5 Renewal processes

5.1 Definition

Let the r.v. T denote the *failure time* or *lifetime* of a component, where the lifetime distribution is continuous with p.d.f. f and corresponding Laplace transform f^*. (The continuity assumption is convenient for avoiding certain complications and for ease of notation, but it is not strictly necessary for the development of the theory.)

Further, consider a sequence of such components. Assume that there is a new component at time 0 with failure time T_1, replaced at time T_1 by a second component with failure time T_2, replaced at time $T_1 + T_2$ by a third component .... The failure time of the nth component in the sequence is represented by T_n, where $\{T_n : n \geq 1\}$ is a sequence of i.i.d. r.v.s, each with p.d.f. f. The n-th failure/renewal occurs at time S_n, where

$$S_n = T_1 + T_2 + \ldots + T_n \qquad n \geq 1.$$

Define $S_0 = 0$ and

$$N(t) = \sup\{n \geq 0 : S_n \leq t\} \qquad t \geq 0.$$

The r.v. $N(t)$ represents the number of renewals up to time t. The stochastic process $\{N(t) : t \geq 0\}$ is a *renewal process* with the given lifetime distribution.

5.2 The distribution of S_n and the distribution of $N(t)$

When the lifetime distribution is an exponential distribution with parameter λ, so that $f(t) = \lambda e^{-\lambda t}$, $t \geq 0$, then $\{N(t) : t \geq 0\}$ is a Poisson process with rate λ. In this case, as was shown in Section 1.3, S_n has a gamma distribution with parameters n and λ, and $N(t)$ has a Poisson distribution with parameter λt. Otherwise $\{N(t) : t \geq 0\}$ is not a Markov process — the exponential distributions are the only distributions with the "memoryless" property.

In general, if f_n denotes the p.d.f. of S_n and f_n^* its Laplace transform then

$$f_n^*(s) = f^*(s)^n \qquad n \geq 0.$$

A relatively simple example occurs when the lifetime distribution is a gamma distribution with parameters ν and λ, in which case

$$f^*(s) = \left(\frac{\lambda}{\lambda + s}\right)^{\nu}$$

and

$$f_n^*(s) = \left(\frac{\lambda}{\lambda + s}\right)^{\nu n}.$$

Hence S_n has the gamma distribution with parameters νn and λ.

Furthermore, if the parameter ν of the gamma distribution is an integer then we may use the *method of stages* to obtain the distribution of $N(t)$. The gamma distribution may be thought of as the sum of ν i.i.d. exponential distributions with parameter λ. Hence the lifetime of each component may be thought of as the sum of ν independent stages, where each stage has an exponential distribution with parameter λ. The renewal process may be constructed from an underlying Poisson process with rate λ, where a renewal occurs at every ν-th arrival in the Poisson process. The event that exactly n renewals have occurred by time t is equivalent to the event that between νn and $\nu n + \nu - 1$ arrivals, inclusive, have occurred in the underlying Poisson process. Thus

$$\mathbb{P}(N(t) = n) = \sum_{i=\nu n}^{\nu n+\nu-1} e^{-\lambda t} \frac{(\lambda t)^i}{i!} \qquad n \geq 0.$$

In the general case, except for a few special cases, it will not be possible to obtain explicit expressions for the distributions of S_n and $N(t)$, but there are general results concerning the behaviour of S_n as $n \to \infty$ and $N(t)$ and $t \to \infty$. Assuming that f has a finite mean μ, where $\mu > 0$, by the Strong Law of Large Numbers, with probability 1,

$$\frac{S_n}{n} \to \mu$$

as $n \to \infty$. Now consider the behaviour of $N(t)$ as $t \to \infty$. Note that, by the definition of $N(t)$,

$$S_{N(t)} \leq t < S_{N(t)+1}\,.$$

Hence, for $N(t) > 0$,

$$\frac{S_{N(t)}}{N(t)} \leq \frac{t}{N(t)} < \frac{S_{N(t)+1}}{N(t)} = \frac{S_{N(t)+1}}{N(t)+1}\,\frac{N(t)+1}{N(t)}\,.$$

As $t \to \infty$, $N(t) \to \infty$ and, by the Strong Law of Large Numbers, with probability 1, both the left hand side and the right hand side in the above inequalities tends to μ. Hence with probability 1, as $t \to \infty$,

$$\frac{t}{N(t)} \to \mu$$

and

$$\frac{N(t)}{t} \to \frac{1}{\mu}\,. \tag{1}$$

So $1/\mu$ is the *rate* of the renewal process, the long-term average number of renewals per unit time.

