CHAPTER TWO

The Random Walk

2.1. Introduction

The simple random walk was described briefly in Example 1.1. In the present section we introduce the general random walk and give some examples. Section 2.2 is devoted to a detailed discussion of the simple random walk and the material is referred to again in Chapter 3 in the larger context of Markov chains. It is therefore recommended that as a minimum Section 2.2 be studied before proceeding to Chapter 3, although not all detailed results are required. In Section 2.3 the general random walk is treated in detail and this section is not essential for a first reading. Some results, however, will be required in Sections 5.2 and 5.3.

Suppose that a particle is initially at a given point X_0 on the x-axis. At time $n = 1$ the particle undergoes a step or jump Z_1, where Z_1 is a random variable having a given distribution. At time $n = 2$ the particle undergoes a jump Z_2, where Z_2 is independent of Z_1 and with the same distribution, and so on. Thus the particle moves along a straight line and after one jump is at the position X_0+Z_1, after two jumps at $X_0+Z_1+Z_2$ and, in general, after n jumps, i.e. at time n, the position of the particle is given by

$$X_n = X_0+Z_1+Z_2+\ldots+Z_n, \tag{1}$$

where $\{Z_i\}$ is a sequence of mutually independent, identically distributed random variables. We say that the particle undergoes a general one-dimensional random walk or more briefly a random walk. Alternatively we may write (1) as

$$X_n = X_{n-1}+Z_n \quad (n = 1,2,\ldots). \tag{2}$$

In the particular case where the steps Z_i can only take the values 1, 0 or -1 with the distribution

$$\mathrm{prob}(Z_i = 1) = p, \quad \mathrm{prob}(Z_i = 0) = 1-p-q, \quad \mathrm{prob}(Z_i = -1) = q,$$

we shall call the process a *simple random walk*. It is usual in the literature to define a simple random walk as one for which each jump is either $+1$ or -1 with $p+q=1$. However, we shall assume that $p+q \leqslant 1$ with $1-p-q$ as the probability of a zero jump. This extra generality is achieved with hardly any cost in analytical complication.

According to our definition, the random walk is a stochastic process in discrete time. The state space will be continuous if the steps $\{Z_i\}$ are continuous random variables and discrete if the steps are restricted to integral values as in the simple random walk. Anticipating the more general terminology of stochastic processes we shall sometimes refer to the position of the particle as its state, e.g. if $X_n = x$ we say that the particle or process is in state x at time n. For simplicity we shall refer to the positive direction as *up* and the negative direction as *down*.

It should be emphasized that it is simply a matter of convenience to use the language and ideas associated with the motion of a particle. While the theory is of course applicable to the study of physical particles undergoing random motion of the type described above, it is also true that several other stochastic processes and physical systems, when viewed appropriately, can be seen to be equivalent to a random walk in all but terminology.

If the particle continues to move indefinitely according to equation (2) the random walk is said to be *unrestricted*. Frequently, however, we consider the motion of the particle to be restricted in some way, usually by the presence of *barriers*. For example, a random walk starting at $X_0 = 0$ may be restricted to within a distance a up and b down from origin in such a way that when the particle reaches or overreaches *either* of the points a or $-b$, the motion ceases. The points a and $-b$ are called *absorbing barriers* in this case and the states $x \geqslant a$ and $x \leqslant -b$ are all absorbing states. Other types of behaviour, such as reflection, are possible at a given barrier and we shall define these when we deal with them.

The following are some examples of stochastic processes arising in dissimilar physical situations which may be represented as random walks.

Example 2.1. *Insurance risk.* Consider an insurance company which starts at period 0 with a fixed capital, X_0. During periods 1, 2, ... it receives sum $Y_1, Y_2, \ldots$ in the form of premium and other income while it pays out sums $W_1, W_2, \ldots$ in the form of claims. Thus at time n, its capital is

$$X_n = X_0 + (Y_1 - W_1) + \ldots + (Y_n - W_n) \tag{3}$$

and if at any time n, $X_n < 0$, then the company is ruined and cannot continue its operations. If we assume, perhaps unrealistically, that $\{Y_r\}$ and $\{W_r\}$ are two sequences of mutually independent and identically distributed random variables, then the capital X_n behaves like a random walk starting at X_0 and with jumps $Z_r = Y_r - W_r$ $(r = 1, 2, \ldots)$. In addition, we have an absorbing barrier at the origin and so we may describe the process as a random walk on the half-line $(0, \infty)$ with an absorbing barrier at 0. The equations defining the process are for $n = 1, 2, \ldots$

$$X_n = \begin{matrix} X_{n-1} + Z_n & (X_{n-1} > 0,\ X_{n-1} + Z_n > 0), \\ 0 & (\text{otherwise}). \end{matrix}$$

A problem of interest here is to determine the probability of ruin for given X_0 and given statistical behaviour of claims and income.

Example 2.2. *The content of a dam.* Let X_n (in suitable units) represent the amount of water in a dam at the end of n time units; for the sake of argument we call the time unit a day. Suppose that during day r, Y_r units flow into the dam in the form of rainfall and supply from rivers, where Y_r has a statistical distribution over days. Water is released from the dam according to the following rule, by opening release gates at the beginning of each day. If the content at the end of day $r-1$ added to the inflow on day r exceeds a given quantity α then α units are released during day r; otherwise the dam becomes drained by the end of day r. If b represents the capacity of the dam, it will be seen that if $X_{r-1}+Y_r-\alpha > b$ then an overflow, of amount $(X_{r-1}+Y_r-\alpha)-b$, occurs on day r and such an overflow is presumed to be lost. The situation may be represented as follows:

$$X_n = \begin{cases} X_{n-1}+Z_n & (0 < X_{n-1}+Z_n < b), & (4)\\ 0 & (X_{n-1}+Z_n \leqslant 0), & (5)\\ b & (X_{n-1}+Z_n \geqslant b), & (6)\end{cases}$$

where $Z_r = Y_r - \alpha$ is the change in the content of the dam on day r, provided such a change does not empty or fill the dam. If the dam is full it remains full until the first negative Z_r, i.e. until the first subsequent day when the amount released exceeds the inflow, while if empty it remains so until the first positive Z_r. If $\{Z_r\}$ is a sequence of mutually independent and identically distributed random variables then we may describe the process X_n as a random walk on the interval $[0,b]$ in the presence of *reflecting* barriers at 0 and b. A reflecting barrier is defined as a state which, when crossed in a given direction, say downwards, holds the particle until a positive jump occurs and allows the particle to move up and resume the random walk. For such a process some problems of interest are

(a) the determination of the long run or equilibrium probability distribution of X_n;
(b) the finding of the probability distribution of empty periods and of non-empty (wet) periods, i.e. periods during which a particle remains continuously on or continuously off a barrier.

The states of the system are the real numbers in $[0,b]$.

Example 2.3. *Gambler's ruin.* The gambler's ruin problem affords a classical illustration of the simple random walk with absorbing barriers. Consider two gamblers A and B who start off with a and b units of capital respectively. The game consists of a sequence of independent turns and at each turn A wins one unit of B's capital with probability q or B wins one unit of A's capital with probability p $(p+q = 1)$. Let X_n denote B's cumulative gain at the end of n turns. Then provided $-b < X_n < a$ we have $X_n = Z_1 + \ldots + Z_n$, where the Z_i are mutually independent with the distribution

$$\text{prob}(Z_i = 1) = p, \quad \text{prob}(Z_i = -1) = q = 1-p.$$

If at any stage $X_n = a$, then B has gained all A's capital and A is ruined, while if $X_n = -b$, B is ruined. Thus X_n is a simple random walk starting at the origin with absorbing barriers at $-b$ and a. One problem of interest is to find the probability of ruin of each of the contestants, i.e. of absorption at each of the barriers a and $-b$, for given p, q, a and b.

Example 2.4. *The escape of comets from the solar system.* Kendall (1961a, b, c) has made an interesting application of the random walk to the theory of comets. We mention briefly one aspect of the theory. During one revolution round the earth the energy of a comet undergoes a change brought about by the disposition of the planets. In successive revolutions the changes in the energy of the comet are assumed to be independent and identically distributed random variables $Z_1, Z_2, \ldots$. If initially the comet has positive energy X_0 then after n revolutions the energy will be

$$X_n = X_0 + Z_1 + \ldots + Z_n.$$

If at any stage the energy X_n becomes zero or negative, the comet escapes from the solar system. Thus the energy level of the comet undergoes a random walk starting at $X_0 > 0$ with an absorbing barrier at 0. Absorption corresponds to escape from the solar system. Problems of interest are to determine the probability of escape and the distribution of the time to escape. For further details the reader is referred to the above-mentioned papers.

It should be noted that the random walk as described in this chapter is a particular kind of Markov process of which more general kinds will be dealt with in later chapters. We may also observe that from one point of view an unrestricted random walk is a sum of independent random variables and we may thus expect an important role to be played by results in probability theory concerning sums of independent random variables, such as the laws of large numbers and the central limit theorem; see, for example, Parzen (1960).

2.2. The simple random walk

We now consider in more detail the simple random walk with independent jumps $Z_1, Z_2, \ldots$, where for $i = 1, 2, \ldots$,

$$\text{prob}(Z_i = 1) = p, \ \text{prob}(Z_i = -1) = q, \ \text{prob}(Z_i = 0) = 1-p-q. \quad (7)$$

(i) UNRESTRICTED

Suppose that the random walk starts at the origin and that the particle is free to move indefinitely in either direction. Then we have

$$X_n = \sum_{r=1}^{n} Z_r.$$

The possible positions of the particle at time n are $k = 0, \pm 1, \ldots, \pm n$. In order to reach the point k at time n the particle has to make r_1 positive jumps, r_2 negative jumps and r_3 zero jumps where r_1, r_2, r_3 may be any non-negative integers satisfying the simultaneous equalities

$$r_1 - r_2 = k, \qquad r_3 = n - r_1 - r_2. \tag{8}$$

Hence the probability that $X_r = k$ is given by the summation of multinomial probabilities,

$$\operatorname{prob}(X_n = k) = \sum \frac{n!}{r_1!\,r_2!\,r_3!} p^{r_1}(1-p-q)^{r_3} q^{r_2}$$

over values of r_1, r_2 and r_3 satisfying (8). It should be noted that if $p+q=1$ then $\operatorname{prob}(X_n = k)$ vanishes for odd k when n is even and for even k when n is odd. The probability generating function (p.g.f.) of the jump Z_r is

$$G(z) = E(z^{Z_r}) = pz + (1-p-q) + qz^{-1}$$

and hence that of X_n is

$$E(z^{X_n}) = \{G(z)\}^n.$$

Since $X_0 = 0$ we define $G_0(z) = 1$ and introduce a generating function

$$G(z,s) = \sum_{n=0}^{\infty} s^n \{G(z)\}^n = \frac{1}{1 - sG(z)} \quad (|sG(z)| < 1)$$

$$= \frac{z}{-spz^2 + z\{1 - s(1-p-q)\} - sq}.$$

Then $G(z,s)$ contains all the information about the process in the sense that $\operatorname{prob}(X_n = k)$ is the coefficient of $z^k s^n$ in $G(z,s)$.

Let μ and σ^2 denote the mean and variance of a jump. Then $\mu = p - q$ and $\sigma^2 = p + q - (p-q)^2$ and hence

$$E(X_n) = n\mu, \qquad V(X_n) = n\sigma^2,$$

where $V(X)$ denotes the variance of the random variable X. It is of interest to calculate the probability that at time n the particle is found in one of the states $j, j+1, \ldots, k$, where j and k are possible values of X_n $(j < k)$. This may involve an inconvenient summation of a large number of multinomial probabilities and so we can resort to an approximation provided by the central limit theorem, according to which X_n

will be approximately normally distributed with mean $n\mu$ and variance $n\sigma^2$ for large n. Thus

$$\text{prob}(j \leqslant X_n \leqslant k) \simeq (2\pi\sigma^2 n)^{-\frac{1}{2}} \int_j^k \exp\left\{-\frac{(x-n\mu)^2}{2n\sigma^2}\right\} dx.$$

A still better approximation is obtained by employing a continuity correction, i.e. by using $j-c$ and $k+c$ as the limits of integration, where $c=\frac{1}{2}$ or $c=1$ according as $p+q<1$ or $p+q=1$. We then have, after transforming the integral to standard form

$$\text{prob}(j \leqslant X_n \leqslant k) \simeq \Phi\left(\frac{k+c-n\mu}{\sigma\sqrt{n}}\right) - \Phi\left(\frac{j-c-n\mu}{\sigma\sqrt{n}}\right), \tag{9}$$

where

$$\Phi(y) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{y} e^{-\frac{1}{2}x^2}\, dx$$

is the standard normal distribution function and is well tabulated.

We can regard (9) and subsequent similar results as asymptotic equivalences if, as $n \to \infty$, $j-\mu n \sim \alpha\sqrt{n}$, $k-\mu n \sim \beta\sqrt{n}$, for some fixed α, β $(\alpha < \beta)$.

Suppose now that $p > q$, i.e. the probability of a jump upwards is greater than that of a jump downwards, and let us investigate where the particle is likely to be found after a large number n of jumps. The mean jump μ is thus positive and again according to the central limit theorem X_n will, with high probability, be within say three standard deviations of its expected value. That is, $\text{prob}(n\mu - 3\sigma\sqrt{n} < X_n < n\mu + 3\sigma\sqrt{n})$ is nearly 1 and we can say that

$$X_n = n\mu + O(\sqrt{n}) = n\mu\{1 + O(n^{-\frac{1}{2}})\}$$

with high probability. Since $\mu > 0$ the probability that X_n is arbitrarily large becomes arbitrarily near one as n increases. More precisely, it can be deduced from (9) or from the weak law of large numbers that

$$\text{prob}(X_n > j) \to 1 \quad \text{as } n \to \infty$$

for any j. By using the strong law of large numbers we can make a stronger statement. For, according to the strong law we can say that for any $\epsilon > 0$ the probability that the particle remains in the region $n(\mu-\epsilon) < X_n < n(\mu+\epsilon)$ for all $n > n_0$ can be made as close to 1 as we please by choosing n_0 sufficiently large. Hence for any j we have that

$$\text{prob}(X_n > j, X_{n+1} > j, \ldots) \to 1 \quad \text{as } n \to \infty,$$

so that with probability one the particle drifts off to $+\infty$. Similarly, if $p < q$ the particle drifts off to $-\infty$ with probability one. But if $p = q$ then the jumps are equally likely to be up or down and $\mu = 0$. The central limit theorem tells us that the particle will be within a distance of order $\sqrt{n}$ from its starting point after n jumps. This does not exclude the possibility of large excursions from the origin and in (iii) the singular fact will be shown that the particle is certain to return to the origin but is also certain to make arbitrarily large excursions.

Example 2.5. *A numerical example.* We examine some of the above results in particular numerical instances. Suppose we take the cases

(a) $p = 0{\cdot}6, \quad q = 0{\cdot}2, \quad 1-p-q = 0{\cdot}2;$
(b) $p = 0{\cdot}4, \quad q = 0{\cdot}3, \quad 1-p-q = 0{\cdot}3;$
(c) $p = 0{\cdot}4, \quad q = 0{\cdot}4, \quad 1-p-q = 0{\cdot}2.$

In (a) the particle takes on average three times as many positive as negative steps so that there is a strong positive drift. We use the normal approximation (9) to examine the relation between n, j and $\text{prob}(X_n \geqslant j)$. In this case (9) becomes

$$\text{prob}(X_n \geqslant j) \simeq 1 - \Phi\left(\frac{j - \frac{1}{2} - n\mu}{\sigma\sqrt{n}}\right),$$

where $\mu = p - q = 0{\cdot}4$ and $\sigma^2 = p + q - (p-q)^2 = 0{\cdot}64$.

Thus for example after exactly 100 steps the probability is about 0·99 that the particle is 24 or more units up from the origin, i.e.

$$\text{prob}(X_{100} \geqslant 24) \simeq 0{\cdot}99.$$

Similarly

$$\text{prob}(X_{10,000} \geqslant 3834) \simeq 0{\cdot}99.$$

In case (b) the drift is positive but weaker than in case (a). We find that with probability 0·99 the particle is above position -8 for $n = 100$ and above position 807 for $n = 10{,}000$. In case (c) there is no drift and we find that with probability 0·99 the particle is within ± 24 units of the origin for $n = 100$ and within ± 231 units for $n = 10{,}000$.

