values

$$S(1) = \begin{cases} S^u & \textit{stock goes up with probablity } p \\ S^d & \textit{stock goes down with probablity } 1-p \end{cases}$$

Note that S^u & S^d are relative to each other, both of them can be greater than $S(0)$.

Hence , $V(1) = \begin{cases} xS^u + yA(1) & \textit{with probablity } p \\ xS^d + yA(1) & \textit{with probablity } 1-p \end{cases}$

e.g; Suppose that $S(0) = \$100$ and

$$S(1) = \begin{cases} \$125 & \textit{with probablity } p \\ \$105 & \textit{with probablity } 1-p \end{cases}$$

where $0 < p < 1$, while the bond prices are $A(0) = \$100 \ \ and \ \ A(1) = \$110$.

$$k_s = \begin{cases} 0.25 \ \ or \ \ 25\% & \textit{with probablity } p \\ 0.05 \ \ or \ \ 5\% & \textit{with probablity } 1-p \end{cases}$$

Thus, the return k_s on stock will be 25% if stock goes up, or 5% if stock goes down. (Observe that both stock prices at time 1 happen to be higher than that at time 0; 'going up' or 'down' is relative to the other price at time 1.) The risk-free return k_A will be 10%.

Proposition: If $S(0) = A(0)$, then $S^d < A(1) < S^u$ or else an arbitrage opportunity would arise.

Proof: Let $A(1) \leq S^d$ ...(1)

In this case, at time $t = 0$:

- Borrow the amount $S(0)$ risk-free.
- Buy one share of stock for $S(0)$.

This way, a portfolio (x, y) created with $x = 1$ shares of stock and $y = -1$ bonds. The time $t = 0$ value of this portfolio is

$$V(0) = 0 = 1.S(0) + (-1)S(0)$$

At time $t = 1$

- Return the risk free borrowed amount $S(0)(1+r) = A(1)$.
- Sell one share of stock for $S(1)$.

And the value of portfolio will become

$$V(1) = \begin{cases} S(1) - A(1) = S^u - A(1) > 0 & \textit{if stock goes up} \\ S(1) - A(1) = S^d - A(1) > 0 & \textit{if stock goes down} \end{cases}$$

Under assumption (1), the first of these two possible values is strictly positive, while the other one is non-negative, that is, $V(1)$ is a non-negative random variable such that V (1) > 0 with probability p > 0. The portfolio provides an arbitrage opportunity, violating the No-Arbitrage Principle.

Hence $A(1) \nleq S^d$ ...(2)

Now suppose that $A(1) \geq S^u$ . ...(3)

If this is the case, then at time $t = 0$

- Sell short one share for $S(0)$.
- Invest $S(0)$ risk-free.

This way, a portfolio (x, y) created with $x = -1$ shares of stock and $y = 1$ bonds and initial value, $V(0) = 0$.

At time $t = 1$

- Receive the risk free invested amount $S(0)(1+r) = A(1)$.
- Buy one share of stock for $S(1)$ and close the short position.

And the value of portfolio will become

$$V(1) = \begin{cases} A(1) - S(1) = A(1) - S^u > 0 & \textit{if stock goes up} \\ A(1) - S(1) = A(1) - S^d > 0 & \textit{if stock goes down} \end{cases}$$

which is non-negative, under assumption (3), with the second value being strictly positive. Thus, $V(1)$ is a non-negative random variable such that $V(1) > 0$

with probability $1 - p > 0$. Once again, this indicates an arbitrage opportunity, violating the No-Arbitrage Principle.

Hence, $A(1) \ngeq S^u$ ...(4)

Using equation (2 & 4) we have the result, $S^d < A(1) < S^u$.

Forward contract

A forward contract is an agreement to buy or sell a risky asset at a specified future time, known as delivery date, for a price F fixed at the present moment, called the forward price.

An investor who agrees to buy the asset is said to enter into a long forward contract or to take a long forward position.

If an investor agrees to sell the asset is said to enter into short forward contract or to take short forward position.

No money is paid at the time when forward contract is exchanged.

Exercise of the contract is mandatory for both the parties.

If asset price on the delivery date $t = 1$ is $S(1)$ and fixed price on the date of contract $t = 0$ is F, then

Payoff for long forward position holder = $S(1) - F$

Payoff for short forward position holder = $F - S(1)$

The value of $S(1) - F$ can be positive, zero, or negative.

Apart from stock and bonds, a Portfolio held by an investor may contain forward contracts, in this case it will be described by a triple (x, y, z). Here x and y are number of stock share and bonds and z is the number of Forward contracts (positive for long forward position and negative for short position).

Since no payment is due when a forward contract is exchanged, initial value of such portfolio is

$$V(0) = xS(0) + yA(0)$$

at the delivery date the value of the portfolio will become

$$V(1) = xS(1) + yA(1) + z(S(1) - F)$$

To remove the ambiguity we shall represent agreed forward price as $F(0, T)$.

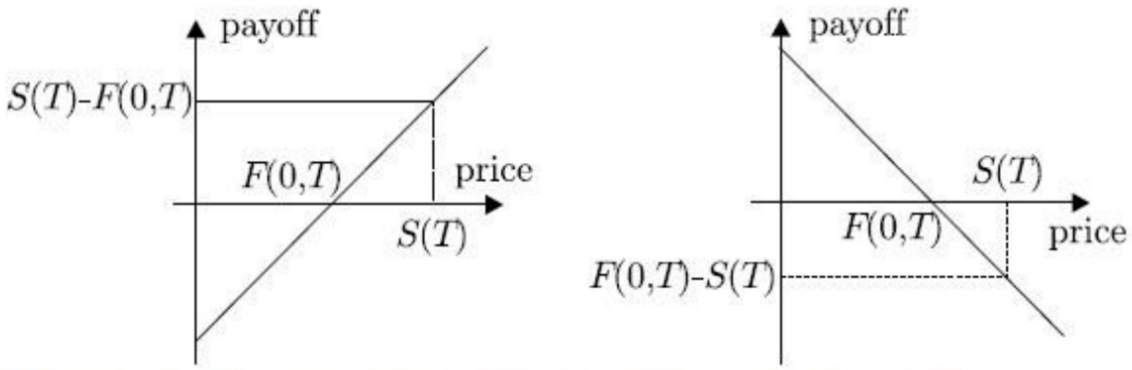
In a forward contract between two parties at the time of delivery one has to bear the loss while other party has gain.

