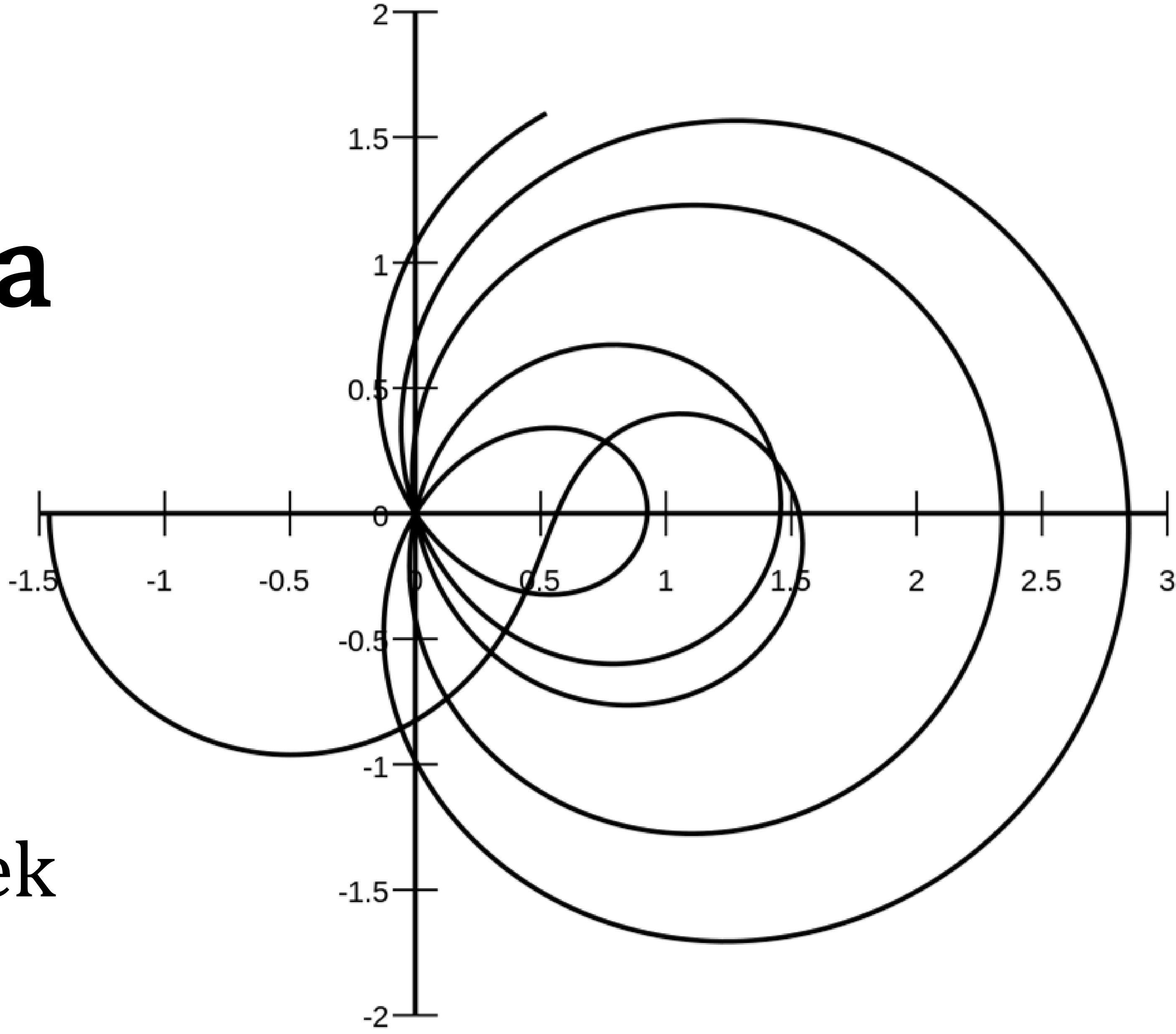
The Riemann Zeta Function

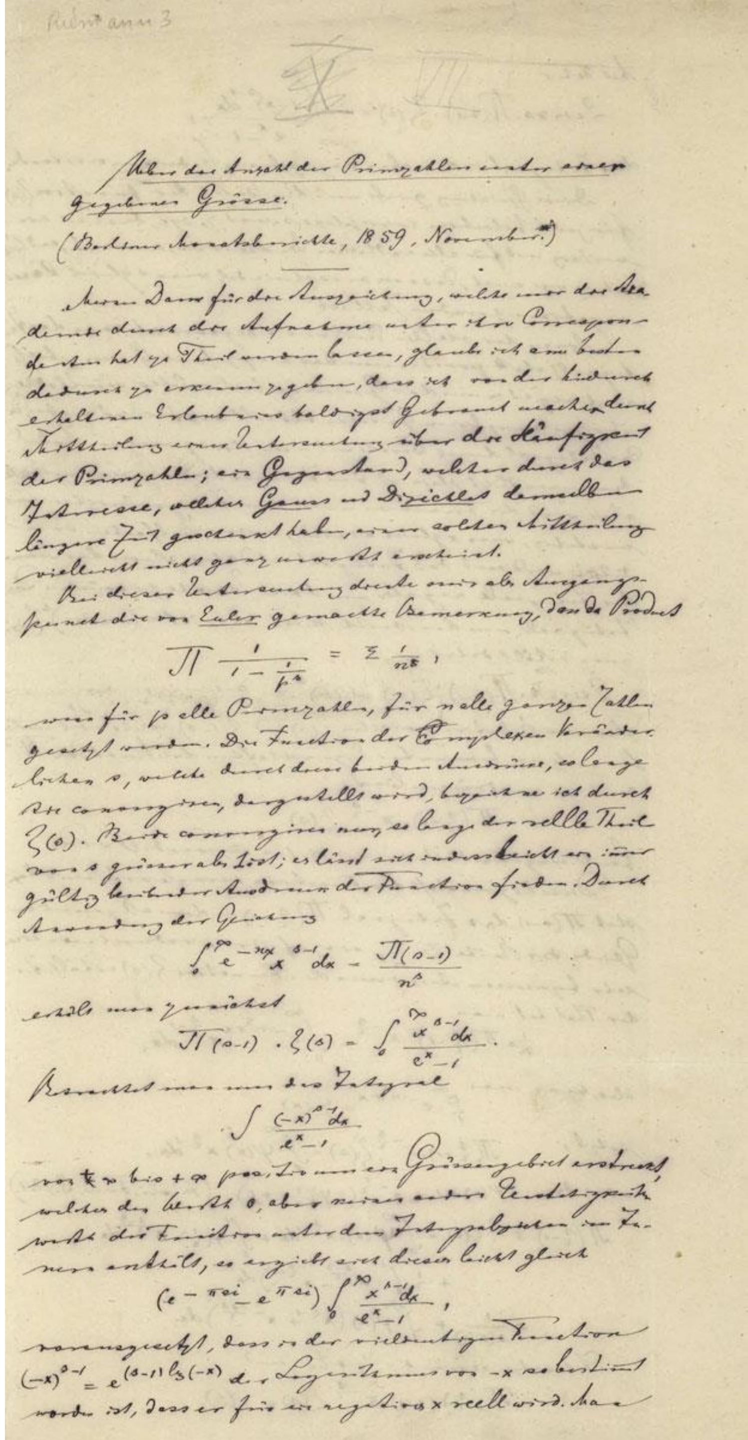
Tasnia Kader

Course: MATH 3138
Professor: Dr. Jeromy Sivek



Agenda

- Definition
- Analytic Continuation
- Functional Equation
- Zeros of $\zeta(s)$
- Basel Problem
- $\zeta(-1)$


Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse.
(Berliner Monatsberichte, 1859, November)

Bernhard Riemann
(1826 – 1866)

