Real & Complex Analysis II

Final Project: The Riemann Zeta Function

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1. Introduction

Although the Riemann zeta function $\zeta(s)$ had been studied in various forms before the work of mathematician Bernhard Riemann, it was Riemann who significantly expanded its definition and explored its properties. In his 1859 paper "On the Number of Primes Less Than a Given Magnitude," Riemann aimed to understand and describe the distribution of prime numbers. To achieve this, he extended the zeta function to complex numbers and introduced an analytic continuation of the function to the entire complex plane, excluding the point $s = 1$. Riemann's key contribution was linking the zeros of the zeta function to the distribution of primes. This exploration also resulted in the famous Riemann Hypothesis, which asserts that all nontrivial zeros lie on the critical line $Re(s) = \frac{1}{2}$. Also, Riemann's work laid the groundwork for a proof strategy concerning Gauss's conjecture on the density of primes. Riemann's extension of the zeta function and his insights into its analytic properties have become crucial to number theory and complex analysis.

2. Riemann Zeta Function

The following is the definition provided by Riemann for the Riemann zeta function. Let $s = \sigma + it$ be a complex number, where $\sigma = Re(s) > 1$. Then, the Zeta Function is defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

Now,

$$\left|\frac{1}{n^s}\right| = \left|\frac{1}{n^{\sigma+it}}\right| = \left|\frac{1}{n^\sigma} \cdot \frac{1}{n^{it}}\right| = \frac{1}{n^\sigma} \cdot \left|e^{-it\ln(n)}\right| = \frac{1}{n^\sigma} \cdot 1 = \frac{1}{n^\sigma}$$

since the complex number $e^{-i(tln(n))}$ has norm 1.

The series $\sum_{n=1}^{\infty} \frac{1}{n^\sigma}$ converges by the p-series test with $p = \sigma > 1$. Hence, the Riemann Zeta function converges absolutely for $\sigma > 1$.

3. Euler Product Formula

Riemann begins his exploration using a formula proven by Euler:

$$\zeta(s) = \prod_p \frac{1}{1 - p^{-s}} \qquad (Re(s) > 1)$$

This formula is an important result as it relates the Riemann Zeta function to the prime numbers p and implies that the series and infinite sum converge to the same value.

Proof

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots$$

First, we compute $\left(1 - \frac{1}{2^s}\right)\zeta(s)$:

$$\frac{1}{2^s}\zeta(s) = \frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \frac{1}{8^s} \dots$$

so

$$\left(1 - \frac{1}{2^s}\right)\zeta(s) = 1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \cdots$$

Now, all terms divisible by 2 have been cancelled.

Next, we compute $\left(1 - \frac{1}{3^s}\right)\left(1 - \frac{1}{2^s}\right)\zeta(s)$:

$$\left(\frac{1}{3^s}\right)\left(1 - \frac{1}{2^s}\right)\zeta(s) = \frac{1}{3^s} + \frac{1}{9^s} + \frac{1}{15^s} + \frac{1}{21^s} + \cdots$$

$$\left(1 - \frac{1}{3^s}\right)\left(1 - \frac{1}{2^s}\right)\zeta(s) = 1 + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{11^s} + \cdots$$

Now, all terms divisible by both 2 and 3 have been cancelled.

We now compute $\left(1 - \frac{1}{5^s}\right)\left(1 - \frac{1}{3^s}\right)\left(1 - \frac{1}{2^s}\right)\zeta(s)$:

$$\left(\frac{1}{5^s}\right)\left(1 - \frac{1}{3^s}\right)\left(1 - \frac{1}{2^s}\right)\zeta(s) = \frac{1}{5^s} + \frac{1}{25^s} + \frac{1}{35^s} + \frac{1}{55^s} + \cdots$$

$$\left(1 - \frac{1}{5^s}\right)\left(1 - \frac{1}{3^s}\right)\left(1 - \frac{1}{2^s}\right)\zeta(s) = 1 + \frac{1}{7^s} + \frac{1}{11^s} + \frac{1}{13^s} + \cdots$$

Again, all terms divisible by 2, 3, and 5 have been cancelled.