There are two possibilities

i. $F(0, T) < S(T)$

ii. $F(0, T) > S(T)$

In case of (i) the party having long position will gain. The Asset will be bought at lower price $F(0,T)$ and sold at market price $S(T)$ getting a profit $S(1) - F$ (which is loss of short position holder)



Payoff for long and short forward positions at delivery

A similar scenario will arise in case (ii).

Sometimes the contract may be initiated at time $t < T$ rather than $t = 0$

In this case the payoff for long and short positions will be $S(T) - F(t,T)$ and $F(t,T) - S(T)$

respectively, where $F(t,T)$ is the forward contract price at time 't'.

Theorem: Forward price formula

Let the price of an asset at $t = 0$ be $S(0)$. Then the forward price $F(0,T)$ is given by

$$F(0,T) = \frac{S(0)}{d(0,T)}$$

Where $d(0,T)$ is the discount factor between $t = 0$ to $t = T$.

Proof: The proof is based on violation of no arbitrage principle. If possible let

$$F(0,T) > \frac{S(0)}{d(0,T)}$$

Then at $t = 0$ we construct the portfolio P as following,

1) Borrow the amount $S(0)$ for time T
2) Buy one unit of underlying stock at $S(0)$ price
3) Take a short forward position with delivery time T and $F(0,T)$

This portfolio is (1,-1,1).

Now the value of portfolio at $t = 0$ will be

$$V_P(0) = S(0) - S(0) = \ 0$$

Now at $t = T$

1) Close the short position by selling asset at price $F(0,T)$
2) Return the borrowed money with interest i.e., $\frac{S(0)}{d(0,T)}$

This gives

$$V_P(T=1) = F(0,T) - \frac{S(0)}{d(0,T)} > 0 \quad \text{, strictly positive.}$$

It is creating and arbitrage opportunity,

Hence , $$F(0,T) \ngtr \frac{S(0)}{d(0,T)}$$

Similarly if possible let $F(0,T) < \frac{S(0)}{d(0,T)}$

Construct a Portfolio P as following;

1) Short sell one unit of underlying asset.
2) Invest in risk free for the time $t = T$
3) Enter into a long forward contract with delivery at T and forward price $F(0,T)$.

This portfolio is (-1,1,1) at $t = 0$.

The value of the portfolio at $t = 0$ will be 0.

$$V_P(0) = -S(0) + S(0) = \ 0$$

at $t = T$,

1) Buy the asset under forward contract for $F(0,T)$
2) Encash risk-free amount $\frac{S(0)}{d(0,T)}$
3) Close shots sell by returning one unit of underlying asset.

The value of the portfolio at T

$$V_P(T=1) = \frac{S(0)}{d(0,T)} - F(0,T) > 0 \quad \text{, strictly positive.}$$

Creating again and arbitrage, hence. $F(0,T) \nless \frac{S(0)}{d(0,T)}$

Combining the conclusion of these two situations we have

$$F(0,T) = \frac{S(0)}{d(0,T)}$$

§§ If the constant interest rate is compounded continuously, we have

$d(0,t) = e^{-rt}$ and hence

$$F(0,T) = S(0)\, e^{rt}$$

§§ If the contract is initiated at intermediate time $0 < t < T$ then

$d(0,t) = e^{-r(T-t)}$, hence $F(t,T) = S(0)\, e^{r(T-t)}$

Since in case of zero coupon Bond $B(0,T)$ represents the discount factor hence the forward price formula becomes

$$F(0,T) = \frac{S(0)}{B(0,T)}$$

Holding of physical assets like gold, sugar oil etc, attract charges like rent insurance etc. These should be included in the cost of Forward price.

Theorem: Forward price formula with carrying cost

Let an asset carry a holding cost of $C(i)$ per unit in period i , $(i = 0,1,2 \ldots (n-1))$. Also let at $t = 0$ the price of this asset be $S(0)$ and short selling be allowed. Then

$$F(0,T) = \frac{S(0)}{d(0,T)} + \sum_{i=0}^{n-1} \frac{C(i)}{d(i,n)}$$

where delivery date is $t = T$ and between $t = 0$ to $t = T$ there are n-periods which have been identified approximately as per the given contract.

Theorem: Forward price formula with dividend

Let an asset be stored at zero cost and also sold short. Let the price of this asset at

$t = 0$ be $S(0)$ and a dividend of Rs. 'div' be paid at time τ , $0 < \tau < T$, Then

$$F(0,T) = \frac{\{S(0) - div.\, d(0,\tau)\}}{d(0,T)}$$

For the case of constant interest rate $'r'$ being compounded continuously. The the above result takes form

$$F(0,T) = \{S(0) - div.\, e^{-r\tau}\}\, e^{rT}$$

In case the asset pays dividend continuously at a rate of r_{div} , Then

$$F(0,T) = S(0)\,.\,e^{(r-r_{div})T}$$

The value of a forward contract

Let a forward contract initiated at $t = 0$ with delivery time at $t = T$. Also let $F(0,T)$ be the forward price of this contract. Consider an intermediate time $0 < \tau < T$ and let $F(\tau,T)$ be the forward price of the contract initiated at $t = \tau$ with delivery at $t = T$.

