Math 345, Project

Complex Analysis

An Introduction to the Mandelbrot set

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1 Goal

My aim is to introduce an interesting subject of fractals. I will focus on the Mandelbrot set and touch upon the related Julia sets. I chose this topic because the Mandelbrot set and the Julia sets are sets of points in the complex plane. Also, their shapes caught my attention. I also provide the codes to generate a Mandelbrot set in R.

This is an introduction to the Mandelbrot set.

2 History

Julia sets were first studied by the French mathematicians Pierre Fatou and Gaston Julia in the early 20th century. Then came Benoit Mandelbrot who was trying to reduce the white noise that disturbed the transmission on telephony lines. Mandelbrot tried to visualise the data. The results showed a fractal - a structure with self-similarity at all scales. The Cantor dust fractal, that is the fractal generated by white noise from the telephony lines, was the first fractal studied by Mandelbrot.

Figure 1: Benoit Mandelbrot.[a]

Robert Brooks and Peter Matelski drew the first picture of the Mandelbrot set in 1978 while working on Kleinian groups. Benoit Mandelbrot started studying Julia sets and discovered Mandelbrot sets in 1980 by making a small change to the Iterated Function System used by Fatou and Julia. However, the mathematicians Adrien Douady and John H. Hubbard really began the mathematical study of the Mandelbrot set. They came up with many of the fundamental properties of the Mandelbrot set. They named the set in honor of Mandelbrot.

Since then, many mathematicians have contributed to the understanding of this set.

3 Introduction

Nearly everyone has seen the beautiful pictures of the Mandelbrot set. They can be seen on T-shirts, posters, album and book covers. What many people may not know is that they are generated by some mathematical formula.

Figure 2: Mandelbrot pancake.[b]

Figure 3: Mandelbrot earrings.[c]

4 Mandelbrot Set

The Mandelbrot set is a fractal. A fractal is a structure with self-similarity at all scales. All fractals can be generated using an Iterated Function System (IFS). An IFS consists of a function f which is executed as a recursive function. The output of the function is fed as input when the function calls itself. The first time the function is called, we input some initial value f(0).

Definition 1.
The Mandelbrot set is the set of all c for which the iteration $z \to z^2 + c$, starting from $z = 0$, does not diverge to infinity.

4.1 Map of the Mandelbrot Set

A very simple way to describe the Mandelbrot set is that it has a cardiod with circles to it. However, we can refine our description to include the secondary bulb, the seahorse valley, the elephant valley and the negative real axis. These regions stem mostly from colorizing the Mandelbrot set which we discuss later.

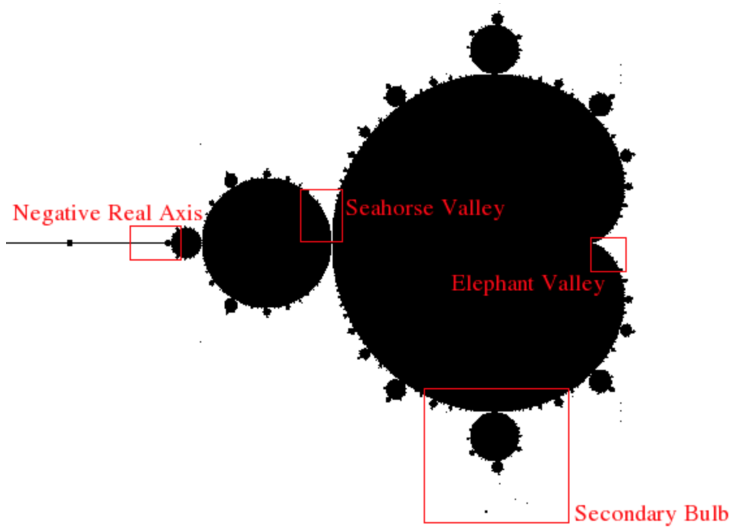

Figure 4: Map

4.1.1 A secondary Bulb

We can find the self-similarity of fractals at all scales. If we look at the secondary bulb, we see the reduplication of circles. If we look closely we can see circles attached to cardiods.

Figure 5: secondary Bulb

4.1.2 Seahorse Valley

At the boundary of the Mandelbrot set, particularly in the seahorse valley, we can see circles on circles in a clear pattern. And if we more closely, we can see a few black pixels around in the white space. And if we zoom in, we can see mini mandelbrots.

