The property (12.24) may be interpreted that *any future state of the process depends upon its present state and is independent of its past.*

Normally, *the one-step transition probabilities*

$$P[X_{n+1} = j \mid X_n = i] = p_{ij}$$

depend upon the index n. However, we are interested primarily in Markov chains for which these probabilities are independent of n. Such a Markov chain is said to be *homogeneous in time*. Here we shall assume the Markov chains to be *time-homogeneous*.

The one-step transition probabilities are completely specified in the form of a *transition probability matrix* **P** given by

$$\mathbf{P} = \begin{bmatrix} p_{00} & p_{01} & p_{02} & \cdots \\ p_{10} & p_{11} & p_{12} & \cdots \\ \vdots & \vdots & \vdots & \\ p_{i0} & p_{i1} & p_{i2} & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix} \qquad \ldots(12.25)$$

with $p_{ij} \geq 0$: $i, j = 0, 1, 2, \ldots,$ and $\sum_j p_{ij} = 1$, for each i.

It is a square matrix. If the number of states is finite, say n, then it will be an $n \times n$ square matrix, otherwise, the matrix will be infinite.

Any such square matrix that has non-negative entries with row sums all equal to unity is called a *stochastic matrix*. Further a chain is said to be *regular* if all the entries of $\mathbf{P}^m$ are positive for some $m \geq 1$.

Consider a communication system that transmits the digits 0 and 1. Each digit must pass through several stages and at each stage there is probability p that digit entered will leave unchanged. If X_n denotes the digit leaving the nth stage of the system and X_o denotes the digit entering the first stage, then $\{X_n, n \geq 0\}$ is a two-state Markov chain with transition probability matrix,

$$\mathbf{P} = \text{State of } X_{n-1} \begin{array}{c} \\ 0 \\ 1 \end{array} \!\!\begin{array}{c} \text{State of } X_n \\ \begin{array}{cc} 0 & 1 \end{array} \\ \begin{bmatrix} p & 1-p \\ 1-p & p \end{bmatrix} \end{array}$$

This matrix **P** can be easily obtained from the *channel diagram*

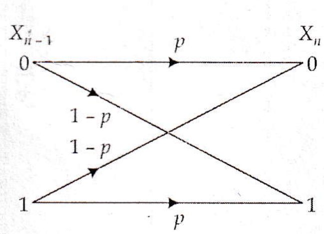

Fig. 12.5

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A graphical representation of this Markov chain is provided by the state diagram as shown in Fig. 12.6.

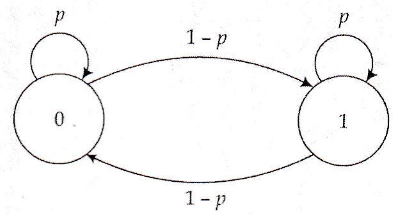


Fig. 12.6

The *joint probability* $[X_n = i_n, X_{n-1} = i_{n-1}, \ldots, X_1 = i_1, X_0 = i_0]$ in case of a Markov chain is given by

$$
\begin{aligned}
P\,[X_n = i_n, X_{n-1} &= i_{n-1}, \ldots, X_1 = i_1, X_0 = i_0] \\
&= P\,[X_n = i_n \mid X_{n-1} = i_{n-1}, \ldots, X_0 = i]\; P\,[X_{n-1} = i_{n-1}, \ldots, X_0 = i_1] \\
&= P\,[X_n = i_n \mid X_{n-1} = i_{n-1}]\; P[X_{n-1} = i_{n-1}, \ldots, X_0 = i_0] \qquad \text{(Markovian property)} \\
&= P\,[X_n = i_n \mid X_{n-1} = i_{n-1}]\; P\,[X_{n-1} = i_{n-1} \mid X_{n-2} = i_{n-2}] \ldots P[X_1 = i_1 \mid X_0 = i_0]\; P[X_0 = i_0] \quad \ldots(12.26)
\end{aligned}
$$

Thus the joint probability is the product of the successive one-step transition probabilities and the initial probability.