We have two forward contracts one initiated at $t = 0$ and another initiated at $t = \tau$ both having delivery date is $t = T$ with the forward price $F(0,T)$ & $F(\tau,T)$ respectively.

As the time passes by the value of the forward contract initiated at $t = 0$ will be changing. Let $f(\tau)$ is the value of the forward contract at $t = \tau$.

Theorem: Value of a forward contract

$$f(\tau) = [F(\tau,T) - F(0,T)]d(\tau,T)$$

Where $d(\tau,T)$ is the risk free discount over the period $t = \tau$ to $t = T$.

Proof: Let $f(\tau) < [F(\tau,T) - F(0,T)]d(\tau,T)$

Now at $t = \tau$ construct a Portfolio P as following;

i. Borrow the amount $f(\tau)$
ii. Enter into long forward contract with forward price $F(0,T)$ (by paying the value $f(\tau)$) and delivery time $t = T$
iii. Enter into short forward position with forward price $F(\tau,T)$ (for which there is no cost as per definition of the forward contract).

The value of the portfolio P at $t = \tau$ is given as

$$V_P(\tau) = 0$$

Now at $t = T$, proceed as following;

i. Close forward contract by collecting (or playing) the amount $S(T) - F(0,T)$ for long forward position and $F(\tau,T) - S(T)$ for short forward position.

ii. Payback the amount of loan with interest that is $\frac{f(\tau)}{d(\tau,T)}$

Therefore the value of the portfolio at $t = T$ is

$$V_P(T) = F(\tau,T) - F(0,T) - \frac{f(\tau)}{d(\tau,T)}$$

Which is strictly positive, violating no arbitrage principle. Hence

$$f(\tau) \nless [F(\tau,T) - F(0,T)]d(\tau,T)$$

Secondly let $f(\tau) > [F(\tau,T) - F(0,T)]d(\tau,T)$

In this case construct a Portfolio at $t = \tau$ as following;

i. Sell short the forward contract which was initiated at it $t = 0$ for the amount $f(\tau)$

ii. Invest $f(\tau)$ in risk free for $t = \tau$ to $t = T$

iii. Enter into a long forward contract with delivery date $t = T$ at $F(\tau,T)$

The worth of the portfolio is zero.

At $t = T$

$$V_P(T) = \{S(T) - F(\tau,T)\} + \{F(0,T) - S(T)\} + \frac{f(\tau)}{d(\tau,T)}$$

$$= F(0,T) - F(\tau,T) + \frac{f(\tau)}{d(\tau,T)}$$

Which is strictly positive, violating no arbitrage principle.

Hence $f(\tau) \ngtr [F(\tau,T) - F(0,T)]d(\tau,T)$

Combining both the conclusions we

$$f(\tau) = [F(\tau,T) - F(0,T)]d(\tau,T)$$

Options

An option is a contract that gives the holder a right to trade, without any obligation to buy or sell an asset at an agreed price on or before a specified period of time.

(each option is usually for 100 shares of specified stock)

There are two types of options;

1. Call option: is the right to buy an asset at a specified price.
2. Put option: is the right to sell an asset at a specified price.

The specified price at which the option holder has the right to trade is called an *exercise price or a strike price*.

The asset on which the put or call option is created is referred to as the underlying asset.

(Note: Options do not come. free. They involve cost. The option premium is the price that the holder of an option has to pay for obtaining a call or a put option. The price will have to be paid, generally in advance, whether or not the holder exercises his option.)

Category of Options:

1. European option: When an option is allowed to be exercised only on the maturity date, it is called a European option.
2. American option: When the option can be exercised any time before its maturity, it is called an American option.

Exercise of the option results into three possibilities:

In-the-money: A put or a call option is said to be in- the- money when it is advantageous for the investor to exercise it.

Out-of-the-money: A put or a call option is out-of-the-money if it is not advantageous for the investor to exercise it.

At-the-money: When the holder of a put or a call option does not lose or gain whether or not he exercises his option, the option is said to be at-the- money.

Buyers are referred to as having long position sellers are referred to as having short positions. Selling is also known as writing the option.

<u>Payoff of a call option C(T)</u>

(1) **<u>Buyer</u>**

Let $S(0)$ be the current asset price and becomes $S(T)$ at maturity date. Let E be the option exercise price. We have three possibilities at maturity;

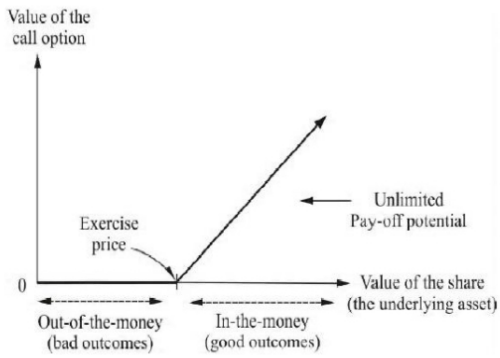
i. $S(T) > E$ call option should we exercised because buy the asset for E and sell for $S(T)$ to get a profit $S(T) - E$ which is payoff of the call option (not including commission and others) .

ii. $S(T) < E$ call option should not be exercised as it will attract loss hence pay office is zero.

iii. $S(T) = E$ again no use to exercise call option hence payoff is zero.