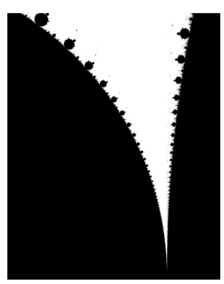

Figure 6: Seahorse Valley

Figure 7: Seahorse Valley

4.1.3 Elephant valley

We again see the self-similarity feature when we look at the elephant valley. We see smaller bulbs attached to larger bulbs in an infinite pattern.

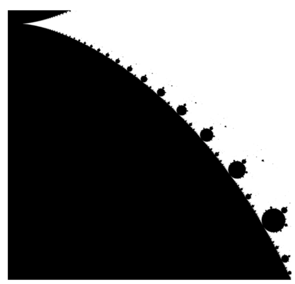

Figure 8: Elephant valley

4.1.4 Negative Real Axis

This part of the Mandelbrot set is from the region along the negative real axis of the complex plane. We can see the repeated mini mandelbrots. This region highlights an important property of the Mandelbrot - it is symmetrical across the real axis.

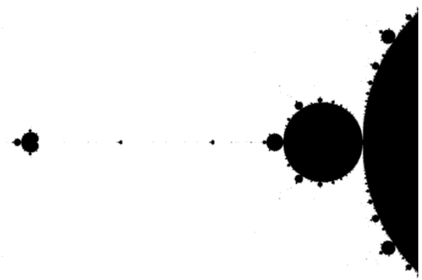

Figure 9: Negative Real Axis

4.2 Generating the set

As stated earlier, the Mandelbrot set is a set of complex numbers. We can take a random complex number, and after a few steps determine whether it is an element of the Mandelbrot set or not. The process of determining which complex numbers we keep is not complicated. We pick some complex number c. We initialize $z_0 = 0$, and let our IFS be $z_{n+1} = z_n^2 + c$. So c is the second number of the sequence. We get the next term by squaring the previous term and adding c to it. And so on. We notice that the input is a complex number and so is the output.

The Mandelbrot set contains all of the complex numbers that remain close to the origin when we apply the above sequence. To be more specific, if no term of the sequence leaves a circle of radius 2 around the origin, then those numbers will be in the Mandelbrot set. We immediately notice two issues. Since we have an infinite number of points on the complex number, it is impossible to examine each number and apply the above sequence. The second issue is that many numbers can bounce around inside our circle for quite some time before 'escaping' the circle. We will have to look at the infinite sequence, which is not possible.

So counter these issues, we use approximations. For the second issue we could pick a number and conclude that this number of terms is sufficient. For instance, $n = 500$ would be more than enough. To tackle the first issue we could build a grid on the plane, i.e. we divide the plane into squares. We can make two by two squares and have the origin be centered by one square. Each square can be labeled by its center point. We then plot the terms of our sequence. Then we can see easily if the sequence quickly escape from the circle of radius 2.

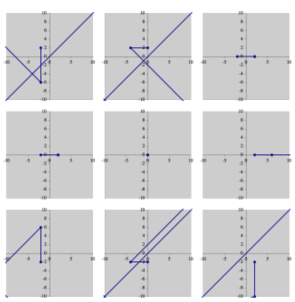

Figure 10: Plotting terms of the sequences

Figure 11: low resolution

For example, we consider squares of 2 units in length. We then get a grid as in Figure 11 above. If we divide a region that is 4 units on the side into 9 squares, we see that two complex numbers (-2,0) and (0,0) are both elements of the Mandelbrot set. We could color the 2 grids black. This is low resolution. In fact by dividing our region into more pieces, we get higher resolution as can be seen in Figure 12 below. The last grid has 400 squares. Clearly we would need a computer to visualize the Mandelbrot set.

