515 - Assignment 5

1] We know $\phi_i(v) = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!\,(n-|S|-1)!}{n!} [v(S \cup \{i\}) - v(S)]$

When i joins S, edges $\{i,j\}, (j \in S)$ get activated

Hence $v(S \cup \{i\}) - v(S) = \sum w(i,j)$

$\therefore \phi_i(v) = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!\,(n-|S|-1)!}{n!} \sum_{\{i,j\} \in E} w(i,j)$

Each subset S that contains j but not i can be writen as, $S = T \cup \{j\}$, where $T \subseteq N \setminus \{i,j\}$

$\therefore |S| = |T| + 1$

$\therefore \frac{|S|!\,(n-|S|-1)!}{n!} = \frac{(|T|+1)!\,(n-|T|-2)!}{n!}$

Number of subsets T of size k = ${}^{n-2}C_k$

$$\therefore \frac{|S|!\,(n-|S|-1)!}{n!} = \sum_{k=0}^{n-2} {}^{n-2}C_k \frac{(k+1)!\,(n-k-2)!}{n!}$$

$$= \frac{1}{n!} \sum_{k=0}^{n-2} \frac{(n-2)!\,(k+1)!\,(n-k-2)!}{k!\,(n-k-2)!}$$

$$= \frac{(n-2)!}{n!} \sum_{k=0}^{n-2} (k+1)$$

$$= \frac{(n-2)!}{n!} \frac{(n-1)n}{2}$$

$$= \frac{n!}{2.n!}$$

$$= \frac{1}{2}$$

$$\therefore \phi_i(v) = \cancel{\sum_{S \subseteq N \setminus \{i\}}} \frac{1}{2} . \sum_{\{i,j\} \in E} w(e)$$

2] a) $x_1 + x_2 + x_3 = 1$

$x_1 + x_2 \geq 1$

$x_2 + x_3 \geq 1$

$x_1 + x_3 \geq 0$

$x_i \geq 0$

Since $x_1 + x_2 \geq 1$ and $x_1 + x_2 + x_3 = 1$,

we get $x_3 \leq 0$. Since $x_3 \geq 0$,

$x_3 = 0$

Since $x_2 + x_3 \geq 1$ and $x_1 + x_2 + x_3 = 1$,

we get $x_1 \leq 0$. However $x_1 \geq 0$

$\therefore x_1 = 0$

Since $x_1 + x_2 + x_3 = 1$, $x_2 = 1$

Hence $(x_1 + x_2 + x_3) = (0, 1, 0)$

$$\phi_i = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!\,(n - |S| - 1)!}{n!} [v(S \cup \{i\}) - v(S)]$$

For $|S| = 0$, $\frac{0!\,2!}{3!} = \frac{2}{6} = \frac{1}{3}$

For $|S| = 1$, $\frac{1!\,1!}{3!} = \frac{1}{6}$

For $|S| = 2$, $\frac{2!\,0!}{3!} = \frac{2}{6} = \frac{1}{3}$

For player 1,

S	v(S)	v(S ∪ {i})	diff
ϕ	0	0	0
{2}	0	1	1
{3}	0	0	0
{2,3}	1	1	0

$\therefore \phi_1 = \frac{1}{3}(0) + \frac{1}{6}(0+1) + \frac{1}{3}(0)$
$= 1/6$

For player 2,

S	$v(S)$	$v(S \cup \{i\})$	diff
ϕ	0	0	0
$\{1\}$	0	1	1
$\{3\}$	0	1	1
$\{1,3\}$	0	1	1

$\therefore \phi_2 = \frac{1}{3}(0) + \frac{1}{6}(1+1) + \frac{1}{3}(1)$
$= \frac{1}{3} + \frac{1}{3}$
$= 2/3$

For player 3,

S	$v(S)$	$v(S \cup \{i\})$	diff
ϕ	0	0	0
$\{1\}$	0	0	0
$\{2\}$	0	1	1
$\{1,2\}$	1	1	0

$\therefore \phi_3 = \frac{1}{3}(0) + \frac{1}{6}(1+0) + \frac{1}{3}(0)$
$= 1/6$

$\therefore$ Shapely value $= (\frac{1}{6}, 2/3, 1/6)$

b) $w(1,2) = 2$, $w(1,3) = 2$, $w(1,4) = 0$, $w(2,3) = 3$, $w(3,4) = 0$

$\phi_i = \frac{1}{2} \sum_{e \in E_i} w(e)$

$\therefore \phi_1 = \frac{1}{2}(2+2+0) = \frac{1}{2}(4) = 2$

$\phi_2 = \frac{1}{2}(2+3) = \frac{1}{2}(5) = 2.5$

$\phi_3 = \frac{1}{2}(2+3+0) = \frac{1}{2}(5) = 2.5$

$\phi_4 = \frac{1}{2}(0+0) = \frac{1}{2}(0) = 0$

$\therefore$ Shapely value $= (2, 2.5, 2.5, 0)$

$x_1 + x_2 + x_3 + x_4 = 7$

$x_1 + x_2 \geq 2$

$x_2 + x_3 \geq 3$

$x_1 + x_3 \geq 2$

We can see that the shapely vector satisfies these conditions.
Hence core = shapely = $(2, 2.5, 2.5, 0)$

c) Total value $= 1+1+1+2+2+3 = 10$
Threshold $= 5$

We know a core exists if and only if there is a veto player

Player 1, ~~and not~~ 2, 3 are not veto players as a winning coalition exists without them (player 4 & player 6)

