Markov Chain (MC)

- A discrete-time stochastic process is a Markov chain if, for t = 0, 1, 2… and all states
  $Pr(X_{t+1} = i_{t+1} | X_t = i_t, X_{t-1}=i_{t-1},…,X_1=i_1, X_0=i_0) = Pr(X_{t+1}=i_{t+1} | X_t = i_t)$
- The probability distribution of the state at time t+1 depends only on the state at time t and does not depend on the states before time t

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Initial Probability Distribution

- q_i: the probability that the chain is in state i at time 0: $Pr(X_0=i) = q_i$
- Let s be the number of states
- We call the vector **q** = $[q_1, q_2, …, q_s]$ the initial probability distribution for the Markov chain

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Stationary Assumption

- $Pr(X_{t+1} = j | X_t = i)$: is the probability that the system will be in state j at time $t+1$, given it is in state i at time t
- A MC is a stationary MC if $Pr(X_{t+1} = j | X_t = i)$ is independent of t (does not change over time)
- $Pr(X_{t+1} = j | X_t = i) = Pr(X_t = j | X_{t-1} = i) = \ldots = Pr(X_1 = j | X_0 = i) = p_{ij}$

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Transition Probabilities

- p_{ij} is the transition probability
- Let s be the number of states
- The transition probabilities are displayed as an s x s matrix

$$\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1s} \\ p_{21} & p_{22} & \cdots & p_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ p_{s1} & p_{s2} & \cdots & p_{ss} \end{bmatrix}$$

- All probabilities in the transition probability matrix are nonnegative
- The probabilities in each row must sum to 1, i.e., for each i

$$\sum_{j=1}^{j=s} p_{ij} = 1$$

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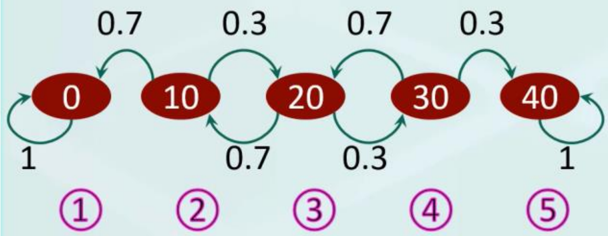
Gambling Example

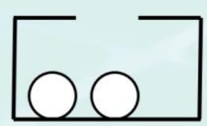
- At time 0, you have \$20. You bet \$10 each time in a gamble
- You win with a probability of 0.3, and lose with probability of 0.7
- If you have \$0 or \$40, the game is over
- X_t is the money you have after each gamble at time t, $X_0=20$



$$\mathbf{q} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

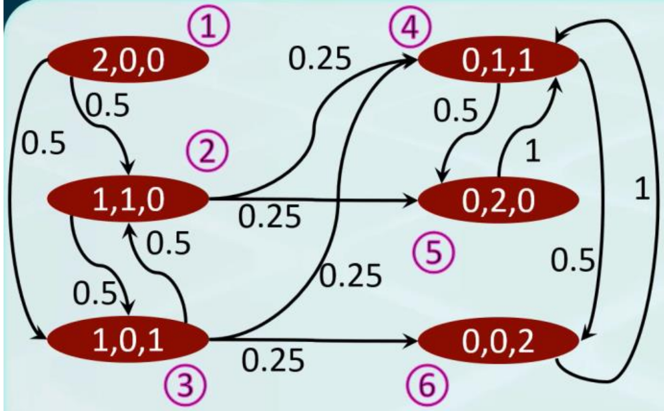
$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.7 & 0 & 0.3 & 0 & 0 \\ 0 & 0.7 & 0 & 0.3 & 0 \\ 0 & 0 & 0.7 & 0 & 0.3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Golf Ball Example



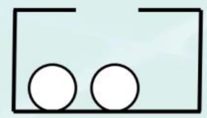
- A box contains two **white** golf balls
- Choose a ball randomly and flip a coin
- If the ball is **white** and the coin shows **heads**, paint the ball **red**
- If the ball is **white** and the coin shows **tails**, paint the ball **blue**
- If the ball is **painted**, then directly **change the color** (red ↔ blue)
- X_t=[w, r, b]: where **w, r, and b** are the numbers of **white, red, and blue balls** in the box at time t, X_0 = [2, 0, 0].

