Assignment - I Stochastic Processes
Submitted by: Prakhar Dhyani 2K22/MC/111

1. What is a stochastic process. Define state space and parameter space of a stochastic process. In how many ways a stochastic process may be classified, explain with examples.

Ans: A stochastic process $X = \{x(t), t \in T\}$ is a collection of random variables. For each $t \in T$ where T is an indexed set $x(t)$ is a random variable and is known as state of the process at time t.
State space is defined as the set of all possible values of state for $t \in T$.
Parameter spaces is defined as set of all possible values of parameters which define a stochastic process (indexing parameter).

State space and parameter space can either be discrete or continous depending on which stochastic process can be classified as:

1). Discrete time Discrete state space stochastic chain
for example: Number of customers in a shop at any hour. (1-24)

2). Continous time Discrete state space stochastic chain.
for example: Number of Jobs in a server at a time t.
then $X = \{x(t) \mid t \in T\}$ is a continous time discrete state process.

3). Discrete time Continous state space stochastic process.
The prices of stock market are observed at discrete time intervals but the prices are computed contoniously

4). Continous time continous state space stochastic process.
During flow of water from a dam, the velocity, potential energy, Kinetic energy vary continously with time.

2. What is a Bernoulli process. Find the probability of getting 7 success in 10 trials. Also calculate the probability of getting 7th success in 10 trials.

Ans: Let i denote the number of successive indipendent bernoulli trials,
Let y_i denote the result of i^{th} trial
then

$$y_i = \begin{cases} 1 & \text{success} \\ 0 & \text{Failure} \end{cases}$$

assuming that probablity of success is p.
Then $P(y_i = 1) = p$ (indipendent of i)
then $\{y_i \mid i = 1, 2, \dots, n\}$ is a discrete time (indexed set)
and $\{0, 1\}$ are discrete state space

then, Bernoulli process is a discrete time discrete space stochastic process.

for ex:
Let us consider a group of n people
They are provided with a spam - e-mail link.

The probablity that a person y_i clicks on the spam email is p.
Each person indipendently decides to click on spam/not spam email
hence $\{y_i \mid i = 1, 2, \dots, n\}$ is a indexed set
with state space $y_i = \begin{cases} 0 & \text{failure} \\ 1 & \text{success.} \end{cases}$ $\{0, 1\}$
hence it is an example of bernoulli process.

$n = 10$
$x = 7$
Let p be the probablity of success

Using $\quad {}^nC_x \, p^x \, (1-p)^{n-x}$.

$$P(7 \text{ successes in } 10 \text{ trials}) = {}^{10}C_7 \, p^7 \, (1-p)^3$$

$$= 120 \, p^7 \, (1-p)^3$$

P (7^{th} success on 10 trials) = P(6 success in 9 trials) x P (success on 10th trial)

$$= \left({}^{9}C_{6}\, p^{6} (1-p)^{3}\right) p$$

$$= 84\, p^{7} (1-p)^{3}$$

3. Explain merging of two independent Bernoulli processes. Let X_t and Y_t are two independent Bernoulli processes then find the probability of an arrival in the merged process given that there is an arrival in either X_t or Y_t.

Ans When two bernoulli processes are merged, the two sequences of independent bernoulli trials are combined into one.

Let X_t and Y_t denote two bernoulli process

Let p_x be probablity of success in X_t
and p_y be probablity of success in Y_t

Then the merged process produces a success in either X_t or Y_t

$\therefore$ P (success in merged process) = $P(X_t \cup Y_t)$

since X_t and Y_t are independent

$$P(X_t \cup Y_t) = P(X_t) + P(Y_t) - P(X_t \cap Y_t)$$
$$= p_x + p_y - p_x p_y$$

4. . Define Poisson process. Let $\{N(t), t \in [0, \infty)\}$ be a Poisson process with rate λ. Let $T_1, T_2, \ldots$ be the

arrival times for this process. Shoe that

$$f_{T_1,T_2,\ldots,T_n}(t_1, t_2, \ldots, t_n) = \lambda^n e^{-\lambda t_n} \text{ for } 0 < t_1 < t_2 < \ldots < t_n.$$

Also, find the probability that there are four arrivals in (0, 2] or three arrivals in (4, 7].

Ans: Possion process can be defined as a renewal counting process
Let events occur successively in time such that the intervals b/w successive events are independent & identically distributed acc^n to an exponential distribution.

Let number of events in an interval (0, t] be N(t)
then the stochastic process N(t) is a possion process.

