The art of proposing a question must be held of higher value than solving it.

Georg Cantor[1]

[1] **Georg Cantor** was a Russian-born mathematician who can be considered as the founder of set theory and introduced the concept of infinite numbers with his discovery of cardinal numbers. He also advanced the study of trigonometric series.

National University of Computer and Emerging Sciences

Assignment 04 | **MT1003 Calculus and Analytical Geometry**
Due Date: November 16th, 2022 | **BS DS (All Sections)**

Applications of Differentiation

Analyze and Sketch the Graph of a function

Introductory Note. It would be difficult to overstate the importance of using graphs in mathematics. Descartes'[2] introduction of analytical geometry contributed significantly to the rapid advance in calculus that began during the mid-seventeenth century. In other words of Lagrange[3], "As long as algebra and geometry traveled separate paths their advances was slow and their applications limited. But when these two sciences joined company, they drew from each other fresh vitality and thenceforth marched on at a rapis pace towards perfection".

So far, you have studied several concepts that are useful in analyzing the graph of a function.

- $x-$intercepts and $y-$intercepts (Chapter 1)
- Domain and Range(Chapter 1 - 1.1)
- Symmetry (Chapter 1 - 1.1)
- Continuity(Chapter 2 - 2.5)
- Horizontal Asymptotes(Chapter 2 - 2.6)
- Vertical Asymptotes(Chapter 2 - 2.6)
- Infinite Limits at Infinity (Chapter 2 - 2.6)
- Differentiability(Chapter 3 – 3.1 & 3.2)
- Relative Extrema(Chapter 4 – 4.1)
- Concavity(Chapter 4 – 4.4)
- Points of Inflection(Chapter 4 – 4.4)

Note: The chapters and sections are listed as per your recommended book of Thomas' Calculus[4]. One can rephrase the above list as follows. In many problems, the properties of interest in the graph of a function are:

- Symmetries
- $x-$intercepts
- Relative extrema
- Intervals of increase or decrease
- Asymptotes
- Periodicity
- $y-$intercepts
- Concavity
- Inflection points
- Behaior as $x \to +\infty$ or as $x \to -\infty$

Some of these properties may not be relevant in certain cases; for example, asymptotes are characteristic of rational functions but not of polynomials, and periodicity is

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[2] **René Descartes** was a French philosopher whose work, *La géométrie*, includes his application of algebra to geometry from which we now have Cartesian geometry. His work had a great influence on both mathematicians and philosophers.

[3] **Joseph-Louis Lagrange** was an Italian-born French mathematician who excelled in all fields of analysis and number theory and analytical and celestial mechanics.

[4] **Roland "Ron" Edwin Larson** is a professor of mathematics at Penn State Erie, The Behrend College, Pennsylvania. He is best known for being the author of a series of widely used mathematics textbooks ranging from middle school through the second year of college.

characteristic of trigonometric functions but not of polynomial or rational functions. Thus, when analyzing the graph of a function f, it helps to know something about the general properties of the family to which it belongs.

In a given problem you will usually have a definite objective for your analysis of a graph. For example, you may be interested in showing all of the important characteristics of the function, you may only be interested in the behavior of the graph as $x \to +\infty$ or as $x \to -\infty$, or you may be interested in some specific feature such as a particular inflection point. **Thus, your objectives in the problem will dictate those characteristics on which you want to focus**[5].

GUIDELINES FOR ANALYZING THE GRAPH OF A FUNCTION

1. Determine the domain andd range of the function.
2. Determine the intercepts, asymptotes, and symmetry of the graph.
3. Locate the $x-$values for which $f'(x)$ and $f''(x)$ either are zero or do not exist. Use the results to determine relative extremas and points of inflection.

Note: In these guidelines, note the importance of algebra (as well as calculus) for solving the rquations $f(x) = 0, f'(x) = 0$, and $f''(x) = 0$.

Sample Problem

Graphing a Rational Function $f(x) = P(x)/Q(x)$ if $P(x)$ and $Q(x)$ have no Common Factors

Problem: Sketch a graph of the equation

$$y = \frac{2x^2 - 8}{x^2 - 16},$$

and identify the locations of the intercepts, relative extrema, inflection points, and asymptotes.

Solution: The numerator and denominator have no common factors, so we will proceed and will outline the procedure at each step.

Step I: Symmeties. **Determine whether there is symmetry about the $y-$axis or the origin**.

