Distance asynchronous Olympiad for volunteers

(Innopolis University, Fall semester 2020, BS-II)

Description

It is distance asynchronous individual written contest for volunteers on Ordinary Differential Equations based on the course by Nikolay V. Shilov for the second year students of Innopolis University in the Fall semester of Academic Year 2020/21 (https://moodle.innopolis.university/course/view.php?id=502).

Every solution of any single problem is welcome! Authors of the best papers (up to 10) will be awarded by additional points for the course (from the reserve up to 6 points in Instructors' Gratitude).

The timeline of the examination follows:

- Contest problem publication on Moodle by Wednesday November 18, 2020.
- Solutions to be uploaded to Moodle for grading by 24:00 Friday November 28, 2020.
- Submissions to be evaluated by Nikolay V. Shilov Friday December 4, 2020.
- Public presentation of solutions and award ceremony (online) – Saturday December 5, 2020.

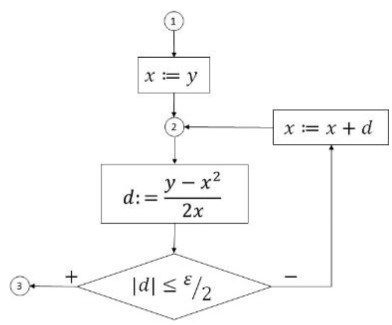
Rules

1. "Proof of individual work" rule means that a participant should be able to present and explain his/her solution(s) at Public presentation and award ceremony on Saturday December 5, 2020.
2. Each submission should consist of a single PDF file named by student first name and surname (for example: NikolayShilov.pdf), but this file should be produced by some document-processing systems (MS Office, LaTeX, etc.)
3. On the top of the front page of each submission should start with student first name and surname.
4. Submissions with scanned or photo images of hand-written solutions will be discarded without consideration!

Problems

Problem 1

Prove *quadratic convergence* of the algorithm $SQRT$(see right), i.e. that every next truncation error is (at least) square of the previous one; using quadratic convergence, evaluate computational complexity of the $SQRT$ algorithm (in terms of number of iterations as a function of y and ε).



Problem 2

Let $f: \boldsymbol{R} \to \boldsymbol{R}$ be a continuous function. Find a solution for the equation $y'' = f(y)$ as general as you can (i.e. if it is the most general then your solution must prove it explicitly).

Problem 3

Let p and q be your individual birthday and birth month (for example, 24 and 4 for April 24), $a < b$ be real numbers and $f: [a, b] \to \boldsymbol{R}$ be a continuous function. Find a solution for the equation $y'' + py' + qy = f(x)$ (where x stays for variable ranging over $[a, b]$) as general as you can (i.e. if it is the most general then your solution must prove it explicitly). Express your answer in terms of elementary functions and integrals (definite or indefinite).

Problem 4

Show that Laplace transform $L(e^{t^2})$ is not defined at any point. In contrast, $L(te^{t^2} \cos e^{t^2})$ is defined in some neighborhood of the infinity $+\infty$.

Problem 5

Prove convergence of the series $\sum_{n\geq 0} \frac{n^{const}}{5^n}$ and then compute $\sum_{n\geq 0} \frac{1}{5^n}$, $\sum_{n\geq 0} \frac{n}{5^n}$, and $\sum_{n\geq 0} \frac{n^2}{5^n}$.

Problem 6

Prove or refute the following statement:

Let $[a, b]$ be an interval of real numbers, and $f_1, f_2, \dots : [a, b] \to \boldsymbol{R}$ *be an infinite sequence of real-valued functions on* $[a, b]$.

- *Assume that the sequence* $f_1(x), f_2(x), \dots$ *converges pointwise to real-valued function* $f_\infty: [a, b] \to \boldsymbol{R}$.
- *Then for every* $\varepsilon > 0$ *there exists a sequence of* $(I_k)_{k\in \boldsymbol{N}}$ *of open intervals with total length less than* ε *(i.e.* $\varepsilon > \sum_{k=1}^{\infty} \ell(I_k)$*) such that the sequence* $f_1(x), f_2(x), \dots$ *uniformly converges to* f_∞ *on* $([a, b] \setminus \bigcup_{k=1}^{\infty} I_k)$ *and* f_∞ *is continuous on* $([a, b] \setminus \bigcup_{k=1}^{\infty} I_k)$.

Problem 7

Let $f(x) = \begin{cases} e^{-1/x}, \text{ if } x > 0 \\ 0, \text{ if } x \leq 0 \end{cases}$.

- Prove that $f^{(n)}(x) = \begin{cases} \frac{p_n(x)}{x^{2n}} f(x), \text{ if } x > 0 \\ 0, \text{ if } x \leq 0 \end{cases}$ for all $n \geq 0$, where $p_n(x)$ is defined recursively as follows: $p_1(x) = 1$ and $p_{(n+1)}(x) = x^2 p_n'(x) + (1 - 2nx)p_n(x)$ for all $n \geq 0$.
- Prove that the function is smooth everywhere on $\boldsymbol{R}$ but nowhere analytical.