Simple Random Walk.

(i) Unrestricted

$p(z_i = 1) = p$, $p(z_i = -1) = q$, $p(z_i = 0) = (1-p-q)$.

where we can denote the value $(0, \pm 1)$ of i^{th} jump by 'Z_i'

If the random walk starts at the origin, provided particle is free to move in either direction., the possible position of particle at time 'n' is

$$X_n = \sum_{r=1}^{n} Z_r = Z_1 + Z_2 + \dots Z_n$$

If we want to reach 'K' at a time 'n', then the particle may have to make r_1 positive jumps, r_2 -ve jumps, and r_3 zero jumps, with the conditions that,

$$r_1 - r_2 = K \quad \& \quad r_3 = n - r_1 - r_2.$$

$$\therefore \text{prob}\{X_n = K\} = \sum \frac{n!}{r_1!\, r_2!\, r_3!} p^{r_1} (1-p-q)^{r_3} q^{r_2}$$

The prob. generating function of a jump Z_r is :

$$G(z) = \sum p(z_i) \cdot z^i = pz^1 + (1-p-q)z^0 + qz^{-1}$$

$$= E(z^{Z_r}) \qquad = q/z + pz + (1-p-q)$$

The prob. generating function of the position of particle at time 'n' is :

$$E(z^{X_n}) = E(z^{X_0 + Z_1 + Z_2 + \dots Z_n}) = E(z^{Z_0}) \cdot E(z^{Z_1}) \dots E(z^{Z_n})$$

$$= (E(z^{Z_r}))^n$$

$$= [G(z)]^n$$

Since $X_0 (= 0$ we define $G_0(z) = 1$ and introduce a generating function for $G(z)$ as :

$$G(z,s) = \sum_{n=0}^{\infty} s^n (G(z))^n = \frac{1}{1-sG(z)} \quad (|sG(z)|<1)$$

$\downarrow$ sum of terms of ∞ G.P.

$$= \frac{z}{-spz^2 + z(1-s(1-p-q)) - sq}$$

Prob. $(X_n = k)$ is the coeff. of $s^n z^k$ in $G(z,s)$.

Let μ and σ^2 denote the mean and variance of a 'single jump', then,

$$\underline{\mu} = p-q \quad \& \quad \underline{\sigma^2} = p+q - (p-q)^2$$

$$\therefore \; E(X_n) = n\mu \quad \text{and} \quad V(X_n) = n\sigma^2$$

We now find the probability of finding the particle in one of the states $j, j+1, \ldots k$ where j & k are two possible values of $X_n (j<k)$.

Using approx. provided by Central Limit theorem, for large 'n', X_n can be normally distributed with mean '$n\mu$' & variance '$n\sigma^2$'. Thus,

$$\text{prob.}(j \le X_n \le k) \simeq (2\pi n\sigma^2)^{-\frac{1}{2}} \cdot \int_j^k e^{\left[-\frac{1}{2}\left(\frac{x-n\mu}{\sqrt{n}\,\sigma}\right)^2\right]} dx$$

Applying continuity correction, ie. use $k+c$ & $j-c$ as the limits of integration, $c = \frac{1}{2}$ (if $p+q<1$) or $c=1$ ($p+q=1$), we tranform the integral to standard form:

$$\text{prob}(j \le X_n \le k) \simeq \Phi\left(\frac{k+c-n\mu}{\sigma\sqrt{n}}\right) - \Phi\left(\frac{j-c-n\mu}{\sigma\sqrt{n}}\right)$$

where $\Phi(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{1}{2}y^2} dy$

case (I): $p > q \Rightarrow \mu > 0$ & $\sigma^2 > 0$

$X_n \sim N(n\mu, n\sigma^2) \Rightarrow \mu \pm 3\sigma > 0.99\bar{9}$

$$\Pr\{n\mu - 3\sqrt{n}\sigma \le X_n \le n\mu + 3\sqrt{n}\sigma\} \simeq 1$$

$$X_n = n\mu + O(\sqrt{n})$$

$$= n\mu\left[1 + o\left(\frac{1}{\sqrt{n}}\right)\right] \to \infty \text{ as } n \to \infty$$

$$\Pr\{X_n > j\} = 1 - \Pr\{X_n \le j\} = 1 - \Phi\left(\frac{j-c-n\mu}{\sqrt{n}\sigma}\right)$$

Now $1 - \Phi\left(\frac{j-c-n\mu}{\sigma\sqrt{n}}\right) \to 1$ as $n \to \infty$

so, particle drifts off to $+\infty$ with prob. 1.

case (II): If $p < q \Rightarrow \mu < 0$

and $X_n = n\mu + O\left(\frac{1}{\sqrt{n}}\right) = n\mu\left(1 + o\left(\frac{1}{\sqrt{n}}\right)\right)$

$\to -\infty$ as $n \to \infty$ $(\because \mu < 0)$

$\therefore$ for large 'n', particle drifts off to $-\infty$ with almost '1' probability.