Example 12.6: Consider a sequence of Bernoulli trials with probability p of success. Let X_n the outcome of the nth trial be k, where $k = 0, 1, 2, \ldots, n$ denotes that there is a run (uninterrupted block) of k successes. Find the transition probability matrix of the process.

Solution: Obviously $\{X_n; n \geq 0\}$ constitutes a Markov chain with one-step transition probabilities

$$p_{jk} = P\,[X_n = k \mid X_{n-1} = j],$$

given by

$$p_{jk} = \begin{cases} p, & k = j+1 \\ q, & k = 0 \\ 0, & \text{otherwise} \end{cases}$$

Thus the transition probability matrix P is an infinite matrix given as

$$
P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ k \\ \vdots \end{array}
\begin{array}{c}
\begin{array}{cccccccc} 0 & 1 & 2 & 3 & \cdots & k & k+1 & \cdots \end{array} \\
\left[\begin{array}{cccccccc}
q & p & 0 & 0 & & 0 & 0 & \\
q & 0 & p & 0 & & 0 & 0 & \\
q & 0 & 0 & p & & 0 & 0 & \\
\vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \\
q & 0 & 0 & 0 & & 0 & p & \\
\vdots & \vdots & \vdots & \vdots & & \vdots & \vdots &
\end{array}\right]
\end{array}
$$

A Markov Chain with three states (valve failure model)

Consider a component such as a valve which is subject to failure. Let the component be inspected each day and classified as per being one of three states

$0 \longrightarrow$ satisfactory

$1 \longrightarrow$ unsatisfactory (functioning but does not pass a specific test)

$2 \longrightarrow$ failed.

$$\begin{array}{cc} & \text{Time } (n+1) \\ & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \text{Time } n \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{bmatrix} p_{00} & p_{01} & p_{02} \\ 0 & p_{11} & p_{12} \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

- A special feature of the Markov property is that the transition probabilities for the transition from time n to time $n+1$ depend on the state given to be occupied at n, but not in addition what happened before time n. Physically this is a very strong restriction on the process.

Modified model of failure

Suppose now that assumptions are modified so that if state 1 (unsatisfactory) is entered, the system remains there for exactly two time periods before passing to state 2. Let X_n denote the state occupied at time n. Then

$$\text{Prob}\{X_{n+1} = 1 \mid X_n = 1, X_{n-1} = 1\} = 0$$

$$\text{Prob}\{X_{n+1} = 1 \mid X_n = 1, X_{n-1} = 0\} = 1.$$

Thus Markov property does not hold.

Self-avoiding random walk

Let the state space be the lattice in the plane i.e. the set of points (i, j) where (i, j) are integers.

325-C

The position of the particle is at (i, j). At time $(n+1)$ the particle moves to one of the points

$$(i, j+1),\ (i+1, j),\ (i-1, j),\ (i, j-1)$$

under the following rules

(a) no point passed through at time $0, 1, \cdots, n-1$ may be revisited.

(b) Each remaining point of the four is chosen with equal probability.

(c) if all the four points have been visited before the walk ends.

Thus the possible transitions from time n to time $(n+1)$ may be affected by the states occupied at all previous times.

Can you convert it to Markovian process.

Example 12.7: Let $\{X_n, n \geq 0\}$ be a three state 0, 1, 2 Markov chain with transition probability matrix

$$\begin{array}{cc} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{bmatrix} 0.75 & 0.25 & 0 \\ 0.25 & 0.50 & 0.25 \\ 0 & 0.75 & 0.25 \end{bmatrix} \end{array}$$

with initial distribution $p_i = P[X_0 = i] = 1/3, \quad i = 0, 1, 2.$

Find $P[X_3 = 1, X_2 = 2, X_1 = 1, X_0 = 2]$.

