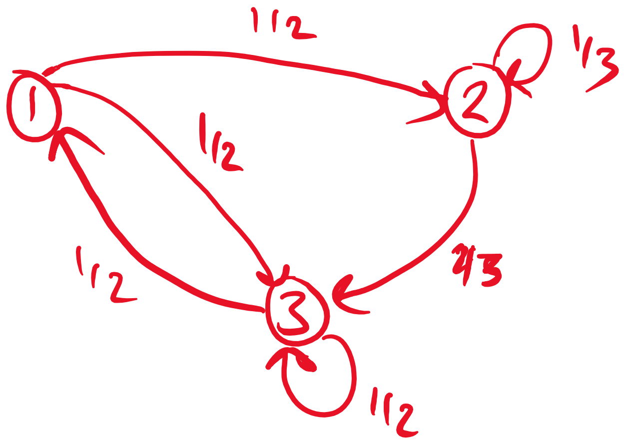
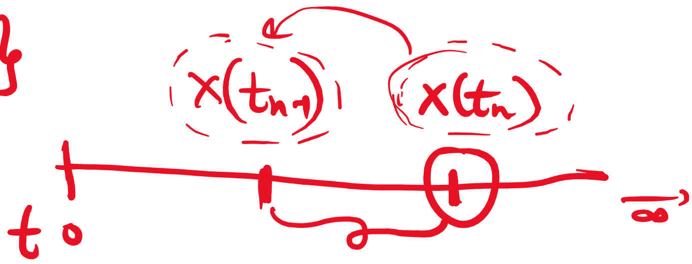
1. Consider a continuous-time Markov chain X(t) with the jump chain shown in following Figure. Assume $v1 = 2, v2 = 3$ *and* $v3 = 4$. Find the generator matrix G for the chain and steady state distribution of the states.



CTMC:

$Z = \{0, 1, 2, \dots, n\}$, T

$\{X(t) : t \geq 0\}$



1. Markov property

$$P(X(t_{n+1}) = i_{n+1} \mid X(t_n) = i_n, \dots, \cancel{X(t_1) = i_1}, \cancel{X(t_0) = i_0})$$

$$= P(X(t_{n+1}) = i_{n+1} \mid X(t_n) = i_n)$$

2. Time Homogeniety

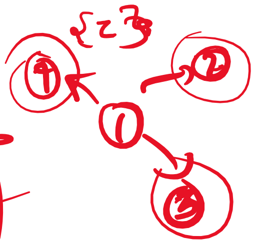
$$P(X(t+s) = j \mid X(s) = i) = P(X(t) = j \mid X(0) = i)$$

$$\Rightarrow P_{ij}(t)$$

$\Rightarrow P_{ij}(t)$

→ Chapman Kolmogorov eq. (*)

$$P_{ij}(t+s) = \sum_{k \in Z} P_{ik}(t)\, P_{kj}(s) = \sum_{k \in Z} P_{ik}(s)\, P_{kj}(t)$$



$\sum_{j \in Z} P_{ij}(t) = 1 \quad \forall t \geq 0 \quad \forall i \in Z$

→ $P(t) = [(P_{ij}(t))] \rightarrow |Z| \times |Z|$

→ Initial dist. / Absolute state probability

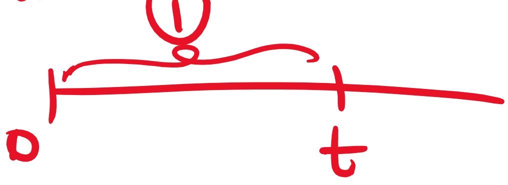
$$P_i(t) = P(X(t) = i)$$

Initial : $P_i(0) = P(X(0) = i)$

Given : $j \rightarrow$ state $\rightarrow$ $P_i(0) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$

→ $P_j(t) = \sum_{i \in Z} P_i(0)\, P_{ij}(t)$ → $\sum_{j \in Z} P_j(t) = 1 \quad \forall t \geq 0$

