Stationary processes

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Stationary stochastic processes

Stationary stochastic processes

- All probabilities are invariant to time shits, i.e., for any s

$$\mathrm{P}\left[X(t_1+s)\geq x_1, X(t_2+s)\geq x_2, \ldots, X(t_K+s)\geq x_K\right] = \mathrm{P}\left[X(t_1)\geq x_1, X(t_2)\geq x_2, \ldots, X(t_K)\geq x_K\right]$$

- If above relation is true process is called strictly stationary (SS)
- First order stationary $\Rightarrow$ probs. of single variables are shift invariant

$$\mathrm{P}\left[X(t+s)\geq x\right] = \mathrm{P}\left[X(t)\geq x\right]$$

- Second order stationary $\Rightarrow$ joint probs. of pairs are shift invariant

$$\mathrm{P}\left[X(t_1+s)\geq x_1, X(t_2+s)\geq x_2\right] = \mathrm{P}\left[X(t_1)\geq x_1, X(t_2)\geq x_2\right]$$

Pdfs and moments of stationary process

- For SS process joint cdfs are shift invariant. Whereby, pdfs also are

$$f_{X(t+s)}(x) = f_{X(t)}(x) = f_{X(0)}(x) := f_X(x)$$

- As a consequence, the mean of a SS process is constant

$$\mu(t) := \mathbb{E}[X(t)] = \int_{-\infty}^{\infty} x f_{X(t)}(x) = \int_{-\infty}^{\infty} x f_X(x) = \mu$$

- The variance of a SS process is also constant

$$\text{var}[X(t)] := \int_{-\infty}^{\infty} (x-\mu)^2 f_{X(t)}(x) = \int_{-\infty}^{\infty} (x-\mu)^2 f_X(x) = \sigma^2$$

- The power of a SS process (second moment) is also constant

$$\mathbb{E}\left[X^2(t)\right] := \int_{-\infty}^{\infty} x^2 f_{X(t)}(x) = \int_{-\infty}^{\infty} x^2 f_X(x) = \sigma^2 + \mu^2$$

Joint pdfs of stationary process

- Joint pdf of two values of a SS stochastic process

$$f_{X(t_1)X(t_2)}(x_1, x_2) = f_{X(0)X(t_2-t_1)}(x_1, x_2)$$

- Have used shift invariance for t_1 shift ($t_1 - t_1 = 0$ and $t_2 - t_1$)
- Result above true for any pair t_1, t_2
  $\Rightarrow$ Joint pdf depends only on time difference $s := t_2 - t_1$
- Writing $t_1 = t$ and $t_2 = t + s$ we equivalently have

$$f_{X(t)X(t+s)}(x_1, x_2) = f_{X(0)X(s)}(x_1, x_2) = f_X(x_1, x_2; s)$$

Stationary processes and limit distributions

- Stationary processes follow the footsteps of limit distributions
- For Markov processes limit distributions exist under mild conditions
  - Limit distributions also exist for some non-Markov processes
- Process somewhat easier to analyze in the limit as $t \to \infty$
- Properties of the process can be derived from the limit distribution
- Stationary process $\approx$ study of limit distribution

- Formally $\Rightarrow$ initialize at limit distribution
- In practice $\Rightarrow$ results true for time sufficiently large

- Deterministic linear systems $\Rightarrow$ transient $+$ steady state behavior
- Stationary systems akin to the study of steady state behavior
- But steady state is in a probabilistic sense (probs., not realizations)

Autocorrelation and wide sense stationarity

Stationary stochastic processes

Autocorrelation function and wide sense stationary processes

Fourier transforms

Linear time invariant systems

Power spectral density and linear filtering of stochastic processes

Autocorrelation function

- From the definition of autocorrelation function we can write

$$R_X(t_1, t_2) = \mathbb{E}\left[X(t_1)X(t_2)\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_{X(t_1)X(t_2)}(x_1, x_2)\, dx_1 dx_2$$