(ii) Two absorbing barriers.

'3 possibilities'

(I) gets absorbed at '$-b$' → has some prob. distribution

(II) continues to wander indefinitely b/w $(-b, a)$ → '0' probability

(III) gets absorbed at 'a' → has some prob. distribution

Q. But why is the probability for (II) to happen is 0?

A. The prob. that the particle is still in motion at time 'n', ie, that it occupies one of the non absorbing states

$-b+1, -b+2 \dots a-1$, cannot exceed the probability that an unrestricted particle occupies one of these states at a time 'n', because while calculating the probab. for the former case, we exclude all possible journeys via the states/points that lie outside of the barriers $(-b, a)$.

Also, it can be deduced for $\text{Prob}\{X_n \geq j, X_n \leq k\}$ in case of unrestricted r.w., that this prob. $\to 0$ as $n \to \infty$. Therefore the prob. in case of 2 absorbing barriers can't exceed 0, and since it can't be smaller than 0 too, it must be 0.

*This also tells us that the probability for absorption is unity.

Let us start at some point 'j' $(-b \leq j \leq a)$ and let $f_{ja}^{(n)}$ $(n = 0, 1 \dots)$ be the distn of prob. of absorption at 'a', among possible times $n = 0, 1, 2 \dots$. $f_{ja}^{(n)}$ is the prob. that particle gets absorbed at 'a' at exactly time 'n'.

The initial condn : —①

$$f_{ja}^{(0)} = \begin{cases} 1, & \text{if } a = j \\ 0 & \text{if } a \neq j \end{cases} \rightarrow \text{prob. for absorption at time '0'.}$$

Also, let A_n denote the event that particle gets absorbed at 'a' after 'n' time has passed. Now based on the 1st step taken by the particle $(+1, 0, -1)$, we can have

$$f_{ja}^{(n)} = \underbrace{(p) \cdot f_{(j+1),a}^{(n-1)}}_{\substack{\text{took +1 step,} \\ \text{now start is} \\ \text{at } j+1 \text{, and rem.} \\ \text{time is 'n-1'.}}} + \underbrace{(1-p-q)\, f_{ja}^{(n-1)}}_{\substack{\text{took 0 step.} \\ \text{Start is } j, \\ \text{time rem is} \\ \text{'n-1'.}}} + \underbrace{(q) \cdot f_{j-1,a}^{(n-1)}}_{\substack{\text{took -1 step,} \\ \text{Start is } j-1, \\ \text{rem. time is 'n-1'.}}} \quad —②$$

boundary condⁿ: — ③

$$f_{aa}^{(n)} = 0 \; ; \; f_{-b,a}^{(n)} = 0 \quad (n \neq 0)$$

Eq. ② is a difference equation in 2 discrete variables 'n' (first order) and 'j' (second order). We can convert this into a single variable difference eqⁿ by using a generating function over the time variable 'n':

$$F_{ja}(s) = \sum_{n=0}^{\infty} f_{ja}^{(n)} \cdot s^{n} \quad \text{(can also be written as } F_j(s)\text{)}. \quad — ④$$

Multiply ② by s^n and sum over $n = 1, 2 \ldots \infty$, we get

$$F_{ja}(s) = s\left[p F_{j+1}(s) + (1-p-q) \cdot F_j(s) + q \cdot F_{j+1}(s)\right] — ⑤$$

This is a second order difference eqⁿ in 'j' with boundary conditions obtained from ① & ③ as:

$$F_a(s) = 1 \quad \& \quad F_{-b}(s) = 0$$

$F_a(s)=1$: absorption at a starting at a

$F_{-b}(s)=0$: absorption at +a starting at −b.

