

# DERIVATIVES OF INVERSES.

GOAL: In this section we will use the derivatives of one-to-one functions to obtain the derivatives of the inverse functions.

↓ This simple fact will help us to find two things.

1. The derivative formula for exponential functions from the derivative formulas for logarithmic functions.
2. The derivative formulas for inverse trigonometric functions from the derivative formulas of trigonometric functions.

Let us begin.

Our first goal in this section is to obtain a formula relating the derivative of an inverse function  $f^{-1}$  to the derivative of an invertible function  $f$  (whose inverse exists).

We will arrive at the required notion by looking at the example.

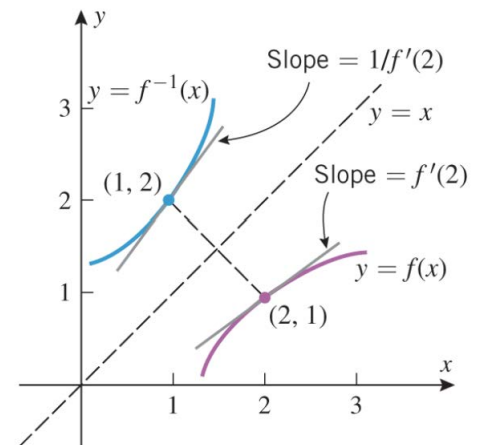
Suppose that  $f$  is a one-to-one differentiable function such that  $f(2) = 1$  and  $f'(2) = \frac{3}{4}$ .

Then tangent line to  $y = f(x)$  at point  $(2, 1)$  has equation

$$y - y_0 = m(x - x_0).$$

Here  $y_0 = 1$ ;  $x_0 = 2$  and  $m = f'(2) = \frac{3}{4}$ . So

$$y - 1 = \frac{3}{4}(x - 2).$$



Since  $f^{-1}$  can be constructed by taking reflection of  $y = f(x)$  about line  $y = x$ . One can obtain the tangent line to  $y = f^{-1}(x)$  at the point  $(1, 2)$  by interchanging  $x$  and  $y$ :

$$x-1 = \frac{3}{4}(y-2) \text{ or } y-2 = \frac{4}{3}(x-1).$$

Notice that the slope of the tangent line to  $y = f^{-1}(x)$  at  $x=1$  is the reciprocal of the slope of the tangent line to  $y = f(x)$  at  $x=2$ .

That  $(f^{-1})'(1) = \frac{1}{f'(2)} = \frac{1}{3/4} = \frac{4}{3} \rightarrow (1)$ .

Notice that  $f^{-1}(1) = 2$  because  $f(2) = 1$ .

Implies,

$$f'(2) = f'(f^{-1}(1)).$$

$$(1) \Rightarrow (f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.$$

In general, if  $f$  is a differentiable and one-to-one function, then

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} \rightarrow (2).$$

Mathematically, the equation  $y = f^{-1}(x)$  is equivalent to  $x = f(y)$ .

By implicit differentiation & differentiating w.r.t  $x$  gives

$$\frac{d}{dx}(x) = \frac{d}{dx} f(y)$$

$$\Rightarrow 1 = f'(y) \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{f'(y)} = \frac{1}{f'(f^{-1}(x))}$$

Also,

$$\Rightarrow \frac{d}{dy}(x) = \frac{d}{dy} f(y) \Rightarrow \frac{dx}{dy} = f'(y) \Rightarrow \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} \quad (3)$$

Is there a link?

Alternative version of (2).

The rest of conclusions will be made.  
during future discussion(s).

Thank you!