(*ii*) TWO ABSORBING BARRIERS

Suppose that the particle starts at the origin and moves in the presence of absorbing barriers at the points $-b$ and a $(a, b > 0)$ so that the motion ceases when the particle enters either of the states $-b$ or a. When the particle enters state a we say that absorption occurs at a, and similarly for state $-b$. Of the three possible ultimate outcomes to the motion, only two, namely absorption at $-b$ and absorption at a, have positive probability in general; we now show that the third possibility, that the particle continues wandering indefinitely between the two barriers, has zero probability. For the probability that the particle is still in motion at time n. i.e. that it occupies one of the non-absorbing states $-b+1$,

$-b+2, \ldots, a-1$, cannot exceed the probability that an unrestricted particle occupies one of these states at time n, for in computing the former probability we must exclude all possible journeys via states outside $(-b+1, a-1)$. Now it can be deduced from (9) that the latter probability tends to zero as $n \to \infty$. Thus the probability that the particle is not yet absorbed at time n also tends to zero as $n \to \infty$, and we shall show later that this vanishing is geometrically fast, i.e.

$$\text{prob}(-b < X_n < a) = O(\rho^n)$$

for some ρ $(0 < \rho < 1)$.

Having deduced that the probability of absorption is unity, let us determine the distribution of this probability between the two absorbing states and among the possible times at which absorption may occur. It is convenient to consider an arbitrary starting point $X_0 = j$ $(-b \leqslant j \leqslant a)$ and let $f_{ja}^{(n)}$ $(n = 0, 1, \ldots)$ be the distribution of the total probability of absorption at a among the possible times $n = 0, 1, 2, \ldots$ at which this event may occur. Thus $f_{ja}^{(n)}$ is the probability that the particle is absorbed at exactly time n. It is instructive to observe that $f_{ja}^{(n)}$ also represents the probability that an unrestricted particle reaches position a for the first time at time n without position $-b$ being occupied at any of the times $1, 2, \ldots, n-1$, all conditional on starting at j. More precisely

$$f_{ja}^{(n)} = \text{prob}(-b < X_1 < a, \ldots, -b < X_{n-1} < a,\; X_n = a \mid X_0 = j) \qquad (n = 1, 2, \ldots),$$

while for $n = 0$ we have the initial conditions

$$f_{ja}^{(0)} = \begin{cases} 1 & (j = a), \\ 0 & (j \neq a), \end{cases} \tag{10}$$

since if the particle starts at a, absorption occurs at time 0 with probability 1 and if it starts at any point other than a absorption cannot occur at time 0.

Let A_n denote the event 'absorption at a at time n'. Then $f_{ja}^{(n)} = \text{prob}(A_n \mid \text{start at } j)$. In order to obtain an equation for $f_{ja}^{(n)}$ we make a decomposition of A_n based on the first step. This method, which we describe here in detail, is used frequently later on to obtain what are called backward equations. If the first step is $+1$ then the particle moves to $j+1$ and all subsequent steps are independent; for A_n now to occur we must have A_{n-1} occurring conditional on starting at $j+1$. The probability that the first step is $+1$ and that A_n occurs is therefore

$$p \,\text{prob}(A_{n-1} \mid \text{start at } j+1).$$

We apply similar arguments to the possibilities of a first step of 0 and -1, and since $+1$, 0 and -1 are mutually exclusive and exhaustive possibilities for the first step, we add the three probabilities and obtain

$$f_{ja}^{(n)} = p\,\text{prob}(A_{n-1}|\text{start at } j+1) + (1-p-q)\,\text{prob}(A_{n-1}|\text{start at } j) + q\,\text{prob}(A_{n-1}|\text{start at } j-1)$$

or

$$f_{ja}^{(n)} = pf_{j+1,a}^{(n-1)} + (1-p-q)f_{ja}^{(n-1)} + qf_{j-1,a}^{(n-1)} \quad (j = -b+1, \ldots, a-1;\ n = 0, 1, \ldots). \tag{11}$$

Together with (11) we must have the initial conditions (10) and the boundary conditions

$$f_{aa}^{(n)} = 0, \qquad f_{-b,a}^{(n)} = 0 \qquad (n = 1, 2, \ldots). \tag{12}$$

In equation (11) the unknown f is a function of the two discrete variables n and j and we have a difference equation of the first order in n and of the second order in j. We therefore need one condition, namely (10), which prescribes all values of f when $n = 0$ and the two boundary conditions (12) which prescribe all values of f when $j = -b$ and $j = a$.

The form of equation (11) suggests the use of generating functions which in effect eliminates one of the variables. Accordingly we define the generating function over the time variable n,

$$F_{ja}(s) = \sum_{n=0}^{\infty} f_{ja}^{(n)} s^n = F_j(s), \tag{13}$$

say, where for simplicity we omit the suffix a. We now multiply (11) by s^n and sum over $n = 1, 2, \ldots$ obtaining

$$F_j(s) = s\{pF_{j+1}(s) + (1-p-q)F_j(s) + qF_{j-1}(s)\}. \tag{14}$$

This is a second-order difference equation for F_j, with boundary conditions, obtained from (10) and (12),

$$F_a(s) = 1, \qquad F_{-b}(s) = 0.$$

We now have only the one variable j in (14). A common method of solving linear difference equations like (14) is to substitute a trial solution $F_j(s) = \lambda^j$. We find that

$$\lambda^j = s\{p\lambda^{j+1} + (1-p-q)\lambda^j + q\lambda^{j-1}\}$$

or

$$ps\lambda^2 - \lambda\{1 - s(1-p-q)\} + qs = 0, \tag{15}$$

a quadratic equation in λ, the two solutions of which are

$$\lambda_1(s), \lambda_2(s) = \frac{1 - s(1-p-q) \pm [\{1 - s(1-p-q)\}^2 - 4pqs^2]^{\frac{1}{2}}}{2ps}. \tag{16}$$

We assume that s is real and positive and that the function inside the square root in (16) is positive, i.e.

$$\{1-s(1-p-q)\}^2 > 4pqs^2,$$

or

$$0 < s < \frac{1}{(1-p-q)+2\sqrt{(pq)}} = \frac{1}{1-(\sqrt{p}-\sqrt{q})^2}$$

Further, we take the positive square root in (16). The general solution of (14) is now given by

$$A\{\lambda_1(s)\}^j + B\{\lambda_2(s)\}^j$$

and A and B, possibly functions of s, must be determined from the boundary conditions to give the solution to our particular problem. We find that

$$F_j(s) = F_{ja}(s) = \frac{\{\lambda_1(s)\}^{j+b} - \{\lambda_2(s)\}^{j+b}}{\{\lambda_1(s)\}^{a+b} - \{\lambda_2(s)\}^{a+b}}. \tag{17}$$

We have thus obtained the solution to (11) in the form of a generating function. In our original problem the particle started at the origin and so by putting $j = 0$ in (17) we obtain

$$F_{0a}(s) = \frac{\{\lambda_1(s)\}^{b} - \{\lambda_2(s)\}^{b}}{\{\lambda_1(s)\}^{a+b} - \{\lambda_2(s)\}^{a+b}}. \tag{18}$$

Equation (26) below gives the corresponding expression for $F_{0,-b}(s)$, the generating function for the probability of absorption at $-b$.

In order to obtain the probabilities $f_{0a}^{(n)}$ we must expand (18) as a power series in s. Before embarking on this by no means trivial calculation we can quite easily obtain the probabilities of the two mutually exclusive and exhaustive outcomes of the process, namely absorption at a or absorption at $-b$. To set $s = 1$ in (18) is equivalent to the summation over n of the probabilities $f_{0a}^{(n)}$, thus obtaining the probability that the particle is absorbed at a, namely $F_{0a}(1)$. This requires initially the substitution $s = 1$ in the roots $\lambda_1(s)$ and $\lambda_2(s)$. Since we have chosen the positive square root in (16) we see that for $s \geqslant 0$, $\lambda_1(s) \geqslant \lambda_2(s)$. Thus, writing $\lambda_1(1) = \lambda_1$ and $\lambda_2(1) = \lambda_2$, we have

$$\begin{aligned}
\lambda_1 &= \frac{q}{p} > \lambda_2 = 1 && (p < q),\\
\lambda_1 &= 1 > \lambda_2 = \frac{q}{p} && (p > q),\\
\lambda_1 &= 1 = \lambda_2 && (p = q).
\end{aligned} \tag{19}$$

Hence, from (18)

$$\text{prob(absorption occurs at } a) = F_{0a}(1) = \begin{cases} p^a \dfrac{p^b - q^b}{p^{a+b} - q^{a+b}} & (p \neq q),\\[2ex] \dfrac{b}{a+b} & (p = q), \end{cases} \tag{20}$$

where for $p = q$ we obtain an indeterminate form from the first expression which may be evaluated by taking the limit as $p \to q$. Thus the function $F_{0a}(s)/F_{0a}(1)$ is the p.g.f. of the time to absorption conditional on absorption occurring at a, and if we denote by N the random variable representing the time to absorption, we may write

$$\begin{aligned} F_{0a}(s) &= F_{0a}(1)\, E(s^N | X_N = a) \\ &= \text{prob}(X_N = a)\, E(s^N | X_N = a). \end{aligned}$$

Since absorption is certain, it immediately follows that

$$\text{prob}(X_N = -b) = \text{prob(absorption occurs at } -b) = 1 - F_{0a}(1)$$

$$= \begin{cases} q^b \dfrac{p^a - q^a}{p^{a+b} - q^{a+b}} & (p \neq q), \\ \dfrac{a}{a+b} & (p = q). \end{cases} \tag{21}$$

We now describe briefly how to obtain the individual probabilities $f_{0a}^{(n)}$ by expanding (18) as a power series in s. This is a special case of an important general method, but nevertheless it may be omitted on a first reading. The method depends on the fact, not immediately obvious, that (18) is a rational function of s, as can be seen by expanding both the numerator and denominator in powers of

$$\phi(s) = [\{1 - s(1-p-q)\}^2 - 4pqs^2]^{\frac{1}{2}}$$

and noting that only odd powers of $\phi(s)$ occur; thus $\phi(s)$ cancels, leaving a ratio of two polynomials in s, each of degree $a+b-1$ in general. (If $p+q = 1$, the degree of the numerator is reduced by 1 if b is even and similarly for the denominator if $a+b$ is even.) An expansion of $F_{0a}(s)$ in partial fractions is obtainable which may be expressed as

$$F_{0a}(s) = (2sp)^a \sum_{\nu=1}^{a+b-1} \frac{\alpha_\nu}{1 - s/s_\nu}, \tag{22}$$

where

$$s_\nu = \frac{1}{1 - p - q + 2(pq)^{\frac{1}{2}} \cos\left(\dfrac{\nu\pi}{a+b}\right)} \quad (\nu = 1, \ldots, a+b-1), \tag{23}$$

are the roots of the denominator (the root given by $\nu = 0$ is also a root of the numerator). It is useful to observe that

$$\lambda_1(s_\nu), \lambda_2(s_\nu) = \left(\frac{q}{p}\right)^{\frac{1}{2}} \exp\left(\pm \frac{\nu\pi i}{a+b}\right) \tag{24}$$

and that

$$\frac{d}{ds}\lambda_j(s) = \frac{\lambda_j(s)}{s^2[q\{\lambda_j(s)\}^{-1} - p\lambda_j(s)]} \quad (j = 1, 2),$$

the latter relation being found by differentiating implicitly the equation (15). The constants α_ν in the partial fraction expansion are given by

$$\begin{aligned}\alpha_\nu &= \lim_{s\to s_\nu}\left(1-\frac{s}{s_\nu}\right)(2ps)^{-a}F_{0a}(s)\\ &= (2ps_\nu)^{-a}\frac{\{\lambda_1(s_\nu)\}^b-\{\lambda_2(s_\nu)\}^b}{-s_\nu\dfrac{d}{ds}[\{\lambda_1(s)\}^{a+b}-\{\lambda_2(s)\}^{a+b}]_{s=s_\nu}}\\ &= \frac{(-1)^{\nu+1}\sin\left(\dfrac{b\nu\pi}{a+b}\right)\sin\left(\dfrac{\nu\pi}{a+b}\right)}{2(a+b)(4pq)^{\frac{1}{2}a}s_\nu^{a-1}} \quad (\nu = 1,\ldots,a+b-1).\end{aligned}$$

Finally, from (22) the coefficient of s^n in $F_{0a}(s)$ is

$$\begin{aligned}f_{0a}^{(n)} &= (2p)^a\sum_{\nu=1}^{a+b-1}\frac{\alpha_\nu}{s_\nu^{n-a}} \quad (n = a, a+1,\ldots)\\ &= \frac{\sqrt{(4pq)}}{(a+b)}\left(\frac{p}{q}\right)^{\frac{1}{2}a}\sum_{\nu=1}^{a+b-1}\frac{(-1)^{\nu+1}\sin\left(\dfrac{b\nu\pi}{a+b}\right)\sin\left(\dfrac{\nu\pi}{a+b}\right)}{s_\nu^{n-1}}\\ &\qquad\qquad (n = a, a+1,\ldots). \qquad (25)\end{aligned}$$

This is the solution which we set out to obtain from the difference equation (11).

It remains to determine the corresponding distribution over time of the probability that absorption occurs at $-b$. By an almost identical argument, in fact by solving the difference equation (14) with boundary conditions

$$F_a(s) = 0, \qquad F_{-b}(s) = 1,$$

we obtain

$$F_{0,-b}(s) = \frac{\{\lambda_2(s)\}^{-a}-\{\lambda_1(s)\}^{-a}}{\{\lambda_2(s)\}^{-a-b}-\{\lambda_1(s)\}^{-a-b}}. \qquad (26)$$

A similar application of partial fractions gives the result

$$f_{0,-b}^{(n)} = \frac{\sqrt{(4pq)}}{(a+b)}\left(\frac{q}{p}\right)^{\frac{1}{2}b}\sum_{\nu=1}^{a+b-1}\frac{(-1)^{\nu+1}\sin\left(\dfrac{a\nu\pi}{a+b}\right)\sin\left(\dfrac{\nu\pi}{a+b}\right)}{s_\nu^{n-1}}$$
$$(n = b, b+1,\ldots).$$

Alternatively this result may be obtained from (25) by interchanging a with b and p with q.

By examining the roots $s_1, s_2, \ldots$ more closely we can see that the root of smallest modulus is

$$s_1 = \frac{1}{1-p-q+2(pq)^{\frac{1}{2}}\cos\left(\dfrac{\pi}{a+b}\right)}, \qquad (27)$$

and further we have the inequalities

$$s_1 > \frac{1}{1-p-q+2(pq)^{\frac{1}{2}}} = \frac{1}{1-(p^{\frac{1}{2}}-q^{\frac{1}{2}})^2} \geqslant 1,$$

so that s_1 always exceeds unity. Now s_1 is the singularity of $F_{0a}(s)$ nearest to the origin and is therefore the radius of convergence of the power series defining $F_{0a}(s)$. Therefore it follows that $f_{0a}^{(n)} = O(\rho^n)$ for some ρ $(0 < \rho < 1)$. More precisely, we can obtain the following inequality from the exact expression (25) for $f_{0a}^{(n)}$:

$$f_{0a}^{(n)} < \frac{1}{2(a+b)}\left(\frac{p}{q}\right)^{\frac{1}{2}a} \frac{a+b-1}{s_1^{n-1}} \leqslant \left(\frac{p}{q}\right)^{\frac{1}{2}a} \frac{1}{s_1^{n-1}}, \tag{28}$$

since $s_1 \geqslant |s_\nu|$ $(\nu = 2, \ldots, a+b-1)$, and similarly

$$f_{0,-b}^{(n)} \leqslant \left(\frac{q}{p}\right)^{\frac{1}{2}b} \frac{1}{s_1^{n-1}}. \tag{29}$$

Further, we have

$$\text{prob}(N = n) = f_{0a}^{(n)} + f_{0,-b}^{(n)},$$

so that we have determined completely the probability distribution of N. Its p.g.f. is clearly

$$E(s^N) = F_{0a}(s) + F_{0,-b}(s).$$

For results on the expected value of N, see Exercise 12. Now the probability that the particle is still in motion at time n is precisely the probability that absorption occurs after time n, i.e.

$$\begin{aligned} \text{prob}(-b < X_n < a) &= \text{prob}(N > n) \\ &= \sum_{r=n+1}^{\infty} \{f_{0a}^{(r)} + f_{0,-b}^{(r)}\} \qquad (30) \\ &= O(s_1^{-n}) \end{aligned}$$

in virtue of the inequalities (28) and (29), and so $\text{prob}(-b < X_n < a)$ tends to zero geometrically in n.

In order to complete the description of the process in probability terms it is necessary to determine the probability $p_k^{(n)}$ that the particle is in a state k at a time n before absorption occurs. For if this is known then the state probabilities of the particle are known for all states, both absorbing and non-absorbing, and for all times. If we define the generating function

$$P_k(s) = \sum_{n=0}^{\infty} p_k^{(n)} s^n \quad \text{with} \quad p_k^{(0)} = \delta_{0k},$$

then we find, by relating the position of the particle at time n with its possible positions at time $n-1$, that $P_k(s)$ satisfies the difference equation

$$P_k(s) - 0k = s\{pP_{k-1}(s) + (1-p-q)P_k(s) + qP_{k+1}(s)\}, \tag{31}$$

an inhomogeneous version of (14) with p and q interchanged. The appropriate boundary conditions are

$$P_a(s) = P_{-b}(s) = 0.$$

However, we shall not proceed with the solution of (31) but rather obtain the result by a more general method in Section 2.3 (iv).