Definition

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots$$

where $s = \sigma + ti$

$$Re(s) \ > \ 1$$

Analytic Continuation

If

$$D_1, D_2\text{: domains}, D_1 \cap D_2 \neq \emptyset$$

$$f_1\text{: analytic on } D_1, f_2\text{: analytic on } D_2$$

$$f_1 = f_2 \text{ on } D_1 \cap D_2$$

Then, f_2 is called an **analytic continuation** of f_1

Extend f_1 to larger domain: $f_1 = f_2 \; \forall z \in D_1 \cup D_2$

Example of Analytic Continuation

$$f_1(z) = 1 + z + z^2 + z^3 + \cdots$$

Converges for $D_1 = \{z \in \mathbb{C}: |z| < 1\}$

$$f_2(z) = \frac{1}{1 - z}$$

Defined for $D_2 = \{z \in \mathbb{C}: z \neq 1\}$

$f_1 = f_2$ on $D_1 \cap D_2$

f_2 analytic continuation of f_1

Define $f_1 = f_2 \; \forall z \in D_1 \cup D_2$

Analytic Continuation of $\zeta(s)$

$$Z(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} = 1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \cdots$$

$$\zeta(s) = \frac{Z(s)}{1 - 2^{1-s}}$$

analytic continuation of $\zeta(s)$ to
$\{s \in \mathbb{C}: Re(s) > 0 \wedge s \neq 1\}$

Proof

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots$$

$$\zeta(s) = 1 + \left(\frac{2}{2^s} - \frac{1}{2^s}\right) + \frac{1}{3^s} + \left(\frac{2}{4^s} - \frac{1}{4^s}\right) + \cdots$$

$$\zeta(s) = \left(1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \cdots\right) + 2\left(\frac{1}{2^s} + \frac{1}{4^s} + \cdots\right)$$

$$\zeta(s) = Z(s) + 2\left(\frac{\zeta(s)}{2^s}\right)$$

Proof Cont.

$$\zeta(s) = Z(s) + 2\left(\frac{\zeta(s)}{2^s}\right)$$

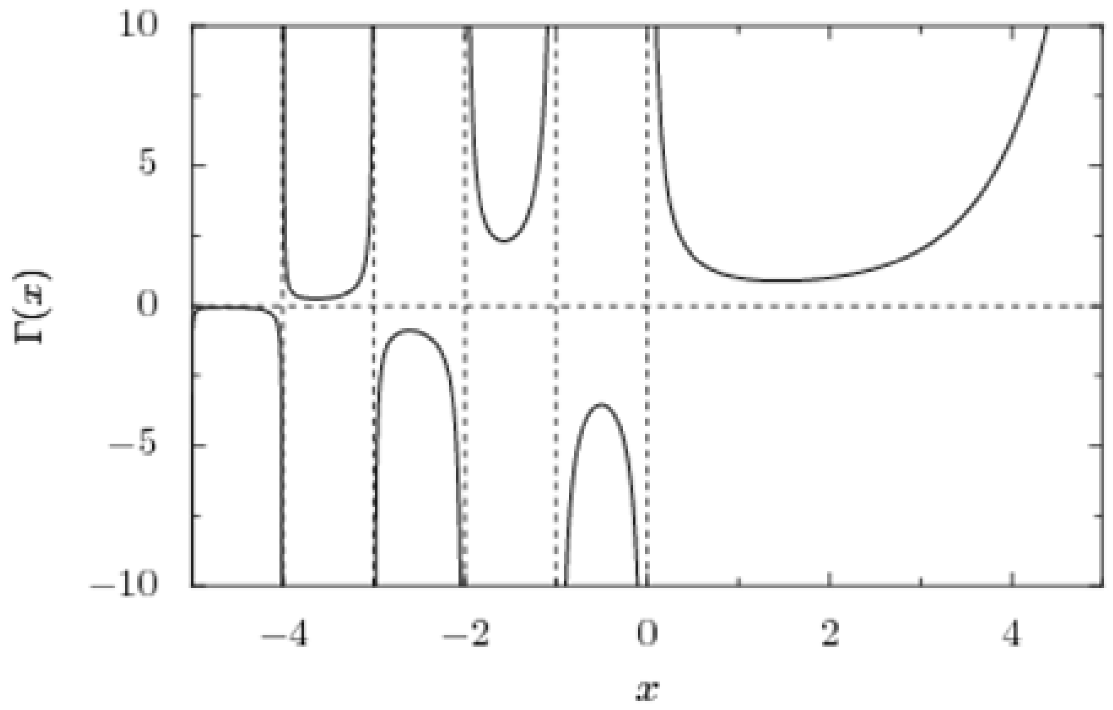
$$\zeta(s) = Z(s) + \frac{\zeta(s)}{2^{s-1}}$$