Replacing x by $-x$ does not change the equation, so the graph is symmetric about the $y-$axis.

$$f(-x) = \frac{2(-x)^2 - 8}{(-x)^2 - 16} = \frac{2x^2 - 8}{x^2 - 16} = f(x), \qquad \textit{even function, symmetric about } y - \textit{axis}.$$

Step II: Intercepts. **Find the $x-$intercepts and $y-$intercepts**.

Setting $y = 0$ yields the $x-$intercepts $x = -2$ and $x = 2$. Setting $x = 0$ yields the $y-$intercept $y = 1/2$.

Step III: Vertical Asymptotes. **Find the values of x for which $Q(x) = 0$. The graph has a vertical asymptote at each such value**.

We observed above that the numerator and denominator of y have no common factors, so the graph has vertical asymptotes at the points where the denominator of y is zero, namely, at $x = -4$ and $x = 4$.

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[5] **Dr. Howard Anton** worked as a mathematician for Burroughs Corporation at Cape Canaveral in the early days of the space program. After returning to graduate school and receiving his Ph.D. in mathematics from the Polytechnic Institute of Brooklyn he accepted a position as research professor of mathematics at Drexel University in Philadelphia, PA. During his tenure at Drexel he wrote several mathematics textbooks that became widely used. After 15 years of teaching and mathematical research, he left academia to work on his textbooks full time. His textbooks are published by John Wiley and Sons, and have been translated into Spanish, Arabic, Portuguese, Italian, Indonesian, French, Japanese, Chinese, Hebrew, and German. There are more than 175 versions of his books used around the world. He is now a Professor Emeritus at Drexel University.

Step IV: Sign of $f(x)$. **The only places where $f(x)$ can change sign are at the x –intercepts or vertical asymptotes. Mark the points on the x –axis at which these occur and calculate a sample value of $f(x)$ in each of the open intervals determined by these points. This will tell you whether $f(x)$ is positive or negative over that interval.**

The set of points where x –intercepts or vertical asymptotes occur is $\{-4,-2,2,4\}$. These points divide the x –axis into the open intervals

$$(-\infty,4), \qquad (-4,-2), \qquad (-2,2), \qquad (2,+\infty).$$

We can find the sign of y on each interval by choosing an arbitrary test point in the interval and evaluating $y=f(x)$ at the test point (Table 1). This analysis is summarized on the first line of Figure 1.

SIGN ANALYSIS OF $y = \dfrac{2x^2-8}{x^2-16}$

INTERVAL	TEST POINT	VALUE OF y	SIGN OF y
$(-\infty,-4)$	-5	$14/3$	$+$
$(-4,-2)$	-3	$-10/7$	$-$
$(-2,2)$	0	$1/2$	$+$
$(2,4)$	3	$-10/7$	$-$
$(4,+\infty)$	5	$14/3$	$+$

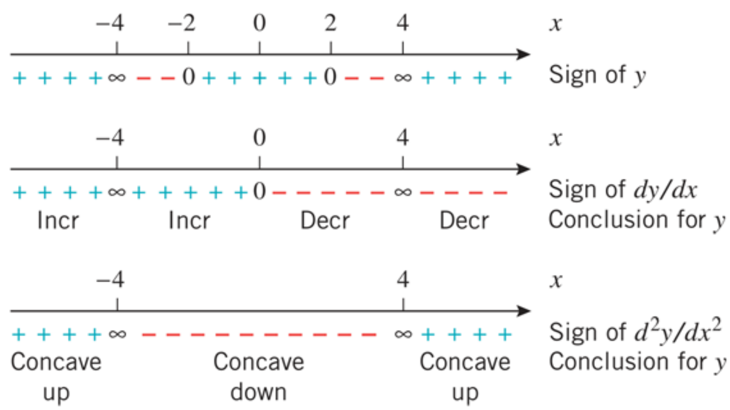

Table 1 **Figure 1**

Step V: End Behavior. **Determine the end behavior of the graph by computing the limits of $f(x)$ as $x \to +\infty$ and as $x \to -\infty$. If either limit has a finite value L, then the line $y=L$ is a horizontal asymptote.**

The limits

$$\lim_{x\to+\infty}\frac{2x^2-8}{x^2-16}=\lim_{x\to+\infty}\frac{2-(8/x^2)}{1-(16/x^2)}=2\,,$$

$$\lim_{x\to-\infty}\frac{2x^2-8}{x^2-16}=\lim_{x\to-\infty}\frac{2-(8/x^2)}{1-(16/x^2)}=2\,,$$

yield the horizontal asymptote $y=2$.