(*iii*) ONE ABSORBING BARRIER

Let us now suppose that the particle starts in the state $X_0 = 0$ and that an absorbing barrier is placed at the point $a > 0$, so that the particle is free to move among the states $x < a$ if and until it reaches the state a which, when once entered, holds the particle permanently. The time to absorption, or the duration of the walk, is clearly also the first passage time from the state 0 to the state a in the unrestricted random walk.

It is of interest to examine the probability that the particle will ever reach an absorbing state at a. Let $f_a^{(n)}$ denote the probability that absorption occurs at a at time n, i.e.

$$f_a^{(n)} = \text{prob}\{X_m < a \quad (m = 1, \ldots, n-1), \quad X_n = a\} \tag{32}$$

and define the generating function

$$F_a(s) = \sum_{n=1}^{\infty} f_a^{(n)} s^n.$$

By letting $b \to \infty$ in the formulae for the two-barrier case we can obtain our results for the present problem. Thus

$$\begin{aligned} F_a(s) &= \lim_{b\to\infty} \frac{\{\lambda_1(s)\}^b - \{\lambda_2(s)\}^b}{\{\lambda_1(s)\}^{a+b} - \{\lambda_2(s)\}^{a+b}} \\ &= \{\lambda_1(s)\}^{-a}, \end{aligned}$$

since $\lambda_1(s) > \lambda_2(s)$. Further, since $\lambda_1(s)\lambda_2(s) = q/p$, we also have

$$F_a(s) = \left(\frac{p}{q}\right)^a \{\lambda_2(s)\}^a. \tag{33}$$

There is, however, an alternative and instructive method of obtaining this result. Let us suppose that $p > q$ so that there is a drift upwards. In order to reach the state a the particle must at some intermediate times occupy each of the intervening states 1, 2, ..., $a-1$, since at each jump the particle can move at most unit distance upwards. The probability of ultimately effecting the passage from state 0 to state 1 is unity, for otherwise, there is a positive probability of forever remaining in the states $0, -1, -2, \ldots$, i.e.

$$\lim_{n\to\infty} \text{prob}(X_1 \leqslant 0, X_2 \leqslant 0, \ldots, X_n \leqslant 0) \geqslant C > 0.$$

But, by a similar argument to that used at the beginning of (ii) of this section, $\text{prob}(X_1 \leqslant 0, \ldots, X_n \leqslant 0)$ cannot exceed the probability that an unrestricted particle lies below the origin at time n, which tends to zero by the central limit theorem. It follows that the passage from state 0 to state 1 is a certain event. Thus the first passage time N_1 from state 0 to state 1 is a random variable taking positive integral values. Let us denote its p.g.f. by $F_1(s)$. Since each jump is independent of the time and the state from which the jump is made, it follows that the first passage time from any given state k to the state $k+1$ is a random variable having the same distribution as N_1 and that the successive first passage times from state 0 to state 1, from state 1 to state 2, and so on, are independent random variables. Thus the first passage time from state 0 to state a is the sum of a independent random variables each with the distribution of N_1. Hence

$$F_a(s) = \{F_1(s)\}^a. \tag{34}$$

We can determine $F_1(s)$ by considering the three possible positions of the particle after making the first jump. Firstly, with probability p the first passage to state 1 is effected at the first step. Secondly, with probability $1-p-q$ the particle remains at 0 after the first step and takes a further time N_1' to reach state 1, where N_1' is a random variable with the distribution of N_1 and independent of the first step. Thirdly, with probability q the particle jumps to -1 at the first stage and then takes time $N_1''+N_1'''$ to reach 1, where N_1'' and N_1''' are independent random variables each with the distribution of N_1. These three statements may be expressed as follows

$$\begin{aligned}
\text{prob}(X_1 = 1)E(s^{N_1}|X_1 = 1) &= ps,\\
\text{prob}(X_1 = 0)E(s^{N_1}|X_1 = 0) &= (1-p-q)sF_1(s),\\
\text{prob}(X_1 = -1)E(s^{N_1}|X_1 = -1) &= qs\{F_1(s)\}^2,
\end{aligned}$$

and, on summing these three expressions, we obtain $E(s^{N_1})$ unconditionally,

$$E(s^{N_1}) = F_1(s) = ps+(1-p-q)sF_1(s)+qs\{F_1(s)\}^2. \tag{35}$$

This is a quadratic equation for $F_1(s)$ and if we take the solution satisfying $F_1(1) = 1$ we find that

$$F_1(s) = \frac{p}{q}\lambda_2(s) = \{\lambda_1(s)\}^{-1},$$

and the result (33), giving the p.g.f. $F_a(s)$, follows from (34).

The above argument can be adapted to cover the cases where $p \leqslant q$. Here we must interpret $F_1(s)$ as the generating function of a probability distribution which does not necessarily add up to 1. More precisely, if A denotes the event that the particle ever reaches state 1 when it starts from state 0 then

$$\text{prob}(A) = \sum_{n=1}^{\infty} f_1^{(n)},$$

where $f_1^{(n)}$ is defined in (32), and

$$\begin{aligned} F_1(s) &= \sum_{n=1}^{\infty} f_1^{(n)} s^n \\ &= \text{prob}(A)\, E(s^{N_1} | A). \end{aligned} \tag{36}$$

The random variable N_1 now has a distribution conditional on A. We obtain the same equation (35) for $F_1(s)$ and again we take the root

$$F_1(s) = \frac{p}{q}\lambda_2(s);$$

the other root, $(p/q)\lambda_1(s)$, is disqualified from being a function of the form (36) by the fact that $\lambda_1(s) \to \infty$ as $s \to 0$.

Thus in all cases we have the result

$$F_a(s) = \left(\frac{p}{q}\right)^a \{\lambda_2(s)\}^a = \{F_1(s)\}^a. \tag{37}$$

Letting N_a denote the time to absorption at a, or equivalently the first passage time to state a from state 0, we may write

$$F_a(s) = \sum_{n=1}^{\infty} s^n \,\text{prob}(N_a = n).$$

For $s = 1$,

$$F_a(1) = \text{prob}(N_a < \infty),$$

which is the probability that absorption occurs at all. Thus

$$\text{prob}(N_a < \infty) = \begin{cases} \left(\dfrac{p}{q}\right)^a & (p < q), \\ 1 & (p \geqslant q). \end{cases} \tag{38}$$

Now N_a also denotes the duration of the random walk and so, when the drift is towards the barrier or when there is no drift, N_a is finite with probability one. When the drift is away from the barrier N_a is finite with probability $(p/q)^a$ while with probability $1-(p/q)^a$ the walk continues indefinitely, the particle remaining forever in states below a.

Example 2.6. *A numerical example.* If $p = 0{\cdot}3$, $q = 0{\cdot}4$, then when the barrier is 10 units from the origin ($a = 10$) the probability (38) that the particle ever reaches the barrier is 0·0563, while if $a = 100$ it is about $10^{-12\cdot 5}$. If $p = 0{\cdot}2$, $q = 0{\cdot}6$ then the corresponding probabilities are about 1/50,000 and $10^{-47\cdot 7}$ respectively.

If absorption is certain then $F_a(s) = \{F_1(s)\}^a$ is the p.g.f. of the number N_a of steps to absorption. In this case we have

$$E(N_a) = aF_1'(1) = \begin{cases} \dfrac{a}{p-q} & (p > q), \\ \infty & (p = q). \end{cases} \tag{39}$$

Thus although absorption is certain when $p = q$ the distribution of time to absorption has infinite mean (and therefore infinite moments of all orders). When $p > q$ we have for the variance

$$\begin{aligned} V(N_a) &= aV(N_1) = a[F_1''(1) + F_1'(1) - \{F_1'(1)\}^2] \\ &= \frac{a\{p+q-(p-q)^2\}}{(p-q)^3} \quad (p > q). \end{aligned} \tag{40}$$

Further N_a is the sum of a independent random variables with finite variance and by the central limit theorem will be approximately normally distributed for large a. In terms of the mean $\mu = p - q$ and variance $\sigma^2 = p + q - (p-q)^2$ of a single jump we have

$$E(N_a) = \frac{a}{\mu}, \qquad V(N_a) = a\frac{\sigma^2}{\mu^3}.$$

Example 2.7. ***A numerical example.*** Taking the numerical values $p = 0{\cdot}4$, $q = 0{\cdot}3$, we find that when $a = 10$, N_a has mean 100 and standard deviation 83 indicating a distribution with a wide spread. When $a = 100$ the mean and standard deviation of N_a are 1000 and 263 respectively; using the normal approximation in this case we see that N_a will lie between 500 and 1500 with probability about 0·95. If $p = 0{\cdot}6$, $q = 0{\cdot}2$ the drift is stronger. For $a = 10$ we find that N_a has mean 33 and standard deviation 31 while for $a = 100$ the corresponding figures are 333 and 97.

When $p = q$ the behaviour of the particle is somewhat singular. It follows from our results that starting from state 0, the particle reaches any other given state with probability one, but that the mean time to achieve this passage is infinite. Having reached the given state it will return to state 0 with probability one, again with infinite mean passage time. Thus an unrestricted particle, if allowed sufficient time, is certain to make indefinitely large excursions from its starting point and is also certain to return to its starting point.

Example 2.8. *Simulation of the symmetrical random walk.* It is simple and interesting to simulate a symmetrical random walk with $p = q = \frac{1}{2}$ and with $a = 1$. Thus we toss an unbiased coin and score $+1$ for heads, -1 for tails and when the cumulative score is $+1$ we stop. Alternatively a table of random digits may be used. In 12 independent realizations it was found that the numbers of steps to absorption were 5, 1, 30, 1, 3, 1, 1, 1, 3, 1, 3, 412. These are observations from a distribution with infinite mean and variance.

(*iv*) FURTHER ASPECTS OF THE UNRESTRICTED RANDOM WALK

We have seen that the imposition of an absorbing barrier is a useful method of studying certain aspects of the unrestricted random walk, in particular the passage times. There is another interesting aspect revealed by the use of absorbing barriers, namely the maximum distance from its starting point reached by the particle in either direction.

More precisely let

$$U_n = \max_{0 \leqslant r \leqslant n} X_r, \qquad L_n = \min_{0 \leqslant r \leqslant n} X_r,$$

so that for a given realization of the process up to time n, U_n and $|L_n|$ are the maximum distances upwards and downwards respectively reached by the particle in the time interval $[0, n]$. We can find the joint probability distribution of U_n and L_n by imposing absorbing barriers at j and $-k$ $(j, k > 0)$ and observing that $\text{prob}(U_n < j, L_n > -k)$ is precisely the probability (30) for $a = j$, $b = k$, i.e., the probability that absorption is later than time n. Hence, using the results for the case of two absorbing barriers, it is possible to calculate explicitly the joint distribution of U_n and L_n, while from the single barrier results the marginal distribution of each may be obtained.

Interesting results are obtained for U_n say, when we let $n \to \infty$, for here there is the possibility of a limiting distribution. Consider the behaviour of U_n for a particular realization. When the mean jump is positive $(p > q)$ then the particle will tend to drift to $+\infty$ and therefore U_n will tend to become indefinitely large as n increases; this will also happen when the mean jump is zero since the particle will make indefinitely large excursions. However, when the mean jump is negative the particle will drift to $-\infty$ after reaching a maximum distance U in the positive direction. We call U the *supremum* of the process and it is denoted by

$$U = \sup_n(X_n);$$

is finite with probability one only when $p < q$. When $p < q$ we have seen that for an absorbing barrier at $j > 0$ the probability is $1-(p/q)^j$ that the particle never reaches the barrier. Hence it follows that

$$\text{prob}(U < j) = 1-\left(\frac{p}{q}\right)^j \quad (j = 1, 2, \ldots)$$

and therefore that

$$\text{prob}(U = j) = \left(\frac{p}{q}\right)^j\left(1-\frac{p}{q}\right) \quad (j = 0, 1, \ldots), \tag{41}$$

a geometric distribution. For $j = 0$, we obtain the probability that the particle never enters the non-negative states, i.e. that the particle remains forever below its starting point. Further, the mean and variance of the supremum are

$$E(U) = \frac{p}{q-p}, \qquad V(U) = \frac{pq}{(q-p)^2}.$$

Similarly we define the *infimum* of the process, $L = \inf_n(X_n)$.

The extrema of the process, i.e. the supremum and infimum, provide a useful illustration of the concept of the sample function of a stochastic process. A sample function, in the case of the simple random walk, is obtained by considering a full realization $x_0, x_1, x_2, \ldots$, and regarding position x as a function of the real variable time. We obtain a step function of a positive real variable with at most unit jump discontinuities at the integers. Figure 1.1 illustrates portions of two sample functions. It is possible to regard the collection of all such step functions, for fixed x_0, as a non-countable sample space which, together with an appropriate probability measure, may be regarded as a definition of the stochastic process. It is on this sample space that the random variables supremum and infimum are defined, for corresponding to each element, i.e. sample function, there is a unique supremum and infimum (not necessarily finite).

Another aspect of the unrestricted walk on which results for the absorbing barrier give information is the phenomenon of *return to the origin*. Conditional on starting at j, let $f_{jj}^{(n)}$ be the probability that the *first* return to j occurs at time n, and let $f_{jk}^{(n)}$ be the probability that the particle first enters position k $(\neq j)$ at time n. Since the steps of the random walk are independent we clearly have

$$f_{jk}^{(n)} = f_{0,k-j}^{(n)}.$$

By a familiar decomposition of the first step we find that

$$f_{00}^{(n)} = pf_{10}^{(n-1)}+qf_{-1,0}^{(n-1)} = pf_{10}^{(n-1)}+qf_{01}^{(n-1)} \quad (n = 2, 3, \ldots)$$

and by definition $f_{00}^{(1)} = 1-p-q$. On taking generating functions we obtain

$$\sum_{n=1}^{\infty} f_{00}^{(n)} s^n = s(1-p-q)+sp\sum_{n=1}^{\infty} f_{10}^{(n)} s^n+sq\sum_{n=1}^{\infty} f_{01}^{(n)} s^n.$$

The second series on the right-hand side is the generating function $F_1(s)$ of absorption probabilities given by equation (36), and the first series is a similar generating function $F_1^*(s)$ for a random walk in which p and q are interchanged. Denoting the series on the left-hand side by $F_{00}(s)$ we obtain

$$F_{00}(s) = s(1-p-q)+spF_1^*(s)+sqF_1(s).$$

On interchanging p and q, $\lambda_2(s)$ is replaced by $\lambda_2^*(s) = (p/q)\lambda_2(s)$ and $F_1(s)$ of equation (37) is replaced by $(q/p)\lambda_2^*(s) = \lambda_2(s) = F_1^*(s)$. Hence

$$\begin{aligned} F_{00}(s) &= s(1-p-q)+sp\lambda_2(s)+sp\lambda_2(s) \\ &= s\{1-p-q+2p\lambda_2(s)\}. \end{aligned}$$

Now $F_{00}(1)$ is the probability of ever returning to the origin and, using equations (19), we have

$$\text{prob(return to the origin)} = F_{00}(1) = \begin{cases} 1 & (p = q), \\ 1+p-q < 1 & (p < q), \\ 1+q-p < 1 & (p > q). \end{cases}$$

Thus return to the origin is certain only if the walk has zero drift ($p = q$) and uncertain if the drift is positive or negative ($p \neq q$). In the case of zero drift $F_{00}(s)$ is the p.g.f. of the time of first return to the origin, and the mean of this distribution, $F'_{00}(1)$, is infinite since $\lambda'_2(1) = \infty$ when $p = q$.

The starting point was chosen to be the origin but clearly the above remarks apply equally to any starting point j. Thus return to j is certain only if the walk has zero drift and is uncertain otherwise. In the case of certain return the mean time to return is infinite.

(*v*) TWO REFLECTING BARRIERS

In Example 2.2 we saw how reflecting barriers arise naturally in a probability model of a dam. We now discuss in more detail the simple random walk in the presence of such barriers.

For a simple random walk we may define a reflecting barrier as follows. Suppose a is a point above the initial position. If the particle reaches a then at the next step it either remains at a or returns to the neighbouring interior state $a-1$ with specified probabilities. In most of what follows we take these probabilities to be $1-q$ and q respectively, so that in effect steps taken when the particle is at a are truncated in the positive direction. However, in particular problems these probabilities may be different from $1-q$ and q as is the case in Example 2.9. Similar remarks apply to a reflecting barrier below the initial position.

