Birth and Death Process

$\lambda \rightarrow$ birth rate (may depend upon the population already present)

$\therefore \lambda \rightarrow \lambda_n$, where n = current population size.

Similarly μ (death rate) $\rightarrow \mu_n$

Consider a stochastic process with some current state denoted by E_n at time 't' (meaning that at 't', population is 'n').

The process will be called a birth-death process if:

(i) State (population) changes are allowed by only '1' at a time ie. from E_n to E_{n+1} or E_n to E_{n-1} and from E_0 to E_1 (not E_{-1} as impractical)

(ii) The probability of state change from E_n to E_{n+1} b/w t to $t+\Delta t$, is $\underline{\lambda_n \Delta t + o(\Delta t)}$ and from E_n to E_{n-1} is $\underline{\mu_n \Delta t + o(\Delta t)}$.

The probability of more than one transition in time interval 't' to '$t+\Delta t$' is $\underline{o(\Delta t) \approx 0}$.

Let $P_n(t) = P[X(t) = n]$ be the prob. that the system is in state E_n (or has a current population of 'n') at time 't'.

If $P_n(t+\Delta t)$, $n \geq 1$ is the prob. that system is in state E_n at time '$t+\Delta t$', then this can happen in the following ways:

(I) syst. was in state E_n at t, and no birth/no death occured b/w t & $t+\Delta t$, with a probability $\underbrace{P_n(t)}_{\text{prob.}[X(t)=n]} \cdot \underbrace{[1 - \lambda_n \Delta t - \mu_n \Delta t]}_{\text{prob. [no birth/death in } \Delta t]}$

(II) At 't', state was E_{n-1} and one birth occurred in Δt to change the state to E_n, with prob $= P_{n-1}(t) \cdot \underbrace{\lambda_{n-1}}_{\text{birth rate when the population was } n-1.} \Delta t$

(III) State was E_{n+1} at 't' and one death occurred to make it E_n at '$t+\Delta t$', with prob. $= p_{n+1}(t).\underbrace{\mu_{n+1}}_{\text{death rate when population was } n+1.}\Delta t$

(IV) The state was already E_n at 't' but 2 or more changes (births/deaths) occurred to revert the state back to E_n with infinitesimal prob $= o(\Delta t)$

So, to find the prob. that system is in state E_n, we add prob. of I, II, III, IV

$$\Rightarrow \quad p_n(t+\Delta t) = p_n(t).[1-\lambda_n \Delta t - \mu_n \Delta t] + p_{n-1}(t).\lambda_{n-1}\Delta t + p_{n+1}(t).\mu_{n+1}\Delta t + o(\Delta t)$$

$$\Rightarrow \frac{P_n(t+\Delta t) - P_n(t)}{\Delta t} = -(\lambda_n + \mu_n)P_n(t) + \lambda_{n-1}.P_{n-1}(t) + \mu_{n+1}.P_{n+1}(t)$$

$$\lim_{\Delta t \to 0} \frac{P_n(t+\Delta t) - P_n(t)}{\Delta t} = \boxed{\frac{dP_n(t)}{dt} = -(\lambda_n + \mu_n)P_n(t) + \lambda_{n-1}.P_{n-1}(t) + \mu_{n+1}.P_{n+1}(t)}$$

$\hookrightarrow$ Eq (A)

For $\underline{n=0} \Rightarrow \dfrac{dP_0(t)}{dt} = -(\lambda_n + \underbrace{0}_{\text{death rate 0 as no person to die}})P_0(t) + \mu_1 P_1(t) + \xcancel{\lambda_{-1} P_{-1}(t)}$ not feasible

$P_i(0) = 1 \} \rightarrow$ at time '0', population is 'i' (given, $\therefore$ certain).

$P_j(0) = 0$, if $i \neq j \} \rightarrow$ meaning that population cannot change without any time passing simultaneously.

Pure Birth Process

We take $\lambda_n = \lambda$ (constant arrival/birth rate) $\forall n$.

since it is a pure birth process, $\mu_n = 0$. (no deaths).

Eq (A) becomes:

$$\frac{dP_n(t)}{dt} = -\lambda P_n(t) + \lambda P_{n-1}(t), \quad n \geq 1$$

$$\frac{dP_0(t)}{dt} = -\lambda P_0(t) + \xcancel{\lambda P_{-1}(t)}$$ not feasible.