Including all three cases we can say that the value of call option at expiration

$$\boldsymbol{C(T) = \max[\, S(T) - E, 0\,]}$$

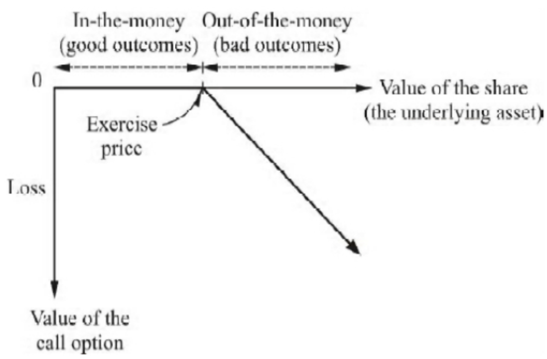


Payoff of a call option buyer

(2) **<u>Seller</u>**

(Depends solely on the buyer) will have just opposite of buyer

$$\boldsymbol{C(T) = min[\, E - S(T), 0\,]}$$

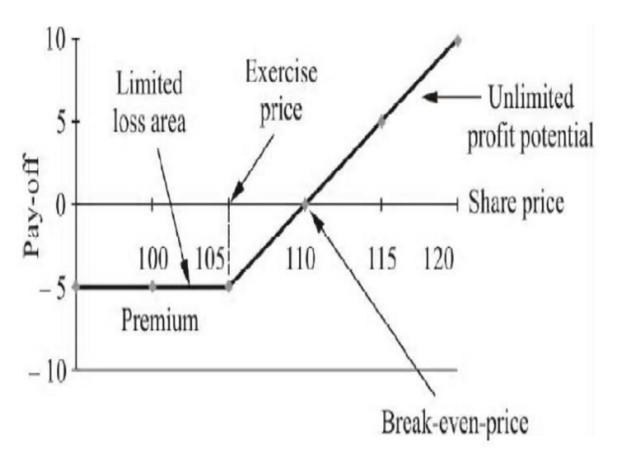


Payoff of a call option seller

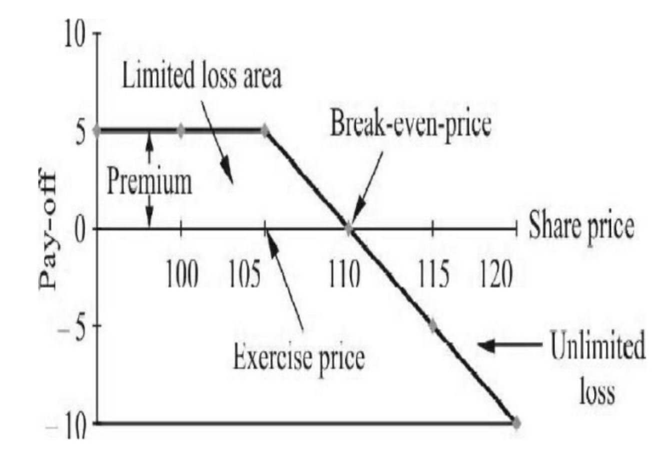
e.g; The share of Telco is selling for Rs. 104. Radhey Acharya buys a 3 months call option at a premium of Rs. 5. The exercise price is Rs. 105. What is Radhey's pay-off if the share price is Rs 100, or Rs. 105, or Rs. 110, or Rs. 115, or Rs. 120 at the time the option is exercised?

The Call Option Holder's Pay-off at Expiration

Share price $S(T)$ ⟶	In Rs. 100	In Rs. 105	In Rs. 110	In Rs. 115	In Rs. 120
Buyer's inflow: Sale of share	-	-	110	115	120
Buyer's out flow: exercise call option	-	-	105	105	105
Call premium	5	5	5	5	5
Net pay-off	-5	-5	0	5	10



For buyer



For seller

The Call Option Seller's Pay-off at Expiration

Share price $S(T)$ ⟶	In Rs. 100	In Rs. 105	In Rs. 110	In Rs. 115	In Rs. 120
Seller's inflow: Call premium	5	5	5	5	5
Seller's inflow: exercise call option	-	-	105	105	105
Seller's out flow:	-	-	110	115	120
Net pay-off	5	5	0	-5	-10

<u>Payoff of a put option</u>

(1) **<u>Buyer</u>**

If $S(0)$ is the current asset price and $S(T)$ is the price at the expiration of the option and E is the exercise price, then we have three possibilities;

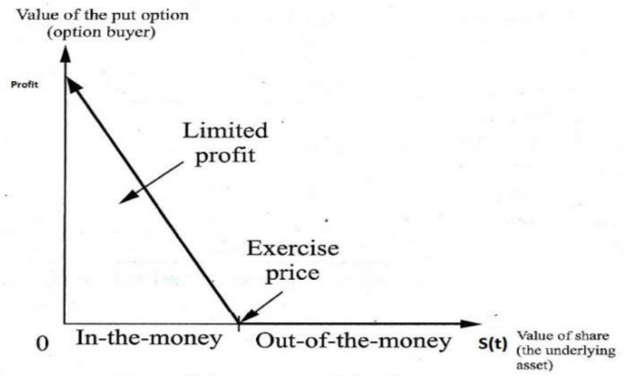
i. If $E > S(T)$ put option should be exercise to gain $E - S(T)$ (pay-off).

ii. If $E < S(T)$ put option not to be exercised that is pay-off is '0'.

iii. If $E = S(T)$ put option should not be exercise that is pay-off is '0'.

Including all three possibilities in one we can say the value of put option

$$\boldsymbol{P(T) = \max[\,E - S(T), 0\,]}$$

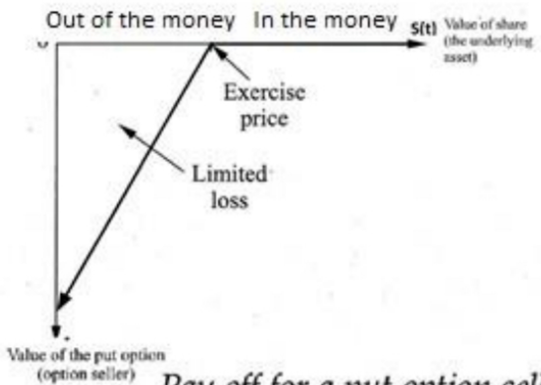


Pay-off for a put option buyer

(2) **<u>Seller</u>**

will have just opposite to that of buyer

$$\boldsymbol{P(T) = \min[\,S(T) - E, 0\,]}$$



Pay-off for a put option seller

(How it will work: if investor anticipate for fall of a share price he will buy put option right to sell at exercise price he will buy a set at low price from the market and sell it at a higher price to the put option seller.)