5.3 The renewal function and the renewal density

Although it is difficult analytically to investigate the distribution of $N(t)$ for finite t, it is relatively easy to obtain general results about $E[N(t)]$.

Define the *renewal function* $H(t)$ by

$$H(t) = E[N(t)] \qquad t \geq 0$$

and the *renewal density* $h(t)$ by

$$h(t) = H'(t) \qquad t \geq 0.$$

The renewal function $H(t)$ is a finite non-decreasing function of t with $H(0) = 0$.

To obtain a formula for $H^*(s)$, first note the identity

$$\{N(t) \geq n\} = \{S_n \leq t\} \qquad n \geq 1,$$

from which it follows that

$$\mathbb{P}(N(t) \geq n) = \mathbb{P}(S_n \leq t) = F_n(t) \qquad n \geq 1,$$

where F_n is the d.f. of S_n. Note also that the Laplace transform F_n^* of F_n is given by

$$F_n^*(s) = \frac{f_n^*(s)}{s} = \frac{f^*(s)^n}{s} \qquad n \geq 1.$$

Hence

$$H(t) = E[N(t)] = \sum_{n=1}^{\infty} \mathbb{P}(N(t) \geq n) = \sum_{n=1}^{\infty} F_n(t)$$

and, taking Laplace transforms,

$$H^*(s) = \sum_{n=1}^{\infty} \frac{f^*(s)^n}{s} \, ,$$

so that

$$H^*(s) = \frac{f^*(s)}{s[1 - f^*(s)]} \, . \tag{2}$$

The Laplace transform of the renewal density is given by

$$h^*(s) = sH^*(s) - H(0) = sH^*(s) \, ,$$

i.e.,

$$h^*(s) = \frac{f^*(s)}{1 - f^*(s)} \, . \tag{3}$$

We may also use what is known as the renewal argument to investigate $H(t)$. Conditioning on the time T_1 of the first renewal and noting that after each renewal the process starts anew,

$$E[N(t)|T_1 = u] = \begin{cases} 0 & u > t \\ 1 + H(t-u) & u \leq t \end{cases}.$$

Hence

$$\begin{aligned} H(t) = E[N(t)] &= \int_0^\infty E[N(t)|T_1 = u]f(u)du \\ &= \int_0^t [1 + H(t-u)]f(u)du \ . \end{aligned}$$

Thus

$$H(t) = F(t) + \int_0^t H(t-u)f(u)du \qquad t \geq 0. \tag{4}$$

This is a version of the *renewal equation*, the integral equation of renewal theory.

Taking Laplace transforms in Equation (4),

$$H^*(s) = \frac{f^*(s)}{s} + H^*(s)f^*(s) \ ,$$

which is equivalent to Equation (2).

Equation (3) may be rewritten as

$$h^*(s) = f^*(s) + h^*(s)f^*(s) \ ,$$

which corresponds to the renewal equation

$$h(t) = f(t) + \int_0^t h(t-u)f(u)du \qquad t \geq 0. \tag{5}$$

Alternatively, Equation (5) may be obtained by differentiating Equation (4).

- For the Poisson process with rate λ, since $N(t)$ has the Poisson distribution with parameter λt, $H(t) = \lambda t$ and $h(t) = \lambda$ for $t \geq 0$, so that $H^*(s) = \lambda/s^2$ and $h^*(s) = \lambda/s$. Here $f(u) = \lambda e^{-\lambda u}$, $u \geq 0$ so that $f^*(s) = \lambda/(\lambda + s)$, and it is easy to verify the truth of Equations (2)-(5).