Solution: Since $\{X_n, n \geq 0\}$ is a Markov chain, thus the joint probability

$$P[X_3 = 1, X_2 = 2, X_1 = 1, X_0 = 2]$$
$$= P[X_3 = 1 | X_2 = 2]\, P[X_2 = 2 | X_1 = 1]\, P[X_1 = 1 | X_0 = 2]\, P[X_0 = 2]$$
$$= p_{21}\, p_{12}\, p_{21}\, p_2$$
$$= (0.75)(0.25)(0.75)\left(\frac{1}{3}\right) = 0.047$$

12.7.1 The *n*-step Transition Probabilities

The n-step transition probability $p_{ij}^{(n)}$ of a Markov chain is the conditional probability given that chain is currently in state i, that it will be in state j after n additional transitions (steps), that is

$$p_{ij}^{(n)} = P[X_{n+m} = j | X_m = i], \quad n \geq 0, i, j \geq 0 \qquad ...(12.27)$$

Obviously, $p_{ij}^{(1)} = p_{ij}$ and we define

$$p_{ij}^{(0)} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

The n-step transition probabilities (12.27) can be computed using the *Chapman-Kolmogorov equations* given by

$$p_{ij}^{(n+m)} = \sum_k p_{ik}^{(n)}\, p_{kj}^{(m)}, \quad n, m \geq 0, i, j \geq 0 \qquad ...(12.28)$$

A justification for Eqs. (12.28) can be obtained as follows.

We have

$$p_{ij}^{(n+m)} = P[X_{n+m} = j \mid X_0 = i]$$
$$= \sum_k P[X_{n+m} = j, X_n = k \mid X_0 = i]$$
$$= \sum_k P[X_{n+m} = j \mid X_n = k, X_0 = i]\, P[X_n = k \mid X_0 = i]$$

$P(AB/C) = P(A/BC)\,P(B/C)$

$$= \sum_k P[X_n = k | X_0 = i]\, P[X_{n+m} = j \mid X_n = k]$$

Markov Property.

$$= \sum_k p_{il}^{(n)} p_{kj}^{(m)}.$$

k

In particular, ~~when $m = 0$~~ we have

$$p_{ij}^{(n)} = \sum_k p_{ik}^{(n-1)} p_{kj}, \quad n = 2, 3, \ldots, \text{ and } i, j \geq 0 \qquad \ldots(12.29)$$

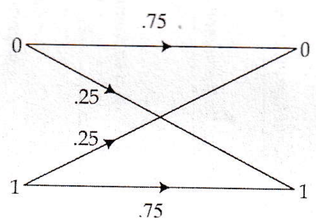
If we denote the stochastic matrix of n-step transition probabilities $p_{ij}^{(n)}$ by $\mathbf{P}^{(n)}$ then from (12.29)

$$\mathbf{P}^{(n)} = \mathbf{P}^{(n-1)}\mathbf{P}$$

that is, $\mathbf{P}^{(n)}$ can be obtained as the matrix product of $\mathbf{P}^{(n-1)}$ by $\mathbf{P}$. Hence $\mathbf{P}^{(n)}$ can be calculated as $\mathbf{P}^n$, the nth power of the stochestic matrix $\mathbf{P}$.

a

Example 12.8: Consider the communication system given by the channel diagram



What is the probability that a 0 entered at the first stage is received as 0 by the fifth stage?

Solution: The transition probability matrix is

$$P = \begin{bmatrix} 0.75 & 0.25 \\ 0.25 & 0.75 \end{bmatrix}$$

We want to find the $p_{00}^{(5)}$, we have

$$\mathbf{P}^{(2)} = \mathbf{P}^2 = \mathbf{PP} = \begin{bmatrix} 0.625 & 0.375 \\ 0.375 & 0.625 \end{bmatrix}, \quad \mathbf{P}^{(4)} = \mathbf{P}^4 = \mathbf{P}^2\,\mathbf{P}^2 = \begin{bmatrix} 0.531 & 0.469 \\ 0.469 & 0.531 \end{bmatrix}$$

and,

$$\mathbf{P}^{(5)} = \mathbf{P}^5 = \mathbf{PP}^4 = \begin{bmatrix} 0.516 & 0.484 \\ 0.484 & 0.516 \end{bmatrix}$$

Thus, $p_{00}^{(5)} = 0.516$.