Valuation of Options price by Replicating Portfolio

The idea behind this valuation is that since the writer has no role to play in the exercise of the option, therefore writer of the option will try to secure against the possible loss. And so a portfolio (with same underlying asset) is made with the amount received as the premium such that the value of the portfolio will match the pay-off at the maturity.

Let $S(0)$ be the current price of the stock and $S(1)$ be the price at maturity. $A(0)$ & $A(1)$ be the price of bond at $t = 0, \& \ 1$ respectively, E is the srike price. Also $C(0)$ be the price of the option and $C(1) = \max[\, S(T) - E, 0\,]$ be the payoff at $t = 1$.

$$S(1) = \begin{cases} S^u & \text{stock goes up} \\ S^d & \text{stock goes down} \end{cases}$$

Let (x, y) the portfolio, then task is to find value of $x \,\&\, y$

Now the value of the portfolio at $t = 0$,

$$V(0) = xS(0) + yA(0) = C(0) \qquad \ldots(1)$$

And $$V(1) = xS(1) + yA(1) = C(1)\ (pay - off)$$
...(2)

Since $S(1)$ takes two values

$$V(1) = \begin{cases} xS^u + yA(1) & if\, stock\ goes\ up \\ xS^d + yA(1) & if\, stock\ goes\ down \end{cases}$$

Therefore the equation (2) reduces into 2-equations

$$xS^u + yA(1) = C(1) \qquad \ldots(3)$$

$$xS^d + yA(1) = C(1) \qquad \ldots(4)$$

Solving equation(3 & 4) we get the value of $x \,\&\, y$, and putting in equation (1) we get price of the option.

Prob: Let $A(0) = \$100,\ A(1) = \$110,\ \&\ S(0) = \$100$ and

$$S(1) = \begin{cases} \$120 & stock\ goes\ up \\ \$80 & stock\ goes\ down \end{cases}$$

find the price of a call option with strike price \$100 and time of maturity at 1.

Sol: Let (x, y) be the portfolio. Then at $t = 0$

$$V(0) = xS(0) + yA(0) = 100x + 100y = C(0)$$

At $t = 1$ the payoff will be $C(1) = \begin{cases} \$20 & stock\ goes\ up \\ 0 & stock\ goes\ down \end{cases}$

$$120x + 110y = 20$$

$$80x + 110y = 0$$

solving these equation we get $x = \frac{1}{2}$ & $y = -\frac{4}{11}$

therefore, $C(0) = \frac{1}{2} * 100 + \left(-\frac{4}{11}\right) * 100 = \$13.6361$

Options general properties

Now to represent the payoff for call (put) option we will use the notation

$$x^+ = \begin{cases} x & if \;\; x > 0 \\ 0 & otherwise \end{cases}$$

For example pay-off of a call option $= \max(S(T) - X, 0) = (S(T) - X)^+$ where X is the exercise price (or strike price).

The gain of an option buyer (or seller) is the pay-off modified by premium $C^E \; or \; P^E$ paid (received) for the option. E: Stands for European option

At time T the gain of the buyer of a European call is $(S(T) - X)^+ - C^E \, e^{rT}$.

And similarly for the buyer of put the gain is $\left(X - S(T)\right)^+ - P^E \, e^{rT}$.

Put - Call Parity

For the stock paying no dividends the following relation holds between the price of European call and put options both with exercise price X and exercise time T ;

$$C^E - P^E = S(0) - Xe^{-rT}$$

Proof: Suppose $C^E - P^E > S(0) - Xe^{-rT}$

Construct a Portfolio as following

i. buy one share for $S(0)$
ii. buy one put option for P^E
iii. write and sell one call option C^E
iv. invest the sum $[\, C^E - P^E - S(0) \,]$ (if positive or borrow if negative) in money market at interest rate 'r'

The balance is zero i.e.; the value of portfolio is zero at $t = 0$ and at $t = T$

i. close the market position collecting (or paying if borrowed) the sum $[\, C^E - P^E - S(0) \,]e^{rT}$
ii. sell the share X to close either by exercise of put if $S(T) \leq X$ or settling the short position in call if $S(T) \geq X$

The balance will be $[\, C^E - P^E - S(0) \,]e^{rT} + X$ which is positive, contradicting no arbitrage principle.

Hence $C^E - P^E \ngtr S(0) - Xe^{-rT}$

Secondly, suppose $C^E - P^E < S(0) - Xe^{-rT}$

Make the following strategy

i. sell short one share for $S(0)$

ii. write and sell a put option for P^E

iii. buy one call option for C^E

iv. invest $S(0) + P^E - C^E$ (or borrow if negative) on the money market at the interest rate 'r'

Now at $t = 0$ the balance of transaction is zero.

At $t = T$

i. close the money market position collecting (or paying) the sum $[S(0) + P^E - C^E]e^{rT}$

ii. buy one share for X if $S(T) > X$ by exercising call are settling put option if $S(T) < X$ and close the short position of stock

The balance will be $[S(0) + P^E - C^E]e^{rT} - X$ which is positive contradicting no arbitrage principle .