- For SS process $f_{X(t_1)X(t_2)}(\cdot)$ depends on time difference only

$$R_X(t_1, t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_{X(0)X(t_2-t_1)}(x_1, x_2)\, dx_1 dx_2 = \mathbb{E}\left[X(0)X(t_2-t_1)\right]$$

- It then follows that $R_X(t_1, t_2)$ is a function of $t_2 - t_1$ only

$$R_X(t_1, t_2) = R_X(0, t_2 - t_1) := R_X(s)$$

- $R_X(s)$ is the autocorrelation function of a SS stochastic process
- Variable s denotes a time difference / shift
- $R_X(s)$ determines correlation between values $X(t)$ spaced s in time

Autocovariance function

- Similarly to autocorrelation, define the autocovariance function as

$$C_X(t_1, t_2) = \mathbb{E}\left[\big(X(t_1) - \mu(t_1)\big)\big(X(t_2) - \mu(t_2)\big)\right]$$

- Expand product to write autocovariance function as

$$C_X(t_1, t_2) = \mathbb{E}[X(t_1)X(t_2)] + \mu(t_1)\mu(t_2) - \mathbb{E}[X(t_1)]\,\mu(t_2) - \mathbb{E}[X(t_2)]\,\mu(t_1)$$

- For SS process $\mu(t_1) = \mu(t_2) = \mu$ and $\mathbb{E}[X(t_1)X(t_2)] = R_X(t_2 - t_1)$

$$C_X(t_1, t_2) = R_X(t_2 - t_1) - \mu^2 = C_X(t_2 - t_1)$$

- Autocovariance depends only on the time shift $t_2 - t_1$
- Most of the time we'll assume that $\mu = 0$ in which case

$$R_X(s) = C_X(s)$$

- If $\mu \neq 0$ can instead study process $X(t) - \mu$ whose mean is null

Wide sense stationary processes

- A process is wide sense stationary (WSS) if it is not stationary but
  - $\Rightarrow$ Mean is constant $\Rightarrow \mu(t) = \mu$ for all t
  - $\Rightarrow$ Autocorrelation is shift invariant $\Rightarrow R_X(t_1, t_2) = R_X(t_2 - t_1)$
- Consequently, autocovariance of WSS process is also shift invariant

$$\begin{aligned} C_X(t_1, t_2) &= \mathbb{E}\left[X(t_1)X(t_2)\right] + \mu(t_1)\mu(t_2) - \mathbb{E}\left[X(t_1)\right]\mu(t_2) - \mathbb{E}\left[X(t_2)\right]\mu(t_1) \\ &= R_X(t_2 - t_1) - \mu^2 \end{aligned}$$

- Most of the analysis of stationary processes is based on the autocorrelation function
- Thus, such analysis does not require stationarity, WSS is sufficient

Wide sense and strict stationarity

- SS processes have shift invariant pdfs
- In particular $\Rightarrow$ constant mean
  $\Rightarrow$ shift invariant autocorrelation
- Then, a SS process is also WSS
- For that reason WSS is also called weak sense stationary

- The opposite is obviously not true
- But if Gaussian, process determined by mean and autocorrelation
- Thus, WSS implies SS for Gaussian process
- WSS and SS are equivalent for Gaussian process (more coming)

Gaussian wide sense stationary process

- WSS Gaussian process $X(t)$ with mean 0 and autocorrelation $R(s)$
- The covariance matrix for $X(t_1+s), X(t_2+s), \ldots, X(t_n+s)$ is

$$\mathbf{C}(t_1{+}s, \ldots, t_n{+}s) = \begin{pmatrix} R(t_1+s, t_1+s) & R(t_1+s, t_2+s) & \ldots & R(t_1+s, t_n+s) \\ R(t_2+s, t_1+s) & R(t_2+s, t_2+s) & \ldots & R(t_2+s, t_n+s) \\ \vdots & \vdots & \ddots & \vdots \\ R(t_n+s, t_1+s) & R(t_n+s, t_2+s) & \ldots & R(t_n+s, t_n+s) \end{pmatrix}$$