Golf Ball Example



X_t= [w, r, b] at time t

Heads: Red; Tails: Blue



$$\mathbf{P} = \begin{bmatrix} 0 & 0.5 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0.25 & 0.25 & 0 \\ 0 & 0.5 & 0 & 0.25 & 0 & 0.25 \\ 0 & 0 & 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\mathbf{q} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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n-Step Transition Probability

- If a Markov chain (MC) is in state i at time t, what is the probability that it will be in state j after n periods?
- For a stationary MC, this probability will be independent of t, so

$$\Pr(X_{t+n} = j \mid X_t = i) = \Pr(X_n = j \mid X_0 = i) = p_{ij}(n)$$

where $p_{ij}(n)$ is called the n-step probability of a transition from state i to state j

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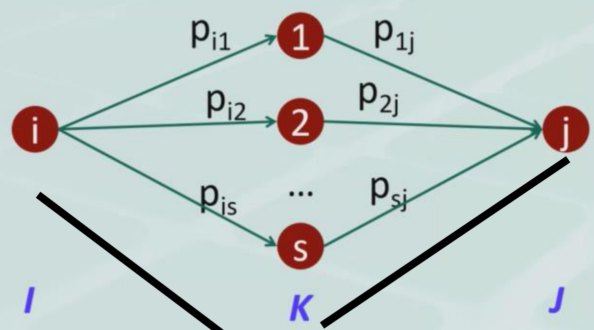
n-Step Transition Probability

$$\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1s} \\ p_{21} & p_{22} & \cdots & p_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ p_{s1} & p_{s2} & \cdots & p_{ss} \end{bmatrix}$$



$$p_{ij}(2) = \sum_{k=1}^{k=s} p_{ik} p_{kj} = ij^{th} \text{ element of } \mathbf{P}^2$$

$$p_{ij}(n) = ij^{th} \text{ element of } \mathbf{P}^n, \quad n = 1, 2, 3, \ldots$$

The Branding Example

- Suppose two toothpaste brands are available on the market
- If an average person last purchased brand 1, there is a 90% chance that their next purchase will still be brand 1
- If an average person last purchased brand 2, there is an 80% chance that their next purchase will still be brand 2
- Define X_t to be the brand purchased by a person on their t^{th} purchase

$$\mathbf{P} = \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix}$$

The Branding Example

- If a person is currently a brand 2 purchaser, what is the probability that he/she will purchase brand 1 two purchases from now?
- $Pr(X_2 = 1|X_0 = 2) = p_{21}(2)$ = element (2,1) of $\mathbf{P}^2$

$$\mathbf{P}^2 = \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix} \times \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix} = \begin{bmatrix} 0.83 & 0.17 \\ 0.34 & 0.66 \end{bmatrix}$$

$$p_{21}(2) = 0.34$$

5

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The Branding Example

- If a person is currently a brand 1 purchaser, what is the probability that he/she will purchase brand 1 three purchases from now?
- $Pr(X_3 = 1|X_0 = 1) = p_{11}(3)$ = element (1,1) of $\mathbf{P}^3$

$$\mathbf{P}^3 = \begin{bmatrix} 0.83 & 0.17 \\ 0.34 & 0.66 \end{bmatrix} \times \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix} = \begin{bmatrix} 0.781 & 0.219 \\ 0.438 & 0.562 \end{bmatrix}$$

$$p_{21}(2) = 0.781$$

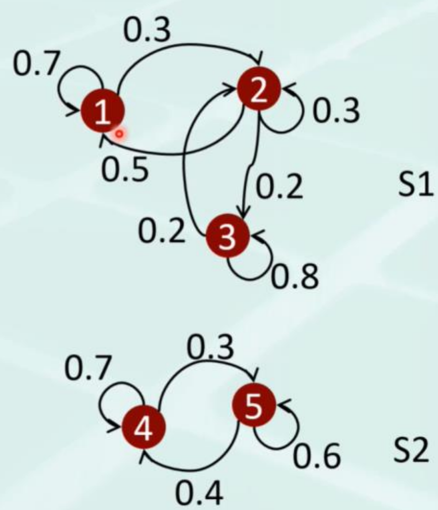
6

Reachable and Communicating States

- A path from i to j is a sequence of transitions that begins in i and ends in j, such that each transition in the sequence has a positive probability
- A state j is reachable from state i if there is a path leading from i to j
- Two states i and j can communicate if j is reachable from i, and i is reachable from j
- A set of states S in a Markov chain is a closed set if no state outside of S is reachable from any state in S