Suppose that the particle is initially in the state j and that the states 0 and a $(a > 0)$ are reflecting barriers. If X_n is the position of the particle immediately after the nth jump Z_n, then we have $X_0 = j$, and

$$X_n = \begin{cases} X_{n-1}+Z_n & (0 \leqslant X_{n-1}+Z_n \leqslant a), \\ a & (X_{n-1}+Z_n > a), \\ 0 & (X_{n-1}+Z_n < 0). \end{cases} \tag{42}$$

Thus the particle remains forever among the states $0, 1, \ldots, a$. On reaching one of the barrier states the particle remains there until a jump of the appropriate sign returns it to the neighbouring interior state.

Since there is now no possibility of the motion ceasing at any stage we have a different type of long-term behaviour from the absorbing barrier case. We shall show in Chapter 3 that after a long time the motion of the particle settles down to a condition of statistical equilibrium in which the occupation probabilities of the various states depend only on the relative position of the barriers but neither on the initial position of the particle nor on the time.

Let $p_{jk}^{(n)}$ be the probability that the particle occupies the state k at time n having started in the state j. Since the jumps are independent the position of the particle at time n depends only on two independent quantities, its position at time $n-1$ and the nth jump. If k is an internal state then during the time interval $(n-1, n)$ it can be reached by one of three mutually exclusive ways, namely by a jump of $+1$, 0 or -1 from states $k-1$, k or $k+1$ respectively. Since the state occupation probabilities at time $n-1$ are independent of the nth jump it follows that the probabilities of these three ways are $pp_{j,k-1}^{(n-1)}$, $(1-p-q)p_{jk}^{(n-1)}$, and $qp_{j,k+1}^{(n-1)}$ respectively. Hence

$$p_{jk}^{(n)} = pp_{j,k-1}^{(n-1)} + (1-p-q)p_{jk}^{(n-1)} + qp_{j,k+1}^{(n-1)} \quad (0 < k < a). \tag{43}$$

Since motion above state a and below state 0 is not permitted, similar reasoning will show that at the barrier states we must have

$$p_{ja}^{(n)} = pp_{j,a-1}^{(n-1)} + (1-q)\,p_{ja}^{(n-1)}, \qquad p_{j0}^{(n)} = (1-p)\,p_{j0}^{(n-1)} + qp_{j1}^{(n-1)}. \tag{44}$$

If there is a limiting equilibrium distribution of the state occupation probabilities then we have as $n \to \infty$

$$p_{jk}^{(n)} \to \pi_k \quad (k = 0, 1, \ldots, a)$$

and from equations (43) and (44) the π_k must satisfy

$$\begin{aligned} \pi_k &= p\pi_{k-1} + (1-p-q)\,\pi_k + q\pi_{k+1} \quad (k = 1, \ldots, a-1), \\ \pi_0 &= (1-p)\,\pi_0 + q\pi_1, \\ \pi_a &= p\pi_{a-1} + (1-q)\,\pi_a. \end{aligned} \tag{45}$$

We may solve for π_1 in terms of π_0, obtaining

$$\pi_1 = \left(\frac{p}{q}\right)\pi_0.$$

Solving (45) recursively, we have that

$$\begin{aligned}\pi_2 &= \{(p+q)\pi_1 - p\pi_0\}/q \\ &= \left(\frac{p}{q}\right)^2 \pi_0\end{aligned}$$

and in general

$$\pi_k = \left(\frac{p}{q}\right)^k \pi_0 \quad (k = 0,\dots,a). \tag{46}$$

We require the solution to be a probability distribution, i.e. $\sum \pi_k = 1$, and this enables us to find π_0. Hence we obtain the truncated geometric distribution

$$\pi_k = \frac{1-\dfrac{p}{q}}{1-\left(\dfrac{p}{q}\right)^{a+1}}\left(\frac{p}{q}\right)^k \quad (k = 0,\dots,a) \tag{47}$$

as the equilibrium set of state occupation probabilities. If $p > q$ then π_k decreases geometrically away from the upper barrier whereas if $p < q$, π_k decreases geometrically away from the lower barrier. If $p = q$ then from (46) we see that $\pi_k = \pi_0 = 1/(a+1)$ for all k, so that in the equilibrium situation all states are equally likely to be occupied by the particle.

In the following example we describe a model of a finite queue in discrete time and we shall see that it has a representation like a random walk with reflecting barriers. It will serve to illustrate both the method of solving these problems and the distinctive property of independent increments possessed by the random walk.

Example 2.9. *A queueing system in discrete time.* Consider the following model of a queueing system. A single server operates a service. Customers may arrive at the service point only at discrete time instants $n = 0, 1, 2, \dots$ and form a queue if the server is occupied. The statistical laws of arrivals and service are such that at any discrete time instant there is a probability α that a customer arrives and independently a probability β that the customer already being served, if any, completes his service. Suppose further that the queue is restricted in size to a customers including the one being served. Thus a customer arriving to find the queue full is turned away.

Let X_n denote the number of customers in the queue, including the one being served, immediately after the nth time instant. Let Z_n denote the number of new arrivals (either 0 or 1) minus the number of customers completing service (also either 0 or 1) at the nth time instant. Then it follows that

$$X_n = X_{n-1} + Z_n \quad (0 \leqslant X_{n-1} + Z_n \leqslant a). \tag{48}$$

If $X_{n-1} = a$, i.e. if the queue is full at time $n-1$, and if at time n there is a new arrival, but no completion of service, i.e. if $Z_n = 1$, then this new arrival is turned away and the queue size remains a at time n. Thus

$$X_n = a \quad (X_{n-1} + Z_n > a). \tag{49}$$

If $X_{n-1} = 0$, i.e. the queue is empty at time $n-1$, then some care is needed in describing the situation. For if there is no arrival at time n, then $X_n = 0$ and if there is an arrival at time n then $X_n = 1$. Since the queue is empty at time $n-1$ there is no possibility of a customer completing service. To describe the distribution of Z_n we have to distinguish two cases. Firstly if X_{n-1} has one of the values $1, 2, \ldots, a$ then, since arrivals and service completions are independent, we have

$$\begin{aligned}
\text{prob}(Z_n = 1) &= \text{prob(new arrival and no service completion)} \\
&= \alpha(1-\beta), \\
\text{prob}(Z_n = 0) &= \text{prob(new arrival and a service completion)} \\
&\qquad + \text{prob(no new arrival and no service completion)} \\
&= \alpha\beta + (1-\alpha)(1-\beta), \\
\text{prob}(Z_n = -1) &= \text{prob(no new arrival and a service completion)} \\
&= (1-\alpha)\beta.
\end{aligned}$$

Secondly if $X_{n-1} = 0$ then

$$\text{prob}(Z_n = 1) = \alpha, \qquad \text{prob}(Z_n = 0) = 1-\alpha.$$

Equations (48) and (49) are the same as the first two equations in (42) but now $\{Z_n\}$ is no longer a sequence of identically distributed random variables, the distribution of Z_n when $X_n = 0$ being different from that when $X_n \neq 0$. This means that the reflecting conditions at the lower barrier are different from those of the process described by equation (42). In all other respects, however, the two processes are the same and we can use the same methods for dealing with them. Thus we may set up equations such as (43) and (44) and these will give rise to the following equations for the equilibrium probability distribution $\pi_0, \ldots, \pi_a$ of queue size.

$$\begin{aligned}
\pi_k &= \alpha(1-\beta)\pi_{k-1} + \{(1-\alpha)(1-\beta) + \alpha\beta\}\pi_k + \\
&\qquad + (1-\alpha)\beta\pi_{k+1} \quad (k = 2, \ldots, a-1), \\
\pi_1 &= \alpha\pi_0 + \{(1-\alpha)(1-\beta) + \alpha\beta\}\pi_1 + (1-\alpha)\beta\pi_2, \\
\pi_0 &= (1-x)\pi_0 + (1-\alpha)\beta\pi_1, \\
\pi_a &= \alpha(1-\beta)\pi_{a-1} + \{1-\beta(1-\alpha)\}\pi_a.
\end{aligned} \tag{50}$$

These equations may be solved in a similar way to (45) and, on writing $\rho = \{\alpha(1-\beta)\}/\{\beta(1-\alpha)\}$, we obtain the solution

$$\pi_k = \frac{\rho^k}{1-\beta}\pi_0 \quad (k = 1, \ldots, a), \qquad \pi_0 = \frac{\beta-\alpha}{\beta-\alpha\rho^a}.$$

(*vi*) ONE REFLECTING BARRIER

Let us now consider a simple random walk over the positive integers starting in the state $j > 0$ with the zero state as a reflecting barrier. If the drift is negative $(q > p)$ then our studies of the absorbing barrier case have shown that the barrier state is certain to be reached. Once in the barrier state the particle remains there for a time T, say, which has a geometric distribution

$$\text{prob}(T = r) = (1-p)^{r-1}p. \tag{51}$$

When the particle leaves the barrier state it is again certain to return and so the particle continues to move on and off the reflecting barrier. Hence after a long time we may expect a condition of statistical equilibrium to be established in which the initial state j plays no part. In fact we can obtain the equilibrium distribution of the position of the particle by letting $a \to \infty$ in equation (47), remembering that $p < q$. Hence we get the geometric distribution

$$\pi_k = \left(1-\frac{p}{q}\right)\left(\frac{p}{q}\right)^k \quad (k = 0, 1, \ldots). \tag{52}$$

Since the steps of a random walk are independent, it follows that successive times spent on and away from the barrier are independent random variables. The periods spent on the barrier have the geometric distribution (51). The periods spent away from the barrier (excluding the initial one) clearly have the same distribution as the first passage time from state 1 to state 0, the p.g.f. of which may be obtained from equation (26) by setting $b = 1$ and letting $a \to \infty$. Thus if U denotes a period spent away from the barrier its p.g.f. is

$$E(s^U) = \lambda_2(s) = \frac{1-s(1-p-q)-[\{1-s(1-p-q)\}^2-4pqs^2]^{\frac{1}{2}}}{2ps}. \tag{53}$$

If N_1, M_1, N_2, M_2, ... denote the successive times at which the particle enters and leaves the reflecting barrier state then (M_1-N_1), (M_2-N_2), ... are independent random variables each with the distribution (51) while (N_2-M_1), (N_3-M_2), ... is a sequence of independent random variables each having the p.g.f. (53), and the two sequences are mutually independent. The sequence of time points N_1, M_1, N_2, M_2, ... form what is called an *alternating renewal process* and such processes will be dealt with in continuous time in Chapter 9.

In the queue of Example 2.9 periods spent on and off the barrier have a definite interpretation. The former correspond to the server's idle periods and the latter to his busy periods.

Let us now consider what happens when the drift is away from the barrier, i.e. $p > q$. We saw in the case of the unrestricted random walk that the particle ultimately drifts off to $+\infty$. It follows that in the present case this will happen *a fortiori*, since corresponding to each sample function of the unrestricted walk there is a sample function of the random walk with the reflecting barrier and the latter never lies below the former. Hence there is no equilibrium distribution in this case.

When $p = q$, i.e. zero drift, there is again no limiting distribution. This can best be seen by examining the solution to the case of two reflecting barriers in which all states are equally likely in the limit, each having probability $1/(a+1)$. Letting $a \to \infty$ we see that each state occupation probability tends to zero. In one respect the particle behaves similarly to one with negative drift; it is always certain to return to the barrier. Periods spent on the barrier still have the geometric distribution (51) while periods spent away from the barrier have the p.g.f. (53) with $p = q$, corresponding to a distribution with infinite mean. We shall see in the theory of the Markov chain (Chapter 3) that when an equilibrium distribution exists the limiting occupation probability for any particular state is inversely proportional to the mean recurrence time, i.e. the mean time between successive visits to that state. In the case of zero drift all recurrence times have infinite mean and so, roughly speaking, visits to a particular state are too infrequent to allow an equilibrium situation to establish itself.

(*vii*) OTHER TYPES OF BARRIER

There are numerous other types of behaviour possible at a given barrier. For example, on reaching a barrier a particle may be immediately returned to the state occupied just before reaching the barrier. This is another type of reflection. Or on reaching a barrier a particle may return to its initial position. Also, in two barrier problems the barriers may be of different types; for example, we may have an absorbing barrier at 0 and a reflecting barrier at a. We shall not go into details here. See, however, Exercise 5.

2.3. The general one-dimensional random walk in discrete time

It is clear that if we are to make any progress in examining the random walk representations of the insurance company and the dam discussed in Examples 2.1 and 2.2 then we must allow the steps Z_1, Z_2, ... to be more general than those of a simple random walk. We consider in this section the general one-dimensional random walk defined by equations (1) or (2) in which the steps, which may be either discrete or continuous, have a given distribution function $F(x)$. We shall find that the mathematical analysis of this process presents more difficult problems.

The simple random walk does however have important uses in the examination of the general random walk, firstly in providing analogy and suggestion and secondly in providing an actual approximation by means of replacing the distribution function $F(x)$ by a discrete distribution over the points $-l$, 0, l having the same mean and variance as $F(x)$ (see Exercise 1).

(*i*) UNRESTRICTED

We suppose the particle starts at the origin. Then, as in the case of the simple random walk, its position X_n after n steps is a sum of n mutually independent random variables each with the distribution function $F(x)$. Suppose that μ and σ^2 are respectively the mean and variance of $F(x)$, assumed finite. Then if n is large, X_n is approximately normally distributed with mean $n\mu$ and variance $n\sigma^2$. We may use similar arguments to those of Section 2.2 (i) to show that when n is large

$$X_n = n\mu + O(\sqrt{n}) \tag{54}$$

with high probability. It follows again that when $\mu > 0$ the particle ultimately drifts off to $+\infty$ and correspondingly to $-\infty$ when $\mu < 0$. When $\mu = 0$ the behaviour is of the same singular type as in the simple random walk, but this fact is somewhat more difficult to demonstrate in the general case. If we wish to compute the probability that after a large number of steps the particle lies in a given interval (y_1, y_2) then we can use the normal approximation

$$\operatorname{prob}(y_1 \leqslant X_n \leqslant y_2) \simeq (2\pi\sigma^2 n)^{-\frac{1}{2}} \int_{y_1}^{y_2} \exp\left\{-\frac{(x-n\mu)^2}{2n\sigma^2}\right\} dx. \tag{55}$$

Statements like (54) and (55) are based on the central limit theorem whose truth depends essentially on the assumption that σ^2 is finite. However, assuming only that the first moment is finite, we may demonstrate, for example, that the particle drifts off to $+\infty$ when $\mu > 0$, by using the strong law of large numbers. If c is an arbitrarily large positive number then we can assert that $\operatorname{prob}(X_n > c, X_{n+1} > c, \ldots)$ can be made as close to 1 as we please by taking n sufficiently large. For according to the strong law of large numbers, if $\epsilon < \mu$ is fixed and η is arbitrarily small then for n sufficiently large

$$\operatorname{prob}\left(\left|\frac{X_n}{n} - \mu\right| < \epsilon, \left|\frac{X_{n+1}}{n+1} - \mu\right| < \epsilon, \ldots\right) > 1 - \eta,$$

i.e. $\qquad \operatorname{prob}\{X_n > n(\mu-\epsilon), X_{n+1} > (n+1)(\mu-\epsilon), \ldots\} > 1-\eta.$

By choosing $n(\mu-\epsilon) > c$ we have the result. Thus if the mean step is positive, then after a sufficiently large number of steps the particle will remain beyond any arbitrarily chosen point in the positive direction with probability as close to unity as we please. When $\mu = 0$, the behaviour is of the same singular type as in the simple random walk, but this fact is more difficult to prove (Chung and Fuchs, 1951); see also Section 2.4(ii).

(*ii*) SOME PROPERTIES OF MOMENT GENERATING FUNCTIONS

In the subsequent discussion of the random walk we require some properties of moment generating functions which we now establish.

Suppose X is a random variable with p.d.f. $f(x)$ such that X is strictly two sided, i.e. X takes both positive and negative values with positive probability. Mathematically we may define this condition by saying that there exists a number $\delta > 0$ such that

$$\text{prob}(X < -\delta) > 0 \quad \text{and} \quad \text{prob}(X > \delta) > 0. \tag{56}$$

The moment generating function (m.g.f.) of X is defined as the two-sided Laplace transform of the p.d.f. of X, namely

$$f^*(\theta) = \int_{-\infty}^{\infty} e^{-\theta x} f(x)\, dx, \tag{57}$$

and we shall suppose that $f(x)$ tends to zero sufficiently fast as $x \to \pm\infty$ to ensure that the integral in (57) converges for all real values of θ. For example, this is true of a normal density function or of a density function which vanishes outside a finite interval. We know that

$$\mu = E(X) = \{-(d/d\theta) f^*(\theta)\}_{\theta=0}$$

and hence if we plot a curve of $f^*(\theta)$ against θ, the slope of the curve at $\theta = 0$ is $-\mu$. Further $f^*(\theta)$ is a convex (downwards) function of θ. This can be seen by considering

$$\frac{d^2 f^*(\theta)}{d\theta^2} = \int_{-\infty}^{\infty} x^2 e^{-\theta x} f(x)\, dx,$$

which is positive for all real values of θ. Also $f^*(\theta) \to \infty$ as $\theta \to \pm\infty$, for, taking $\theta < 0$ as an example, we have that

$$\begin{aligned} f^*(\theta) &> \int_{\delta}^{\infty} e^{-\theta x} f(x)\, dx \\ &> e^{-\theta\delta} \int_{\delta}^{\infty} f(x)\, dx \\ &= e^{-\theta\delta}\, \text{prob}(X > \delta) \to \infty \end{aligned}$$

as $\theta \to -\infty$ in virtue of (56). By a similar argument it can be seen that

$f^*(\theta) \to \infty$ as $\theta \to \infty$. Thus if we plot a graph of $f^*(\theta)$ against θ we obtain a shape illustrated in Fig. 2.1.