Example Consider a renewal process with a lifetime distribution that is a mixture of two exponential distributions, so that its p.d.f. is given by

$$f(t) = \theta\lambda e^{-\lambda t} + (1-\theta)\kappa e^{-\kappa t} \qquad t \geq 0,$$

where $0 < \theta < 1$ and $\lambda \neq \kappa$.

$$f^*(s) = \frac{\theta\lambda}{\lambda + s} + \frac{(1-\theta)\kappa}{\kappa + s} \ .$$

Substitution into Equation (3) leads to

$$\begin{aligned} h^*(s) &= \frac{[\theta\lambda + (1-\theta)\kappa]s + \lambda\kappa}{s[(1-\theta)\lambda + \theta\kappa + s]} \\ &= \frac{A}{s} + \frac{B}{(1-\theta)\lambda + \theta\kappa + s} \ , \end{aligned}$$

where A and B must satisfy

$$[(1-\theta)\lambda + \theta\kappa]A = \lambda\kappa$$

and

$$A + B = \theta\lambda + (1-\theta)\kappa\ .$$

Thus

$$A = \frac{\lambda\kappa}{(1-\theta)\lambda + \theta\kappa}$$

and

$$B = \frac{\theta(1-\theta)(\lambda-\kappa)^2}{(1-\theta)\lambda + \theta\kappa}\ .$$

Hence

$$h(t) = \frac{\lambda\kappa}{(1-\theta)\lambda + \theta\kappa} + \frac{\theta(1-\theta)(\lambda-\kappa)^2}{(1-\theta)\lambda + \theta\kappa}\ e^{-[(1-\theta)\lambda+\theta\kappa]t}\ .$$

Integrating and using the fact that $H(0) = 0$,

$$H(t) = \frac{\lambda\kappa t}{(1-\theta)\lambda + \theta\kappa} + \frac{\theta(1-\theta)(\lambda-\kappa)^2}{[(1-\theta)\lambda + \theta\kappa]^2}\ \left[1 - e^{-[(1-\theta)\lambda+\theta\kappa]t}\right]\ .$$

5.4 The limiting value of the renewal density

Given an arbitrary lifetime distribution, it is not in general possible to obtain explicit expressions for $H(t)$ or $h(t)$. However, there is a simple result for the limiting value of $h(t)$ as $t \to \infty$.

If the lifetime distribution has a finite mean μ then

$$f^*(s) = 1 - \mu s + o(s)$$

as $s \to 0$. Hence, substituting into Equation (3),

$$h^*(s) = \frac{1-\mu s + o(s)}{\mu s + o(s)} \sim \frac{1}{\mu s}$$

as $s \to 0$. Using an asymptotic result about Laplace transforms from Section 3.3.2,

$$h(t) \to \frac{1}{\mu} \tag{6}$$

as $t \to \infty$, which is a version of the so-called *Renewal Theorem*.

For instance, in the example at the end of Section 5.3, as $t \to \infty$,

$$h(t) \to \frac{\lambda\kappa}{(1-\theta)\lambda + \theta\kappa} = \frac{1}{\mu}\ .$$

5.5 Interpretation of the renewal density

$$\begin{aligned} h(t) &= \lim_{\delta t \to 0} \frac{H(t+\delta t) - H(t)}{\delta t} \\ &= \lim_{\delta t \to 0} \frac{E[N(t+\delta t) - N(t)]}{\delta t} \ . \end{aligned}$$

Now $N(t+\delta t) - N(t)$ represents the number of renewals in the time interval $(t, t+\delta t]$, and, as $\delta t \to 0$,

$$\mathbb{P}[N(t+\delta t) - N(t) \geq 2] = o(\delta t) \ .$$

Hence

$$E[N(t+\delta t) - N(t)] = \mathbb{P}[N(t+\delta t) - N(t) = 1] + o(\delta t) \ ,$$