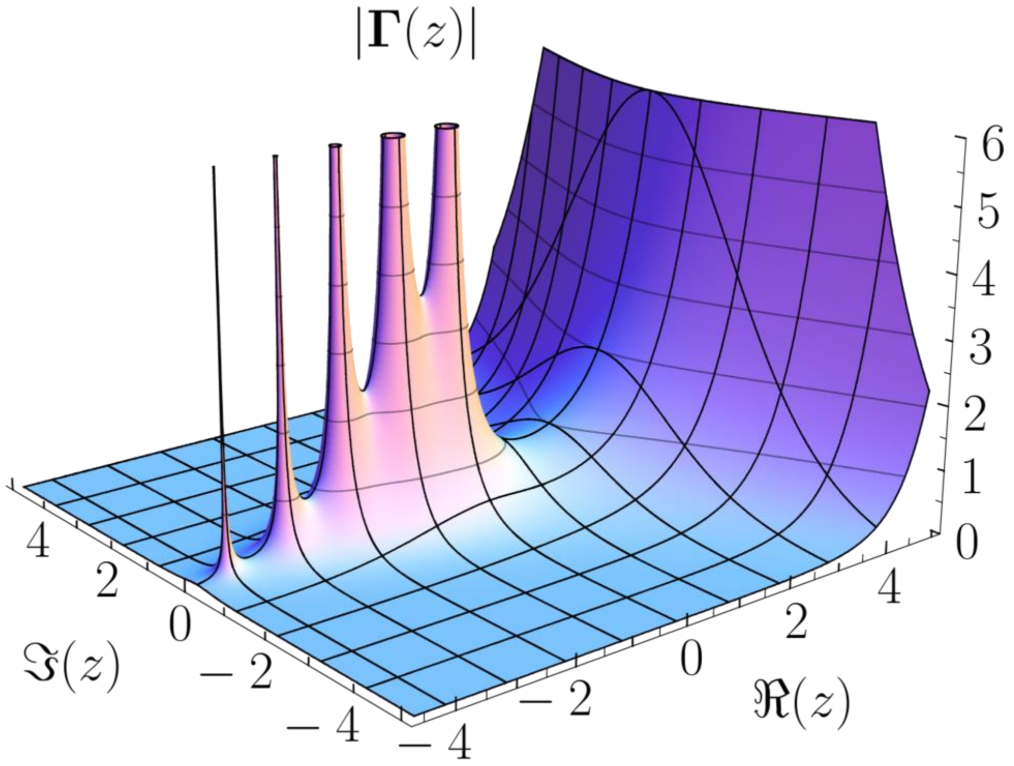
$$\vdots$$

$$\zeta(s) = \frac{Z(s)}{1 - 2^{1-s}}$$

Gamma Function Γ

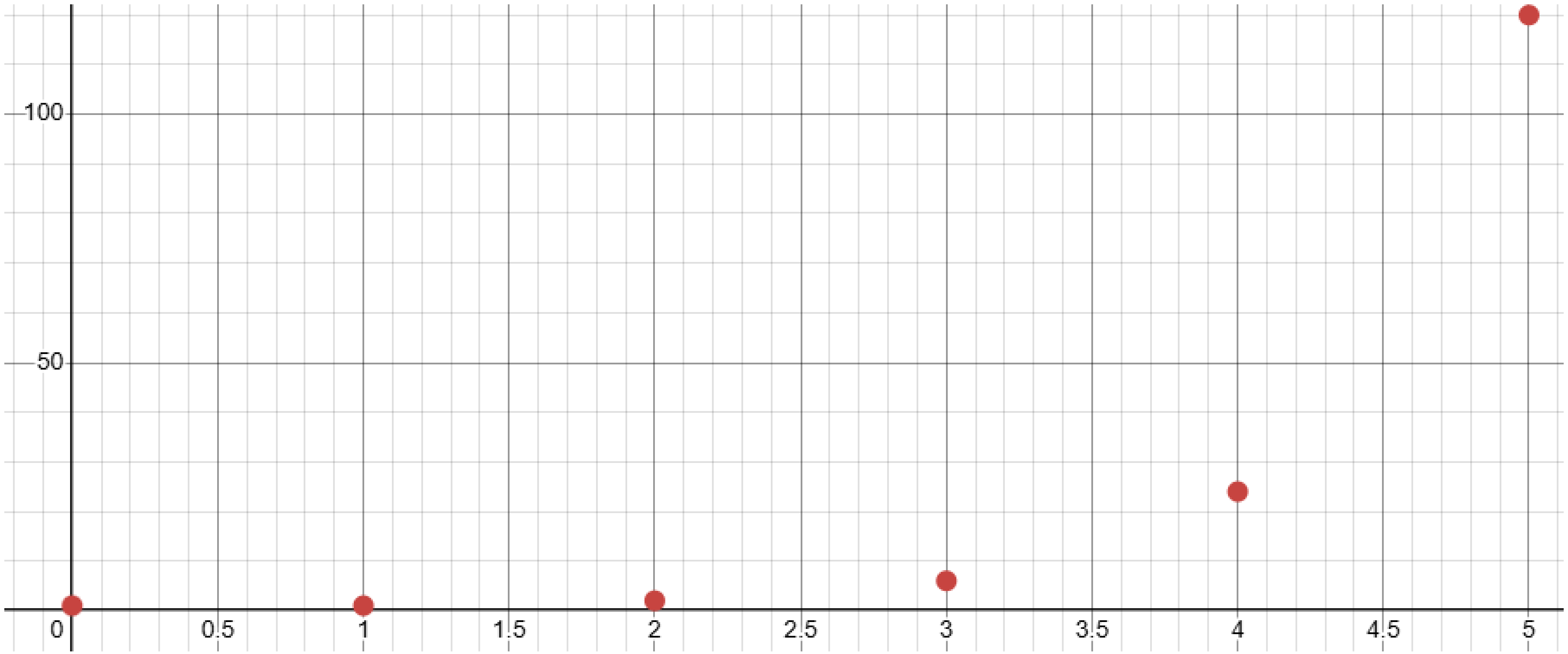
$$\Gamma(s) = \int_0^\infty e^{-t}\, t^{s-1} dt \qquad Re(s) > 1$$