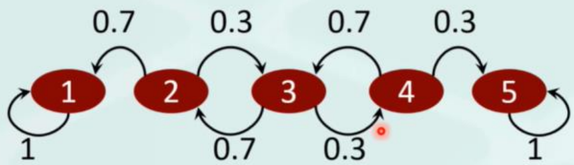
Example

$$\mathbf{P} = \begin{bmatrix} 0.7 & 0.3 & 0 & 0 & 0 \\ 0.5 & 0.3 & 0.2 & 0 & 0 \\ 0 & 0.2 & 0.8 & 0 & 0 \\ 0 & 0 & 0 & 0.7 & 0.3 \\ 0 & 0 & 0 & 0.4 & 0.6 \end{bmatrix}$$



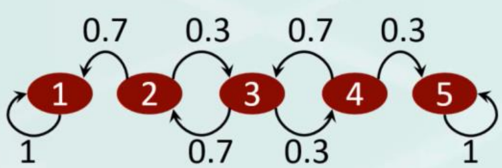
Absorbing State

- A state *i* is an absorbing state if p_{ii}=1
- Whenever we enter an absorbing state, we never leave the state
- An absorbing state is also a closed set containing only one state

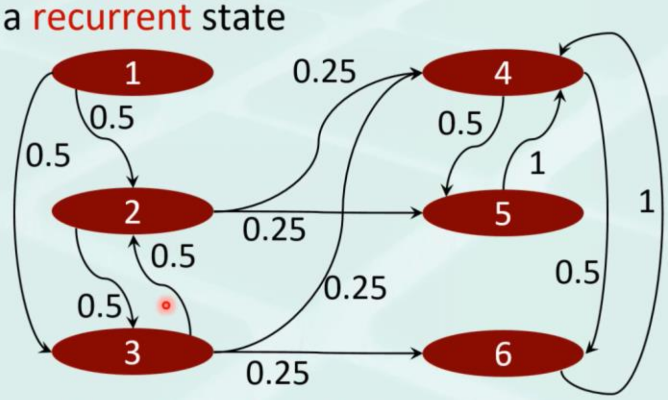


Transient and Recurrent States

- A state *i* is a transient state **if there exists a state *j*** that is reachable from *i*, but the state *i* is not reachable from state *j*
- If a state is not transient, it is a recurrent state