Let rate be λ

$$f_{T_1,T_2,\ldots T_n}(t_1,t_2,\ldots,t_n) = \lambda^n e^{-\lambda t_n}$$

$$\text{for } (0 < t_1 < t_2 < \cdots < t_n)$$

Let inter arrival times $S_i = T_{i+1} - T_i$

The Probablity distribution function

$$f_{S_i}(s_i) = \lambda e^{-\lambda s_i}$$

$\therefore T_1 = S_1, T_2 = S_1 + S_2, \ldots T_n = \sum_{i=1}^{n} S_i$

$$\therefore f_{T_1,T_2,\ldots T_n}(t_1,t_2,\ldots,t_n) = f_{S_1,S_2,\ldots,S_n}(t_1, t_2-t_1, t_3-t_2, \ldots (t_n-t_{n-1}))$$

$$f_{T_1,T_2,\ldots,T_n}(t_1,t_2,\ldots,t_n) = \lambda e^{-\lambda t_1} \times \lambda e^{-\lambda(t_2-t_1)} \times \lambda e^{-\lambda(t_3-t_2)} \times \cdots \lambda e^{-\lambda(t_n-t_{n-1})}$$

$$= \lambda e^{-\lambda(t_1+t_2-t_1+t_3-t_2+\cdots+t_n-t_{n-1})}$$

$$= \lambda e^{-\lambda t_n}$$

using,

$$P(N(x+y)-N(y)=n) = \frac{e^{-\lambda x}(\lambda \times x)^n}{n!}$$

(a) 4 arrivals (success) in $(0,2]$

$$P(N(2)=4) = \frac{(\lambda \times 2)^4 e^{-\lambda \times 2}}{4!}$$

(b) 3 arrivals in $(4,7]$

$$P(N(4,7)) = P(N(7)-N(4)=3) = \frac{(\lambda \times 3)^3 e^{-\lambda \times 3}}{3!}$$

5. What is a Renewal process. Define renewal function and renewal density function. Show that a Poisson process is a renewal process whose inter arrival times are mutually independent random variables and the random variable follows exponential distribution.

Ans: Let the random variable T denote the failure time or lifetime of a component.

Assume that there is a new component at time 0 with failure time T_1, at / after T_1 it is replaced by a second component with failure time T_2 replaced at T_1+T_2 by 3rd component.

Similarly the failure time of n^{th} component in sequence represented by T_n.

and n^{th} failure occours at

$$S_n = T_1 + T_2 + \cdots + T_n. \quad (\text{and } S_0 = 0)$$

then $N(t) = \sup\{n \geq 0 : S_n \leq t\}$
where $N(t)$ is renewals up to time t
and the process $\{N(t) : t \geq 0\}$ is a renewal process.

Renewal function and density:
Renewal function is defined as
$$H(t) = E[N(t)] \quad t \geq 0$$
Renewal density
$$h(t) = H'(t)$$
where $H(t)$ is a non decreasing function of t.
with $H(0) = 0$

In a Poisson process time b/w consecutive events is denoted by $S_1, S_2, S_3, \ldots, S_n$ are indipendent random variables.
The inter arrival times are exponentially distributed with a parameter $\lambda > 0$
and have pdf
$$f_X(x) = \lambda e^{-\lambda x}.$$

since, inter arrival times are indipendent and identically distributed random variables and follow an exponential distribution the occurance of each event in possion process renews the process. making it a renewal process.

6. Define Gaussian process. Show that wide sense stationary and strict sense stationary are equivalent for a Gaussian process.

Ans A gaussian process is collection of random variables any finite number of which have a joint gaussian distribution.
$\therefore$ $\{x(t) : t \in T\}$ is a gaussian process if $x(t)$ $\forall\, t \in T$ follows gaussian distribution

Strict Sense Stationary
as gaussian process $\{x(t) \mid t \in T\}$ is strict sense stationary when $(x(t_1), x(t_2), \ldots, x(t_n))$ has a gaussian distribution similar to $(x(t_1+h), x(t_2+h), \ldots, x(t_n+h))$
hence by definition, $E[X(t)] = E[x(t+h)] = \ldots$
$= \ldots E[x(t+nh)]$
$\therefore$ $E[x(t)]$ is constant.
and $Cov(X[t), X[t+h))$ depends on value of h

Wide sense stationary
A stochastic process is wide sense stationary if mean $E[x(t)]$ is constant and $cov(x(t), x(t+h))$ depends on h not t.

hence Strict sense stationary and wide sense stationary are similar for gaussian processes.

Define standard Brownian motion and Brownian motion with variance σ^2 and drift μ. Show that a standard Brownian motion is a unique Gaussian process with $m(t) = 0$ and $a(s, t) = min\{s, t\}$, $0 \le s \le t$.

Ans: In a fluid, tiny organic and inorganic particles move randomly along a zig-zag path. such a moment is known as brownian motion.
It is a continous time stochastic process $\{B(t) \mid t \geq 0\}$ with state space $\mathbb{Z} = (-\infty, \infty)$ and has following properties.