- For WSS process, autocorrelations depend only on time differences

$$\mathbf{C}(t_1+s, \ldots, t_k+s) = \begin{pmatrix} R(t_1-t_1) & R(t_2-t_1) & \ldots & R(t_n-t_1) \\ R(t_1-t_2) & R(t_2-t_2) & \ldots & R(t_n-t_2) \\ \vdots & \vdots & \ddots & \vdots \\ R(t_1-t_n) & R(t_2-t_n) & \ldots & R(t_n-t_n) \end{pmatrix} = \mathbf{C}(t_1, \ldots, t_k)$$

- Covariance matrices $\mathbf{C}(t_1, \ldots, t_k)$ are shift invariant

Gaussian wide sense stationary process (continued)

- The joint pdf of $X(t_1+s), X(t_2+s), \ldots, X(t_n+s)$ is

$$f_{X(t_1+s),\ldots,X(t_n+s)}(x_1,\ldots,x_n) = \mathcal{N}(\mathbf{0}, \mathbf{C}(t_1+s,\ldots,t_n+s); [x_1,\ldots,x_n]^T)$$

- Completely determined by $\mathbf{C}(t_1+s,\ldots,t_n+s)$
- Since covariance matrix is shift invariant can write

$$f_{X(t_1+s),\ldots,X(t_n+s)}(x_1,\ldots,x_n) = \mathcal{N}(\mathbf{0}, \mathbf{C}(t_1,\ldots,t_n); [x_1,\ldots,x_n]^T)$$

- Expression on the right is the pdf of $X(t_1), X(t_2), \ldots, X(t_n)$. Then

$$f_{X(t_1+s),\ldots,X(t_n+s)}(x_1,\ldots,x_n) = f_{X(t_1),\ldots,X(t_n)}(x_1,\ldots,x_n)$$

- Joint pdf of $X(t_1), X(t_2), \ldots, X(t_n)$ is shift invariant

$\Rightarrow$ Proving that WSS is equivalent to SS for Gaussian processes

Properties of autocorrelation function

For WSS processes:

(i) The autocorrelation for $s = 0$ is the energy of the process

$$R_X(0) = \mathbb{E}\left[X^2(t)\right] = \mathbb{E}\left[X(t)X(t+0)\right]$$

(ii) The autocorrelation function is symmetric $\Rightarrow R_X(s) = R_X(-s)$

Proof: Commutative property of product & shift invariance of $R_X(t_1, t_2)$

$$\begin{aligned}
R_X(s) &= R_X(t, t+s) \\
&= \mathbb{E}\left[X(t)X(t+s)\right] = \mathbb{E}\left[X(t+s)X(t)\right] \\
&= R_X(t+s, t) \\
&= R_X(t, t-s) \\
&= R_X(-s)
\end{aligned}$$

Properties of autocorrelation function (continued)

For WSS processes:

(iii) Maximum absolute value of the autocorrelation function is for $s = 0$

$$|R_X(s)| \leq R_X(0)$$

Proof: Expand the square $\mathbb{E}\left[\left(X(t+s) \pm X(t)\right)^2\right]$

$$\mathbb{E}\left[\left(X(t+s) \pm X(t)\right)^2\right] = \mathbb{E}\left[X^2(t+s)\right] + \mathbb{E}\left[X^2(t)\right] \pm 2\mathbb{E}\left[X^2(t+s)X^2(t)\right]$$
$$= R_X(0) + R_X(0) \pm 2R_X(s)$$

Square $\mathbb{E}\left[\left(X(t+s) \pm X(t)\right)^2\right]$ is always positive, then

$$0 \leq \mathbb{E}\left[\left(X(t+s) \pm X(t)\right)^2\right] = 2R_X(0) \pm 2R_X(s)$$