Two further results follow from the convexity property of $f^*(\theta)$. Firstly there is a unique value of θ, say θ_1, at which $f^*(\theta)$ attains its minimum value; θ_1 has the same sign as μ and $\theta_1 = 0$ if $\mu = 0$. Also

$$f^*(\theta_1) \begin{matrix} < 1 & (\mu \neq 0), \\ = 1 & (\mu = 0). \end{matrix} \tag{58}$$

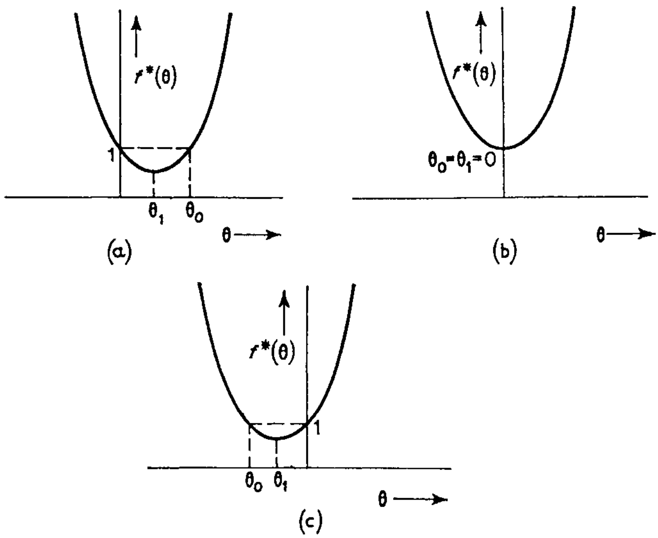

Fig. 2.1. A moment generating function $f^*(\theta)$ of a random variable X for (a) $\mu > 0$, (b) $\mu = 0$, (c) $\mu < 0$, where $\mu = E(X) = \{-(d/d\theta)\, f^*(\theta)\}_{\theta=0}$.

Secondly let us consider the roots of the equation

$$f^*(\theta) = 1. \tag{59}$$

Clearly $\theta = 0$ is one root. If $\mu \neq 0$ there is a unique second real root $\theta = \theta_0 \neq 0$ also having the same sign as μ (see Fig. 2.1). If $\mu = 0$ then $\theta = 0$ is a double root of (59).

We have assumed in the above discussion that the p.d.f. $f(x)$ of the random variable X exists and, moreover, that the integral (57) defining $f^*(\theta)$ converges for all real θ. For a general distribution function $F(x)$, we define the m.g.f. as the two-sided Laplace–Stieltjes transform

$$\int_{-\infty}^{\infty} e^{-\theta x}\, dF(x),$$

provided that the integral converges in a real θ-interval of which $\theta = 0$ is an interior point. The convexity property still holds in this interval and for a fairly general class of distributions we can still assert the existence of the numbers θ_0 and θ_1 (see Exercises 6 and 7).

For a random variable discrete over the integers, instead of the m.g.f. we usually use the p.g.f., for which similar results may be obtained (see Exercise 8).

Example 2.10. *The normal distribution.* Consider the normal distribution with mean μ and variance σ^2. The moment generating function is given by

$$f^*(\theta) = \exp(-\mu\theta + \tfrac{1}{2}\sigma^2\theta^2).$$

The real roots of $f^*(\theta) = 1$ are given by

$$-\mu\theta + \tfrac{1}{2}\sigma^2\theta^2 = 0,$$

i.e. are $\theta = 0$ and $\theta = 2\mu/\sigma^2$. Thus we have

$$\theta_0 = 2\mu/\sigma^2.$$

The unique real root θ_1, of $(d/d\theta)f^*(\theta) = 0$ is given by

$$-\mu + \sigma^2\theta = 0,$$

i.e. $\theta_1 = \mu/\sigma^2$.

(*iii*) ABSORBING BARRIERS

In the simple random walk it was sufficient to define two absorbing states $-b$ and a without paying any attention to possible states beyond these points. In the more general random walk, however, the particle may jump over or land on a barrier and we shall define absorption to include both these possibilities. Accordingly we suppose that all states in the open interval $(-b, a)$ $(a, b > 0)$ are non-absorbing, whereas all states in the two semi-infinite intervals $(-\infty, -b]$ and $[a, \infty)$ are absorbing states. If the particle starts at the origin then the number N of steps to absorption is the time of first exit or first passage from the interval (a, b), i.e. the time at which the particle first crosses one or other of the barriers. When absorption occurs the process ceases.

The first thorough investigation of the general random walk with absorbing barriers was made by Wald (1947) in the theory of a statistical technique known as sequential analysis. We shall give a somewhat more general version of Wald's results.

We shall confine our discussion for the present to continuous random variables although the method we use applies also to the discrete case. Accordingly we suppose that the random variables $Z_1, Z_2, \ldots$ are mutually independent each with the same p.d.f. $f(x)$ and m.g.f. $f^*(\theta)$.

Let $f_n(x)dx$ denote the probability that the particle remains inside the interval $(-b, a)$ for the first $n-1$ steps and that at the nth step it lies in the interval $(x, x+dx)$, where x may be either an absorbing or a non-absorbing state, i.e.

$$f_n(x)\,dx = \text{prob}(-b < X_1, X_2, \ldots, X_{n-1} < a, x < X_n < x+dx)$$
$$(n = 1, 2, \ldots;\ -\infty < x < \infty). \quad (60)$$

We define

$$f_0(x) = \delta(x),$$

$\delta(x)$ representing the p.d.f. of a probability distribution located entirely at the point $x = 0$, i.e. $\delta(x)$ is the so-called Dirac delta function. It should be noted that $f_n(x)$ depends on a and b, but to avoid a cumbersome notation such as $f_n(x; a, b)$ we write simply $f_n(x)$. In the single barrier case, e.g. $b = \infty$, we would define

$$f_n(x)\,dx = \text{prob}(X_1, X_2, \ldots, X_{n-1} < a, x < X_n < x+dx)$$
$$(n = 1, 2, \ldots;\ -\infty < x < \infty). \quad (61)$$

The function $f_n(x)$ is an important one in the present discussion and so we shall examine it a little more closely. It represents the probability distribution of the particle's position at time n but it does not in general integrate to unity because the particle may be absorbed before time n. The sample functions of a general unrestricted random walk are step functions with discontinuities at the integers. On the graph of a sample function the barriers are represented by horizontal lines at a distance a above and b below the time axis. If we terminate these lines at time $n-1$ thus forming a 'pipe' then

$$\int_{-\infty}^{\infty} f_n(x)\,dx$$

is the probability measure of all sample functions emanating from the origin and passing within the 'pipe'.

Further we note that

$$\text{prob}(N > n) = \int_{-b}^{a} f_n(x)\,dx, \quad (62)$$

for the right-hand side is the probability that the particle is still between the barriers at time n, i.e. that absorption occurs after time n. Let p_A denote the probability that absorption ultimately occurs. We shall use (62) to show that $p_A = 1$ if a and b are both finite. We have

$$p_A = \text{prob}(N < \infty) = \lim_{n\to\infty} \text{prob}(N < n)$$

$$= 1 - \lim_{n\to\infty} \int_{-b}^{a} f_n(x)\,dx.$$

Now by an argument we used for the simple random walk the right-hand side of (62) cannot exceed the probability that an unrestricted particle lies in the interval (a, b) at time n, which by the normal approximation (55), tends to zero as $n \to \infty$, provided that $\sigma^2 < \infty$. For a more general argument, not depending on the assumption of finite variance, see Exercise 10, which shows in addition that the probability of the particle being between the barriers at time n tends to zero geometrically fast in n.

By using the moment generating function we can obtain an upper bound for (62) when $\mu \neq 0$. Let $g_n(x)$ be the p.d.f. of an unrestricted particle at time n, so that

$$\int_{-\infty}^{\infty} e^{-\theta x} g_n(x)\, dx = \{f^*(\theta)\}^n.$$

Now for $\theta > 0$ we have

$$\begin{aligned}\int_{-b}^{a} f_n(x)\, dx \leqslant \int_{-b}^{a} g_n(x)\, dx &\leqslant \int_{-b}^{a} e^{-\theta(x-a)} g_n(x)\, dx \\ &\leqslant e^{\theta a} \int_{-\infty}^{\infty} e^{-\theta x} g_n(x)\, dx \\ &= e^{\theta a}\{f^*(\theta)\}^n.\end{aligned}$$

This holds for all $\theta > 0$ and the left-hand side is independent of θ. When $\mu > 0$ then $\theta_1 > 0$, so that

$$\int_{-b}^{a} f_n(x)\, dx \leqslant e^{\theta_1 a}\{f^*(\theta_1)\}^n, \tag{63}$$

again showing that (62) tends to zero geometrically in n, since $f^*(\theta_1) < 1$. The above argument does not depend on b being finite, so that (63) also holds for the case of a single barrier at a with a drift towards the barrier. The corresponding inequality for $\mu < 0$ (see Exercise 9) is

$$\int_{-b}^{a} f_n(x)\, dx \leqslant e^{-\theta_1 b}\{f^*(\theta_1)\}^n. \tag{64}$$

A further property of $f_n(x)$ is obtained by observing that

$$f_n(x)\, dx = \text{prob}(N = n,\ x < X_N < x+dx) \quad (x < -b \text{ or } x > a,\ n = 1, 2, \ldots),$$

i.e. $f_n(x)dx$ ($x < -b$ or $x > a$) is the joint probability that the time N to absorption is n and that the position reached when absorption occurs is between x and $x+dx$. Hence if we take a moment generating function with respect to n and with respect to x over absorbing states, we have

$$E(e^{-\theta X_N} s^N) = \sum_{n=1}^{\infty} s^n \left\{ \int_{-\infty}^{-b} e^{-\theta x} f_n(x)\, dx + \int_{a}^{\infty} e^{-\theta x} f_n(x)\, dx \right\} \qquad (65)$$

Now define the generating function $K(\theta, s)$ of $f_n(x)$ taken with respect to x over the non-absorbing states and with respect to n over $0, 1, 2, \ldots$, i.e.

$$K(\theta, s) = \sum_{n=0}^{\infty} s^n \int_{-b}^{a} e^{-\theta x} f_n(x)\, dx. \qquad (66)$$

We shall establish the following identity:

$$E(e^{-\theta X_N} s^N) = 1 - \{1 - s f^*(\theta)\} K(\theta, s). \qquad (67)$$

First we obtain a recurrence relation between $f_n(x)$ and $f_{n-1}(x)$. If at time $n-1$ the particle is in the position y $(-b < y < a)$ then to reach the position x at time n the jump Z_n must take the value $x-y$, the distribution of Z_n being independent of y. Hence

$$\begin{aligned} \operatorname{prob}(x < X_n < x+dx \mid X_{n-1} = y) &= \operatorname{prob}(x-y < Z_n < x-y+dx) \\ &= f(x-y)\, dx \end{aligned}$$

and

$$\begin{aligned} f_n(x)\, dx &= \int_{-b}^{a} \operatorname{prob}(x < X_n < x+dx \mid X_{n-1} = y) f_{n-1}(y)\, dy \\ &= \left\{ \int_{-b}^{a} f(x-y) f_{n-1}(y)\, dy \right\} dx. \end{aligned}$$

Thus we have the recurrence relation

$$f_n(x) = \int_{-b}^{a} f_{n-1}(y) f(x-y)\, dy \quad (n = 1, 2, \ldots). \qquad (68)$$

Taking the Laplace transform of (68), we have that

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-\theta x} f_n(x)\, dx &= \int_{-\infty}^{\infty} e^{-\theta x} \left\{ \int_{-b}^{a} f_{n-1}(y) f(x-y)\, dy \right\} dx \\ &= \int_{-b}^{a} e^{-\theta y} f_{n-1}(y) \left\{ \int_{-\infty}^{\infty} e^{-\theta(x-y)} f(x-y)\, dx \right\} dy \\ &= f^*(\theta) \int_{-b}^{a} e^{-\theta y} f_{n-1}(y)\, dy. \end{aligned} \qquad (69)$$

The product of generating functions on the right-hand side of (69) arises because the position of the particle at time n is given by the addition of an independent random variable Z_n to its position at time $n-1$, provided that it is still between the barriers at time $n-1$.

Second, the expression in parentheses on the right-hand side of (65) may be written

$$\int_{-\infty}^{\infty} e^{-\theta x} f_n(x)\,dx - \int_{-b}^{a} e^{-\theta x} f_n(x)\,dx. \tag{70}$$

Finally combining (65), (66), (69) and (70), we obtain

$$\begin{aligned} E(e^{-\theta X_N} s^N) &= s f^*(\theta) K(\theta, s) - \{K(\theta, s) - 1\} \\ &= 1 - \{1 - s f^*(\theta)\} K(\theta, s), \end{aligned}$$

which is the identity (67).

This identity provides a somewhat general expression for $E(e^{-\theta X_N} s^N)$. The barriers do not enter explicitly and (67) still holds if one of the barriers is placed at ∞, thus producing a single absorbing barrier. In such a case, however, the expectation may be with respect to a probability distribution whose total mass is less than unity.

If the random walk is discrete over the integers, i.e. if the steps $Z_1, Z_2, \ldots$ take on only integral values with p.g.f. $G(z)$, then (67) takes the form

$$E(z^{X_N} s^N) = 1 - \{1 - sG(z)\} L(z, s), \tag{71}$$

where $L(z,s)$ is the discrete analogue of $K(\theta, s)$.

(*iv*) APPLICATION TO THE SIMPLE RANDOM WALK

The identity (71) is particularly useful for the simple random walk because absorption occurs exactly on a barrier and X_N can take only the values $-b$ and a. If we use the notation of Section 2.2, (71) takes the form

$$z^a F_{0a}(s) + z^{-b} F_{0,-b}(s) = 1 - \{1 - s(pz + 1 - p - q + qz^{-1})\} L(z, s), \tag{72}$$

where

$$\begin{aligned} L(z, s) &= \sum_{k=-b+1}^{a-1} z^k \sum_{n=0}^{\infty} s^n p_k^{(n)} \\ &= \sum_{k=-b+1}^{a-1} z^k P_k(s), \end{aligned} \tag{73}$$

where $P_k(s)$ satisfies equation (31). Now we have already determined $F_{0a}(s)$ and $F_{0,-b}(s)$ in (18) and (26). Thus from (72) we can find $L(z,s)$ and hence $P_k(s)$ by picking out the coefficient of z^k.

We could alternatively use the identity (71) to determine explicitly $E(z^{X_N} s^N)$. For we know that it must be of the form given in the left-hand side of (72). Now $\lambda_1(s)$ and $\lambda_2(s)$ are the roots of

$$1 - sG(z) = 0.$$

If we set $z = \lambda_1(s)$ and $z = \lambda_2(s)$ in turn in (72) the second term on the right-hand side vanishes; we obtain the two equations

$$\{\lambda_1(s)\}^a F_{0a}(s) + \{\lambda_1(s)\}^{-b} F_{0,-b}(s) = 1,$$
$$\{\lambda_2(s)\}^a F_{0a}(s) + \{\lambda_2(s)\}^{-b} F_{0,-b}(s) = 1,$$

which may be solved for $F_{0a}(s)$ and $F_{0,-b}(s)$. Hence in the simple random walk the identity (71) gives us all the generating functions of interest and hence all the results of Section 2.2(ii).

(*v*) WALD'S IDENTITY

Suppose in (67) that we set $s = \{f^*(\theta)\}^{-1}$. Then the second term on the right-hand side vanishes and we obtain

$$E[e^{-\theta X_N}\{f^*(\theta)\}^{-N}] = 1. \tag{74}$$

This is Wald's fundamental identity of sequential analysis and is a general result for random walks with absorbing barriers having some useful applications. If there is only a single absorbing barrier and if the drift is away from the barrier, then we saw for the simple random walk that ultimate absorption is not a certain event. Hence in such a case the total probability mass in the joint distribution of N and X_N is less than unity and we have to interpret the expectation sign in (74) as being taken over such a distribution. This is quite in order since in all cases the function $f_n(x)$ is well defined and so therefore is the generating function on the right-hand side of (67). If absorption is uncertain, say with a single barrier at a, then

$$\sum_{n=1}^{\infty} \int_a^{\infty} f_n(x)\,dx < 1.$$

Alternatively, if we denote the probability of absorption by p_A and let E_A denote expectation conditional upon the occurrence of absorption, then we may write (74) quite generally in the form

$$p_A E_A[e^{-\theta X_N}\{f^*(\theta)\}^{-N}] = 1. \tag{75}$$

The setting of $s = \{f^*(\theta)\}^{-1}$ in (67) in order to obtain Wald's identity requires justification, for if $K(\theta, s)$ becomes infinite when s takes this value the substitution would not be valid.