Remark: We must note that $p_{ij}^{(n)}$ is the probability that the state after n additional stages is j subject to the condition that the current state is i. In case we need to calculate the unconditional distribution of the state after n stages (time), that is $P\,[X_n = j], j = 1, 2, \ldots$ we must have the initial probability distribution of the states $0, 1, 2, \ldots$ If $P[X_0 = i] = p_i, i \geq 0, \sum_i p_i = 1$ is the initial probability distribution, then for any j,

$$p_j^{(n)} = P[X_n = j] = \sum_i P[X_n = j \mid X_0 = i]\, P[X_0 = i] = \sum_i p_{ij}^{(n)} p_i, \quad j = 1, 2, \ldots, \qquad \ldots(12.30)$$

p^n

This can be obtained by multiplying the row vector $\boldsymbol{p} = (p_0, p_1, p_2, \ldots)$ by the jth column of the stochastic matrix P using matrix multiplication, and this defines the probability distribution of X_n.

Example 12.9: Suppose that whether it rains today depends on previous weather conditions only from the last two days and let

P [If it has rained for the past two days, then it will rain tomorrow] = 0.7

P [If it has rained today but not yesterday, then it will rain tomorrow] = 0.5

P [If it rained yesterday but not today, then it will rain tomorrow] = 0.4

P [If it has not rained past two days, then it will rain tomorrow] = 0.2

Assuming the system to be homogeneous, write it as Markov chain. Let it rained on both Monday and Tuesday. What is the probability that it will rain on Thursday?

Solution: Define

State 0: If it rained both today and yesterday

State 1: If it rained today but not yesterday

State 2: If it rained yesterday but not today

State 3: If it rained neither today nor yesterday

$A \rightarrow$ Rain today $B \rightarrow$ Rain yesterday

0 $A \cap B$

1 $A \cap B^c$

2 $A^c \cap B$ 3 $A^c \cap B^c$

On the basis of the probabilities given for different possibilities, the situation can be represented by a four-state Markov chain with transition probability matrix given as

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \quad P = \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.7 & 0.0 & 0.3 & 0.0 \\ 0.5 & 0.0 & 0.5 & 0.0 \\ 0.0 & 0.4 & 0.0 & 0.6 \\ 0.0 & 0.2 & 0.0 & 0.8 \end{bmatrix} \end{array}$$

0 RR

1 DR

2 RD

3 DD

The two-step transition probability matrix is

$$P^2 = \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.49 & 0.12 & 0.21 & 0.18 \\ 0.35 & 0.20 & 0.15 & 0.30 \\ 0.20 & 0.12 & 0.20 & 0.48 \\ 0.10 & 0.16 & 0.10 & 0.64 \end{bmatrix} \end{array}$$

Now the chain is in state 0 on Tuesday and it will rain on Thursday if the chain is in either state 0 or state 1 on that day. Hence, the desired probability is

$$p_{00}^{(2)} + p_{01}^{(2)} = 0.49 + 0.12 = 0.61.$$

Example 12.10: Given a two state Markov chain with transition probability matrix

$$P = \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix}, 0 \le p, q \le 1, \ |1 - p - q| < 1$$

Show that the n-step transition probability matrix $P^{(n)}$ is given by

$$P^{(n)} = \begin{bmatrix} \dfrac{q+p(1-p-q)^n}{p+q} & \dfrac{p-p(1-p-q)^n}{p+q} \\ \dfrac{q-q(1-p-q)^n}{p+q} & \dfrac{p+q(1-p-q)^n}{p+q} \end{bmatrix}$$

Solution: We prove it by induction. Obviously the result is true for $n = 1$.