Rearranging terms $\Rightarrow R_X(0) \geq \mp R_X(s)$

Fourier transforms

Stationary stochastic processes

Autocorrelation function and wide sense stationary processes

Fourier transforms

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Power spectral density and linear filtering of stochastic processes

Definition of Fourier transform

- The Fourier transform of a function (signal) $x(t)$ is

$$X(f) = \mathcal{F}(x(t)) := \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}\, dt$$

- where the complex exponential is

$$\begin{aligned} e^{-j2\pi ft} &= \cos(-j2\pi ft) + j\sin(-j2\pi ft) \\ &= \cos(j2\pi ft) - j\sin(j2\pi ft) \end{aligned}$$

- The Fourier transform is complex (has a real and a imaginary part)
- The argument f of the Fourier transform is referred to as frequency

Examples

- Fourier transform of a constant $X(t) = c$

$$\mathcal{F}(c) = \int_{-\infty}^{\infty} c e^{-j2\pi ft}\, dt = c\delta(f)$$

- Fourier transform of scaled delta function $x(t) = c\delta(t)$

$$\mathcal{F}(c\delta(t)) = \int_{-\infty}^{\infty} c\delta(t) e^{-j2\pi ft}\, dt = c$$

- For a complex exponential $X(t) = e^{j2\pi f_0 t}$ with frequency f_0 we have

$$\mathcal{F}(e^{j2\pi f_0 t}) = \int_{-\infty}^{\infty} e^{j2\pi f_0 t} e^{-j2\pi ft}\, dt = \int_{-\infty}^{\infty} e^{-j2\pi(f-f_0)t}\, dt = \delta(f - f_0)$$

- For a shifted delta $\delta(t - t_0)$ we have

$$\mathcal{F}(\delta(t - t_0)) = \int_{-\infty}^{\infty} \delta(t - t_0) e^{-j2\pi ft}\, dt = e^{-j2\pi f t_0}$$

- Note the symmetry in the first two and last two transforms

Fourier transform of a cosine

- Begin noticing that we may write $\cos(2\pi f_0 t) = \frac{1}{2}e^{j2\pi f_0 t} + \frac{1}{2}e^{-j2\pi f_0 t}$
- Fourier transformation is a linear operation (integral), then

$$\mathcal{F}\big(\cos(2\pi f_0 t)\big) = \int_{-\infty}^{\infty} \left(\frac{1}{2}e^{j2\pi f_0 t} + \frac{1}{2}e^{-j2\pi f_0 t}\right) e^{-j2\pi f t}\, dt$$
$$= \frac{1}{2}\delta(f - f_0) + \frac{1}{2}\delta(f + f_0)$$

- A pair of delta functions at frequencies $f = \pm f_0$
- Since f_0 is the frequency of the cosine it (somewhat) justifies the name frequency for the variable f

Inverse Fourier transform

- If $X(f)$ is the Fourier transform of $x(t)$, $x(t)$ can be recovered as

$$x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}\, df$$

- Above transformation is the inverse Fourier transform
- Sign in the exponent changes with respect to Fourier transform
- To show that $x(t)$ can be expressed as above integral, substitute $X(f)$ for its definition

$$\int_{-\infty}^{\infty} X(f)e^{j2\pi ft}\, df = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x(u)e^{-j2\pi fu}\, du \right) e^{j2\pi ft}\, df$$

Inverse Fourier transform

- Nested integral can be written as double integral

$$\int_{-\infty}^{\infty} X(f)e^{j2\pi ft}\, df = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(u)e^{-j2\pi fu}e^{j2\pi ft}\, du\, df$$

- Rewrite as nested integral with integration with respect to f carried on first

$$\int_{-\infty}^{\infty} X(f)e^{j2\pi ft}\, df = \int_{-\infty}^{\infty} x(u)\left(\int_{-\infty}^{\infty} e^{-j2\pi f(t-u)}\, df\right) du$$

