Bayesian Decision Theory

Bayesian Decision Theory

- Design classifiers to recommend decisions that minimize some total expected ”risk”.
  - The simplest risk is the classification error (i.e., costs are equal).
  - Typically, the risk includes the cost associated with different decisions.

Terminology

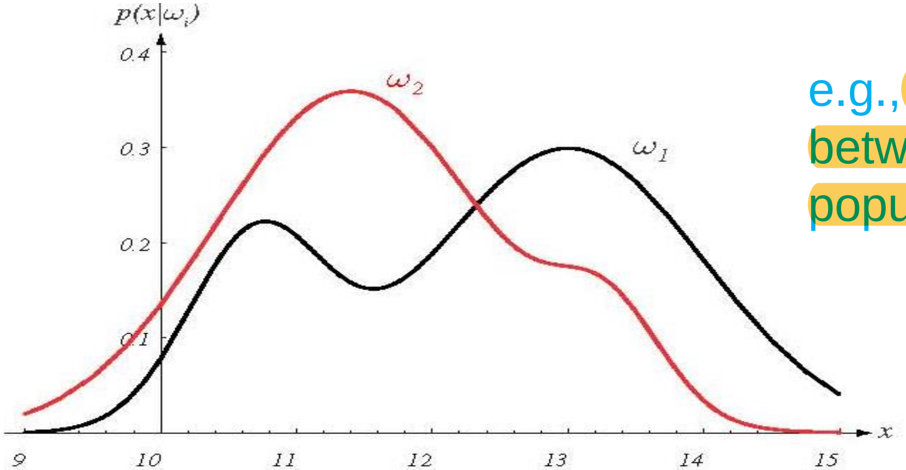
- State of nature ω *(random variable):*
  - e.g., ω_1 for sea bass, ω_2 for salmon

- Probabilities $P(\omega_1)$ and $P(\omega_2)$ *(priors):*
  - e.g., prior knowledge of how likely is to get a sea bass or a salmon

- Probability density function $p(x)$ *(evidence):*
  - e.g., how frequently we will measure a pattern with feature value x (e.g., x corresponds to lightness)

Terminology (cont'd)

- Conditional probability density $p(x/\omega_j)$ (*likelihood*) :
  - e.g., how frequently we will measure a pattern with feature value *x* given that the pattern belongs to class ω_i



e.g., lightness distributions between salmon/sea-bass populations

FIGURE 2.1. Hypothetical class-conditional probability density functions show the probability density of measuring a particular feature value x given the pattern is in category ω_i. If x represents the lightness of a fish, the two curves might describe the difference in lightness of populations of two types of fish. Density functions are normalized, and thus the area under each curve is 1.0. From: Richard O. Duda, Peter E. Hart, and David G. Stork, *Pattern Classification*. Copyright © 2001 by John Wiley & Sons,

Terminology (cont'd)

- Conditional probability $P(\omega_j/x)$ *(posterior)* :
  - e.g., the probability that the fish belongs to class ω_j given measurement x.

Decision Rule Using Prior Probabilities

Decide ω_1 if $P(\omega_1) > P(\omega_2)$; otherwise **decide** ω_2

$$P(error) = \begin{cases} P(\omega_1) & if\ we\ decide\ \omega_2 \\ P(\omega_2) & if\ we\ decide\ \omega_1 \end{cases}$$

or $P(error) = min[P(\omega_1), P(\omega_2)]$

- Favours the most likely class.
- This rule will be making the same decision all times.
  - i.e., optimum if no other information is available

Decision Rule Using Conditional Probabilities

- Using Bayes' rule, the posterior probability of category ω_j given measurement x is given by:

$$P(\omega_j / x) = \frac{p(x/\omega_j)P(\omega_j)}{p(x)} = \frac{likelihood \times prior}{evidence}$$

where $$p(x) = \sum_{j=1}^{2} p(x/\omega_j)P(\omega_j)$$ (i.e., scale factor – sum of probs = 1)

Decide ω_1 if $P(\omega_1/x) > P(\omega_2/x)$; otherwise **decide** ω_2

or

Decide ω_1 if $p(x/\omega_1)P(\omega_1) > p(x/\omega_2)P(\omega_2)$ otherwise **decide** ω_2