Gamma Function Γ

$$\Gamma(n+1) = n! \quad \text{where } n \in \mathbb{Z}_{\geq 0}$$



Functional Equation of $\zeta(s)$

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s)$$

defined for all complex s, except $s = 1$

Zeros of $\zeta(s)$

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s)$$

Let $s = -2n,\ n \in \mathbb{Z}^+$

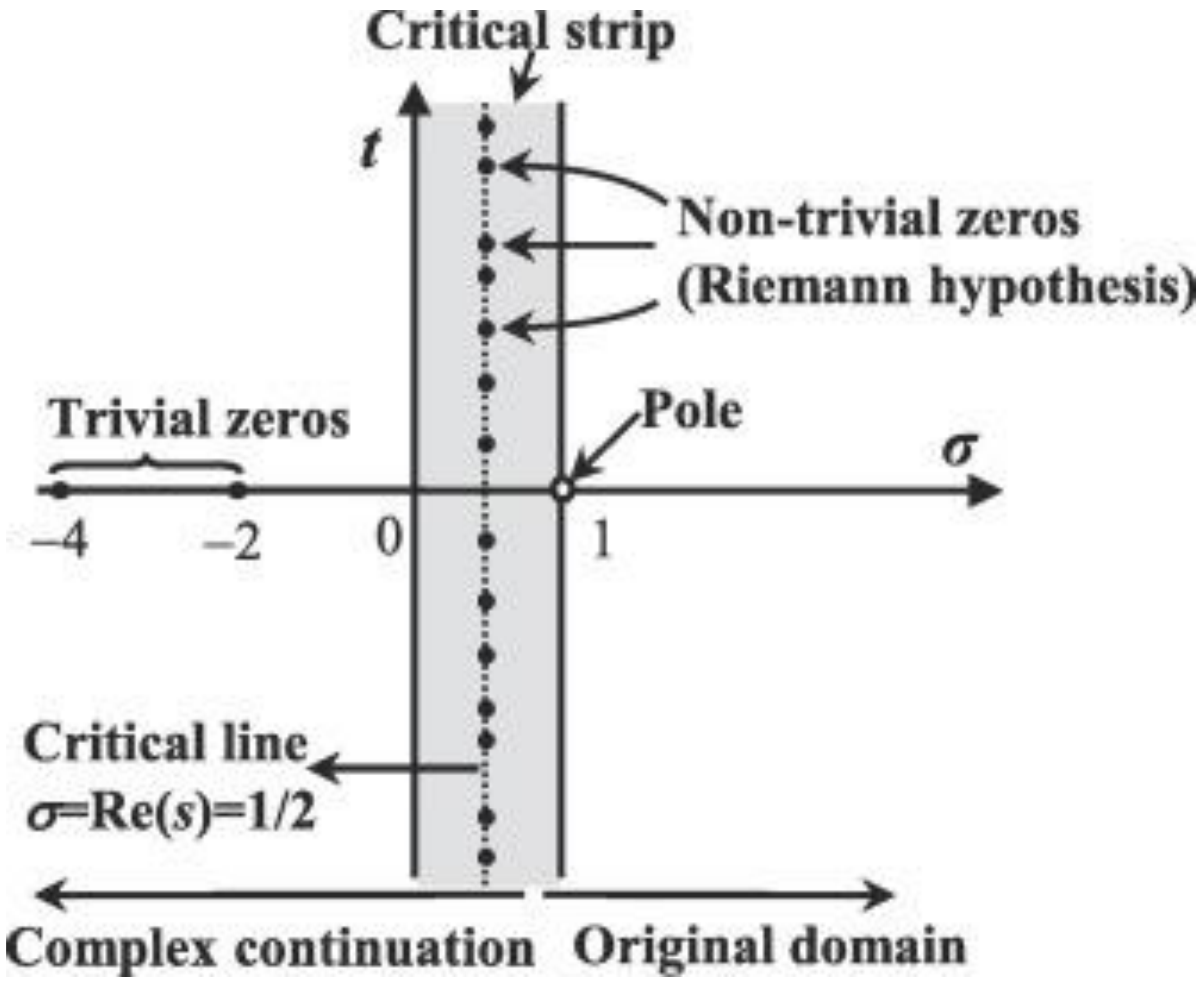
$$\sin\left(\frac{\pi \cdot -2n}{2}\right) = \sin(-\pi n) = 0$$