- Innermost integral is a delta function

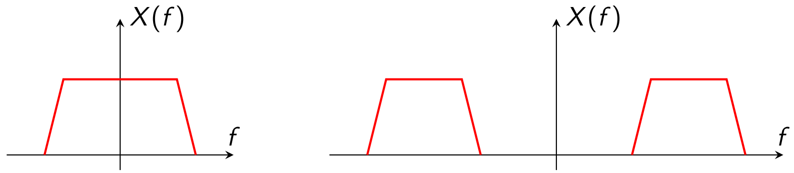
$$\int_{-\infty}^{\infty} X(f)e^{j2\pi ft}\, df = \int_{-\infty}^{\infty} x(u)\delta(t-u)\, du = x(t)$$

Frequency components of a signal

- Inverse Fourier transform permits interpretation of Fourier transform

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft}\, df \approx (\Delta f) \sum_{n=\infty}^{\infty} X(f_n) e^{j2\pi f_n t}$$

- Signal $x(t)$ written as linear combination of complex exponentials
- $X(f)$ determines the weight of frequency f in the signal $x(t)$

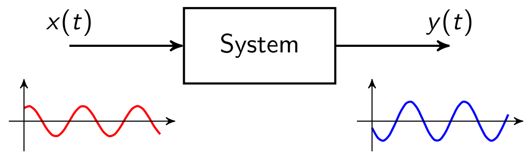


- Signal on the left contains low frequencies (changes slowly)
- Signal on the right contains high frequencies (changes fast)

Linear time invariant systems

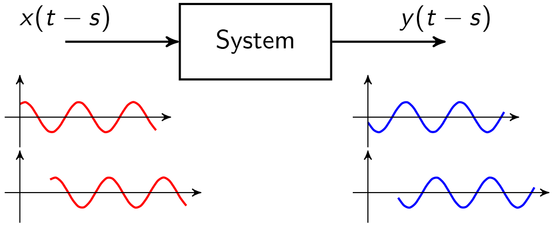
Systems

- A system is characterized by an input ($x(t)$) output ($y(t)$) relation
- This relation is between functions, not values
- Each output value $y(t)$ depends on all input values $x(t)$



Time invariant system

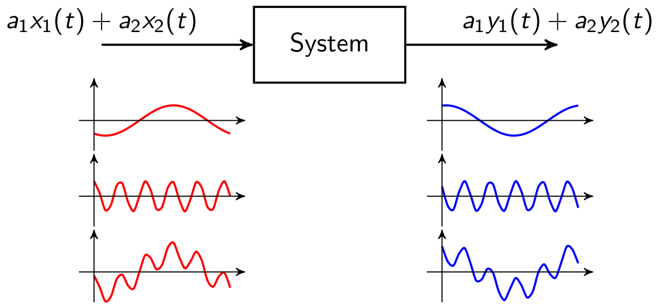
- A system is time invariant if a delayed input yields a delayed output
- I.e., if input $x(t)$ yields output $y(t)$ then input $x(t-s)$ yields $y(t-s)$
- Think of output applied s time units later



Linear system

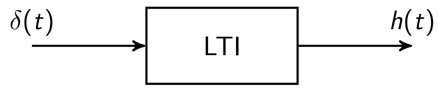
- A system is linear if the output of a linear combination of inputs is the same linear combination of the respective outputs
- That is if input $x_1(t)$ yields output $y_1(t)$ and $x_2(t)$ yields $y_2(t)$, then

$$a_1x_1(t) + a_2x_2(t) \quad \Rightarrow \quad a_1y_1(t) + a_2y_2(t)$$



Linear time invariant system

- Linear + time invariant system = linear time invariant system (LTI)
- Denote as $h(t)$ the system's output when the input is $\delta(t)$
- $h(t)$ is the impulse response of the LTI system