If this method is used iteratively, we are only left with a 1 on the right-hand side:

$$\ldots\left(1-\frac{1}{5^s}\right)\left(1-\frac{1}{3^s}\right)\left(1-\frac{1}{2^s}\right)\zeta(s)=1$$

$$\prod_p\left(1-\frac{1}{p^s}\right)\zeta(s)=1$$

$$\prod_p\left(1-\frac{1}{p^s}\right)=\frac{1}{\zeta(s)}$$

$$\zeta(s)=\prod_p\left(1-\frac{1}{p^s}\right)^{-1}$$

$$\zeta(s)=\prod_p\frac{1}{1-p^{-s}}\quad Re(s)>1$$

□

The formula also reveals something profound about the number of prime numbers. Consider $\zeta(1)=\sum_{n=1}^{\infty}\frac{1}{n}$ which is the harmonic series and is known to diverge. However, each factor $\frac{1}{1-p^{-1}}$ is finite, so for the entire product $\prod_p\frac{1}{1-p^{-s}}$ to diverge, there must be infinitely many such factors. In other words, the divergence of the harmonic series implies that there must be infinitely many prime numbers for the Euler product formula to hold.

4. Analytic Continuation

The Riemann Zeta function is defined over the domain $Re(s)>1$ because otherwise the series does not converge. However, using analytic continuation, it is possible to extend this definition to the entire complex plane except at $s=1$. While the idea of extending the domain of a function beyond its region of convergence existed before Riemann, he was the first to rigorously formalize this concept in the context of complex analysis.

Definition (Analytic Function): A function f is called analytic on an open set if it is differentiable everywhere in that set.

Definition (Analytic Continuation): Let f_1 be an analytic function defined on a complex domain D_1, and let f_2 be an analytic function defined on another complex domain D_2. Suppose the domains overlap ($D_1\cap D_2\neq\emptyset$), and on this overlapping region, the functions are equal: $f_1(z)=f_2(z)\ \forall z\in D_1\cap D_2$. Then, f_2 is called an **analytic continuation** of f_1, and we can extend f_1 to the larger domain $D_1\cup D_2$ by defining it to be equal to f_2: $f_1=f_2\ \forall z\in D_1\cup D_2$.

To understand the concept of analytic continuation, we can consider a simple example:

$$f(z)=1+z+z^2+z^3+\cdots$$

This series converges in the disk $D = \{z \in \mathbb{C}: |z| < 1\}$ and defines an analytic function within this domain.

We already are familiar with the formula

$$f(z) = \frac{1}{z-1}$$

for $z \in D$, which describes the convergent sum of the geometric series.

Notice that the function $\frac{1}{z-1}$ is defined on a much larger set of z values.

Hence, we can define a new analytic function $F(z) = \frac{1}{z-1}$ on the domain $D' = \mathbb{C}/\{1\}$. In this case, $F(z)$ is the analytic continuation of $f(z)$.

Analytic Continuation of $\zeta(s)$

We now consider an analytic continuation of the Riemann Zeta function.

Consider the alternating zeta function: $Z(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} = 1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \cdots$

Then, $\frac{Z(s)}{1-2^{1-s}}$ is an analytic continuation of $\zeta(s)$ to the domain $\{s \in \mathbb{C}: Re(s) > 0 \wedge s \neq 1\}$.

Proof

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} \ldots$$

$$\zeta(s) = 1 + \left(\frac{2}{2^s} - \frac{1}{2^s}\right) + \frac{1}{3^s} + \left(\frac{2}{4^s} - \frac{1}{4^s}\right) + \cdots$$

Now, group the terms in the following way:

$$\zeta(s) = \left(1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \cdots\right) + 2\left(\frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \cdots\right)$$

$$\zeta(s) = Z(s) + 2\left(\frac{\zeta(s)}{2^s}\right)$$

$$\zeta(s) = Z(s) + \frac{\zeta(s)}{2^{s-1}}$$

$$\zeta(s)\left(1 - \frac{1}{2^{s-1}}\right) = Z(s)$$

$$\zeta(s) = \frac{Z(s)}{1 - \frac{1}{2^{s-1}}}$$

$$\zeta(s) = \frac{Z(s)}{1-2^{1-s}}$$

$\square$

$\frac{Z(s)}{1-2^{1-s}}$ gives us an analytic continuation of $\zeta(s)$ over the domain $\{s \in \mathbb{C}: Re(s) > 0 \land s \neq 1\}$

However, Riemann's approach to analytic continuation extends the domain much further, specifically to the entire complex plane, excluding $s = 1$. To achieve this, we must consider Euler's Gamma function.

5. Euler's Gamma Function

In the 1720s, mathematicians Daniel Bernoulli and Christian Goldbach proposed the problem of extending the factorial function to non-integer values. Euler presented his solution to the problem by introducing the Gamma Function or Euler Integral:

$$\Gamma(z) = \int_0^\infty e^{-t}\, t^{z-1} dt, \quad Re(z) > 0$$

where $z = \sigma + it$ is a complex number.