Self transition X



↳ Holding times ✓

↓ periodicity X

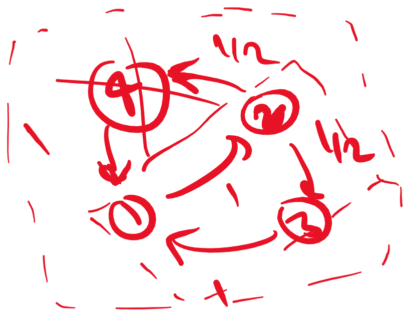
→ Recurrence, Transience

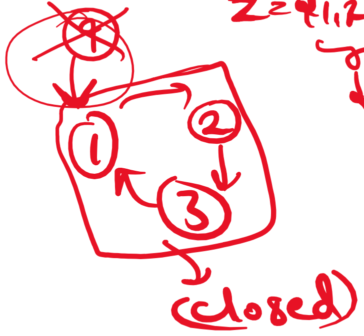
↳ (i) ... after ... (i)

↳ $i \to j$ after some t i

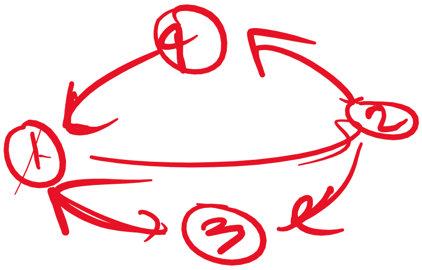
$Z = \{1,2,3,4\}$

→ Reducible, Irreducible





$\sum_{j \in C} p_{ij} = 1$, $i \in C$ → closed set



→ Irreducible

$Z \to C$ ✗

$C \to$ ~~closed~~

(*) Finite State space → Irreducible
↓
Recurrent

(*) Steady state: (Irreducible & positive recurrent)

$t \to \infty$, $P_{ij}(t) = p_j(t) = \boxed{\pi_j}$, $\forall j \in Z$

→ Stationary distribution.

$\pi = [\pi_0, \pi_1, \dots, \pi_n]$

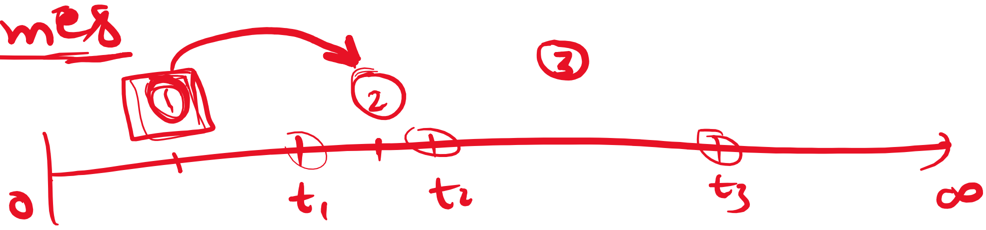
$P(t)$

$$\boxed{\pi P(t) = \pi}$$

$$\boxed{\sum_j \pi_j = 1}$$ → Normalization cond^{n}

(*) Holding Times

⊛ Holding Times



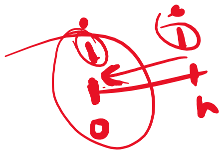
→ Memoryless
- → Geometric (DTMC)
- → exponential (CTMC)

→ Y_i → expontially distributed.

⊛ Transition rates

→ Unconditional rate of leaving state i (q_i / v_i)

→ conditional rate of transitioning from state i to state j (q_{ij})



$P_{ij}(t)$

$$\lim_{h \to 0^+} P_{ii}(h) = 1$$

$$v_i = q_i = \lim_{h \to 0} \frac{1 - P_{ii}(h)}{h} = -P_{ii}'(0)$$

$$q_{ij} = \lim_{h \to 0} \frac{P_{ij}(h)}{h} = P_{ij}'(0)$$

→ $$\sum_{\substack{j \in Z \\ j \neq i}} q_{ij} = q_i \ (v_i)$$

$$\sum_{j \in Z} P_{ij}(t) = 1$$

$$-P_{ii}'(0) + P_{ij}'(0) = 0$$

$j \in Z'$ ...