Assume the result is true for $n = k$, then for $n = k + 1$

$$P^{(k+1)} = P^{(k)}\,P = \begin{bmatrix} \dfrac{q+p(1-p-q)^k}{p+q} & \dfrac{p-p(1-p-q)^k}{p+q} \\ \dfrac{q-q(1-p-q)^k}{p+q} & \dfrac{p+q(1-p-q)^k}{p+q} \end{bmatrix} \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix}$$

The element at the (1, 1) place of the matrix $P^{(k+1)}$ is

$$\frac{[q+p(1-p-q)]^k(1-p)+[p-p(1-p-q)^k]q}{p+q} = \frac{q(1-p)+pq+(1-p-q)^k(p-p^2-pq)}{p+q}$$

$$= \frac{q+p(1-p-q)^{k+1}}{p+q}$$

Similarly the elements at the places (1, 2), (2, 1) and (2, 2) can be obtained, and we have

$$P^{(k+1)} = \begin{bmatrix} \dfrac{q+p(1-p-q)^{k+1}}{p+q} & \dfrac{p-p(1-p-q)^{k+1}}{p+q} \\ \dfrac{q-q(1-p-q)^{k+1}}{p+q} & \dfrac{p+q(1-p-q)^{k+1}}{p+q} \end{bmatrix}$$

This proves the induction step.

Hence, the result is true for all integral values of n provided $|1-p-q| < 1$.

We note here that as n tends to large, $P^{(n)}$ tends to

$$\begin{bmatrix} \dfrac{q}{p+q} & \dfrac{p}{p+q} \\ \dfrac{q}{p+q} & \dfrac{p}{p+q} \end{bmatrix}$$

Thus, the n step probabilities become independent of n.

Example 12.11: A gambler has a fortune of Rs. 2. He bets Re 1 at a time and wins Re 1 with probability 1/2. He stops playing if he loses all his fortune or doubles it. Write the transition probability matrix. What is the probability that he loses his fortune at the end of three plays?

Solution: If X_n denotes the fortune of the gambler, then state space of X_n is 0, 1, 2, 3, 4. The transition probability matrix is

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

We want $p_{20}^{(3)} = P[X_3 = 0 \mid X_0 = 2]$

Using Chapman-Kolmogorov equation (12.29) for $n = 3$, we have

$$p_{20}^{(3)} = \sum_{k=0}^{4} p_{2k}^{(2)} p_{k0}$$

$$= p_{20}^{(2)} p_{00} + p_{21}^{(2)} p_{10} + p_{22}^{(2)} p_{20} + p_{23}^{(2)} p_{30} + p_{24}^{(2)} p_{40}$$

$$= p_{20}^{(2)} + \frac{1}{2} p_{21}^{(2)}, \quad \text{since } p_{00} = 1, p_{10} = 1/2, p_{20} = p_{30} = p_{40} = 0$$

Also,
$$p_{20}^{(2)} = \sum_{k=0}^{4} p_{2k} p_{k0}$$

$$= p_{20} p_{00} + p_{21} p_{10} + p_{22} p_{20} + p_{23} p_{30} + p_{24} p_{40}$$

$$= p_{20} + \frac{1}{2} p_{21} = 0 + \frac{1}{2}\left(\frac{1}{2}\right) = \frac{1}{4}$$

and,
$$p_{21}^{(2)} = \sum_{k=0}^{4} p_{2k} p_{k1}$$

$$= p_{20} p_{01} + p_{21} p_{11} + p_{22} p_{21} + p_{23} p_{31} + p_{24} p_{41}$$

$$= 0 + \frac{1}{2}(0) + 0 + \frac{1}{2} + (0) + 0$$

$$= 0$$

Thus,
$$p_{20}^{(3)} = \frac{1}{4}$$

12.7.2 Classification of States

First we define *periodic* and *aperiodic* states.