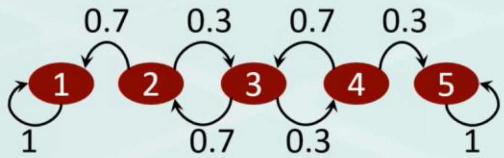


Transient: 2, 3, 4
Recurrent: 1, 5

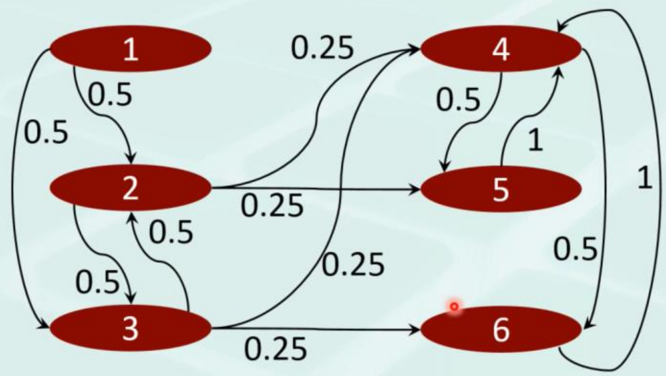


Transient: 1, 2, 3
Recurrent: 4, 5, 6

Periodic and Aperiodic States



None is periodic



1, 2, 3 aperiodic

4, 5, 6 periodic, period k =2

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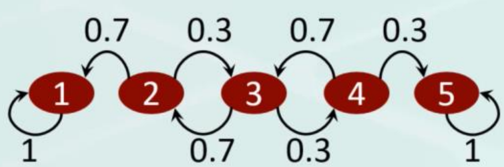
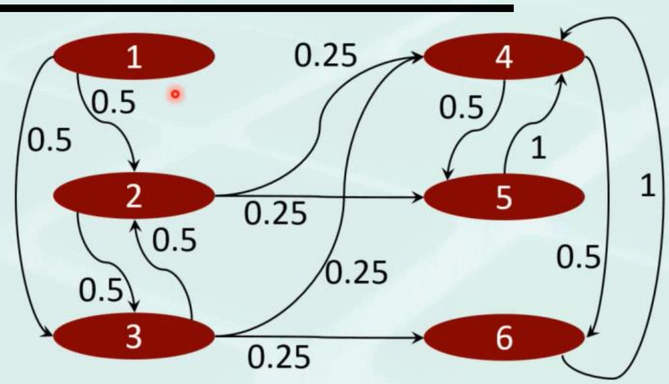
Ergodic Markov Chain

- If **all states** in a Markov chain are recurrent (not transient), aperiodic (not periodic), and communicate with each other, the chain is said to be ergodic



Nonergodic



Nonergodic

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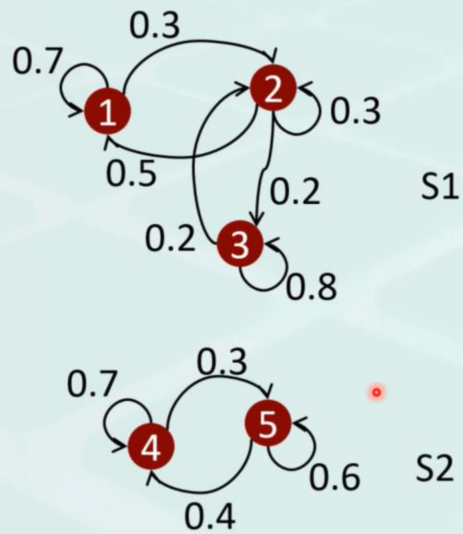
Ergodic Markov Chain



S1: Ergodic
S2: Ergodic
S1+S2: Nonergodic

Steady State of the Toothpaste Brand Example

- Suppose two toothpaste brands are available on the market

$$\mathbf{P} = \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix}$$

Operations Research 13D: Markov Chain Steady-State Theorem

Steady State of the Toothpaste Brand Example

- Suppose two toothpaste brands are available on the market

$$\mathbf{P} = \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix} \qquad \mathbf{P}^{10} = \begin{bmatrix} 0.68 & 0.32 \\ 0.65 & 0.35 \end{bmatrix}$$

$$\mathbf{P}^{2} = \begin{bmatrix} 0.83 & 0.17 \\ 0.34 & 0.66 \end{bmatrix} \qquad \mathbf{P}^{20} = \begin{bmatrix} 0.67 & 0.33 \\ 0.67 & 0.33 \end{bmatrix}$$

$$\mathbf{P}^{3} = \begin{bmatrix} 0.78 & 0.22 \\ 0.44 & 0.56 \end{bmatrix} \qquad \mathbf{P}^{30} = \begin{bmatrix} 0.67 & 0.33 \\ 0.67 & 0.33 \end{bmatrix}$$

$$\mathbf{P}^{5} = \begin{bmatrix} 0.72 & 0.277 \\ 0.55 & 0.45 \end{bmatrix} \qquad \mathbf{P}^{40} = \begin{bmatrix} 0.67 & 0.33 \\ 0.67 & 0.33 \end{bmatrix}$$

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Steady-State Probabilities

$$\mathbf{P}^{40} = \begin{bmatrix} 0.67 & 0.33 \\ 0.67 & 0.33 \end{bmatrix}$$

- After a long time, the probability that a person's next toothpaste purchase would be brand 1 with a probability 0.67 and brand 2 with a probability 0.33
- These probabilities do not depend on whether the person initially purchased brand 1 or brand 2
- These are called the steady state probabilities, which can be used to describe the long-run behavior of a Markov chain

