Random Walk

Assume that, the particle is initially at a point x_0 on the x axis at time $t=1$, the particle undergoes a jump/step Z_1, where Z_1 is a random variable.
at time $t=2$, the particle undergoes a jump/step Z_2, where Z_2 and Z_1 are indipendent.

$t=0$ $\quad x_0$
$t=1$ $\quad x_0+Z_1$
$t=2$ $\quad x_0+Z_1+Z_2$

after n steps/jumps, at time $t=n$
position of particle is

$$t=n,\ x_n = x_0 + Z_1 + Z_2 + \dots + Z_n \quad \text{———} (1)$$

where Z_i are a sequence of mutually indipendent and identically distributed random varibles

$$x_n = x_{n-1} + Z_n. \quad \text{———} (2)$$

here, Z_i can take values $\{1, 0, -1\}$

with the distribution

$$P(Z_i=1) = p$$

$$P(Z_i=0) = 1-q-p$$

$$P(Z_i=-1) = q$$

This process is known as simple random walk.

* The state space will be continous if steps Z_i are continous random varibale, and discrete if the steps $Z_i \in \mathbb{Z}$ (set of int)

We consider +ve as up, -ve as down, 0 is stationary

If particle continues to move indefinitely then the random walk is unrestricted
But if we consider motion of particle to be restricted. Then there is a presence of some barrier.

A random walk starting at $x_0 = 0$ may be restricted to move within a distance a up and b down. (cannot go beyond from a, b) from the origin in such a way, that when a object reaches point a or -b then the random walk stops.

The point a and -b are absorbing barriers, and states $x \geq a$ and $x \leq -b$ are called absorbing states.

Examples:

- Insurance risk

consider an insurance company which start at period 0 with a fixed capital x_0.
during period 1, 2, ... the company recives some $y_1, y_2, y_3, \ldots$ in form of premium while it pays out some $w_1, w_2, w_3, \ldots$ in form of claims.
therefor at period n,

$$x_n = x_0 + (y_1 - w_1) + (y_2 - w_2) + \ldots + (y_n - w_n)$$

now z_i can be represented as $y_i - w_i$

$$z_i = y_i - w_i$$

$$x_n = x_0 + z_1 + z_2 + \ldots + z_n$$

if at time n, $x_n < 0$, then company is running into losses and will not continue operation

This is an example of random walk where $z_j = y_j - w_j$ and y_j and w_j are sequence of mutually indipendent randomly distributed random variables.

here $x_n = 0$ is the absorbing barrier

2). The content of a dam.

Let x_n denote the amount of water at end of n time suppose that during day 'r' Y_r units flow of water flow into the dam..
Water is released acc^n to following rule. If the content at end of day '$r-1$' is added to the inflow on day 'r' exceeds a given quantity 'α'. Then α units of water is released during day r.
otherwise the dam becomes drained by end of day r
if 'b' is capacity of the dam then we note that if $X_{r-1} + Y_r - \alpha > b$, then the dam will overflow leading to floods. The amount of overflow is $X_{r-1} + Y_r - \alpha - b$ on day r

This situation can be written as:

$$x_n = x_{n-1} + Z_n \quad (0 < x_{n-1} + Z_n < b)$$
$$x_n = 0 \quad (x_{n-1} + Z_n < 0)$$
$$x_n = b \quad (x_{n-1} + Z_n > 0)$$

where $Z_n = \{Y_n - \alpha, Y_n\}$
$\downarrow$
water flowing in

where Z_n is change in content of dam on day 'r' provided such a change does not empty/fill the dam. here Z_n is a sequence of mutually independent & identically distrib. random variables.
$x_n \in [0, b]$
It consists of reflecting barriers on $x_n = 0$ & $x_n = b$

Reflecting Barrier: State which when crossed in a given direction holds the particle until a positive jump occours. and allows it to move up to resume the random walk.

Gambler's Ruin:

This is an example of random walk with absorbing barrier
consider two gamblers 'H' and 'K' starding with 'a' and 'b'.
A sequence of indipendent turns takes place. At each turn 'H' wins 1 unit of 'K''s capital with a probablity 'q' and 'K' wins 1 unit of 'H''s capital with a probablity 'p'. such that $p+q=1$

Let x_n denote 'H''s profit after n turns

$$x_n = z_1 + z_2 + z_3 + \cdots + z_n$$

where z_i are mutually indipendent and
$P(z_i = 1) = p$
and $P(z_i = -1) = q$

at any given stage $x_n = b$ implies 'H' has gained all of 'K''s capital. and 'K' is bankrupt
Similarly, at $x_n = a$
implies 'K' as gained all of 'H''s capital and 'H' is bankrupt.

This implies x_n is a random walk starding at origin with absorbing barriers at '$-b$' and 'a'.

Unrestricted Random walk.

Here, suppose random walk starts at the origin and the particle is free to move indefinately in either direction.
The possible positions of particle at time n are

$$x_n = \sum_{r=1}^{n} z_r \qquad [K = 0, \pm 1, \pm 2, \ldots, \pm n]$$

in order to reach a point K at the time n, it has to take positive jumps (r_1), negative jumps (r_2) and no jump (r_3). where $r_1, r_2, r_3 \geq 0$

such that $r_1 - r_2 = K$ and
$r_3 = n - r_2 - r_1$

hence the probablity x_n is at k position on time 'n'. is

$$P(X_n = K) = \sum \frac{n!}{r_1! \, r_2! \, r_3!} \underbrace{p^{r_1}}_{\text{Jump } (+)} \underbrace{(1-p-q)^{r_3}}_{\text{Jump } (0)} \underbrace{(q)^{r_2}}_{\text{Jump } (-)}$$

Two absorbing barriers.
suppose a particle starts at origin and is moving in presence of absorbing barriers at $x=-b$ and $x=a$

When a particle enters state a, absorption occours at point a, and similar for $-b$
There are 3 possible outcomes of which there are 2 absorption at 'a' and '$-b$' and 3rd is that particle moves indefinitely b/w a and $-b$

we will show that when a particle is still in motion then it occupies one of the non-absorbing states $-b+1, -b+2, \cdots, a-2, a-1$

and the probablity of particle in motion cannot exceed the probablity that a unrestricted particle occupies one of the states at time 'n'

The probablity is given by

$$P(j \leq x_n \leq K) \simeq \phi\left(\frac{K+c-n\mu}{\sigma\sqrt{n}}\right) - \phi\left(\frac{j-c-n\mu}{\sigma\sqrt{n}}\right)$$