The *period* of a state i is the greatest common divisor (g.c.d.) of the set of all positive integers n such that $p_{ii}^{(n)} > 0$. If the g.c.d. is greater than 1, the state i is said to be *periodic*, otherwise *aperiodic*. A Markov chain is said to be *aperiodic* if every state has period 1.

i

For example, the Markov chain with transition probability matrix

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

is aperiodic since every power of P is the identity matrix. On the other hand, the Markov chain with transition probability matrix

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

is periodic with period 2, since every even power of P is the identity matrix and every odd power of P is the matrix itself.

Next, a state j of a Markov chain $\{X_n, n \geq 0\}$ is said to be *accessible* from state i if $p_{ij}^{(n)} > 0$ for some $n \geq 0$. If state j is accessible from state i, and state i is accessible from state j, then we say that the two states i and j *communicate*, and symbolically it is expressed as $i \leftrightarrow j$.

We can very easily check that the relation *communication* satisfies the following three properties:

1. $i \leftrightarrow i$ (*reflexivity*)
2. If $i \leftrightarrow j$, then $j \leftrightarrow i$ (*symmetry*)
3. If $i \leftrightarrow j$ and $j \leftrightarrow k$, then $i \leftrightarrow k$ (*transitivity*) (Prove it)

Thus, 'communication' is an *equivalence relation* and hence partitions the states of a Markov chain in *equivalence classes* (mutually exclusive and exhaustive) of states such that the two states i and j are in the same *class* if, and only if $i \leftrightarrow j$.

A Markov chain is said to be *irreducible* if and only if there is only one equivalence class. In this case every state is accessible from every other state, that is, all states communicate with each other. Further, if $i \leftrightarrow j$, then i and j have the same period. (Prove it)

For example, the Markov chain with transition probability matrix

$$P = \begin{bmatrix} 0.75 & 0.25 \\ 0.25 & 0.75 \end{bmatrix}$$

is irreducible since every state is accessible from every other state.

Consider a Markov chain with four states 0, 1, 2, 3 with transition probability matrix.

$$\begin{array}{cc} & \begin{array}{cccc} 0 & \;\;1 & \;\;2 & \;\;3 \end{array} \\ P = & \begin{bmatrix} 0.5 & 0.5 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 \\ 0.25 & 0.25 & 0.25 & 0.25 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

We note that

$$P^2 = \begin{bmatrix} 0.5 & 0.5 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 \\ 5/16 & 5/16 & 1/16 & 5/16 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The classes of this Markov chain are $\{0, 1\}$, $\{2\}$, and $\{3\}$. We observe that

1. States 0 and 1 communicate.
2. States 0 and 1 are accessible from the state 2 but converse is not true.
3. State 3 is an absorbing state that is, $p_{33} = 1$, no other state is accessible from it.

Next, we define *transient* and *recurrent* states.

Let for each state i of a Markov chain $\{X_n, n \geq 0\}$, $f_i^{(n)}$ be the probability that the first return to the state i occurs in n transitions after leaving i. That is

$$f_i^{(n)} = P[X_n = i, X_k \neq i \text{ for } k = 1, 2, \ldots, n-1 \mid X_0 = i],$$

where we define $f_i^{(0)} = 1$, for all i.

The probability of ever returning to the state i is given by

$$f_i = \sum_{n=1}^{\infty} f_i^{(n)} \qquad \ldots(12.31)$$

The state i is said to be *transient* (or *non-recurrent*) state, if $f_i < 1$, and it is said to be *recurrent* state, if $f_i = 1$. Thus, in case state i is transient, then there is a positive probability $1 - f_i$ that the process will never visit the state i again.

Further if the state i is a recurrent state, then

$$m_i = \sum_{n=1}^{\infty} n f_i^{(n)} \qquad \ldots(12.32)$$

is defined as the *mean recurrence time of the state* i. If m_i is infinite, then state i is said to be *recurrent null*; in case of finite m_i, the state i is said to be *positive recurrent*, or *recurrent non-null*.