and

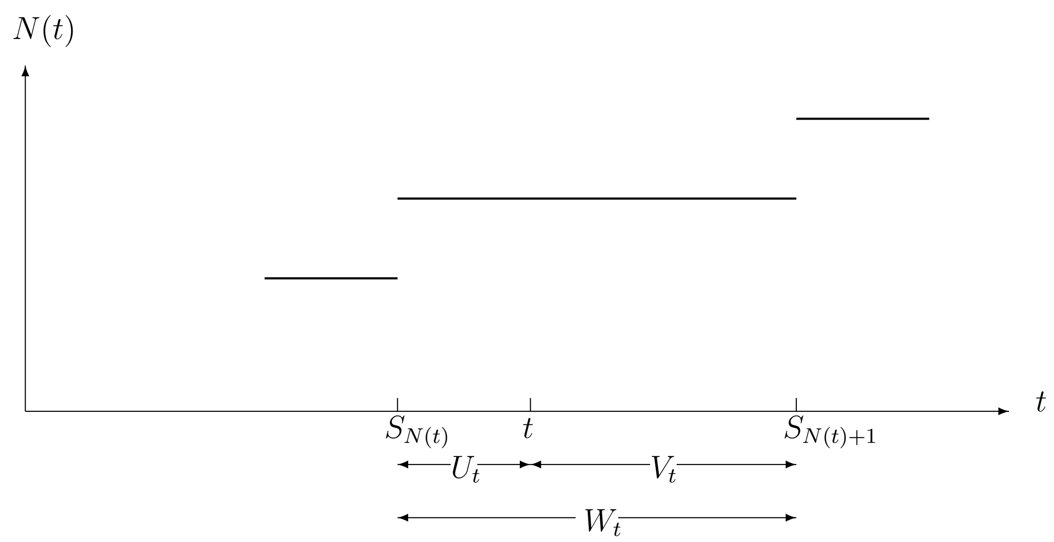
$$h(t) = \lim_{\delta t \to 0} \frac{\mathbb{P}(\text{a renewal occurs in } (t, t+\delta t])}{\delta t} \ .$$

From the result of Equation (6), $1/\mu$ is the asymptotic rate at which renewals occur. If the renewal process has been in operation for a long time then the probability of a renewal occurring in a given time interval $(t, t+\delta t]$ is

$$\frac{\delta t}{\mu} + o(\delta t).$$

- Note that the result of Equation (6) is similar to but not identical with the result of Equation (1).

5.6 Backward and forward recurrence times



The *backward recurrence time (current life)*, $U_t \equiv t - S_{N(t)}$, is the age of the component currently in use at a given time t. For $0 \leq u < t$,

$$\begin{aligned}
\mathbb{P}(U_t \in (u, u+\delta u)) &= \mathbb{P}(\text{a renewal in } (t-u-\delta u, t-u), \\
&\qquad \text{next failure time greater than u}) + o(\delta u) \\
&= h(t-u)\ \delta u\ Q(u) + o(\delta u),
\end{aligned}$$

where $Q(t)$ is the survivor function of the lifetime distribution, as defined in Section 1.1. For any given u, as $t \to \infty$, by the result of Equation (6), $h(t-u) \to 1/\mu$. Hence, as $t \to \infty$, the distribution of U_t converges to a distribution with p.d.f. g, where

$$g(u) = \frac{Q(u)}{\mu} \qquad u \geq 0. \tag{7}$$

(Note that the expression in Equation (7) really does specify a p.d.f., since, by Theorem 1.1.2 of Section 1.1, $\int_0^\infty Q(u)du = \mu$.)

The *forward recurrence time (excess life)*, $V_t \equiv S_{N(t)+1} - t$, is the length of time remaining until failure of the component currently in use at time t. Let T denote the lifetime of an arbitrary component.

$$\begin{aligned}
\mathbb{P}(V_t > v | U_t = u) &= \mathbb{P}(T > u+v | T > u) \\
&= \frac{Q(u+v)}{Q(u)}\ .
\end{aligned} \tag{8}$$