$$\sum_{j \neq i} P'_{ij}(0) + P'_{ii}(0) = 0$$

$$\sum_{j \neq i} q_{ij} = q_i \;(v_i)$$

CTMC: → Transition graph ✓

→ **Infinitesimal Generator** (G) (Q)

$$G = [(g_{ij})] \rightarrow |Z| \times |Z|$$

$$g_{ij} = -q_i, \text{ for } i = j$$

$$g_{ij} = q_{ij}, \text{ for } i \neq j$$

$$\boxed{\sum_{j \in Z} g_{ij} = 0}$$

→ **Kolmogorov equations:**

⊛ **Backward:** $P_{ij}(t+h) = \sum_{k \in Z} P_{ik}(h) \cdot P_{kj}(t)$

$$P'_{ij}(t) = \lim_{h \to 0} \frac{P_{ij}(t+h) - P_{ij}(t)}{h}$$

$P_{ii}(h) \cdot P_{ij}(t) - P_{ij}(t)$

$$= \lim_{h \to 0} \frac{\sum_{k \in Z} P_{ik}(h) \cdot P_{kj}(t) - P_{ij}(t)}{h}$$

$$= \lim_{h \to 0} \sum_{\substack{k \in Z \\ k \neq i}} \frac{P_{ik}(h) \cdot P_{kj}(t)}{h} - \frac{(1 - P_{ii}(h))}{h} P_{ij}(t)$$

$$\boxed{P'_{ij}(t) = \sum q_{ik} P_{kj}(t) - q_i P_{ij}(t)}$$

$$\boxed{p_{ij}'(t) = \sum_{k \neq i} q_{ik} P_{kj}(t) - q_i P_{ij}(t)}$$

$$P_{ij}'(t) = \sum_{k \neq i} g_{ik} P_{kj}(t) + g_{ii} P_{ij}(t) = \sum_{k \in Z} g_{ik} P_{kj}(t)$$

$$\boxed{P'(t) = G\, P(t)} \longrightarrow P(t) = e^{tG}$$

* **Forward**: $P_{ij}(t+h) = \sum_{k \in Z} P_{ik}(t) \cdot P_{kj}(h)$

$$P'_{ij}(t+h) = \lim_{h \to 0} \frac{P_{ij}(t+h) - P_{ij}(t)}{h}$$

$$\Rightarrow \boxed{P'(t) = P(t)\, G} \qquad \boxed{P(0) = I}$$

$$\longrightarrow P(t) = P(0)\, e^{tG} = e^{tG}$$

$$= \sum_{n=0}^{\infty} \frac{(tG)^n}{n!}$$

→ **Steady State** (CTMC)

$$\pi = \pi P(t)$$

$$\Rightarrow \pi = \pi e^{tG}$$

$$= \pi \sum_{n=0}^{\infty} \frac{(tG)^n}{n!}$$

$$\pi = \pi \left[I + G + \frac{G^2}{2!} + \cdots \right]$$

$$\pi = \pi \left[I + G + \frac{G^2}{2!} + \cdots \right]$$

$$\Rightarrow \cancel{\pi} = \cancel{\pi} + \pi G + \frac{\pi G^2}{2!} + \cdots$$

$$\Rightarrow 0 = \boxed{\pi G} + \frac{\pi G^2}{2!} + \cdots$$

$$\Rightarrow \boxed{\pi G = 0} \rightarrow \text{Global Balance equations}$$

$$\boxed{\sum_j \pi_j = 1}$$

$\pi_0 \quad \pi_1, \pi_2, \pi_j, \ldots \pi_n$

q_{0j}

q_{1j}

q_{2j}

$\vdots$

$-q_j$

$\vdots$

q_{nj}

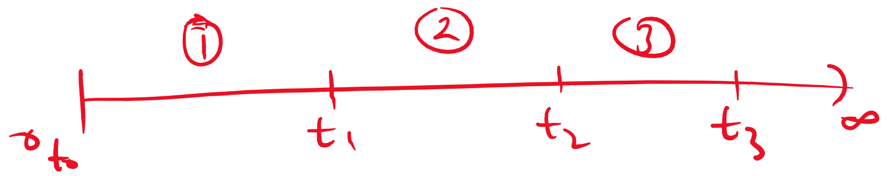
$$\Rightarrow \sum_{k \neq j} \pi_k q_{kj} - \pi_j q_j = 0$$