This function extends the definition of factorials to complex numbers.

When $s = n + 1$ with $n > 0$ and n is a natural number, we can show that this formula gives $n!$ using integration by parts.

Proof

We start with

$$\boxed{\begin{aligned} u &= t^n \\ du &= nt^{n-1}dt \\ dv &= e^{-t}dt \\ v &= -e^{-t} \end{aligned}}$$

$$\int_0^\infty e^{-t}\, t^{z-1} dt = \int_0^\infty e^{-t}\, t^n dt$$

$$= (-t^n e^{-t})|_0^\infty + n\int_0^\infty e^{-t} t^{n-1} dt$$

Now, $-t^n e^{-t} = -\frac{t^n}{e^t}$ and $\lim_{t\to\infty} \frac{-t^n}{e^t} = \lim_{t\to\infty} \frac{-n!}{e^t} = 0$ (using L'Hopital's Rule).

Hence,$(-t^n e^{-t})|_0^\infty$ simplifies to 0 and we get

$$\boxed{\begin{aligned} u &= t^{n-1} \\ du &= (n-1)t^{n-2}dt \\ dv &= e^{-t}dt \\ v &= -e^{-t} \end{aligned}}$$

$$= n\int_0^\infty e^{-t} t^{n-1} dt$$

$$= n\left[(-t^{n-1}e^{-t})|_0^\infty + (n-1)\int_0^\infty e^{-t} t^{n-2} dt\right]$$

Now, again $(-t^{n-1}e^{-t})|_0^\infty = 0$, so we get

$$= n(n-1)\int_0^\infty e^{-t}t^{n-2}dt$$

Now, if we continue applying this process, we will ultimately get

$$= n(n-1)(n-2)\cdots 1 \cdot \int_0^\infty e^{-t}dt$$

$$= n!\,(-e^{-t})|_0^\infty$$

$$= n!\,(1) = n!$$

Therefore, $\int_0^\infty e^{-t}\,t^n dt = n!$ which proves that $\Gamma(n+1) = n!$

$\square$

Functional Equation

The zeta function satisfies the functional equation:

$$\zeta(s) = 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s)$$

This equation helps us define the Riemann zeta Function over the entire complex plane.

6. Zeros and the Riemann Hypothesis

Trivial Zeros

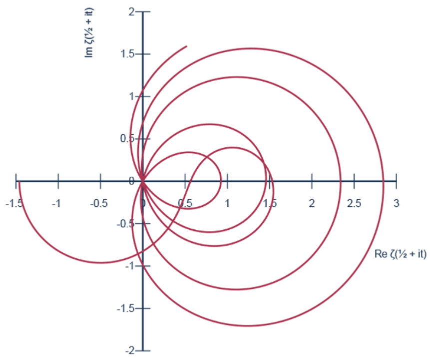
The trivial zeros of the Riemann zeta function occur at negative even integers. This can be demonstrated by analyzing the functional equation from earlier.

When $s = -2n$ where $n \in \mathbb{N}$, we get $\sin\left(\frac{\pi\cdot -2n}{2}\right) = \sin(-\pi n) = 0$, so $\zeta(-2n) = 0$. Thus, the trivial zeros of $\zeta(s)$ are at $s = -2, -4, -6, \ldots$. Note that for even positive integers, the zeta function is not zero, and the original definition of $\zeta(s)$ can be used to show this.

Riemann Hypothesis

The Riemann Hypothesis concerns the nontrivial zeros of $\zeta(s)$. Riemann conjectured that the nontrivial zeros of $\zeta(s)$ lie on the critical line in the complex plane which is where the real part of the complex numbers is $\frac{1}{2}$. In other words, the hypothesis implies that the nontrivial zeros of $\zeta(s)$ are of the form $s = \frac{1}{2} + it$.

Consider the graph below of $\zeta\left(\frac{1}{2}+it\right)$ plotted in polar coordinates. Each time the graph passes through the origin (meaning both the real and imaginary parts are zero) it represents a nontrivial zero of the Riemann zeta function.



This has yet to be proven. The Riemann Hypothesis is one of the seven Millennium Prize Problems for which the Clay Mathematics Institute has offered a prize of one million dollars to anyone who can provide a correct proof. Proving this hypothesis is crucial because it would allow for a more accurate estimate of the number of primes less than or equal to an integer x with a smaller margin of error.