For fixed $\theta > 0$ the function $K(\theta, s)$ satisfies

$$K(\theta, s) < \sum_{n=0}^{\infty} s^n e^{\theta b} \int_{-b}^{a} f_n(x)\,dx$$

and in virtue of the inequality (63), the series on the right-hand side converges, provided that

$$sf^*(\theta_1) < 1, \quad \text{i.e. } s < \{f^*(\theta_1)\}^{-1}. \tag{76}$$

The same is easily shown to be true for $\theta < 0$, and so $K(\theta, s)$ is finite for any real θ and for s satisfying (76). It follows that, provided a and b are finite, we may safely make the substitution $s = \{f^*(\theta)\}^{-1}$; for then (76) is automatically satisfied for all real s, since θ_1 is the point at which $f^*(\theta)$ attains its unique minimum. It follows that for finite a and b, (74) is valid for all values of θ for which $|f^*(\theta)| > f^*(\theta_1)$. When either a or b is infinite, the single barrier case, a little more care is needed in order to justify (74), but in practice such cases can be examined by proceeding to the limit.

We have seen that $p_A = 1$ when a and b are both finite. Let p_a denote the probability that the particle is absorbed at the upper barrier and p_{-b} the corresponding probability for the lower barrier. We shall use Wald's identity to obtain an approximation to these quantities. We may write (74) as

$$p_a E[e^{-\theta X_N}\{f^*(\theta)\}^{-N} | X_N \geqslant a] + p_{-b} E[e^{-\theta X_N}\{f^*(\theta)\}^{-N} | X_N \leqslant -b] = 1, \tag{77}$$

since $X_N \geqslant a$ and $X_N \leqslant -b$ are mutually exclusive and exhaustive events. Now when $\mu \neq 0$ then $\theta_0 \neq 0$ and $f^*(\theta_0) = 1$, so that

$$p_a E[e^{-\theta_0 X_N} | X_N \geqslant a] + p_{-b} E[e^{-\theta_0 X_N} | X_N \leqslant -b] = 1.$$

If we neglect the excess over the barrier and write

$$X_N \simeq a \text{ when } X_N \geqslant a \quad \text{and} \quad X_N \simeq -b \text{ when } X_N \leqslant -b, \tag{78}$$

then we have

$$p_a e^{-\theta_0 a} + p_{-b} e^{\theta_0 b} \simeq 1.$$

Finally, using the fact that

$$p_a + p_{-b} = 1,$$

we obtain the approximations for the absorption probabilities when $\mu \neq 0$

$$p_a \simeq \frac{1 - e^{\theta_0 b}}{e^{-\theta_0 a} - e^{\theta_0 b}}, \qquad p_{-b} \simeq \frac{e^{-\theta_0 a} - 1}{e^{-\theta_0 a} - e^{\theta_0 b}}. \tag{79}$$

These ought to be good approximations for large a and b, for with reasonable distributions any excess over the barrier will then be small compared with a and b. When $\mu = 0$ then $\theta_0 = 0$ and the expressions in (79) become indeterminate. However, if we let $\theta_0 \to 0$ we obtain when $\mu = 0$

$$p_a \simeq \frac{b}{a+b}, \qquad p_{-b} \simeq \frac{a}{a+b}. \tag{80}$$

The approximations (80) are independent of the particular distribution $f(x)$ involved.

Using the same method, we can obtain approximately the p.g.f. of N, the number of steps to absorption. Let us reconsider the substitution

$$s = \{f^*(\theta)\}^{-1} \quad \text{or} \quad 1/s = f^*(\theta).$$

By the convexity property of $f^*(\theta)$ this equation will have two real roots in θ, provided that $1/s > f^*(\theta_1)$. Let us denote the roots by $\lambda_1(s)$ and $\lambda_2(s)$, where $\lambda_1(1) = 0$ and $\lambda_2(1) = \theta_0$. We now obtain Wald's identity in another form, expressed by the following two equations, by setting $\theta = \lambda_1(s)$ and $\theta = \lambda_2(s)$ in turn in the identity (67). Thus

$$E[\exp\{-\lambda_i(s)X_N\}s^N] = 1 \quad (i = 1, 2). \tag{81}$$

Again, using the approximation (78), we have

$$p_a e^{-\lambda_i(s)a} E_a(s^N) + p_{-b} e^{\lambda_i(s)b} E_{-b}(s^N) \simeq 1 \quad (i = 1, 2), \tag{82}$$

where E_a denotes expectation conditional on absorption at the upper barrier a, etc. Using the expressions (79) and (80) for p_a and p_{-b}, the two equations (82) may be solved to obtain approximately the p.g.f. of N, namely

$$E(s^N) = p_a E_a(s^N) + p_{-b} E_{-b}(s^N).$$

Further interesting results can be obtained by observing that the expression on the left of Wald's identity is a function of θ, whereas the right-hand side is a constant. We may therefore expand the expression inside the expectation sign in powers of θ and equate the expected value of the coefficients to zero. This is equivalent to differentiating inside the expectation sign and then setting $\theta = 0$. Remembering that the cumulant generating function $\log f^*(\theta) = -\mu\theta + \frac{1}{2}\sigma^2\theta^2 - \ldots$, we may write (74) as

$$E[\exp\{-(X_N - N\mu)\theta - \tfrac{1}{2}N\sigma^2\theta^2 + \ldots\}] = 1.$$

Hence

$$E(X_N - N\mu) = 0$$

and

$$E\{(X_N - N\mu)^2 - N\sigma^2\} = 0,$$

from the coefficients of θ and θ^2 respectively. Hence

$$E(X_N) = \mu E(N) \tag{83}$$

and

$$E\{(X_N - N\mu)^2\} = \sigma^2 E(N). \tag{84}$$

Thus

$$E(N) = \begin{cases} \dfrac{1}{\mu} E(X_N) & (\mu \neq 0), \\[2ex] \dfrac{1}{\sigma^2} E(X_N^2) & (\mu = 0). \end{cases} \tag{85}$$

Now $E(X_N) \simeq ap_a - bp_{-b}$ and $E(X_N^2) \simeq a^2 p_a + b^2 p_{-b}$, so that from (85) we obtain the following approximation for the expected number of steps to absorption:

$$E(N) \simeq \begin{cases} \dfrac{(a+b) - a\,e^{\theta_0 b} - b\,e^{-\theta_0 a}}{e^{-\theta_0 a} - e^{\theta_0 b}} & (\mu \neq 0), \\[2ex] \dfrac{ab}{\mu\sigma^2} & (\mu = 0). \end{cases} \tag{86}$$

These formulae have important statistical applications in sequential analysis.

Example 2.11. ***Random walk with normally distributed steps.*** Let us apply some of these results to a random walk in which the steps have a normal distribution with mean μ and variance σ^2. Then

$$f^*(\theta) = \exp(-\mu\theta + \tfrac{1}{2}\sigma^2\theta^2)$$

and we saw in Example 2.10 that $\theta_0 = 2\mu/\sigma^2$. Then applying the results (79) we find that the probability of absorption at the upper barrier is given by

$$p_a \simeq \frac{1 - \exp(2\mu b/\sigma^2)}{\exp(-2\mu a/\sigma^2) - \exp(2\mu b/\sigma^2)} \quad (\mu \neq 0)$$

and that at the lower barrier by

$$p_{-b} \simeq \frac{\exp(-2\mu a/\sigma^2) - 1}{\exp(-2\mu a/\sigma^2) - \exp(2\mu b/\sigma^2)} \quad (\mu \neq 0).$$

For the expected number of steps to absorption we have

$$E(N) \simeq \frac{(a+b) - a\exp(2\mu b/\sigma^2) - b\exp(-2\mu a/\sigma^2)}{\exp(-2\mu a/\sigma^2) - \exp(2\mu b/\sigma^2)} \quad (\mu \neq 0).$$

When $\mu = 0$, p_a and p_{-b} are approximately independent of σ^2 and are given by (80), whereas $E(N)$ is given by (86).

Example 2.12. ***Mixture of two exponential distributions.*** We consider a random walk in which the distribution of each step is a mixture of two exponential distributions, one on $(0, \infty)$ and the other on $(-\infty, 0)$. We choose this somewhat unrealistic example since it illustrates a case where Wald's identity can be used to obtain the exact probabilities of absorption and the exact p.g.f. of N. The simplification in this case arises from the fact, which we shall demonstrate below, that X_N and N are independently distributed. We define for $0 < \alpha < 1$

$$f(x) = \begin{cases} \alpha\nu\, e^{-\nu x} & (x > 0), \\ (1-\alpha)\lambda\, e^{x\lambda} & (x < 0). \end{cases}$$

Thus

$$f^*(\theta) = \alpha\left(\frac{\nu}{\nu+\theta}\right) + (1-\alpha)\left(\frac{\lambda}{\lambda-\theta}\right).$$

In particular, if $\alpha = \lambda/(\lambda+\nu)$ then

$$f^*(\theta) = \frac{\nu\lambda}{(\nu+\theta)(\lambda-\theta)}, \tag{87}$$

and each step is then the sum of two independent components, one having an exponential p.d.f. $\nu e^{-\nu x}$ on $(0, \infty)$ and the other having an exponential p.d.f. $\lambda e^{\lambda x}$ on $(-\infty, 0)$. Alternatively, each step is the difference of two independent random variables, each with an exponential distribution.

If absorption occurs at, say, the upper barrier, then the step which carries the particle over the barrier must arise from the positive component of the mixture. The excess over the barrier, namely $X_N - a$, is the excess of the exponentially distributed random variable Z_N over the quantity $a - X_{N-1}$ conditional on this excess being positive and, by a well-known property of the exponential distribution, this excess has the same exponential distribution. Hence

$$E(e^{-\theta(X_N-a)} | X_N \geqslant a) = \frac{\nu}{\nu+\theta}$$

and

$$E(e^{-\theta X_N} | X_N \geqslant a) = \frac{\nu e^{-\theta a}}{\nu+\theta}$$

independently of N. Similar remarks apply to the lower barrier and Wald's identity takes the form

$$p_a\left(\frac{\nu e^{-\theta a}}{\nu+\theta}\right) E[\{f^*(\theta)\}^{-N} | X_N \geqslant a] + p_{-b}\left(\frac{\lambda e^{\theta b}}{\lambda-\theta}\right) E[\{f^*(\theta)\}^{-N} | X_N \leqslant -b] = 1. \tag{88}$$

When $f^*(\theta)$ is given by (87), then we find that $\theta_0 = \lambda - \nu$ and on setting $\theta = \lambda - \nu$ in (88) we obtain, remembering that $f^*(\theta_0) = 1$,

$$\left(\frac{\nu}{\lambda}\right) p_a e^{-a(\lambda-\nu)} + \left(\frac{\lambda}{\nu}\right) p_{-b} e^{b(\lambda-\nu)} = 1.$$

Because $p_a + p_{-b} = 1$, we have

$$p_a = \begin{cases} \dfrac{1 - (\lambda/\nu)\, e^{b(\lambda-\nu)}}{(\nu/\lambda)\, e^{-a(\lambda-\nu)} - (\lambda/\nu)\, e^{b(\lambda-\nu)}} & (\lambda \neq \nu), \\[2ex] \dfrac{b + \lambda^{-1}}{a + b + 2\lambda^{-1}} & (\lambda = \nu), \end{cases} \tag{89}$$

where the second expression is obtained from the first by letting $\nu \to \lambda$. This exact expression for the probability of absorption at the upper barrier may be compared with the approximation derived from (79), namely

$$p_a \simeq \begin{cases} \dfrac{1-e^{b(\lambda-\nu)}}{e^{-a(\lambda-\nu)}-e^{b(\lambda-\nu)}} & (\lambda \neq \nu), \\[2ex] \dfrac{b}{a+b} & (\lambda = \nu). \end{cases}$$

For further details, see Exercise 14.

(*vi*) ONE ABSORBING BARRIER

Suppose that the random walk starts at the origin and that there is an absorbing barrier at a. We may examine this case by letting $b \to \infty$ in the two-barrier situation discussed above. Before doing this we can infer from the properties of the unrestricted random walk discussed in (*i*) that the barrier is certain to be crossed if $\mu = E(Z_i) \geqslant 0$. Hence, letting p_a denote the probability of absorption, i.e. the probability that the barrier is crossed at all, we have, for $\mu \geqslant 0$,

$$p_a = 1.$$

If $\mu < 0$, then we can obtain an approximation by letting $b \to \infty$ in (79). Thus, for $\mu < 0$,

$$p_a \simeq e^{\theta_0 a}, \tag{90}$$

where $\theta_0 < 0$, and a is assumed to be large.

When absorption is certain we can use (82) to obtain approximately the p.g.f. of N, the number of steps to absorption. Setting $p_a = 1$ and $p_{-b} = 0$ in (82), we obtain

$$E(s^N) = E_a(s^N) \simeq e^{a\lambda_i(s)} \quad (i = 1, 2).$$

We choose the root $\lambda_1(s)$ since $\lambda_1(1) = 0$, thus making $E(s^N)$ a proper p.g.f. Hence

$$E(s^N) \simeq e^{a\lambda_1(s)}.$$

Again the approximation involved is that of neglecting the excess over the barrier, i.e. writing $X_N = a$ whenever $X_N \geqslant a$. By differentiating the above expression for $E(s^N)$,we obtain for the first two moments of N when $\mu > 0$

$$E(N) \simeq \frac{a}{\mu}, \qquad V(N) \simeq \frac{\sigma^2 a}{\mu^3}.$$

When $\mu = 0$ these moments become infinite. Since in fact $E(X_N) \geqslant a$, the approximation for $E(N)$ is really an inequality $E(N) \geqslant a/\mu$.

Example 2.13. *Insurance risk.* Consider the insurance company described in Example 2.1. We saw that its capital X_n at time n can be represented as a random walk starting at X_0 (the company's initial capital) with an absorbing barrier at 0. Absorption corresponds to the exhaustion of the capital which implies the ruin of the company.

Let us assume that the net income in each period is constant. Hence let

$$Y_r = \gamma \quad (r = 1, 2, \ldots).$$

The claim totals W_1, W_2, ... may each be regarded as the sum of a large number of independent claims and it is therefore reasonable to assume that W_r is normally distributed with, say, positive mean μ and variance σ^2, although in practice this assumption is likely to be vitiated by occasional very large claims. Hence $Z_r = Y_r - W_r$, the change in capital in period r, is normally distributed with mean $\gamma - \mu$ and variance σ^2. Now if $\gamma - \mu \leqslant 0$, then ruin is certain. Therefore the company must arrange its liabilities so that $\gamma > \mu$, i.e. its policies must be arranged so that the net income per period exceeds the average claim. Let $\pi(x_0)$ denote the probability of ruin for an initial capital x_0. Then from (90) it follows that

$$\pi(x_0) \simeq e^{-\theta_0 x_0},$$

where $\theta_0 > 0$ since the barrier is now on the negative side of the starting point. From Example 2.10

$$\theta_0 = \frac{2(\gamma - \mu)}{\sigma^2}$$

and therefore

$$\pi(x_0) \simeq \exp\left\{-\frac{2(\gamma-\mu)x_0}{\sigma^2}\right\}. \tag{91}$$

Using this formula we could find, for example, what initial capital a company would have to borrow to achieve a sufficiently small probability of ruin for given γ, μ and σ^2 (see Exercise 15).

The results for the single-barrier case have been derived by letting one of the barriers in the two-barrier case approach infinity. However, Wald's identity in the general form (75) holds for a single absorbing barrier although the values of θ for which it is valid are not quite so general as in the case of two barriers. See Exercise 11. All the results for the single-barrier case may be obtained directly from Wald's identity.