Transient Analysis

- The problem reaches the steady state after around 20 transitions
- If **P** contains very few entries that are near 0 or 1, the steady state is usually reached very quickly
- The behavior of a Markov chain before the steady state is reached is often called **transient** (or short-run) **behavior**

Steady-State Theorem

- Let **P** be the transition matrix for an *s*-state ergodic Markov chain. Then there exists a vector $\boldsymbol{\pi} = \begin{bmatrix} \pi_1 & \pi_2 & \cdots & \pi_s \end{bmatrix}$ (steady-state distribution)

such that $\lim_{n\to\infty} \mathbf{P}^n = \begin{bmatrix} \pi_1 & \pi_2 & \cdots & \pi_s \\ \pi_1 & \pi_2 & \cdots & \pi_s \\ \vdots & \vdots & \ddots & \vdots \\ \pi_1 & \pi_2 & \cdots & \pi_s \end{bmatrix}$ and $\lim_{n\to\infty} p_{ij}(\mathrm{n}) = \pi_j$

- The steady-state distribution

$$\pi_1 + \pi_2 + \cdots + \pi_s = 1$$

$$\boldsymbol{\pi}(n+1) = \boldsymbol{\pi}(n) \cdot \mathbf{P} \qquad \boldsymbol{\pi} = \boldsymbol{\pi}\mathbf{P}$$

Steady-State Distribution

$$\mathbf{P} = \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix}$$

$$[\pi_1 \quad \pi_2] = [\pi_1 \quad \pi_2] \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix}$$

$$\boxed{\begin{gathered} \pi_1 = 0.9\pi_1 + 0.2\pi_2 \\ \pi_2 = 0.1\pi_1 + 0.8\pi_2 \\ \pi_1 + \pi_2 = 1 \end{gathered}}$$

$$\boldsymbol{\pi} = [\pi_1 \quad \pi_2] = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

Use of Steady-State Probabilities in Decision Making

- There are 100 million customers
- Each customer purchases one tube of toothpaste every month
- One tube of toothpaste costs \$1 to produce and is sold for \$2
- For \$50 million/year, an advertising firm guarantees to decrease the fraction of brand 1 customers who switch to brand 2 after a purchase from 10% to 5%
- Should the company that makes brand 1 toothpaste hire the firm?

Steady-State Profits Before and After Advertising

$$\mathbf{P} = \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix} \qquad \boldsymbol{\pi} = \begin{bmatrix} \pi_1 & \pi_2 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

<u>2/3</u> × (100 million customers) × (1 tube/month × 12 months) × ($2 - $1)
= **<u>$800 million</u>**

$$\mathbf{P} = \begin{bmatrix} 0.95 & 0.05 \\ 0.2 & 0.8 \end{bmatrix} \qquad \boldsymbol{\pi} = \begin{bmatrix} \pi_1 & \pi_2 \end{bmatrix} = \begin{bmatrix} 0.8 & 0.2 \end{bmatrix}$$

<u>0.8</u> × (100 million customers) × (1 tube/month × 12 months) × ($2 - $1)
<u>– $50 million</u> = **<u>$910 million</u>**

Mean First Passage Time

- For an ergodic chain, the mean first passage time from state i to state j (denoted by m_{ij}) is the expected number of transitions before we first reach state j, given that we are currently in state i

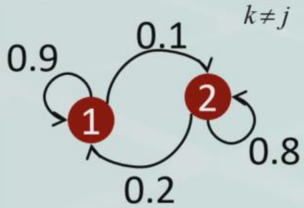
Mean First Passage Time

- We are currently in state i
- Case 1: With probability p_{ij}, it will take one transition to go from state i to state j;
- Case 2: With probability p_{ik} it will take one transition to go from i to state k ($k \neq j$); Now we are in state k. It will to m_{kj} transitions to go from k to j. Therefore, it will take an average of $1 + m_{kj}$ transitions to go from i to j

$$m_{ij} = \underbrace{\boxed{p_{ij} \cdot 1}}_{\text{Case 1}} + \underbrace{\boxed{\sum_{k \neq j} p_{ik}(1 + m_{kj})}}_{\text{Case 2}} = 1 + \sum_{k \neq j} p_{ik} m_{kj} \qquad m_{ii} = 1/\pi_i$$