Player 4 and 5 are not veto players as a winning coalition exists without them (player 1, 2, and 6)

Player 6 is not a veto player as a winning coalition exists without them (player 1, 4, and 5)

Thus core is empty

By law of symmetry, ~~ϕ_1~~ $\phi_1 = \phi_2 = \phi_3$, $\phi_4 = \phi_5$

For coalition size k,

$(|s|-1)!\,(n-|s|)! = \begin{cases} 1!\cdot 4! = 24 \\ 2!\cdot 3! = 12 \\ 3!\cdot 2! = 12 \end{cases}$

S	MWC_s	Weighted Sum
1	5	3 → size 3, 2 → size 4 → $3\times 12 + 2\times 12 = 60$
4	5	1 → size 2, 3 → size 3, 1 → size 4 = $24 + 3\times 12 + 12 = 72$
6	5	2 → size 2, 3 → size 3 = $48 + 36 = 84$

$\therefore\ \phi_1 = \frac{60}{720} = \frac{1}{12}$

$\phi_4 = \frac{72}{720} = \frac{1}{10}$

$\phi_6 = \frac{84}{720} = \frac{7}{60}$

$\therefore$ Shapely values = $\left(\frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{10}, \frac{1}{10}, \frac{7}{60}\right)$

3] a) No it is not necessary that a superadditive game has a non-empty core

eg. $v(1) = v(2) = v(3) = 0$, $v(\{1,2\}) =$ $v(\{1,3\}) = v(\{2,3\}) = 1$, $v(\{1,2,3\}) = 1$

$v(\{1,2,3\}) \geq v(\{1,2\}) + v(\{3\})$. Hence it is superadditive.

$x_1 + x_2 + x_3 = 1$

$x_1 + x_2 \geq 1$, $x_2 + x_3 \geq 1$, $x_1 + x_3 \geq 1$

$\therefore 2(x_1 + x_2 + x_3) \geq 3$

$\therefore x_1 + x_2 + x_3 \geq 1.5$

Hence the core is empty

b) In order to show this we need to prove $\alpha x + (1-\alpha) y$ satisfies both efficiency and rationality of coalition

$$\sum_i \alpha x_i + (1-\alpha) y_i = \alpha \sum_i x_i + (1-\alpha) \sum_i y_i = \alpha V(N) + (1-\alpha) V(N) = V(N)$$

Hence efficiency satisfied

For any S, As x, y belong to the core,

$$\sum_{i \in S} \alpha x_i + (1-\alpha) \sum_{i \in S} y_i = \alpha \sum_i x_i + (1-\alpha) \sum_i y_i \geq \alpha v(S) + (1-\alpha) v(S)$$
$$\geq v(S)$$

Hence rationality of coalition is true

Thus if $x, y \in \text{core}(G)$, $\alpha x + (1-\alpha) y \in$ Core as well

c) The definition of dummy player is if $\forall S \subseteq N$, $v(S \cup \{i\}) = v(S)$

$\therefore V(N) = v(N \setminus \{i\})$ — (1)

We know that in the core, $\sum_{j \in N} x_j = v(N)$

$\therefore \sum_{j \in N} x_j = \sum_{j \in N \setminus \{i\}} x_j$ — substituting in (1)

$\therefore$ This is only possible if $x_i = 0$

Hence the payoff for player i is 0.

4] a) $\phi_i = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!\,(n-|S|-1)!}{n!} \left[v(S \cup \{i\}) - v(S)\right]$

For coalitions, where i is pivitol, $v(S \cup \{i\}) = 1$ and $(v(S) = 0)$

$\therefore v(S \cup \{i\}) - v(S) = 1$

$\therefore \phi_i = \sum_{k=0}^{n-1} \sum_{S \in Piv_i k} \frac{|k|!\,(n-|k|-1)!}{n!}$

$= \frac{1}{n!} \sum_{k=0}^{n-1} |k|!\,(n-|k|-1)! \cdot |Piv_i(k)|$ → Since there are $Piv_i(k)$ such coalitions

b) Since player i is pivitol, ~~S~~ $\forall$, $S \subseteq Piv_i(k)$

$v(S \setminus \{i\}) < q$ → By definition of pivitol we know that the coalition was not winning before i joins and wins with i.