- System is completely characterized by impulse response

$$x(t) = \int_{-\infty}^{\infty} x(u)h(t-u)\,du = (x * h)(t)$$

- The output is the convolution of the input with the impulse response

Frequency response of linear time invariant system

- The frequency response of a LTI system is

$$H(f) := \mathcal{F}(h(t)) = \int_{-\infty}^{\infty} h(t)e^{-j2\pi ft}\, dt$$

- I.e., the Fourier transform of the impulse response $h(t)$

- If a signal with spectrum $X(f)$ is input to a LTI system with freq. response $H(f)$ the spectrum of the output is

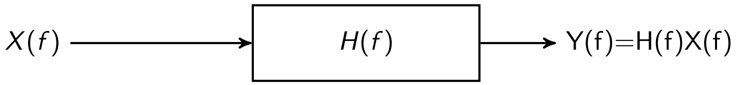
$$Y(f) = H(f)X(f)$$

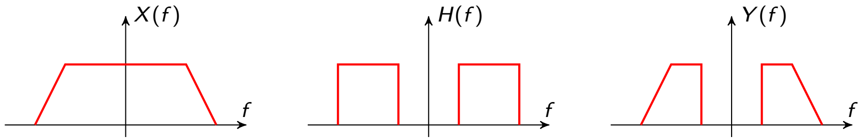
$$X(f) \longrightarrow \boxed{H(f)} \longrightarrow \text{Y(f)=H(f)X(f)}$$

More on frequency response

- Frequency components of input get "scaled" by $H(f)$
  - Since $H(f)$ is complex, scaling is a complex number
  - It represents a scaling part (amplitude) and a phase shift (argument)
- Effect of LTI on input easier to analyze

  $\Rightarrow$ Product instead of convolution





Power spectral density and linear filtering

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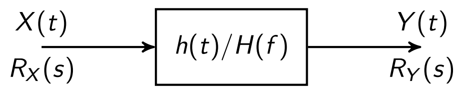
Linear filters

- Linear filter (system) with $\Rightarrow$ impulse response $h(t)$
  $\Rightarrow$ frequency response $H(f)$
- Input to filter is wide sense stationary (WSS) stochastic process $X(t)$
- Process is 0 mean with autocorrelation function $R_X(s)$
- Output is obviously another stochastic process $Y(t)$
- Describe $Y(t)$ in terms of $\Rightarrow$ properties of $X(t)$
  $\Rightarrow$ filters impulse and/or frequency response
- Is $Y(t)$ WSS? Mean of $Y(t)$? Autocorrelation function of $Y(t)$?
- Easier and more enlightening in the frequency domain



Power spectral density

- The power spectral density (PSD) of a stochastic process is the Fourier transform of the autocorrelation function

$$S_X(f) = \mathcal{F}\big(R_X(s)\big) = \int_{-\infty}^{\infty} R_X(s) e^{-j2\pi fs}\, ds$$

- Does $S_X(f)$ carry information about frequency components of $X(t)$?
- Not clear, $S_X(f)$ is Fourier transform of $R_X(s)$, not $X(t)$
- But yes. We'll see $S_X(f)$ describes spectrum of $X(t)$ in some sense

- Is it possible to relate PSDs at the input and output of a linear filter?

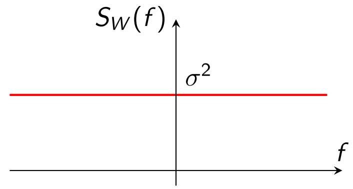
$$\xrightarrow{S_X(f)} \boxed{H(f)} \longrightarrow S_Y(f) = \ldots$$

Example: Power spectral density of white noise

- Autocorrelation of white noise $W(t)$ is $\Rightarrow R_W(s) = \sigma^2\delta(s)$
- PSD of white noise is Fourier transform of $R_W(s)$

$$S_W(f) = \int_{-\infty}^{\infty} \sigma^2\delta(s)e^{-j2\pi fs}\, ds = \sigma^2$$