Hence $C^E - P^E \nless S(0) - Xe^{-rT}$

Combining both the situations, we have

$$C^E - P^E = S(0) - Xe^{-rT}$$

Bounds on the option prices

We have

$$C^E < S(0) \qquad \ldots(1)$$

If the reverse inequality were satisfied, i.e.; $C^E > S(0)$ then we could write and sell the option and buy the stock investing the balance on the money market. On the exercise date T, we would sell this stock for $\min\{S(T), X\}$. Setting the option our arbitrage profit would be

$$\left(C^E - S(0)\right)e^{rT} + \min\{S(T), X\} > 0$$

Thus (1) holds which is upper bound.

On the other hand we have lower bound

$$S(0) - Xe^{-rT} \leq C^E \qquad \ldots(2)$$

Which directly follow from put call parity relation.

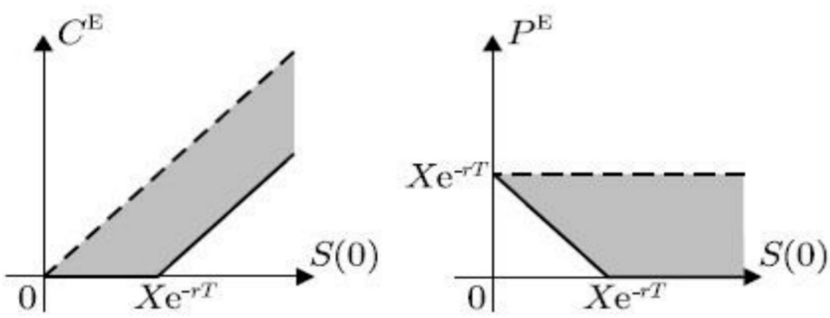
We therefore have upper bound for put option

$$P^E < Xe^{-rT} \qquad \ldots(3)$$

(since $C^E - P^E = S(0) - Xe^{-rT}$ & $C^E < S(0)$)

Also the lower bound is

$$-S(0) + Xe^{-rT} \leq P^E \qquad (\text{ since } C^E \geq 0) \quad \ldots(4)$$



Bounds on European call and put prices

Finally combining the results

$$max(0, S(0) - Xe^{-rT}) \leq C^E < S(0)$$

$$max(0, -S(0) + Xe^{-rT}) \leq P^E < Xe^{-rT}$$

Variables determining Option price (European option)

The option prices depends upon the number of variables, viz; the strike price X, expiry time T, the variables describing underlying assets, risk free interest etc

Dependence on the Strike price

Consider options on the same underlying asset and with the same exercise time T but with different values of the strike price X, and the underlying asset price $S(0)$ will be kept fixed (for time being)

§§ If $X_1 < X_2$ then

$$C^E(X_1) > C^E(X_2) \qquad \& \qquad P^E(X_1) < P^E(X_2) \qquad \dots(1)$$

This means that $C^E(X)$ is strictly decreasing and $P^E(X)$ is strictly increasing function of X .

§§ If $X_1 < X_2$ then

$$\left.\begin{array}{l} C^E(X_1) - C^E(X_2) < e^{-rT}(X_2 - X_1) \\ P^E(X_2) - P^E(X_1) < e^{-rT}(X_2 - X_1) \end{array}\right\} \dots(2)$$

Proof: Using put-call parity

$$C^E(X_1) - P^E(X_1) = S(0) - X_1 e^{-rT}$$

$$C^E(X_2) - P^E(X_2) = S(0) - X_2 e^{-rT}$$

Subtracting

$$\{C^E(X_1) - C^E(X_2)\} + \{P^E(X_2) - P^E(X_1)\} = (X_2 - X_1)e^{-rT}$$

Since both the term in left are positive, hence (2) holds.

§§ let $X_1 < X_2$ and let $\alpha \in (0,1)$ then

$$C^E(\alpha X_1 + (1-\alpha)X_2) \le \alpha C^E(X_1) + (1-\alpha)C^E(X_2)$$

$$P^E(\alpha X_1 + (1-\alpha)X_2) \le \alpha P^E(X_1) + (1-\alpha)P^E(X_2)$$

This means that C^E & P^E are convex function of X .

Proof: For brevity, we put

$$\hat{X} = \alpha X_1 + (1-\alpha)X_2 \qquad \dots(3)$$

suppose that $C^E(\hat{X}) > \alpha C^E(X_1) + (1-\alpha)C^E(X_2)$

we can write and sell call option with strike price $\hat{X}$, and purchase α call options with strike price X_1 and $(1-\alpha)$ options with strike price X_2 investing the balance

$$C^E(\hat{X}) - \{\alpha C^E(X_1) + (1-\alpha)C^E(X_2)\} > 0 \text{ in risk free.}$$

If the option with a strike price $\hat{X}$ is exercised at expiry, then we shall have to pay

$\left(S(T) - \hat{X}\right)^+$. We can collect the amount $(that\ means\ S(T) > \hat{X})$

$\alpha(S(T) - X_1)^+ + (1-\alpha)(S(T) - X_2)^+$

by exercising the options which will lead to arbitrage because

$$\left(S(T) - \hat{X}\right)^+ \le \alpha(S(T) - X_1)^+ + (1-\alpha)(S(T) - X_2)^+$$

$$\left(\begin{gathered} since\ \left(S(T) - \hat{X}\right)^+ = [\alpha S(T) + (1-\alpha)S(T) - \{\alpha X_1 + (1-\alpha)X_2\}]^+ = [\alpha\{S(T) - X_1\} + (1-\alpha)\{S(T) - X_2\}]^+ \\ also \\ |aX_1 + bX_2| \le a|X_1| + b|X_2| \end{gathered}\right)$$

Dependency on the underlying asset price

Assuming that all remaining variables are fixed. The current price of the underlying asset is given by the market, and cannot be altered. We can consider and option on portfolio consisting of $'x'$ shares worth $S = xS(0)$ With the strike price X on such portfolio and expiry time T .

The payoff will be $(xS(T) - X)^+$ for call option and $\left(X - xS(T)\right)^+$ for put option.

