# Renewal Process

Poisson process ⇒ times b/w successive events are i.i.d. random variables which are distributed exponentially.

Renewal process ⇒ times b/w successive events are i.i.d. random variables which are not neccessarily exponentially distributed.

Renewal process is a discrete parameter process $\{X_n, n \geq 1\}$ where $X_1, X_2, X_3 \ldots$ are i.i.d non-negative r.v. where $X_i$ is defined as the time elapsed b/w $(i-1)^{st}$ event and $i^{th}$ event, removing the restriction of exponential distribution.

An example of a renewal process can be a duration for which the candle remains lit and durations after which it has to be lit again (renewed).

If $\{X_n, n \geq 1\}$ is the renewal process, the C.D.F (common distribution function) of the random variables $X_i$ is

$$F(x) = P[X_n \leq x], \quad n = 1, 2, 3 \ldots$$

N(t) is the count of the number of events that have occurred till time 't'. Here, an event means a renewal happening (ie. the candle being lit again).

The sum $W_n$,

$$W_n = X_1 + X_2 + X_3 \ldots + X_n, \quad n \geq 1$$

is called the waiting time until $n^{th}$ renewal.

$$W_0 = 0.$$

If $N(t)$ defines the number of renewals in time interval $[0,t]$, then,

$M(t) = E(N(t))$ $\}\rightarrow$ renewal function (avg. no. of renewals b/w $[0,t]$.)

$m(t) = \dfrac{dM(t)}{dt}$ is called the renewal density. This is the probability of occurrence of a renewal in the interval $[t, t+\Delta t]$.

[In case of a poisson process $(\lambda)$, $M(t) = \lambda t$ and $m(t) = \lambda$]

Now let $X_i$'s be the i.i.d r.v. with $F(x)$ as the C.D.F and $W_n = \sum_{i=1}^{n} X_i$ be the waiting time till $n^{th}$ renewal, with distribution function as $F^{(n)}(t)$.

For $F^{(n)}(t)$, we define,

$$F^{(0)}(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

If $N(t)$ = number of renewals till $t$, then

$P[N(t) = n]$ = Prob ['n' renewals till time 't']

= P ['t' is somewhere b/w the $n^{th}$ renewal and $n+1^{th}$ renewal]

$= P[W_n \leq t \leq W_{n+1}]$

$= \underbrace{P[W_n \leq t]}_{\text{prob. that 't' is greater than the waiting time of } n^{th} \text{ renewal, cause it has already occurred}} - \underbrace{P[W_{n+1} \leq t]}_{\text{prob. that 't' is greater than waiting time of 'n+1'}^{th} \text{ renewal, which means } n+1^{th} \text{ renewal has also occurred}}$

$= F^{(n)}(t) - F^{(n+1)}(t)$.

Now, $M(t) = E(N(t))$ (= expectation of 'n' renewals)

$$= \sum_{n=0}^{\infty} n \cdot P[N(t)=n]$$

$$= \sum_{n=0}^{\infty} n \cdot F^{(n)}(t) - \sum_{n=0}^{\infty} n \cdot F^{(n+1)}(t)$$

$$= F^{(1)}(t) + 2F^{(2)}(t) + \dots - [F^{(2)}(t) + 2F^{(3)}(t) + \dots]$$

$$= F^{(1)}(t) + F^{(2)}(t) + F^{(3)}(t) + \dots$$

$$= \boxed{\sum_{n=1}^{\infty} F^{(n)}(t)} \quad \text{— (A)}$$

$$= F^{(1)}(t) + \sum_{n=1}^{\infty} F^{(n+1)}(t)$$

$F^{(n+1)}$ is the convolution of $F^{(n)}$ and $F$.

If 'f' is the density function, then

$\frac{dF}{dt}$

$$F^{(n+1)}(t) = \int_0^t F^{(n)}(t-x) \cdot f(x)\, dx.$$

$$\therefore F^{(1)}(t) + \sum_{n=1}^{\infty} F^{(n+1)}(t)$$

$$= M(t) = F^{(1)}(t) + \sum_{n=1}^{\infty} \left[ \int_0^t F^{(n)}(t-x) \cdot f(x) \cdot dx \right]$$

$$= F^{(1)}(t) + \int_0^t \left[ \sum_{n=1}^{\infty} F^{(n)}(t-x) \right] \cdot f(x) \cdot dx$$

$$\boxed{M(t) = F^{(1)}(t) + \int_0^t M(t-x) \cdot f(x) \cdot dx} \quad \text{— (B)}$$

$\left[ \sum_{n=1}^{\infty} F^{(n)}(t-x) = M(t-x) \text{ from Eq}^n \text{ (A)} \right]$

Fundamental Renewal Equation

Also from Eq$^n$ (A) $\Rightarrow$ $M(t) = \sum_{n=1}^{\infty} F^{(n)}(t)$

differentiating (A) term by term $\Rightarrow$

$$\frac{dM(t)}{dt} = m(t) = \sum_{n=1}^{\infty} f^{(n)}(t).$$

$f^{(n)}$ can be treated as the convolution of $f^{(n-1)}$ and $f$.

From Eq$^n$. (B) $\Rightarrow$

$$\boxed{m(t) = f^{(1)}(t) + \int_0^t m(t-x) . f(x) . dx}$$

$\hookrightarrow$ Renewal Equation

Solution of Renewal Equation is obtained by using laplace Transform.

Take Laplace transform on both sides $\Rightarrow$

$$L(m(t)) = L(f(t)) + L\left[\int_0^t m(t-x) . f(x) . dx\right]$$

$$L_m(s) = L_f(s) + L_m(s) . L_f(s)$$

$$\text{where } \left\{ L(f(t)) = \int_0^{\infty} e^{-st} f(t)\, dt \right\}$$

This gives $L_m(s) = \dfrac{L_f(s)}{1 - L_f(s)}$

$$L_f(s) = \frac{L_m(s)}{1 + L_m(s)}$$

now using these 2 equations, $m(t)$ can be found using $f(t)$ or vice-versa.