Decision Rule Using Conditional pdf (cont'd)

$p(x/\omega_j)$

$$P(\omega_1) = \frac{2}{3} \qquad P(\omega_2) = \frac{1}{3}$$

$P(\omega_j/x)$

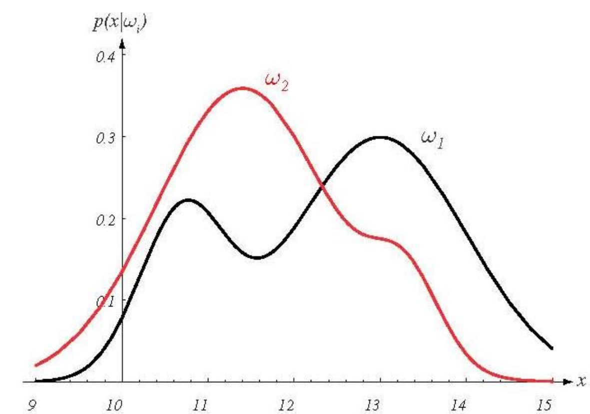


FIGURE 2.1. Hypothetical class-conditional probability density functions show the probability density of measuring a particular feature value x given the pattern is in category ω_i. If x represents the lightness of a fish, the two curves might describe the difference in lightness of populations of two types of fish. Density functions are normalized, and thus the area under each curve is 1.0. From: Richard O. Duda, Peter E. Hart, and David G. Stork, *Pattern Classification*. Copyright © 2001 by John Wiley & Sons,

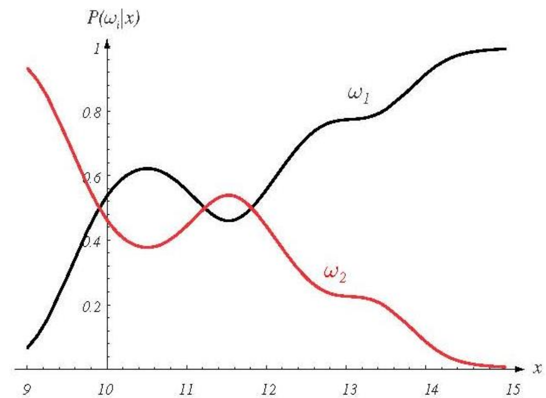


FIGURE 2.2. Posterior probabilities for the particular priors $P(\omega_1) = 2/3$ and $P(\omega_2) = 1/3$ for the class-conditional probability densities shown in Fig. 2.1. Thus in this case, given that a pattern is measured to have feature value $x = 14$, the probability it is in category ω_2 is roughly 0.08, and that it is in ω_1 is 0.92. At every x, the posteriors sum to 1.0. From: Richard O. Duda, Peter E. Hart, and David G. Stork, *Pattern Classification*. Copyright © 2001 by John Wiley & Sons, Inc.

Probability of Error

- The probability of error is defined as:

$$P(error/x) = \begin{cases} P(\omega_1/x) \;\; if \;\; we\, decide\, \omega_2 \\ P(\omega_2/x) \;\; if \;\; we\, decide\, \omega_1 \end{cases}$$

or $$P(error/x) = min[P(\omega_1/x), P(\omega_2/x)]$$

- What is the average probability error?

$$P(error) = \int_{-\infty}^{\infty} P(error, x)dx = \int_{-\infty}^{\infty} P(error/x)p(x)dx$$

- The Bayes rule is optimum, that is, it minimizes the average probability error!

Where do Probabilities Come From?

- There are two competitive answers to this question:

(1) **Relative frequency** (objective) approach.

– Probabilities can only come from experiments.

(2) **Bayesian** (subjective) approach.

– Probabilities may reflect degree of belief and can be based on opinion.

