M303 Assignment-3

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① Given:

System with two components (identical)

Probability of failing $= p$

Probability of being repaired $= r$

X_n: Number of components in operation at n

$\{X_n \mid n = 0, 1, \ldots\} \rightarrow$ DTMC

State space, $I = \{0, 1, 2\}$

For the transition matrix P, we can obtain the entries as follows:

$p_{00} = (1-r)^2, \quad p_{01} = 2r(1-r), \quad p_{02} = r^2$

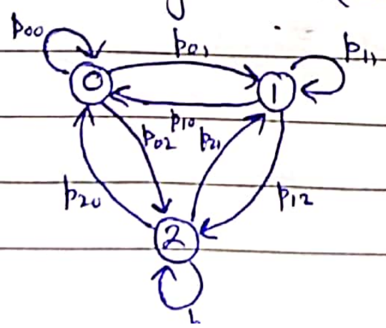
$p_{10} = p(1-r), \quad p_{11} = pr + (1-p)(1-r), \quad p_{12} = (1-p)r$

$p_{20} = p^2, \quad p_{21} = 2p(1-p), \quad p_{22} = (1-p)^2$

Hence,

$$P = \begin{bmatrix} (1-r)^2 & 2r(1-r) & r^2 \\ p(1-r) & pr+(1-p)(1-r) & (1-p)r \\ p^2 & 2p(1-p) & (1-p)^2 \end{bmatrix}$$

and the state diagram is:



It is apparent from the state diagram that the Markov Chain is irreducible, aperiodic and positive recurrent.
Hence, steady state distribution exists.

For the steady state distribution,

$$\pi P = \pi$$

where $\pi = [\pi_0 \ \pi_1 \ \pi_2] \rightarrow$ row vector

and $\sum_{i=0}^{2} \pi_i = 1$

Using the above balance equations and the normalization conditions, the steady state distribution can be obtained.

(2) Given:

Computer system with 3 states:

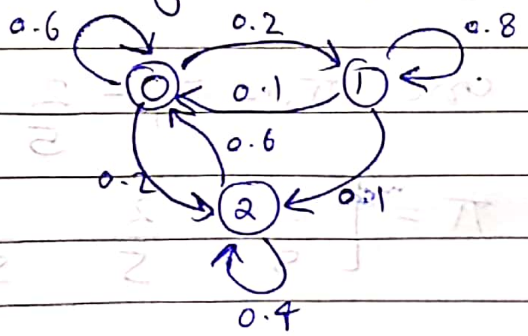
Busy (0), Idle (1), Repair (2)

System $\longrightarrow$ DTMC

and Transition matrix:

$$P = \begin{bmatrix} 0.6 & 0.2 & 0.2 \\ 0.1 & 0.8 & 0.1 \\ 0.6 & 0.0 & 0.4 \end{bmatrix}$$

The state diagram is:



The given chain is irreducible, that is, ~~Ix~~ $Z=\{0,1,2\}$ is a minimal closed set of states. It is easy to see as for any proper subset S of Z, $\exists\ i \in S$, such that

$$\sum_{j\in S} p_{ij} \neq 1$$

Hence, As the chain is finite and irreducible, So, it is recurrent (positive).
Hence, steady state distribution exists.
So, solving

$$\pi P = \pi$$

$$\text{and} \quad \sum_{j \in Z} \pi_j = 1$$

we have the following equations:

$$\Rightarrow \quad 6\pi_0 + \pi_1 + 6\pi_2 = 10\pi_0$$

$$2\pi_0 + 8\pi_1 = 10\pi_1$$

$$2\pi_0 + \pi_1 + 4\pi_2 = 10\pi_2$$

$$\pi_0 + \pi_1 + \pi_2 = 1$$

On solving, $\pi_0 = \pi_1 = 2\pi_2$

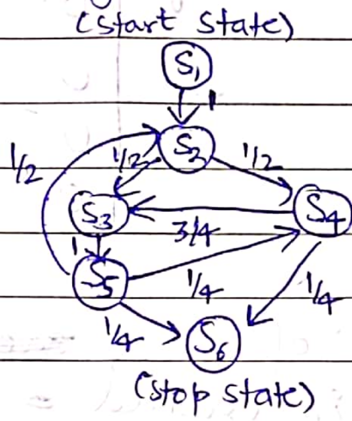
So, $2\pi_2 + 2\pi_2 + \pi_2 = 1 \Rightarrow \pi_2 = \frac{1}{5}$

and $\pi_0 = \pi_1 = \frac{2}{5}$

So, $\pi = \begin{bmatrix} \frac{2}{5} & \frac{2}{5} & \frac{1}{5} \end{bmatrix}$

③ <u>Given:</u>

Transition graph:



So, Transition matrix:

$$P = \begin{array}{c} \begin{array}{cccccc} & & & Q & & R \end{array} \\ \left[\begin{array}{ccccc|c} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cancel{0}1 & 0 \\ 0 & 0 & 3/4 & 0 & 0 & 1/4 \\ 0 & 1/2 & 0 & 1/4 & 0 & 1/4 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \\ \begin{array}{cccccc} & & O & & & I \end{array} \end{array}$$

So, $Q = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 3/4 & 0 & 0 \\ 0 & 1/2 & 0 & 1/4 & 0 \end{bmatrix}$

and $(I-Q) = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1/2 & -1/2 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & -3/4 & 1 & 0 \\ 0 & -1/2 & 0 & -1/4 & 1 \end{bmatrix}$

So, fundamental matrix,

$$M = (I-Q)^{-1} = \frac{1}{6}\begin{bmatrix} 6 & 13 & 14 & 10 & 14 \\ 0 & 13 & 14 & 10 & 14 \\ 0 & 8 & 14 & 8 & 16 \\ 0 & 6 & 12 & 12 & 12 \\ 0 & 8 & 10 & 8 & 16 \end{bmatrix}$$

So, Average no. of times ~~each~~ vertex S_j (transient) is visited $= m_{ij}$

So, for S_1, $m_{11} = 1$

for S_2, $m_{12} = 13/6 = 2.167$

for S_3, $m_{13} = \cancel{14}/6 = 2.333$

for S_4, $m_{14} = 10/6 = 1.667$

for S_5, $m_{15} = 14/6 = 2.333$

(Even for S_6, $m_{16} = 1$, obvious)

Now, execution time for vertex S_i is $t_i = 2i + 1$

Hence, Average execution time

$$= \sum_{j=1}^{5} m_{ij} t_j + t_6$$

$$= 3 + (2.167)(5) + (2.333)(7) + (1.667)(9) + (2.333)(11) + 13$$

$$= 3 + 10.835 + 16.331 + 17.973 + 25.663 + 13$$

$$= 86.802 \text{ units.}$$