7. Evaluating the Zeta Function

Basel Problem

The challenge of evaluating $\zeta(2)=\sum_{n=1}^{\infty}\frac{1}{n^2}$ in a closed form is known as the **Basel Problem.** It was first introduced in 1644 by Italian mathematician Pietro Mengoli. The problem gained more attention later in the 1600s, thanks to Jacob Bernoulli from the University of Basel. However, it wasn't until 1735 that Euler found the exact value of the sum, showing that $\zeta(2)=\frac{\pi^2}{6}$. The proof of this result is shown below.

Proof

First, we look at the Taylor series expansion for $\sin x$

$$\sin(x)=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots$$

Now, we use the **Weierstrass Factorization Theorem**, which states that any function that is analytic over the entire complex plane (entire) can be written as a product involving its zeros.

The function $\sin(x)$ is entire and its zeros occur at $x=0,\pm\pi,\pm 2\pi,\pm 3\pi,\ldots$

Using the theorem, we can write $\sin(x)$ as

$$\sin(x) = x\left(1 - \frac{x}{\pi}\right)\left(1 + \frac{x}{\pi}\right)\left(1 - \frac{x}{2\pi}\right)\left(1 + \frac{x}{2\pi}\right)\cdots$$

$$\sin\,(x) = x\left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{4\pi^2}\right)\cdots$$

Now, we examine the x^2 terms in the expansion.

$$\sin\,(x) = x\left[\ldots - \frac{1}{\pi^2}x^2\left(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots\right)\ldots\right]$$

After distributing the x, we get

$$\sin(x) = \cdots - \frac{1}{\pi^2}x^3\sum_{n=1}^{\infty}\frac{1}{n^2} + \cdots$$

The coefficient of x^3 from the Taylor series expansion of $\sin\,(x)$ is $-\frac{1}{3!} = -\frac{1}{6}$.

Thus, we can set the coefficients of x^3 equal to each other

$$-\frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2} = -\frac{1}{6}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$

$\square$

Evaluating $\boldsymbol{\zeta(-1)}$

The value of $\zeta(-1)$ was first suggested by Euler in the 18th century. However, it was Riemann's concept of analytic continuation that made this result rigorously valid.

To compute $\zeta(-1)$, we start with the original definition of the zeta function

$$\sum_{n=1}^{\infty}\frac{1}{n^{-1}} = \sum_{n=1}^{\infty} n = 1 + 2 + 3 + \cdots$$

This sum diverges.

However, we can still evaluate $\zeta(-1)$ using the functional equation

$$\zeta(-1) = 2^{-1}\pi^{-1-1}\sin\left(\frac{\pi\cdot -1}{2}\right)\Gamma(1+1)\zeta(1+1)$$

$$\zeta(-1) = \frac{1}{2\pi^2}\sin\,(-\frac{\pi}{2})\Gamma(2)\zeta(2)$$

$$\zeta(-1) = \frac{1}{2\pi^2}(-1)(1!)\left(\frac{\pi^2}{6}\right)$$

$$\zeta(-1) = \frac{-1}{12}$$

Now if we consider the definition of $\zeta(-1)$ through its infinite summation form

$$1 + 2 + 3 + \cdots = \frac{-1}{12}$$

This is truly a surprising statement! Both Euler and Ramanujan gave their own derivations of this result. However, it's important to clarify that this does not mean adding up all the positive integers will somehow give us $-\frac{1}{12}$. Instead, the equal sign here functions more like an assignment than a traditional equation. While the series $1 + 2 + 3 + \cdots$ is divergent in the usual sense, analytic continuation of the Riemann zeta function allows us to assign it the finite value $-\frac{1}{12}$. This shows how analytic continuation can extend the concept of summation, giving meaningful values to otherwise divergent series.

8. Conclusion

The study of the Riemann Zeta function has given rise to powerful mathematical techniques, such as analytic continuation, which allows us to extend the domain of functions beyond their original limits. It has also led to astonishing results, such as $1 + 2 + 3 + \cdots = -\frac{1}{12}$, which challenges conventional intuition. Riemann's insights and analysis of this function were instrumental in laying the foundation for the Prime Number Theorem, which describes the asymptotic distribution of prime numbers. His work revolutionized how we understand infinite series and complex functions. It also continues to inspire deep inquiry into one of the most important unsolved problems in mathematics: the Riemann Hypothesis. Indeed, Riemann's legacy will continue to inspire and influence generations of mathematicians to come.

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