Suyash Goyal 2K22/MC/161

Assignment - 4

Stochastic Processes (MC303)

(A1) Transition Matrix

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 1/2 & 1/2 \\ 0 & 1/3 & 2/3 \\ 1/2 & 0 & 1/2 \end{bmatrix} \end{array}$$

given $v_1=2$, $v_2=3$, $v_3=4$ are holding rates for all states,

$q_{ij} = v_i \cdot P_{ij}$ (when $i \neq j$)

$q_{12} = v_1 \cdot P_{12} = 2(1/2) = 1$; $q_{13} = v_1 \cdot P_{13} = 2(1/2) = 1$; $q_{21} = v_2 \cdot P_{21} = 0$;

$q_{23} = v_2 \cdot P_{23} = 3(2/3) = 2$; $q_{31} = v_3 \cdot P_{31} = 4(1/2) = 2$; $q_{32} = v_3 \cdot P_{32} = 0$;

$q_{ii} = -\sum_{j \neq i} q_{ij}$

$q_{11} = -(1+1) = -2$; $q_{22} = -(0+2) = -2$; $q_{33} = -(0+2) = -2$; $\therefore Q = \begin{bmatrix} -2 & 1 & 1 \\ 0 & -2 & 2 \\ 2 & 0 & -2 \end{bmatrix}$

$$\frac{dp_0(t)}{dt} = -2p_0(t) + 2p_2(t) \qquad p_2(0) = 1$$

$$\frac{dp_1(t)}{dt} = p_0(t) - 2p_1(t) \qquad p_1(0) = 0, \quad p_0(0) = 0$$

$$\frac{dp_2(t)}{dt} = p_0(t) + 2p_1(t) - 2p_2(t)$$

$$sp_0(s) = -2p_0(s) + 2p_2(s) + p_0(s) = \frac{2}{s+2} p_2(s)$$

$$sp_1(s) = p_0(s) - 2p_1(s) \qquad p_1(s) = \frac{p_0(s)}{s+2}$$

$$sp_2(s) - 1 = \frac{2}{s+2} p_2(s) + \frac{2p_2(s)}{(s+2)^2} - 2p_2(s)$$

$$\frac{p_2(s)}{(s+2)^2}\left(s(s+2)^2 + 2(s+2) + 2(s+2)^2\right) = 1$$

$$p_2(s) = \frac{(s+2)}{(s+2)^2+2} \qquad p_0(s) = \frac{2}{(s+2)^2+2} \qquad p_1(s) = \frac{2}{(s+2)((s+2)^2+2)}$$

Taking inverse laplace,

$p_0(t) = \sqrt{2}e^{-2t}\sin(\sqrt{2}t)$

$p_1(t) = (1-\cos(\sqrt{2}t))e^{-2t}$

$p_2(t) = \sqrt{2}\cos\sqrt{2}te^{-2t}$

(A2) Let $d(x) = \lambda e^{-\lambda}$ be breakdown distribution.

$r(x) = \mu e^{-\mu}$ be repair distribution.

Making the generator matrix

$Q = [q_{ij}]$ $\qquad q_{ij}$ is moving from i to j

given $X(0)=2$ both are Functional.

Let $X(t)=0$ denote both are damaged.

$X(t)=1$ denote one is Functional.

$X(t)=2$ denote both are Functional.

$q_{00} = -2\mu$ (no repairs)

$q_{01} = 2\mu$ (one repaired)

$q_{02} = 0$

$q_{10} = \lambda$ (one damaged)

$q_{11} = -(\lambda+\mu)$

$q_{12} = \mu$

$q_{20} = 0$

$q_{21} = 2\lambda$ (one damaged)

$q_{22} = -2\lambda$ (none damaged)

$$Q = \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} -2\mu & 2\mu & 0 \\ \lambda & -(\mu+\lambda) & \mu \\ 0 & 2\lambda & -2\lambda \end{bmatrix} \end{array}$$

forming Forward Kolmogorov equation.

$$\frac{dp_2(t)}{dt} = -2\lambda p_2(t) + \mu p_1(t) \quad \text{——— (1)}$$

$$\frac{dp_1(t)}{dt} = 2\lambda p_2(t) - (\mu+\lambda) p_1(t) + 2\mu p_0(t) \quad \text{——— (2)}$$

$$d p_0(t) = -2\mu p_0(t) + \lambda p_1(t) \quad \text{——— (3)}$$

$p_2(0)=1 \qquad p_1(0)=0 \qquad p_0(0)=0$

Taking Laplace Transform of eq^n (1),

$$sp_2(s) - p_2(0) = -2\lambda p_2(s) + \mu p_1(s) \Rightarrow (s+2\lambda) p_2(s) - 1 = \mu p_1(s)$$