Prior and Posterior Probabilities

- – P(A) and P(B) are called prior probabilities
- – P(A|B), P(B|A) are called posterior probabilities

Example 8.6: Prior versus Posterior Probabilities

- This table shows that the event Y has two outcomes namely A and B, which is dependent on another event X with various outcomes like x_1, x_2 and x_3.
- **Case1:** Suppose, we don't have any information of the event A. Then, from the given sample space, we can calculate $P(Y = A) = \frac{5}{10} = 0.5$
- **Case2:** Now, suppose, we want to calculate $P(X = x_2 | Y = A) = \frac{2}{5} = 0.4$.

The later is the conditional or posterior probability, where as the former is the prior probability.

X	Y
x_1	A
x_2	A
x_3	B
x_3	A
x_2	B
x_1	A
x_1	B
x_3	B
x_2	B
x_2	A

Example (objective approach)

- Classify cars whether they are more or less than \$50K:
  - Classes: C_1 if price > \$50K, C_2 if price <= \$50K
  - Features: x, the height of a car

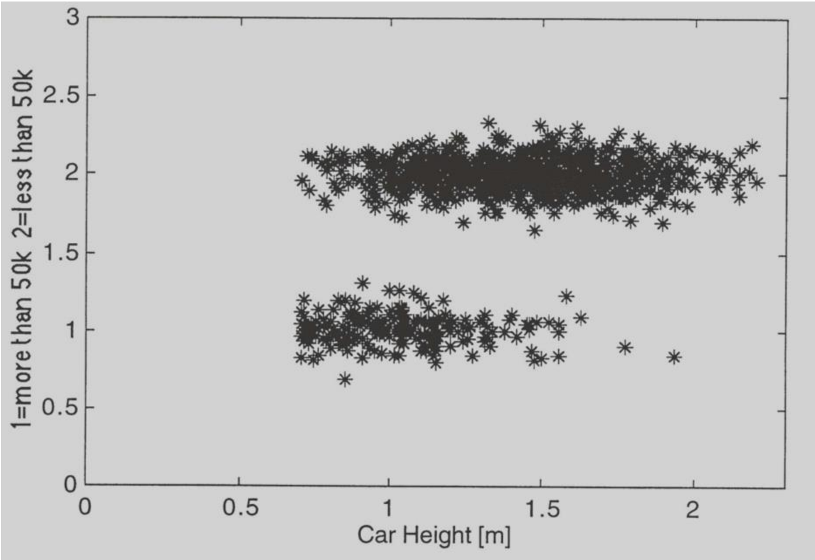
- Use the Bayes' rule to compute the posterior probabilities:

$$P(C_i / x) = \frac{p(x / C_i) P(C_i)}{p(x)}$$

- We need to estimate $p(x/C_1)$, $p(x/C_2)$, $P(C_1)$, $P(C_2)$

Example (cont'd)

- Collect data
  - Ask drivers how much their car was and measure height.
- Determine prior probabilities $P(C_1), P(C_2)$
  - e.g., 1209 samples: $\#C_1=221$ $\#C_2=988$