The solution of these equations, satisfying the initial conditions: $P_i(0) = 1$ and $P_j(0) = 0, i \neq j$ is:

$$\boxed{P_n(t) = e^{-\lambda t} \cdot \frac{(\lambda t)^n}{n!}}, \quad n \geq 0, t \geq 0.$$

Hence, pure birth process is a Poisson process with avg. arrival rate = birth rate λ.

Pure Death Process

We assume that system starts with 'N' people at $t = 0$ and no births are allowed, ie. $\lambda_n = 0$ (since it is a pure death process).
death rate $= \mu_n = \mu \ \forall n$ (constant death rate).

Eq^n (A) becomes:

$$\frac{dP_N(t)}{dt} = -\mu P_N(t)$$

$$\frac{dP_n(t)}{dt} = -\mu P_n(t) + \mu P_{n+1}(t), \quad \underline{0 < n < N}$$

$$\frac{dP_0(t)}{dt} = \mu P_1(t)$$ } → Meaning 1 death occurred when population was 1, to reach the final state of 0 people.

with initial conditions:

$P_N(0) = 1$ } → at $t=0$, we started off with N people

$P_k(0) = 0$ } → without any passage of time, it is impossible for the population to change from N to k. $(0 < k < N)$.

The solution to this system is a truncated poisson distribution, with rate μ:

$$P_n(t) = \frac{(\mu t)^{N-n} \cdot e^{-\mu t}}{(N-n)!}, \quad n = 1, 2 \ldots N.$$

$$P_0(t) = 1 - \sum_{n=1}^{N} P_n(t). \quad \left(\text{since } \sum_{n=0}^{N} P_n(t) = 1\right)$$

Steady state solution of Birth & Death Process.

At steady state ($t=\infty$), we assume $\frac{d\,p_n(t)}{dt}=0$. Also,

at $t=\infty$, we consider that the probability $p_n(t)$ becomes time independent and we set $p_n = \lim_{t\to\infty} p_n(t)$, $\forall n$.

So, Eq^n (A) becomes $\Rightarrow$

$$0 = -(\lambda_n+\mu_n)p_n + \lambda_{n-1} p_{n-1} + \mu_{n+1} p_{n+1}, \quad n\geq 1 \quad \text{— (B)}$$

and $$0 = -\lambda_0 p_0 + \mu_1 p_1, \quad n=0 \quad \text{— (C)}$$

Rearranging (B) $\Rightarrow$ $\lambda_n p_n - \mu_{n+1} \cdot p_{n+1} = \lambda_{n-1} p_{n-1} - \mu_n \cdot p_n$

similarly put $n \to n-1 \Rightarrow$ $\lambda_{n-1} p_{n-1} - \mu_n p_n = \lambda_{n-2} p_{n-2} - \mu_{n-1} p_{n-1}$

$$\therefore \lambda_n p_n - \mu_{n+1} p_{n+1} = \lambda_{n-1} p_{n-1} - \mu_n \cdot p_n = \lambda_{n-2} \cdot p_{n-2} - \mu_{n-1} p_{n-1} = \cdots$$

so on

$$= \lambda_0 p_0 - \mu_1 p_1 = 0 \quad \text{(from (C))}$$

$$\therefore \lambda_n p_n - \mu_{n+1} \cdot p_{n+1} = 0 \Rightarrow \boxed{p_{n+1} = \frac{\lambda_n}{\mu_{n+1}} p_n}, \quad n\geq 0$$

or $$\boxed{p_n = \frac{\lambda_{n-1}}{\mu_n} \cdot p_{n-1}}, \quad n \geq 1$$

This gives $$\boxed{p_n = \frac{\lambda_0 \lambda_1 \lambda_2 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n} \cdot p_0 = p_0 \prod_{i=0}^{n-1}\left(\frac{\lambda_i}{\mu_{i+1}}\right)}, \quad n\geq 1$$

Also, since sum of prob. distr. $=1 \Rightarrow \sum_{n=0} p_n = 1 \Rightarrow$

$$p_0 = \frac{1}{1+\sum_{n\geq 1} \prod_{i=0}^{n-1}\left(\frac{\lambda_i}{\mu_{i+1}}\right)}.$$