(1) $B(0) = 0$
(2) $\{B(t) \mid t \geq 0\}$ has homogenous and indipendent increments

(3) It has normal distribution b/w
$E[B(t)] \geq 0$
$Var[B(t)] = \sigma^2 t$

Brownian motion with variance σ^2 and drift μ is defined as.
$\{X(t) \mid t \geq 0\}$
and $X(t) = \mu t + \sigma B(t)$
where $B(t)$ is standard brownian motion

Brownion Motion as a Gaussian process.
The vector $(B(t_1), B(t_2), \ldots, B(t_n))$ has multivariate normal distribution because event
$$\{B(t_1) = x_1, B(t_2) = x_2, \ldots, B(t_n) = x_n\}$$
can be re written as

$$\{B(t_1) = x_1, B(t_2) - B(t_1) = x_2 - x_1, \ldots, B(t_n) - B(t_{n-1}) = x_n - x_{n-1}\}$$

yeilding Joint density as

$$f(x_1, \ldots, x_n) = f_{t_1}(x_1)\, f_{t_2 - t_1}(x_2 - x_1) \cdots f_{t_n - t_{n-1}}(x_n)$$

$f_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}$ is density of normal distribution

Since multivariate normal distributions are completely determined by its mean and covariance.
For standard BM with $0 \leq s \leq t$.

$m(t) = 0 \qquad a(s,t) = s$

$$\begin{aligned} cov(B(s), B(t)) &= cov(B(s), B(s) + B(t) - B(s)) \\ &= cov(B(s), B(s)) + cov(B(s), B(t) - B(s)) \\ &= var(B(s)) \\ &= s \end{aligned}$$

Thus $m(t) = 0$ & $a(s,t) = \min(s,t)$

8. What is a random walk. Give real work processes which can be realized as random walk. Discuss different types of random walk and individual question of interest one can find in terms of probabilities.

Ans: Assume that a particle is initially at x_0 on the axis at time $t=1$, the particle undergoes an jump/step z_1 where z_1 is a random variable.
at time $t=2$, the particle undergoes an jump/step z_2 where z_2 is a random variable indipendent of z_1

$\therefore t_0 = x_0$

$t_1 = x_0 + z_1$

$t_2 = x_0 + z_1 + z_2$

$\vdots$

$t_n = x_0 + z_1 + z_2 + \dots + z_n$

where z_i are mutually indipendent and identically distributed,
where z_i can take values $\{1, 0, -1\}$
with distribution

$P(z_i = 1) = p$

$P(z_i = 0) = 1 - p - q$

$P(z_0 = -1) = q$

This is known as a simple random walk.

Real life examples
Stock market, a stocks price may increace with probab. p, decrease with probab q and stay the same with probab $1-p-q$.

Diffusion of a particle, particles may diffuse in any direction, Insurance Risk, company may lose the money as claims or gain money as premiums

Gamblers Ruin; A gambler may win more money or lose all of his money

Types of random walk:

1). Simple random walk: steps are taken in a fixed set of direction with equal probablity.

2). Random walk with absorbing barriers:
once a absorbing state is hit in a random walk the walk ends.

3). Random walk with reflecting barriers
Instead of the walk being over the walker is sent back into allowed range.

4). Unrestricted random walk.
The walker is free to move in any direction indefinately

9. Consider a random walk with two absorbing barriers $a = 5$ *and* $b = -4$ with probability of moving one forward step is $p = 0.3$ and probability of moving one forward step $q = 0.4$ with initial position of the particle at the origin. Find the probability of absorption at $a = 5$ and $b = -4$. Also, find the probability that absorption occurs for $N = 10$.

$P(Z_i = 1) = 0.3 = p$

$P(Z_i = -1) = 0.4 = q$

$a = 5$

$b = -4$

$N = 10$

10. Consider a simple random walk $Xn = \sum zi$, and suppose it starts from 0. As usual, $P(z_i = 1) = p$, $P(z_i = -1) = q = 1 - p$. Compute $E(e^{\alpha X}n)\alpha \in R$. Comment on the asymptotic behavior of X_n.

$$X_n = \sum_{i=1}^{n} z_i \qquad z_i = \{1, -1\} \qquad P(z_i = 1) = p$$
$$P(z_i = -1) = 1 - p$$

$$E(z_i) = p \times 1 + (1-p)(-1) = 2p - 1$$

$$E(x_n) = n(2p-1)$$

$$E(e^{\alpha x_n}) \text{ for } \alpha \in \mathbb{R} = E(e^{a \sum_{i=1}^{n} z_i}) = (E(e^{a z_1}))^n$$

[z_i are indipendent]

$$E[e^{\alpha z_1}) = p e^{a(1)} + (1-p)e^{-a}$$

$$(E(e^{a z_1}))^n = (pe^{a} + (1-p)e^{-a})^n$$

Asymptotic behaviour:

for $p = 0.5$,

$$E(x_n) = n(2(0.5)-1) = 0$$

$\Rightarrow$ for $x_n > 0.5$, x_n will tend to $+\infty$ as n increases
$x_n < 0.5$, x_n will tend to $-\infty$ as n increases
$x_n = 0$, x_n will oscillate around 0.

$E(X_m | X_n) = (m/n)\, X_n$ *for* $m < n$
otherwise X_n *for* $m > n$