Differentiating the expression in Equation (8) with respect to v, we find that the conditional p.d.f. of V_t given $U_t = u$ is $g(v|u)$, where

$$g(v|u) = \frac{f(u+v)}{Q(u)} \qquad v \geq 0. \tag{9}$$

Using Equations (7) and (9), we find that for large t the joint p.d.f. $g(u,v)$ of U_t and V_t is given by

$$\begin{aligned}
g(u,v) &= g(v|u)g(u) \\
&= \frac{f(u+v)}{Q(u)}\ \frac{Q(u)}{\mu} \\
&= \frac{f(u+v)}{\mu} \qquad u \geq 0, v \geq 0.
\end{aligned} \tag{10}$$

Note the symmetry in u and v of the joint p.d.f. in Equation (10), so that the marginal p.d.f. of V_t is the same as the marginal p.d.f. of U_t. We may check this by integrating out u in Equation (10): the p.d.f. of V_t is given by

$$\int_0^\infty \frac{f(u+v)}{\mu}\ du = \frac{Q(v)}{\mu} \qquad v \geq 0.$$

Let $W_t = U_t + V_t$, the lifetime (*total life*) of the component in use at time t. For large t, from Equation (10), the joint p.d.f. of U_t and W_t is given by

$$\frac{f(w)}{\mu} \qquad 0 \le u \le w.$$

(The Jacobian of the corresponding transformation is 1.) Integrating out u, the marginal p.d.f. of W_t is

$$\int_0^w \frac{f(w)}{\mu}\, du \;=\; \frac{wf(w)}{\mu} \qquad w \ge 0. \tag{11}$$

Assuming that the lifetime distribution has a finite variance σ^2,

$$\begin{aligned} E(W_t) &= \frac{1}{\mu}\int_0^\infty w^2 f(w)\, dw \\ &= \frac{1}{\mu}(\sigma^2 + \mu^2) \\ &= \mu(1 + C^2)\ , \end{aligned} \tag{12}$$

where C is the *coefficient of variation*,

$$C = \frac{\sigma}{\mu}\ .$$

The inspection paradox

Equation (12) demonstrates the *inspection paradox*, that the component in use at a given time t tends to have a longer lifetime than a component chosen at random.

To obtain an intuitive feel for the inspection paradox, imagine the time-axis split up into intervals corresponding to the successive lifetimes of the components. If the point t is chosen at random on the time-axis, the probability of it falling in a given interval is proportional to the length of the interval, so that we have what is known as *length-biased sampling* of the intervals and hence of the lifetimes. In the long run, the proportion of components with failure times in $(t, t+dt)$ is $f(t)dt$, but the proportion of time for which components with failure times in $(t, t+dt)$ are in use is proportional to $tf(t)dt$. Hence the p.d.f. of Equation (11).

Since, in the limit as $t \to \infty$, U_t and V_t are identically distributed and $W_t = U_t + V_t$, it follows from Equation (12) that, in the limit,

$$E(U_t) = E(V_t) = \frac{1}{2}E(W_t) = \frac{1}{2}\mu(1 + C^2)\ . \tag{13}$$

The waiting-time paradox

Perhaps an even more startling result than the inspection paradox is the *waiting-time paradox* that the expectation $E(V_t)$ of the forward recurrence time, i.e., the expected excess life, can be greater than the expectation μ of the lifetime distribution. From Equation (13) we see that this occurs if $C > 1$, i.e., the coefficient of variation of the lifetime distribution exceeds one.

For example, we may suppose that buses arrive at a bus-stop according to a renewal process. (This is an approximation, since inter-arrival times are unlikely to be independently distributed.) If the average inter-arrival time for buses is 10 minutes, say, it may be the case that, when we arrive at the bus-stop, the expected waiting time until the next arrival of a bus is greater than 10 minutes.

The equilibrium renewal process

We may consider what is known as an *equilibrium renewal process*, where $\{T_n : n \geq 2\}$ is a sequence of i.i.d. r.v.s, each with p.d.f. f, and T_1 is independently distributed of $\{T_n : n \geq 2\}$, but has p.d.f. $Q(t)/\mu,\ t \geq 0$, which corresponds to the limiting distribution of U_t as $t \to \infty$. Results similar to those for ordinary renewal processes, but with slight modifications, may be obtained. Some results are much simpler. For example, $H(t) = t/\mu$ and $h(t) = 1/\mu,\ t \geq 0$.