- PSD of white noise is constant for all frequencies
- That's why it's white $\Rightarrow$ Contains all frequencies in equal measure



Process's power

- The power of process $X(t)$ is its (constant) second moment

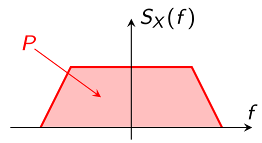
$$P = \mathbb{E}\left[X^2(t)\right] = R_X(0)$$

- Use expression for inverse Fourier transform evaluated at $t = 0$

$$R_X(s) = \int_{-\infty}^{\infty} S_X(f)e^{-j2\pi f s}\, df \ \Rightarrow R_X(0) = \int_{-\infty}^{\infty} S_X(f)e^{-j2\pi f 0}\, df$$

- Since $e^0 = 1$, can write $R_X(0)$ and therefore process's power as

$$P = \int_{-\infty}^{\infty} S_X(f)\, df$$



- Area under PSD is the power of the process

Autocorrelation of filter's output

- Let us start with second question
- Compute autocorrelation function $R_Y(s)$ of filter's output $Y(t)$
- Start noting that for any times t and s filter's output is

$$Y(t) = \int_{-\infty}^{\infty} h(u_1)X(t-u_1)\,du_1, \quad Y(t+s) = \int_{-\infty}^{\infty} h(u_2)X(t+s-u_2)\,du_2$$

- The autocorrelation function $R_Y(s)$ of the process $Y(t)$ is

$$R_Y(s) = R_Y(t, t+s) = \mathbb{E}\left[Y(t)Y(t+s)\right]$$

- Substituting $Y(t)$ and $Y(t+s)$ by their convolution forms

$$R_Y(s) = \mathbb{E}\left[\int_{-\infty}^{\infty} h(u_1)X(t-u_1)\,du_1 \int_{-\infty}^{\infty} h(u_2)X(t+s-u_2)\,du_2\right]$$

Autocorrelation of filter's output (continued)

- Product of integrals is double integral of product

$$R_Y(s) = \mathbb{E}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(u_1)X(t-u_1)h(u_2)X(t+s-u_2)\,du_1du_2\right]$$

- Exchange order of integral and expectation

$$R_Y(s) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(u_1)\mathbb{E}\Big[X(t-u_1)X(t+s-u_2)\Big]h(u_2)\,du_1du_2$$

- Expectation in the integral is autocorrelation function of input $X(t)$

$$\mathbb{E}\Big[X(t-u_1)X(t+s-u_2)\Big] = R_X\Big(t-u_1-(t+s-u_2)\Big) = R_X(s-u_1+u_2)$$

- Which upon substitution in expression for $R_Y(s)$ yields

$$R_Y(s) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(u_1)R_X(s-u_1+u_2)h(u_2)\,du_1du_2$$

Power spectral density of filter's output

- Power spectral density of $Y(t)$ is Fourier transform of $R_Y(s)$

$$S_Y(f) = \mathcal{F}\big(R_Y(s)\big) = \int_{-\infty}^{\infty} R_Y(s) e^{-j2\pi fs}\, ds$$

- Substituting $R_Y(s)$ for its value

$$S_Y(f) = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(u_1) R_X\big(s - u_1 + u_2\big) h(u_2)\, du_1 du_2 \right) e^{-j2\pi fs}\, dv$$

- Change variable s by variable $v = s - u_1 + u_2$ $(dv = ds)$

$$S_Y(f) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(u_1) R_X(v) h(u_2) e^{-j2\pi f(v+u_1-u_2)}\, du_1 du_2 dv$$

- Rewrite exponential as $e^{-j2\pi f(v+u_1-u_2)} = e^{-j2\pi fv} e^{-j2\pi fu_1} e^{+j2\pi fu_2}$