In a Markov chain if a state i is recurrent, then starting in state i, the process will re-enter the state i again and again and in fact infinitely often. The expected number of time periods that the process is in state i is infinite. In case, we define

$$I_n = \begin{cases} 1, & \text{if } X_n = i \\ 0, & \text{if } X_n \neq i \end{cases}$$

then $\sum_{n=0}^{\infty} I_n$ represents the number of periods that the process is in state i. Consider

$$E\left[\sum_{n=0}^{\infty} I_n \mid X_0 = i\right] = \sum_{n=0}^{\infty} E[I_n \mid X_0 = i]$$

$$= \sum_{n=0}^{\infty} P[X_n = i \mid X_0 = i]$$

$$= \sum_{n=0}^{\infty} p_{ii}^{(n)}$$

Thus, we have the following result:

A state i is recurrent, if $\sum_{n=0}^{\infty} p_{ij}^{(n)} = \infty$. *In case the state i is transient, then* $\sum_{n=0}^{\infty} p_{ij}^{(n)} < \infty$.

We have observed that while a recurrent state will be visited infinitely many times but transient state will be visited only a finite number of times. *Thus, in case a Markov chain has finite number of states, then not all the states can be transient; at least one of the states must be recurrent.* Also, recurrence is a class property, that is, if state is recurrent and communicates with state j, then state j is also recurrent. Similarly, transient is also a class property.

Further in a finite-state Markov chain all recurrent states are positive recurrent and positive recurrent aperiodic states are called *ergodic*.

Another important result is that *a finite-state Markov chain that is irreducible and aperiodic is ergodic and the n-step transition probabilities* $p_{ij}^{(n)}$ *for an irreducible, ergodic Markov chain become independent of* i *as* $n \to \infty$, *and this is called steady-state probability distribution.*

For example, the finite-state Markov chain considered in *Example 12.10* is irreducible and aperiodic for $0 < p,\ q < 1$ and hence the steady-state distribution exists, and is given by $\left[\frac{q}{p+q}, \frac{p}{p+q}\right]$.

However, the steady-state distribution $v = [v_0\ v_1]$ in case it exists, can be obtained directly also as the solution to the system of equations.

$$v = vP \qquad \text{...(12.33)}$$

For example, in case of the transition probability matrix

$$P = \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix}, \quad 0 < p, q < 1,\ |1-p-q| < 1$$

The system of Eq. (12.33) is given by

$$[v_0\ v_1] = [v_0\ v_1] \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix}$$

or,

$$v_0 = (1-p)\,v_0 + qv_1$$
$$v_1 = pv_0 + (1-q)\,v_1$$

Rearranging these as

$$pv_0 - qv_1 = 0$$
$$-pv_0 + qv_1 = 0$$

which is a set of two linearly dependent equations. Taking one of these and solving it along with the constraint $v_0 + v_1 = 1$, we obtain

$$v_0 = \frac{q}{p+q} \quad \text{and} \quad v_1 = \frac{p}{p+q},$$

the steady-state solution as obtained earlier.

$0 < p < 1$
$\Rightarrow 0 > -p > -1$
$0 > -q > -1$
$0 > -p-q > -2$
$1 > 1-p-q > -1$
$|1-p-q| < 1$

$0 < p+q < 1$
$-1 < 1-p-q < 1$
$-2 < -p-q < 0$
$2 > p+q > 0$
$0 < p+q < 2$

Example 12.12: Consider a Markov chain with transition probability matrix

$$P = \begin{bmatrix} 0.7 & 0.0 & 0.3 & 0.0 \\ 0.5 & 0.0 & 0.5 & 0.0 \\ 0.0 & 0.4 & 0.0 & 0.6 \\ 0.0 & 0.2 & 0.0 & 0.8 \end{bmatrix}$$

Determine whether or not this Markov chain is irreducible.