To solve ⑤, we substitute a trial solⁿ, $F_j(s) = \lambda^j$:

$$\lambda^j = s\left(p\lambda^{j+1} + (1-p-q)\lambda^j + q\lambda^{j-1}\right)$$

or $$ps\lambda^2 - \lambda(1 - s(1-p-q)) + qs = 0 \quad — ⑥$$

which is a quad. eqⁿ in λ, with 2 solⁿ as:

$$\lambda_1(s), \lambda_2(s) = \frac{1 - s(1-p-q) \pm \sqrt{[1-s(1-p-q)]^2 - 4pqs^2}}{2ps} \quad — ⑦$$

we take s as $$0 < s < \frac{1}{1-(\sqrt{p}-\sqrt{q})^2}$$

and we take $\lambda_1(s) > \lambda_2(s)$

$F_j(s) = A[\lambda_1(s)]^j + B[\lambda_2(s)]^j$ (A & B are arbitrary constants)

$F_a(s) = 1$ & $F_{-b}(s) = 0$ (boundary cond^{n}s).

using these, we obtain:

$$F_j(s) = F_{ja}(s) = \frac{\{\lambda_1(s)\}^{j+b} - \{\lambda_2(s)\}^{j+b}}{\{\lambda_1(s)\}^{a+b} - \{\lambda_2(s)\}^{a+b}} \quad - ⑧$$

If particle starts from 0, $j=0 \Rightarrow$

$$F_{0a}(s) = \frac{\{\lambda_1(s)\}^b - \{\lambda_2(s)\}^b}{\{\lambda_1(s)\}^{a+b} - \{\lambda_2(s)\}^{a+b}} \quad — ⑨$$

To obtain the prob. that ultimately particle gets absorbed at a, which is $F_{0a}(s) = \sum_{n=0}^{\infty} f_{0a}^{(n)} s^n$, we put $s=1$:

$$\Rightarrow \left[\begin{array}{l} \lambda_1(1) = \lambda_1 = \frac{q}{p} > \lambda_2 = 1 \quad (\because p<q). \\ \qquad\qquad\qquad\quad \lambda_2(1) = \lambda_2 \\ \lambda_1 = 1 > \lambda_2 = \frac{q}{p} \quad (p>q) \\ \lambda_1 = 1 = \lambda_2 \quad (p=q) \end{array}\right.$$

From ⑨ $\Rightarrow$ prob.(absorption at 'a') $= F_{0a}(1) =$

$$\begin{cases} p^a \cdot \dfrac{p^b - q^b}{p^{a+b} - q^{a+b}}, & (p \neq q) \\ \dfrac{b}{a+b}, & (p=q) \end{cases} \quad - ⑩$$

$F_{0,-b} = 1 - F_{0a}$ ($\because$ $F_{0,-b}$ & F_{0a} are mutually exclusive & exhaustive.)

$$\therefore F_{0,-b}(1) = \begin{cases} q^b \cdot \dfrac{p^a - q^a}{p^{a+b} - q^{a+b}} & , (p \neq q) \\ \dfrac{a}{a+b} & , (p = q) \end{cases}$$

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let N be the random variable that gives the prob. of absorption at N^{th} step :

$$\text{prob}\{N = n\} = f_{0a}^{(n)} + f_{0,-b}^{(n)} \quad (n = 0, 1, --)$$

$$\sum_{n=0}^{\infty} \left[f_{0a}^{(n)} + f_{0,-b}^{(n)} \right] = F_{0a}(1) + F_{0,-b}(1) = 1$$

$\therefore N$ is a r.v.

(iii) One absorbing barrier

let there be an absorbing barrier placed at 'a'. $(a > 0)$.

we want to examine the probability that if the particle will ever reach an absorbing state at 'a'.

let $f_a^{(n)}$ denote the probability of absorption at 'a' at a time 'n'.

$$f_a^{(n)} = \text{prob.}\{X_m < a \; (m = 1, --- n-1), \; X_n = a\}$$

with a generating function as :

$$F_a(s) = \sum_{n=1}^{\infty} f_a^{(n)} \cdot s^n$$

we have already obtained the solution of the above equation in the case of 2 absorbing barriers. we can thus use the same result just by plugging $b \to \infty$.