(*vii*) REFLECTING BARRIERS

Suppose now that the barriers at $-b$ and a are reflecting barriers. By this we mean that once the particle crosses a barrier, it instantaneously returns to that barrier and remains there until a step of the appropriate sign allows it to move into the region between the barriers. We assume $X_0 = 0$. The equations defining such a process are

$$X_n = \begin{cases} X_{n-1}+Z_n & (-b < X_{n-1}+Z_n < a), \\ a & (X_{n-1}+Z_n \geqslant a), \\ -b & (X_{n-1}+Z_n \leqslant -b), \end{cases}$$

where $Z_1, Z_2, \ldots$, the steps of the particle, are independent random variables. If the steps are continuously distributed, then after n steps the particle will have a probability distribution consisting of discrete probabilities at $-b$ and a, representing the probabilities that the particle will be located on the respective barriers, and a continuous distribution of probability in the interval $(-b, a)$. In view of this fact it is more convenient to work with the distribution function rather than the density function. Thus let

$$H_n(x) = \text{prob}(X_n \leqslant x).$$

Then $H_n(x)$ is non-decreasing in $-b < x < a$, continuous on the right and

$$\begin{aligned} H_n(x) &= 0 \quad (x < -b), & H_n(-b) &= \text{prob}(X_n = -b), \\ 1-H_n(a-0) &= \text{prob}(X_n = a), & H_n(x) &= 1 \quad (x \geqslant a). \end{aligned} \tag{92}$$

The jump discontinuities of $H_n(x)$ at $-b$ and a represent the discrete probabilities of locating the particle on the respective barriers. To obtain a recurrence relation for $H_n(x)$ we note that if it is given that the particle is at the position y $(-b \leqslant y \leqslant a)$ at time $n-1$ then

$$\text{prob}(X_n \leqslant x | X_{n-1} = y) = \begin{cases} 0 & (x < -b), \\ F(x-y) & (-b \leqslant x < a), \\ 1 & (x \geqslant a), \end{cases}$$

where $F(x)$ is the distribution function of Z_n, the nth step. Hence $H_n(x) = 0$ $(x < -b)$, $H_n(x) = 1$ $(x \geqslant a)$, whereas for $-b \leqslant x \leqslant a$

$$\text{prob}(X_n \leqslant x) = H_n(x) = \int_{-b-0}^{a+0} F(x-y)\, dH_{n-1}(y).$$

Integrating by parts on the right-hand side, we obtain, assuming $F'(x) = f(x)$,

$$\begin{aligned} H_n(x) &= F(x-a)\, H_{n-1}(a+0) - F(x+b)\, H_{n-1}(-b-0) \\ &\quad + \int_{-b}^{a} H_{n-1}(y)\, f(x-y)\, dy \\ &= F(x-a) + \int_{-b}^{a} H_{n-1}(y)\, f(x-y)\, dy \quad (-b \leqslant x < a), \end{aligned} \tag{93}$$

in virtue of (92). As in the case of the simple random walk we expect an equilibrium situation to become established after a long time has elapsed, i.e. as $n \to \infty$, $H_n(x) \to H(x)$, the equilibrium distribution function. Letting $n \to \infty$ in (93), we see that $H(x)$ must satisfy the integral equation

$$H(x) = F(x-a) + \int_{-b}^{a} H(y) f(x-y)\, dy \quad (-b \leqslant x < a). \tag{94}$$

If the steps are not continuously distributed then (94) holds in the form

$$H(x) = F(x-a-0) + \int_{-b}^{a} H(y)\, dF(x-y) \quad (-b \leqslant x < a). \tag{95}$$

In particular, if the steps Z_n have a discrete distribution $\{f_j\}$ over the integers, then there will be an equilibrium probability distribution $\{h_k\}$ over the integers $k = -b, -b+1, \ldots, a$. Such a process is a Markov chain with discrete states in discrete time and is more appropriately discussed in the context of the next chapter.

The equilibrium distribution has the properties (a) that it is independent of the point at which the random walk starts, since the known quantities in the equation (94) are a, b and $f(x)$ and the only unknown is $H(x)$, and (b) that if the initial position of the particle is random with the probability distribution $H(x)$, then at each subsequent stage its distribution is $H(x)$, i.e. if we set $n = 1$ in (93) and $H_0(x) = H(x)$ then we see that $H_1(x) = H(x)$, and similarly $H_n(x) = H(x)$ for all $n \geqslant 1$.

It is not known in general how to obtain a solution in reasonably explicit terms to the equation (94) and thus it is not possible in general to determine explicitly the equilibrium distribution. However, if one of the barriers is moved to infinity, e.g. if we set $b = 0$, $a = \infty$ then (94) gives that $H(x) = 0$ $(x < 0)$ and, for $x \geqslant 0$,

$$H(x) = \int_{0}^{\infty} H(y) f(x-y)\, dy, \tag{96}$$

an equation of the *Wiener–Hopf* type which can be solved explicitly in certain special cases; see, for example, Titchmarsh (1948, p. 339) and Noble (1958). It has been shown by Lindley (1952) that there is a unique probability distribution $H(x)$ satisfying (96) if $E(Z_n) < 0$ (negative drift) and no solution if $E(Z_n) \geqslant 0$ (positive or zero drift). This is plausible by the same sort of general argument which was used for the simple random walk in Section 2.2(vi). Spitzer (1957) has shown that the equilibrium distribution $H(x)$ exists under somewhat more general conditions on the density function $f(x)$ and has given a rather complicated explicit expression for the characteristic function of $H(x)$.

Example 2.14. *Waiting time in the single-server queue.* We give here an example of a process which, on close examination, is seen to have the structure of a random walk although initially it appears to be quite unlike one. The process in question is the waiting time process for the queue with a single server and the formulation is due to Lindley (1952).

Suppose that the queue is initially empty and that the server serves the customers in order of arrival. Customers arrive at times T_0, T_0+T_1, $T_0+T_1+T_2$, ... Thus T_n is the time interval between the arrival of the nth and $(n+1)$th customer. We have here a process in continuous time but we use the method of imbedding mentioned in Example 1.4 and examine the process only at the time instants when successive customers commence their service. Essentially, our discrete time variable now represents the serial number of customers in order of arrival. We assume that the inter-arrival times T_n form a sequence of mutually independent, identically distributed random variables. Let S_n denote the service-time of the nth customer and we assume that these also form a sequence of mutually independent, identically distributed random variables, independent of the T_n.

Let W_n denote the waiting time of the nth customer so that W_n is the time that elapses between the arrival of the nth customer and the commencement of his service. If the queue is empty when the nth customer arrives, then $W_n = 0$. Finally let $Z_n = S_n - T_n$. Then we shall show that

$$W_{n+1} = \begin{array}{ll} W_n+Z_n & (W_n+Z_n > 0), \\ 0 & (W_n+Z_n \leqslant 0). \end{array} \tag{97}$$

This is the equation of a random walk on the positive half-axis with a reflecting barrier at 0.

To prove (97), consider the nth customer. He arrives at time $Y_n = T_0+T_1+\ldots+T_{n-1}$, waits for a time W_n and is served for a time S_n. Thus he completes his service at time $Y_n+W_n+S_n$. If by this time the next customer has already arrived, i.e. if $Y_n+T_n < Y_n+W_n+S_n$, the nth customer now commences his service and

$$Y_n+T_n+W_{n+1} = Y_n+W_n+S_n \quad (Y_n+T_n < Y_n+W_n+S_n),$$

$$\text{i.e.} \quad W_{n+1} = W_n+(S_n-T_n) \qquad \{W_n+(S_n-T_n) > 0\}. \tag{98}$$

On the other hand if the $(n+1)$th customer has not arrived by the time that the nth customer completes his service, i.e. if $Y_n+T_n \geqslant Y_n+W_n+S_n$, then the queue will be empty when the $(n+1)$th customer does arrive and he will not have to wait. Thus

$$W_{n+1} = 0 \quad (Y_n+T_n \geqslant Y_n+W_n+S_n),$$

$$\text{i.e.} \qquad W_{n+1} = 0 \quad \{W_n+(S_n-T_n) \leqslant 0\}. \tag{99}$$

Equations (98) and (99) are the same as (97).

Thus the structure of the waiting time process is that of a random walk with a reflecting barrier. This illustrates the usefulness of trying to identify the process under consideration, or some aspect of it, with a common or more basic process. It follows from the results on the random walk that W_n will have an equilibrium distribution if $E(S_n - T_n) < 0$, i.e. if $E(S_n) < E(T_n)$ or, in words, the mean service-time is less than the mean arrival interval. This means that the queue will be stable if the average rate at which customers are served is greater than the average rate at which they arrive. On the other hand if $E(S_n) \geqslant E(T_n)$ then W_n will not have an equilibrium distribution as $n \to \infty$; as more customers arrive their waiting times will tend to become longer and not to settle down to a state of statistical equilibrium.

(*viii*) EQUIVALENCE BETWEEN ABSORBING AND REFLECTING BARRIER PROBLEMS

Hitherto we have considered separately the cases of absorbing and reflecting barriers. We shall show now that these two types of random walk are in a sense mathematically equivalent. In particular, we shall show that if we have determined the absorption probability at one of the barriers (and hence at the other) for an arbitrary starting point then we can immediately write down the equilibrium distribution for the corresponding reflecting barrier situation. More specifically, suppose we have absorbing barriers at $-a$ and a (we can without loss of generality choose the origin midway between the barriers) and let $Q(x)$ denote the probability that absorption occurs at $-a$ when the walk starts at x $(-a \leqslant x \leqslant a)$. Let $H(x)$ denote the equilibrium distribution function when the barriers are reflecting. Then we shall show that

$$H(x) = Q(-x), \tag{100}$$

a result due to Lindley (1959).

Consider first the case where the steps are continuously distributed with p.d.f. $f(x)$. Suppose that the barriers are reflecting. Let $H_n(x)$ $(-a \leqslant x \leqslant a)$ be the distribution function of the particle's position after n steps when initially it is at the upper barrier. Then

$$H_0(x) = \begin{matrix} 1 & (x \geqslant a), \\ 0 & (x < a), \end{matrix} \tag{101}$$

and $H_n(x)$ satisfies equation (93), namely

$$H_n(x) = F(x-a) + \int_{-a}^{a} H_{n-1}(y) f(x-y)\, dy \quad (n \geqslant 1,\ -a \leqslant x < a). \tag{102}$$

Next suppose that the barriers are absorbing and for a walk starting at x let $Q_n(x)$ be the probability that absorption occurs at the barrier $-a$

at or before the nth step. Then in order to find a recurrence relation for $Q_n(x)$ we note that absorption can occur in two mutually exclusive ways: either it occurs at the first step with probability $F(-x-a)$ or the particle moves to y $(-a < y < a)$ at the first step and absorption occurs at one of the steps $2, \ldots, n$. Thus

$$Q_n(x) = F(-x-a) + \int_{-a}^{a} f(y-x)\, Q_{n-1}(y)\, dy \quad (n \geqslant 1,\ -a < x < a),$$

and

$$Q_0(x) = \begin{matrix} 1 & (x = -a), \\ 0 & (x > -a). \end{matrix}$$

Rewriting in terms of $-x$, we have

$$Q_0(-x) = \begin{matrix} 1 & (x = a), \\ 0 & (x < a), \end{matrix} \tag{103}$$

and

$$Q_n(-x) = F(x-a) + \int_{-a}^{a} f(x-y)\, Q_{n-1}(-y)\, dy \quad (n \geqslant 1,\ -a < x < a).$$

From (101), (102) and (103) we see that $H_n(x)$ and $Q_n(-x)$ satisfy the same initial conditions and the same recurrence relation. It follows that for all n

$$Q_n(-x) = H_n(x) \tag{104}$$

and the result (100) follows on taking the limit as $n \to \infty$.

Some care is needed when interpreting the results (100) and (104) for $x = a$ and $x = -a$ since $H(x)$ has discontinuities at these points. We can overcome the difficulty by interpreting (100) and (104) as holding for $-a < x < a$. The discontinuities of $H_n(x)$ at $-a$ and a will then be given by

$$\begin{aligned} H_n(-a+0) &= \lim_{x \to a-0} Q_n(x), \\ 1 - H_n(a-0) &= 1 - \lim_{x \to -a+0} Q_n(x). \end{aligned} \tag{105}$$

When the steps of the random walk are discrete over the integers then we must define $Q_n(x)$ in terms of absorbing barriers at $a+1$ and $-a$ and $H_n(x)$ (right-continuous) in terms of reflecting barriers at a and $-a$. Then the results (100) and (104) hold for $x = -a, \ldots, a$.

Example 2.15. *Simple random walk.* In the simple random walk with reflecting barriers at $-a$ and a the equilibrium probability distribution is by equation (47)

$$\pi_k = \frac{1-\left(\frac{p}{q}\right)}{1-\left(\frac{p}{q}\right)^{2a+1}}\left(\frac{p}{q}\right)^{a+k} \quad (k = -a,\ldots,a; p \neq q).$$

Now define

$$H(x) = \sum_{k=-a}^{x} \pi_k$$

$$= \frac{1-\left(\frac{p}{q}\right)^{a+1+x}}{1-\left(\frac{p}{q}\right)^{2a+1}} \quad (x = -a,\ldots,a).$$

Now consider the simple random walk with absorbing barriers at $-a$ and $a+1$ and let $Q(x)$ denote the probability that absorption occurs at $-a$. Then from the result (20) we have

$$Q(x) = \frac{1-\left(\frac{p}{q}\right)^{a+1-x}}{1-\left(\frac{p}{q}\right)^{2a+1}} \quad (x = -a,\ldots,a)$$

and hence $Q(-x) = H(x)$, illustrating the general result (100) in this instance.

Example 2.16. *Mixture of two exponential distributions.* Consider the random walk discussed in Example 2.12 where each step is the sum of independent positive and negative exponential variates. Adapting equation (89) to absorbing barriers at $-a$ and a and initial position x we have

$$Q(x) = \begin{cases} \dfrac{1-\left(\dfrac{\lambda}{\nu}\right)e^{(a-x)(\lambda-\nu)}}{1-\left(\dfrac{\lambda}{\nu}\right)^2 e^{2a(\lambda-\nu)}} & (\lambda \neq \nu), \\[2ex] \dfrac{a+\lambda^{-1}-x}{2(a+\lambda^{-1})} & (\lambda = \nu). \end{cases} \tag{106}$$

This is obtained from (89) by replacing a and b by $a-x$ and $b+x$ respectively and then putting $Q(x) = 1-p_a$. The general result (100) now enables us to obtain the equilibrium distribution $H(x)$ when the barriers are reflecting by simply writing $H(x) = Q(-x)$. The discrete probabilities of locating the particle on the lower and upper barriers in the equilibrium situation are given respectively by $H(-a)$ and $1-H(a)$. The continuous distribution between the barriers is thus a truncated exponential one if $\lambda \neq \nu$ and uniform if $\lambda = \nu$.

It might seem that the approximation afforded by Wald's identity in the case of absorbing barriers ought to give an approximation to the equilibrium distribution function in the case of reflecting barriers. However, Wald's approximation holds when the barriers are far from the starting point and if we use it in (100) we can only expect to approximate to the equilibrium distribution function $H(x)$ when x is roughly midway between the barriers. But we get no approximation to the discrete probabilities on the barriers and these are important in the reflecting barrier case.

For a single barrier there is also a duality between the absorbing and reflecting barrier cases. Consider a random walk on the non-negative half-line with identically distributed steps $Z_1, Z_2, \ldots$ and suppose that zero is a *reflecting* barrier. If $E(Z_n) < 0$ then the equilibrium distribution $H(x)$ exists and satisfies equation (96) where $f(x)$ is the p.d.f. of the steps Z_n. Now consider a random walk with the same $f(x)$ on the non-positive half-line starting at $-x$ $(x > 0)$ with zero as an *absorbing* barrier. Let $Q(x)$ be the probability that absorption at zero ultimately occurs. Then

$$H(x) = 1-Q(x) \quad (x > 0), \tag{107}$$

and

$$H(0+) = \lim_{x \to 0+} \{1-Q(x)\}$$

is the discrete probability at zero in the reflecting barrier case. The proof of (107), which is similar to that of (100), is left as an exercise for the reader.

Equation (107) holds quite generally, even without assumptions concerning the continuity of the steps, provided we define $H(x)$ as being left-continuous, i.e. define $H(x) = \text{prob}(X_\infty < x)$, not $\text{prob}(X_\infty \leqslant x)$.

Example 2.17. *Waiting time in the single-server queue.* We can use the result (107) to examine the equilibrium distribution of the waiting time in a single-server queue, Example 2.14. We shall show that when the service-times are exponentially distributed then for any distribution of inter-arrival times the equilibrium distribution of the waiting time is also an exponential distribution with a discrete probability at zero, provided of course that the queue is stable, i.e. the mean service-time is less than the mean inter-arrival time. This result is due to Smith (1953).

In the notation of Example 2.14, let

$$E(e^{-\theta S_n}) = \frac{\beta}{\beta+\theta}$$

$$E(e^{-\theta T_n}) = g^*(\theta)$$

where we assume that

$$\frac{1}{\beta} < -g^{*\prime}(0),$$

i.e. that $E(S_n) < E(T_n)$. Let $f^*(\theta)$ be the m.g.f. of the distribution of $Z_n = S_n - T_n$, i.e.

$$f^*(\theta) = \frac{\beta}{\beta+\theta} g^*(-\theta), \tag{108}$$

under the assumptions of Example 2.14 concerning the independence of the S_n and T_n. If we consider a random walk on the non-negative half-line whose steps have a distribution with m.g.f. (108), and with a reflecting barrier at zero, then we showed in Example 2.14 that the equilibrium distribution of this random walk is the equilibrium waiting time distribution of the queue; denote this distribution by $H(x)$.