$\therefore v(S \setminus \{i\}) \leq q-1$

Since ~~ℓ~~ i can only increase the coalition with its own weight,

$V(S \setminus \{i\}) + v(i) \geq q$

$\therefore v(S \setminus \{i\}) \geq q - w_i$

$\therefore q-1 \geq v(S \setminus \{i\}) \geq q - w_i$

c) Let A be the coalitions not containing player j.

$\therefore A = S \subseteq \{1, \dots, j-1\}$ where $|S| = k$, $\sum_{i \in S} w_i = r$

$\therefore A = x(k, r, j-1)$

Let B be the coalitions containing player j

$\therefore$ $B = \{S \subseteq \{1 \dots j\} \mid j \in S, |S| = k, \sum_{i \in S} w_i = r\}$

For any $S \in B$, $S' = S \setminus \{j\}$, Then $S' \subseteq \{1, \dots, j-1\}$, $|S'| = k-1$ and $\sum_{i \in S'} w_i = r - w_j$

The converse is true as well
Thus there is a bijection btwn B and S'

$\therefore$ $|B| = x(k-1, r-w_j, j-1)$

Since for every coalition $S \subseteq \{1, \dots, j\}$ of size k and weight r is either in A or B. Since $A \cap B = \emptyset$,

$$x(k,r,j) = |A| + |B| = x(k,r,j-1) + x(k-1, r-w_j, j-1)$$

c) Let $T = S \setminus \{i\} \subseteq N$ s.t. $T \subseteq N \setminus \{i\}$, $|T| = k-1$, $w(T) \in [q-w_i, q-1]$

For player n,

$Piv_n(k) = \{S \subseteq \{1 \dots n\} \mid |S| = k, n \in S, w(T) < q, w(T) + w_n \geq q\}$

$x(k-1, r, n-1) = \{T \subseteq \{1, \dots, n-1\} \mid |T| = k-1, w(T) = r\}$

$\therefore$ For $q - w_n \leq r \leq q-1$,

$$|Piv_n(k)| = \sum_{r=q-w_n}^{q-1} x(k-1, r, n-1)$$

Computing $x(k,r,j)$ can be done in ~~constant~~ linear time using dynamic programming

$\therefore$ Computing $x(k,r,j) = O(n \cdot k_{max} \cdot r_{max})$

Computing $Piv_i(k)$ from the table can be done in $O(w_n)$

Thus computing $Piv_i(k) = O(n \cdot k_{max} \cdot r_{max})$

e) From part (a), $\phi_i = \frac{1}{n!} \sum_{k=0}^{n-1} k!(n-k-1)! |P_{iv_i}(k)|$

From part (c), $|P_{iv_i}(k)| = \sum_{q-w_i}^{q-1} x(k,r,n-1)$

In the DP table, each entry can be computed in $O(1)$ time using recurrence, so total time to fill table $= O(n^2 q)$

If we repeat for each player $= O(n^3 q)$
Thus this not mean P=NP. It is said to be P-complete when weights are encoded in binary.
The dynamic programming method described here runs in pseudo-polynomial time, polynomial in n and q, but exponential in the size of the input

5] $\sum_{i\in N}\phi_i = \sum_{i\in N} \sum_{S\subseteq N\setminus\{i\}} \frac{|S|!(|N|-|S|-1)!}{|N|!} (v(S\cup\{i\}) - v(S))$

$= \sum_{S\subseteq N} \sum_{i\in N\setminus S} \frac{|S|!(|N|-|S|-1)!}{|N|!} (v(S\cup\{i\}) - v(S))$

Each inner sum corresponds to the total value of the grand coalition thus summing over all players contribution gives $v(N)$

$\therefore \sum_{i\in N} \phi(i) = v(N)$

$\phi_i = \sum_{S\subseteq N\setminus\{i\}} \frac{|S|!(|N|-|S|-1)!}{|N|!} (v(S\cup\{i\}) - v(S))$

Since $v(\{i\})$ is atleast the marginal contribution of player i,
$\phi_i = \sum_{S\subseteq N\setminus\{i\}} \frac{|S|!(|N|-|S|-1)!}{|N|!} (v(S\cup\{i\}) - v(S)) \geq v(\{i\})$

Thus ϕ_i satisfies individual rationality

$$\sum_{i \in S} \phi_i = \sum_{i \in S} \sum_{T \subseteq N \setminus \{i\}} \frac{|T|!\,(|N|-|T|-1)!}{|N|!} (v(T \cup \{i\}) - v(T))$$

$$= \sum_{T \subseteq N} \sum_{i \in S \cap (N \setminus T)} \frac{|T|!\,(|N|-|T|-1)!}{|N|!} (v(T \cup \{i\}) - v(T))$$

$\therefore$ For any coalition $T \subseteq N$ and players $i \in S$, supermodality tells us,

$$v(T \cup \{i\}) - v(T) \geq v(S \cup \{i\}) - v(S)$$

Thus summing over all players $i \in S$, and applying this property,

$$\sum_{i \in S} \phi_i \geq v(S)$$

Since shapely value satisfies efficiency, individual rationality and conditional rationality, shapely value is a core for the game