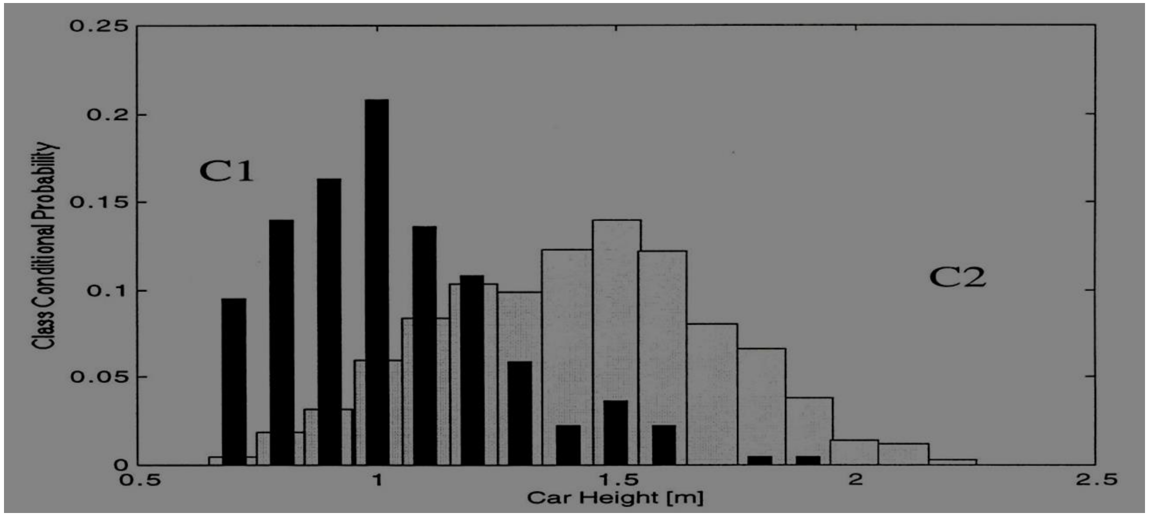
$$P(C_1) = \frac{221}{1209} = 0.183$$

$$P(C_2) = \frac{988}{1209} = 0.817$$

Example (cont'd)

- Determine class conditional probabilities (*likelihood*)
  - Discretize car height into bins and use normalized histogram

$p(x / C_i)$



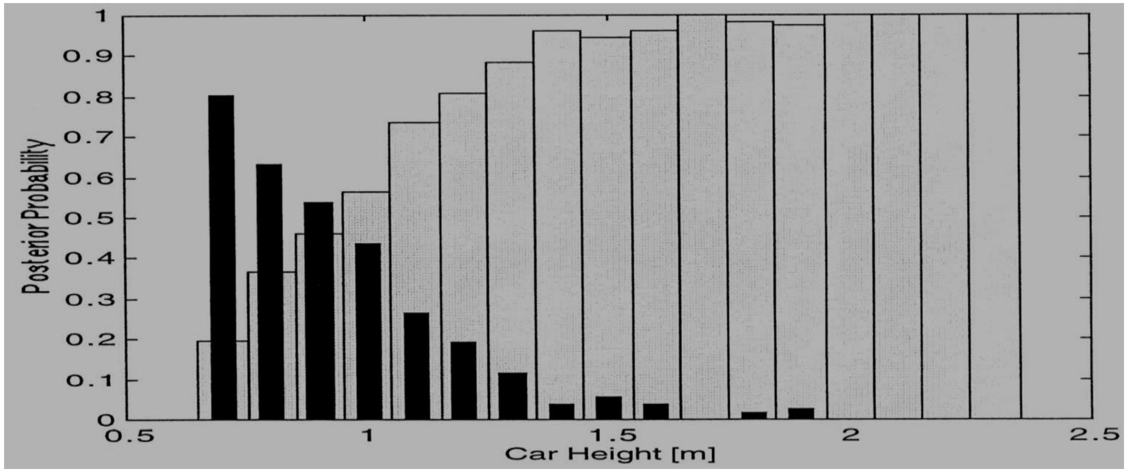
Example (cont'd)

- Calculate the posterior probability for each bin:

$$P(C_1 / x = 1.0) = \frac{p(x = 1.0 / C_1) P(C_1)}{p(x = 1.0 / C_1) P(C_1) + p(x = 1.0 / C_2) P(C_2)} =$$

$$= \frac{0.2081 * 0.183}{0.2081 * 0.183 + 0.0597 * 0.817} = 0.438$$

$P(C_i / x)$



A More General Theory

- Use more than one features.
- Allow more than two categories.
- Allow actions other than classifying the input to one of the possible categories (e.g., rejection).
- Employ a more general error function (i.e., “risk” function) by associating a “cost” (“loss” function) with each error (i.e., wrong action).

Terminology

- Features form a vector $\mathbf{x} \in R^d$
- A finite set of c categories $\omega_1, \omega_2, \ldots, \omega_c$
- Bayes rule (i.e., using vector notation):

$$P(\omega_j / \mathbf{x}) = \frac{p(\mathbf{x} / \omega_j) P(\omega_j)}{p(\mathbf{x})} \quad \textit{where} \quad p(\mathbf{x}) = \sum_{j=1}^{c} p(\mathbf{x} / \omega_j) P(\omega_j)$$

- A finite set of *l* actions $\alpha_1, \alpha_2, \ldots, \alpha_l$
- A *loss* function $\lambda(\alpha_i / \omega_j)$
  - the cost associated with taking action α_i when the correct classification category is ω_j.
  - For example, in case of binary classification if we take an action of classifying an input feature vector into class 1 when it should have been in class 2, we incur the loss $\lambda(\alpha_1/\omega_2)$.
  - If ***i* = *j***, then we get a smaller value of the loss as compared to the alternative cases because it corresponds to a correct decision.