$$\Rightarrow \boxed{\sum_{k \neq j} \pi_k q_{kj} = \pi_j q_j}$$

$$\longrightarrow \lim_{t \to \infty} P_{ij}'(t) \longrightarrow 0$$

$$P_{ij}(t) = P_j(t) = \pi_j$$

(*) Jump Chain / Embedded DTMC



Jump chain $\rightarrow \{ X(t_0), \; X(t_1), \; X(t_2), \; X(t_3), \; \ldots \}$

1 (Initial) $\qquad$ 1 $\rightarrow$ 2 $\qquad$ 2 $\rightarrow$ 3 $\qquad$ 3 $\rightarrow$?

$Y_i \rightarrow i$, $\boxed{X(Y_i)} \rightarrow$ Jump chain (DTMC) transition out of i (self transition X)

$[Y_i, X(Y_i)]$

$$\boxed{P_{ij} = \frac{q_{ij}}{q_i}}$$

or

$$\boxed{P_{ij} = \frac{q_{ij}}{-q_{ii}}}$$

Q: $v_1 = 2$, $v_2 = 3$, $v_3 = 4$

$P_{11} = 0$, $P_{12} = 1/2$, $P_{13} = 1/2$,

$P_{21} = 0$, $P_{22} = \frac{1}{3}$, $P_{23} = \frac{2}{3}$

$P_{31} = \frac{1}{2}$, $P_{32} = 0$, $P_{33} = \frac{1}{2}$

$v_i P_{ij} = q_{ij}$

$q_{11} = 0$, $q_{12} = 1$, $q_{13} = 1$

$q_{21} = 0$, $q_{22} = 1$, $q_{23} = 2$

$q_{31} = 2$, $q_{32} = 0$, $q_{33} = 2$

$$G = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -3 & 2 \\ 2 & 2 & -4 \end{bmatrix}, \quad \boxed{\pi G = 0}$$

30 November 2024 00:08

A hospital owns two identical and independent power generators. The time to breakdown for each is exponential with parameter λ and the time for repair of a malfunctioning one is exponential with parameter μ. Let $X(t)$ be the Markov process which is the number of operational generators at time $t \geq 0$. Assume $X(0) = 2$. Prove that the probability that both generators are functional at time $t > 0$ is

$\exp(\lambda)$ $\exp(\mu)$

$z = \{0, 1, 2\}$

$P_0(0) = 0$
$P_1(0) = 0$
$P_2(0) = 1$

$$\boxed{P_2(t) = \frac{\mu^2}{(\lambda+\mu)^2} + \frac{\lambda^2 e^{-2(\mu+\lambda)t}}{(\lambda+\mu)^2} + \frac{2\lambda\mu e^{-(\lambda+\mu)t}}{(\lambda+\mu)^2}}$$

(*) Holding times / Sojourn times & transition rate

$\hookrightarrow \exp(\lambda_i) \longleftrightarrow q_i$

i: $0 \to h$

$$q_i = \lim_{h\to 0} \frac{1 - P_{ii}(h)}{h} = \lim_{h\to 0} \frac{1 - P(Y_i \geq h)}{h}$$

$\text{pdf} = \lambda e^{-\lambda t}$

$\text{cdf} = 1 - e^{-\lambda t} = P(T \leq t)$

$$q_i = \lim_{h\to 0} \frac{P(Y_i < h)}{h} = \lim_{h\to 0} \frac{1 - e^{-\lambda_i h}}{h}$$

$$= \lim_{h\to 0} \lambda_i e^{-\lambda_i h} = \lambda_i$$

(*) Unconditional rate of leaving i (q_i/ν_i) is equal to the parameter (λ_i) of the exponentially distributed holding times