Example

$$m_{ij} = p_{ij} \cdot 1 + \sum_{k \neq j} p_{ik}(1 + m_{kj}) = 1 + \sum_{k \neq j} p_{ik} m_{kj} \qquad m_{ii} = 1/\pi_i$$

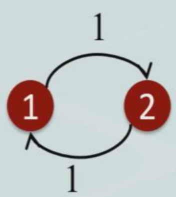


$$\mathbf{P} = \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix} \qquad [\pi_1 \quad \pi_2] = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

- $m_{11}=1/(2/3)=1.5$
- $m_{22}=1/(1/3)=3$
- $m_{12}=1+p_{11}m_{12}=1+0.9m_{12} \rightarrow m_{12}=10$
- $m_{21}=1+p_{22}m_{21}=1+0.8m_{21} \rightarrow m_{21}=5$

Extreme Example

$$m_{ij} = p_{ij} \cdot 1 + \sum_{k \neq j} p_{ik}(1 + m_{kj}) = 1 + \sum_{k \neq j} p_{ik} m_{kj} \qquad m_{ii} = 1/\pi_i$$



$$\mathbf{P} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad [\pi_1 \quad \pi_2] = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

- $m_{11}=1/(1/2)=2$
- $m_{22}=1/(1/2)=2$
- $m_{12}=1+p_{11}m_{12}=1+0\cdot m_{12} \rightarrow m_{12}=1$
- $m_{21}=1+p_{22}m_{21}=1+0\cdot m_{21} \rightarrow m_{21}=1$

Absorbing Markov Chain

- If a Markov chain has at least one absorbing state and the rest are transient states, it is called an absorbing chain
- If we begin in a transient state, eventually we are sure to leave the transient state and end up in one of the absorbing states

Fundamental Matrix

$$\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1s} \\ p_{21} & p_{22} & \cdots & p_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ p_{s1} & p_{s2} & \cdots & p_{ss} \end{bmatrix}$$

$$\begin{array}{ccc} & \text{s−m} & \text{m} & \\ \mathbf{P} = & \left[\begin{array}{c|c} \mathbf{Q} & \mathbf{R} \\ \hline \mathbf{0} & \mathbf{I} \end{array}\right] & \begin{array}{c} \text{s−m} \\ \text{m} \end{array} \end{array}$$

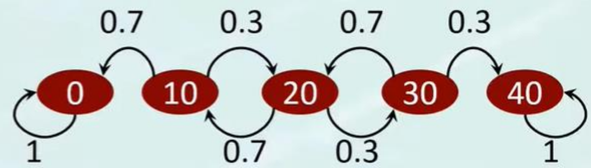
m absorbing states ($a_1, a_2, \ldots, a_m$)

s – m transient states ($t_1, t_2, \ldots, t_{s-m}$)

- If we are currently in transient state t_i
  - the **expected number of periods** that we will spend in transient state t_j before being absorbed is equal to the ij th element of the fundamental matrix $(\mathbf{I} - \mathbf{Q})^{-1}$
  - the **probability** that we will eventually be absorbed in absorbing state a_j is equal to the ij th element of the matrix $(\mathbf{I} - \mathbf{Q})^{-1}\mathbf{R}$

Gambling Example

- At time 0, you have \$20.
- You bet \$10 each time in a gamble
- You win with a probability of 0.3, and lose with probability of 0.7
- If you have \$0 or \$40, the game is over



$$
\mathbf{P} = \begin{array}{c} \\ \left[\begin{array}{ccc|cc} 0 & 0.3 & 0 & 0.7 & 0 \\ 0.7 & 0 & 0.3 & 0 & 0 \\ 0 & 0.7 & 0 & 0 & 0.3 \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \end{array}
\begin{array}{c} \\ \$10 \\ \$20 \\ \$30 \\ \$0 \\ \$40 \end{array}
$$