Similarly for eq^n(2),

$$sp_1(s) - p_1(0) = 2\lambda P_2 - (\mu+\lambda) P_1 + 2\mu P_0$$

$$(s+\mu+\lambda) P_1 = 2\lambda P_2 + 2\mu P_0$$

Eq^n (3),

$$s p_0(s) = -2\mu p_0(s) + \lambda p_1(s)$$

$$p_0 = \frac{2\mu p_1}{s+2\mu}$$

Putting values of p_0 and p_2,

$$(s+\mu+\lambda) P_1(s) = 2\lambda \frac{(\mu P_1(s)+1)}{s+2\lambda} + \frac{2\mu\lambda P_1(s)}{s+2\mu}$$

$$\Rightarrow P_1(s) \left[(s+\mu+\lambda) - \frac{2\lambda\mu}{s+2\lambda} - \frac{2\mu\lambda}{s+2\mu} \right] = \frac{2\lambda}{s+2\lambda}$$

$$\Rightarrow P_1(s) = \frac{2\lambda(s+2\mu)}{s(2\lambda^2+4\lambda\mu+3\lambda s+2\mu^2+3\mu s+s^2)}$$

we get $$P_2(s) = \frac{2\lambda\mu(s+2\mu)P_1(s)+1}{s(2\lambda^2+4\lambda\mu+3\lambda s+2\mu^2+3\mu s+s^2)}$$

$$P_2(t) = \mathcal{L}^{-1}(P_2(s)) = \frac{2\lambda\mu e^{-(\lambda+\mu)t}}{(\lambda+\mu)^2} + \frac{\lambda^2 e^{-2(\lambda+\mu)t}}{(\lambda+\mu)^2} + \frac{\mu^2}{(\lambda+\mu)^2}$$ Hence Proved.

(A5) constructing generator matrix

$$q_{i,i+1} = \frac{\alpha}{1+\alpha} \qquad q_{i,i-1} = \frac{1}{1+\alpha} \qquad q_{ii} = 0$$

$$\begin{bmatrix} 0 & \alpha/1+\alpha & 0 & 0 & 0 & \cdots \\ 1/1+\alpha & 0 & \alpha/1+\alpha & 0 & 0 & \cdots \\ 0 & 1/1+\alpha & 0 & \alpha/1+\alpha & 0 & \cdots \\ 0 & 0 & 1/1+\alpha & 0 & \alpha/1+\alpha & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$\pi_0 = \frac{\alpha}{1+\alpha}\pi_1 \Rightarrow \pi_1 = (1+\alpha)\pi_0$$

$$\pi_1 = \frac{\pi_0}{1+\alpha} + \frac{\alpha\pi_2}{1+\alpha} \Rightarrow \pi_i = \frac{\pi_{i-1}}{1+\alpha} + \frac{\alpha\pi_{i+1}}{1+\alpha}$$

and $\sum_i \pi_i = 1$

Let $\pi_i = c\lambda^i$

$$c\lambda^i = \frac{c\lambda^{i-1}}{1+\alpha} + \frac{\alpha c\lambda^{i+1}}{1+\alpha} \Rightarrow \lambda = \alpha \Rightarrow \pi_i = c\alpha^i$$

$$\sum \pi_i = 1 \Rightarrow \sum c\alpha^i = 1 \Rightarrow c\sum \alpha^i = 1$$

For $\alpha < 1$,

$$c\left(\frac{1}{1-\alpha}\right) = 1 \Rightarrow c = 1-\alpha \Rightarrow \pi_i = (1-\alpha)\alpha^i$$

For $\alpha > 1$,

The series $\sum \alpha^i$ is increasing and converges to ∞ ∴ no longer steady state.

When exponentially distributed with rate μ_j, the holding time is introduced, making it an Embedded DTMC. Since distribution is exponential, it becomes a CTMC.

The generator matrix G.

$$g_{i,i-1} = \frac{\mu_i}{1+\alpha} \qquad g_{i,i+1} = \frac{\alpha\mu_i}{1+\alpha} \qquad g_{ii} = -\mu_i\frac{(1+\alpha)}{1+\alpha} = -\mu_i \qquad g_{01} = \frac{1}{1+\alpha} \qquad g_{00} = \frac{-1}{1+\alpha}$$

$$G = \begin{bmatrix} -1/1+\alpha & 1/1+\alpha & 0 & 0 & \cdots \\ \mu_1/1+\alpha & -\mu_1 & \mu_1\alpha/1+\alpha & 0 & \cdots \\ 0 & \mu_2/1+\alpha & -\mu_2 & \mu_2\alpha/1+\alpha & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

P.T.O.

Since $\pi G = 0 \Rightarrow -\pi_i \mu_i + \pi_{i-1} \frac{\mu_{i-1}}{1+\alpha} + \pi_{i+1} \frac{\mu_{i+1}\alpha}{1+\alpha} = 0$

Assuming $\mu_i = \mu$ (constant), we get a solution,

$\pi_i = (1-\alpha)\alpha^i \quad (\alpha < 1)$

(A4) $X = \{X(t) : t \geq 0\}$ is CTMC with distribution π.

$S_1, S_2, \ldots$ event of poisson process with rate λ.

$Y_n = X(S_n)$

Prove that $Y = \{Y_n : n \geq 1\}$ is a DTMC with stationary distribution π.

Since we observe $X(S_n)$, the time intervals $S_1, S_2, \ldots,$ are no longer continuous. They are random times from poisson distribution, hence discrete.

For the CTMC, we know,

$P(X_i = t \mid X_{i-1} = S, X_{i-2} = S_{i-2}, \ldots X_0 = S_0) = P(X_i = t \mid X_{i-1} = S)$

The same can be said,

$P(X_i = S_i \mid X_{i-1} = S_{i-1}, X_{i-2} = S_{i-2}, \ldots, X_1 = S_1) = P(X_i = S_i \mid X_{i-1} = S_{i-1})$

Hence it is a DTMC.

Now, since steady state distribution for CTMC is π and it was exponentially distributed with a parameter λ and the random times are sampled from the same distribution for DTMC, we can say the stationary distribution would be same and stay λ.