In order to find $H(x)$ we consider a dual absorbing barrier problem, i.e. we consider a random walk starting at $-x$ with an absorbing barrier at zero and suppose the distribution of the steps has m.g.f. (108). Equivalently we may assume that the starting point is 0 and the absorbing barrier is at x. Let $Q(x)$ be the probability that absorption ultimately occurs. Then using Wald's identity in the form (75), we have

$$Q(x)\, E_A[e^{-\theta X_N}\{f^*(\theta)\}^{-N}] = 1. \tag{109}$$

By an argument similar to that used in Example 2.12 it will be seen that $X_N - x$, the excess over the barrier, must have the same exponential distribution as the positive component of the Z_n, conditional on crossing the barrier at all, i.e.

$$E_A\{e^{-\theta(X_N - x)}\} = \frac{\beta}{\beta+\theta},$$

independently of N. Thus

$$E_A\{e^{-\theta X_N}\} = \frac{\beta\, e^{-\theta x}}{\beta+\theta}$$

and (109) becomes

$$Q(x)\frac{\beta\, e^{-\theta x}}{\beta+\theta} E_A[\{f^*(\theta)\}^{-N}] = 1. \tag{110}$$

Now let θ_0 be the real non-zero root of $f^*(\theta) = 1$, i.e. of the equation

$$g^*(-\theta) = 1 + \frac{1}{\beta}\theta. \tag{111}$$

This root always exists since, for real θ, $g^*(-\theta)$ is a convex function satisfying $g^*(0) = 1$ and $g^{*\prime}(0) > 1/\beta$. Thus the curve $g^*(-\theta)$ and the line $1+\theta/\beta$ must intersect at some value $\theta_0 < 0$. Setting $\theta = \theta_0$ in (110) we obtain

$$Q(x) = \frac{\beta+\theta_0}{\beta} e^{\theta_0 x} \quad (\theta_0 < 0).$$

Hence by the result (107), the equilibrium distribution of the waiting time is

$$H(x) = 1-Q(x) = 1-\frac{\beta+\theta_0}{\beta} e^{\theta_0 x} \quad (x > 0),$$

an exponential distribution with the discrete probability $H(0) = -\theta_0/\beta$ at $x = 0$.

2.4. Further topics

We now consider some further topics on the subject of the random walk. These are more advanced and may be omitted in a first reading. In any case, much of this section is of a fairly general and discursive nature rather than detailed and mathematical.

(*i*) THE MULTIDIMENSIONAL RANDOM WALK

So far in this chapter we have been concerned only with the random walk on a line, i.e. the one-dimensional random walk. From the point of view of the motion of a particle it is of course more natural to consider a random walk on a plane or in three-dimensional Euclidean space or even in non-Euclidean spaces, for example on a sphere. We shall confine our brief remarks to Euclidean space and for the sake of generality to the case of m dimensions.

When we leave one dimension the possibilities become much richer and the mathematical problems much more difficult, especially so when there are boundaries present. Consider for example a random walk in the plane. A particle starts at the origin and undergoes steps $\mathbf{Z}_1$, $\mathbf{Z}_2$, ... where the $\mathbf{Z}_i$ are independent two-dimensional random vectors with a given bivariate distribution. After n steps the position of the particle is given by the two-dimensional vector

$$\mathbf{X}_n = \mathbf{Z}_1+\ldots+\mathbf{Z}_n. \tag{112}$$

If we try to generalize from one dimension the idea of absorbing barriers we find that any region Ω containing the origin may be defined as a non-absorbing region, while the remainder $\bar{\Omega}$ of the plane is an absorbing region. The possible kinds of boundaries are much richer. Even the simplest boundaries, e.g. a square or rectangle, produce intractable mathematical problems if the steps are other than of the simplest kind. Some boundaries reduce the problem to the one-dimensional case. For example, for continuous steps in the plane with a single straight line absorbing boundary we may take axes perpendicular and parallel to the boundary. From the point of view of the time to absorption, only motion along the axis perpendicular to the boundary is relevant and this is a one-dimensional problem. This is not so, however, if the random walk takes place on a lattice unless the boundary is parallel to one of the axes.

In m dimensions we can generalize directly the results given by the central limit theorem and the laws of large numbers. The central limit theorem gives an approximation to the distribution of the position vector after a large number of steps. If the individual steps are independent, identically distributed m-dimensional vectors $\mathbf{Z}_1$, $\mathbf{Z}_2$, ... with finite second moments then (112) again gives the position vector $\mathbf{X}_n$ after n steps. By the central limit theorem $\mathbf{X}_n$ has asymptotically the multivariate normal distribution with mean vector $n\boldsymbol{\mu}$ and dispersion matrix $n\boldsymbol{\Sigma}$ where $\boldsymbol{\mu}$ is the mean and $\boldsymbol{\Sigma}$ the dispersion matrix of the individual steps. If the $\mathbf{Z}_i$ have finite mean $\boldsymbol{\mu}$ but not necessarily finite second moments, then according to the strong law of large numbers we have, with probability as close to unity as we please and n_0 sufficiently large,

$$\mathbf{X}_n = n\boldsymbol{\mu} + o(n_0)$$

for all $n \geqslant n_0$, where $o(n_0)$ depends only on n_0 and not on n. More specifically if $\mathbf{X}_n = (X_n^{(1)}, \ldots, X_n^{(m)})$, $\boldsymbol{\mu} = (\mu^{(1)}, \ldots, \mu^{(m)})$ then

$$\text{prob}\left(\left|\frac{X_n^{(1)}}{n} - \mu_1\right| > \epsilon_1, \left|\frac{X_n^{(2)}}{n} - \mu_2\right| > \epsilon_2, \ldots, \left|\frac{X_n^{(m)}}{n} - \mu_m\right| > \epsilon_m \right.$$
$$\left. \text{for all } n \geqslant n_0\right) < \delta$$

for any positive $\epsilon_1, \ldots, \epsilon_m$, δ and n_0 sufficiently large.

We may note also that the identity (67), developed for the case of a one-dimensional random walk with absorbing barriers, extends to the random walk in several dimensions with an absorbing region. Suppose that we consider the random walk

$$\mathbf{X}_n = \mathbf{X}_0 + \mathbf{Z}_1 + \ldots + \mathbf{Z}_n; \tag{113}$$

where $\mathbf{X}_0$ is a given point in m-dimensional space (the initial position) and the steps $\mathbf{Z}_i$ are mutually independent, identically distributed m-dimensional random variables. We denote the non-absorbing region of m-dimensional space by Ω, where we suppose that $\mathbf{X}_0$ is a point in Ω, while the complementary region $\bar{\Omega}$ is the absorbing region. Let N denote the time at which the walk first enters $\bar{\Omega}$. Then if $F_n(d\mathbf{x})$ denotes the probability

$$F_n(d\mathbf{x}) = \text{prob}(\mathbf{X}_1 \in \Omega, \ldots, \mathbf{X}_{n-1} \in \Omega, \mathbf{X}_n \in d\mathbf{x}) \tag{114}$$

we have

$$E(e^{-\boldsymbol{\theta}'\mathbf{X}_N} s^N) = 1-\{1-sF^*(\boldsymbol{\theta})\}\left\{\sum_{n=0}^{\infty} s^n \int_{\Omega} e^{-\boldsymbol{\theta}'\mathbf{x}} F_n(d\mathbf{x})\right\}, \quad (115)$$

where $F^*(\boldsymbol{\theta})$ is the m.g.f. of the $\mathbf{Z}_i$. The result may be proved in a similar manner to the one-dimensional identity (67).

(*ii*) THE RESULTS OF CHUNG AND FUCHS

We have seen in Section 2.2(iii) that a particle undergoing a simple symmetrical random walk in one dimension is certain to reach any given position. In the next chapter on Markov chains we describe such behaviour by saying that the process is *recurrent.* Chung and Fuchs (1951) investigated the corresponding problem for general random variables in one, two and three dimensions. If

$$\mathbf{X}_n = \mathbf{Z}_1+\ldots+\mathbf{Z}_n$$

is a random walk in one, two or three dimensions then the value $\mathbf{b}$ is said to be *possible* if, for any $\epsilon > 0$, the region $|\mathbf{X}-\mathbf{b}| < \epsilon$ can be reached with non-zero probability. Here $|\mathbf{Y}|$ denotes the maximum of the absolute values of the components of $\mathbf{Y}$. The value $\mathbf{b}$ is said to be *recurrent* if the region $|\mathbf{X}-\mathbf{b}| < \epsilon$ is certain to be reached. Chung and Fuchs showed in the one-dimensional case that if

$$E(|Z_i|) < \infty, \qquad E(Z_i) = 0,$$

then every possible value is recurrent and in the two-dimensional case that if

$$E(\mathbf{Z}_i^2) < \infty, \qquad E(\mathbf{Z}_i) = 0,$$

where $\mathbf{Z}_i^2$ is the square length of the vector $\mathbf{Z}_i$, then again every possible value is recurrent. However in the three-dimensional random walk no value is recurrent.

(*iii*) SPITZER'S IDENTITY

Returning to the one-dimensional random walk, we consider the distribution of the maximum distance reached by a particle, say in the positive direction, during n steps. Accordingly let

$$U_n = \max(0, X_1, \ldots, X_n).$$

In Section 2.2(iv) we considered the limiting distribution of U_n for the simple random walk. Using purely combinatorial methods, Spitzer (1956) established a relation between the sequence of distributions of U_n and the sequence of distributions of the unrestricted sums X_n. Let

$$V_n = \max(0, X_n).$$

Then Spitzer proved, for an arbitrary distribution of the steps, the identity

$$\sum_{n=0}^{\infty} s^n E(e^{-\theta U_n}) = \exp\left\{\sum_{n=1}^{\infty} \frac{s^n}{n} E(e^{-\theta V_n})\right\}, \tag{116}$$

and various results concerning the limiting distribution of U_n and the number of positive X_n's follow from (116). For details the reader is referred to Spitzer's paper. Later authors, for example Wendel (1958), have proved (116) using purely analytical methods.

Bibliographic Notes

The insurance risk problem, Example 2.1, and more general forms of it, have been extensively studied, especially by Scandinavian mathematicians; for a discussion, historical note and further references, see Cramér (1954). Stochastic models for the content of storage systems, such as Example 2.2 are discussed in the monograph of Moran (1959) and the review papers of Gani (1957) and Prabhu (1964). The simple random walk, Section 2.2, is discussed extensively by Feller (1957, Chapter 14) for the case $p+q = 1$. The discussion of the general random walk with absorbing barriers, Section 2.3(iii), follows Miller (1961a). Further discussion of the general random walk and, in particular, the connexion with the Wiener–Hopf equation (96), may be found in the monograph of Kemperman (1961). In recent years, research has been directed towards exploring the connexion between the random walk and potential theory; for a discussion and references see Spitzer (1964).

Exercises

1. Let $F(x)$ be a distribution function with moments μ_1, μ_2', μ_3', ... about the origin. Examine how far it is possible to approximate to $F(x)$ by a discrete distribution over the three points $-l$, 0, l by means of equating moments.

2. Show in Example 2.9 that the time intervals between the arrivals of successive customers and the service times of successive customers each have a geometric distribution.

3. If $a \to \infty$ in Example 2.9 show that there is an equilibrium distribution of queue size if $\alpha < \beta$ and find this distribution.

4. Use the methods of Section 2.2(iii) in Example 2.9 to find the p.g.f. of the server's busy period when $a = \infty$.

5. Consider a simple random walk with $p+q = 1$. The walk starts at j

($0 < j \leqslant a$) where 0 is an absorbing barrier and a a reflecting barrier. Let

$$F_j(s) = \sum_{n=0}^{\infty} s^n f_j^{(n)}$$

be the p.g.f. of N, the time to absorption and let

$$\lambda_1(s), \lambda_2(s) = \frac{1}{2ps}\{1 \pm (1-4pqs^2)^{\frac{1}{2}}\}.$$

Show that

(i) $$F_j(s) = \left(\frac{q}{p}\right)^j \frac{\lambda_1^{a-j+1} - \lambda_2^{a-j+1} - \lambda_1^{a-j} + \lambda_2^{a-j}}{\lambda_1^{a+1} - \lambda_2^{a+1} - \lambda_1^{a} + \lambda_2^{a}}$$

(ii) $$E(N) = \begin{cases} \dfrac{j}{q-p} + \dfrac{p^{a+1}}{q^a(q-p)^2}\left\{1-\left(\dfrac{q}{p}\right)^j\right\} & (p \neq q), \\ j + j(2a-j) & (p = q = \frac{1}{2}) \end{cases}$$

(Weesakul, 1961).

6. Suppose X is a two-sided random variable (i.e. satisfying the condition (56)) with p.d.f. $f(x)$ and d.f. $F(x)$, where $F(x)$ satisfies

$$1 - F(x) = O(e^{-\mu x}) \quad (x \to \infty)$$
$$F(x) = O(e^{\lambda x}) \quad (x \to -\infty)$$

for some $\lambda > 0$, $\mu > 0$ (roughly speaking the distribution of X has exponentially small 'tails'). Show that the integral (57) defining $f^*(\theta)$ converges at least in the interval $-\mu < \theta < \lambda$.

7. Let $X = Y + Z$ where Y is a positive random variable with p.d.f. $\alpha e^{-\alpha y}$ and Z a negative random variable with p.d.f. $\beta e^{\beta z}$. Find the m.g.f. $f^*(\theta)$ of X and show that the interval of convergence of (57) is $-\alpha < \theta < \beta$. Find the roots θ_0 and θ_1 and evaluate $f^*(\theta_1)$, verifying (58). Sketch $f^*(\theta)$.

8. Translate the results of Section 2.3(ii) concerning m.g.f.'s into results for discrete random variables concerning p.g.f.'s by making the substitution $z = e^{-\theta}$, so that, for example, $\theta = 0$ corresponds to $z = 1$.

9. Obtain the inequality (64).

10. Let N denote the time to absorption in a general random walk between two absorbing barriers a and $-b$. Without assuming any conditions on the moments of the steps $Z_1, Z_2, \ldots$, use (62) and (68) to prove that prob$(N > n) = O(\rho^n)$ for some ρ satisfying $0 < \rho < 1$.

11. Using inequalities similar to (63) and (64) show that the substitution $s = \{f^*(\theta)\}^{-1}$ in (67), and hence Wald's identity (75), is valid in the case of a single absorbing barrier at a provided that θ satisfies

$$|f^*(\theta)| > f^*(\theta_1) \quad \text{and} \quad \mathscr{R}(\theta) < \theta_1.$$

12. In the simple random walk starting at 0 with absorbing barriers at $-b$ and a use (20) and (21) to show that the moments of X_N are given by

$$E(X_N^k) = \begin{cases} \dfrac{a^k p^a(p^b-q^b)+(-b)^k q^b(p^a-q^a)}{p^{a+b}-q^{a+b}} & (p \neq q), \\[2ex] \dfrac{ba^k+a(-b)^k}{a+b} & (p = q). \end{cases}$$

Hence, using (85), prove that

$$E(N) = \begin{cases} \dfrac{ap^a(p^b-q^b)-bq^b(p^a-q^a)}{(p-q)(p^{a+b}-q^{a+b})} & (p \neq q), \\[2ex] \dfrac{ab}{p+q} & (p = q). \end{cases}$$

13. Show how Wald's identity may be used to find the exact p.g.f. of N in Example 2.12.

14. Suppose we let $b \to \infty$ in Example 2.12 so that we have a single barrier at a. Show that absorption is certain if $\nu \leqslant \lambda$ and that in this case the p.g.f. of N is

$$E(s^N) = \frac{[\lambda+\nu-\{(\lambda+\nu)^2-4\lambda\nu s\}^{\frac{1}{2}}]}{2\nu}\exp\left[\frac{\lambda-\nu-\{(\lambda+\nu)^2-4\lambda\nu s\}^{\frac{1}{2}}}{2}\right].$$

Find the probability of absorption when $\nu > \lambda$.

15. A man wishing to start a certain type of insurance business assesses that he has m prospective policy holders. From past statistical data on policies of this type the yearly claim per policy can be assumed to have mean γ and standard deviation σ. Running expenses are δ per policy per annum and interest charges on borrowed capital are 100α per cent per annum. If the man is willing to accept a probability β of ruin then by using (91), show that in order to minimize the average annual premium per policy he must borrow the initial sum

$$x_0 = \sigma\left\{\frac{m}{2\alpha}\log\left(\frac{1}{\beta}\right)\right\}^{\frac{1}{2}}.$$

Find the minimum average annual premium.