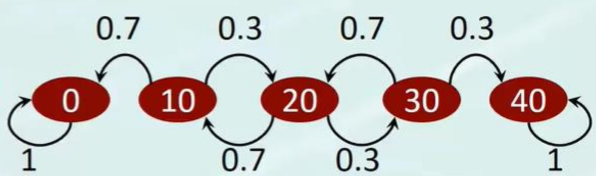
Column labels: \$10, \$20, \$30, \$0, \$40 (blocks: Q, R / 0, I)

4

Gambling Example

$$
(\mathbf{I}-\mathbf{Q})^{-1} = \begin{array}{c|ccc} & \$10 & \$20 & \$30 \\ \hline \$10 & 1.36 & 0.52 & 0.16 \\ \$20 & 1.21 & 1.72 & 0.52 \\ \$30 & 0.84 & 1.21 & 1.36 \end{array}
$$

$$
(\mathbf{I}-\mathbf{Q})^{-1}\mathbf{R} = \begin{array}{c|cc} & \$0 & \$40 \\ \hline \$10 & 0.95 & 0.05 \\ \$20 & 0.84 & 0.16 \\ \$30 & 0.59 & 0.41 \end{array}
$$



$$
\mathbf{P} = \left[\begin{array}{ccc|cc} 0 & 0.3 & 0 & 0.7 & 0 \\ 0.7 & 0 & 0.3 & 0 & 0 \\ 0 & 0.7 & 0 & 0 & 0.3 \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]
\begin{array}{c} \$10 \\ \$20 \\ \$30 \\ \$0 \\ \$40 \end{array}
$$

Column labels: \$10, \$20, \$30, \$0, \$40 (blocks: Q, R / 0, I)

5

Gambling Example

$$
(\mathbf{I}-\mathbf{Q})^{-1} = \begin{array}{c} \\ \\ \\ \end{array}
\begin{array}{ccc} \$10 & \$20 & \$30 \\ \left[\begin{matrix} 1.36 & 0.52 & 0.16 \\ 1.21 & 1.72 & 0.52 \\ 0.84 & 1.21 & 1.36 \end{matrix}\right] \end{array}
\begin{array}{c} \\ \$10 \\ \$20 \\ \$30 \end{array}
$$

$$
(\mathbf{I}-\mathbf{Q})^{-1}\mathbf{R} = \begin{array}{cc} \$0 & \$40 \\ \left[\begin{matrix} 0.95 & 0.05 \\ 0.84 & 0.16 \\ 0.59 & 0.41 \end{matrix}\right] \end{array}
\begin{array}{c} \\ \$10 \\ \$20 \\ \$30 \end{array}
$$

- If you have \$20 now, what is the <u>expected number of periods</u> that you will have \$30?

  [element 23 of $(\mathbf{I} - \mathbf{Q})^{-1}$] = 0.52

- If you have \$20 now, what is the <u>expected number of periods</u> that you will be playing the game (game is not over)?

  [element 21 + element 22 + element 23 of $(\mathbf{I} - \mathbf{Q})^{-1}$] = 1.21+1.72+0.52 =3.45

Gambling Example

$$
(\mathbf{I}-\mathbf{Q})^{-1} = \begin{array}{ccc} \$10 & \$20 & \$30 \\ \left[\begin{matrix} 1.36 & 0.52 & 0.16 \\ 1.21 & 1.72 & 0.52 \\ 0.84 & 1.21 & 1.36 \end{matrix}\right] \end{array}
\begin{array}{c} \\ \$10 \\ \$20 \\ \$30 \end{array}
$$

$$
(\mathbf{I}-\mathbf{Q})^{-1}\mathbf{R} = \begin{array}{cc} \$0 & \$40 \\ \left[\begin{matrix} 0.95 & 0.05 \\ 0.84 & 0.16 \\ 0.59 & 0.41 \end{matrix}\right] \end{array}
\begin{array}{c} \\ \$10 \\ \$20 \\ \$30 \end{array}
$$

- If you have \$20 now, what is the probability that you will lose them all?

  [element 21 of $(\mathbf{I} - \mathbf{Q})^{-1}\mathbf{R}$] = 0.84

- If you have \$20 now, what is the probability that you will end up having \$40?

  [element 22 of $(\mathbf{I} - \mathbf{Q})^{-1}\mathbf{R}$] = 0.16