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2K22/MC/44

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Assignment 5
Stochastic processes

(Q1) $\mu = 120$
$\sigma = 24$

lifetime of systems: $X \sim N(\mu = 120, \sigma = 24)$
normal distributed

Total time interval : $T = 1000$ hours.
No. of systems used: N
Spare systems needed: $N-1$

expected no. of failures :-

$$E(N) = \frac{T}{\mu} = \frac{1000}{120} = 8.33$$ minimum needed.

Therefore $N = 9$
$N - 1 = 8$.

* variability due to standard deviation

(1) Mean of $N = \frac{T}{\mu}$

$$SD \text{ of } N = \sqrt{\frac{T\sigma^2}{\mu^3}}$$ putting $T = 1000$, $\sigma = 24$, $\mu = 120$

$SD = 2.83$
$N \sim N(8.33, 2.83^2)$

process continues with probability
(i) 0.60.
find N st $P(N \leq K) = 0.60$.
$$Z = \frac{K - \mu_N}{\sigma_N}$$ $\mu_N = 8.33$, $\sigma_N = 2.83$

from standard normal table $z = 0.253$

$K = \mu_N + z\sigma_N = 9.05$

Thus $K \approx 9$

thus 9 systems needed

8 spare.

2(ii) 0.79

$z = 0.805$

$K = \mu_N + z\sigma_N = 10.61$

$K \approx 11$

Spare needed 10.

(Q2) $H(t) = ? =$ Expected no. of renewals in $[0, t]$

erlang distribution

$n = 3 \quad \lambda = 1$

PDF of cycle length (T):

$$f_T(t) = \frac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!} \quad t \geq 0$$

$n = 3 \; \lambda = 1$

$$f_T(t) = \frac{t^2 e^{-t}}{2} \quad t \geq 0$$

laplace transform

$$L(f_T(t)) = \tilde{f}_T(s)$$

$$\tilde{f}_T(s) = \int_0^\infty f_T(t)\, e^{-st}\, dt$$

$$= \int_0^\infty \frac{t^2 e^{-t}}{2} e^{-st}\, dt$$

$$L(t^n e^{-at}) = \frac{n!}{a^{n+1}}$$

$$\tilde{f}_T(s) = \frac{1}{2}\int_0^\infty t^2 e^{-(s+1)t}\,dt$$

$$= \left(\frac{1}{2}\right)\left[\frac{2!}{(s+1)^3}\right] = \frac{1}{(s+1)^3}$$

renewal fun^x

$$\tilde{H}(s) = \frac{\tilde{f}_T(s)}{1-\tilde{f}_T(s)} = \frac{\frac{1}{(s+1)^3}}{1-\frac{1}{(s+1)^3}}$$

$$= \frac{1}{(s+1)^3 - 1}$$

inverse laplace

$$H(t) = L^{-1}\left(\frac{1}{(s+1)^3-1}\right)$$

$$(s+1)^3 - 1 = (s+1-\omega)(s+1-\omega^2)(s+1-\omega^3)$$

$$= L^{-1}\left(\frac{A}{s+1-\omega} + \frac{B}{s+1-\omega^2} + \frac{C}{s+1-\omega^3}\right)$$

$H(t) = t/\mu$ for large t

$\mu = 3$ $\boxed{H(t) = t/3}$

(Q3) $H(t) = t/15$

at $t = 15$

$H(15) = 1$

no. of renewals by $t = 15$ = 1

for renewal process N(t) follows Poisson distribution

$\lambda = H(t)$ at $t = 15$ $N(15) \sim \text{Poisson}(1)$

$P(N(15) \geq 2) = 1 - P(N(15) < 2)$

$= 1 - P(N(15) = 0) - P(N(15) = 1)$

$P(N = k) = \dfrac{\lambda^k e^{-\lambda}}{k!}$ $\lambda = 1$.

$= 1 - \dfrac{e^{-1}}{0!} - \dfrac{e^{-1}}{1!}$

$= 1 - 2e^{-1}$

$= 0.2642.$

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(Q4) M/M/1 queue model

$1/\mu$ ~~service time~~ = 3

$\mu = 1/3$ jobs/min

job arrival avg rate $\lambda = 1/4$ jobs/min.

✱ FCFS

① ρ (utilization factor)

$\rho = \lambda/\mu < 1$ (system stable)

(a) probability (TAT > 20 mins).

TAT = service time + waiting time

for M/M/1 Queue TAT T follows exp distn

$$E[T] = \frac{1}{\mu(1-\rho)}$$

$\rho = \lambda/\mu = 3/4 = 0.75$

$$E[T] = \frac{1}{1/3(1-0.75)} = 12 \text{ minutes}$$

$P(T>20) = ?$

$$P(T>t) = e^{-\frac{t}{E[T]}} = e^{-20/12} = e^{-5/3}$$

≈ 0.2231

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(b) no. of jobs waiting

$$L_q = \frac{\rho^2}{1-\rho}$$

$$L_q = \frac{(0.75)^2}{0.25} = 2.25$$

(Q5) M/M/2 queue (2 servers)
exp distributed time
Poisson arrivals
FCFS.

(a) State space → no. of customers in system including in service + waiting.

X(t) = no. of customers in system.
state space
$S = \{0, 1, 2 \ldots \}$

X(t) = 0 sys. empty
X(t) = 1 one customer
X(t) = 2 2 customer
X(t) ≥ 3 { 2 serving, X(t) − 2 queue }.

Generator Q
arrival rate: λ
service rate: μ per server
if K ≤ 2 service rate is Kμ
K servers work simultaneously

$q_{i,i+1} = \lambda \quad \forall\, i \geq 0$ (arrival rates)

$q_{i,i-1} = \begin{cases} i\mu & \text{if } i \leq 2 \\ 2\mu & \text{if } i \geq 2 \end{cases}$ (departure rates)

$q_{i,i} = -\sum_{j \neq i} q_{i,j}.$

(b) Stationary distribution.

$\pi = \{\pi_0, \pi_1, \pi_2 \dots\}$ satisfies

$\sum_{i=0}^{\infty} \pi_i = 1 \;\; \& \;\; \pi Q = 0$

$i=0$

$\lambda \pi_0 = \mu \pi_1$

$\boxed{\pi_1 = \frac{\lambda \pi_0}{\mu}}$

$i=1$

$\lambda \pi_1 = 2\mu \pi_2$

$\pi_2 = \frac{\lambda \pi_1}{2\mu}.$

$i \geq 2$

$\lambda \pi_i = 2\mu \pi_{i-1}$

$\pi_i = \frac{\lambda}{2\mu} \pi_{i-1}$

$\pi_i = \frac{\lambda^i}{i!\, \mu^i} \pi_0 . \quad i \leq 2$

$\pi_i = \frac{\lambda^i}{2^{i-2}\, 2!\, \mu^i} \pi_0 \quad i \geq 3.$

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(c) probability of overtaking.

occurs in state $i \geq 2$ (both server busy)

$$P(\text{overtake}) = \sum_{i=2}^{\infty} \pi_i \, P(\text{overtake in state } i)$$

for $i \geq 2$, $P(\text{overtaking in state } i) = 1/2$

$$P(\text{overtake}) = \frac{1}{2} \sum_{i=2}^{\infty} \pi_i$$

$$= \frac{1}{2}\left(1 - \pi_0 - \pi_1\right)$$