Thus $F_0(s) = \lim_{b\to\infty} \dfrac{[\lambda_1(s)]^b - [\lambda_2(s)]^b}{[\lambda_1(s)]^{a+b} - [\lambda_2(s)]^{a+b}}$

$= [\lambda_1(s)]^{-a}$ (since we assumed that $\lambda_1(s) > \lambda_2(s) \Rightarrow \left(\frac{\lambda_2(s)}{\lambda_1(s)}\right)^b \to 0$) as $b \to \infty$

$\therefore F_a(s) = [\lambda_1(s)]^{-a}$

Let N_a be the random variable denoting the time to get absorbed at 'a',

$$F_a(s) = \sum_{n=1}^{\infty} s^n . \text{prob}\{N_a = n\}$$

$$F_a(1) = \sum_{n=1}^{\infty} \text{Prob}\{N_a = n\} = \text{prob}\{N_a < \infty\}$$

this is the prob. that which means that the particle will surely get absorbed befor ∞ time, ie. at a definite time.

$$\Rightarrow \text{Prob}\{N_a < \infty\} = F_a(1) = [\lambda_1(1)]^{-a}$$

$$= \begin{cases} \left(\frac{p}{q}\right)^a, & (p < q) \\ 1, & (p \geq q) \end{cases}$$

Also, the probability that the walk will continue indefinitely is $1 - \left(\frac{p}{q}\right)^a$.

(iv) Two Reflecting Barriers

let us suppose there are 2 reflecting barriers at '0' and 'a' and the particle starts the walk from 'j'.

If X_n denotes the position of particle after n^{th} jump,

Z_n, we can write :

$$X_n = \begin{cases} X_{n-1} + Z_n & , \ 0 \le X_{n-1} + Z_n \le a \text{ (within barriers)} \\ a & , \ X_{n-1} + Z_n > a \text{ (outside a)} \\ 0 & , \ X_{n-1} + Z_n < 0 \text{ (outside '0')} \end{cases}$$

Thus, particle will remain confined within the states/ positions $0, 1, 2 --- a$.

In these types of motions, we want to evaluate the behaviour of the particle in a long run. Basically, the motion of such a particle settles down to a condition.

let $P_{jK}^{(n)}$ be the probab. that particle occupies the posn 'K' at a time 'n', having started at 'j'.

The current ~~jump~~ posn. of a particle depends upon only 2 things: the position of particle just a moment before (ie. at 'n-1') and the n^{th} jump to reach current posn.

If we take 'K' as an internal state, then during the time interval $(n-1, n)$, it can be reached by any one of the 3 mutually exclusive jumps, ie. (i) a +1 jump from K-1, (ii) a '0' jump from K and (iii) a -1 jump from K+1.

$$\therefore \ P_{jK}^{(n)} = \underbrace{p}_{\substack{+1 \text{ prob.} \\ \text{jump}}} . P_{j,K-1}^{(n-1)} + \underbrace{(1-p-q)}_{\substack{0 \text{ prob.} \\ \text{jump}}} . P_{j,K}^{(n-1)} + \underbrace{q}_{-1 \text{ jump prob.}} . P_{j,K+1}^{(n-1)} \quad — ①$$

Similarly for barrier state 'a' and '0' : — ②

$$P_{ja}^{(n)} = p . P_{j,a-1}^{(n-1)} + (1-q) . P_{ja}^{(n-1)}$$

and

$$P_{j0}^{(n)} = (1-p) . P_{j0}^{(n-1)} + q . P_{j1}^{(n-1)}$$

Also, the occupational probabilities of various states ('K') depend only upon the 'relative posⁿ of the two barriers' & neither on the initial posⁿ., nor the time, as $n \to \infty$.

$$\therefore \text{ as } n \to \infty \Rightarrow p_{jK}^{(n)} \to \Pi_K \text{ (say)} \quad (K = 0, 1, 2 \ldots a)$$

From ① and ② $\Rightarrow$

$$\Pi_K = p\Pi_{K-1} + (1-p-q)\Pi_K + q.\Pi_{K+1} \quad (K = 1, 2 \ldots a-1)$$

and $\Pi_0 = (1-p)\Pi_0 + q.\Pi_1$

$$\Pi_a = p\Pi_{a-1} + (1-q)\Pi_a$$

Thus, we can write Π_1 in terms of Π_0 as

$$\Pi_1 = \left(\frac{p}{q}\right)\Pi_0$$

and $\Pi_2 = \left(\frac{p}{q}\right)^2 \Pi_0$

⋮

and so on

In general,

$$\Pi_K = \left(\frac{p}{q}\right)^K \Pi_0 \quad (K = 0, \ldots a)$$

But since we need Π_K to be a prob. dist., then $\sum \Pi_K$ must be 1. Upon solving this, we get

$$\Pi_K = \frac{1 - \frac{p}{q}}{1 - \left(\frac{p}{q}\right)^{a+1}} \cdot \left(\frac{p}{q}\right)^K \quad (K = 0, \ldots a).$$

This is a truncated geometric distribution.