Proposition: If $S_1 < S_2 \quad for \quad x_1 < x_2$ then

$$C^E(S_1) < C^E(S_2)$$

$$P^E(S_1) > P^E(S_2)$$

That means $C^E(S)$ is strictly increasing function and $P^E(S)$ strictly decreasing function .

Proof: Suppose that $C^E(S_1) \ge C^E(S_2) \quad for\ some \quad S_1 < S_2$.
we can write and sell a call option on a Portfolio with x_1 − share and buy a call option on a Portfolio with x_2 − share having the same strike price and expiry time T . Invest $C^E(S_1) - C^E(S_2) \ge 0$ in risk free.

Since $x_1 < x_2$, hence payoff

$$(x_1 S(T) - X)^+ \leq (x_2 S(T) - X)^+$$

With strict inequality whenever $X < x_2 S(T)$.

After covering our liability we are left with arbitrage profit .

Similarly a proof can be given for put option.

Proposition: If $S_1 < S_2$ then

$$C^E(S_2) - C^E(S_1) < S_2 - S_1$$

$$P^E(S_1) - P^E(S_2) < S_2 - S_1$$

Proof: Using put-call parity formula

$C^E(S_2) - P^E(S_2) = S_2 - Xe^{-rT}$ and

$C^E(S_1) - P^E(S_1) = S_1 - Xe^{-rT}$

Subtracting we get

$$[\,C^E(S_2) - C^E(S_1)\,] + [\,P^E(S_1) - P^E(S_2)\,] = S_2 - S_1$$

Since both the bracket in left are positive, hence the result.

Proposition: Let $S_1 < S_2$ & $\alpha \in (0,1)$, then

$$C^E(\,\alpha S_1 + (1-\alpha)S_2\,) \leq \alpha C^E(S_1) + (1-\alpha)C^E(S_2)$$

$$P^E(\,\alpha S_1 + (1-\alpha)S_2\,) \leq \alpha P^E(S_1) + (1-\alpha)P^E(S_2)$$

i.e.; call and put prices are convex function of S .

Risk Neutral Probability Measure (RNPM)

Let Ω represents the state space of economic scenario i.e;

$\Omega = \{ \omega_1, \omega_2, \dots, \omega_m \}$ where ω_j , $j = 1,2, \dots m$ represents the economy scenario.

e.g.; one period binomial tree contains only 2-scinario $\{\omega_1, \omega_2\}$ one up-tick and

other down-tick movements.

While 2-period binomial tree may contain 4-scinario.

Apart from bond and under lying stock there could be securities (derivative) so we represent price by

$$S_i^k(\omega_j) , \qquad for \ k = 0,1,2 \dots n \ , \quad i = 1,2, \dots l \quad \& \quad j = 1.2 \dots m$$

For convention, $k = 0$ is for bond, $k = 1$ for stock, and $k = 2,3 \dots n$ for other securities. $'j'$ is the state of economy, and $'i'$ is the time period.

e.g.; $S_1^0(\omega_1) \rightarrow price\ of\ bond\ at\ t = 1\ in\ \omega_1\ scinario.$

$S_1^4(\omega_2) \rightarrow price\ of\ security\ no.3\ \ at\ t = 1\ in\ \omega_2\ scinario.$

***Definition:* Risk Neutral Probability Measure**

A RNPM is a vector $p^* = (p_1^*, p_2^*, \dots, p_m^*)$ such that

i. $p_j^* \geq 0$, $j = 1,2 \dots m$
ii. $\sum_{j=1}^{m} p_j^* = 1$

and for every security k $(0,1, \dots n)$ we have

$$S_0^k = \frac{1}{R}\left[\sum_{j=1}^{m} p_j^* \, S_1^k(\omega_j)\right]$$

§§ First fundamental theorem of asset pricing

A risk neutral probability measure p^* exists iff no arbitrage principle holds.

§§ Second fundamental theorem of asset pricing

RNPM is unique iff the market is complete.

Portfolio Optimization

Definition: Portfolio

A portfolio is a collection of two or more assets say, $a_1, a_2, .. a_n$, represented by an ordered $n-tuple$ $\Theta = \Theta(x_1, x_2, .. x_n)$, where $x_i \in R, i = 1 \dots n$ is the number of units of the asset a_i $(i = 1, :.., n)$ owned by the investor.

We consider only a single period model, ie, in between the initial time taken as $t = 0$ and the final transaction time taken as $t = T$, no transaction ever takes place.

Let $V_i(0)$ and $V_i(T)$ be the values of the i$^{\text{th}}$ asset at $t = 0$ and $t = T$. respectively. Let $V_\Theta(0)$ and $V_\Theta(T)$ denote the values of the portfolio $\Theta = \Theta(x_1, x_2, .. x_n)$ at $t = 0$ and $t = T$, respectively. Then,

$$V_\Theta(0) = \sum_{i=1}^{n} x_i V_i(0) \qquad \& \qquad V_\Theta(T) = \sum_{i=1}^{n} x_i V_i(T)$$

Then the quantity

$$r_\Theta(T) = \frac{V_\Theta(T) - V_\Theta(0)}{V_\Theta(0)}$$

is referred as the return of the portfolio $\Theta(x_1, x_2, .. x_n)$.

Definition: Asset Weights

The weight w_i of the asset a_i is the proportion of the value of the asset in the portfolio for $(i = 1, :.., n)$ at $t = 0$, i.e.

$$w_i = \frac{x_i V_i(0)}{V_\Theta(0)} = \frac{x_i V_i(0)}{\sum_{i=1}^{n} x_i V_i(0)} \quad , \quad i = 1, :.., n$$

It can be observed that $w_1 + w_2 + \cdots + w_n = 1$.

Therefore, a portfolio can now be represented by the weights as $(w_1, w_2, .. w_n)$.