Step VI: Derivatives. **Find $f'(x)$ and $f''(x)$.**

$$\frac{dy}{dx}=\frac{(x^2-16)(4x)-(2x^2-8)(2x)}{(x^2-16)^2}=\frac{48x}{(x^2-16)^2}\,.(Verify)$$

$$\frac{d^2y}{dx^2}=\frac{48(16+3x^2)}{(x^2-16)^3}\,.(Verify)$$

Step VII: Conclusions and Graph. **Analyze the sign changes of $f'(x)$ and $f''(x)$ to determine the**

intervals where $f(x)$ is increasing, decreasing, concave up, and concave down. Determine the locations of all stationary points, relative extrema, and inflection points. Use the sign analysis of $f(x)$ to determine the behavior of the graph in the vicinity of the vertical asymptotes. Sketch a graph of f that exhibits these conclusions.

- The sign analysis of y in Figure 1 reveals the behavior of the graph in the vicinity of the vertical asymptotes: The graph increases without bound as $x \to -4^-$ and decreases without bound as $x \to -4^+$; and the graph decreases without bound as $x \to -4^-$ and increases without bound as $x \to -4^+$ (Figure 2).

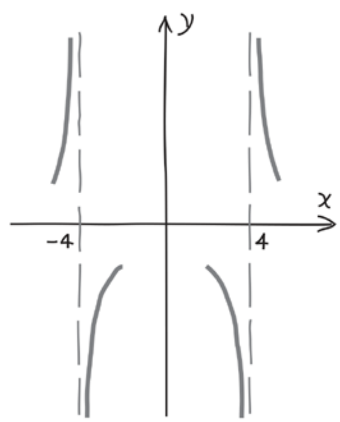

Figure 2

- The sign analysis of dy/dx in Figure 1 shows that the graph is increasing to the left of $x = 0$ and is decreasing to the right of $x = 0$. Thus, there is a relative maximum at the stationary point $x = 0$. There are no relative minima.

- The sign analysis of d^2y/dx^2 in Figure 1 shows that the graph is concave up to the left of $x = -4$, is concave down between $x = -4$ and $x = 4$, and is concave up to the right of $x = 4$. There are no inflection points.

The graph is shown in Figure 3.

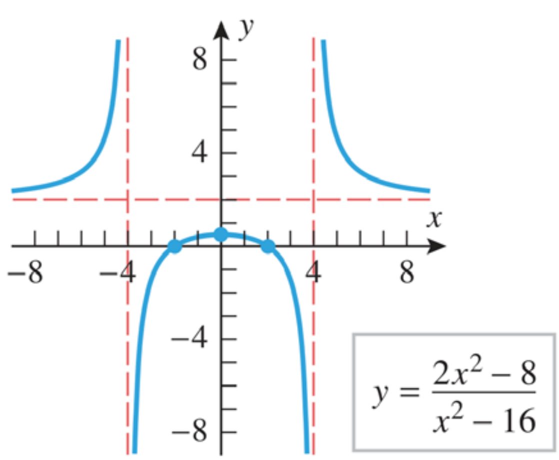

Figure 3

Q. 1 **Sketching the Graph of a Polynomial Function.** Analyze and sketch the graph of $f(x) = x^4 - 12x^3 + 48x^2 - 64x$.

Q. 2 **Sketching the Graph of a Rational Function.** Analyze and sketch the graph of $f(x) = 2(x^2 - 9)/x^2 - 4$.

Q. 3 **Sketching the Graph of a Rational Function.** Analyze and sketch the graph of $f(x) = x^2 - 2x + 4/x - 2$.

Q. 4 **Sketching the Graph of a Radical Function.** Analyze and sketch the graph of $f(x) = 2x^{5/3} - 5x^{4/3}$.

Q. 5 **Sketching the Graph of a Radical Function.** Analyze and sketch the graph of $f(x) = x/\sqrt{x^2 + 2}$.

Q. 6 **Sketching the Graph of a Trigonometric Function.** Analyze and sketch the graph of $f(x) = \cos x\,/(1 + \sin x)$.

Q. 7 **Graphing Where Some Features Are Difficult to See.** Draw a graph of

$$f(x) = \frac{1}{x} + \sqrt{\frac{1}{x^2} + 4},$$

and show all significant features.

Recommended Readings

1. Calculus *Early Transcendentals (*International Student Version or Available One) by **Howard Anton** et al.
2. Calculus *Early Transcendentals (*Metric Version or Available One) by **James Stewart** et al.
3. Calculus by **Smith** and **Minton**.

Good Luck