Conditional Risk (or Expected Loss)

- Suppose we observe **x** and take action α_i
- Suppose that the cost associated with taking action α_i with ω_j being the correct category is $\lambda(\alpha_i / \omega_j)$
- The **conditional risk** (or **expected loss**) with taking action α_i is:

$$R(a_i / \mathbf{x}) = \sum_{j=1}^{c} \lambda(a_i / \omega_j) P(\omega_j / \mathbf{x})$$

Overall Risk

- Suppose $\alpha(\boldsymbol{x})$ is a general decision rule that determines which action $\alpha_1, \alpha_2, \ldots, \alpha_l$ to take for every **x**; then the overall risk is defined as:

$$R = \int R(a(\mathbf{x}) / \mathbf{x})\, p(\mathbf{x})\, d\mathbf{x}$$

- The optimum decision rule is the *Bayes rule*

Overall Risk (cont'd)

- The *Bayes decision rule* minimizes R by:
  - (i) Computing $R(\alpha_i/\mathbf{x})$ for every α_i given an $\mathbf{x}$
  - (ii) Choosing the action α_i with the minimum $R(\alpha_i/\mathbf{x})$

- The resulting minimum overall risk is called *Bayes risk* and is the best (i.e., optimum) performance that can be achieved:

$$R^* = \min R$$

Example: Two-category classification

- Define
  - α_1: decide ω_1
  - α_2: decide ω_2
  - $\lambda_{ij} = \lambda(\alpha_i / \omega_j)$

(c=2)

- The conditional risks are:

$$R(a_i / \mathbf{x}) = \sum_{j=1}^{c} \lambda(a_i / \omega_j) P(\omega_j / \mathbf{x})$$

$$R(a_1/\mathbf{x}) = \lambda_{11} P(\omega_1/\mathbf{x}) + \lambda_{12} P(\omega_2/\mathbf{x})$$
$$R(a_2/\mathbf{x}) = \lambda_{21} P(\omega_1/\mathbf{x}) + \lambda_{22} P(\omega_2/\mathbf{x})$$

Example: Two-category classification (cont'd)

- Minimum risk decision rule:

$$\textbf{Decide } \omega_1 \text{ if } R(a_1/\mathbf{x}) < R(a_2/\mathbf{x}); \text{ otherwise decide } \omega_2$$

or

$$\textbf{Decide } \omega_1 \text{ if } (\lambda_{21} - \lambda_{11})P(\omega_1/\mathbf{x}) > (\lambda_{12} - \lambda_{22})P(\omega_2/\mathbf{x}); \text{ otherwise decide } \omega_2$$

or (i.e., using likelihood ratio)

$$\textbf{Decide } \omega_1 \text{ if } \frac{p(\mathbf{x}/\omega_1)}{p(\mathbf{x}/\omega_2)} > \frac{(\lambda_{12} - \lambda_{22})}{(\lambda_{21} - \lambda_{11})} \frac{P(\omega_2)}{P(\omega_1)}; \text{ otherwise decide } \omega_2$$

likelihood ratio threshold

Special Case: Zero-One Loss Function

- Assign the same loss to all errors:

$$\lambda(a_i/\omega_j) = \begin{cases} 0 & i = j \\ 1 & i \neq j \end{cases}$$

- The conditional risk corresponding to this loss function:

$$R(a_i/\mathbf{x}) = \sum_{j=1}^{c} \lambda(a_i/\omega_j)P(\omega_j/\mathbf{x}) = \sum_{i \neq j} P(\omega_j/\mathbf{x}) = 1 - P(\omega_i/\mathbf{x})$$

Special Case: Zero-One Loss Function (cont'd)

