Suyash Goyal 2K22/MC/161

Stochastic Processes (MC303)

Assignment - 2

(A1) (a) $p_n(x) = \dfrac{A_n}{x_n^2 + x^2}$ where $x_n > 0 \ \forall\, n$.

The characteristic function for this PDF is given by,

$$\hat{p}_n(K) = \langle e^{iKx} \rangle = \int_{-\infty}^{+\infty} p_n(x)\, e^{iKx} dx = A_n \int_{-\infty}^{\infty} \frac{e^{iKx}}{x_n^2 + x^2} dx$$

Now we know that,

$$\oint_{C_R^+} \frac{e^{iKz}}{z^2 + x_n^2} dz = 2i\pi \operatorname*{Res}_{z = ix_n} \left(\frac{e^{iKz}}{z^2 + x_n^2} \right) \quad \text{(Residue Theorem)}$$

Also,

$$\oint_{C_R^+} \frac{e^{iKz}}{z^2 + x_n^2} dz = \int_{-R}^{+R} \frac{e^{iKx}}{x_n^2 + x^2} dx + \int_{\Gamma_R^+} \frac{e^{iKz}}{z^2 + x_n^2} dz$$

where Γ_R^+ is the open contour over which z is imaginary.

On this contour, we have,

$$\left| \int_{\Gamma_R^+} \frac{e^{iKz}}{z^2 + x_n^2} dz \right| \le \pi R \frac{e^{-KR}}{R^2 - x_n^2} \xrightarrow[R \to \infty]{} 0$$

We conclude that when $R \to \infty$, we have,

$$\oint_{C_\infty^+} \frac{e^{iKz}}{z^2 + x_n^2} dz = \int_{-\infty}^{+\infty} \frac{e^{iKx}}{x_n^2 + x^2} dx$$

Hence, calculating the residue, we obtain,

$$\int_{-\infty}^{\infty} \frac{e^{iKx}}{x_n^2 + x^2} dx = \frac{\pi}{x_n} e^{-Kx_n} \quad \text{for } K > 0$$

similarly,

$$\int_{-\infty}^{\infty} \frac{e^{iKx}}{x_n^2 + x^2} dx = \frac{\pi}{x_n} e^{Kx_n} \quad \text{for } K < 0$$

Thus we can write,

$$\hat{p}_n(K) = \frac{\pi A_n}{x_n} e^{-|K| x_n} ;$$

The normalization condition gives,

$$\hat{p}_n(0) = 1 \iff A_n = \frac{x_n}{\pi}$$

$$\Rightarrow \hat{p}_n(K) = e^{-x_n |K|}$$

We can clearly see $\hat{p}_n(K)$ is continuous at $K = 0$ but $\hat{p}_n'(K)$ is not and hence characteristic function is not analytic at the origin.

(b) We know that,

$$P_n(x) = \int_{-\infty}^{+\infty} e^{-ikx}\left[\prod_{j=1}^{n} \hat{p}_j(k)\right]\frac{dk}{2\pi}$$

$$= \int_{-\infty}^{+\infty} e^{-ikx} e^{-|k|X_n}\frac{dk}{2\pi} \quad \text{where } X_n = \sum_{j=1}^{n} x_j$$

$$= \frac{1}{2\pi}\left(\frac{1}{ix + X_n} - \frac{1}{ix - X_n}\right) \quad \text{since } X_n > 0$$

Thus we obtain,

$$P_n(x) = \frac{1}{\pi}\frac{X_n}{X_n^2 + x^2} \quad \text{with } X_n = \sum_{j=1}^{n} x_j$$

For case of identical steps $x_n = a$ we get,

$$p(x) = \frac{a}{\pi}\frac{1}{a^2+x^2} \quad \text{and} \quad P_n(x) = \frac{na}{\pi}\frac{1}{n^2a^2+x^2}.$$

By scaling $\xi = \frac{x}{na}$, the normalized PDF for the variable ξ is,

$$\tilde{P}_n(\xi) = na\, P_n(x) = \frac{1}{\pi}\frac{1}{1+\xi^2} \quad \text{independent of } n.$$

(A2) To find MGF of X_n, $E(e^{\alpha X_n})$, we will first calculate MGF of single step z_i:

$$M_{z_i}(\alpha) = E(e^{\alpha z_i}) = e^{\alpha}p + e^{-\alpha}q$$

Since X_n is sum of n IID random variables z_i, MGF of X_n is,

$$E(e^{\alpha X_n}) = \left(E(e^{\alpha z_i})\right)^n = (e^{\alpha}p + e^{-\alpha}q)^n$$

The asymptotic behaviour of X_n largely depends on value of p and q :-

(i) If $p = q$ then X_n behaves like a normal distribution by CLT for large n. with mean 0 and var n.

(ii) If $p \neq q$ then X_n will drift with mean $n(2p-1)$ and its variance remains n.