Solution: The answer is not obvious from looking at P. Consider $P^{(2)}$, we have

$$P^{(2)} = P^2 = \begin{bmatrix} 0.49 & 0.12 & 0.21 & 0.18 \\ 0.35 & 0.20 & 0.15 & 0.30 \\ 0.20 & 0.12 & 0.20 & 0.48 \\ 0.10 & 0.16 & 0.10 & 0.64 \end{bmatrix}$$

Since each element of $P^{(2)}$ is positive, thus every state is accessible from any other state in two steps, hence the Markov chain is irreducible.

Example 12.13: A salesman's territory consists of three cities A, B and C. He never visit in the same city on two consecutive days. If he visits city A, then next day he visits city B. However, if he visits either B or C, then next day he is twice as likely to visit city A as other city. Find how often does he visit each of the cities in the long run.

Solution: The state space of the Markov chain $\{X_n\}$ is A, B, and C. The transition probability matrix is

$$P = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 2/3 & 0 & 1/3 \\ 2/3 & 1/3 & 0 \end{bmatrix} \end{array}$$

It is easy to see that this is a finite state irreducible aperiodic Markov chain. Hence, the steady state probabilities exist. Let $v = [v_A \; v_B \; v_C]$ be the steady state probability distribution. It is given by probability distribution

$$[v_A \; v_B \; v_C] = [v_A \; v_B \; v_C] \begin{bmatrix} 0 & 1 & 0 \\ 2/3 & 0 & 1/3 \\ 2/3 & 1/3 & 0 \end{bmatrix}$$

This gives

$$v_A = 2/3\, v_B + 2/3\, v_C$$
$$v_B = v_A + 1/3\, v_C$$
$$v_C = 1/3\, v_B$$

From the first two equations, $8v_C = 3v_A$

$V_B = \frac{1}{3} V_C = \frac{2}{3} V_B + \frac{2}{3} V_C$

$\frac{1}{3} V_B$:

$V_A = \frac{2}{3} V_A + \frac{2}{9} V_C + \frac{2}{3} V_C$

$\frac{1}{3} V_A = \frac{2}{9} V_C + \frac{2}{3} V_C$

$V_A = \frac{2}{3} V_C + 2 V_C$

From the third equation, $3v_C = v_B$.

Solving these two along with the constraint, $v_A + v_B + v_C = 1$, we get

$$v_A = 2/5, \quad v_B = 9/20, \quad \text{and} \quad v_C = 3/20$$

Thus, the salesman visits the cities A, B, and C respectively 40%, 45% and 15% times.

Example 12.14: A fair coin is tossed until 3 heads occur in a row. Let X_n denote the longest string of heads ending at the nth trial. Find the transition probability matrix and classify the states.

Solution: The state space for X_n is {0, 1, 2, 3}.

The transition probability matrix is

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

The chain is not irreducible, here the state 3 is absorbing state and all other states are aperiodic.

Example 12.15: Describe the nature of the states of the Markov chain with the transition probability matrix

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix}$$

Solution: Let the states be described by $i = 0, 1, 2$.

We have

$$P^2 = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix}, \quad P^3 = P^2P = \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix} = P$$

Thus, $P^4 = P^2$ and in general $P^{2n} = P^2$ and $P^{2n+1} = P$.

From P and P^2, we observe that

$$p_{00}^{(2)}, p_{01}^{(1)}, p_{02}^{(2)} > 0$$
$$p_{10}^{(1)}, p_{11}^{(2)}, p_{12}^{(1)} > 0$$
$$p_{20}^{(2)}, p_{21}^{(1)}, p_{22}^{(2)} > 0$$

Thus, the chain is irreducible.

Also $p_{ii}^{(2)} = p_{ii}^{(4)} = p_{ii}^{(6)} \ldots > 0$, for all i, all the states are periodic with period 2. Further, since the chain is finite and irreducible, all its states are non-null persistent, hence all its states are non-ergodic.