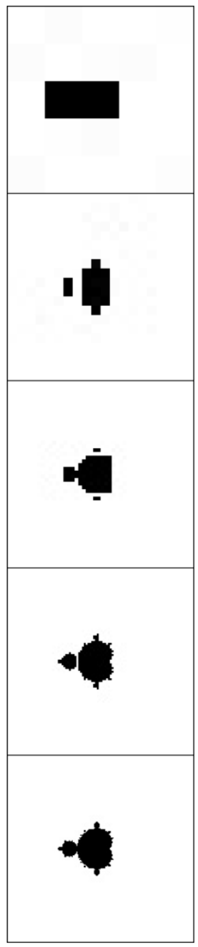

Figure 12: Higher resolution

4.3 Coloring the Set

The Escape Time Algorithm

When determining whether a complex number lies in the Mandelbrot set, we start with the sequence. We then examine the sequence over a relatively large number of terms. The sequence could then remain in an orbit near the origin or escape the circle. If it does eventually escape, how long does it take to escape?

We established that a sequence escapes if there is a term that is greater than 2 units from the origin. The longer we wait for a sequence to escape, the more the image generated looks like the Mandelbrot set as can be seen in the following figures.

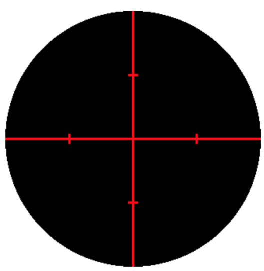

Figure 13: Initially

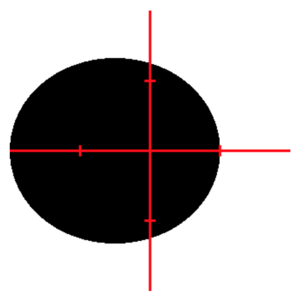

Figure 14: Escape after first term

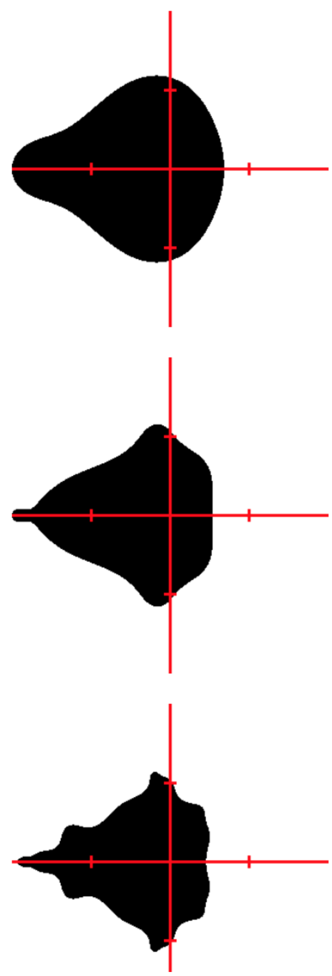

Figure 15: More Escapes

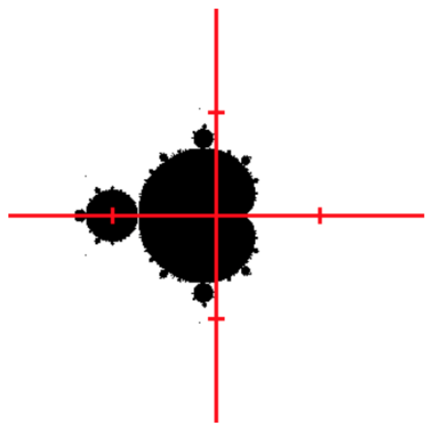

Figure 16: 256th term

For instance, we can take the images for an escape time of 0 to 4 terms and color them differently. We could color the white areas of the first image blue, the white areas of the second image aqua and so on. We could then stack these images to get the following image.

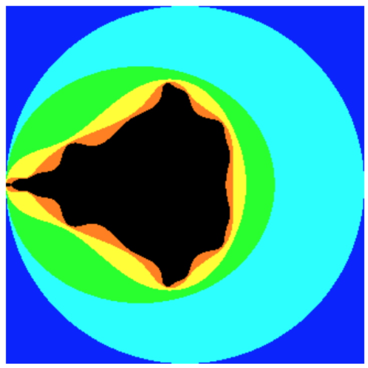

Figure 17: 0-4 escape time

If we apply the same rainbow scheme to a maximum escape time of 256 terms, we get an image like this one:

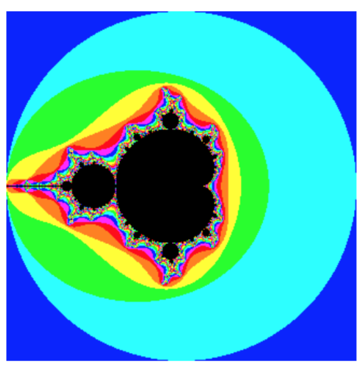

Figure 18: 256th term

Those regions with the same escape time will have the same color. We can change the colors to get different images. The colorful points are not actually elements of the Mandelbrot set. They are rather the complement of the Mandelbrot set.