(*) Markov System

$$Y_1 = \min(Y_{10}, Y_{12})$$

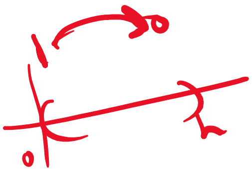
Y_{10}: λ, Y_{12}: μ

$1 - e^{-\mu h}$

$1 - P(Y_{12} \leq h)$

$\boxed{Y_1 - Y_{10}} \to \boxed{1 \to 0}$

$1 \to 0$



$$q_{10} = \lim_{h\to 0} \frac{P_{10}(h)}{h} = \lim_{h\to 0} \frac{P(Y_{10}\le h)\, P(Y_{12}>h)}{h}$$

$$= \lim_{h\to 0} \frac{(1-e^{-\lambda h})\cdot(e^{-\mu h})}{h}$$

$$= \lim_{h\to 0} \frac{(1-e^{-\lambda h})}{h}\cdot \cancel{\lim_{h\to 0} e^{-\mu h}}$$

$$\boxed{q_{10} = \lambda}$$

$\rightarrow Y_1 = Y_{12} \Rightarrow \boxed{q_{12} = \mu}$

$\rightarrow q_1 = (\lambda+\mu)$

(*) $Y_i = \min_{j\neq i}\{Y_{ij}\}$ λ_{ij} $\left(= \min(Y_{i0}, Y_{i1}, \ldots, Y_{in_k})\right)$

$\rightarrow \boxed{q_{ij} = \lambda_{ij}} \Rightarrow \boxed{q_i = \sum_{j\neq i} \lambda_{ij}}$

$$\longrightarrow G = \begin{bmatrix} -\mu & \mu & 0 \\ \lambda & -(\lambda+\mu) & \mu \\ 0 & \lambda & -\lambda \end{bmatrix}$$

$q_{01} = \mu$

$P_{i1} \quad P_{i2} \; - P_{ij} - \; P_{in}$

q_{1j}
q_{2j}
$\vdots$
$-q_j$
$\vdots$
q_{nj}

(*) <u>forward eq.</u> $P'(t) = P(t)\, G$

$$\Rightarrow P'_{ij}(t) = \sum_{k\neq j} P_{ik}(t)\, q_{kj} - q_j P_{ij}(t)$$

$$\Rightarrow P_j(t) = \sum_{i} P_i(0)\cdot P_{ij}(t)$$

$$\Rightarrow P_j(t) = \sum_{i\in Z} P_i(0)\cdot P_{ij}(t)$$

$$\Rightarrow P_j'(t) = \sum_{i\in Z} P_i(0)\cdot P_{ij}'(t)$$

$$\Rightarrow P_j'(t) = \sum_{k\neq j} P_k(t)\, q_{kj} - q_j P_j(t)$$

Abs.   Abs. prob. vectors

$$\Rightarrow \boxed{p'(t) = p(t)Q} \Rightarrow \boxed{\sum_j P_j(t) = 1}$$

$$\longrightarrow P_0(t),\ P_1(t),\ P_2(t) \Rightarrow \sum_{j=1}^{3} P_j(t) = 1$$

$$[P_0'(t)\ \ P_1'(t)\ \ P_2'(t)] = [P_0(t)\ \ P_1(t)\ \ P_2(t)]\begin{bmatrix} -\mu & \mu & 0 \\ \lambda & -(\lambda+\mu) & \mu \\ 0 & \lambda & -\lambda \end{bmatrix}$$

$$\Rightarrow P_0'(t) = -\mu P_0(t) + \lambda P_1(t) \quad — ①$$

$$\Rightarrow P_1'(t) = \mu P_0(t) - (\lambda+\mu)P_1(t) + \lambda P_2(t) \quad — ②$$