so $\zeta(-2n) = 0$

$s = -2n$ known as **trivial zeros**

Riemann Hypothesis

The **nontrivial zeros** of $\zeta(s)$ have real part equal to $\frac{1}{2}$



Basel Problem $\zeta(2)$

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

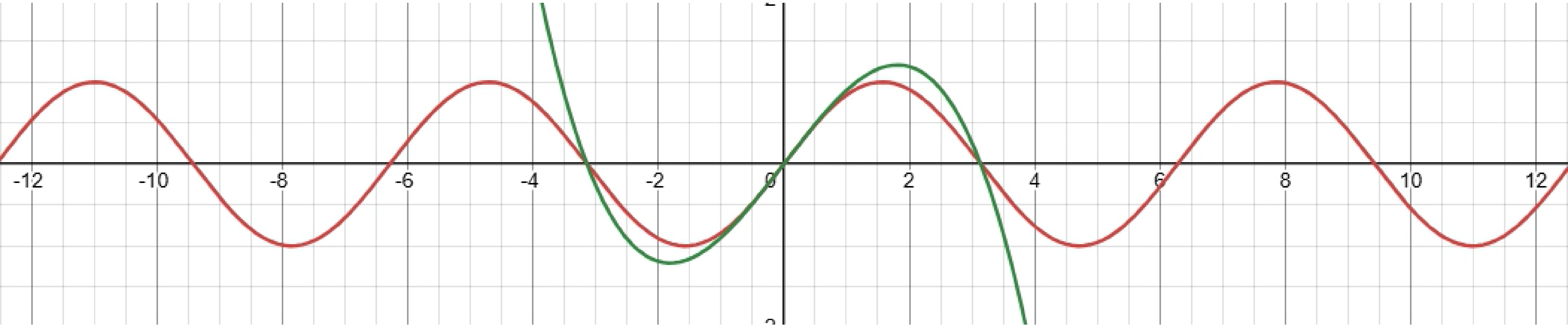
Weierstrass Factorization Theorem

Any function that is analytic over the entire complex plane can be written as a product involving its zeros