5 Implementation

Here is a code snippet to create a Mandelbrot gif in R. We iterate the complex equation $z_{n+1} = z_n^2 + c$ (starting with $z_0 = 0$) for various values of c, and if doesn't go to infinity then c is part of the Mandelbrot Set.

“‘

library(caTools) — external package providing write.gif function

$jet.colors = colorRampPalette(c(\text{''}\#00007F\text{'''},\text{''}blue\text{''},\text{''}\#007FFF\text{'''},\text{''}cyan\text{''},\text{''}\#7FFF7F\text{'''},$ $\text{''}yellow\text{''},\text{''}\#FF7F00\text{''},\ \text{''}red\text{''},\text{''}\#7F0000\text{''}))$

m = 600 # define size

C = complex(real=rep(seq(-1.8,0.6, length.out=m), each=m),

imag=rep(seq(-1.2,1.2, length.out=m), m))

C = matrix(C,m,m) # reshape as square matrix of complex numbers

Z = 0 # initialize Z to zero

X = array(0, c(m,m,20)) # initialize output 3D array

for (k in 1:20) { # loop with 20 iterations

$Z = Z^2 + C$ # the central difference equation

X[,,k] = exp(-abs(Z)) # capture results }

write.gif(X, ”Mandelbrot.gif”, col=jet.colors, delay=100)

‘‘‘

Example 1.

Here is what the process looks like with 20 iterations.

Mandelbrot Example

A breakdown of the iteration can also be found in the appendices.

6 Julia Set

When we generate an image of the Mandelbrot set, the value of c is different for every point we examine. If we fix a single constant, and write a function of that constant, we get the Julia sets. The Julia sets and the Mandelbrot set are related in the sense that they are both created using the same formula.

While the Julia Set is just mentioned in this paper to help understand the history of the Mandelbrot set and how they are related, it is useful to know its formal definition.

Definition 2.

Let $R(z)$ be a rational function

$R(z) \equiv \frac{P(z)}{Q(z)}$,

where $z \in \mathbb{C}^*, \mathbb{C}^*$ is the Riemann sphere $\mathbb{C}^* \cup \{\infty\}$, and P and Q are polynomials without common divisors. The filled-in Julia set J_R is the set of points

z which do not approach infinity after $R(z)$ is repeatedly applied. The true Julia set J is the boundary of the filled-in set.

Here is an example of a Julia set formed by $f(z) = z^2 + c$ where $c = -0.7269 + 0.1889i$ [1]

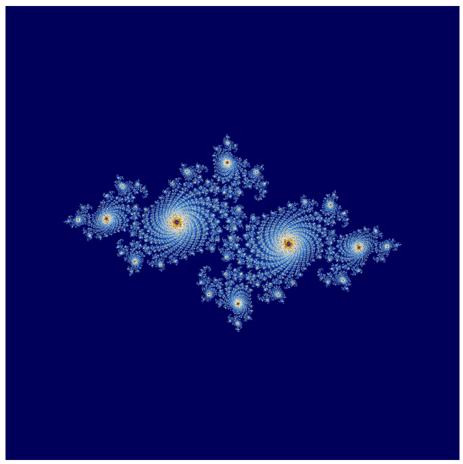

Figure 19: The Julia set for c = -0.7269 + 0.1889i

7 Conclusion

The Mandelbrot set illustrates how mathematical sequences can create great computer graphics. The equation used to find the Mandelbrot set depends on complex numbers. Although the Mandelbrot set as such does not have many uses, fractals have many applications.

[1]By IkamusumeFan-Own work, CC BY-SA 4.0, https://commons.wikimedia.org/w/index.php?curid=48176546

8 Sources

- The Universal Mandelbrot Set [Dolotin Morozov]
- The Mandelbrot Set [Xander Henderson]
- An introduction to the Mandelbrot set [Bastian Fredriksson]
- R code by David Smith (link)
- Julia Set [Wolfram] (link)