$$\Rightarrow P_2'(t) = \mu P_1(t) - \lambda P_2(t) \quad — ③$$

$$\boxed{\mathcal{L}\{P_j(t)\} = \hat{P}_j(s)}$$

$$P_0(t) + P_1(t) + P_2(t) = 1 \quad — ④$$

$$\left.\begin{aligned} &\Rightarrow s\hat{P}_0 - 0 = -\mu\hat{P}_0 + \lambda\hat{P}_1 \\ &\Rightarrow s\hat{P}_1 - 0 = \mu\hat{P}_0 - (\lambda+\mu)\hat{P}_1 + \lambda\hat{P}_2 \\ &\Rightarrow s\hat{P}_2 - 1 = \mu\hat{P}_1 - \lambda\hat{P}_2 \\ &\quad \hat{P}_0 + \hat{P}_1 + \hat{P}_2 = 1 \end{aligned}\right\} \Rightarrow \hat{P}_2 = ?$$

$$\Rightarrow (s+\mu)\hat{P}_0 - \lambda\hat{P}_1 = 0$$

$$\Rightarrow -\mu\hat{P}_0 + (s+\lambda+\mu)\hat{P}_1 - \lambda\hat{P}_2 = 0 = \begin{bmatrix} s+\mu & -\lambda & 0 \\ -\mu & s+\lambda+\mu & -\lambda \\ 0 & -\mu & s+\lambda \\ 1 & 1 & 1 \end{bmatrix}_{4\times 3}\begin{bmatrix}\hat{P}_0 \\ \hat{P}_1 \\ \hat{P}_2\end{bmatrix}_{3\times 1}$$

$$\Rightarrow -\mu\hat{P}_1 + (s+\lambda)\hat{P}_2 = 1$$

$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad -\mu\hat{P}_1 + (s+\lambda)\hat{P}_2 = 1$$

$$\Rightarrow \quad \hat{P}_0 + \hat{P}_1 + \hat{P}_2 = 1$$

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}_{4\times 1}$$

$$R \Rightarrow \left[\begin{array}{ccc|c} s+\mu & -\lambda & 0 & 0 \\ -\mu & s+\lambda+\mu & -\lambda & 0 \\ 0 & -\mu & s+\lambda & 1 \\ 1 & 1 & 1 & 1 \end{array}\right]$$

$$1 + \left(\frac{s+\lambda+\mu}{\mu}\right)$$

$$\boxed{1 - \frac{\lambda}{\mu}}$$

$$R_4 \to R_4 + \frac{1}{\mu}(R_2) \Rightarrow \left[\begin{array}{ccc|c} s+\mu & -\lambda & 0 & 0 \\ -\mu & s+\lambda+\mu & -\lambda & 0 \\ 0 & -\mu & s+\lambda & 1 \\ 0 & \frac{s+\lambda+2\mu}{\mu} & 1-\frac{\lambda}{\mu} & 1 \end{array}\right]$$

$$\frac{1}{\mu^2}(s+\lambda$$

$$R_4 \longrightarrow R_4 + \frac{1}{\mu^2}(s+\lambda+2\mu)R_3$$

$$\Rightarrow \left[\begin{array}{ccc|c} s+\mu & -\lambda & 0 & 0 \\ -\mu & s+\lambda+\mu & -\lambda & 0 \\ 0 & -\mu & s+\lambda & 1 \\ 0 & 0 & 1-\frac{\lambda}{\mu}+\frac{1}{\mu^2}(s+\lambda+2\mu) & 1+\frac{s+\lambda+2\mu}{\mu^2} \end{array}\right]$$

$$\hat{P}_2 = \frac{1+\dfrac{s+\lambda+2\mu}{\mu^2}}{1-\dfrac{\lambda}{\mu}+\dfrac{1}{\mu^2}(s+\lambda+2\mu)}$$

$$\hat{P}_2 = \frac{\mu^2+2\mu+\lambda+s-(\lambda\mu)+\lambda\mu}{\mu^2-\lambda\mu+2\mu+\lambda+s}$$

$$\hat{P}_2 = 1 + \frac{\lambda\mu}{\mu^2-\lambda\mu+2\mu+\lambda+s}$$