Power spectral density of filter's output

- Write triple integral as product of three integrals

$$S_Y(f) = \int_{-\infty}^{\infty} h(u_1)e^{-j2\pi f u_1}\,du_1 \int_{-\infty}^{\infty} R_X(v)e^{-j2\pi f v}\,dv \int_{-\infty}^{\infty} h(u_2)e^{j2\pi f u_2}\,du_2$$

- Integrals are Fourier transforms

$$S_Y(f) = \mathcal{F}\big(h(u_1)\big) \times \mathcal{F}\big(R_X(v)\big) \times \mathcal{F}\big(h(-u_2)\big)$$

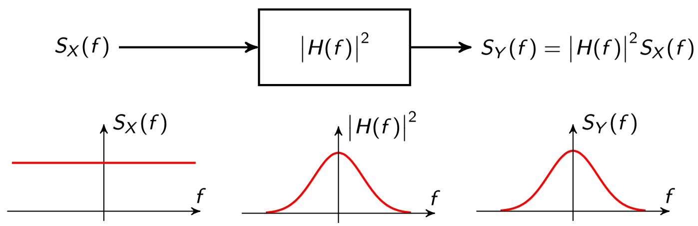
- Note definitions of $\Rightarrow$ $X(t)$'s PSD $\Rightarrow S_X(f) = \mathcal{F}\big(R_X(s)\big)$
  $\Rightarrow$ Filter's frequency response $\Rightarrow H(f) := \mathcal{F}\big(h(t)\big)$
  Also note that $\Rightarrow H^*(f) := \mathcal{F}\big(h(-t)\big)$

- Latter three observations yield (also use $H(f)H^*(f) = |H(f)|^2$)

$$S_Y(f) = H(f)S_X(f)H^*(f) = |H(f)|^2 S_X(f)$$

Example: White noise filtering

- Input process $X(t) = W(t) =$ white noise with variance σ^2
- Filter with frequency response $H(f)$. PSD of output $Y(t)$?
- PSD of input $\Rightarrow S_W(f) = \sigma^2$
- PSD of output $\Rightarrow S_Y(f) = \left|H(f)\right|^2 S_W(f) = \left|H(f)\right|^2 \sigma^2$
- Output's spectrum is the filter's frequency response scaled by σ^2



- Systems identification $\Rightarrow$ LTI system with unknown response
- Input white noise $\Rightarrow$ PSD of output is frequency response of filter

Interpretation of PSD

- Consider a narrowband filter with frequency response centered at f_0

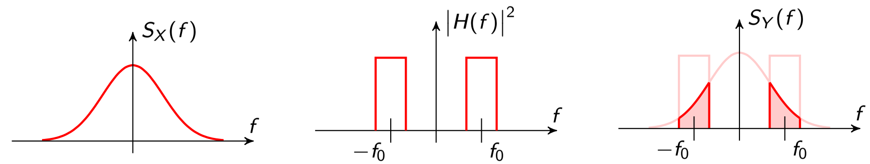
$$H(f) = 1 \quad \text{for: } \begin{array}{l} f_0 - h/2 \le f \le f_0 + h/2 \\ f_0 - h/2 \le f \le f_0 + h/2 \end{array}$$

- Input is WSS process with PSD $S_X(f)$. Output's power P_Y is

$$P_Y = \int_{-\infty}^{\infty} S_Y(f)\, df = \int_{-\infty}^{\infty} S_X(f) \left|H(f)\right|^2 df \approx h\Big(S_X(f_0) + S_X(-f_0)\Big)$$

- $S_X(f)$ is the power density the process $X(t)$ contains at frequency f

$$S_X(f) \longrightarrow \boxed{\left|H(f)\right|^2} \longrightarrow S_Y(f) = \left|H(f)\right|^2 S_X(f)$$



Thanks

- It has been my pleasure. I am very happy abut how things turned out
- If you need my help at some point in the next 30 years, let me know
- I will be retired after that