(A3) (This and (A2) were already done in Assignment-1)

(A4) (a) The transition probability matrix P is,

$$P = \begin{bmatrix} (1-r)^2 & 2r(1-r) & r^2 \\ p(1-r) & pr + (1-p)(1-r) & r(1-p) \\ p^2 & 2p(1-p) & (1-p)^2 \end{bmatrix}$$

(b) The steady-state conditions are given by system of equations,

$$\pi P = \pi \quad \text{where } \pi = (\pi_0, \pi_1, \pi_2)$$

This leads to following system of linear equations :

$$\pi_0 = \pi_0 (1-2r) + \pi_1 p(1-p) + \pi_2 p^2$$
$$\pi_1 = \pi_0 (2r) + \pi_1 (p(1-p) + (1-p)(1-r)) + \pi_2 (2p)(1-p)$$
$$\pi_2 = \pi_1 (1-p) r + \pi_2 (1-p)^2$$

Additionally, $\pi_0 + \pi_1 + \pi_2 = 1$.

Solving these equations, we obtain,

$$\pi_0 = \left[\frac{r}{p+r}\right]^2, \quad \pi_1 = 2\left[\frac{r}{p+r}\right]\left[\frac{p}{p+r}\right], \quad \pi_2 = \left[\frac{p}{p+r}\right]^2$$

(A5) A Markov Chain is irreducible if it is possible to get to any state from any state in a finite number of steps.

Looking at the entries of the matrix; from state state 0, one can reach state 1 and state 2 (both directly); from state 1, one can reach both state 0 and state 2 (both directly); from state 2, one can reach state 0 directly and state 1 via state 0 (indirectly). Thus we can [redacted] conclude the system is irreducible.

For steady state probabilities, $\pi P = \pi$ and $\pi_0 + \pi_1 + \pi_2 = 1$.

From matrix P, we set up the equations :-

$$\pi_0 = 0.6\pi_0 + 0.1\pi_1 + 0.6\pi_2$$
$$\pi_1 = 0.2\pi_0 + 0.8\pi_1$$
$$\pi_2 = 0.2\pi_0 + 0.1\pi_1 + 0.4\pi_2$$

Solving, we get,

$\pi_0 = 3/7$; $\pi_1 = 3/7$ and $\pi_2 = 1/7$.

(A6) The transition matrix is :-

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 3/4 & 0 & 0 & 1/4 \\ 0 & 1/2 & 0 & 1/4 & 0 & 1/4 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The execution times for each node is given by $t_i = 2i+1$.

$t_1 = 3$; $t_2 = 5$; $t_3 = 7$; $t_4 = 9$; $t_5 = 11$; $t_6 = 13$

System of equations based on matrix is :-

$$\pi_1 = \pi_2$$
$$\pi_2 = \tfrac{1}{2}\pi_3 + \tfrac{1}{2}\pi_4$$
$$\pi_3 = \pi_5$$
$$\pi_4 = \tfrac{3}{4}\pi_3 + \tfrac{1}{4}\pi_6$$

$\pi_5 = \frac{1}{2}\pi_2 + \frac{1}{4}\pi_4 + \frac{1}{4}\pi_6$

$\pi_1 + \pi_2 + \pi_3 + \pi_4 + \pi_5 + \pi_6 = 1$

Solving these equations, we obtain,

$\cancel{\pi_1 = \pi_2 = 5/24,\ \pi_3 = \pi_5 = 1/6,\ \pi_4 = 1/2,\ \pi_6 = 0}$

$\pi_1 = \pi_2 = \pi_3 = \pi_4 = \pi_5 = \pi_6 = 1/6$

Average Total Execution Time $= 3 \cdot 1/6 + 5 \cdot 1/6 + 7 \cdot 1/6 + 9 \cdot 1/6 + 11 \cdot 1/6 + 13 \cdot 1/6$

$= 48/6 = 12$

(A7) The states represent the pages that could be in memory. We assume that there are 3 pages in total $(n=3)$ and 2 pages in memory $(m=2)$.

Transition Matrix $Q = \begin{bmatrix} 0 & e & 1-e \\ 1/2-d & 0 & 1/2+d \\ 0 & 1 & 0 \end{bmatrix}$

$\pi P = P$; $\pi_1 + \pi_2 + \pi_3 = 1$;

$\pi_1 = \pi_2 (1/2 - d) + \pi_3$

$\pi_2 = \pi_1 (1-e) + \pi_2 (0) + \pi_3 (0)$

$\pi_3 = \pi_1 (e) + \pi_2 (1/2 + d) + \pi_3 (0)$

Solving, we get,

$\pi_1 = \dfrac{1}{2.5 - e(1 - d/2) + d}$

$\pi_2 = \pi_1 (1-e)$

$\pi_3 = \pi_1 (1/2 + d + e(1/2 - d))$

Page-Fault Rate $= \pi_1 \cdot P(1 \to 2) + \pi_2 \cdot P(2 \to 1) + \pi_3 \cdot P(3 \to 1)$

$= \pi_1 (1-e) + \pi_2 (1/2 - d) + \pi_3$