Weierstrass Factorization Theorem

$\sin(x)$ is analytic over the entire complex plane
zeros of $\sin(x)$ occur at $0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$

$$\sin(x) = x\left(1 - \frac{x}{\pi}\right)\left(1 + \frac{x}{\pi}\right)\left(1 - \frac{x}{2\pi}\right)\left(1 + \frac{x}{2\pi}\right)\cdots$$



Basel Problem $\zeta(2)$ Cont.

$$\sin(x) = x\left(1 - \frac{x}{\pi}\right)\left(1 + \frac{x}{\pi}\right)\left(1 - \frac{x}{2\pi}\right)\left(1 + \frac{x}{2\pi}\right)\cdots$$

$$\sin(x) = x\left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{4\pi^2}\right)\cdots$$

$$\sin(x) = x\left[\ldots - \frac{x^2}{\pi^2} - \frac{x^2}{4\pi^2} - \frac{x^2}{9\pi^2} - \cdots\right]$$

Basel Problem $\zeta(2)$ Cont.

$$\sin(x) = x\left[\dots - \frac{x^2}{\pi^2} - \frac{x^2}{4\pi^2} - \frac{x^2}{9\pi^2} - \cdots\right]$$

$$\sin(x) = \cdots - \frac{1}{\pi^2}x^3\left(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots\right)\cdots$$

$$\sin(x) = \cdots - \frac{1}{\pi^2}x^3\sum_{n=1}^{\infty}\frac{1}{n^2}\dots$$

Basel Problem $\zeta(2)$ Cont.

$$\sin(x) = \cdots - \frac{1}{\pi^2} x^3 \sum_{n=1}^{\infty} \frac{1}{n^2} \ldots$$

$$-\frac{1}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} = -\frac{1}{6}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

$\boldsymbol{\zeta(-1)}$

$$\sum_{n=1}^{\infty} \frac{1}{n^{-1}} = \sum_{n=1}^{\infty} n = 1 + 2 + 3 + \cdots$$

$\boldsymbol{\zeta(-1)}$ Cont.

$$\zeta(-1) = 2^{-1}\pi^{-1-1}\sin\left(\frac{\pi \cdot -1}{2}\right)\Gamma(1-(-1))\zeta(1-(-1))$$

$$\zeta(-1) = \frac{1}{2\pi^2} \cdot -1 \cdot \Gamma(2) \cdot \zeta(2)$$

$$\zeta(-1) = \frac{1}{2\pi^2} \cdot -1 \cdot 1! \cdot \zeta(2)$$

$\boldsymbol{\zeta(-1)}$ Cont.

$$\zeta(-1) = \frac{1}{2\pi^2} \cdot -1 \cdot 1! \cdot \zeta(2)$$

$$\zeta(-1) = -\frac{1}{2\pi^2} \cdot \zeta(2)$$

$$\zeta(-1) = -\frac{1}{2\pi^2} \cdot \frac{\pi^2}{6}$$

$\boldsymbol{\zeta(-1)}$ Cont.

$$\zeta(-1) = -\frac{1}{2\pi^2} \cdot \frac{\pi^2}{6}$$

$$\zeta(-1) = -\frac{1}{12}$$

$$1 + 2 + 3 + \cdots = -\frac{1}{12}$$

$$1 + 2 + 3 + \cdots = -\frac{1}{12}$$

References

https://math.iisc.ac.in/seminar-slides/2019/2019-12-18-KenOno-1.pdf

https://math.uchicago.edu/~may/REU2014/REUPapers/Moros.pdf

https://www.math.ucdavis.edu/~temple/MAT21B/Riemann-Biography.pdf

http://sections.maa.org/epadel/awards/studentpaper/winners/2011_Yoder.pdf

https://www.math.cmu.edu/~bwsulliv/basel-problem.pdf

Thank you!

Any Questions?