- The decision rule becomes:

$$\textbf{Decide } \omega_1 \text{ if } R(a_1/\mathbf{x}) < R(a_2/\mathbf{x})\text{; otherwise decide } \omega_2$$

or $$\textbf{Decide } \omega_1 \text{ if } 1 - P(\omega_1/\mathbf{x}) < 1 - P(\omega_2/\mathbf{x})\text{; otherwise decide } \omega_2$$

or $$\textbf{Decide } \omega_1 \text{ if } P(\omega_1/\mathbf{x}) > P(\omega_2/\mathbf{x})\text{; otherwise decide } \omega_2$$

- In this case, the overall risk is the average probability error!

Example

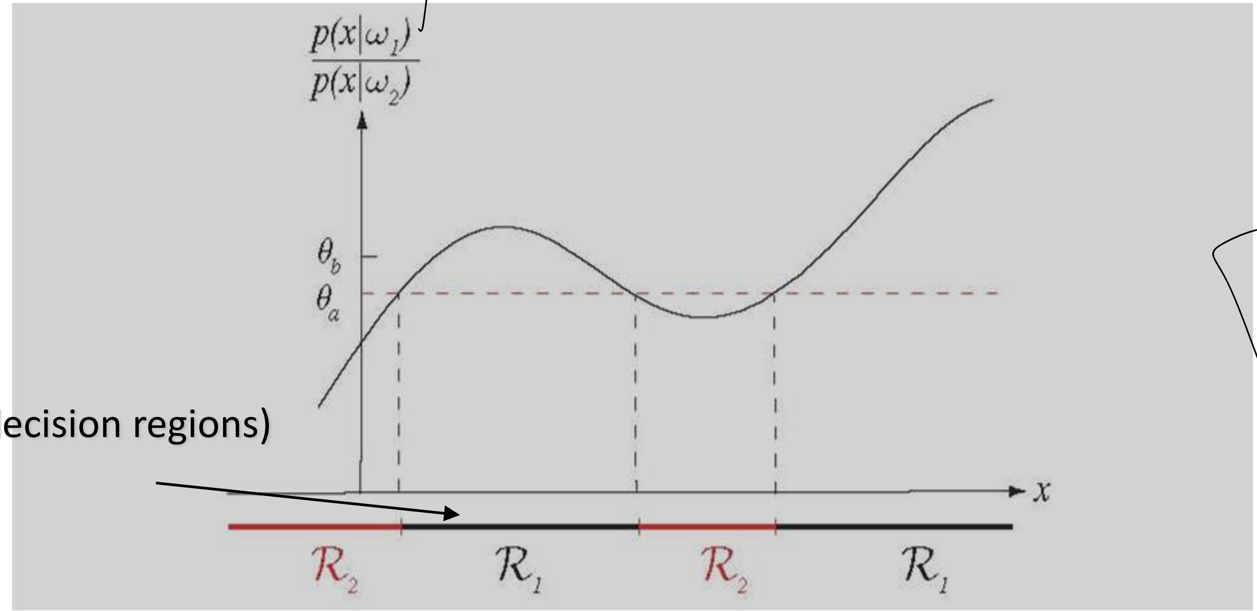
Assuming general loss:

$$\textbf{Decide } \omega_1 \text{ if } \frac{p(\mathbf{x}/\omega_1)}{p(\mathbf{x}/\omega_2)} > \frac{(\lambda_{12}-\lambda_{22})}{(\lambda_{21}-\lambda_{11})}\frac{P(\omega_2)}{P(\omega_1)}; \text{ otherwise decide } \omega_2$$

Assuming zero-one loss:

Decide ω_1 if $p(x/\omega_1)/p(x/\omega_2)>P(\omega_2)/P(\omega_1)$ otherwise **decide** ω_2

(decision regions)



$$\theta_a = P(\omega_2)/P(\omega_1)$$

$$\theta_b = \frac{P(\omega_2)(\lambda_{12}-\lambda_{22})}{P(\omega_1)(\lambda_{21}-\lambda_{11})}$$

assume: $\lambda_{12} > \lambda_{21}$