§§ $w_i < 0$ for some 'i' is also possible in a portfolio, it indicates that the investor has taken a short position on the $i-th$ asset a_i.

Let r_i be the return on the $i-th$ asset. Then

$$r_i = \frac{V_i(T) - V_i(0)}{V_i(0)} \quad , \quad i = 1, .. n$$

Definition: Mean of the Portfolio Return

(*referred as return of the portfolio*)

Let $(w_1, w_2, .. w_n)$ be a portfolio of 'n' assets $a_1, a_2, .. a_n$. Let r_i, $(i = 1, .. n)$ be the return on the ith asset a_i and $E(r_i) = \mu_i$, $(i = 1, :.., n)$, be its expected value . Then the mean of the portfolio return is defined as

$$\mu = E\left(\sum_{i=1}^{n} w_i r_i\right) = \sum_{i=1}^{n} w_i E(r_i) = \sum_{i=1}^{n} w_i \mu_i$$

[Since (total return of the portfolio) $R_i = \sum_{i=1}^{n} w_i r_i$, hence $\mu = E(R_i)$]

Definition: Variance of the Portfolio

(*referred as risk of the portfolio*)

Let $(w_1, w_2, .. w_n)$ be a portfolio of 'n' assets $a_1, a_2, .. a_n$. Then the variance of the portfolio defined as

$$\sigma^2 = Var\left(\sum_{i=1}^{n} w_i r_i\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{ij} \quad , where \quad \sigma_{ij} = Cov(r_i, r_j)$$

Also, $\sigma^2{}_i = Var(r_i)$ & $\sigma^2{}_j = Var(r_j)$

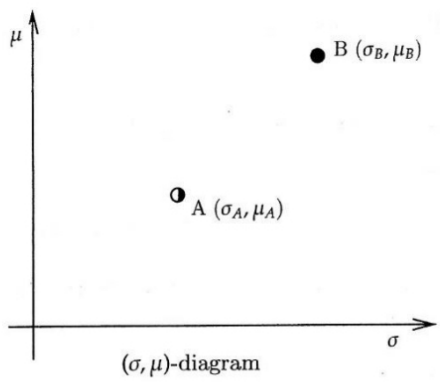
If ρ_{ij} is correlation coefficient between r_i & r_j , then

$$\rho_{ij} = \frac{Cov(r_i, r_j)}{\sigma_i \sigma_j} = \frac{\sigma_{ij}}{\sigma_i \sigma_j} \Rightarrow \sigma_{ij} = \rho_{ij} \sigma_i \sigma_j$$

Hence variance is also expressed as

$$\sigma^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \rho_{ij} \sigma_i \sigma_j$$

Therefore given a portfolio $A: (w_1, w_2, .. w_n)$, we can compute its mean μ_A and standard deviation σ_A and therefore get the point A (σ_A, μ_A) in $(\sigma, \mu) - plane$. Thus irrespective of the number assets, a portfolio can always be identified as a point in the $(\sigma, \mu) - plane$.



(σ, μ)-diagram

The portfolio optimization problem refers to the problem of determining weights $w_i\,,(\,i\ =\ 1,..n)$ such that the return of the portfolio is maximum and the risk of the portfolio is minimum. Thus we aim to solve the following optimization problem,

i. Minimize the risk, ie, $min\ \sum_{i=1}^{n}\sum_{j=1}^{n} w_i\, w_j \sigma_{ij}$
ii. Maximize the return, ie, $max \sum_{i=1}^{n} w_i \mu_i$

subject to, $w_1 + w_2 + \cdots + w_n = 1$

[*the search of combination of weights* $(w_1, w_2, .. w_n)$]

Two Assets Portfolio Optimization

Consider a portfolio with two assets , say, $a_1\ \&\ a_2$ with weights $w_1\ \&\ w_2$ returns $r_1\ \&\ r_2$ and standard deviations $\sigma_1\ \&\ \sigma_2$ respectively. Then the portfolio expected return μ and portfolio variance σ^2 are given by

$$\mu = E(w_1 r_1 + w_2 r_2) = w_1\mu_1 + w_2\mu_2 \quad .....(1)$$

$$\sigma^2 = Var(w_1 r_1 + w_2 r_2) = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1 w_2\,\rho\,\sigma_1\sigma_2 \quad .....(2)$$

Here ρ is the coefficient of correlation between $r_1\ \&\ r_2$ and lies in $[-1, 1]$.

The value of ρ provides a measure of the extent of diversification of the portfolio so as to reduce risk. Larger the value of ρ with negative sign, smaller will be the value of σ^2 .

Since , $w_1 + w_2 = 1$

Moreover, in case of short selling, the weights can be negative, hence

Let $w_2 = s \quad then \quad w_1 = 1 - s\ , \quad s \in R.$

$$\mu = (1-s)\mu_1 + s\mu_2 \quad .....(3)$$

$$\sigma^2 = (1-s)^2\sigma_1^2 + s^2\sigma_2^2 + 2s(1-s)\,\rho\,\sigma_1\sigma_2 \quad .....(4)$$

or, $$\sigma^2 = (\sigma_1^2 + \sigma_2^2 - 2\,\rho\,\sigma_1\sigma_2)s^2 - 2(\sigma_1 - \rho\sigma_2)\sigma_1 s + \sigma_1^2$$

Without loss of generality we assume that $0 < \sigma_1 \le \sigma_2$.

We discuss equ. (3 & 4) under two cases

(1) $\rho = \pm 1$ (2) $-1 < \rho < 1$

Case (1): for $\rho = \pm 1$, equ. (3 & 4) reduces to

$$\mu = (1-s)\mu_1 + s\mu_2$$

$$\sigma = |(1-s)\sigma_1 \pm s\sigma_2\,|$$